amit

X-932-74-43
PREPRINT

NASA TM X- 70610

ESTIMATION STRATEGIES FOR ORBIT DETERMINATION OF APPLICATIONS SATELLITES

P. ARGENTIERO
J. J. LYNN

(NASA-TM-X-70610) ESTIMATION STRATEGIES FOR ORBIT DETERMINATION OF APPLICATIONS SATELLITES (NASA) 27 p HC $4.50 CSCL 22C

N74-19501
Unclas
G3/30 32730

FEBRUARY 1974

GSFC — GODDARD SPACE FLIGHT CENTER —
GREENBELT, MARYLAND

For information concerning availability
of this document contact:

Technical Information Division, Code 250
Goddard Space Flight Center
Greenbelt, Maryland 20771

(Telephone 301-982-4488)

X-932-74-43

ESTIMATION STRATEGIES FOR ORBIT DETERMINATION OF APPLICATIONS SATELLITES

P. Argentiero
Goddard Space Flight Center

J. J. Lynn
Old Dominion Systems Inc.

February 1974

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

ESTIMATION STRATEGIES FOR ORBIT DETERMINATION OF APPLICATIONS SATELLITES

P. Argentiero
Goddard Space Flight Center

J. J. Lynn
Old Dominion Systems Inc.

ABSTRACT

A procedure for applying covariance analysis to determine the most efficient estimation strategy for satisfying the stringent mission requirements of long arc orbit determination of applications satellites is presented. The procedure is applied to the problem of satisfying mission requirements with respect to altitude determination of GEOS-C. It is shown that requirements are met when twelve dominant geopotential coefficients are estimated along with satellite state. This application of covariance analysis is general and can be applied to future applications satellites. Recommendations for future studies are also given.

CONTENTS

ILLUSTRATIONS

PRECEDING PAGE BLANK NOT FILMED

ESTIMATION STRATEGIES FOR ORBIT DETERMINATION OF APPLICATIONS SATELLITES

INTRODUCTION

The use of satellite based observations for earth and ocean physics applications has resulted in unprecedented demands for orbit determination accuracy. For example, the GEOS-C satellite, scheduled for launch in late 1974, will be equipped with an altimeter capable of one meter precision. In order to make effective use of such altimeter data it is advisable to determine the GEOS-C altitude with one meter or better precision. Other applications such as marine position determination and navigation and polar motion determination also call for high accuracy orbit determination. See (1) for details.

This paper presents a procedure for utilizing covariance analysis and propagation techniques to obtain optimal estimation strategies for long arc orbit determination. The procedure is applied to the problem of long arc orbit determination of GEOS-C and it is shown that the resultant accuracies are consistent with the needs of an earth and ocean physics program. These results are significant in themselves with respect to the requirements of the GEOS-C mission and they are also significant in that they suggest what proper estimation strategies can accomplish in determining orbits of future applications satellites.

COVARIANCE ANALYSIS AND OPTIMAL ESTIMATION STRATEGIES FOR LONG ARC ORBIT DETERMINATION

It is convenient to think of the errors in an orbit determination as arising from two distinguishable sources. First, the data which is processed in the orbit determination procedure is corrupted with high frequency noise. The second error source is due to a misrepresentation of parameters in the functional relationship between the estimated state and the data. These parameters fall into two categories—measurement parameters and dynamic parameters. Measurement parameters are involved in defining the geometrical relationship between the satellite at a given instant and an observation of the satellite obtained at that instant. The most significant measurement parameters are observation biases and station location coordinates. Dynamic parameters are used in defining the relationship between the state of the satellite at one instant and the state of the satellite at a later instant. The most commonly mentioned dynamic parameters are those which determine atmospheric drag and solar pressure and the coefficients of the spherical harmonic expansion of the geopotential field. The standard deviation of the estimate of a given component of satellite state is the root sum square of a contribution due to data noise, a contribution due to uncertain measurement parameters, and a contribution due to uncertain dynamic parameters. To obtain an intelligent estimation strategy for determining a given orbit it is necessary to know the major contributions to orbit estimation error. If for instance the dominant error source for a given orbit and a given tracking configuration were due to data noise, one could recommend the use of a more accurate data type or the addition of extra tracking stations as the effective way to improve the orbital estimate. If, however, the dominant error source were due to uncertain parameters the addition of more data would be of little use. In fact, the error contribution due to data noise is seldom significant in orbit determination problems. Generally for short arc orbit determinations, uncertain measurement parameters provide the dominant error source and for long arc orbit determinations, uncertain dynamic parameters are the major contributors. See, for instance (2) for a study of this phenomenon.

How can one determine which misrepresented parameters are significant sources of error for a specific orbit determination problem? The significance of a given parameter as an error source, is a function both of its analytical impact on the functional relationship between the data and the estimated state and of the extent to which the parameter is likely to be misrepresented. Both factors must be taken into account. This can be done by simulations in the following manner: First simulate data for a specific orbit determination problem using the nominal value of the parameter under investigation. Next, process the data in an orbit determination program taking care that the same model used to simulate the data is used in the program with the exception that the given parameter be perturbed by what is believed to be the standard deviation of its estimate.

The resultant estimate is then propagated and compared at various time points with the assumed true orbit. The result is a time history of the effect of a one sigma perturbation of the parameters on the estimate of satellite state. Obviously this procedure can be very expensive. Covariance techniques provide a better method of obtaining the same end. The difference in approach between a simulation study and a covariance analysis can be described as follows: in a simulation, data are generated and a least squares adjustment process is actually performed. The estimated state is then compared to an assumed true state and conclusions are drawn. In a covariance analysis mode, the least squares adjustment process is postulated rather than actually performed and only its associated covariance matrix is computed. Sophisticated covariance analysis software permits one to list at any point in an orbit the contribution to the error in each component of state due to each uncertain parameter and at a small fraction of the cost that would be incurred in obtaining the same information by means of simulations. The mathematical details of this procedures are provided in the appendix.

