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ANALYSIS OF FINITE-SIZE PHASED ARRAYS OF CIRCULAR WAVEGUIDE ELEMENTS

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16. Abstract

A derivation is presented for the calculation of the interelement mutual coupling in a finite-size planar array of waveguide-fed apertures covered by a multilayered dielectric and/or plasma. The general mutual admittance expression is evaluated for circular apertures and the mutual coupling calculations are verified experimentally for two transverse electric (TE_{11}) circular waveguide mode excited apertures. A parametric study of higher order mode aperture fields indicates that the only significant change in the circular aperture mutual coupling is due to the transverse magnetic (TM_{11}) mode, which introduces an additional phase shift. Qualitative agreement between calculations for a 183-element array of circular apertures and an infinite array establishes the validity of the finite-array theoretical model.

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SUMMARY

A derivation is presented for the calculation of the interelement mutual coupling in a finite-size planar array of waveguide-fed apertures covered by a multilayered dielectric and/or plasma. The general mutual admittance expression is evaluated for circular apertures and the mutual coupling calculations are verified experimentally for two transverse electric (TE_{11}) circular waveguide mode excited apertures. A parametric study of higher order mode aperture fields indicates that the only significant change in the circular aperture mutual coupling is due to the transverse magnetic (TM_{11}) mode, which introduces an additional phase shift. Qualitative agreement between calculations for a 183-element array of circular apertures and an infinite array establishes the validity of the finite-array theoretical model.

INTRODUCTION

The wide flexibility available in the design of antenna arrays is very useful in applications where factors such as beam shaping, side lobe level control, and rapid beam steering are of prime consideration; however, the design is complicated by the effects of mutual interaction between the radiating elements. These interactions are principally evident as (1) a distortion of the radiation pattern, (2) an element driving impedance which varies as the array is phased to point the beam in different directions, and (3) a polarization variation with scan angle in an array with elements which can support more than one sense of polarization. The degree to which the interelement coupling affects the performance of the array will depend upon the element type, the polarization and excitation of each element, the geometry of the array, and the surrounding environment. In order to study the effects of mutual interelement coupling in an array, the analysis must include all these factors.

The work reported here is an analysis of the mutual coupling in a planar array of circular waveguide-fed apertures in an infinite conductor as typically illustrated in figure 1.

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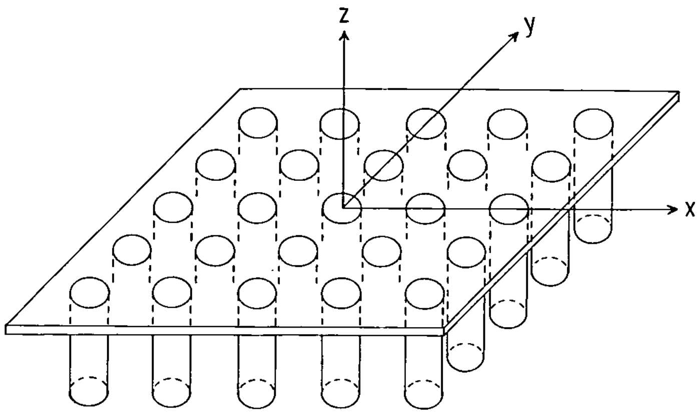

Figure 1.- Planar array of circular waveguide-fed apertures.

The analysis is expected to yield good results for planar arrays on finite-size ground planes which are electrically large; however, for small ground planes or for array elements near the edge of a finite ground plane, the electromagnetic scatter from the ground plane edges may be significant in some cases.

The problem is first formulated for arbitrary waveguide apertures radiating into a multilayered region and then specialized to circular apertures excited in either TE or TM modes. The effects of mutual coupling are determined by first computing the self and mutual admittances among all the elements of the array to form a complex admittance matrix, which is then operated on to determine the complex scattering matrix for the array. The scattering matrix gives the relationship between the amplitudes and phases of the waveguide modal fields which are incident on and reflected from the apertures. This relationship then allows one to determine the reflection coefficient and coupling coefficients of all the elements of the array for any phasing or amplitude taper.

SYMBOLS

$\vec{A}$	magnetic vector potential
A or $A(x_i, y_i, z_i)$	functional form for z_i component of $\vec{A}$
$A(k_x, k_y, z_i)$	bidimensional Fourier transform of $A(x_i, y_i, z_i)$
$A'(k_x, k_y, z_i)$	derivative of $A(k_x, k_y, z_i)$ with respect to z_i
$A(\alpha, \beta)$	undetermined quantity used in equation (69)
$A(\alpha, \beta, 0)$	bidimensional Fourier transform of $A(x_i, y_i, z_i)$ in cylindrical coordinates evaluated at $z_i = 0$

A_j	parameter defined by equation (79)
A'_j	parameter defined by equation (80)
a	dummy parameter used in equations (132) and (133)
a_i	radius of ith circular aperture
a_j	radius of jth circular aperture
a_{p_i}	complex amplitude of pth waveguide mode incident on ith aperture
a_{q_j}	complex amplitude of qth waveguide mode incident on jth aperture
[a]	complex column matrix whose elements consist of all a_{p_i}
$B(\alpha, \beta)$	undetermined quantity used in equation (69)
b_{p_i}	complex amplitude of pth waveguide mode reflected from ith aperture
[b]	complex column matrix whose elements consist of all the b_{p_i}
$C(\alpha,\beta)$, $C_1(\alpha,\beta)$, $C_2(\alpha,\beta)$	undetermined quantities used in equations (70), (53), and (54), respectively
C_j^{TE}	quantity defined by equation (81)
C_j^{TM}	quantity defined by equation (82)
$D(\alpha, \beta)$	undetermined quantity used in equation (70)
d	thickness of one dielectric layer
$\left.\begin{matrix} d_1, d_2, \ldots, d_{p-1}, d_p, \\ d_{p+1}, \ldots, d_{N'-1}, d_{N'} \end{matrix}\right\}$	distances from aperture plane to outer surfaces of layers 1, 2, . . ., p-1, p, p+1, N'-1, N', respectively
$\vec{E}$	electric field vector

$\vec{E}(x_i, y_i, z_i)$	functional form of $\vec{E}$
$\vec{E}(k_x, k_y, z_i)$	bidimensional Fourier transform of $\vec{E}(x_i, y_i, z_i)$
$\hat{e}'_q$	normalized electric vector mode function for qth TE waveguide mode
$\hat{e}''_q$	normalized electric vector mode function for qth TM waveguide mode
$\vec{F}$	electric vector potential
F or $F(x_i, y_i, z_i)$	functional form for z_i component of $\vec{F}$
$F(k_x, k_y, z_i)$	bidimensional Fourier transform of $F(x_i, y_i, z_i)$
$F'(k_x, k_y, z_i)$	derivative of $F(k_x, k_y, z_i)$ with respect to z_i
$f(\alpha, \beta, z_i)$	normalized form of $F(k_x, k_y, z_i)$ defined in equation (56)
$f'(\alpha, \beta, z_i)$	derivative of $f(\alpha, \beta, z_i)$ with respect to z_i
$g(\alpha, \beta, z_i)$	normalized form of $A(k_x, k_y, z_i)$ defined in equation (57)
$g'(\alpha, \beta, z_i)$	derivative of $g(\alpha, \beta, z_i)$ with respect to z_i
$\vec{H}$	magnetic field vector
$\vec{H}(x_i, y_i, z_i)$	functional form of $\vec{H}$
$\vec{H}(k_x, k_y, z_i)$	bidimensional Fourier transform of $\vec{H}(x_i, y_i, z_i)$
$\hat{h}'_q$	normalized magnetic vector mode function for qth TE waveguide mode
$\hat{h}''_q$	normalized magnetic vector mode function for qth TM waveguide mode
I	reaction integral (eq. (21))

$I_1, I_2, \ldots, I_{12}$	intermediate quantities used in derivation of equations (167) to (170)
I_{p_i}	equivalent current for pth mode in ith aperture
I'_q	equivalent current for qth TE waveguide modal fields
I''_q	equivalent current for qth TM waveguide modal fields
$[I]$	complex column vector whose elements consist of all I_{p_i}
$J_m(z)$	Bessel function of the first kind of order m and argument z
$J'_m(z)$	derivative of $J_m(z)$ with respect to z
$j = \sqrt{-1}$	
k_0	wave propagation constant in free space, $2\pi/\lambda$
k_x	Fourier transform variable with respect to x_i
k_y	Fourier transform variable with respect to y_i
k_z	complex wave propagation constant in z_i direction
k'_x, k'_y	dummy variables for integration
k'_{cq}	cutoff wave number for qth TE waveguide mode
k''_{cq}	cutoff wave number for qth TM waveguide mode
M	number of waveguide modes in each aperture
M_j	total number of modes assumed in the jth aperture
m_i, m_j, m'_i, m'_j	order of Bessel functions and cyclic variation of fields
N	number of apertures in array

N'	number of dielectric layers outside of aperture plane
R	center-to-center spacing between two apertures or between origins of ith and jth aperture coordinate systems
S_i	area of ith aperture
S_{p_i,q_j}	complex coupling coefficient between pth mode in ith aperture and qth mode in jth aperture
$[S]$	complex square matrix whose elements consist of all S_{p_i,q_j}
T_x, T_y	beam-pointing directional cosines
t	time, sec
TE	transverse electric
TM	transverse magnetic
$U_{ij}(\beta)$	quantity for simplification of admittance expression (see eqs. (167) to (170))
$V_{ij}(\beta)$	quantity for simplification of admittance expression (see eq. (167))
V_i	equivalent voltage of ith aperture field
V_j	equivalent voltage of jth aperture field
V_{p_i}	equivalent voltage for pth waveguide mode in ith aperture
V_{q_j}	equivalent voltage for qth waveguide mode in jth aperture
V_{q_k}	equivalent voltage for qth waveguide mode in kth aperture
V'_q	equivalent voltage for qth TE waveguide modal fields
V''_q	equivalent voltage for qth TM waveguide modal fields

[V]	complex column matrix whose elements consist of all V_{p_i}
$W_1(\beta)$, $W_2(\beta)$	quantities defined by equations (162) and (163)
w	dummy variable
x, y, z	variables in reference Cartesian coordinate system
x_i, y_i, z_i	variables in ith aperture Cartesian coordinate system
x_j, y_j, z_j	variables in jth aperture Cartesian coordinate system
$\hat{x}$, $\hat{y}$, $\hat{z}$	unit vectors in x, y, and z directions
$\hat{x}_i$, $\hat{y}_i$, $\hat{z}_i$	unit vectors in x_i, y_i, and z_i directions
$\hat{x}_j$, $\hat{y}_j$, $\hat{z}_j$	unit vectors in x_j, y_j, and z_j directions
x_i', y_i'	translation of x_i, y_i, z_i coordinate system in x and y directions
x_j', y_j'	translation of x_j, y_j, z_j coordinate system in x and y directions
Y_{p_i}	characteristic admittance of pth waveguide mode in ith aperture
$Y_{i,j}$	element in ith row and jth column of [Y] or mutual admittance between ith aperture electric field and magnetic field produced by jth aperture field for one mode apertures
Y_{p_i,q_j}	mutual admittance between pth waveguide mode electric field in ith aperture and magnetic field produced by qth waveguide mode in jth aperture
[Y]	complex square matrix whose elements consist of all Y_{p_i,q_j}
$[Y_0]$	complex diagonal matrix whose nonzero elements consist of all Y_{p_i}
z'	dummy variable used in definition of delta function (see eq. (28))

α	angular Fourier transform variable in cylindrical coordinate system of ith aperture
β	normalized radial Fourier transform variable in cylindrical coordinate system of ith aperture
$\delta(z - z')$	delta function defined by equation (28)
ϵ or $\epsilon(z_i)$	permittivity of dielectric region
ϵ'	permittivity of medium outside of layered region
ϵ_0	permittivity of free space
$\epsilon_1(0)$	permittivity for $z_i = 0^+$ immediately adjacent to aperture plane
$\zeta_i^{TE}(\beta)$	quantity defined by equation (165)
$\zeta_j^{TE}(\beta)$	quantity defined by equation (165) with i replaced by j
λ	wavelength in free space
μ or $\mu(z_i)$	permeability of dielectric region
μ'	permeability of medium outside of layered region
μ_0	permeability of free space
$\mu_1(0)$	permeability for $z_i = 0^+$ immediately adjacent to aperture plane
$\xi_i^{TE}(\beta)$, $\xi_i^{TM}(\beta)$	quantities defined by equations (164) and (166)
$\xi_j^{TE}(\beta)$, $\xi_j^{TM}(\beta)$	quantities defined by equations (164) and (166) with i replaced by j
θ, γ	dummy variables
ρ	dummy variable for integration in equations (132) and (133)

