ESL-P-529   January 1974

ON THE DESIGN OF OPTIMAL INPUT SIGNALS IN SYSTEM IDENTIFICATION*

by

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ABSTRACT

The problem of designing optimal inputs in the identification of linear
systems with unknown random parameters is considered using a Bayesian
approach. The information matrix, which is positive definite for the
class of systems analyzed, gives a measure of performance for the sy-
stem inputs. The computation of the optimal closed-loop input mappings
is shown to be a nontrivial exercise in adaptive control. Deterministic
optimal inputs are shown to be easily computable. Numerical examples
are given. A Kalman filter is used to estimate the parameters. A
necessary condition for the Kalman filter not to diverge when applying
linear feedback, is also given.

INTRODUCTION

The problem of designing deterministic system inputs to en-
hance the performance of some identification algorithms has received
some attention in the literature. The formulations are based on the
Fisher Information Matrix and on least squares identification. Continuous-
time systems have been discussed by Levadi [1], Nahi et al [2,3] and
Mehra [4]. Deterministic systems observed through additive Gaussian
noise are considered in those works. Discrete-time single-input single-
output systems are considered by Gagliardi [5], Aoki and Staley [6],
Goodwin et al [7] and more recently by Mehra [8] whose frequency-domain
approach allows multi-output systems.

The more general results have been obtained for formulations
that lead to a linear model [1],[8], where linear least-squares
techniques can be applied. The Fisher Information approach usually
yields equations that depend on the unknown parameters [2,3,4,6,7] and
thus some suboptimal scheme has to be used.

In this paper, we shall be concerned with a particular class of
linear systems, namely those whose unknown random parameters appear
linearly in the system description. Using a Bayesian approach, we
will attempt to design optimal closed-loop inputs via the Information
Matrix criterion. In Section II we indicate that this is a difficult
problem in adaptive control. Section III gives the solution to the
problem for deterministic inputs and some examples are given in
Section IV. Finally, Section V gives an asymptotic result for iden-
tification of closed-loop systems.

II. OPTIMAL CLOSED-LOOP INPUT MAPPINGS

In this section we will derive conditions that the optimal feed-back input mappings, maximizing the sensitivity of the system output with respect to the parameters, must satisfy. This sensitivity criterion is also shown to be equivalent to maximizing the trace of the Information Matrix (which is positive definite for the class of systems considered).

Consider the discrete-time linear system[1]

$$\underline{y}(k) = \sum_{i=1}^{n} \underline{A}_i(k-i)\underline{y}(k-i) + \sum_{i=1}^{m} \underline{B}_i(k-i)\underline{u}(k-i) + \underline{\theta}(k) \qquad (2.1)$$

where $\underline{y}(k) \in R^r$ and $\underline{u}(k) \in R^s$ are the observed system output and input, respectively, $\underline{\theta}(k) \in R^r$ is a white Gaussian noise with known statistics (δ_{kj} is Kronecker's delta) $E\{\underline{\theta}(k)\} = 0$; $E\{\underline{\theta}(k)\underline{\theta}^T(j)\} = \underline{\Theta}(k)\delta_{kj}$; $\underline{\Theta}(k) > \underline{0}$ all k, and where $\underline{A}_i(k) \in R^{r \times r}$ and $\underline{B}_i(k) \in R^{r \times s}$ are partially known random matrices.

It is assumed that the initial conditions $\underline{y}(0), \ldots, \underline{y}(-n+1)$ and $\underline{u}(-1) = \ldots = \underline{u}(-m+1) = \underline{0}$ are known.

By an appropriate rearrangement of Eq. (2.1) as noisy observations on the unknown parameters, one can apply the results of Kalman filtering theory. Assume that all elements of the $\underline{A}_i(k)$ and $\underline{B}_i(k)$ matrices are unknown random parameters varying according to a Gauss-Markov process of the following form. Let $\underline{a}_i^{j}(k)$ and $\underline{b}_i^{j}(k)$ be the j-th row of $\underline{A}_i(k)$

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[1]Although this system has a very particular structure, it can be used in a large class of realistic problems. See e.g. [11], [12], [13].

and $\underline{B}_i(k)$ respectively, the vector $\underline{\gamma}(k) \in R^{nr^2+msr}$ of unknown parameters

at the k-step is formed by placing the rows of $\underline{A}_i(k-i)$ one after another

in consecutive order $j = 1, \ldots r$ and for $i = 1, \ldots n$, followed by the

rows of $\underline{B}_i(k-i)$ in the same ordering fashion, i.e.

$$\underline{\gamma}(k) = [\underline{a}_1^{\,1}(k-1)\underline{a}_1^{\,2}(k-1)\ldots\underline{a}_n^{\,1}(k-n)\ldots\underline{a}_n^{\,n}(k-n)\underline{b}_1^{\,1}(k-1)\ldots b_m^{\,m}(k-m)]^T$$

then $\underline{\gamma}(k)$ is assumed to vary according to

$$\underline{\gamma}(k+1) = \underline{F}(k)\underline{\gamma}(k) + \underline{\xi}(k) \tag{2.2}$$

where $\underline{F}(k)$ is known and $\underline{\gamma}(0)$ and $\underline{\xi}(k)$ are mutually independent (also

independent of $\theta(j)$) Gaussian variables with known statistics,

$$E\{\underline{\gamma}(0)\} = \overline{\underline{\gamma}} , \qquad E\{(\underline{\gamma}-\overline{\underline{\gamma}})(\underline{\gamma}-\overline{\underline{\gamma}})^T\} = \underline{\Sigma}_\gamma$$

$$E\{\underline{\xi}(k)\} = \underline{0} , \qquad E\{\underline{\xi}(k)\underline{\xi}^T(j)\} = \underline{\Xi}(k)\delta_{kj}$$