Granted that covariance techniques are effective in identifying significant error sources in an orbit determination process, what is to be done once this information is obtained? The straightforward approach is to place in an estimation mode along with satellite state all those parameters whose uncertainties are significant error sources. Another procedure which is often more sparing in terms of computer time is the so-called "lumped parameter approach." With this approach one invents an essentially fictitious set of parameters whose combined error signature in the data is similar to that which is left by the combined effect of the authentic misrepresented parameters. If this is possible then by placing the fictitious parameters in an estimation mode the errors produced by the authentic misrepresented parameters will be "absorbed" by the fictitious parameters and a good orbit determination will be the result. And if the fictitious parameters are much less in number than the authentic parameters much is gained in terms of computational efficiency. See (3), (4) and (5) for interesting applications of this technique.

Without doubt the dominant error source in long arc orbit determination of an applications type satellite will be the misrepresentation of geopotential coefficients. To find a small number of fictitious parameters whose combined error signature resembles that left by misrepresented geopotential terms is far from a trivial task. Consequently the authors have opted for the straightforward approach to long arc orbit determination in which they utilize covariance techniques to identify those geopotential terms whose uncertainties are significant error sources in the orbit determination process and then place these terms in an estimation mode along with satellite state. Properly weighted a priori estimates of the estimated geopotential coefficients are assumed in the process in order to avoid the normal matrix ill-conditioning which otherwise may well result when several geopotential coefficients are estimated from tracking data associated with a single satellite. In the next section the procedure is utilized in developing estimation strategies for long arc orbit determination of the GEOS-C satellite.

ESTIMATION STRATEGIES FOR LONG ARC ORBIT DETERMINATION OF GEOS-C

The results of an error analysis of GEOS-C long arc and short arc orbit determination are reported in (2). In that study it was determined that in the long arc case the only significant errors are due to uncertainties in geopotential coefficients. Hence the most direct way to improve the orbital estimate is to place selected geopotential coefficients in an estimation mode along with satellite state. To determine the most efficient use of computer time we first arrange the geopotential terms in the order of their significance as error sources. Then we investigate the resultant orbit determination accuracy, say, when the first three terms in the list are estimated, the first six terms, etc.

The data postulated for the covariance analysis is collected over a seven day span from nine laser stations. The stations were located at Antigua, Bermuda, Canal Zone, Cape Canaveral, Grand Turk, Goddard, Rosman, Athens, and Carnarvon. The first seven of these stations are located in the Caribbean region. For several passes per day tracking geometry is excellent in this region. Figure 1 demonstrates this geometry for one such pass. Two such passes are obtained in successive orbits. In the next six orbits visibility is poor and then the pattern is repeated. Since the GEOS-C period is 102 min, this implies a strong 13 hour observability pattern. This is the explanation for the fact that from both simulations and covariance analysis the orbital errors are periodic through the entire 7 days tracking interval with period approximately 13 hours (2). Consequently it is adequate to view the orbital errors during any 13 hour interval in order to understand the behavior of the errors for the entire 7 days.

The laser data was assumed to be corrupted by 10 cm white noise and the data acquisition rate was one per minute. The only systematic errors considered were those due to geopotential errors. Attention was focused on those geopotential terms which are capable of producing at least a twenty-five meter perturbation of the GEOS-C orbit. Zonal terms were excluded and this left twenty-four geopotential coefficients which were considered as error sources. The standard deviations of these coefficients are determined by differencing the values of the coefficients in the S.A.O. 69 geopotential with corresponding coefficients in the G.E.M. 4 field. The significance of a geopotential term as an error source in state estimation is, of course, a function of the point in the orbit that one considers. By utilizing mathematical procedures outlined in the appendix it was possible to determine at various points along the orbit the root sum square contribution to the uncertainty in each component of satellite state due to each considered geopotential coefficient. These contributions are called aliasing terms and the geopotential coefficients were ordered according to their maximum altitude aliasing along the seven day arc. The results are displayed in Table 1. Columns one and two contain the order and degree of the geopotential

