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AN IMPROVED EXCEEDANCE THEORY FOR COMBINED RANDOM STRESSES

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16. Abstract

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SUMMARY

This paper presents an extension of Rice's classic solution for the exceedances of a constant level by a single random process to its counterpart for an n-dimensional vector process. An interaction boundary, analogous to the constant level considered by Rice for the one-dimensional case, is assumed in the form of a hypersurface. The theory for the numbers of boundary exceedances is developed by using a joint statistical approach which fully accounts for all cross-correlation effects. An exact expression is derived for the n-dimensional exceedance density function, which is valid for an arbitrary interaction boundary. For application to biaxial states of combined random stress, the general theory is reduced to the two-dimensional case. An elliptical stress-interaction boundary is assumed and the exact expression for the density function is presented. The equations are expressed in a format which facilitates calculating the exceedances by numerically evaluating a line integral. The paper concludes with a brief discussion of the behavior of the density function for the two-dimensional case.

INTRODUCTION

In the last two decades considerable progress has been made in the application of random process theory to structural dynamics problems. Typical areas of application for some of these statistical methods have been (1) response of structural components such as shells and panels to boundary-layer turbulence and acoustical noise, (2) response of vehicles to surface (runway, rail, and road) roughness, and (3) response of aircraft and launch vehicles to atmospheric turbulence (wind gusts). With regard to this last area, that is, aircraft gust response, statistical methods have been developed to the extent of acceptance in aircraft design procedures. (See refs. 1 to 4.) Reference 1 contains an extensive bibliography. Contributions which are particularly noteworthy in this area are the advances which have been made in representing atmospheric turbulence as a continuous random process (ref. 5) and the accompanying refinement of power spectral methods for predicting aircraft gust response statistics (refs. 1 and 6, for example). Concurrently, advances in computer technology have made it possible to determine accurately and economically the significant flexible vibration modes of an airframe and to include these degrees of freedom in the structural model.

In designing a structure, such as an airframe, to withstand the continuously varying stresses induced by a random loading environment, an important statistical response parameter is the number of exceedances. The exceedances are the expected (average) number of times per unit time that a single stationary random stress crosses a critical level (refs. 1 to 4, and 7). The solution to this classic problem in statistical mechanics was developed by S. O. Rice (ref. 8). Rice's one-dimensional exceedance theory finds frequent application in fatigue-related studies, particularly for predicting service life under random loadings. The number of exceedances enables approximations to be made for the first-passage probability density function, mean time to failure, and the standard deviation of the time to failure. If the process is narrow band, then the probability distribution for the peak stresses can also be calculated. (See ref. 7.)

Another typical design situation (refs. 1 to 3, and 9) requires the consideration of two (or more) stress components acting simultaneously in combination. This situation is commonly referred to as a state of combined random stress. The individual components are interpreted as defining a random stress vector. For the two-dimensional case, the service life calculation is based on the exceedances, by the stress vector, of a closed curve which is called a stress-interaction boundary.

Prior to reference 10, only the two-dimensional theory for combined random stresses had been developed. The first papers (refs. 2, 11, and 12) approximated the stress-interaction boundary as a connected union of straight-line segments. A questionable feature of these earliest exceedance theories is that they are based on relations which neglect some of the off-diagonal elements of the covariance matrix. The off-diagonal elements of this matrix account for all the cross-correlation effects between the stress and stress-rate components. Cooper (ref. 13) first recognized the potential importance of retaining the full covariance matrix and he accordingly derived an exact expression for the exceedance density function for a linear stress boundary.

In the first part of this paper a general theory is presented for calculating the exceedances of interaction boundaries in an n-dimensional stress space. A joint-statistical approach, based on the full covariance matrix, is adopted. The stress boundary is taken to be a hypersurface. An exact expression is developed for the n-dimensional exceedance density function. This analysis is followed by the specialization of the general theory to two dimensions. The exact two-dimensional exceedance density is then presented for an elliptical stress-interaction boundary. The equations for the density are expressed in a format which facilitates calculating the number of exceedances by numerically evaluating a line integral. (In ref. 10, an approximate closed-form expression is derived for this line integral by application of Laplace's method.) The paper concludes with a brief discussion of the behavior of the density function for the two-dimensional case. Some useful properties relating to integration of quadratic forms are presented in the appendix.

SYMBOLS

ds	element of surface area on stress-interaction boundary
$d\dot{X} = d\dot{x}_1 d\dot{x}_2 \cdot\ \cdot\ \cdot d\dot{x}_n$	
$d\dot{Y} = d\dot{y}_1 d\dot{y}_2 \cdot\ \cdot\ \cdot d\dot{y}_n$	
$E\{\ \}$	expected value
E_i	unit vectors, $i = 1, 2, \ldots, n$
$\mathrm{erf}(\xi)$	error function (eq. (40))
$G(X), G(Y), G(x_1,x_2), G(\theta)$	exceedance density functions
$\tilde{G}(\theta)$	defined by equation (73)
$h(\theta)$	defined by equation (77)
J	Jacobian determinant
K(Y)	defined by equation (25)
$L, \tilde{L}$	orthogonal transformation matrices, equations (11) and (17), respectively
ℓ_{ij}	element in ith row and jth column of L matrix
M	submatrix of $\tilde{M}$
$\tilde{M}$	covariance matrix (eq. (10))
$\hat{M}$	submatrix of $\hat{P}_y$ (eq. (31))

N	average number of times per unit time stress vector crosses stress-interaction boundary
N^+	exceedances, average number of times per unit time stress vector crosses stress-interaction boundary in outward sense
n	number of stress components
P	submatrix of $\tilde{M}^{-1}$ (eq. (23b))
$\hat{P}$	submatrix of P_y^{-1} (eq. (32))
$p(X;\dot{X})$	joint probability density function
Q	submatrix of $\tilde{M}^{-1}$ (eq. (23b))
$\hat{Q}$	submatrix of P_y^{-1} (eq. (32))
R	submatrix of $\tilde{M}^{-1}$ (eq. (23b))
$\hat{R}$	submatrix of P_y^{-1} (eq. (32))
$S(\omega)$	power spectrum
$\hat{S}$	submatrix of P_y (eq. (31))
T	submatrix of $\tilde{M}$ (eq. (23a))
$\hat{T}$	submatrix of P_y (eq. (31))
t	time
V	vector (eq. (26))
$\tilde{V}$	scalar (eq. (34))
V_n	component of stress-rate vector normal to stress-interaction boundary
v_i	ith component of V_y

W vector defined by equation (A5)

w_n component of vector W

$X(t)$ stress vector, subvector of $\tilde{X}(t)$

$\tilde{X}(t)$ augmented stress vector (eq. (8))

$X'(t)$ vector (eq. (55))

x_i coordinates of stress space where $i = 1, 2, \ldots, n$

$x_i(t)$ ith component of $X(t)$ where $i = 1, 2, \ldots, n$

x_{1c}, x_{2c} critical stress levels for $x_1(t)$ and $x_2(t)$, respectively

$Y(t)$ stress vector, subvector of $\tilde{Y}(t)$

$\tilde{Y}(t)$ augmented stress vector

y_i stress coordinates where $i = 1, 2, \ldots, n$

$y_i(t)$ ith component of $Y(t)$ where $i = 1, 2, \ldots, n$

Z vector (eq. (30))

$\tilde{Z}$ subvector of Z (eq. (30))

z_i ith component of Z where $i = 1, 2, \ldots, n$

$\alpha(X), \alpha(Y)$ defined by equation following equation (54) and equation (42), respectively

$\alpha(\theta)$ defined by equation (74)

α_{ij} element in ith row and jth column of P matrix

β_i root mean square of ith stress-rate component where $i = 1, 2, \ldots, n$

β_{ij} element in ith row and jth column of P^{-1} matrix

γ correlation coefficient for $\dot{x}_1(t)$ and $\dot{x}_2(t)$

η dummy variable of integration

θ coordinate defined by equation (69)

$\lambda(\theta)$ defined by equation (78)

$\bar{\mu}$ mean stress vector (eq. (55))

μ_i ith component of $\bar{\mu}$ where $i = 1, 2, \ldots, n$

ν defined by equation (76)

ρ correlation coefficient for $x_1(t)$ and $x_2(t)$

ρ'_o correlation parameter for $x_1(t)$ and $\dot{x}_2(t)$ and also for $\dot{x}_1(t)$ and $x_2(t)$ (eq. (59g))

σ_i root mean square of ith stress component where $i = 1, 2, \ldots, n$

$\tilde{\sigma}(\theta)$ defined by equation (75)

$\Phi(U)$ defined by equation (A2)

$\phi(\theta)$ defined by equation (72)

$\psi(X), \psi\left(x_1, x_2\right)$ stress-interaction boundary expressed as $\psi(X) = 0$

ω frequency, radians per unit time

Capital letters generally denote vectors or matrices. Bars over Greek letters denote vectors. $(\ )^T$ denotes matrix transpose. $(\ )^{-1}$ denotes matrix inverse. $(\dot{\ })$ denotes a differentiation with respect to time t. $(\ )_{,x}$ denotes ( ) referred to X space coordinates. $(\ )_{,y}$ denotes ( ) referred to Y space coordinates. $(\ )^+$ indicates that only outward crossings are counted.