With the above definitions, Eq. (2.1) can be rewritten as the

linear model

$$\underline{y}(k) = \underline{C}^T(k)\underline{\gamma}(k) + \underline{\theta}(k) \overset{\Delta}{=} \underline{h}(k) + \underline{\theta}(k) \tag{2.3}$$

where $\underline{C}(k) \in R^{(nr^2+msr)xr}$ is given by

$$\underline{C}^T(k) = \begin{bmatrix} \underline{y}^T(k-1) & 0 & . & . & 0 & \underline{y}^T(k-2)\ldots\ldots.\underline{y}^T(k-n). .0 & \underline{u}^T(k-1)\ldots \\ 0 & \underline{y}^T(k-1) & & . & & & . & . & . & . \\ . & . & . & . & & . & . & . & . \\ . & . & . & . & & . & . & . & . \\ . & . & . & . & & . & . & . & . \\ 0 & 0 & \ldots.\underline{y}^T(k-1) & 0 & \ldots\ldots0\ldots\underline{y}^T(k-n) & 0 & \ldots \\ & & & \underline{u}^T(k-m) & \ldots & 0 \\ & & & . & . & . \\ & & & . & . & . \\ & & & . & . & . \\ & & & 0 & & \underline{u}^T(k-m) \end{bmatrix}$$

The following lemma is merely a statement of the Kalman filter and will be used in the proof of Theorem 2.1 below. Its proof requires a modification of Kalman's original proof to take into account that $\underline{C}(k)$ is random. A formal proof can be obtained paralleling Striebel's [9] derivation of the conditional distribution for a class of systems.

Lemma 2.1

Let $Y_k = \{\underline{y}(1),\ldots,\underline{y}(k)\}$ be the observed outputs of system (2.1) and assume that the inputs are of the feedback type (nonanticipative) $\underline{u}(k) = \underline{u}(k,Y_k)$. Then the a posteriori density $p(\underline{\gamma}(k)|Y_k)$ is Gaussian with mean and covariance given respectively by

$$\hat{\underline{\gamma}}(k) = \underline{F}(k-1)\hat{\underline{\gamma}}(k-1) + \underline{\Sigma}(k|k)\underline{C}(k)\underline{\Theta}^{-1}(k)(\underline{y}(k)-\underline{c}^T(k)\underline{F}(k-1)\hat{\underline{\gamma}}(k-1))$$

$$(2.4)$$

and

$$\underline{\Sigma}(k|k) = (\underline{\Sigma}^{-1}(k|k-1) + \underline{C}(k)\underline{\Theta}^{-1}(k)\underline{c}^T(k))^{-1} \qquad (2.5)$$

where $\underline{\Sigma}(k|k-1) = \underline{F}(k-1)\underline{\Sigma}(k-1|k-1)\underline{F}^T(k-1) + \underline{\Xi}(k-1)$ and $\underline{\Sigma}(0|0) = \underline{\Sigma}_\gamma > \underline{0}$, $\hat{\underline{\gamma}}(0) = \bar{\underline{\gamma}}$.

If some of the parameters are known, one can write $\underline{y}(k) = \underline{z}(k) + \underline{c}_1^T(k)\underline{\gamma}_1(k) + \underline{\theta}(k)$, where $\underline{\gamma}_1(k)$ is the vector of unknown parameters. Once again Striebel's procedure can be applied to derive $p(\underline{\gamma}_1(k)|y_k)$. The result is to substitute $\underline{y}(k)$ by $\underline{y}(k)-\underline{z}(k)$ and $\underline{C}(k)$ by $\underline{C}_1(k)$ in the above equations.

Equations (2.4) and (2.5) provide a recursive estimator for the unknown parameters, which is known to be unbiased $(E\{\hat{\underline{\gamma}}(k)\} = E\{\underline{\gamma}(k)\})$ and minimizes quadratic estimation errors. Moreover, if the parameters

are mutually independent or are simple unknown constants, Eqs. (2.4) and (2.5) can be broken down to yield r Kalman filters of smaller dimension [10].

We now introduce some more definitions in order to state the main theorem of this section. Let $\underline{Q} \in R^{NsxNs}$ be the diagonal matrix.

$$\underline{Q} = \text{Diag}[(q_1+\ldots+q_n)\underline{I}^s, (q_2+\ldots+q_{n+1})\underline{I}^s, \ldots (q_{N-m}+\ldots+q_{n-1})\underline{I}^s,$$

$$(q_{N-m+1}+\ldots+q_N)\underline{I}^s, \ldots, (q_{N-1}+q_N)\underline{I}^s, q_N\underline{I}^s)] \text{ where } q_k = \text{tr } \underline{\theta}^{-1}(k) \text{ and}$$

$\underline{I}^s$ is the s x s identity matrix. Similarly, let $\underline{P} \in R^{(N-1)rx(N-1)r}$

be the diagonal matrix $\underline{P} = \text{Diag}[(q_2+\ldots+q_{n+1})\underline{I}^r, \ldots, (q_{N-n}+\ldots+q_{N-1})\underline{I}^r,$

$$(q_{N-n+1}+\ldots+q_n)\underline{I}^r, \ldots (q_{N-1}+q_N)\underline{I}^r, q_N\underline{I}^r] \text{ and define the submatrices}$$

$\underline{Q}_i \in R^{sxs}$ and $\underline{P}_i \in R^{rxr}$ in correspondence with $\underline{Q}$ and $\underline{P}$ as follows

$$\underline{Q} = \text{Diag}[\underline{Q}_0, \ldots, \underline{Q}_{N-1}]$$

$$\underline{P} = \text{Diag}[\underline{P}_1, \ldots, \underline{P}_{N-1}] , \quad \underline{P}_0 = 0$$

Theorem 2.1

Let $\underline{\gamma}(k+1) = \underline{\gamma}(k) = \underline{\gamma}$ with $\underline{\gamma}(0)$ a Gaussian variable with mean $\overline{\underline{\gamma}}$ and covariance $\underline{\Sigma}_\gamma > \underline{0}$ and let $\underline{M}_N$ be the Information Matrix

$$\underline{M}_N = \iint (\nabla_\gamma \log p(\underline{Y}_N,\underline{\gamma}))^T (\nabla_\gamma \log p(\underline{Y}_N,\underline{\gamma})) d\underline{Y}_N d\underline{\gamma} \qquad (2.6)$$

where ∇_γ denotes gradient and $\underline{Y}_N = [\underline{y}^T(1)\ldots y^T(N)]^T$.