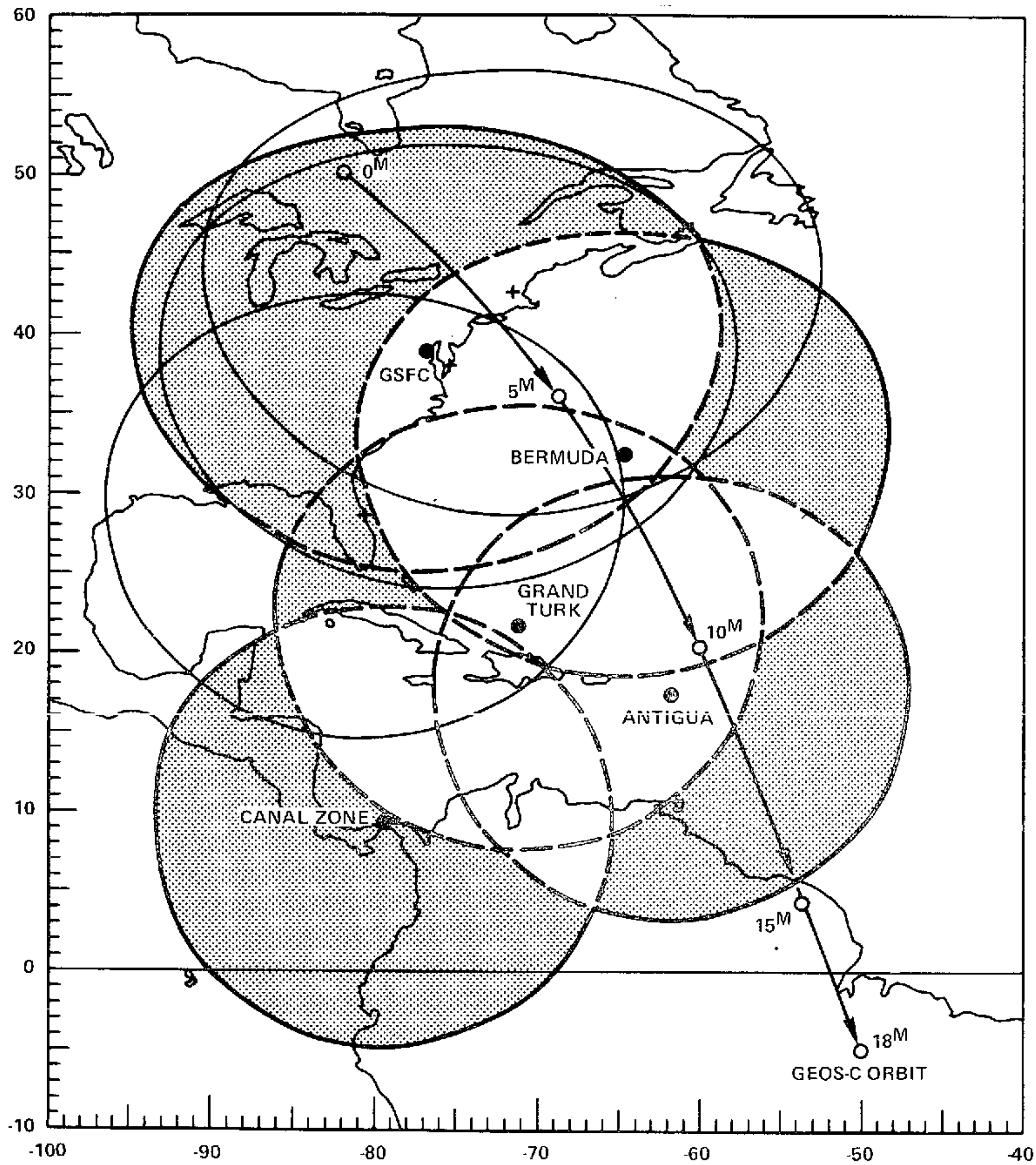

Figure 1. Visibility for GEOS-C

Table 1

Maximum radial, along track, and cross track errors (in meters) of orbit determination of GEOS-C due to uncertainties of dominant geopotential terms.

Geopotential Coefficient	σ	Maximum Height σ	Maximum Along Track σ	Maximum Cross Track σ
S(6,6)	$0.5(10)^{-10}$	4.88	19.72	7.16
C(6,5)	$0.1(10)^{-9}$	3.47	14.12	6.19
C(4,4)	$0.2(10)^{-8}$	3.0	18.15	4.9
C(6,6)	$0.16(10)^{-10}$	1.88	5.62	2.26
C(3,2)	$0.1(10)^{-7}$	1.4	2.9	0.52
C(4,3)	$0.2(10)^{-8}$	1.3	9.0	2.13
S(4,3)	$0.3(10)^{-8}$	1.3	11.00	3.7
S(5,2)	$0.3(10)^{-8}$	0.96	2.18	0.28
S(6,2)	$0.3(10)^{-8}$	0.91	7.20	1.20
C(2,2)	$0.2(10)^{-7}$	0.80	9.9	2.3
S(6,5)	$0.2(10)^{-10}$	0.62	3.76	1.23
S(4,2)	$0.1(10)^{-7}$	0.6	3.6	5.00
S(3,1)	$0.4(10)^{-8}$	0.43	1.0	0.083
C(3,3)	$0.15(10)^{-8}$	0.37	0.96	0.26
C(5,2)	$0.1(10)^{-8}$	0.29	0.65	0.11
S(3,3)	$0.1(10)^{-8}$	0.26	0.64	0.29
C(6,2)	$0.1(10)^{-8}$	0.22	2.2	0.38
S(4,4)	$0.1(10)^{-9}$	0.14	0.66	0.21
S(3,2)	$0.1(10)^{-8}$	0.13	0.27	0.064
C(4,2)	$0.3(10)^{-8}$	0.13	0.96	1.3
C(3,1)	$0.1(10)^{-8}$	0.11	0.22	0.026
C(4,1)	$0.1(10)^{-8}$	0.064	0.51	0.1
S(2,2)	$0.6(10)^{-9}$	0.03	0.009	0.08
S(4,1)	$0.1(10)^{-8}$	0.023	0.23	0.08

coefficients and their associated standard deviations. Columns three, four and five contain the maximum aliasing contributed respectively in the radial, along track, and cross track direction by each term.

It is interesting to examine the aliasing of a geopotential coefficient as it varies along an arc. Figures 2, 3 and 4 are plots of the contributions to the standard deviations of the radial, along track, and cross track components of GEOS-C state due to coefficient S(6,6) for the first fifteen hours after epoch. What is seen at least in the radial and cross track direction is a periodic occilation whose period is the rectification of the orbital period and whose amplitude is modulated by a complex observability pattern. The pattern in the along track direction cannot be interpreted so simply.

Attention is focused on the altitude determination requirements of GEOS-C imposed by its on board altimeter. Since the instrument is capable of one to two meter precision it is required to determine GEOS-C altitude to a standard deviation of one to two meters. If, say, n parameters are to be added to the estimated state in order to improve altitude determination accuracy, then the maximum benefit from the added computational cost is obtained when the estimated parameters are the first n geopotential coefficients in the list of Table 1. Hence the cheapest way to satisfy altitude determination requirements for GEOS-C is to determine how many consecutive terms in the list of Table 1 must be estimated in order to meet the requirements. Using covariance techniques we determined the total standard deviation of altitude estimate for the GEOS-C spacecraft assuming the postulated data was used to estimate state and, no geopotential terms, the first 3 terms of the list, the first 8 terms, the first 12 terms, and finally all 24 terms. The altitude errors were propagated along the entire seven day arc and the percentage of time that the standard deviation was under one meter and under two meters was computed. The results are shown in Table 2. The simultaneous estimation of the twelve dominant geopotential coefficients appears to be a logical strategy for satisfying mission requirements at the smallest possible computation cost. A more detailed picture is provided by Figure 5 which shows the altitude determination standard deviation for the first 15 hours after epoch when no geopotential terms are estimated and when the dominant 12 geopotential terms are estimated. The change in the maximum standard deviation in the along track error is 26 meters when no geopotential terms are estimated and 12 meters when the dominant 12 geopotential errors are estimated. In the case of the maximum standard deviation in the cross track direction the improvement is from 10 meters to 5 meters.