REVIEW OF EXCEEDANCE THEORY

This section reviews previous one- and two-dimensional exceedance theories and notes two needed results from the literature.

Rice's One-Dimensional Theory

Consider, as shown in figure 1, a typical randomly varying function of time $x_1(t)$, for example, a stress induced in a structure by a random loading. It is a useful concept to think of this stress time history as a single realization of a large collection or ensemble of random stress time histories. Let x_{1c} be a constant critical stress level as illustrated by the dashed line. Of interest is the expected number of times per second N^+ that the random stress crosses the critical level in the upward sense as shown, for example, at points a and b.

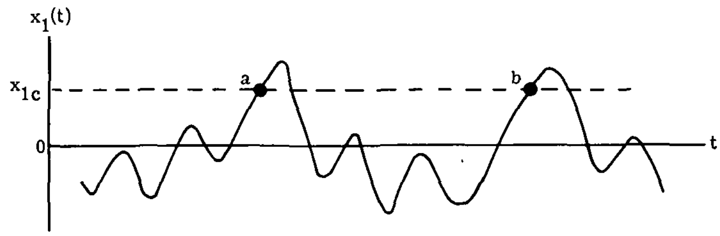

Figure 1.- Typical random stress time history showing exceedances of critical stress level x_{1c}.

Rice (ref. 8) found that for a stationary normal random process with zero mean,

$$N^+ = \frac{1}{2\pi} \frac{\beta_{1,x}}{\sigma_{1,x}} \exp\left(-\frac{x_{1c}^2}{2\sigma_{1,x}^2}\right) \tag{1}$$

in which $\sigma_{1,x}$ is the root-mean-square level of the stress $x_1(t)$, and $\beta_{1,x}$ is the root-mean-square level of the stress rate $\dot{x}_1(t)$. The root-mean-square values $\sigma_{1,x}$ and $\beta_{1,x}$ may be determined from the power spectral density $S(\omega)$ of the process $x_1(t)$ by integrations as follows:

$$\sigma_{1,x}^2 = \int_{-\infty}^{+\infty} S(\omega)\, d\omega \tag{2a}$$

$$\beta_{1,x}^2 = \int_{-\infty}^{+\infty} \omega^2 S(\omega)\, d\omega \tag{2b}$$

The statistical average N^+ is commonly referred to as the exceedances and is a constant independent of time by virtue of the assumption of stationarity.

Two-Dimensional Theory

When two or more random stress components act in combination, a state of combined random stress is said to occur and, in general, Rice's one-dimensional exceedance theory is no longer applicable. An appreciation of the two-dimensional case may be acquired with the aid of figure 2. As shown in the figure, $x_1(t)$ and $x_2(t)$ are two random stress time histories, perhaps a shear stress and an axial stress, respectively, and are interpreted to be the components of a two-dimensional stress vector $X(t)$. When a design condition depends upon both stress components, it is possible to define a closed boundary curve $\psi\left(x_1,x_2\right) = 0$ as illustrated in the figure. The closed boundary curve is referred to as a "stress-interaction boundary." In this case the exceedances N^+ are defined to be the expected number of times per second that the tip of the random stress vector crosses the boundary in the outward direction. Such crossings are illustrated by points a, b, and c of the figure.

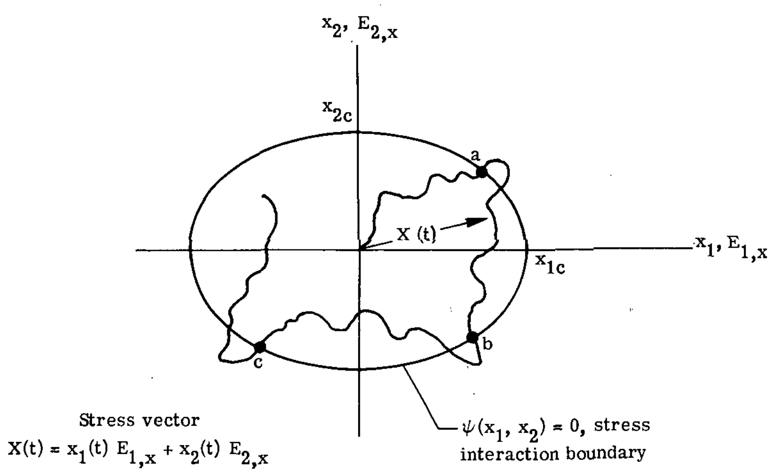

Figure 2.- Combined random stress showing exceedances of a stress-interaction boundary.

Let N (note that the + superscript is missing) be the expected number of both inward and outward crossings. For the two-dimensional case, references 2, 10, 11, and 12 present expressions for this statistical average which are equivalent to

$$N = \oint G\left(x_1,x_2\right)\,ds \tag{3a}$$

$$G\left(x_1,x_2\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|V_n\right| p\left(x_1,x_2;\dot{x}_1,\dot{x}_2\right)\,d\dot{x}_1\,d\dot{x}_2 \tag{3b}$$

where

(1) $G(x_1,x_2)$ is the exceedance density function, crossings per unit time per unit of arc length on the boundary curve $\psi(x_1,x_2) = 0$

(2) $\oint (\ )\, ds$ indicates a line integral over the boundary curve

(3) V_n is the component of the stress-rate vector $\dot{X}(t)$ along the normal to the boundary curve

(4) $p(x_1,x_2;\dot{x}_1,\dot{x}_2)$ is the joint probability density function for the components of $X(t)$ and $\dot{X}(t)$

EXCEEDANCE DENSITY FUNCTION IN n-DIMENSIONS

The goals of this section are to note an extension of the preceding ideas to an n-dimensional stress space and to obtain an exact expression for the exceedance density function.

The stress vector is assumed to have n components $\left\{x_1(t), x_2(t), \ldots, x_n(t)\right\}$ in a Cartesian space with a basis of orthogonal unit vectors $\left\{E_{1,x}, E_{2,x}, \ldots, E_{n,x}\right\}$

$$X(t) = x_1(t)\, E_{1,x} + x_2(t)\, E_{2,x} + \ldots + x_n(t)\, E_{n,x} \tag{4}$$

The components are random functions of time and are assumed to be jointly stationary with zero mean. Extension to random vectors with nonzero mean will be discussed subsequently.

The arguments ordinarily given to support equations (3) may be paralleled by use of the geometry of n-dimensional vector spaces. In an n-space the stress-interaction boundary becomes a hypersurface

$$\psi(x_1, x_2, \ldots, x_n) = 0 \tag{5}$$

The exceedance density function and joint probability function take the forms

$$G(X) = G(x_1, x_2, \ldots, x_n)$$

$$p(X;\dot{X}) = p\left(x_1, x_2, \ldots, x_n; \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n\right)$$

The expression, analogous to equations (3), for the total number of expected crossings may then be expressed in the form (ref. 10)

$$N = \oint G(X)\, ds \tag{6a}$$

$$G(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left|V_n\right| p(X;\dot{X})\, d\dot{X} \tag{6b}$$

where

(1) $d\dot{X} = d\dot{x}_1 d\dot{x}_2 \ldots d\dot{x}_n$

(2) $\oint (\;)\, ds$ indicates an integration over the interaction boundary, now a hypersurface

(3) The expression $\left|V_n\right|$ appearing in equation (6b) is given by (refs. 14 and 15)

$$\left|V_n\right| = \left| \frac{\dot{x}_1 \dfrac{\partial\psi}{\partial x_1} + \dot{x}_2 \dfrac{\partial\psi}{\partial x_2} + \ldots + \dot{x}_n \dfrac{\partial\psi}{\partial x_n}}{\sqrt{\left(\dfrac{\partial\psi}{\partial x_1}\right)^2 + \left(\dfrac{\partial\psi}{\partial x_2}\right)^2 + \ldots + \left(\dfrac{\partial\psi}{\partial x_n}\right)^2}} \right| \tag{7}$$

The stress and stress-rate components will be assumed to satisfy a joint normal (Gaussian) distribution. It is convenient to define a new vector $\tilde{X}(t)$ (call it an augmented stress vector) by

$$\tilde{X}(t) = \begin{Bmatrix} X(t) \\ \hdashline \dot{X}(t) \end{Bmatrix} \tag{8}$$