Then, for system (2.1) we have that

(i) $\quad \underline{M}_N > \underline{0} \quad (\underline{M}_N \geq \underline{\Sigma}_\gamma^{-1})$ $\qquad\qquad\qquad\qquad$ (2.7)

(ii)   The following problems are equivalent (maximization is with

respect to input mappings $\underline{u}(k,Y_k)$)

(a)   Max tr $\underline{M}_N$ (2.8)

(b)   Max of the sensitivity index

$$E\left\{\sum_{k=0}^{N-1} \sum_{i=1}^{r} p_i(k) \left(\frac{\partial h_i(k)}{\partial \underline{Y}}\right)^T \left(\frac{\partial h_i(k)}{\partial \underline{Y}}\right)\right\} \qquad (2.9)$$

where $p_i(k) = [\underline{\theta}^{-1}(k)]_{ii}$ is the i-th diagonal element

of $\underline{\theta}^{-1}(k)$ and h_i is the i-th element of $\underline{h}$ (Eq. 2.3).

(c)   Max

$$E\left\{\sum_{k=0}^{N-1} (\underline{y}^T(k)\underline{P}_k\underline{y}(k) + \underline{u}^T(k)\underline{Q}_k\underline{u}(k))\right\} \qquad (2.10)$$

Proof.   See Appendix A.

Remarks.

It is known that $\underline{M}_N$ is related to the error covariance matrix
$\underline{\Sigma}_N$ of any unbiased estimator of $\underline{Y}$ by the Cramer-Rao inequality

$$\underline{\Sigma}_N - \underline{M}_N^{-1} \geq \underline{0} \qquad (2.11)$$

Eq. (2.7) indicates that $\underline{M}_N$ has only nonzero eigenvalues and since

$1/\text{tr } \underline{M}_N$ and tr $\underline{M}_N^{-1}$ go to zero at the rate $\theta(1/\lambda \text{ min})$ as $\lambda_{min} \rightarrow \infty$ [6],

where λ_{min} is the minimum eigenvalue of $\underline{M}_N$, Eqs. (2.8) and (2.11) give

a relationship of the optimal inputs with mean square estimation errors.

Eqs. (2.9) and (2.3) show that the inputs should maximize the sensitivity

of the system output with respect to the parameters, where $p_i(k)$ has the

interpretation that if the noise is "high", the output contains "little information" about the parameters. Eq. (2.10) is derived from either (2.8) or (2.9) and remains unchanged if one considers time-varying Markov parameters (Eq. (2.2)) with the index (2.9) changed to

$$E\left\{\sum_{k=0}^{N-1} \sum_{i=1}^{r} p_i(k) \left(\frac{\partial h_i(k)}{\partial \underline{\gamma}(k)}\right)^T \left(\frac{\partial h_i(k)}{\partial \underline{\gamma}(k)}\right)\right\} \qquad (2.11)$$

Eq. (2.10) implies that this type of inputs for identification is conflicting with any input aimed to regulate the system.

III. OPTIMAL OPEN-LOOP INPUTS.

It is not difficult to check that, unfortunately, the maximization of Eq. (2.10), subject to an energy constraint, cannot be performed analytically. Some suboptimal schemes, developed in the theory of adaptive controllers may however be applicable. In this section, we give the solution to the problem for the case of deterministic inputs. We will consider only the case of constant random parameters, the extension to time-varying parameters with index (2.11) is completely straightforward.

Let us define $\underline{A} \in R^{(N-1)r \times (N-1)r}$ and $\underline{B} \in R^{(N-1)r \times (N-1)s}$ by

$$\underline{A} = \begin{bmatrix} -\underline{I} & & & & \cdot & \cdot & 0 \\ \underline{A}_1 & -\underline{I} & & & & & \\ \cdot & & \cdot & & & & \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \underline{A}_n & & & & & \cdot & \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \\ 0 & \cdot & \cdot & \underline{A}_n & \cdot & \cdot & \cdot & \underline{A}_1 & -\underline{I} \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} \underline{B}_1 & & & & \cdot & \cdot & 0 \\ \underline{B}_2 & \cdot & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \underline{B}_m & & & & \cdot & \cdot & \\ \cdot & & & & & \cdot & \cdot \\ \cdot & & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \underline{B}_m & \cdot & \cdot & \cdot & \underline{B}_2 & \underline{B}_1 \end{bmatrix} \qquad (3.1)$$

Let $\underline{h} \in R^{(N-1)r}$ be a vector depending on the initial conditions through

$$\underline{h} = \left[\left(\sum_{i=1}^{n} \underline{A}_i \underline{y}(1-i) \right)^T \left(\sum_{i=2}^{n} \underline{A}_i \underline{y}(2-i) \right)^T \cdots (\underline{A}_n \underline{y}(0))^T \underbrace{0 \cdots 0}_{(N-n-1)r \text{ zeros}} \right]$$

(3.2)

Let $\underline{H} \in R^{Ns \times Ns}$ be the symmetric positive definite matrix

$$\underline{H} = \underline{Q} + E\left\{ \begin{bmatrix} \underline{B}^T \underline{A}^{-T} \underline{P} \, \underline{A}^{-1} \underline{B} & \underline{0} \\ \underline{0} & \end{bmatrix} \right\}$$

(3.3)

where $\underline{A}^{-T} = (\underline{A}^{-1})^T$ always exists since det $\underline{A} = 1$, and let $\underline{d} \in R^{Ns}$

$$\underline{d} = E\{ [(\underline{B}^T \underline{A}^{-T} \underline{P} \, \underline{A}^{-1} \underline{h})^T \underbrace{0 \cdots 0}_{s \text{ zeros}}]^T \}$$

(3.4)

Then we can state the following theorem.

Theorem 3.1

Let $\underline{U}_N = [\underline{u}^T(0) \cdots \underline{u}^T(N-1)]^T \in R^{Ns}$ be the vector of the first

N system inputs. The optimal input sequence $\underline{u}^*(0), \ldots, \underline{u}^*(N-1)$ that

solves part (ii) of Theorem 2.1, subject to the energy constraint

$\sum_{i=0}^{N-1} \underline{u}^T(i) \underline{u}(i) \leq W$, is given by the vector

$$\underline{U}_N^* = [-\underline{H} + \lambda \underline{I}]^{-1} \underline{d} \qquad \text{if } \underline{d} \neq \underline{0}$$

(3.5)

where $\underline{I}$ is the Ns x Ns identity matrix and λ is such that

$$(\underline{U}_N^*)^T (\underline{U}_N^*) = W$$

(3.6)

If $\underline{d} = \underline{0}$, then $\underline{U}_N^*$ is the eigenvector corresponding to the

maximum eigenvalue of $\underline{H}$, normalized according to Eq. (3.6).