Within the context of this study it was not possible to determine the extent to which the estimates of geopoential coefficients obtained as a byproduct of the orbit determination procedure are physically meaningful. In general, geopotential coefficients of the same or of adjacent degrees are highly non orthogonal

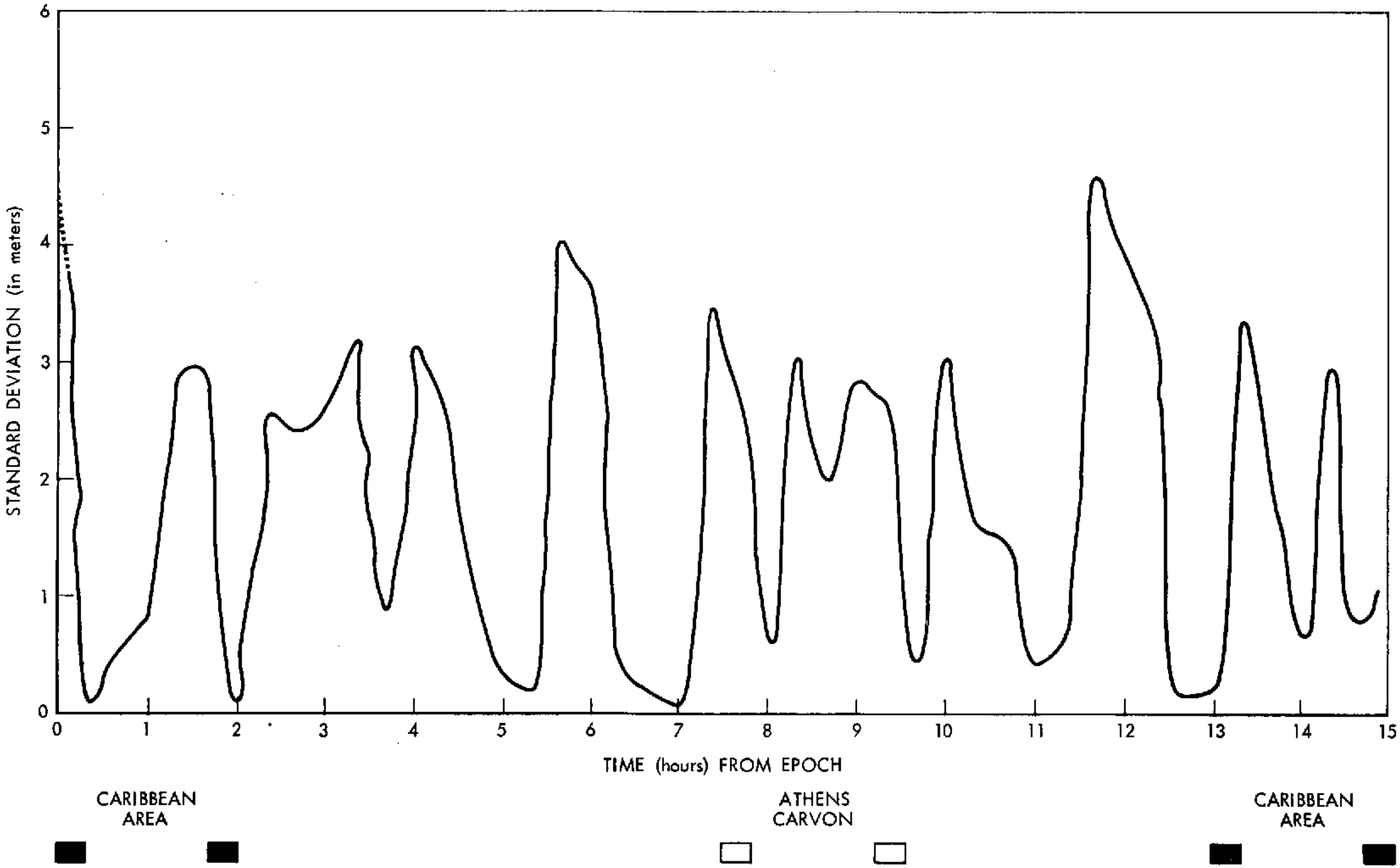

Figure 2. Contribution to Standard Deviation of Radial Component of GEOS-C due to S(6,6)