By virtue of the assumption of stationarity, it can be shown that $\dot{X}(t)$ has zero mean (ref. 16, p. 316). Since $X(t)$ has been assumed to have zero mean, it then follows that $\tilde{X}(t)$ has zero mean. The joint normal probability density then takes the special form (ref. 17)

$$p(X;\dot{X}) = \frac{(2\pi)^{-n}}{\sqrt{|\tilde{M}_X|}} \exp\left[-\frac{1}{2}\tilde{X}^T\tilde{M}_X^{-1}\tilde{X}\right] \tag{9}$$

In equation (9) $\tilde{M}_X$ is the covariance matrix for $\tilde{X}(t)$ defined by

$$\tilde{M}_X = E\left\{\tilde{X}(t)\ \tilde{X}^T(t)\right\} \tag{10}$$

where $E\{\ \}$ denotes expected value. By this definition covariance matrices are symmetric and nonnegative definite. As discussed, for example, in references 16 and 18, a singular covariance matrix implies that one or more linear combinations of the random variables $\{x_1, x_2, \ldots, x_n\}$ is zero with probability 1. In such cases the problem should be reposed in terms of a lesser number of random variables. Throughout this study $\tilde{M}_X$ will be assumed positive definite. The covariance matrix is therefore nonsingular and the inverse operation $\tilde{M}_X^{-1}$ in equation (9) is legitimate.

Consider an arbitrary point on the stress-interaction boundary and visualize the unit outward normal at this point. Now define a new vector Y with components $\{y_1, y_2, \ldots, y_n\}$ produced by the transformation

$$Y = LX \tag{11}$$

The transformation is constructed so that the y_1-axis is parallel to the unit outward normal. The matrix L is a square orthogonal matrix of order n, and its elements are functions of position on the boundary. The transformation may be thought of as a rotation of axes. The orthogonality of the matrix L may be expressed as follows:

$$\left.\begin{aligned} |L| &= 1 \\ L^{-1} &= L^T \end{aligned}\right\} \tag{12}$$

The components of $\dot{X}$ also transform according to equation (11) so that

$$\dot{Y} = L\dot{X} \tag{13}$$

The normal component of the stress-rate vector can now be written as

$$V_n = \dot{y}_1 \tag{14}$$

An augmented vector $\tilde{Y}$ is defined as

$$\tilde{Y} = \left\{ \begin{array}{c} Y \\ \hline \dot{Y} \end{array} \right\} \tag{15}$$

and it readily follows that

$$\tilde{Y} = \tilde{L}\tilde{X} \tag{16}$$

in which $\tilde{L}$ is the $2n$ by $2n$ matrix:

$$\tilde{L} = \left[\begin{array}{c|c} L & \\ \hline & L \end{array} \right] \tag{17}$$

It follows directly from equations (12) and (17) that $\tilde{L}$ is an orthogonal matrix

$$\left. \begin{array}{l} |\tilde{L}| = 1 \\ \tilde{L}^{-1} = \tilde{L}^T \end{array} \right\} \tag{18}$$

Substituting equations (8), (9), and (16) into equation (6b) yields an expression for the exceedance density function in terms of the coordinates of the Y stress space. This expression is

$$G(Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{(2\pi)^{-n}}{\sqrt{|\tilde{M}_y|}} |\dot{y}_1| \exp\left[-\frac{1}{2} \tilde{Y}^T \tilde{M}_y^{-1} \tilde{Y} \right] d\dot{Y} \tag{19}$$

where

$$d\dot{Y} = d\dot{y}_1 d\dot{y}_2 \cdot \, . \, . d\dot{y}_n \tag{20}$$

$$\tilde{M}_y = \tilde{L}\tilde{M}_x\tilde{L}^T \tag{21}$$

In determining equation (19), use was made of the fact that $d\dot{X} = d\dot{Y}$ which follows from the orthogonality of L (ref. 19).

For outward crossings $\dot{y}_1 > 0$ and therefore $\left|\dot{y}_1\right| = \dot{y}_1$. Modifying equation (19) accordingly gives for $G^+(Y)$, the exceedance density for outward crossings,

$$G^+(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cdots \int_{0}^{+\infty} \frac{(2\pi)^{-n}}{\sqrt{\left|\tilde{M}_y\right|}} \dot{y}_1 \exp\left[-\frac{1}{2}\tilde{Y}^T\tilde{M}_y^{-1}\tilde{Y}\right] d\dot{Y} \tag{22}$$

The covariance matrix $\tilde{M}_y$ and its inverse $\tilde{M}_y^{-1}$ may now be partitioned (see eqs. (A13); also ref. 17, p. 5):

$$\tilde{M}_y = \left[\begin{array}{c|c} M_y & T_y \\ \hline T_y^T & S_y \end{array}\right] \tag{23a}$$

$$\tilde{M}_y^{-1} = \left[\begin{array}{c|c} Q_y^{-1} & R_y \\ \hline R_y^T & P_y^{-1} \end{array}\right] \tag{23b}$$

where the various submatrices are all of order n by n.

In order to place equation (22) into a form for exact integration, repeated use will be made of the ideas leading to equation (A17). With the use of this equation, equation (22) can be rewritten as

$$G^+(Y) = K(Y)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \dot{y}_1 \exp\left[-\frac{1}{2}\left(\dot{Y} - V_y\right)^T P_y^{-1}\left(\dot{Y} - V_y\right)\right] d\dot{y}_1 d\dot{y}_2 \ldots d\dot{y}_n \tag{24}$$

where

$$K(Y) = \frac{(2\pi)^{-n}}{\sqrt{\left|\tilde{M}_y\right|}} \exp\left[-\frac{1}{2} Y^T M_y^{-1} Y\right] \tag{25}$$

$$V_y = -P_y R_y^T Y \tag{26}$$

and (eq. (A14c)):

$$P_y = S_y - T_y^T M_y^{-1} T_y \tag{27}$$

The components of V_y are denoted as $\{v_{1,y}, v_{2,y}, \ldots, v_{n,y}\}$.

Let Z be the vector with components $\{z_1, z_2, \ldots, z_n\}$ defined by

$$Z = \dot{Y} - V_y \tag{28}$$

Equation (24) then becomes

$$G^+(Y) = K(Y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-v_{1,y}}^{+\infty} \left(z_1 + v_{1,y}\right) \exp\left[-\frac{1}{2} Z^T P_y^{-1} Z\right] dz_1 dz_2 \cdots dz_n \tag{29}$$

Partition once more and write

$$Z = \left\{\begin{array}{c} z_1 \\ \hline \tilde{Z} \end{array}\right\} \tag{30}$$

$$P_y = \left[\begin{array}{c:c} \hat{M} & \hat{T} \\ \hdashline \hat{T}^T & \hat{S} \end{array}\right] = \left[\begin{array}{c:ccc} \alpha_{11,y} & \alpha_{12,y} & \cdots & \alpha_{1n,y} \\ \hdashline \alpha_{21,y} & \alpha_{22,y} & \cdots & \alpha_{2n,y} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1,y} & \alpha_{n2,y} & \cdots & \alpha_{nn,y} \end{array}\right] \tag{31}$$

$$P_y^{-1} = \left[\begin{array}{c|c} \hat{Q}^{-1} & \hat{R} \\ \hline \hat{R}^T & \hat{P}^{-1} \end{array}\right] = \left[\begin{array}{c|ccc} \beta_{11,y} & \beta_{12,y} & \cdots & \beta_{1n,y} \\ \hline \beta_{21,y} & \beta_{22,y} & \cdots & \beta_{2n,y} \\ \vdots & \vdots & & \vdots \\ \beta_{n1,y} & \beta_{n2,y} & \cdots & \beta_{nn,y} \end{array}\right] \tag{32}$$

As indicated z_1, $\hat{M}$, and $\hat{Q}^{-1}$ are scalars (order 1). The vector $\tilde{Z}$ is of order $n - 1$. The matrices $\hat{T}$ and $\hat{R}$ are of order 1 by $n - 1$. The matrices $\hat{S}$ and $\hat{P}^{-1}$ are square matrices of order $n - 1$.