ρ_i	radial variable in ith aperture cylindrical coordinate system
ρ_j	radial variable in jth aperture cylindrical coordinate system
$\sum_{q}$	sum of terms for all values of q
$\sum_{j=1}^{N}$	sum of terms for all values from j = 1 through j = N
$\sum_{q_j=1}^{M_j}$	sum of terms for all values from $q_j = 1$ through $q_j = M_j$
Φ	magnetic scalar potential
ϕ	angle defined by equation (89)
ϕ_i	angular variable in ith aperture cylindrical coordinate system
ϕ_j	angular variable in jth aperture cylindrical coordinate system
ϕ_i'	rotation of x_i, y_i coordinates with respect to x, y coordinates
ϕ_j'	rotation of x_j, y_j coordinates with respect to x, y coordinates
$\phi_p = \phi_j' - \phi_i'$	relative polarization angle between i and j aperture fields
ϕ_q	solution to equation (5)
$\chi_{m_j' n_j'}$	n_j'th zero of $J_{m_j'}(x)$
$\chi'_{m_j n_j}$	n_jth zero of $J'_{m_j}(x)$
Ψ	electrical scalar potential
ψ	temporary variable used in derivation of equations (167) to (170) to represent quantity in equation (95)
ψ_q	solution to equation (6)

ψ_x phase shift between array elements for H-plane scan

ψ_y phase shift between array elements for E-plane scan

ω angular frequency, rad/sec

$$\vec{\nabla} = \frac{\partial}{\partial x_i} \hat{x}_i + \frac{\partial}{\partial y_i} \hat{y}_i + \frac{\partial}{\partial z_i} \hat{z}_i$$

$$\nabla^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}$$

Subscripts:

i ith aperture

j jth aperture

ij either the ith row and jth column of matrix or the interaction of the jth aperture fields upon the ith aperture fields

m order of Bessel function (see eq. (116))

m_i first subscript of transverse electric waveguide mode in ith aperture and order of Bessel function in field equations

m_j first subscript of transverse electric waveguide mode in jth aperture and order of Bessel function in field equations

m_i' first subscript of transverse magnetic waveguide mode in ith aperture and order of Bessel function in field equations

m_j' first subscript of transverse magnetic waveguide mode in jth aperture and order of Bessel function in field equations

N' outermost dielectric layer

N'+1 medium outside the layered region

n_i	second subscript of transverse electric waveguide mode in ith aperture
n_j	second subscript of transverse electric waveguide mode in jth aperture
n_i'	second subscript of transverse magnetic waveguide mode in ith aperture
n_j'	second subscript of transverse magnetic waveguide mode in jth aperture
p	pth dielectric layer, except when used in conjunction with $\hat{e}_p'$ or $\hat{e}_p''$ to denote pth waveguide mode functions
p+1	adjacent dielectric layer outside pth layer
p_i	pth mode in ith aperture
q	qth waveguide mode
q_j	qth mode in jth aperture
t	transverse component of field vectors
x_i, y_i, z_i	components in ith aperture Cartesian coordinate system
ρ_j, ϕ_j	components in jth aperture polar coordinate system

Superscripts:

(i)	electric field in ith aperture
(j)	either the electric field in jth aperture or the magnetic field produced at ith aperture due to an electric field excited in jth aperture
p	pth dielectric layer

(p_i)	pth waveguide mode in ith aperture
TE	transverse electric waveguide mode
TM	transverse magnetic waveguide mode
TE, TE	mutual admittance between TE modes in apertures i and j
TM, TM	mutual admittance between TM modes in apertures i and j
TE, TM	mutual admittance between TE mode in ith aperture and TM mode in jth aperture
TM, TE	mutual admittance between TM mode in ith aperture and TE mode in jth aperture

REVIEW OF THE LITERATURE

There are many approaches to the analysis of mutual coupling effects upon the performance of phased arrays and each one has its own inherent advantages and disadvantages. It is not the intention of the author to present an exhaustive review of all the previous work that has been accomplished in the analysis of phased arrays; however, a summary will be given of the more pertinent work of which the author is presently aware, and more specifically that which is applicable to planar arrays of apertures. This summary is presented in order to acquaint the reader with the scope, depth, and variety of attention which phased arrays have received during the past decade.

The theoretical analyses can generally be divided into two broad categories such as infinite arrays and finite arrays. The infinite-array approach is very useful in the analysis of the impedance and radiation characteristics of the elements near the center of a very large array, but breaks down when applied to the elements near the edge. The finite-array approach yields good results for all the elements of the array, but the analysis is more complicated, requires more computer time to obtain results, and is generally restricted to arrays of no more than about 200 to 300 elements because of the necessity of inverting a large matrix or solving a set of simultaneous equations.

Much effort has been devoted to the analysis of a variety of infinite arrays of periodically spaced identical elements (refs. 1 to 57). These have included the more common aperture elements such as infinite slots (refs. 23 to 31), rectangular (refs. 32 to 47), and circular (refs. 48 to 54) as well as the ridged waveguide aperture (ref. 55) and multiple frequency interleaved arrays (refs. 56 and 57). Some authors have also considered the effects of dielectric loading such as plugs in the waveguide apertures (refs. 29, 31, 40, 42, 51, and 54) or dielectric sheets covering the aperture plane (refs. 11, 21, 24, 26, 28, 29, 30, 40, 41, 49, 51, and 54), and the effects of higher order aperture fields (refs. 31, 38 to 40, 42, 44 to 46, and 50 to 54).

These analyses have been very useful in the study of certain resonance phenomena which have been observed in large phased arrays and array simulators. (See refs. 1, 4, 11, 38, 39, 43, and 58 to 64.) This resonance is manifested by a null in the array element pattern or a large reflection coefficient at specific scan angles closer to broadside than the angle at which a grating lobe can occur; thus, the angular scan range of a large phased array is limited. This resonance could be considered as the electromagnetic analogy of the Woods "anomalies" (ref. 65) for the diffraction of light from optical gratings. This resonance in infinite arrays is generally attributed to the excitation of surface waves on the periodic structure (refs. 4, 11, 30, 51, 52, 54, 58, 59, and 66) or higher order mode aperture fields (refs. 31, 38, 39, 45, 53, 67, and 68).

Several techniques are available for the elimination of this resonance or of improving the wide-angle matching capability of large arrays (ref. 69). These techniques involve the use of such things as conducting fences or corrugations between the radiating elements (refs. 70 to 74), irises in the apertures (refs. 74 to 77), proper design of the dielectric loading (refs. 78 to 81), separate matching networks for each element (refs. 82 and 83), interconnecting circuits (refs. 84 and 85), selective mode excitation (refs. 86 and 87), or possibly disrupting the periodicity of the array (refs. 88 to 92). The wide-angle matching is achieved either by a reduction in the interelement mutual coupling or by proper compensation. In either case, a detailed knowledge of the interelement coupling or terminating impedance is required.

The theoretical analyses for infinite arrays and measurement techniques (refs. 93 to 97) have proven useful in the study of the radiation and impedance characteristics of the "typical" elements of large arrays; however, the "nontypical" elements near the edge or the elements of a small array must be analyzed by other means.

The characteristics of the edge elements in large arrays have been analyzed by perturbation (ref. 98) and modifications (refs. 99 and 100) of infinite-array techniques. An integral equation method has been used to study the radiation properties of a finite parallel-plate waveguide array (refs. 101 to 103). These studies indicate that the impedance and radiation properties of the edge elements of an array can be vastly different from those near the center.

Much effort has also been devoted to the determination of the mutual coupling between pairs of waveguide apertures. The most comprehensive study of the coupling between various antennas was performed by a group at the University of Michigan (ref. 104); however, others have also made significant contributions in this area by using a variety of techniques.

Graf (ref. 105) investigated the effect of mutual coupling between half-wave slots by using the electromagnetic duality of slots and dipoles. Tartakovskiy and Rubinshteyn (ref. 106) introduced a numerical method for solving the system of Wiener-Hopf-Fok equations which occur in the diffraction at a finite or infinite number of equidistant half-planes and applied it to the coupling between two waveguides. Others (refs. 107 to 111) have used Keller's geometrical theory of diffraction (ref. 112) to compute the coupling between parallel-plate waveguides. Others (refs. 113 to 121 and 123) have used variational techniques to determine the mutual coupling between rectangular (refs. 114 to 121 and 123), parallel-plate (refs. 121 and 122), and annular slot apertures (ref. 113). Some have also considered the effects of a dielectric or plasma outside the aperture plane (refs. 118 to 121). Fante (ref. 121) used the concept of an impedance sheet to represent the plasma layer under certain restrictions. Galejs (ref. 118) approximated the external plasma layers by a large dielectric-filled waveguide. Golden and Stewart (refs. 119 and 120) analyzed the coupling between rectangular slots under an inhomogeneous plasma by using an integrated electron density and a stepped approximation for the plasma profile. Previous work (ref. 122) has indicated that stepped plasma profiles can sometimes yield erroneous resonance effects which are not present in a practical plasma. Sugio and Makimoto (ref. 123) formulated a variational expression for the scattering coefficients of a finite array of rectangular waveguides with dielectric plugs; however, no results were given.

The work to be presented in this paper is a variational formulation for the mutual admittance of two waveguide apertures which need not be identical in shape nor excitation. The formulation is general enough to include the effect of an arbitrary number of dielectric and/or plasma layers, each of which may be inhomogeneous; however, no stepped approximation to the plasma profile is made nor is an integrated electron density approximation used.

Since no results have been published for finite arrays of circular waveguides, the general formulation for mutual admittance is evaluated for circular apertures excited in either TE or TM circular waveguide modes and numerical as well as experimental data are presented for mutual coupling with either free space or a dielectric sheet outside the aperture plane.

The approach used in the general formulation parallels that for the self admittance of one aperture (ref. 124).

THEORY

General Theory

It is assumed that each aperture in the array is fed by a uniform waveguide, the cross section of which coincides with the aperture. The electromagnetic fields in the apertures will be represented as the sum of the waveguide modal fields; therefore, the total transverse fields in each aperture are given by

$$\vec{E}_t = \sum_q V'_q \hat{e}'_q + \sum_q V''_q \hat{e}''_q \tag{1}$$

$$\vec{H}_t = \sum_q I'_q \hat{h}'_q + \sum_q I''_q \hat{h}''_q \tag{2}$$

where $\hat{e}'_q$, $\hat{h}'_q$ and $\hat{e}''_q$, $\hat{h}''_q$ represent the normalized vector mode functions for the TE and TM modes, respectively, defined so that in Cartesian coordinates

$$\left.\begin{aligned} \hat{e}'_q &= -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\phi_q \\ \hat{h}'_q &= \hat{z} \times \hat{e}'_q \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \hat{e}''_q &= \hat{z} \times \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)\psi_q \\ \hat{h}''_q &= \hat{z} \times \hat{e}''_q \end{aligned}\right\} \tag{4}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors in the x, y, and z directions, and ϕ_q and ψ_q are scalar functions which satisfy the differential equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_q + (k'_{cq})^2 \phi_q = 0 \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi_q + (k''_{cq})^2 \psi_q = 0 \tag{6}$$

subject to the appropriate boundary conditions of the waveguide modal fields.

The equivalent modal voltages and currents are defined as

$$\left.\begin{aligned} V'_q &= \iint \vec{E}_t \cdot \hat{e}'_q \, dx\, dy \\ I'_q &= \iint \vec{H}_t \cdot \hat{h}'_q \, dx\, dy \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} V''_q &= \iint \vec{E}_t \cdot \hat{e}''_q \, dx\, dy \\ I''_q &= \iint \vec{H}_t \cdot \hat{h}''_q \, dx\, dy \end{aligned}\right\} \tag{8}$$

where the integrals are taken over the cross section of the waveguide.