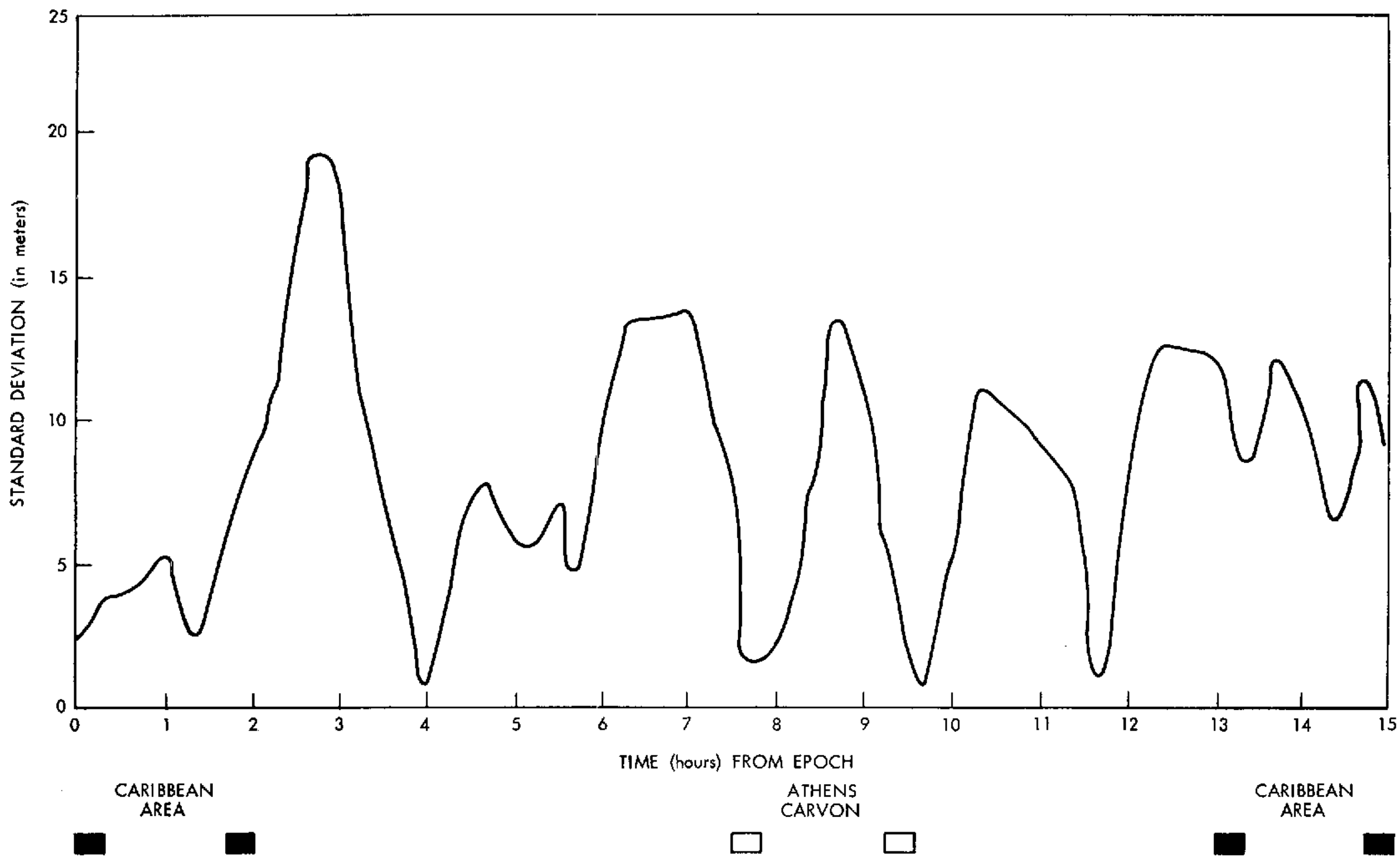

Figure 3. Contribution to Standard Deviation of Along Track Component of GEOS-C due to S(6,6)

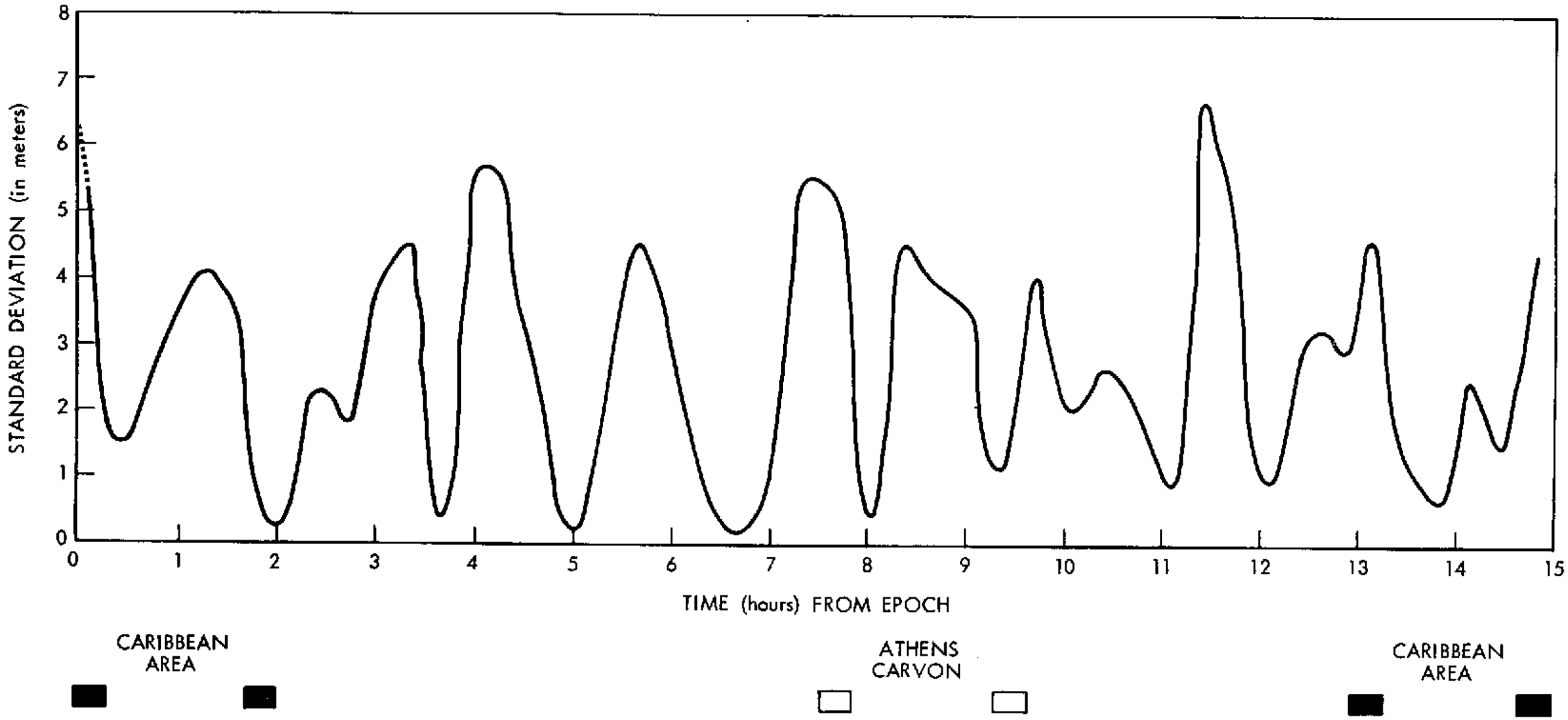

Figure 4. Contribution to Standard Deviation of Cross Track Component of GEOS-C due to S(6,6)

Table 2

Altitude determination accuracy for GEOS-C as a function of number of estimated geopotential terms.

Number of Geopotential Terms Estimated	% of time under 1 meter	% of time under 2 meters
0	0	0
3	8	55
8	50	100
12	80	100
24	100	100

in data obtained from a single satellite. Consequently the estimated geopotential terms are likely to be badly aliased by uncertainties in unestimated geopotential coefficients. But it is precisely this aliasing effect on the estimated terms that permits them to be effective in functioning as lumped parameters which absorb the effect of uncertainties in unestimated geopotential coefficients.