Proof   See Appendix B.

Remarks.

In Eq. (3.5) λ is a Lagrange multiplier. If the initial conditions are zero, or $E\{\underline{B}\} = \underline{0}$ and $\underline{B}$ is independent of $\underline{A}$ (element-wise) then $\underline{d} = \underline{0}$. These conditions are however not necessary for $\underline{d}$ to be zero. From Eqs. (3.1) through (3.4) it follows that $\underline{H}$ and $\underline{d}$ depend on the noise statistics, on the initial conditions and on $E\{\gamma_i{}^k\}$, $k = 1,\ldots,2N-4$, where γ_i is the i-th unknown parameter. The only difficulty in the computation of explicit expressions for $\underline{H}$ and $\underline{d}$ is that of notation. As an example, the single-input single-output (SISO) case is considered next.

IV. EXAMPLES

Example 1

The SISO system is described by the equation

$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{m} b_i u(k-i) + \theta(k) \qquad (4.1)$$

Proposition

If $\underline{\gamma} = [a_1 .. a_n b_1 .. b_n]^T$ is a vector of Gaussian mutually independent random variables, the expectations required in $\underline{H}$ and $\underline{d}$ can be obtained as follows

$$E\{\underline{B}^T \underline{A}^{-T} \underline{P} \ \underline{A}^{-1} \underline{B}\} = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} E\{b_i\}E\{b_j\}\underline{Q}_{ij} + \sum_{i=1}^{m} E\{b_i^2\}\underline{Q}_{ii}$$

$$+ \sum_{k=1}^{2N-4} \sum_{\substack{k_1+\ldots+k_n=k \\ k_\alpha \epsilon I}} E\{a_1^{k_1}\}\ldots E\{a_n^{k_n}\} \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{m} E\{b_i\}E\{b_j\}\underline{R}_{ijk} \right.$$

$$\left. + \sum_{i=1}^{m} E\{b_i^2\}\underline{R}_{iik} \right]$$

$$E\{\underline{B}^T \underline{A}^{-T} \underline{P} \ \underline{A}^{-1} \underline{h}\} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{\ell=1}^{i} E\{a_i\}E\{b_j\}y_{\ell-i}\underline{f}_{\ell j}$$

$$+ \sum_{k=1}^{2N-4} \sum_{j=1}^{m} \sum_{\substack{k_1+\ldots+(k_i+1)+\ldots+k_n=k+1 \\ i=1,\ldots,n \\ k_\alpha \epsilon I}} \sum_{\ell=1}^{i}$$

$$E\{a_1^{k_1}\}\ldots E\{a_i^{k_i+1}\}\ldots E\{a_n^{k_n}\}E\{b_j\}y_{\ell-i}\underline{p}_{\ell k j}$$

where $k_\alpha \epsilon I$ reads "for all $k_\alpha \epsilon \{0,1,\ldots\}$, $\alpha = 1,\ldots n$" and the matrices

$\underline{R}_{ijk}, \underline{Q}_{ij} \epsilon R^{(N-1) \times (N-1)}$ and $\underline{f}_{\ell j}, \underline{p}_{\ell k j} \epsilon R^{N-1}$ can be precomputed for

fixed N(they depend only on the noise statistics $\Theta(k)$, $k = 2,\ldots,N$,

and if $\Theta(k) = \Theta$ all k, they can be replaced by constant matrices

independent of Θ).

If $\underline{Y}$ is a vector of jointly Gaussian variables, with mean $\overline{\underline{Y}}$

and covariance $\underline{\Sigma}_\gamma$, the transformation $\underline{Y}_1 = \underline{\Gamma}\underline{Y}$ where $\underline{\Gamma}$ satisfies

$\underline{\Gamma}\underline{\Sigma}_\gamma\underline{\Gamma}^T$ = Diagonal Matrix and $\underline{\Gamma}\underline{\Gamma}^T = \underline{I}$, yields a set γ_1 of mutually

independent Gaussian variables. The expressions for the expectations

are then similar to those above.

Proof   See Appendix C.

Example 2 (Numerical example).

    Consider the two-input one-output system

$y(k) = ay(k-1) + b_1 u_1(k-1) + b_2 u_2(k-1) + \theta(k)$ where $\theta(k)$ is white

Gaussian $N(0,1)$. The following two cases of random parameters are

analyzed.

Case (i) a, b_1 and b_2 are jointly Gaussian with

$$
\begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} .5 & 0 & 0 \\ 0 & .1 & .1 \\ 0 & .1 & 1 \end{pmatrix} \right) \tag{4.2}
$$

The inputs, computed according to Section III are (N=10) shown in Fig. 4.1.

Case (ii) a, b_1 and b_2 are jointly Gaussian with

$$
\begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} .1 & 0 & 0 \\ 0 & .1 & .1 \\ 0 & .1 & 1 \end{pmatrix} \right) \tag{4.3}
$$

The optimal inputs for this case are shown in Fig. 4.2.

    The inputs of Fig. 4.3 with the same energy (W = 85.23) that

those of Figs. 4.1 and 4.2 were used as comparison.

    Mean square errors for 2500 Monte Carlo runs were computed,

the results are shown in Table 4.1.

System	Input	tr $\underline{\Sigma}_{10}$	
(4.2)	Fig. 4.1	.1281	*
	Fig. 4.3	.2510	
(4.3)	Fig. 4.2	.1163	*
	Fig. 4.3	.2062	
	Fig. 4.1	.1287	

Table 4.1   Mean Square Errors for Different Inputs

From the figures we can see that u_2* contains most of the energy of the optimal inputs. This agrees with our intuition, since b_2, the parameter multiplying u_2 has the largest uncertainty. The results of the simulation (Table 4.1) indicate that the mean square estimation errors corresponding to the optimal inputs (indicated by *) are approximately only 50% of those corresponding to constant inputs.