Because of orthogonality properties of the vector mode functions

$$\iint \hat{e}'_q \cdot \hat{e}''_p \, dx\, dy = 0 \tag{9}$$

$$\iint \hat{e}'_q \cdot \hat{e}'_p \, dx\, dy = \left\{\begin{matrix} 1 & (p = q) \\ 0 & (p \neq q) \end{matrix}\right\} \tag{10}$$

$$\iint \hat{e}''_q \cdot \hat{e}''_p \, dx\, dy = \left\{\begin{matrix} 1 & (p = q) \\ 0 & (p \neq q) \end{matrix}\right\} \tag{11}$$

energy propagates along a uniform waveguide in each mode independently; therefore, for computational purposes, each modal field in each aperture of the array is assumed to be fed by a separate waveguide which can only be excited by that single mode. This assumption corresponds to treating an array of N waveguide-fed apertures as an N times M microwave equivalent network, where M is the total number of modes needed in each aperture to represent the total field distribution adequately. This assumption restricts the analysis to apertures of relatively simple shapes (such as rectangular, circular, elliptical, etc.) for which the corresponding waveguide modal fields can be determined.

The transverse electric and magnetic fields of the p_ith mode can be represented either as the superposition of an incident (a_{p_i}) and reflected (b_{p_i}) wave, or as an equivalent voltage (V_{p_i}) and current (I_{p_i}). For TE modes

$$\left.\begin{aligned} \vec{E}_t^{(p_i)} &= (a_{p_i} + b_{p_i})\, \hat{e}'_{p_i} \\ \vec{H}_t^{(p_i)} &= Y_{p_i}\, (a_{p_i} - b_{p_i})\, \hat{h}'_{p_i} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \vec{E}_t^{(p_i)} &= V_{p_i}\, \hat{e}'_{p_i} \\ \vec{H}_t^{(p_i)} &= I_{p_i}\, \hat{h}'_{p_i} \end{aligned}\right\} \tag{13}$$

where Y_{p_i} is the characteristic admittance of the p_ith mode. The corresponding expressions for TM modes are obtained by replacing the primes by double primes in equations (12) and (13).

Because of the coupling or the mutual interaction of the external fields, the equivalent aperture voltages and currents will not be independent, but will be related by a set of simultaneous equations such as

$$I_{p_i} = \sum_{j=1}^{N} \sum_{q_j=1}^{M_j} Y_{p_i,q_j} V_{q_j} \tag{14}$$

where N is the total number of apertures in the array and M_j is the total number of modes in the jth aperture necessary to represent the aperture field adequately.

The amplitudes of the incident and reflected modal fields are related by a similar set of simultaneous equations such as

$$b_{p_i} = \sum_{j=1}^{N} \sum_{q_j=1}^{M_j} S_{p_i,q_j} a_{q_j} \tag{15}$$

If each aperture requires M modes to represent the aperture fields adequately, then N times M equations such as equations (14) and (15) would be needed to describe the coupling mechanism of the array. In matrix notation these equations are written as

$$[I] = [Y] \; [V] \tag{16}$$

$$[b] = [S] \; [a] \tag{17}$$

By algebraic manipulation of equations (16) and (17), the wave scattering matrix $[S]$ will be related to the aperture admittance matrix $[Y]$ as

$$[S] = \Big[[Y_0] - [Y]\Big] \; \Big[[Y_0] + [Y]\Big]^{-1} \tag{18}$$

where $[Y_0]$ is a diagonal matrix whose elements are the characteristic admittances of the waveguide modes, and $[\;]^{-1}$ indicates matrix inversion. Thus, the number of apertures N and/or the number of modes M per aperture is limited by the ability of the available computer to invert an $N \times M$ complex square matrix.

The coupling problem then reduces to the determination of the elements of the aperture admittance matrix which are the mutual admittances between each aperture modal field and all others of the array.

Mutual Admittance Between Apertures

General.- In order to compute the coupling between apertures, the components of the admittance matrix must be determined. As seen from equation (14), the component Y_{p_i,q_j} (where p_i refers to the pth mode in the ith aperture and q_j refers to the qth mode in the jth aperture) is the mutual admittance between modes p_i and q_j with all other modal voltages set equal to zero; that is,

$$Y_{p_i,q_j} = \frac{I_{p_i}}{V_{q_j}} \tag{19}$$

with all $V_{q_k} = 0$ except V_{q_j}.

In order to simplify the subscript notation, and since each modal field will be treated as a separate aperture, the notation $Y_{i,j}$ will be used to represent the (i,j)th element of the (N times M) by (N times M) admittance matrix.

A stationary expression for mutual impedance for linear antennas or its dual for aperture self admittance can be obtained from the electromagnetic reaction of the assumed equivalent electric or magnetic currents (ref. 125, sections 7-9 and 8-12). The mutual admittance between two apertures also can be determined from a consideration of

$$Y_{i,j} = \frac{1}{V_i V_j} \iint_{S_i} [\vec{E}^{(i)} \times \vec{H}^{(j)}] \cdot \hat{z}_i \, dS_i \tag{20}$$

where V_i and V_j are the normalized modal voltages (see eqs. (7) and (8)), $\vec{E}^{(i)}$ is the assumed electric field of the ith aperture, $\vec{H}^{(j)}$ is the magnetic field produced in aperture i by an assumed electric field $\vec{E}^{(j)}$ in aperture j. The integral in equation (20) is taken over the area (S_i) of the ith aperture.

Borgiotti (ref. 115) used equation (20) to show that the mutual admittance of identical apertures radiating into free space can be expressed as the Fourier transform of a function which is obtained from the plane-wave spectrum of the field radiated by the aperture. He also showed that this formalism can be used to determine the "grating lobe series" for the driving-point admittance of an element in an infinite periodic array of identical apertures.

A more general expression is developed here which is applicable to apertures which are not identical in shape or excitation. The mutual admittance expression will also include the influence of a planar stratified region outside the aperture plane as indicated in figure 2. This expression, which is not presently available in the literature, is then used to compute the near-field coupling between circular apertures in a finite planar array.

Since the tangential component of the assumed aperture field $\vec{E}^{(i)}$ is zero over the remaining surface of the infinite aperture plane (all other aperture voltages are temporarily set equal to zero, see eq. (19)), the surface integral in equation (20) can be extended to infinity

$$I = \iint_{S_i} \left[\vec{E}^{(i)} \times \vec{H}^{(j)}\right] \cdot \hat{z}_i \, dS_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\vec{E}^{(i)} \times \vec{H}^{(j)}\right] \cdot \hat{z}_i \, dx_i \, dy_i \tag{21}$$

where x_i and y_i are the coordinate variables of the ith aperture. Taking Fourier transforms so that

$$\vec{E}^{(i)} (k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{E}^{(i)} (x_i, y_i) \, e^{jk_x x_i} \, e^{jk_y y_i} \, dx_i \, dy_i \tag{22}$$

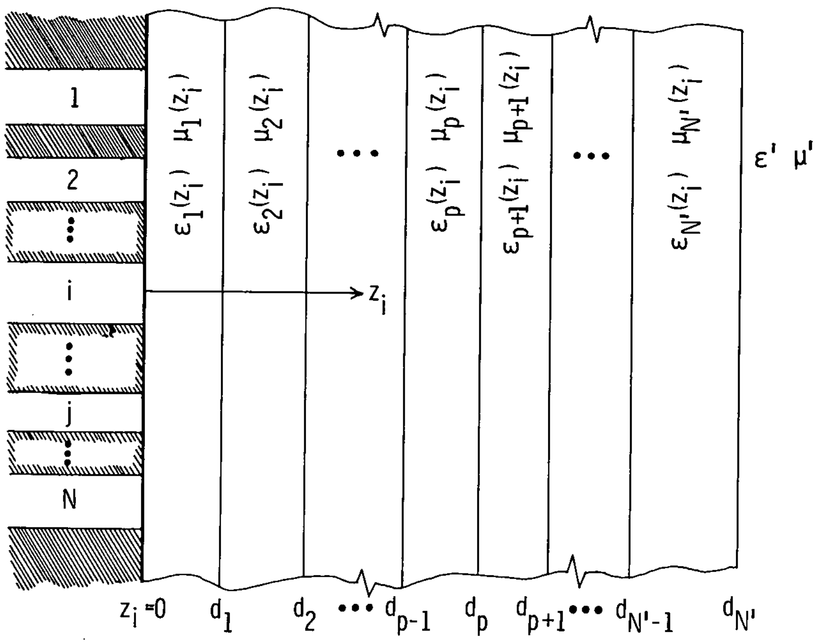

Figure 2.- Cross section of N waveguides radiating into N' dielectric layers.

$$\vec{H}^{(j)}(k_x', k_y') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{H}^{(j)}(x_i, y_i)\, e^{jk_x'x_i}\, e^{jk_y'y_i}\, dx_i\, dy_i \tag{23}$$

and inversely,

$$\vec{E}^{(i)}(x_i, y_i) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{E}^{(i)}(k_x, k_y)\, e^{-jk_x x_i}\, e^{-jk_y y_i}\, dk_x\, dk_y \tag{24}$$

$$\vec{H}^{(j)}(x_i, y_i) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{H}^{(j)}(k_x', k_y')\, e^{-jk_x'x_i}\, e^{-jk_y'y_i}\, dk_x'\, dk_y' \tag{25}$$

and substituting equations (24) and (25) into equation (21) gives

$$I = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left[\vec{E}^{(i)} (k_x, k_y) \times \vec{H}^{(j)} (k_x', k_y') \right] \right. \right.$$

$$\left. \left. \cdot \hat{z}_i e^{-jk_x x_i} e^{-jk_y y_i} e^{-jk_x' x_i} e^{-jk_y' y_i} \right\} dk_x \, dk_y \, dk_x' \, dk_y' \right\} dx_i \, dy_i \tag{26}$$

Interchanging the order of integration yields

$$I = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left[\vec{E}^{(i)} (k_x, k_y) \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{H}^{(j)} (k_x', k_y') \right. \right.$$

$$\left. \left. \left\{ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(k_x + k_x')x_i} e^{-j(k_y + k_y')y_i} dx_i \, dy_i \right\} dk_x' \, dk_y' \right] \hat{z}_i \right\} dk_x \, dk_y \tag{27}$$

and using the definition of the delta function $\delta(z - z')$, (eq. (C-19), ref. 125)

$$\delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(z-z')w} \, dw \tag{28}$$

equation (27) can be written as

$$I = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \left[\vec{E}^{(i)} (k_x, k_y) \right. \right.$$

$$\left. \left. \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{H}^{(j)} (k_x', k_y') \, \delta(k_x + k_x') \, \delta(k_y + k_y') \, dk_x' \, dk_y' \right] \cdot \hat{z}_i \right\} dk_x \, dk_y \tag{29}$$

which yields

$$I = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\vec{E}^{(i)} (k_x, k_y) \times \vec{H}^{(j)} (-k_x, -k_y) \right] \cdot \hat{z}_i \, dk_x \, dk_y \tag{30}$$

Equation (30) is recognized as a form of Parseval's theorem (ref. 125, eq. (C-15)). If k_x and k_y are the wave propagation numbers in the x_i and y_i directions, then one could visualize $\vec{H}^{(j)} (-k_x, -k_y)$ as the bidimensional Fourier transform of a wave whose direction of propagation in the x_i, y_i plane is reversed. The problem now reduces to the determination of the tangential component of $\vec{H}^{(j)}(-k_x, -k_y)$ at the ith aperture because of an assumed electric field in the jth aperture.