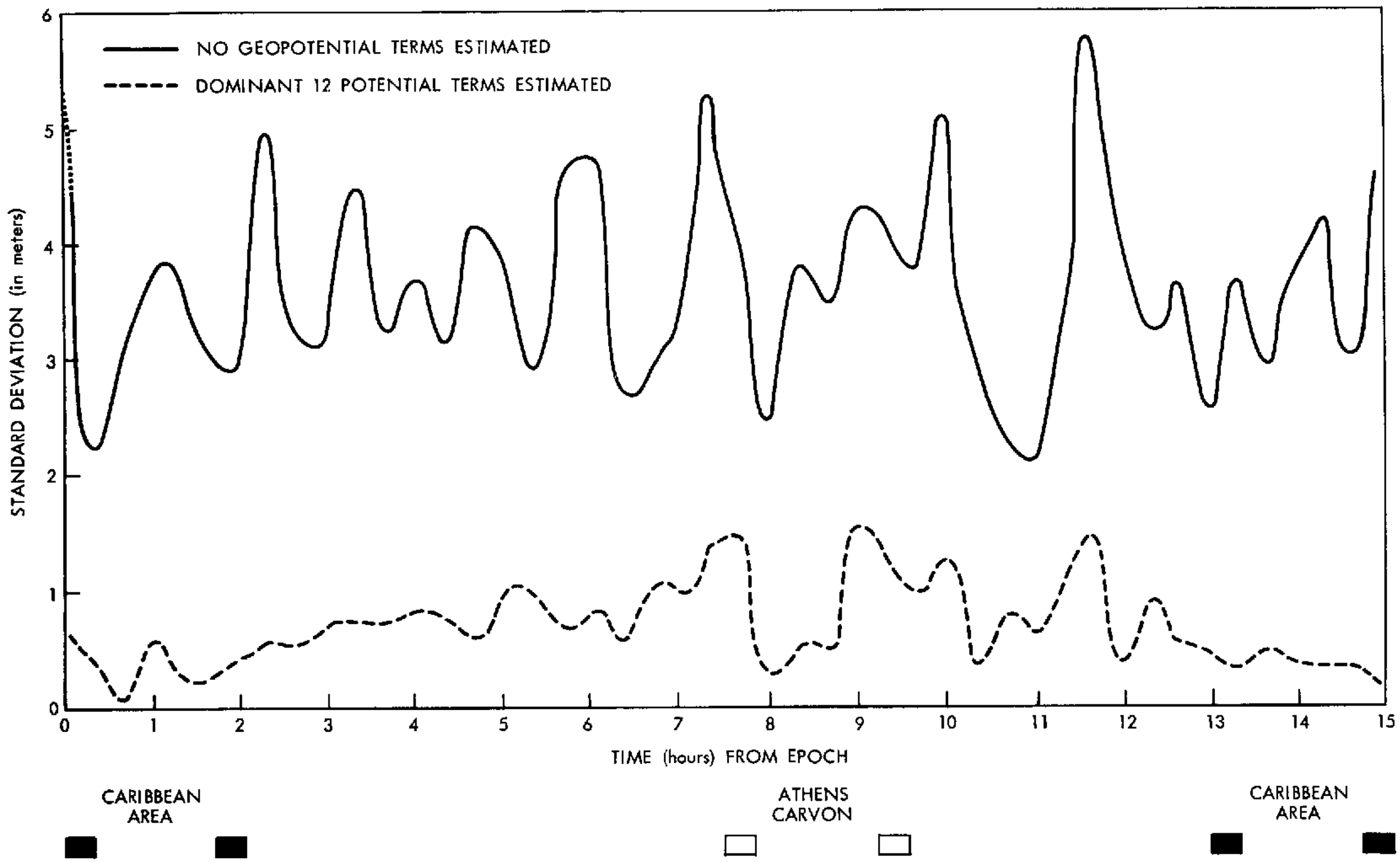

Figure 5. Altitude Standard Deviation for GEOS-C

CONCLUSIONS AND RECOMMENDATIONS

This paper has suggested that the systematic use of covariance analysis techniques is effective in identifying the cheapest estimation strategy which will satisfy a given set of mission requirements for orbit determination accuracy. The techniques were used to determine an estimation strategy which satisfied mission requirement of one to two meter altitude resolution for GEOS-C orbit determination. The procedures outlined in the paper should be useful in meeting the stringent orbit determination requirements of future applications satellites.

Although the results which have been obtained are encouraging, the present study has several limitations. The most obvious limitation is that we consider only 24 geopotential coefficients as error sources for GEOS-C orbit determination. A full field up to degree and order 8 together with certain resonant terms should be considered. When this is done, covariance techniques will answer the question of whether the estimated geopotential coefficients have physical significance or are functioning as lumped parameters. Another limitation is that covariance techniques, powerful as they are, cannot provide information about certain kinds of difficulties which may arise when the least squares computational algorithm is fully implemented. The covariance results should be supported by full simulations. The final limitation which will be mentioned relates to the fact that as the orbit precesses the set of geopotential coefficients whose uncertainties are significant error sources will change. This means that at certain intervals it may become necessary to repeat the entire procedure which is outlined in the paper. We have not considered this complication in the study. It is a difficulty which should be thoroughly explored in future studies.

REFERENCES

(1) Earth and Ocean Physica Applications Program, Vol. II, Rationale and Program Plans, NASA, Sept. 1972.

(2) Koch, D., Argentiero, P., and Kahn, W., "Long and Short Arc Altitude Determination for GEOS-C;" NASA, X-document X-591-73-368, Nov. 1973.

(3) Jazwinski, A., Hipkins, C., "Adaptive Estimation for Model Enhancement in Satellite Orbit Determination," Final Report BTS TR-72-1, prepared for Goddard Space Flight Center, July 1972.

(4) Jazwinski, A., "Adaptive Sequential Estimation with Applications," presented at the Fifth IFAC Symposium on Automatic Control in Space, Genoa, Italy, June 1973 (to appear in March 1974 issue of IFAC Journal-Automatica).

(5) Jazwinski, A., Hipkins, C., "J-Adaptive Estimation with Estimated Noise Statistics," Final Report BTS TR-73-5, prepared for Goddard Space Flight Center, November 1973.