The optimal input for system (4.2), Fig. 4.1, is of the impulse type. An interesting result follows by comparing the optimal inputs for system (4.3) with the impulse type input Fig. 4.1 (3th vs. last row in Table 4.1). The former gives a 10 % reduction on mean-square estimation errors corresponding to the latter. Thus impulse type inputs are not always the best.

V.   ASYMPTOTIC ANALYSIS

Most identification algorithms have been designed on the basis of an open-loop operation of the plant and erroneous results may be obtained if one uses such algorithms for closed-loop system identification [14][15]. Nevertheless, if one is careful about the type of feedback being used, then correct results are obtained.

Consider for instance the system [16]

$$x(k) + ax(k-1) = bu(k-1) + \theta(k)$$
$$u(k) = \alpha x(k) \tag{5.1}$$

It is well known that the Kalman filter on least squares estimate can be obtained minimizing

$$J(a,b) = \sum_{k=1}^{N} (x(k+1) + ax(k) - bu(k))^2 + (a-\bar{a} \quad b-\bar{b})\underline{\Sigma}_{ab}^{-1}\begin{pmatrix} a-\bar{a} \\ b-\bar{b} \end{pmatrix} \tag{5.2}$$

where $\bar{a}, \bar{b}, \underline{\Sigma}_{ab}$ is the a priori information. As $N \to \infty$ the a priori information becomes negligible. Thus Eq. (5.2) becomes

$$J(a,b) = \sum_{k=1}^{N} (x(k+1) + ax(k) - bu(k))^2$$

$$= \sum_{k=1}^{N} (x(k+1) + (a+\alpha\lambda)x(k) - (b+\lambda)u(k))^2$$

$$= J(a + \alpha\lambda, \ b + \lambda) \tag{5.3}^2$$

and the parameters are not identifiable. They become identifiable if the feedback loop is changed to $u(k) = \alpha_1 x(k) + \alpha_2 x(k-1)$.

The following theorem gives a necessary condition for the Kalman filter (Eqs. (2.4),(2.5)) not to diverge when linear feedback is being used. Only the SISO case is considered. The extension to multi-input multi-output systems is conceptually straightforward.

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[2] Note that this cost function can also correspond to a maximum likelihood estimate.

Theorem 5.1

Let the system be

$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{i=1}^{m} b_i u(k-i) + \theta(k) \qquad (5.4)$$

where $\theta(k)$ is white Gaussian $N(0,\sigma^2)$. Let $\underline{\gamma} = [a_1 \ldots a_n \; b_1 \ldots b_m]^T$

be a Gaussian random vector $N(\bar{\underline{\gamma}}, \underline{\Sigma}_\gamma)$, and assume that linear feedback

$$u(k) = \sum_{i=0}^{\ell} \alpha_i y(k-i) \qquad (5.5)$$

is being used. Then, if $\ell < n$ we have that the mean square errors

satisfy

$$\lim_{N \to \infty} E\{(\underline{\gamma} - \hat{\underline{\gamma}}(N))^T (\underline{\gamma} - \hat{\underline{\gamma}}(N))\} \to \infty \qquad (5.6)$$

Proof. See Appendix D.

VI. CONCLUSION

The design of inputs for system identification has been con-
sidered using a Bayesian approach. The optimal identifying inputs are
shown to be conflicting with a system regulator. Optimal deterministic
inputs have been shown to be easily computable and digital simulation
shows that they yield smaller mean square estimation errors than arbi-
trary inputs. The divergence of the Kalman filter-estimator for
closed-loop systems has been analyzed.

APPENDIX A

From the definition of the Information Matrix given by Eq. (2.6),

$$\underline{M}_N = E\{(\nabla_\gamma \log p(\underline{Y}_N, \underline{\gamma}))^T (\nabla_\gamma \log p(\underline{Y}_N, \underline{\gamma}))\} \qquad (A.1)$$

Using Baye's rule we get

$$\underline{M}_N = E\{(\nabla_\gamma \log p(\underline{\gamma})\underline{Y}_N) + \nabla_\gamma \log p(\underline{Y}_N))^T (\nabla_\gamma \log p(\underline{\gamma})\underline{Y}_N)$$

$$+ \nabla_\gamma \log p(\underline{Y}_N))\}$$

i.e. $\underline{M}_N = E\{(\nabla_\gamma \log p(\underline{\gamma}|\underline{Y}_N))^T (\nabla_\gamma \log p(\underline{\gamma}|\underline{Y}_N))\}$

From the results of Lemma 2.1, the density $p(\underline{\gamma}|\underline{Y}_N)$ for any set of admissible (nonanticipative) input mappings $\underline{u}(k, Y_k)$ is Gaussian with mean and covariance given by $\hat{\underline{\gamma}}(N)$ and $\underline{\Sigma}(N|N)$, thus

$$\nabla_\gamma \log p(\underline{\gamma}|\underline{Y}_N) = - (\underline{\gamma} - \hat{\underline{\gamma}}(N))^T \underline{\Sigma}^{-1}(N|N)$$

and using the properties of conditional expectations, $\underline{M}_N$ can be written as

$$\underline{M}_N = E\{E\{\underline{\Sigma}^{-1}(N|N)(\underline{\gamma} - \hat{\underline{\gamma}}(N))(\underline{\gamma} - \hat{\underline{\gamma}}(N))^T \underline{\Sigma}^{-1}(N|N)|\underline{Y}_N\}\}$$

or

$$\underline{M}_N = E\{\underline{\Sigma}^{-1}(N|N)\} \qquad (A.2)$$

Now, from Eq. (2.5) of the lemma

$$\underline{\Sigma}^{-1}(k|k) = \underline{\Sigma}^{-1}(k-1|k-1) + \underline{c}(k)\underline{\theta}^{-1}(k)\underline{c}^T(k)$$

then

$$\underline{\Sigma}^{-1}(N|N) = \underline{\Sigma}_\gamma^{-1} + \sum_{k=1}^{N} \underline{c}(k)\underline{\theta}^{-1}(k)\underline{c}^T(k) \qquad (A.3)$$

Substitution of (A.3) into (A.2) proves part (i) of the Theorem.