By use once again of the results of the appendix, this partitioning allows equation (29) to be written as follows:

$$G^+(Y) = K(Y) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-v_{1,y}}^{+\infty} \left(z_1 + v_{1,y}\right) \exp\left(-\frac{z_1^2}{2\alpha_{11}}\right) \exp\left[-\frac{1}{2}\left(\tilde{Z} - \tilde{V}\right)^T \hat{P}^{-1}\left(\tilde{Z} - \tilde{V}\right)\right] dz_1 dz_2 \ldots dz_n \tag{33}$$

where the quantity $\tilde{V}$ is a scalar given by

$$\tilde{V} = -\hat{P}\hat{R}^T z_1 \tag{34}$$

The multidimensional integral on $dz_2 dz_3 \ldots dz_n$ is performed first and this integral is (eq. (A8))

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\left(\tilde{Z} - \tilde{V}\right)^T \hat{P}^{-1}\left(\tilde{Z} - \tilde{V}\right)\right] dz_2 dz_3 \ldots dz_n = (2\pi)^{\frac{n-1}{2}} \sqrt{\left|\hat{P}\right|} \tag{35}$$

in which (eq. (A14d))

$$\left|\hat{P}\right| = \left|P_y\right| \left|\hat{M}\right|^{-1} = \frac{\left|P_y\right|}{\alpha_{11,y}} \tag{36}$$

Substituting equations (35) and (36) into equation (33) produces

$$G^{+}(Y) = (2\pi)^{\frac{n-1}{2}} \sqrt{\frac{|P_y|}{\alpha_{11,y}}}\, K(Y) \int_{-v_{1,y}}^{\infty} \left(z_1 + v_{1,y}\right) \exp\left(- \frac{z_1^2}{2\alpha_{11,y}}\right) dz_1 \tag{37}$$

The remaining two integrations on z_1 may now be performed and the results are

$$\int_{-v_{1,y}}^{+\infty} z_1 \exp\left(- \frac{z_1^2}{2\alpha_{11,y}}\right) dz_1 = \alpha_{11,y} \exp\left(- \frac{v_{1,y}^2}{2\alpha_{11,y}}\right) \tag{38}$$

$$\int_{-v_{1,y}}^{+\infty} \exp\left(- \frac{z_1^2}{2\alpha_{11,y}}\right) dz_1 = \frac{1}{2}\sqrt{2\pi\alpha_{11,y}} \left[1 + \operatorname{erf}\left(\frac{v_{1,y}}{\sqrt{2\alpha_{11,y}}}\right)\right] \tag{39}$$

The error function $\operatorname{erf}(\xi)$ is defined by the integral

$$\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \exp(-\eta^2)\, d\eta \tag{40}$$

and can be found in tabulated or series form in the literature (ref. 18).

Substituting equations (38) and (39) into equation (37) gives for the exact integral

$$G^{+}(Y) = \frac{(2\pi)^{-\frac{n+1}{2}} \sqrt{\alpha_{11,y}}}{\sqrt{|M_y|}} \left[\exp\left(-\alpha^2(Y)\right) + \sqrt{\pi}\alpha(Y)\left\{1 + \operatorname{erf}\left[\alpha(Y)\right]\right\}\right] \exp\left(- \frac{1}{2}\, Y^T M_y^{-1} Y^T\right) \tag{41}$$

in which

$$\alpha(Y) = \frac{v_{1,y}}{\sqrt{2\alpha_{11,y}}} \tag{42}$$

In determining equation (41), use has been made of equation (26) and the following relation which comes from equation (A14d):

$$\frac{|P_y|}{|\tilde{M}_y|} = \frac{1}{|M_y|} \tag{43}$$

Equation (41) constitutes an exact expression for the exceedance density function and is valid for an arbitrary stress-interaction boundary. The density $G^+(Y)$ is a point function of the Y space stress coordinates. It is desirable, therefore, to reexpress the density function in terms of the original X stress coordinates.

By substituting equation (11) into the homogeneous quadratic form $Y^T M_y^{-1} Y$ (eq. (41)), it is quite easy to show that

$$Y^T M_y^{-1} Y = X^T M_x^{-1} X \tag{44a}$$

$$|M_y| = |M_x| \tag{44b}$$

An alternate form of equation (26), which defines the vector V_y, will be used:

$$V_y = T_y^T M_y^{-1} Y \tag{45}$$

This form follows from equation (A14a). The $\tilde{M}_x$ and $\tilde{M}_x^{-1}$ matrices (eq. (10)) are now partitioned in a like manner to the partitioning of the $\tilde{M}_y$ and $\tilde{M}_y^{-1}$ matrices in equations (23):

$$\tilde{M}_x = \left[\begin{array}{c|c} M_x & T_x \\ \hline T_x^T & S_x \end{array}\right] \tag{46a}$$

$$\tilde{M}_x^{-1} = \left[\begin{array}{c|c} Q_x^{-1} & R_x \\ \hline R_x^T & P_x^{-1} \end{array}\right] \tag{46b}$$

Next, by using equations (10), (16), and (46), it can be shown that

$$\left.\begin{aligned} M_y &= LM_xL^T \\ T_y^T &= LT_x^TL^T \end{aligned}\right\} \tag{47}$$

Substituting equations (16) and (47) into equation (45) gives

$$V_y = LV_x \tag{48}$$

where

$$V_x = T_x^TM_x^{-1}X \tag{49}$$

The components of the vector V_x are denoted as $\{v_{1,x}, v_{2,x}, \ldots, v_{n,x}\}$.

Let ℓ_{ij} be the scalar element in the ith row and jth column of the transformation matrix L. The elements ℓ_{1j} (j = 1, 2, . . ., n) are the components of the unit outward normal vector in the X system. From equation (48) the first component of V_y is

$$v_{1,y} = \sum_{j=1}^{n} \ell_{1j}v_{j,x} \tag{50}$$

The remaining quantity to be transformed is $\alpha_{11,y}$ (eq. (31)). The elements of the matrix P_x are denoted by

$$P_x = S_x - T_x^TM_x^{-1}T_x = \begin{bmatrix} \alpha_{11,x} & \alpha_{12,x} & \cdots & \alpha_{1n,x} \\ \alpha_{21,x} & \alpha_{22,x} & \cdots & \alpha_{2n,x} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1,x} & \alpha_{n2,x} & \cdots & \alpha_{nn,x} \end{bmatrix} \tag{51}$$

and, as before,

$$P_y = LP_xL^T \tag{52}$$

Substituting equations (31) and (51) into equation (52) gives

$$\alpha_{11,y} = \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_{1i}\ell_{1j}\alpha_{ij,x} \tag{53}$$

By collecting results

$$G^+(X) = \frac{(2\pi)^{-\frac{n+1}{2}} \sqrt{\alpha_{11,y}}}{\sqrt{|M_x|}} \left[\exp\left(-\alpha^2(X)\right) + \sqrt{\pi}\,\alpha(X)\left\{1 + \mathrm{erf}(\alpha(X))\right\}\right] \exp\left(-\frac{1}{2} X^T M_x^{-1} X\right) \tag{54}$$

in which (from eqs. (42), (50), and (53))

$$\alpha(X) = \frac{v_{1,y}}{\sqrt{2\alpha_{11,y}}}$$

$$v_{1,y} = \sum_{i=1}^{n} \ell_{1i} v_{i,x}$$

$$\alpha_{11,y} = \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_{1i}\ell_{1j}\alpha_{ij,x}$$

This section is concluded by commenting on the modifications when the random vector process has nonzero mean. Assume now that $X(t)$ has a nonzero-mean vector $\bar{\mu}_x$ with components $\left\{\mu_{1,x}, \mu_{2,x}, \ldots, \mu_{n,x}\right\}$. The components of the mean vector are constants for stationary processes. Define a new random vector process

$$X'(t) = X(t) - \bar{\mu}_x \tag{55}$$

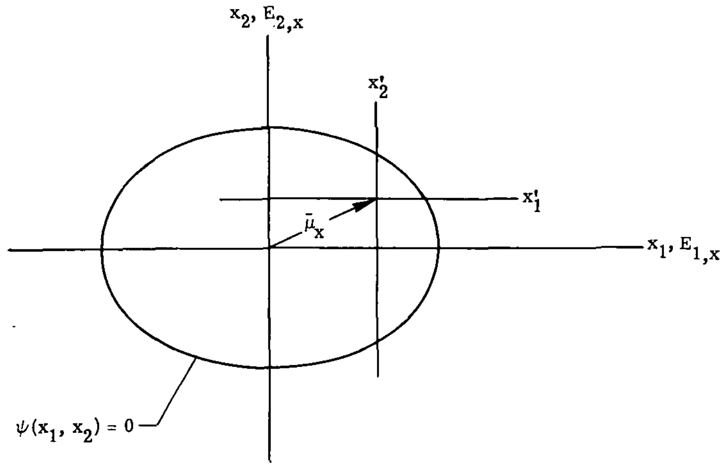

Figure 3.- Stress-interaction boundary showing effect of nonzero-mean stress vector.