PRECEDING PAGE BLANK NOT FILMED

APPENDIX

COVARIANCE ANALYSIS AS APPLIED TO ORBIT DETERMINATION

COMPUTING COVARIANCE MATRICES

Let $\tilde{y}(m)$ be an m dimensional vector consisting of the differences between the correct values of observations of a satellite and nominal values of the observations as determined from a nominal orbit. Also let $\tilde{z}(n)$ be an n dimensional vector of differences between actual and nominal values of the state of the satellite at an epoch and differences between actual and nominal values of parameters in the dynamic and measurement models whose associated uncertainties may limit our ability to estimate satellite state from the data. The sensitivity matrix $c(m, n)$ is defined as that matrix whose element in the ith row and the jth column is the partial derivative of $\tilde{y}(i)$ with respect to $\tilde{z}(j)$. A first order Taylor series expansion of the functional relationship between $\tilde{y}$ and $\tilde{z}$ about the nominal value of $\tilde{z}$ yields

$$\tilde{y} = c\tilde{z} \tag{A-1}$$

An orbit determination program in processing observations y of $\tilde{y}$ to obtain a least square adjustment to $\tilde{z}$ computes a so-called normal matrix defined as

$$\eta(n, n) \equiv c^T wc \tag{A-2}$$

where w is a weighting matrix and is usually the inverse of the covariance matrix of the observations y of $\tilde{y}$. Once an orbit determination program computes and stores the normal matrix, a number of questions can be raised and answered at very little cost in terms of computation time.

The best estimate of the state of the satellite at epoch is obtained by performing a least squares adjustment of the state at epoch and all other parameters with which are associated significant uncertainties. But frequently this straightforward approach leads to severe core storage requirements. In practice some of the parameters in the dynamic and measurement models are estimated along with state and others are fixed at their nominal values and left unadjusted in the least squares process. In order to determine the consequences of estimating some parameters and ignoring others it is useful to compute the covariance matrix of such a least squares estimation procedure.

Let $\tilde{z}$ be decomposed into two disjoint parameter sets as follows

$$\tilde{z} = \begin{bmatrix} \tilde{x}_1(n_1) \\ \\ \tilde{x}_2(n_2) \end{bmatrix} \tag{A-3}$$

where $\tilde{x}_1$, is a set of n_1, parameters which are to be estimated in a least squares process and $\tilde{x}_2$ is a set of n_2 parameters whose nominal values are left unadjusted by the least squares process but whose uncertainties are to be considered in computing the covariance matrix of the resulting estimator. Define a matrix $A(m, n_1)$ as a matrix whose element in the ith row and jth column is the partial derivative of $\tilde{y}(i)$ with respect to x_1 (j). Analogously define $B(m, n_2)$ as the matrix whose element in the ith row and jth column is the partial derivative of $\tilde{y}(i)$ with respect to x_2 (j). For future reference notice that the normal matrix η of $\tilde{z}$ as computed and stored by an orbit determination program and defined by Equation A-2 can be written as

$$\eta = \begin{bmatrix} A^T wA & A^T wB \\ \\ B^T wA & B^T wB \end{bmatrix} \tag{A-4}$$

Assume that there exists a priori estimates of $\tilde{x}_1$ and $\tilde{x}_2$ with properties

$$x_1' = \tilde{x}_1 + \alpha_1, \qquad E(\alpha_1) = \bar{0}, \qquad E(\alpha\alpha^T) = P_1$$

$$x_2' = \tilde{x}_2 + \alpha_2, \qquad E(\alpha_2) = \bar{0}, \qquad E(\alpha_2\alpha_2^T) = P_2$$

and assume that the observation vector y or $\tilde{y}$ has properties

$$y = \tilde{y} + \nu, \qquad E(\nu) = \bar{0}, \qquad E(\nu\nu^T) = w^{-1}$$

The least squares estimate of $\tilde{x}_1$ is obtained as the value of $\tilde{x}_1$ which minimizes the loss function

$$L(x_1) = (y - Ax_1 - Bx_2')^T w(y - Ax_1 - Bx_2') + (x_1' - x_1)^T P_1^{-1}(x_1' - x_1) \tag{A-5}$$

The resulting least squares estimator of $\tilde{x}_1$ is well known to be

$$\hat{x}_1 = (A^T wA + P_1^{-1})^{-1} \left[A^T w(y - Bx_2') + P_1^{-1} x_1' \right] \tag{A-6}$$

Define

$$P = \left[E \ (\hat{x}_1 - \tilde{x}_1) \ (\hat{x}_1 - \tilde{x}_1)^T \right] \tag{A-7}$$

A series of substitutions reveals that

$$\hat{x}_1 - \tilde{x}_1 = (A^T wA + P_1^{-1})^{-1} \ (- A^T wB\alpha_2 + A^T w\nu + P_1^{-1} \alpha_1) \tag{A-8}$$

Equation 8 yields

$$P = (A^T wA + P_1^{-1})^{-1} + (A^T wA + P_1^{-1})^{-1} A^T wBP_2B^T wA(A^T wA + P_1^{-1})^{-1} \tag{A-9}$$

Notice that the right side of Equation 9 can be computed if one has a priori covariance matrices P_1 and P_2, and the upper right and upper left portions of the normal matrix. To determine the covariance matrix of an estimator which estimates some subset of $\tilde{z}$ other than $\tilde{x}_1$, all that is necessary is to permute the rows and columns of η in the appropriate fashion and proceed as before. Thus if one assumes that the normal matrix defined by Equation 2 is precomputed it becomes an easy matter to obtain the resultant covariance matrix when any subset of the $\tilde{z}$ parameters are estimated in a least squares sense and the rest are ignored.