The electric and magnetic fields external to the aperture plane can be uniquely determined from a set of vector potentials

$$\left.\begin{aligned}\vec{A} &= A(x_i, y_i, z_i)\,\hat{z}_i\\ \vec{F} &= F(x_i, y_i, z_i)\,\hat{z}_i\end{aligned}\right\} \tag{31}$$

as follows (see appendix A)

$$E_{x_i}(x_i, y_i, z_i) = \frac{1}{j\omega\epsilon}\frac{\partial^2}{\partial x_i\,\partial z_i}\left(\frac{A}{\mu}\right) - \frac{1}{\epsilon}\frac{\partial F}{\partial y_i} \tag{32}$$

$$E_{y_i}(x_i, y_i, z_i) = \frac{1}{j\omega\epsilon}\frac{\partial^2}{\partial y_i\,\partial z_i}\left(\frac{A}{\mu}\right) + \frac{1}{\epsilon}\frac{\partial F}{\partial x_i} \tag{33}$$

$$E_{z_i}(x_i, y_i, z_i) = \frac{1}{j\omega}\frac{\partial}{\partial z_i}\left[\frac{1}{\epsilon}\frac{\partial}{\partial z_i}\left(\frac{A}{\mu}\right)\right] - j\omega A \tag{34}$$

$$H_{x_i}(x_i, y_i, z_i) = \frac{1}{j\omega\mu}\frac{\partial^2}{\partial x_i\,\partial z_i}\left(\frac{F}{\epsilon}\right) + \frac{1}{\mu}\frac{\partial A}{\partial y_i} \tag{35}$$

$$H_{y_i}(x_i, y_i, z_i) = \frac{1}{j\omega\mu}\frac{\partial^2}{\partial y_i\,\partial z_i}\left(\frac{F}{\epsilon}\right) - \frac{1}{\mu}\frac{\partial A}{\partial x_i} \tag{36}$$

$$H_{z_i}(x_i, y_i, z_i) = \frac{1}{j\omega}\frac{\partial}{\partial z_i}\left[\frac{1}{\mu}\frac{\partial}{\partial z_i}\left(\frac{F}{\epsilon}\right)\right] - j\omega F \tag{37}$$

where ϵ and μ are the permittivity and permeability of the external medium and ω is the angular frequency of the signal. A time harmonic variation of the form $e^{j\omega t}$ has been suppressed.

Substituting the inverse Fourier transforms (eqs. (A27), (A28), (A29), and (A30)) into equations (32), (33), (35), and (36) gives, after interchanging orders of integration and differentiation,

$$E_{x_i}(k_x, k_y, z_i) = -\frac{k_x}{\omega\epsilon(z_i)} A'(k_x, k_y, z_i) + jk_y F(k_x, k_y, z_i) \tag{38}$$

$$E_{y_i}(k_x, k_y, z_i) = -\frac{k_y}{\omega\epsilon(z_i)} A'(k_x, k_y, z_i) - jk_x F(k_x, k_y, z_i) \tag{39}$$

$$H_{x_i}(k_x, k_y, z_i) = -jk_y A(k_x, k_y, z_i) - \frac{k_x}{\omega\mu(z_i)} F'(k_x, k_y, z_i) \tag{40}$$

$$H_{y_i}(k_x, k_y, z_i) = jk_x A(k_x, k_y, z_i) - \frac{k_y}{\omega\mu(z_i)} F'(k_x, k_y, z_i) \tag{41}$$

where the primes denote differentiation with respect to z_i.

Then equation (30) becomes

$$I = \frac{j}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{k_x^2 + k_y^2}{\omega\epsilon(0)} A'(k_x, k_y, 0) A(-k_x, -k_y, 0) + \frac{k_x^2 + k_y^2}{\omega\mu(0)} F(k_x, k_y, 0) F'(-k_x, -k_y, 0) \right\} dk_x \, dk_y \tag{42}$$

where

$$A'(k_x, k_y, 0) = \left[\frac{d}{dz_i} A(k_x, k_y, z_i)\right]_{z_i=0} \tag{43}$$

$$F'(-k_x, -k_y, 0) = \left[\frac{d}{dz_i} F(-k_x, -k_y, z_i)\right]_{z_i=0} \tag{44}$$

If all apertures except the jth are short circuited, then continuity of tangential electric fields over the aperture plane gives, from equations (38) and (39),

$$A'(k_x, k_y, 0) = -\frac{\omega\epsilon(0)}{k_x^2 + k_y^2}\left[k_x E_{x_i}^{(j)}(k_x, k_y, 0) + k_y E_{y_i}^{(j)}(k_x, k_y, 0)\right] \tag{45}$$

$$F(k_x, k_y, 0) = \frac{1}{j(k_x^2 + k_y^2)}\left[k_y E_{x_i}^{(j)}(k_x, k_y, 0) - k_x E_{y_i}^{(j)}(k_x, k_y, 0)\right] \tag{46}$$

where $E_{x_i}^{(j)}(k_x, k_y, 0)$ and $E_{y_i}^{(j)}(k_x, k_y, 0)$ are the bidimensional Fourier transforms of the assumed modal electric field in the jth aperture.

Likewise, if all apertures except the ith are short circuited,

$$A'(-k_x, -k_y, 0) = \frac{\omega\epsilon(0)}{k_x^2 + k_y^2}\left[k_x E_{x_i}^{(i)}(-k_x, -k_y, 0) + k_y E_{y_i}^{(i)}(-k_x, -k_y, 0)\right] \tag{47}$$

$$F(-k_x, -k_y, 0) = \frac{-1}{j(k_x^2 + k_y^2)}\left[k_y E_{x_i}^{(i)}(-k_x, -k_y, 0) - k_x E_{y_i}^{(i)}(-k_x, -k_y, 0)\right] \tag{48}$$

where $E_{x_i}^{(i)}(-k_x, -k_y, 0)$ and $E_{y_i}^{(i)}(-k_x, -k_y, 0)$ are the bidimensional Fourier transforms of the assumed modal electric fields in the ith aperture with the direction of propagation in the x_i and y_i directions being reversed.

Note that the transformed wave equations (eqs. (A35) and (A36)) are even functions of k_x and k_y; therefore, by using equations (45), (46), (47), and (48) in equation (42), the mutual admittance becomes

$$\begin{aligned} Y_{ij} = \frac{1}{V_i V_j (2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Bigg\{ & \left[\frac{-j\omega\epsilon(0)}{k_x^2 + k_y^2}\right] \left[\frac{A(k_x, k_y, 0)}{A'(k_x, k_y, 0)}\right] \left[k_x E_{x_i}^{(j)}(k_x, k_y, 0) + k_y E_{y_i}^{(j)}(k_x, k_y, 0)\right] \left[k_x E_{x_i}^{(i)}(-k_x, -k_y, 0)\right. \\ & \left. + k_y E_{y_i}^{(i)}(-k_x, -k_y, 0)\right] + \left[\frac{j}{\omega\mu(0)\,(k_x^2 + k_y^2)}\right] \left[\frac{F'(k_x, k_y, 0)}{F(k_x, k_y, 0)}\right] \left[k_y E_{x_i}^{(j)}(k_x, k_y, 0) - k_x E_{y_i}^{(j)}(k_x, k_y, 0)\right] \\ & \times \left[k_y E_{x_i}^{(i)}(-k_x, -k_y, 0) - k_x E_{y_i}^{(i)}(-k_x, -k_y, 0)\right] \Bigg\} \, dk_x \, dk_y \end{aligned} \tag{49}$$

If a change of variables is made in the transform domain to cylindrical coordinates so that $k_x = k_0\beta\cos\alpha$ and $k_y = k_0\beta\sin\alpha$, then

$$Y_{ij} = \frac{k_0^2\sqrt{\frac{\epsilon_0}{\mu_0}}}{V_iV_j(2\pi)^2}\int_{\beta=0}^{\infty}\int_{\alpha=0}^{2\pi}\left\{\left[\frac{k_0\frac{\epsilon(0)}{\epsilon_0}A(\alpha,\beta,0)}{jA'(\alpha,\beta,0)}\right]\left[E_{x_i}^{(j)}(\alpha,\beta,0)\cos\alpha + E_{y_i}^{(j)}(\alpha,\beta,0)\sin\alpha\right]\right.$$

$$\times\left[E_{x_i}^{(i)}(\alpha,-\beta,0)\cos\alpha + E_{y_i}^{(i)}(\alpha,-\beta,0)\sin\alpha\right] + \left[\frac{jF'(\alpha,\beta,0)}{k_0\frac{\mu(0)}{\mu_0}F(\alpha,\beta,0)}\right]\left[E_{x_i}^{(j)}(\alpha,\beta,0)\sin\alpha - E_{y_i}^{(j)}(\alpha,\beta,0)\cos\alpha\right]$$

$$\left.\times\left[E_{x_i}^{(i)}(\alpha,-\beta,0)\sin\alpha - E_{y_i}^{(i)}(\alpha,-\beta,0)\cos\alpha\right]\right\}\beta\,d\beta\,d\alpha \tag{50}$$

where $A(\alpha,\beta,0)$ and $F(\alpha,\beta,0)$ now satisfy the differential equations

$$\frac{d^2}{dz_i^2}A(\alpha,\beta,z_i) - \frac{1}{\epsilon(z_i)}\frac{d\epsilon(z_i)}{dz_i}\frac{d}{dz_i}A(\alpha,\beta,z_i) + k_0^2\left[\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\,\epsilon_0} - \beta^2\right]A(\alpha,\beta,z_i) = 0 \tag{51}$$

$$\frac{d^2}{dz_i^2}F(\alpha,\beta,z_i) - \frac{1}{\mu(z_i)}\frac{d\mu(z_i)}{dz_i}\frac{d}{dz_i}F(\alpha,\beta,z_i) + k_0^2\left[\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\,\epsilon_0} - \beta^2\right]F(\alpha,\beta,z_i) = 0 \tag{52}$$

subject to the boundary conditions (eqs. (A37), (A38), (A39), and (A40)) at each boundary $(z_i = d_p)$ in figure 2.

Assume that the region outside the aperture plane $(z_i > 0)$ consists of N' layers whose total thickness is $d_{N'}$. Also assume that the remaining space outside the layered region $(z_i > d_{N'})$ is filled with a homogeneous material whose permittivity and permeability are ϵ' and μ', respectively. The solutions to equations (51) and (52) outside the layered region $(z_i > d_{N'})$ will then be of the form

$$A_{N'+1}(\alpha,\beta,z_i) = C_1(\alpha,\beta)\,e^{-jk_z z_i} \tag{53}$$

$$F_{N'+1}(\alpha,\beta,z_i) = C_2(\alpha,\beta)\,e^{-jk_z z_i} \tag{54}$$

where k_z is defined to satisfy the radiation condition at infinity, that is,

$$k_z = k_0 \sqrt{\frac{\epsilon'\mu'}{\epsilon_0\mu_0} - \beta^2} \qquad \left(\beta^2 \leq \frac{\epsilon'\mu'}{\epsilon_0\mu_0}\right)$$

$$k_z = -jk_0 \sqrt{\beta^2 - \frac{\epsilon'\mu'}{\epsilon_0\mu_0}} \qquad \left(\beta^2 > \frac{\epsilon'\mu'}{\epsilon_0\mu_0}\right) \tag{55}$$

For convenience, the solutions to equations (51) and (52) for each layer p will be normalized to the solutions in the outer region evaluated at the outer surface of the layered region $(z_i = d_{N'})$ according to

$$f_p(\alpha, \beta, z_i) = \frac{F_p(\alpha, \beta, z_i)}{F_{N'+1}(\alpha, \beta, d_{N'})} \tag{56}$$

$$g_p(\alpha, \beta, z_i) = \frac{A_p(\alpha, \beta, z_i)}{A_{N'+1}(\alpha, \beta, d_{N'})} \tag{57}$$

which are solutions of

$$\frac{d^2}{dz_i^2} g_p(\alpha, \beta, z_i) - \frac{1}{\epsilon(z_i)} \frac{d\epsilon(z_i)}{dz_i} \frac{dg_p(\alpha, \beta, z_i)}{dz_i} + k_0^2 \left[\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\epsilon_0} - \beta^2\right] g_p(\alpha, \beta, z_i) = 0 \tag{58}$$

$$\frac{d^2}{dz_i^2} f_p(\alpha, \beta, z_i) - \frac{1}{\mu(z_i)} \frac{d\mu(z_i)}{dz_i} \frac{df_p(\alpha, \beta, z_i)}{dz_i} + k_0^2 \left[\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\epsilon_0} - \beta^2\right] f_p(\alpha, \beta, z_i) = 0 \tag{59}$$