The process $X'(t)$ has mean zero by definition. As illustrated in figure 3, for the two-dimensional case, the transformation equation (55) may be viewed as a shift of origin. Therefore, to apply equation (54) to this situation, it is necessary only to

(1) Replace $X(t)$ by $X'(t)$

(2) Replace the covariance matrix $\tilde{M}_X$ by the covariance matrix $\tilde{M}_{X'}$ for the (zero-mean) $X'(t)$ process

(3) Express the stress-interaction boundary in terms of the x_i' stress coordinates as follows:

$$\psi\left(x_1' + \mu_{1,x}, x_2' + \mu_{2,x}, \ldots, x_n' + \mu_{n,x}\right) = 0$$

THE TWO-DIMENSIONAL CASE

In this section an elliptical stress-interaction boundary is assumed and the previous results are specialized to two dimensions. The characteristic behavior of the exceedance density function is then discussed.

Reduction of n-Dimensional Theory

For the two-dimensional case the augmented stress vector has the form

$$
\mathbf{X}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \hline \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \tag{56}
$$

in which $x_1(t)$ and $x_2(t)$ have zero mean. As previously discussed, the bias imposed by nonzero-mean stress components is accounted for by expressing the stress-interaction boundary as

$$
\psi\left(x_1, x_2\right) = \left(\frac{x_1 + \mu_{1,x}}{x_{1c}}\right)^2 + \left(\frac{x_2 + \mu_{2,x}}{x_{2c}}\right)^2 - 1 = 0 \tag{57}
$$

The constants $\mu_{i,x}$ are the mean stresses for the original random processes $x_i(t) + \mu_{i,x}$ $(i = 1, 2)$.

The covariance matrix for $\tilde{X}(t)$ will be written as

$$
\tilde{M}_x = \left[\begin{array}{cc|cc} \sigma^2_{1,x} & \rho\sigma_{1,x}\sigma_{2,x} & 0 & -\rho'_o\sigma_{1,x}\sigma_{2,x} \\ \rho\sigma_{1,x}\sigma_{2,x} & \sigma^2_{2,x} & \rho'_o\sigma_{1,x}\sigma_{2,x} & 0 \\ \hline 0 & \rho'_o\sigma_{1,x}\sigma_{2,x} & \beta^2_{1,x} & \gamma\beta_{1,x}\beta_{2,x} \\ -\rho'_o\sigma_{1,x}\sigma_{2,x} & 0 & \gamma\beta_{1,x}\beta_{2,x} & \beta^2_{2,x} \end{array}\right] \tag{58}
$$

in which

$$
\sigma^2_{1,x} = E\left\{x_1^2(t)\right\} \tag{59a}
$$

$$
\sigma^2_{2,x} = E\left\{x_2^2(t)\right\} \tag{59b}
$$

$$
\beta^2_{1,x} = E\left\{\dot{x}_1^2(t)\right\} \tag{59c}
$$

$$\beta_{2,x}^2 = E\left\{\dot{x}_2^2(t)\right\} \tag{59d}$$

$$\rho = \frac{1}{\sigma_{1,x}\sigma_{2,x}} E\left\{x_1(t)\ x_2(t)\right\} \tag{59e}$$

$$\gamma = \frac{1}{\beta_{1,x}\beta_{2,x}} E\left\{\dot{x}_1(t)\ \dot{x}_2(t)\right\} \tag{59f}$$

$$\rho_o' = \frac{1}{\sigma_{1,x}\sigma_{2,x}} E\left\{\dot{x}_1(t)\ x_2(t)\right\} = -\frac{1}{\sigma_{1,x}\sigma_{2,x}} E\left\{x_1(t)\ \dot{x}_2(t)\right\} \tag{59g}$$

The zero off-diagonal terms appearing in $\tilde{M}_X$ follow from the well-known results for stationary processes:

$$E\left\{x_i(t)\ \dot{x}_i(t)\right\} = 0 \qquad (i = 1, 2)$$

The correlation coefficients ρ and γ and the cross-correlation parameter ρ_o' can be shown to satisfy the following inequalities (ref. 16):

$$\left.\begin{array}{l} -1 \leqq \rho \leqq 1 \\ \\ -1 \leqq \gamma \leqq 1 \\ \\ -\dfrac{\beta_{1,x}}{\sigma_{1,x}} \leqq \rho_o' \leqq \dfrac{\beta_{1,x}}{\sigma_{1,x}} \\ \\ -\dfrac{\beta_{2,x}}{\sigma_{2,x}} \leqq \rho_o' \leqq \dfrac{\beta_{2,x}}{\sigma_{2,x}} \end{array}\right\} \tag{60}$$

It should be noted that ρ_o' is the cross-correlation parameter which is neglected in the exceedance theories of references 2, 11, and 12. If $\rho_o' = 0$, then the covariance matrix $\tilde{M}_X$ (eq. (58)) assumes a block-diagonal form and the calculation of the exceedances is considerably simplified.

The elements ℓ_{1j} (j = 1, 2) of the transformation matrix L are the components of the unit outward normal vector to the elliptical boundary:

$$\left.\begin{aligned}
\ell_{11} &= \frac{\dfrac{x_1 + \mu_{1,x}}{x_{1c}^2}}{\sqrt{\left(\dfrac{x_1 + \mu_{1,x}}{x_{1c}^2}\right)^2 + \left(\dfrac{x_2 + \mu_{2,x}}{x_{2c}^2}\right)^2}} \\
\ell_{12} &= \frac{\dfrac{x_2 + \mu_{2,x}}{x_{2c}^2}}{\sqrt{\left(\dfrac{x_1 + \mu_{1,x}}{x_{1c}^2}\right)^2 + \left(\dfrac{x_2 + \mu_{2,x}}{x_{2c}^2}\right)^2}}
\end{aligned}\right\} \tag{61}$$

By use of equations (46a) and (58),

$$M_x^{-1} = \frac{1}{1 - \rho^2}\begin{bmatrix} 1/\sigma_{1,x}^2 & -\rho/\sigma_{1,x}\sigma_{2,x} \\ -\rho/\sigma_{1,x}\sigma_{2,x} & 1/\sigma_{2,x}^2 \end{bmatrix} \tag{62}$$

The determinant for M_x is

$$\left|M_x\right| = \sigma_{1,x}\sigma_{2,x}\left(1 - \rho^2\right) \tag{63}$$

From equations (46) and equation (A14c)

$$P_x = \begin{bmatrix} \beta_{1,x}^2 - \tilde{\rho}^2\sigma_{1,x}^2 & \gamma\beta_{1,x}\beta_{2,x} - \rho\tilde{\rho}^2\sigma_{1,x}\sigma_{2,x} \\ \gamma\beta_{1,x}\beta_{2,x} - \rho\tilde{\rho}^2\sigma_{1,x}\sigma_{2,x} & \beta_{2,x}^2 - \tilde{\rho}^2\sigma_{2,x}^2 \end{bmatrix} \tag{64}$$

where

$$\tilde{\rho}^2 = \frac{\rho_o'^2}{1 - \rho^2} \tag{65}$$

Now from equations (46a), (49), and (62)

$$\left.\begin{aligned} v_{1,x} &= \frac{\rho_o'}{1 - \rho^2}\left(-\rho x_1 + \frac{\sigma_{1,x}}{\sigma_{2,x}} x_2\right) \\ v_{2,x} &= \frac{\rho_o'}{1 - \rho^2}\left(-\frac{\sigma_{2,x}}{\sigma_{1,x}} x_1 + \rho x_2\right) \end{aligned}\right\} \tag{66}$$

Equation (50) becomes

$$v_{1,y} = \ell_{11} v_{1,x} + \ell_{12} v_{2,x} \tag{67}$$

Next, with the aid of equations (51) to (53) and (64)

$$\alpha_{11,y} = \left(\beta_{1,x}^2 - \tilde{\rho}^2\sigma_{1,x}^2\right)\ell_{11}^2 + 2\left(\gamma\beta_{1,x}\beta_{2,x} - \rho\tilde{\rho}^2\sigma_{1,x}\sigma_{2,x}\right)\ell_{11}\ell_{12} + \left(\beta_{2,x}^2 - \tilde{\rho}^2\sigma_{2,x}^2\right)\ell_{12}^2 \tag{68}$$

In order to facilitate numerical integration a point on the stress-interaction boundary is expressed in terms of a single parameter θ as follows:

$$\left.\begin{aligned} x_1 + \mu_{1,x} &= x_{1c} \cos \theta \\ x_2 + \mu_{2,x} &= x_{2c} \sin \theta \end{aligned}\right\} \tag{69}$$