THE ALIAS MATRIX

Assume that all the data has the same variance. Hence

$$w = (I\sigma_0^2)^{-1} \tag{A-10}$$

where σ_0^2 is the common variance of each data point. Also assume that the a priori estimates of the unadjusted parameters are independent. Under this assumption the covariance matrix P_2 of x_2' can be written as

$$P_2 = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix} \tag{A-11}$$

where σ_i^2 is the a priori variance of the ith unadjusted parameter. Also define a matrix $K(n_1, n_2)$ as

$$K = (A^T w A)^{-1} A^T w B \tag{A-12}$$

With these assumptions Equation 9 yields the following expression for the ith diagonal element of P

$$P(I, I) = \sum_{j=0}^{n_2} (\beta_{i,j}\, \sigma_j)^2 \tag{A-13}$$

where $\beta_{i,0}$ is the ith diagonal element of the matrix $(A^T\ A)^{-1}$ (this assumes that diagonal elements of the matrix P_1^{-1} are relatively small) and

$$\beta_{i,j} = K(i, j), \quad j \geq 1 \tag{A-14}$$

The standard deviation of the ith estimated parameter is given by

$$\sigma_i = \left(\sum_{j=0}^{n_2} (\beta_{i,j}\, \sigma_j)^2 \right)^{1/2} \tag{A-15}$$

Define the error sensitivity matrix as

$$S = \{ \beta_{i,j} \}, \qquad i = 1, 2, \cdots n_1, \qquad j = 0, 1, \cdots n_2 \tag{A-16}$$

And finally define the Alias Matrix as

$$L = \bar{S}\bar{\sigma} \tag{A-17}$$

where

$$\bar{\sigma} = \begin{bmatrix} \sigma_0 & & & 0 \\ & \sigma_1 & & \\ & & \ddots & \\ 0 & & & \sigma_{n_2} \end{bmatrix} \tag{A-18}$$

The standard deviation of the ith estimated parameter is seen to be the root sum square of the terms in the ith row of the alias matrix. The elements in the first column of the alias matrix represent the RSS contribution to the standard deviation of each estimated parameter due to the data noise. The elements in the jth column, $j \geq 2$, represent the RSS contribution to the standard deviation of each estimated parameter due to the $j - 1$st unadjusted parameter.

Possession of the alias matrix reveals much of the probability structure of the postulated least squares estimator. With this information one can quickly determine which error sources are significant with regard to the estimation of a given parameter.

Propagating Covariance Matrices

Equation 9 provides the covariance matrix of the state $\tilde{x}_1$ at some specified epoch. In many cases it is important to determine how accurately the state can be determined at some time other than epoch. In order to do this correctly it is necessary to take into proper account uncertainties in dynamic parameters. These parameters may be in an estimated mode or in an unadjusted mode and to incorporate their effect one resorts to state transition matrices which presumably have been precomputed by an orbit determination program. Let $\tilde{x}_1$ (T) be the estimated state at time T. Assume as output from an orbit determination program the state transition matrices

$$\bar{v}_1 (T) = \frac{\partial \tilde{x}_1 (T)}{\partial \tilde{x}_1}, \qquad \bar{v}_2 (T) = \frac{\partial \tilde{x}_1 (T)}{\partial \tilde{x}_2} \tag{A-19}$$

If there are no dynamic parameters in the estimation vector $\tilde{x}_1$, the matrix $\bar{v}_1$ (T) takes on the particularly simple form,

$$\bar{v}_1 (T) = \begin{bmatrix} \delta & 0 \\ 0 & I \end{bmatrix} \tag{A-20}$$

where δ is the six by six matrix defined as the partial derivative matrix of the state of the satellite at time T with respect to the state of the satellite at epoch. If dynamic parameters are included in the estimated state, the off diagonal matrices become non-zero and $\bar{v}_1$ (T) assumes a more complicated form. The matrix $\bar{v}_2$ (T) is the matrix of partial derivatives of the state $\tilde{x}_1$ (T) with respect to the unadjusted parameters $\tilde{x}_2$. If no dynamic parameters are in the unadjusted mode, $\bar{v}_2$ (T) is the null matrix. A first order Taylor series expansion of the function which describes the time evolution of the state $\tilde{x}_1$ (T) yields

$$\tilde{x}_1 (T) = \bar{v}_1 (T) \tilde{x}_1 + \bar{v}_2 (T) \tilde{x}_2 \tag{A-21}$$

Substituting $\hat{x}_1$ as obtained from Equation 6 for $\tilde{x}_1$ and x_2' for $\tilde{x}_2$ provides the best estimate $\hat{x}_1$ (T), of $\tilde{x}_1$ (T)

$$\hat{x}_1 (T) = \bar{v}_1 (T) \hat{x}_1 + \bar{v}_2 (T) x_2' \tag{A-22}$$

The covariance matrix of $\hat{x}_1$ (T) is given by

$$P (T) = \bar{v}_1 (T) P \bar{v}_1^T (T) + \bar{v}_2 (T) P_2 \bar{v}_2^T (T) + \bar{v}_1 (T) E [\hat{x}_1 x_2'] \bar{v}_2 (T)$$
$$+ \bar{v}_2 (T) E [x_2' \hat{x}_1^T] \bar{v}_1^T (T) \tag{A-23}$$

Equation 23 in conjunction with Equations 6 and 9 yields

$$P (T) = \bar{v}_1 (T) (A^T w A + P_1^{-1})^{-1} \bar{v}_1^T (T) + \left[\bar{v}_1 (T) (A^T w A + P_1^{-1})^{-1} A^T w B \right.$$
$$\left. - \bar{v}_2 (T) \right] P_2 \left[\bar{v}_1 (T) (A^T w A + P_1^{-1})^{-1} A^T w B - \bar{v}_2 (T) \right]^T \tag{A-24}$$

Finally notice that in much the same fashion that Equation 9 was used to develop an alias matrix at epoch, Equation 24 can be utilized to develop an alias matrix for any time T.

REMARKS

If one possesses a functioning orbit determination program it becomes a relatively easy matter to add covariance analysis capability to the system. A computer program can be written which assumes as input a normal matrix and state transition matrices as generated by the orbit determination program. By permuting the rows and columns of the normal matrix and completing the matrix operations defined by Equation 9, the covariance matrix of a least square process which adjusts any subset of the parameters and ignores the rest can be computed. An alias matrix can be obtained and significant error sources can be identified. By utilizing the precomputed state transition matricies, the covariance matrix of the estimate of the state can be propagated from epoch to any other time. These operations are very simple and they consume little computer time.

Since the normal matrix and state transition matrices are computed once and permanently stored, it is possible to investigate a large number of possible estimation strategies. This can be done conveniently and cheaply. For many applications such a program is a useful and quickly developed addition to an orbit determination system.

NASA–GSFC