Then by using the boundary conditions at each interface $(z_i = d_p)$

$$f_p(\alpha, \beta, d_p) = f_{p+1}(\alpha, \beta, d_p) \tag{60}$$

$$g_p(\alpha, \beta, d_p) = g_{p+1}(\alpha, \beta, d_p) \tag{61}$$

$$f'_p(\alpha, \beta, d_p) = \frac{\mu_p(d_p)}{\mu_{p+1}(d_p)} f'_{p+1}(\alpha, \beta, d_p) \tag{62}$$

$$g'_p(\alpha, \beta, d_p) = \frac{\epsilon_p(d_p)}{\epsilon_{p+1}(d_p)} \; g'_{p+1}(\alpha, \beta, d_p) \tag{63}$$

starting with the initial conditions

$$f_{N'}(\alpha, \beta, d_{N'}) = 1 \tag{64}$$

$$g_{N'}(\alpha, \beta, d_{N'}) = 1 \tag{65}$$

$$f'_{N'}(\alpha, \beta, d_{N'}) = -jk_z \; \frac{\mu_{N'}(d_{N'})}{\mu'} \tag{66}$$

$$g'_{N'}(\alpha, \beta, d_{N'}) = -\, jk_z \; \frac{\epsilon_{N'}(d_{N'})}{\epsilon'} \tag{67}$$

and solving the differential equations (58) and (59) for each layer in turn beginning with the outermost layer and working back toward the aperture plane $(z_i = 0)$, the mutual admittance for two assumed aperture field distributions $(\vec{E}^{(i)}$ and $\vec{E}^{(j)})$ radiating into a plane multilayered region can be determined by performing the following integrations:

$$\begin{aligned} Y_{ij} = \frac{k_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}}}{V_i V_j (2\pi)^2} \int_{\beta=0}^{\infty} \int_{\alpha=0}^{2\pi} \Bigg\{ & \left[\frac{k_0 \dfrac{\epsilon_1(0)}{\epsilon_0} \; g_1(\alpha, \beta, 0)}{jg'_1(\alpha, \beta, 0)} \right] \\ & \times \left[E_{x_i}^{(j)}(\alpha, \beta, 0) \cos\alpha + E_{y_i}^{(j)}(\alpha, \beta, 0) \sin\alpha \right] \left[E_{x_i}^{(i)}(\alpha, -\beta, 0) \cos\alpha + E_{y_i}^{(i)}(\alpha, -\beta, 0) \sin\alpha \right] \\ & + \left[\frac{jf'_1(\alpha, \beta, 0)}{k_0 \dfrac{\mu_1(0)}{\mu_0} \; f_1(\alpha, \beta, 0)} \right] \left[E_{x_i}^{(j)}(\alpha, \beta, 0) \sin\alpha - E_{y_i}^{(j)}(\alpha, \beta, 0) \cos\alpha \right] \\ & \times \left[E_{x_i}^{(i)}(\alpha, -\beta, 0) \sin\alpha - E_{y_i}^{(i)}(\alpha, -\beta, 0) \cos\alpha \right] \Bigg\} \beta \, d\beta \, d\alpha \end{aligned} \tag{68}$$

Only a limited number of dielectric profiles have so far been investigated whereby the solutions to equations (58) and (59) can be expressed in terms of well-known functions. A few of these solutions are found in reference 126. No attempt is made here to cover this class of problems. It suffices to point out that once the available solutions are evaluated in the aperture plane $(z_i = 0)$, the mutual admittance can then be determined.

For the most general case, the differential equations (58) and (59) must be solved numerically, but for the special case of a homogeneous dielectric layer $(\epsilon_p(z_i) = \epsilon_p,\ \mu_p(z_i) = \mu_0)$, the solutions to equations (58) and (59) take the form

$$f_p(\alpha, \beta, z_i) = A(\alpha, \beta)\ e^{-jk_z^p z_i} + B(\alpha, \beta)\ e^{jk_z^p z_i} \tag{69}$$

$$g_p(\alpha, \beta, z_i) = C(\alpha, \beta)\ e^{-jk_z^p z_i} + D(\alpha, \beta)\ e^{jk_z^p z_i} \tag{70}$$

where the unknown coefficients are determined from the boundary conditions (eqs. (60) to (67)). If the external region consists of only one homogeneous layer of thickness d, the ratios of the functions in equation (68) become

$$\frac{k_0 \dfrac{\epsilon_1(0)}{\epsilon_0} g_1(\alpha, \beta, 0)}{jg_1'(\alpha, \beta, 0)} = \frac{\left(\dfrac{\epsilon_1}{\epsilon_0}\right)}{\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}} \left\{ \frac{\dfrac{\epsilon'}{\epsilon_0}\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2} + j\,\dfrac{\epsilon_1}{\epsilon_0}\sqrt{\dfrac{\epsilon'}{\epsilon_0} - \beta^2}\ \tan\left(k_0 d\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\right)}{\dfrac{\epsilon_1}{\epsilon_0}\sqrt{\dfrac{\epsilon'}{\epsilon_0} - \beta^2} + j\,\dfrac{\epsilon'}{\epsilon_0}\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\ \tan\left(k_0 d\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\right)} \right\} \tag{71}$$

$$\frac{jf_1'(\alpha, \beta, 0)}{k_0 \dfrac{\mu_1(0)}{\mu_0} f_1(\alpha, \beta, 0)} = \sqrt{\frac{\epsilon_1}{\epsilon_0} - \beta^2} \left\{ \frac{\sqrt{\dfrac{\epsilon'}{\epsilon_0} - \beta^2} + j\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\ \tan\left(k_0 d\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\right)}{\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2} + j\sqrt{\dfrac{\epsilon'}{\epsilon_0} - \beta^2}\ \tan\left(k_0 d\sqrt{\dfrac{\epsilon_1}{\epsilon_0} - \beta^2}\right)} \right\} \tag{72}$$

where ϵ_1 and ϵ' are the permittivities of the dielectric layer and the medium outside the layer, respectively.

If the thickness of the dielectric layer is allowed to go to zero, equations (71) and (72) reduce to

$$\frac{k_0 \frac{\epsilon_1(0)}{\epsilon_0} g_1(\alpha, \beta, 0)}{jg_1'(\alpha, \beta, 0)} = \frac{\left(\frac{\epsilon'}{\epsilon_0}\right)}{\sqrt{\frac{\epsilon'}{\epsilon_0} - \beta^2}} \tag{73}$$

$$\frac{jf_1'(\alpha, \beta, 0)}{k_0 \frac{\mu_1(0)}{\mu_0} f_1(\alpha, \beta, 0)} = \sqrt{\frac{\epsilon'}{\epsilon_0} - \beta^2} \tag{74}$$

for a homogeneous half space over the apertures.

If the permittivity ϵ' is real, the radical $\sqrt{\frac{\epsilon'}{\epsilon_0} - \beta^2}$ represents a branch point at $\beta^2 = \frac{\epsilon'}{\epsilon_0}$; therefore, to account for this properly in the integration, the radical must be replaced by $-j\sqrt{\beta^2 - \frac{\epsilon'}{\epsilon_0}}$ for $\beta^2 > \frac{\epsilon'}{\epsilon_0}$, which corresponds to the radiation condition (eq. (55)).

If both ϵ_1 and ϵ' are real (lossless dielectric layer), it can be seen from equations (71) and (72) that the integrand in equation (68) will be infinite for discrete values of β. These poles on the real axis of a complex β-plane correspond to the excitation of surface wave modes and must be properly accounted for by residues for the integration on β in the vicinity of these poles; however, this problem can be circumvented by assuming the dielectric to be slightly lossy, and this assumption causes the poles to move off the real β-axis. In most cases, a dielectric loss tangent of 0.001 is sufficient to eliminate the numerical integration difficulty near these poles while maintaining a 3 or 4 significant figure accuracy when compared with calculations for a lossless dielectric.

Circular apertures.- If the apertures are round holes, the fields in the apertures can be described by the set of circular waveguide modes whose transverse electric fields (normalized according to eqs. (7) to (11)) are given (ref. 127)

For TE:

$$E_{\rho_j}^{(j)TE}(\rho_j, \phi_j) = C_j^{TE} \frac{m_j J_{m_j}(A_j \rho_j)}{A_j \rho_j} \sin(m_j \phi_j) \tag{75}$$

$$E_{\phi_j}^{(j)TE}(\rho_j, \phi_j) = C_j^{TE} J'_{m_j}(A_j \rho_j) \cos(m_j \phi_j) \tag{76}$$

For TM:

$$E_{\rho_j}^{(j)TM}(\rho_j, \phi_j) = - C_j^{TM} J'_{m'_j}(A'_j \rho_j) \cos(m'_j \phi_j) \tag{77}$$

$$E_{\phi_j}^{(j)TM}(\rho_j, \phi_j) = C_j^{TM} \frac{m'_j J_{m'_j}(A'_j \rho_j)}{A'_j \rho_j} \sin(m'_j \phi_j) \tag{78}$$

where

$$A_j = \frac{\chi'_{m_j n_j}}{a_j} \tag{79}$$

$$A'_j = \frac{\chi_{m'_j n'_j}}{a_j} \tag{80}$$

$$C_j^{TE} = \frac{V_j^{TE} \sqrt{\frac{\epsilon_{m_j}}{\pi}} A_j}{J_{m_j}(\chi'_{m_j n_j}) \sqrt{(\chi'_{m_j n_j})^2 - m_j^2}} \tag{81}$$

$$C_j^{TM} = \frac{V_j^{TM} \sqrt{\frac{\epsilon_{m'_j}}{\pi}}}{a_j J_{m'_j+1}(\chi_{m'_j n'_j})} \tag{82}$$

with

$$\epsilon_{m_j} = 1 \qquad (m_j = 0)$$

$$\epsilon_{m_j} = 2 \qquad (m_j \neq 0)$$

$$\epsilon_{m_j'} = 1 \qquad (m_j' = 0)$$

$$\epsilon_{m_j'} = 2 \qquad (m_j' \neq 0)$$

and where $\chi_{m_j' n_j'}$ are the zeros of the Bessel function of the first kind, that is,

$$J_{m_j'}(\chi_{m_j' n_j'}) = 0 \tag{83}$$

and $\chi'_{m_j n_j}$ are the zeros of the derivative of $J_{m_j}(\chi)$

$$J'_{m_j}(\chi)\Big|_{\chi=\chi'_{m_j n_j}} = 0 \tag{84}$$

where the prime on $J_{m_j}(\chi)$ indicates differentiation with respect to the argument.

From figure 3, a transformation of variables is made so that

$$x_i = R \cos \phi + \rho_j \cos (\phi_j + \phi_p) \tag{85}$$

$$y_i = R \sin \phi + \rho_j \sin (\phi_j + \phi_p) \tag{86}$$

where

$$R = \sqrt{(y_j' - y_i')^2 + (x_j' - x_i')^2} \tag{87}$$

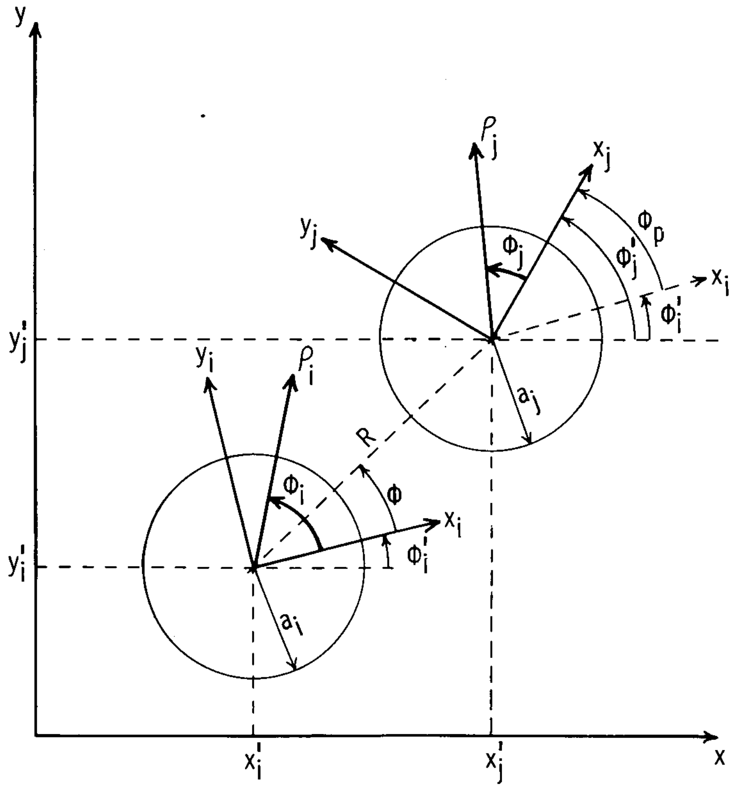

Figure 3.- Coordinate geometry for the ith and jth elements of a planar array of circular waveguide-fed apertures.

is the center-to-center spacing between the apertures,

$$\phi_p = \phi_j' - \phi_i' \tag{88}$$

is the polarization angle of the fields in the jth aperture relative to those in the ith aperture, the angle ϕ is defined as

$$\phi = \arctan\left(\frac{y_j' - y_i'}{x_j' - x_i'}\right) - \phi_i' \tag{89}$$

and the angles ϕ_i' and ϕ_j' are the polarization angles of the ith and jth aperture fields with respect to a fixed x, y coordinate system.