The exceedances N^+ (see eq. (3a)) can now be determined from

$$N^+ = \int_0^{2\pi} G^+(\theta)\, d\theta \tag{70}$$

whereby using the transformation equations (69) and previous results, $G^+(\theta)$ can be computed from the following relations:

$$G^+(\theta) = \phi(\theta)\ \exp(\nu h(\theta)) \tag{71}$$

$$\phi(\theta) = \frac{\left(\frac{x_{1c}}{\sigma_{1,x}}\right)\left(\frac{x_{2c}}{\sigma_{2,x}}\right)}{(2\pi)^{3/2}\sqrt{1-\rho^2}}\ \tilde{G}(\theta)\ \tilde{\sigma}(\theta) \tag{72}$$

$$\tilde{G}(\theta) = \exp(-\alpha^2(\theta)) + \sqrt{\pi}\,\alpha(\theta)\left[1 + \operatorname{erf}\left\{\alpha(\theta)\right\}\right] \tag{73}$$

$$\alpha(\theta) = \frac{\rho_o'}{\sqrt{2}\left(1-\rho^2\right)\tilde{\sigma}(\theta)}\left\{-\rho\cos 2\theta + \frac{1}{2}\frac{x_{2c}/\sigma_{2,x}}{x_{1c}/\sigma_{1,x}}\left[1 - \left(\frac{x_{1c}/\sigma_{1,x}}{x_{2c}/\sigma_{2,x}}\right)^2\right]\sin 2\theta + \left(\rho\frac{\mu_{1,x}}{x_{1c}}\right.\right.$$

$$\left.\left. - \frac{\mu_{2,x}}{x_{2c}}\frac{x_{2c}/\sigma_{2,x}}{x_{1c}/\sigma_{1,x}}\right)\cos\theta - \left(\rho\frac{\mu_{2,x}}{x_{2c}} - \frac{\mu_{1,x}}{x_{1c}}\frac{x_{1c}/\sigma_{1,x}}{x_{2c}/\sigma_{2,x}}\right)\sin\theta\right\} \tag{74}$$

$$\tilde{\sigma}(\theta) = \left[\frac{\left(\beta_{1,x}/\sigma_{1,x}\right)^2 - \tilde{\rho}^2}{\left(x_{1c}/\sigma_{1,x}\right)^2}\cos^2\theta + 2\,\frac{\gamma\,\frac{\beta_{1,x}}{\sigma_{1,x}}\frac{\beta_{2,x}}{\sigma_{2,x}} - \rho\tilde{\rho}^2}{\frac{x_{1c}}{\sigma_{1,x}}\frac{x_{2c}}{\sigma_{2,x}}}\sin\theta\cos\theta + \frac{\left(\beta_2/\sigma_{1,x}\right)^2 - \tilde{\rho}^2}{\left(x_{2c}/\sigma_{2,x}\right)^2}\sin^2\theta\right]^{1/2} \tag{75}$$

$$\nu = -\frac{1}{2\left(1-\rho^2\right)} \tag{76}$$

$$h(\theta) = \left(\frac{x_{1c}}{\sigma_{1,x}}\right)^2\cos^2\theta - 2\rho\frac{x_{1c}}{\sigma_{1,x}}\frac{x_{2c}}{\sigma_{2,x}}\sin\theta\cos\theta + \left(\frac{x_{2c}}{\sigma_{2,x}}\right)^2\sin^2\theta + 2\left(1-\rho^2\right)\lambda(\theta) \tag{77}$$

$$\lambda(\theta) = \frac{1}{1-\rho^2}\left\{\frac{x_{1c}}{\sigma_{1,x}}\left(\rho\frac{\mu_{2,x}}{x_{2c}}\frac{x_{2c}}{\sigma_{2,x}} - \frac{\mu_{1,x}}{x_{1c}}\frac{x_{1c}}{\sigma_{1,x}}\right)\cos\theta + \frac{x_{2c}}{\sigma_{2,x}}\left(\rho\frac{\mu_{1,x}}{x_{1c}}\frac{x_{1c}}{\sigma_{1,x}} - \frac{\mu_{2,x}}{x_{2c}}\frac{x_{2c}}{\sigma_{2,x}}\right)\sin\theta\right.$$

$$\left. + \frac{1}{2}\left[\left(\frac{\mu_{1,x}}{x_{1c}}\right)^2\left(\frac{x_{1c}}{\sigma_{1,x}}\right)^2 + \left(\frac{\mu_{2,x}}{x_{2c}}\right)^2\left(\frac{x_{2c}}{\sigma_{2,x}}\right)^2 - 2\rho\frac{\mu_{1,x}}{x_{1c}}\frac{\mu_{2,x}}{x_{2c}}\frac{x_{1c}}{\sigma_{1,x}}\frac{x_{2c}}{\sigma_{2,x}}\right]\right\} \tag{78}$$

Characteristic Behavior of Exceedance Density Function

This section presents some trends in the behavior of the exceedance density function for the two-dimensional case. The elliptical stress-interaction boundary will be specialized to a unit circle. The cross-correlation parameters ρ, γ, and ρ_o' and the root-mean-square parameters $\sigma_{1,x}$, $\sigma_{2,x}$, $\beta_{1,x}$, and $\beta_{2,x}$ will be varied from the following base values:

$$\left.\begin{aligned} \sigma_{1,x} &= \sigma_{2,x} = 1 \\ \beta_{1,x} &= \beta_{2,x} = 1 \\ \rho &= \gamma = \rho_o' = 0 \\ \mu_{1,x} &= \mu_{2,x} = 0 \\ x_{1c} &= x_{2c} = 1 \end{aligned}\right\} \tag{79}$$

The main purpose is to indicate the manner in which these parameters individually affect the behavior of the density function. For this reason results will be shown for some parameter combinations which are not physically realizable. For example, with stationary processes $\rho = 1$ implies that $\gamma = 1$ and also that $\rho_o' = 0$, whereas results will be presented for $\rho \to 1$, $\gamma = 0$ and $\rho = 0$, $\gamma \to 1$. The bounds imposed by equations (60) will be maintained.

Figure 4 shows the effect of varying ρ, the cross-correlation coefficient for the stress components $x_1(t)$ and $x_2(t)$. For a value of $\rho = 0$, the density is a constant function. This condition is expected because with zero cross correlation and a circular boundary, there is no preferred direction for the stress vector. As ρ approaches unity

($\rho \to 1$), prominent peaks emerge in the density at $\theta = 45^{o}$ and $\theta = 225^{o}$. This is also expected. As $\rho \to 1$ the stress vector tends to fall more and more along the line $x_2 = x_1$ shown in the inset, top right of figure 4. For the limiting value of $\rho = 1$, it can be shown (ref. 16) that $x_2(t) = Cx_1(t)$ with probability 1. For the present case, the constant $C = 1$. It thus appears plausible that as $\rho \to 1$, the density would take on the characteristics of two delta functions located 180^{o} apart.

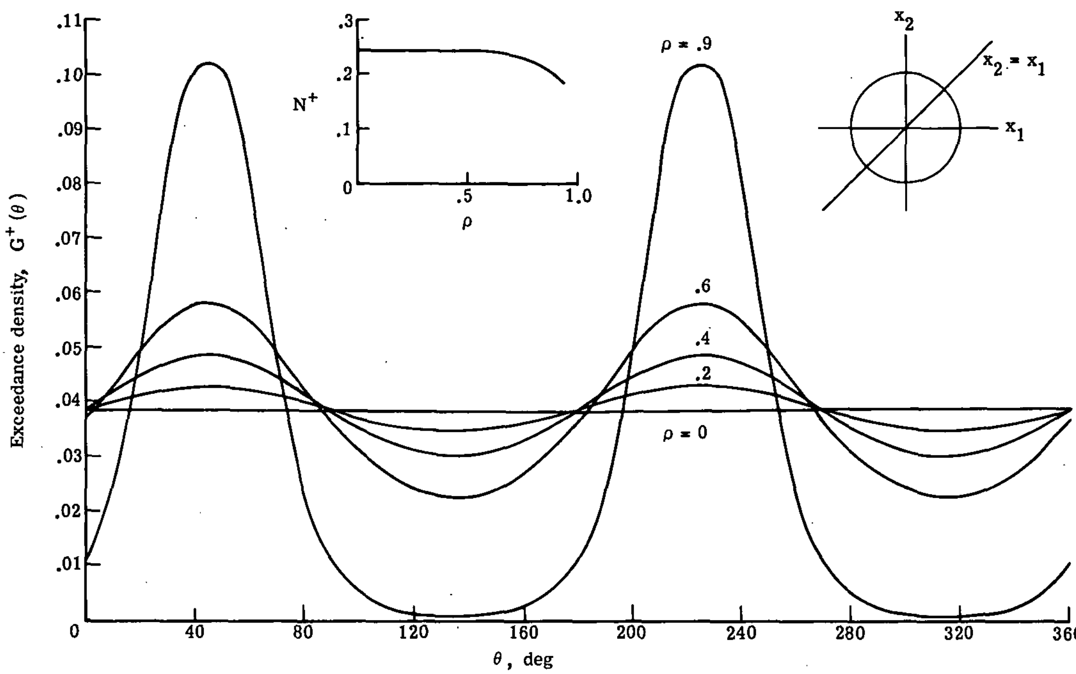

Figure 4.- Exceedance density function showing effect of cross-correlation coefficient ρ. (For base case $\sigma_{1,x} = \sigma_{2,x} = \beta_{1,x} = \beta_{2,x} = 1$; $\gamma = \rho_o' = 0$; $\mu_{1,x} = \mu_{2,x} = 0$; $x_{1c} = x_{2c} = 1$.)