Maximization of tr $\underline{M}_N$ is, by (A.2) and (A.3) equivalent to maximization of

$$J_1 = \text{tr } E\left\{\sum_{k=1}^{N} \underline{C}^T(k)\underline{C}(k)\underline{\theta}^{-1}(k)\right\} \tag{A.4}$$

Using the definitions of $\underline{C}(k)$, $\underline{h}(k)$ and $p_i(k)$, it follows after some algebra that Eqs. (2.9) and (A.4) are equivalent. This proves parts (ii-a) and (ii-b) of the theorem.

We also have that

$$\underline{C}^T(k)\underline{C}(k) = \left(\sum_{i=1}^{n} \underline{y}^T(k-i)\underline{y}(k-i) + \sum_{i=1}^{m} \underline{u}^T(k-i)\underline{u}(k-i)\right)\underline{I}$$

where $\underline{I}$ is the $r \times r$ identity matrix. Thus, using the definition of q_k we get

$$\text{tr}[\underline{C}^T(k)\underline{C}(k)\underline{\theta}^{-1}(k)] = q_k\left(\sum_{i=1}^{n} \underline{y}^T(k-i)\underline{y}(k-i) + \sum_{i=1}^{m} \underline{u}^T(k-i)\underline{u}(k-i)\right)$$

Adding N of these terms as indicated by (A.4) and using the definitions of $\underline{P}$ and $\underline{Q}$ we have

$$J_1 = E\{\underline{Y}_{N-1}^T \underline{P} \ \underline{Y}_{N-1} + \underline{U}_N^T \ \underline{Q} \ \underline{U}_N\} \tag{A.5}$$

where $\underline{Y}_{N-1} = [\underline{y}^T(1)\ldots\underline{y}^T(N-1)]$, $\underline{U}_N = [\underline{u}^T(0)\ldots\underline{u}^T(N-1)]$.

Using the partitions of $\underline{P}$ and $\underline{Q}$, from (A.5) we get

$$J = E\left\{\sum_{k=0}^{N-1} (\underline{y}^T(k)\underline{P}_k\underline{y}(k) + \underline{u}^T(k)\underline{Q}_k u(k))\right\}$$

This completes the proof.

APPENDIX B

For deterministic inputs, (A.5) can be written as

$$J_1 = \text{tr } \underline{P} \; E\{E\{\underline{Y}_{N-1}\underline{Y}_{N-1}^T/\underline{Y}\}\} + \underline{U}_N^T \underline{Q} \; \underline{U}_N$$

or

$$J_1 = \text{tr } \underline{P} \; E\{\overline{\underline{Y}}_{N-1}\overline{\underline{Y}}_{N-1}^T + \underline{\Sigma}_y\} + \underline{U}_N^T \underline{Q} \; \underline{U}_N \qquad (B.1)$$

where

$$\overline{\underline{Y}}_{N-1} = E\{\underline{Y}_{N-1}/\underline{Y}\}$$

$$\underline{\Sigma}_y = E\{(\underline{Y}_{N-1} - \overline{\underline{Y}}_{N-1})(\underline{Y}_{N-1} - \overline{\underline{Y}}_{N-1})^T/\underline{Y}\}$$

It is easy to see that the first N-1 system outputs satisfy

$$\underline{A} \; \underline{Y}_{N-1} + \underline{B} \; \underline{U}_{N-1} + \underline{I} \; \underline{\theta}_{N-1} + \underline{h} = \underline{0} \qquad (B.2)$$

where $\underline{A}$, $\underline{B}$ and $\underline{h}$ are defined in the Theorem, $\underline{I} \in R^{(N-1)r \times (N-1)r}$ is the identity matrix and

$$\underline{\theta}_{N-1} = [\underline{\theta}^T(1)....\underline{\theta}^T(N-1)]^T$$

Since $\underline{A}^{-1}$ always exists, the vector

$$\underline{Y}_{N-1} = - \underline{A}^{-1}(\underline{B} \; \underline{U}_{N-1} + \underline{\theta}_{N-1} + \underline{h})$$

has a Gaussian conditional (on $\underline{Y}$) density, with

$$\overline{\underline{Y}}_{N-1} = - \underline{A}^{-1}(\underline{B} \; \underline{U}_{N-1} + \underline{h})$$

$$\underline{\Sigma}_y = \underline{A}^{-1}\underline{\Theta} \; \underline{A}^{-T}$$

where $\underline{\Theta} = \text{Diag } (\underline{\Theta}(1),...,\underline{\Theta}(N-1))$.

Therefore, substituting into (B.1) and considering only terms that depend on $\underline{U}_N$, we must maximize

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$$J_1' = \underline{U}_{N-1}^T E\{\underline{B}^T \underline{A}^{-T} \underline{P} \ \underline{A}^{-1}\underline{B}\}\underline{U}_{N-1} + 2\underline{U}_{N-1}^T E\{\underline{B}^T \underline{A}^{-T} \underline{P} \ \underline{A}^{-1}\underline{h}\} + \underline{U}_N^T \underline{Q} \ \underline{U}_N$$

Using $\underline{H}$ and $\underline{d}$ as defined in the Theorem we have

$$J_1' = \underline{U}_N^T \underline{H} \ \underline{U}_N + 2\underline{U}_N^T \underline{d}$$

To maximize J_1' subject to the energy constraint

$$\underline{U}_N^T \underline{U}_N \leq W$$

notice that since J_1' is strictly convex, the optimum occurs when $\underline{U}_N^T \underline{U}_N = W$, the optimization is thus equivalent to

$$\text{Min} \ -\underline{U}_N^T \underline{H} \ \underline{U}_N - 2\underline{U}_N^T \underline{d} + \lambda(\underline{U}_N^T \underline{U}_N - W)$$

where λ is a Lagrange multiplier. The necessary condition gives

$$\underline{H} \ \underline{U}_N^* + \underline{d} - \lambda\underline{U}_N^* = \underline{0} \tag{B.3}$$

If $\underline{d} \neq \underline{0}$, $\underline{U}_N^* = [-\underline{H} + \lambda\underline{I}]^{-1}\underline{d}$, which proves the first part of the Theorem.