From figure 3, the x_i and y_i components of the aperture fields in the jth aperture are

$$E_{x_i}^{(j)}(\rho_j, \phi_j) = E_{\rho_j}^{(j)}(\rho_j, \phi_j)\cos(\phi_j + \phi_p) - E_{\phi_j}^{(j)}(\rho_j, \phi_j)\sin(\phi_j + \phi_p) \tag{90}$$

$$E_{y_i}^{(j)}(\rho_j, \phi_j) = E_{\rho_j}^{(j)}(\rho_j, \phi_j)\sin(\phi_j + \phi_p) + E_{\phi_j}^{(j)}(\rho_j, \phi_j)\cos(\phi_j + \phi_p) \tag{91}$$

Then with the change of variables in equations (85) and (86), and the definitions $k_x = k_0\beta\cos\alpha$ and $k_y = k_0\beta\sin\alpha$,

$$\left[k_x x_i + k_y y_i\right] = \left\{k_0\beta R\cos(\alpha - \phi) + k_0\beta\rho_j\cos\left[\phi_j - (\alpha - \phi_p)\right]\right\} \tag{92}$$

and the transforms of the aperture fields (eq. (22)) become

$$E_{x_i}^{(j)}(\beta, \alpha) = e^{j\psi}\int_{\rho_j=0}^{a_j}\int_{\phi_j=0}^{2\pi}\left\{\left[E_{\rho_j}^{(j)}(\rho_j, \phi_j)\cos(\phi_j + \phi_p)\right.\right.$$

$$\left.\left. - E_{\phi_j}^{(j)}(\rho_j, \phi_j)\sin(\phi_j + \phi_p)\right] e^{jk_0\beta\rho_j\cos\left[\phi_j - (\alpha - \phi_p)\right]}\right\}\rho_j\, d\rho_j\, d\phi_j \tag{93}$$

$$E_{y_i}^{(j)}(\beta, \alpha) = e^{j\psi}\int_{\rho_j=0}^{a_j}\int_{\phi_j=0}^{2\pi}\left\{\left[E_{\rho_j}^{(j)}(\rho_j, \phi_j)\sin(\phi_j + \phi_p)\right.\right.$$

$$\left.\left. + E_{\phi_j}^{(j)}(\rho_j, \phi_j)\cos(\phi_j + \phi_p)\right] e^{jk_0\beta\rho_j\cos\left[\phi_j - (\alpha - \phi_p)\right]}\right\}\rho_j\, d\rho_j\, d\phi_j \tag{94}$$

where

$$\psi = k_0\beta R\cos(\alpha - \phi) \tag{95}$$

Then for $TE_{m_j n_j}$ modes,

$$E_{x_i}^{(j)TE}(\beta, \alpha) = C_j^{TE} e^{j\psi} \left\{ \int_0^{a_j} \left[\frac{m_j J_{m_j}(A_j \rho_j)}{A_j \rho_j} \int_0^{2\pi} \cos(\phi_j + \phi_p) \sin(m_j \phi_j) \right.\right.$$

$$\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j$$

$$- \int_0^{a_j} \left[J'_{m_j}(A_j \rho_j) \int_0^{2\pi} \sin(\phi_j + \phi_p) \cos(m_j \phi_j) \right.$$

$$\left.\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j \right\} \tag{96}$$

$$E_{y_i}^{(j)TE}(\beta, \alpha) = C_j^{TE} e^{j\psi} \left\{ \int_0^{a_j} \left[\frac{m_j J_{m_j}(A_j \rho_j)}{A_j \rho_j} \int_0^{2\pi} \sin(\phi_j + \phi_p) \sin(m_j \phi_j) \right.\right.$$

$$\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j$$

$$+ \int_0^{a_j} \left[J'_{m_j}(A_j \rho_j) \int_0^{2\pi} \cos(\phi_j + \phi_p) \cos(m_j \phi_j) \right.$$

$$\left.\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j \right\} \tag{97}$$

and for $TM_{m_j' n_j'}$ modes,

$$E_{x_i}^{(j)TM}(\beta,\alpha) = -C_j^{TM} e^{j\psi} \left\{ \int_0^{a_j} \left[\frac{m_j' J_{m_j'}(A_j' \rho_j)}{A_j' \rho_j} \int_0^{2\pi} \sin(\phi_j + \phi_p) \sin(m_j' \phi_j) \right. \right.$$

$$\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j$$

$$+ \int_0^{a_j} \left[J'_{m_j'}(A_j' \rho_j) \int_0^{2\pi} \cos(\phi_j + \phi_p) \cos(m_j' \phi_j) \right.$$

$$\left. \left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j \right\} \tag{98}$$

$$E_{y_i}^{(j)TM}(\beta,\alpha) = C_j^{TM} e^{j\psi} \left\{ \int_0^{a_j} \left[\frac{m_j' J_{m_j'}(A_j' \rho_j)}{A_j' \rho_j} \int_0^{2\pi} \cos(\phi_j + \phi_p) \sin(m_j' \phi_j) \right. \right.$$

$$\left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j$$

$$- \int_0^{a_j} \left[J'_{m_j'}(A_j' \rho_j) \int_0^{2\pi} \sin(\phi_j + \phi_p) \cos(m_j' \phi_j) \right.$$

$$\left. \left. \times e^{jk_0 \beta \rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} d\phi_j \right] \rho_j \, d\rho_j \right\} \tag{99}$$

Then by using trigonometric identities, the integrals over ϕ_j in equations (96), (97), (98), and (99) can be expressed in terms of integrals of the form

$$I_1 = \int_0^{2\pi} \sin \phi_j \sin m\phi_j \, e^{jk_0 \beta\rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} \, d\phi_j \tag{100}$$

$$I_2 = \int_0^{2\pi} \sin \phi_j \cos m\phi_j \, e^{jk_0 \beta\rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} \, d\phi_j \tag{101}$$

$$I_3 = \int_0^{2\pi} \cos \phi_j \sin m\phi_j \, e^{jk_0 \beta\rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} \, d\phi_j \tag{102}$$

$$I_4 = \int_0^{2\pi} \cos \phi_j \cos m\phi_j \, e^{jk_0 \beta\rho_j \cos\left[\phi_j - (\alpha - \phi_p)\right]} \, d\phi_j \tag{103}$$

where $m = m_j$ for TE modes or $m = m_j'$ for TM modes. Then

$$E_{x_i}^{(j)TE}(\beta, \alpha) = C_j^{TE} e^{j\psi} \int_0^{a_j} \left[\frac{m_j J_{m_j}(A_j \rho_j)}{A_j \rho_j} (I_3 \cos \phi_p - I_1 \sin \phi_p) - J'_{m_j}(A_j \rho_j)(I_2 \cos \phi_p + I_4 \sin \phi_p) \right] \rho_j \, d\rho_j \tag{104}$$

$$E_{y_i}^{(j)TE}(\beta, \alpha) = C_j^{TE} e^{j\psi} \int_0^{a_j} \left[\frac{m_j J_{m_j}(A_j \rho_j)}{A_j \rho_j} (I_1 \cos \phi_p + I_3 \sin \phi_p) + J'_{m_j}(A_j \rho_j)(I_4 \cos \phi_p - I_2 \sin \phi_p) \right] \rho_j \, d\rho_j \tag{105}$$

$$E_{x_i}^{(j)TM}(\beta, \alpha) = -C_j^{TM} e^{j\psi} \int_0^{a_j} \left[\frac{m_j' J_{m_j'}(A_j' \rho_j)}{A_j' \rho_j} (I_1 \cos \phi_p + I_3 \sin \phi_p) + J'_{m_j'}(A_j' \rho_j)(I_4 \cos \phi_p - I_2 \sin \phi_p) \right] \rho_j \, d\rho_j \tag{106}$$

$$E_{y_i}^{(j)TM}(\beta, \alpha) = C_j^{TM} e^{j\psi} \int_0^{a_j} \left[\frac{m_j' J_{m_j'}(A_j' \rho_j)}{A_j' \rho_j} (I_3 \cos \phi_p - I_1 \sin \phi_p) - J'_{m_j'}(A_j' \rho_j)(I_2 \cos \phi_p + I_4 \sin \phi_p) \right] \rho_j \, d\rho_j \tag{107}$$

By writing the trigonometric functions in equations (100) to (103) as exponentials, the integrals on ϕ_j become

$$I_1 = -\frac{1}{4}(I_5 - I_6 - I_7 + I_8) \tag{108}$$

$$I_2 = \frac{1}{4j}(I_5 + I_6 - I_7 - I_8) \tag{109}$$

$$I_3 = \frac{1}{4j}(I_5 - I_6 + I_7 - I_8) \tag{110}$$

$$I_4 = \frac{1}{4}(I_5 + I_6 + I_7 + I_8) \tag{111}$$

where

$$I_5 = e^{j(m+1)(\alpha - \phi_p)} \int_{-(\alpha-\phi_p)}^{2\pi - (\alpha - \phi_p)} e^{j(m+1)\theta} \, e^{jk_0 \beta \rho_j \cos \theta} \, d\theta \tag{112}$$

$$I_6 = e^{-j(m-1)(\alpha - \phi_p)} \int_{-(\alpha-\phi_p)}^{2\pi - (\alpha - \phi_p)} e^{-j(m-1)\theta} \, e^{jk_0 \beta \rho_j \cos \theta} \, d\theta \tag{113}$$

$$I_7 = e^{j(m-1)(\alpha-\phi_p)} \int_{-(\alpha-\phi_p)}^{2\pi-(\alpha-\phi_p)} e^{j(m-1)\theta} e^{jk_0\beta\rho_j \cos\theta} d\theta \tag{114}$$

$$I_8 = e^{-j(m+1)(\alpha-\phi_p)} \int_{-(\alpha-\phi_p)}^{2\pi-(\alpha-\phi_p)} e^{-j(m+1)\theta} e^{jk_0\beta\rho_j \cos\theta} d\theta \tag{115}$$

where a change of variables has been made so that $\theta = \phi_j - (\alpha - \phi_p)$. The integrals in equations (112) to (115) are recognized as a form of the Bessel function of the first kind, that is (see p. 367, ref. 128)

$$J_m(z) = \frac{j^{-m}}{2\pi} \int_{-\gamma}^{2\pi-\gamma} e^{jm\theta} e^{jz\cos\theta} d\theta \tag{116}$$

where γ is any arbitrary angle.

Then with the relationship (p. 128 of ref. 129)

$$J_{-m}(z) = (-1)^m J_m(z) \tag{117}$$

equations (112) to (115) become

$$I_5 = 2\pi(j)^{m+1} J_{m+1}(k_0\beta\rho_j)\ e^{j(m+1)(\alpha-\phi_p)} \tag{118}$$

$$I_6 = 2\pi(j)^{1-m} (-1)^{m-1} J_{m-1}(k_0\beta\rho_j)\ e^{-j(m-1)(\alpha-\phi_p)} \tag{119}$$

$$I_7 = 2\pi(j)^{m-1} J_{m-1}(k_0\beta\rho_j)\ e^{j(m-1)(\alpha-\phi_p)} \tag{120}$$

$$I_8 = 2\pi(j)^{-m-1} (-1)^{m+1} J_{m+1}(k_0\beta\rho_j)\ e^{-j(m+1)(\alpha-\phi_p)} \tag{121}$$

Substituting equations (118) to (121) into equations (108) to (111) and combining terms yields

$$I_1 = -\pi(j)^{m+1} \left\{ J_{m+1} (k_0 \beta\rho_j) \cos \left[(m + 1) (\alpha - \phi_p)\right] \right.$$

$$\left. + J_{m-1} (k_0\beta\rho_j) \cos \left[(m - 1) (\alpha - \phi_p)\right] \right\} \tag{122}$$

$$I_2 = \pi(j)^{m+1} \left\{ J_{m+1} (k_0 \beta\rho_j) \sin \left[(m + 1) (\alpha - \phi_p)\right] \right.$$

$$\left. + J_{m-1} (k_0\beta\rho_j) \sin \left[(m - 1) (\alpha - \phi_p)\right] \right\} \tag{123}$$

$$I_3 = \pi(j)^{m+1} \left\{ J_{m+1} (k_0 \beta\rho_j) \sin \left[(m + 1) (\alpha - \phi_p)\right] \right.$$

$$\left. + J_{m-1} (k_0 \beta\rho_j) \sin \left[(m - 1) (\alpha - \phi_p)\right] \right\} \tag{124}$$

$$I_4 = \pi(j)^{m+1} \left\{ J_{m+1} (k_0\beta\rho_j) \cos \left[(m + 1) (\alpha - \phi_p)\right] \right.$$

$$\left. - J_{m-1} (k_0 \beta\rho_j) \cos \left[(m - 1) (\alpha - \phi_p)\right] \right\} \tag{125}$$