The inset plot, top center of figure 4, shows the variation of the number of exceedances N^+ with ρ. The numerical value of N^+ was obtained by numerically evaluating the area under the corresponding density curve. For values of ρ up to about 0.6, the curve is flat and then gradually falls off for higher values of ρ.

Figure 5 shows the bias imposed on the density by unequal root-mean-square stress levels. Here an off-nominal correlation coefficient of $\rho = 0.6$ has been arbitrarily selected and the densities plotted for $\sigma_{1,x} = 1$ and $\sigma_{1,x} = 2$. As might be expected the peak at 45^{o} for $\sigma_{1,x} = 1$ shifts toward the x_1-axis when $\sigma_{1,x} = 2$. A similar shift is noticed for the peak originally occurring at 225^{o}. Exceedance values for the two cases are shown in the table at the top of the figure. Doubling $\sigma_{1,x}$ while holding the root-mean-square stress rate $\beta_{1,x}$ fixed has the effect of reducing the number of exceedances. The same trend is apparent in Rice's one-dimensional theory. (See eq. (1).)

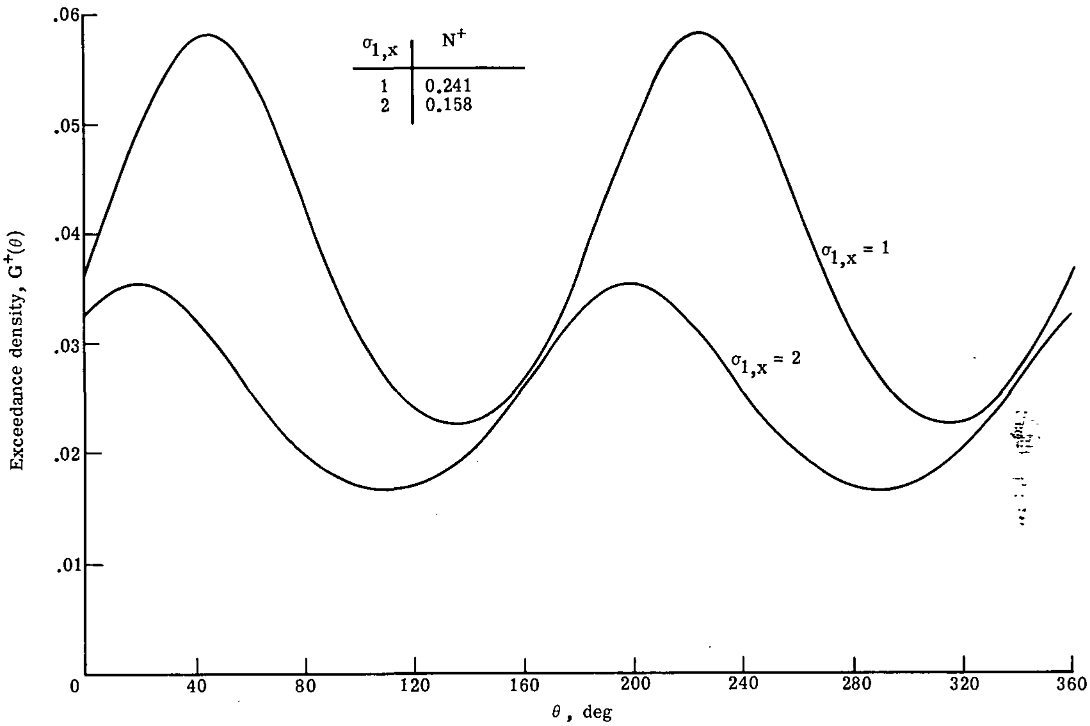

Figure 5.- Exceedance density function for $\rho = 0.6$ showing effect of root-mean-square level $\sigma_{1,x}$. $\left(\text{For base case, } \sigma_{2,x} = \beta_{1,x} = \beta_{2,x} = 1;\ \gamma = \rho_o' = 0;\ \mu_{1,x} = \mu_{2,x} = 0;\ x_{1c} = x_{2c} = 1.\right)$

In figure 6 the effect of varying γ, the cross-correlation coefficient for the stress-rate components $\dot{x}_1(t)$ and $\dot{x}_2(t)$, is shown. As the correlation is increased from $\gamma = 0$ (the base case for which $G^+(\theta)$ is a constant function) the density again begins to peak at 45^o and 225^o. The peaks, however, are not as sharp as in the previous cases (fig. 4). This behavior is also expected. For $\gamma = 1$, reflecting perfect correlation between the stress-rate components, $\dot{x}_2 = \dot{x}_1$ with probability 1. Therefore, the velocity vector remains parallel to the line $x_2 = x_1$. For this limiting situation the density is zero only at the tangent points of 135^o and 315^o.

The exceedances N^+ for this case are shown plotted against γ in the inset of figure 6. The behavior is similar to that discussed previously.

The effect of the cross-correlation parameter ρ_o' on the exceedance density function is illustrated in figure 7. It is this parameter which is neglected in the exceedance theories presented in references 2, 10, and 11. For the present choice of parameters the

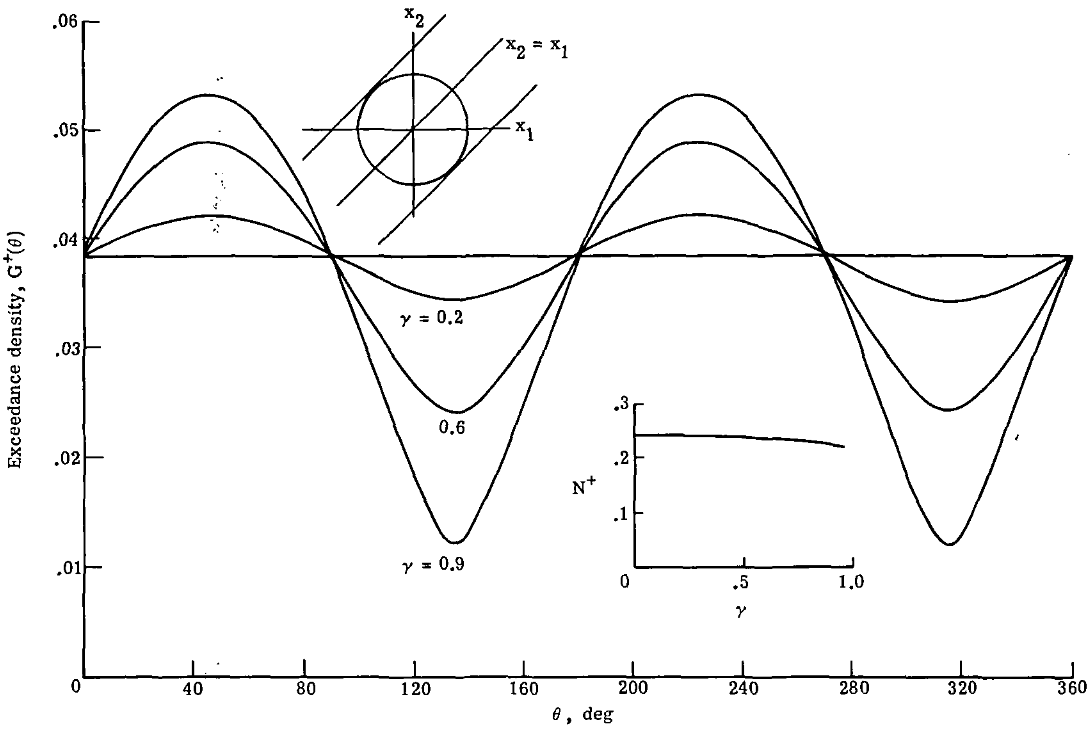

Figure 6.- Exceedance density function showing effect of cross-correlation coefficient γ. (For base case, $\sigma_{1,x} = \sigma_{2,x} = \beta_{1,x} = \beta_{2,x} = 1$; $\rho = \rho_0' = 0$; $\mu_{1,x} = \mu_{2,x} = 0$; $x_{1c} = x_{2c} = 1.$)

exceedance density is a constant function throughout its domain $0 \leqq \theta \leqq 2\pi$. Reference to equations (71) to (78) reveals that this is indeed the case. As the cross-correlation parameter is increased (from $\rho_0' = 0$), the level of the density function is gradually reduced as illustrated in the figure. This same trend appears in the inset to figure 7, which shows the variation of the exceedances N^+ with ρ_0'.