If $\underline{d} = \underline{0}$, then (B.3) gives $\underline{H} \ \underline{U}_N^* = \lambda\underline{U}_N^*$, i.e. $\underline{U}_N^*$ is an eigenvector of $\underline{H}$. Since for all $\underline{x} \in R^{Ns}$

$$\lambda_{min}(\underline{H})\underline{x}^T\underline{x} \leq (J_1' = \underline{x}^T\underline{H} \ \underline{x}) \leq \lambda_{max}(\underline{H})\underline{x}^T\underline{x}$$

the cost attains its maximum when $\underline{U}_N^*$ is the eigenvector corresponding to the maximum eigenvalue of $\underline{H}$. This completes the proof.

APPENDIX C

The inverse $\underline{A}^{-1} \in R^{(N-1) \times (N-1)}$ can be obtained via

$$\underline{A}^{-1} = - (\underline{I} + \underline{G} + \ldots + \underline{G}^{N-2})$$

where

$$\underline{G} = \sum_{i=1}^{n} a_i \underline{I}_i$$

where

$\underline{I}_k \in R^{(N-1) \times (N-1)}$ is the shift matrix with i-jth element

$[\underline{I}_k]_{ij} = \delta_{i-j,k}.$

Since $\underline{G}^k = \sum_{i_1,\ldots,i_k=1} a_{i_1} \ldots a_{i_k} \underline{I}_{i_1} \ldots \underline{I}_{i_k}$

we have that

$$\underline{A}^{-T} \underline{P} \, \underline{A}^{-1} = \underline{P} + \sum_{i_1=1}^{n} a_{i_1} (\underline{P} \, \underline{I}_{i_1} + \underline{I}_{i_1}^{T} \underline{P})$$

$$+ \sum_{i_1,i_2=1}^{n} a_{i_1} a_{i_2} (\underline{P} \, \underline{I}_{i_1} \underline{I}_{i_2} + \underline{I}_{i_1}^{T} \underline{P} \, \underline{I}_{i_2} + \underline{I}_{i_2}^{T} \underline{I}_{i_1}^{T} \underline{P})$$

$$+ \ldots$$

$$+ \sum_{i_1,\ldots,i_{2N-5}=1}^{n} a_{i_1} \ldots a_{i_{2N-5}} (\underline{I}_{i_{N-3}}^{T} \ldots \underline{I}_{i_1}^{T} \underline{P} \, \underline{I}_{i_{N-2}} \ldots \underline{I}_{i_{2N-5}}$$

$$+ \underline{I}_{i_{N-2}}^{T} \ldots \underline{I}_{i_1}^{T} \underline{P} \, \underline{I}_{i_{N-1}} \ldots \underline{I}_{i_{2N-5}})$$

$$+ \sum_{i_1,\ldots,i_{2N-4}=1}^{n} a_{i_1} \ldots a_{i_{2N-4}} (\underline{I}_{i_{N-2}}^{T} \ldots \underline{I}_{i_1}^{T} \underline{P} \, \underline{I}_{i_{N-1}} \ldots \underline{I}_{i_{2N-4}})$$

or

$$\underline{A}^{-T}\underline{P}\,\underline{A}^{-1} = \underline{P} + \sum_{k=1}^{2N-4} \sum_{i_1,\ldots,i_k=1}^{n} a_{i_1}\ldots a_{i_k}\underline{N}_k$$

where $\underline{N}_k$ is defined by correspondence.

Expressing $\underline{B}$ as $\underline{B} = \sum_{i=1}^{m} b_i\underline{I}_{i-1}$ we have that

$$\underline{B}^T\underline{A}^{-T}\underline{P}\,\underline{A}^{-1}\underline{B} = \sum_{i,j=1}^{m} (b_ib_j\underline{Q}_{ij} + \sum_{k=1}^{2N-4}\sum_{i_1,\ldots,i_k=1}^{n} a_{i_1}\ldots a_{i_k}b_ib_j\underline{R}_{ijk}) \qquad (C.1)$$

where

$$\underline{Q}_{ij} = \underline{I}_{i-1}^T\underline{P}\,\underline{I}_{j-1}$$

$$\underline{R}_{ijk} = \underline{I}_{i-1}\underline{N}_k\underline{I}_{j-1}$$

The vector $\underline{h}$ can also be written as

$$\underline{h} = \sum_{i=1}^{n}\sum_{\ell=1}^{i} a_i Y_{\ell-i}\underline{I}_{\ell-1}\begin{bmatrix}1\\0\\ \cdot\\ \cdot\\0\end{bmatrix}$$

therefore

$$\underline{B}^T\underline{A}^{-T}\underline{P}\,\underline{A}^{-1}\underline{h} = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{\ell=1}^{i} a_ib_j Y_{\ell-1}\underline{f}_{\ell j}$$

$$+ \sum_{k=1}^{2N-4}\sum_{i,i_1,\ldots,i_k=1}^{n}\sum_{j=1}^{m}\sum_{\ell=1}^{i} a_i a_{i_1}\ldots a_{i_k}b_j Y_{\ell-i}\underline{p}_{\ell kj} \qquad (C.2)$$

where

$$\underline{f}_{\ell j} = \underline{I}_{j-1}{}^{T} \underline{P} \, \underline{I}_{\ell-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\underline{P}_{\ell k j} = \underline{I}_{j-1}{}^{T} \underline{N}_{k} \underline{I}_{\ell-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The expected values of (C.1) and (C.2) will lead to the expressions stated in the Proposition. If the vector $\underline{Y}$ has correlated variables, it can be substituted in (C.1) and (C.2) by $\underline{Y} = \underline{\Gamma}^{T} \underline{Y}_{1}$, where $\underline{Y}_{1}$ has independent variables. The resulting expectations will therefore be similar, in terms of expected values of $\underline{Y}_{1}$. This completes the proof.

APPENDIX D

The conditional error covariance matrix Eq. (2.5) corresponding to the system (5.4) is

$$\underline{\Sigma}^{-1}(N/N) = \underline{\Sigma}_{\gamma}^{-1} + \frac{1}{\sigma^2} \sum_{k=1}^{N} \underline{c}(k)\underline{c}^T(k) \qquad (D.1)$$

where $\underline{c}(k) = [y(k-1) \ldots y(k-n)u(k-1) \ldots u(k-m)]^T$. Equation (5.5) can also be written as

$$\underline{U}(N) = \underline{F}(N)\underline{Y}(N) \qquad (D.2)$$

where $\underline{U}(N) = [u(0) \ldots u(N)]^T$, $\underline{Y}(N) = [y(0) \ldots y(N)]^T$ and

$$\underline{F}(N) = \sum_{i=0}^{\ell} \alpha_i \underline{I}_i \qquad (D.3)$$

where $\underline{I}_i$ is the $(N+1) \times (N+1)$ shift matrix defined in a previous appendix.