By substituting equations (122) to (125) into equations (104) to (107) and using the recurrence equations for Bessel functions

$$J_{m-1}(z) = \frac{mJ_m(z)}{z} + J'_m(z) \tag{126}$$

$$J_{m+1}(z) = \frac{mJ_m(z)}{z} - J'_m(z) \tag{127}$$

the transforms of the aperture electric fields become

$$E_{x_i}^{(j)TE}(\beta, \alpha) = \pi(j)^{m_j+1} C_j^{TE} e^{j\psi}$$

$$\times \left\{ \sin\left[(m_j + 1)\alpha - m_j\phi_p\right] \int_0^{a_j} J_{m_j+1}(A_j\rho_j)\, J_{m_j+1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right.$$

$$\left. - \sin\left[(m_j - 1)\alpha - m_j\phi_p\right] \int_0^{a_j} J_{m_j-1}(A_j\rho_j)\, J_{m_j-1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right\} \qquad (128)$$

$$E_{y_i}^{(j)TE}(\beta, \alpha) = -\pi(j)^{m_j+1} C_j^{TE} e^{j\psi}$$

$$\times \left\{ \cos\left[(m_j + 1)\alpha - m_j\phi_p\right] \int_0^{a_j} J_{m_j+1}(A_j\rho_j)\, J_{m_j+1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right.$$

$$\left. + \cos\left[(m_j - 1)\alpha - m_j\phi_j\right] \int_0^{a_j} J_{m_j-1}(A_j\rho_j)\, J_{m_j-1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right\} \qquad (129)$$

$$E_{x_i}^{(j)TM}(\beta, \alpha) = \pi(j)^{m_j'+1} C_j^{TM} e^{j\psi}$$

$$\times \left\{ \cos\left[(m_j' + 1)\alpha - m_j'\phi_p\right] \int_0^{a_j} J_{m_j'+1}(A_j'\rho_j)\, J_{m_j'+1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right.$$

$$\left. + \cos\left[(m_j' - 1)\alpha - m_j'\phi_p\right] \int_0^{a_j} J_{m_j'-1}(A_j'\rho_j)\, J_{m_j'-1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right\} \qquad (130)$$

$$E_{y_i}^{(j)TM}(\beta, \alpha) = \pi(j)^{m_j'+1} C_j^{TM} e^{j\psi}$$

$$\times \left\{ \sin\left[(m_j' + 1)\alpha - m_j'\phi_p\right] \int_0^{a_j} J_{m_j'+1}(A_j'\rho_j)\, J_{m_j'+1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right.$$

$$\left. - \sin\left[(m_j' - 1)\alpha - m_j'\phi_p\right] \int_0^{a_j} J_{m_j'-1}(A_j'\rho_j)\, J_{m_j'-1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j \right\} \qquad (131)$$

The integrals over ρ_j in equations (128) to (131) can now be evaluated in closed form (see p. 146 of ref. 129)

$$\int_0^a J_{m+1}(A\rho)\, J_{m+1}(k_0\beta\rho)\, \rho\, d\rho$$

$$= \left[\frac{1}{A^2 - (k_0\beta)^2}\right] \left[(k_0\beta a)\, J_{m+1}(aA)\, J_m(k_0\beta a) - aAJ_m(aA)\, J_{m+1}(k_0\beta a)\right] \tag{132}$$

$$\int_0^a J_{m-1}(A\rho)\, J_{m-1}(k_0\beta\rho)\, \rho\, d\rho$$

$$= \left[\frac{1}{A^2 - (k_0\beta)^2}\right] \left[(k_0\beta a)\, J_{m-1}(aA)\, J_{m-2}(k_0\beta a) - aAJ_{m-2}(aA)\, J_{m-1}(k_0\beta a)\right] \tag{133}$$

which gives for the $TE_{m_jn_j}$ modes

$$\int_0^{a_j} J_{m_j+1}(A_j\rho_j)\, J_{m_j+1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j$$

$$= \left(\frac{1}{k_0^2}\right) \frac{J_{m_j}(\chi'_{m_jn_j})}{\chi'_{m_jn_j}} \left[\frac{(\chi'_{m_jn_j})^2\, J'_{m_j}(k_0a_j\beta)}{\left(\frac{\chi'_{m_jn_j}}{k_0a_j}\right)^2 - \beta^2} - \frac{m_j(k_0a_j)\, J_{m_j}(k_0a_j\beta)}{\beta}\right] \tag{134}$$

$$\int_0^{a_j} J_{m_j-1}(A_j\rho_j)\, J_{m_j-1}(k_0\beta\rho_j)\, \rho_j\, d\rho_j$$

$$= \left(\frac{1}{k_0^2}\right) \frac{J_{m_j}(\chi'_{m_jn_j})}{\chi'_{m_jn_j}} \left[\frac{(\chi'_{m_jn_j})^2\, J'_{m_j}(k_0a_j\beta)}{\left(\frac{\chi'_{m_jn_j}}{k_0a_j}\right)^2 - \beta^2} + \frac{m_j(k_0a_j)\, J_{m_j}(k_0a_j\beta)}{\beta}\right] \tag{135}$$

and for the $TM_{m_j' n_j'}$ modes

$$\int_0^{a_j} J_{m_j'+1}(A_j' \rho_j)\, J_{m_j'+1}(k_0 \beta \rho_j)\, \rho_j \, d\rho_j = \left(\frac{a_j}{k_0}\right) J_{m_j'+1}(\chi_{m_j' n_j'}) \left[\frac{\beta J_{m_j'}(k_0 a_j \beta)}{\left(\frac{\chi_{m_j' n_j'}}{k_0 a_j}\right)^2 - \beta^2}\right] \tag{136}$$

$$\int_0^{a_j} J_{m_j'-1}(A_j' \rho_j)\, J_{m_j'-1}(k_0 \beta \rho_j)\, \rho_j \, d\rho_j = \left(\frac{a_j}{k_0}\right) J_{m_j'+1}(\chi_{m_j' n_j'}) \left[\frac{\beta J_{m_j'}(k_0 a_j \beta)}{\left(\frac{\chi_{m_j' n_j'}}{k_0 a_j}\right)^2 - \beta^2}\right] \tag{137}$$

The transforms of the aperture electric fields then become

$$E_{x_i}^{(j)TE}(\alpha, \beta) = 2(j)^{m_j+1}\, e^{j\psi}\, V_j^{TE} \sqrt{\pi} \sqrt{\epsilon_{m_j}} \left[\frac{1}{k_0 \sqrt{\left(\chi'_{m_j n_j}\right)^2 - m_j^2}}\right]$$

$$\times \left\{ \left[\frac{\left(\frac{\chi'_{m_j n_j}}{k_0 a_j}\right) \left(\chi'_{m_j n_j}\right) J'_{m_j}(k_0 a_j \beta)}{\left(\frac{\chi'_{m_j n_j}}{k_0 a_j}\right)^2 - \beta^2}\right] \sin\alpha \cos\left[m_j(\alpha - \phi_p)\right] \right.$$

$$\left. \left[\frac{m_j J_{m_j}(k_0 a_j \beta)}{\beta}\right] \cos\alpha \sin\left[m_j(\alpha - \phi_p)\right] \right\} \tag{138}$$

$$E_{y_i}^{(j)TE}(\alpha,\beta) = -2(j)^{m_j+1} e^{j\psi} V_j^{TE} \sqrt{\pi} \sqrt{\epsilon_{m_j}} \left[\frac{1}{k_0 \sqrt{(\chi'_{m_j n_j})^2 - m_j^2}}\right]$$

$$\times \left\{ \left[\frac{\left(\frac{\chi'_{m_j n_j}}{k_0 a_j}\right) (\chi'_{m_j n_j}) J'_{m_j}(k_0 a_j \beta)}{\left(\frac{\chi'_{m_j n_j}}{k_0 a_j}\right)^2 - \beta^2} \right] \cos\alpha \cos\left[m_j(\alpha - \phi_p)\right] \right.$$

$$\left. + \left[\frac{m_j J_{m_j}(k_0 a_j \beta)}{\beta}\right] \sin\alpha \sin\left[m_j(\alpha - \phi_p)\right] \right\} \quad (139)$$

$$E_{x_i}^{(j)TM}(\alpha,\beta) = 2(j)^{m'_j+1} e^{j\psi} V_j^{TM} \sqrt{\pi} \sqrt{\epsilon_{m'_j}} \left(\frac{1}{k_0}\right)$$

$$\times \left[\frac{\beta J_{m'_j}(k_0 a_j \beta)}{\left(\frac{\chi_{m'_j n'_j}}{k_0 a_j}\right)^2 - \beta^2}\right] \cos\alpha \cos\left[m'_j(\alpha - \phi_p)\right] \quad (140)$$

$$E_{y_i}^{(j)TM}(\alpha,\beta) = 2(j)^{m'_j+1} e^{j\psi} V_j^{TM} \sqrt{\pi} \sqrt{\epsilon_{m'_j}} \left(\frac{1}{k_0}\right)$$

$$\times \left[\frac{\beta J_{m'_j}(k_0 a_j \beta)}{\left(\frac{\chi_{m'_j n'_j}}{k_0 a_j}\right)^2 - \beta^2}\right] \sin\alpha \cos\left[m'_j(\alpha - \phi_p)\right] \quad (141)$$

The transform fields in equations (138) to (141) are even functions of β when m_j or m_j' is odd and odd functions of β when m_j or m_j' is even; therefore,

$$E_{x_i}^{(i)TE}(\alpha, -\beta) = (-1)^{m_i+1} E_{x_i}^{(i)TE}(\alpha, \beta) \tag{142}$$

$$E_{y_i}^{(i)TE}(\alpha, -\beta) = (-1)^{m_i+1} E_{y_i}^{(i)TE}(\alpha, \beta) \tag{143}$$

$$E_{x_i}^{(i)TM}(\alpha, -\beta) = (-1)^{m_i'+1} E_{x_i}^{(i)TM}(\alpha, \beta) \tag{144}$$

$$E_{y_i}^{(i)TM}(\alpha, -\beta) = (-1)^{m_i'+1} E_{y_i}^{(i)TM}(\alpha, \beta) \tag{145}$$

where $E_{x_i}^{(i)TE}(\alpha, \beta)$, $E_{y_i}^{(i)TE}(\alpha, \beta)$, $E_{x_i}^{(i)TM}(\alpha, \beta)$, and $E_{y_i}^{(i)TM}(\alpha, \beta)$ are obtained from equations (138) to (141) by setting $\psi = \phi_p = 0$ and replacing the j subscripts by i. This will complete the evaluation of the Fourier transforms of the aperture electric fields of an open-end circular waveguide.