The case with $\rho_0' = 0$ has been discussed previously (fig. 4). It is of interest, however, to discuss the limiting case as $\rho_0' \rightarrow 1.0$. For $\rho_0' = 1.0$, there is perfect correlation between x_1 and $\dot{x}_2$ and between x_2 and $\dot{x}_1$. Therefore, with probability 1

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}\right\} \tag{80}$$

As a consequence of equations (80), the outward normal component V_n of velocity

$$V_n = x_1\dot{x}_1 + x_2\dot{x}_2 \tag{81}$$

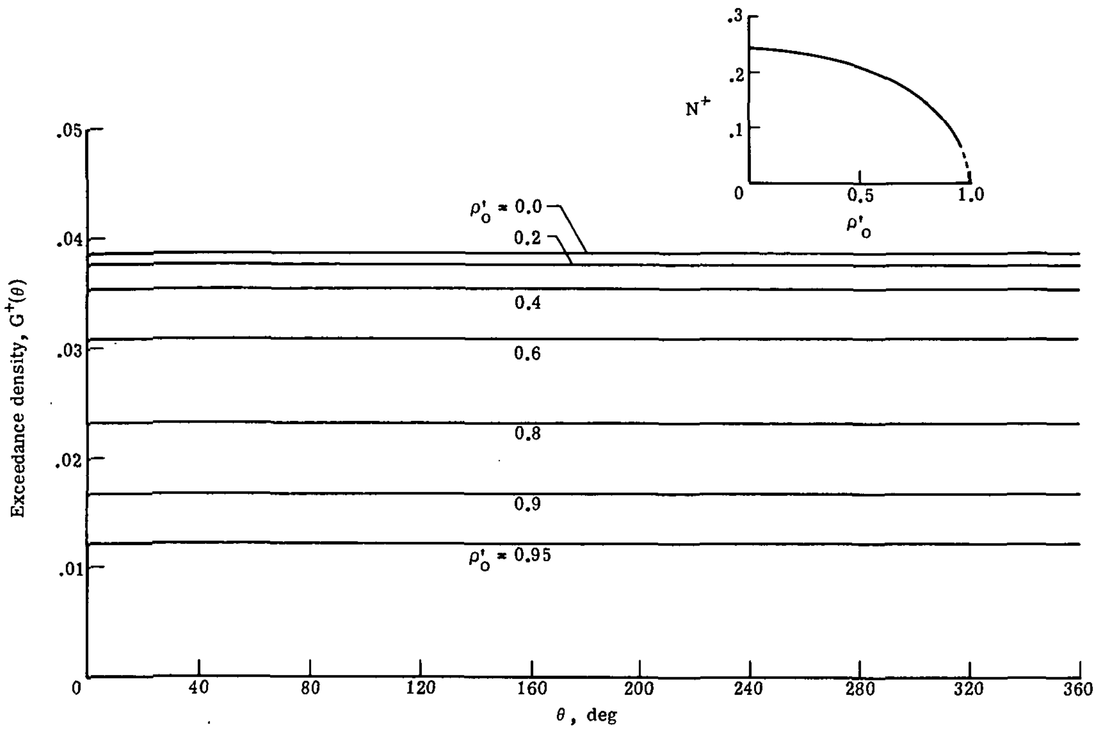

Figure 7.- Exceedance density function showing effect of cross-correlation parameter ρ'_o. (For base case, $\sigma_{1,x} = \sigma_{2,x} = \beta_{1,x} = \beta_{2,x} = 1$; $\rho = \gamma = 0$; $\mu_{1,x} = \mu_{2,x} = 0$; $x_{1c} = x_{2c} = 1$.)

vanishes with probability 1 and $G^+(\theta) = 0$. This trend is indicated by the dashed line in the inset of figure 7. Therefore, the parameter ρ'_o, which is customarily nulled, may be significant.

CONCLUDING REMARKS

A theory for combined random stresses has been developed for predicting the expected number of times per unit time that an n-dimensional random stress vector crosses a stress-interaction boundary (hypersurface). A joint statistical approach is employed and accounts for all cross-correlation effects between the stress components. An exact expression is obtained for the n-dimensional exceedance density function for an arbitrary stress boundary.

The general n-dimensional theory is reduced for application to biaxial states of combined random stress. An elliptical stress-interaction boundary is assumed and the density function is expressed in a form convenient for calculating the exceedances by numerically evaluating a line integral. The characteristic behavior of the exceedance density function for a circular stress boundary is discussed for some nominal parameter combinations.

Of particular theoretical interest is the influence of a cross-correlation parameter ρ_0' which has usually been neglected in previous exceedance theories. This parameter is representative of the measure of cross correlation between the stress and stress-rate components. For some combinations of parameters this cross-correlation effect was found to have a significant influence on the exceedances. It remains to be seen, however, whether the conditions under which this cross correlation becomes important are realized in practical combined stress design applications. The improved exceedance theory presented herein should provide a mathematical basis for further research in this area.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., December 28, 1973.

APPENDIX

INTEGRALS OF QUADRATIC FORMS

For the convenience of the reader this appendix presents some useful properties relating to integration of quadratic forms. The notation follows very closely that of the main text and of reference 17. The same results can also be found elsewhere in the literature.

It is desired to evaluate the integral

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Phi(U)\, dU \tag{A1}$$

where the vector U has the components $\{u_1, u_2, \ldots, u_n\}$ and

$$\left.\begin{aligned} \Phi(U) &= \exp\left[-\frac{1}{2} U^T C^{-1} U\right] \\ dU &= du_1 du_2 \ldots du_n \end{aligned}\right\} \tag{A2}$$

It will be assumed herein that C is a constant symmetric positive-definite matrix. There then exists a nonsingular n by n matrix Q such that

$$C = Q^T Q \tag{A3}$$

$$|C| = |Q|^2 \tag{A4}$$

Now make a transformation of coordinates

$$W = \left(Q^{-1}\right)^T U \tag{A5}$$

and denote the components of W as $\{w_1, w_2, \ldots, w_n\}$. By substituting equation (A4) into equation (A1), it follows that

$$I = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[w_1^2 + w_2^2 + \ldots + w_n^2\right] J\, dw_1 dw_2 \ldots dw_n \tag{A6}$$

where J is the Jacobian determinant of the transformation (eq. (A5)). From the well-known result of reference 17,

$$\int_{-\infty}^{+\infty} \exp\left(-\eta^2/2\right)d\eta = \sqrt{2\pi} \tag{A7}$$

and by using equation (A4), it follows that

$$I = (2\pi)^{n/2}\sqrt{|C|} \tag{A8}$$

Partition U as

$$U = \left\{\begin{array}{c} U_{(a)} \\ \hdashline U_{(b)} \end{array}\right\} \tag{A9}$$

in which

$$U_{(a)} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_k \end{Bmatrix} \tag{A10a}$$

and

$$U_{(b)} = \begin{Bmatrix} u_{k+1} \\ u_{k+2} \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{Bmatrix} \tag{A10b}$$

APPENDIX

Consider the integral

$$I_{(b)} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Phi(U)\, dU_{(b)} \tag{A11}$$

where

$$dU_{(b)} = du_{k+1} du_{k+2} \cdots du_n \tag{A12}$$

The matrix C and the inverse C^{-1} are partitioned as follows:

$$C = \left[\begin{array}{c|c} M & T \\ \hline T^T & S \end{array}\right] \tag{A13a}$$

$$C^{-1} = \left[\begin{array}{c|c} Q^{-1} & R \\ \hline R^T & P^{-1} \end{array}\right] \tag{A13b}$$

The various submatrices appearing in equations (A13) have the following orders:

M and Q^{-1} . . . . k by k
T and R . . . . k by $(n - k)$
S and P^{-1} . . . . $(n - k)$ by $(n - k)$

The submatrices can be shown to satisfy the following relations (see, for example, ref. 17):

$$PR^T = -T^T M^{-1} \tag{A14a}$$

$$Q^{-1} = M^{-1} + RPR^T \tag{A14b}$$

$$P = S - T^T M^{-1} T \tag{A14c}$$

$$|P| = |C||M|^{-1} \tag{A14d}$$

With some manipulation, equation (A11) can be written as

$$I_{(b)} = \exp\left[-\frac{1}{2} U_{(a)}^T M^{-1} U_{(a)}\right] \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\left(U_{(b)} - V\right)^T P^{-1}\left(U_{(b)} - V\right)\right] dU_{(b)} \tag{A15}$$

in which

$$V = -PR^T U_{(a)} \tag{A16}$$

Applying equation (A8) gives

$$I_{(b)} = (2\pi)^{\frac{n-k}{2}} \sqrt{\frac{|C|}{|M|}} \exp\left[-\frac{1}{2} U_{(a)}^T M^{-1} U_{(a)}\right] \tag{A17}$$

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