Let $< \underline{X},\underline{Y} >$ denote the usual inner product in R^{N+1}. Thus using (D.2) in (D.1) we have that as $N \to \infty$,

$$\underline{\Sigma}^{-1}(N/N) = \begin{bmatrix} <\underline{Y}(N),\underline{Y}(N)> & <\underline{Y}(N),\underline{I}_1\underline{Y}(N)>\ldots<\underline{Y}(N),\underline{I}_{n-1}\underline{Y}(N)> & <\underline{Y}(N),\underline{F}(N)\underline{Y}(N)>\ldots \\ \vdots & & \\ <\underline{I}_{n-1}\underline{Y}(N),\underline{Y}(N)> & & \vdots \\ <\underline{F}(N)\underline{Y}(N),\underline{Y}(N)> & & \\ \vdots & & \\ <\underline{I}_{m-1}\underline{F}(N)\underline{Y}(N),\underline{Y}(N)> \quad \cdot \quad \cdot \quad \quad \cdot \quad \cdot \quad \cdot \quad <\underline{I}_{m-1}\underline{F}(N)\underline{Y}(N),\underline{F}(N)\underline{Y}(N)>\ldots \end{bmatrix}$$

$$
\left.\begin{array}{c}
<\underline{Y}(N), \underline{I}_{m-1}\underline{F}(N)\underline{Y}(N)> \\
\cdot \\
\cdot \\
\cdot \\
\\
<\underline{I}_{m-1}\underline{F}(N)\underline{Y}(N), \underline{I}_{m-1}\underline{F}(N)\underline{Y}(N)>
\end{array}\right] \frac{1}{\sigma^2} \quad \text{a.s.} \qquad (D.4)
$$

To check positive definiteness one can use Silvester's rule. By the properties of determinants and Eq. (D.3) it follows that if $\ell < n$ the matrix (D.4) is singular. This implies that as $N \to \infty$, at least one eigenvalue of $\underline{\Sigma}(N/N)$ will go to infinity a.s. and $\text{tr} = \underline{\Sigma}(N/N)$ will not be integrable. Thus $\lim_{N\to\infty} E\{(\underline{\gamma}-\hat{\underline{\gamma}}(N))^T(\underline{\gamma}-\hat{\underline{\gamma}}(N))\} = \lim_{N\to\infty} E\{\text{tr } \underline{\Sigma}(N/N)\} \to \infty.$

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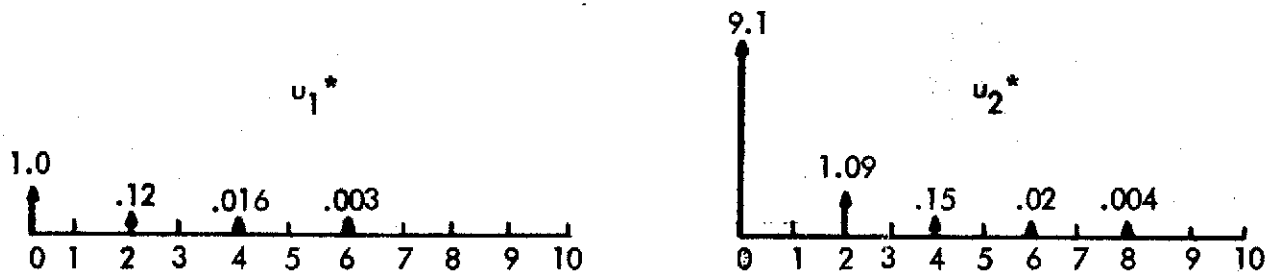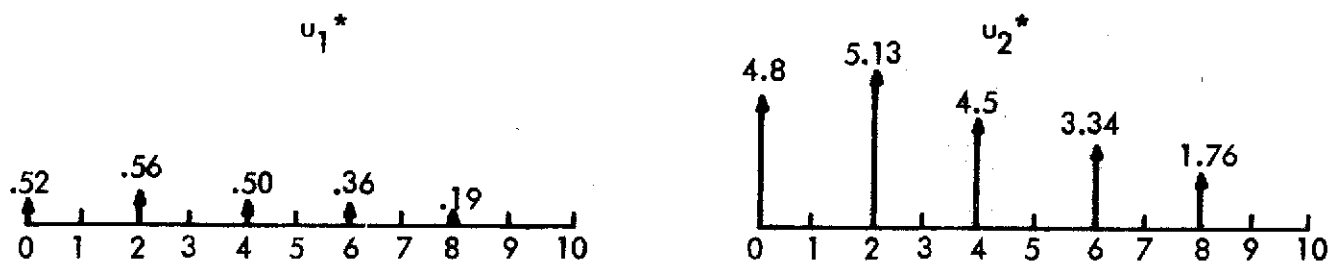
Fig. 4.1 Optimal Inputs for System (4.2)

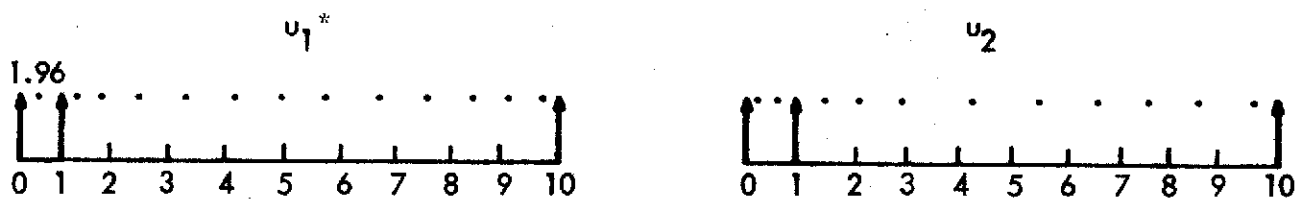Fig. 4.2 Optimal Inputs for System (4.3)

Fig. 4.3 Constant Inputs