Since the ratios of the solutions to the wave equation and their derivatives are independent of α (see eqs. (71) to (74)), the mutual admittance between a TE_{m_i,n_i} mode in the ith aperture and a TE_{m_j,m_j} mode in the jth aperture can be expressed as

$$
\begin{aligned}
Y_{ij}^{TE,TE} = {} & \left(\frac{1}{4\pi}\right)(-1)^{m_i+1}(j)^{m_j+m_i}\left\{\frac{\sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_{m_i}\epsilon_{m_j}}}{\sqrt{(x'_{m_in_i})^2 - m_i^2}\sqrt{(x'_{m_jn_j})^2 - m_j^2}}\right\}\int_0^\infty\left\{\left[\frac{m_iJ_{m_i}(k_0a_i\beta)}{\beta}\right]\left[\frac{m_jJ_{m_j}(k_0a_j\beta)}{\beta}\right]\right. \\
& \times\left[\frac{k_0\frac{\epsilon_1(0)}{\epsilon_0}g_1(\beta, 0)}{jg_1'(\beta, 0)}\right]\left[(I_9 - I_{10} - I_{11} + I_{12})\cos m_j\phi_p - j(I_9 + I_{10} - I_{11} - I_{12})\sin m_j\phi_p\right] \\
& \left[\frac{\left(\frac{x'_{m_in_i}}{k_0a_i}\right)(x'_{m_in_i})J'_{m_i}(k_0a_i\beta)}{\left(\frac{x'_{m_in_i}}{k_0a_i}\right)^2 - \beta^2}\right]\left[\frac{\left(\frac{x'_{m_jn_j}}{k_0a_j}\right)(x'_{m_jn_j})J'_{m_j}(k_0a_j\beta)}{\left(\frac{x'_{m_jn_j}}{k_0a_i}\right)^2 - \beta^2}\right]\left[\frac{jf_1'(\beta, 0)}{k_0\frac{\mu_1(0)}{\mu_0}f_1(\beta, 0)}\right] \\
& \left.\times\left[(I_9 + I_{10} + I_{11} + I_{12})\cos m_j\phi_p - j(I_9 - I_{10} + I_{11} - I_{12})\sin m_j\phi_p\right]\right\}\beta\, d\beta
\end{aligned}
\tag{146}
$$

where

$$I_9 = \int_0^{2\pi} e^{j(m_j+m_i)\alpha} e^{jk_0\beta R \cos(\alpha-\phi)} d\alpha \tag{147}$$

$$I_{10} = \int_0^{2\pi} e^{-j(m_j-m_i)\alpha} e^{jk_0\beta R \cos(\alpha-\phi)} d\alpha \tag{148}$$

$$I_{11} = \int_0^{2\pi} e^{j(m_j-m_i)\alpha} e^{jk_0\beta R \cos(\alpha-\phi)} d\alpha \tag{149}$$

$$I_{12} = \int_0^{2\pi} e^{-j(m_j+m_i)\alpha} e^{jk_0\beta R \cos(\alpha-\phi)} d\alpha \tag{150}$$

and by replacing m_j by m_j' and m_i by m_i' in equations (147) to (150), the mutual admittance between a $TM_{m_i'n_i'}$ mode in the ith aperture and a $TM_{m_j'n_j'}$ mode in the jth aperture becomes

$$Y_{ij}^{TM,TM} = \left(\frac{1}{4\pi}\right)(-1)^{m_i'+1}(j)^{m_j'+m_i'}\sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_{m_j'}\epsilon_{m_i'}}$$

$$\times \int_0^{\infty} \left\{ \left[\frac{\beta J_{m_j'}(k_0 a_j \beta)}{\left(\frac{\chi_{m_j'n_j'}}{k_0 a_j}\right)^2 - \beta^2}\right] \left[\frac{\beta J_{m_i'}(k_0 a_i \beta)}{\left(\frac{\chi_{m_i'n_i'}}{k_0 a_i}\right)^2 - \beta^2}\right] \left[\frac{k_0 \frac{\epsilon_1(0)}{\epsilon_0} g_1(\beta, 0)}{j g_1'(\beta, 0)}\right] \right.$$

$$\left. \times \left[-(I_9 + I_{10} + I_{11} + I_{12}) \cos m_j' \phi_p + j(I_9 - I_{10} + I_{11} - I_{12}) \sin m_j' \phi_p\right] \right\} \beta \, d\beta$$

(151)

By replacing m_j by m_j' in equations (147) to (150), the mutual admittance between a $TE_{m_i n_i}$ mode in the ith aperture and a $TM_{m_j' n_j'}$ mode in the jth aperture can be written as

$$Y_{ij}^{TE,TM} = \left(\frac{1}{4\pi}\right)(-1)^{m_i+1}(j)^{m_j'+m_i}\sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_{m_i}\epsilon_{m_j'}}\left(\frac{1}{\sqrt{(x_{m_i n_i}')^2 - m_i^2}}\right)$$

$$\times\int_0^\infty\left\{\left[\frac{m_i J_{m_i}(k_0 a_i \beta)}{\beta}\right]\left[\frac{\beta J_{m_j'}(k_0 a_j \beta)}{\left(\frac{x_{m_j' n_j'}}{k_0 a_j}\right)^2 - \beta^2}\right]\left[\frac{k_0 \frac{\epsilon_1(0)}{\epsilon_0} g_1(\beta, 0)}{j g_1'(\beta, 0)}\right]\right.$$

$$\left.\times\left[-(I_9 - I_{10} - I_{11} + I_{12})\sin m_j'\phi_p - j(I_9 + I_{10} - I_{11} - I_{12})\cos m_j'\phi_p\right]\right\}\beta\, d\beta \qquad (152)$$

and the mutual admittance between a $TM_{m_i' n_i'}$ mode in the ith aperture and a $TE_{m_j n_j}$ mode in the jth aperture becomes (with m_i in equations (147) to (150) replaced by m_i')

$$Y_{ij}^{TM,TE} = \left(\frac{1}{4\pi}\right)(-1)^{m_i'+1}(j)^{m_j+m_i'}\sqrt{\frac{\epsilon_0}{\mu_0}}\sqrt{\epsilon_{m_i'}\epsilon_{m_j}}\left(\frac{1}{\sqrt{(x_{m_j n_j}')^2 - m_j^2}}\right)$$

$$\times\int_0^\infty\left\{\left[\frac{\beta J_{m_i'}(k_0 a_i \beta)}{\left(\frac{x_{m_i' n_i'}}{k_0 a_i}\right)^2 - \beta^2}\right]\left[\frac{m_j J_{m_j}(k_0 a_j \beta)}{\beta}\right]\left[\frac{k_0 \frac{\epsilon_1(0)}{\epsilon_0} g_1(\beta, 0)}{j g_1'(\beta, 0)}\right]\right.$$

$$\left.\times\left[-(I_9 - I_{10} - I_{11} + I_{12})\sin m_j\phi_p - j(I_9 + I_{10} - I_{11} - I_{12})\cos m_j\phi_p\right]\right\}\beta\, d\beta \qquad (153)$$

In the evaluation of the integrals on α, equations (147) to (150) become

$$I_9 = I_{10} = I_{11} = I_{12} = 2\pi \qquad (R = 0; \quad m_j = m_i = 0) \tag{154}$$

$$I_9 = I_{10} = I_{11} = I_{12} = 0 \qquad (R = 0; \quad m_j \neq m_i) \tag{155}$$

$$I_9 = I_{12} = 0 \qquad (R = 0; \quad m_j = m_i; \quad m_i \neq 0) \tag{156}$$

$$I_{10} = I_{11} = 2\pi \qquad (R = 0; \quad m_j = m_i; \quad m_i \neq 0) \tag{157}$$

and for $R \neq 0$, a change of variables is made such that $\theta = \alpha - \phi$, and equations (147) to (150) can be expressed in the form of a Bessel function (see eq. (116))

$$I_9 = 2\pi(j)^{m_j+m_i} J_{m_j+m_i}(k_0\beta R) \; e^{j(m_j+m_i)\phi} \tag{158}$$

$$I_{10} = 2\pi(j)^{m_i-m_j} (-1)^{m_j-m_i} J_{m_j-m_i}(k_0\beta R) \; e^{-j(m_j-m_i)\phi} \tag{159}$$

$$I_{11} = 2\pi(j)^{m_j-m_i} J_{m_j-m_i}(k_0\beta R) \; e^{j(m_j-m_i)\phi} \tag{160}$$

$$I_{12} = 2\pi(j)^{-m_j-m_i} (-1)^{m_j+m_i} J_{m_j+m_i}(k_0\beta R) \; e^{-j(m_j+m_i)\phi} \tag{161}$$

By substituting equations (154) to (161) into equations (146), (151), (152), and (153), and using the following definitions:

$$W_1(\beta) = \frac{k_0 \dfrac{\epsilon_1(0)}{\epsilon_0} g_1(\beta, 0)}{jg_1'(\beta, 0)} \tag{162}$$

$$W_2(\beta) = \frac{jf_1'(\beta, 0)}{k_0 \dfrac{\mu_1(0)}{\mu_0} f_1(\beta, 0)} \tag{163}$$

$$\xi_i^{TE}(\beta) = \frac{m_i J_{m_i}(k_0 a_i \beta)}{\beta \sqrt{(\chi'_{m_i n_i})^2 - m_i^2}} \tag{164}$$

$$\zeta_i^{TE}(\beta) = \frac{\left(\frac{\chi'_{m_i n_i}}{k_0 a_i}\right) (\chi'_{m_i n_i}) J'_{m_i}(k_0 a_i \beta)}{\left[\left(\frac{\chi'_{m_i n_i}}{k_0 a_i}\right)^2 - \beta^2\right] \sqrt{(\chi'_{m_i n_i})^2 - m_i^2}} \tag{165}$$

$$\xi_i^{TM}(\beta) = \frac{\beta J_{m'_i}(k_0 a_i \beta)}{\left(\frac{\chi_{m'_i n'_i}}{k_0 a_i}\right)^2 - \beta^2} \tag{166}$$

and defining $\xi_j^{TE}(\beta)$, $\zeta_j^{TE}(\beta)$, and $\xi_j^{TM}(\beta)$ by changing the subscript i in equations (164) to (166) to j, the mutual admittance becomes: For the $TE_{m_i n_i}$ and $TE_{m_j n_j}$ modes

$$Y_{ij}^{TE,TE} = -\sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{\epsilon_{m_i} \epsilon_{m_j}} \int_0^{\infty} \Big[W_1(\beta)\, \xi_i^{TE}(\beta)\, \xi_j^{TE}(\beta)\, U_{ij}^{TE,TE}(\beta)$$

$$- W_2(\beta)\, \zeta_i^{TE}(\beta)\, \zeta_j^{TE}(\beta)\, V_{ij}^{TE,TE}(\beta) \Big] \beta \, d\beta \tag{167}$$

where

$$U_{ij}^{TE,TE}(\beta) = V_{ij}^{TE,TE}(\beta) = 0, \qquad (R = 0;\quad m_j \neq m_i)$$

$$\left.\begin{aligned} U_{ij}^{TE,TE}(\beta) &= -(\epsilon_{m_i} - 1) \cos m_i \phi_p \\ V_{ij}^{TE,TE}(\beta) &= \frac{2}{\epsilon_{m_i}} \cos m_i \phi_p \end{aligned}\right\} \qquad (R = 0;\quad m_j = m_i)$$

$$U_{ij}^{TE,TE}(\beta) = (-1)^{m_j} \left\{ J_{m_j+m_i}(k_0 \beta R) \cos\left[(m_j + m_i)\phi - m_j \phi_p\right] \right.$$

$$\left. - (-1)^{m_i} J_{m_j-m_i}(k_0 \beta R) \cos\left[(m_j - m_i)\phi - m_j \phi_p\right] \right\} \qquad (R \neq 0)$$

$$V_{ij}^{TE,TE}(\beta) = (-1)^{m_j} \left\{ J_{m_j+m_i}(k_0 \beta R) \cos\left[(m_j + m_i)\phi - m_j \phi_p\right] \right.$$

$$\left. + (-1)^{m_i} J_{m_j-m_i}(k_0 \beta R) \cos\left[(m_j - m_i)\phi - m_j \phi_p\right] \right\} \qquad (R \neq 0)$$

For the $TM_{m_i' n_i'}$ and $TM_{m_j' n_j'}$ modes

$$Y_{ij}^{TM,TM} = -\sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{\epsilon_{m_i'} \epsilon_{m_j'}} \int_0^\infty \left[W_1(\beta)\, \xi_i^{TM}(\beta)\, \xi_j^{TM}(\beta)\, U_{ij}^{TM,TM}(\beta)\right] \beta \, d\beta \qquad (168)$$

where

$$U_{ij}^{TM,TM}(\beta) = 0 \qquad (R = 0;\ m_j' \neq m_i')$$

$$U_{ij}^{TM,TM}(\beta) = -\left(\frac{2}{\epsilon_{m_i'}}\right) \cos m_i' \phi_p \qquad (R = 0;\ m_j' = m_i')$$

$$U_{ij}^{TM,TM}(\beta) = -(-1)^{m_j'} \left\{ J_{m_j'+m_i'}(k_0 \beta R) \cos\left[(m_j' + m_i')\phi - m_j' \phi_p\right] \right.$$

$$\left. + (-1)^{m_i'} J_{m_j'-m_i'}(k_0 \beta R) \cos\left[(m_j' - m_i')\phi - m_j' \phi_p\right] \right\} \qquad (R \neq 0)$$

For the $TE_{m_i n_i}$ and $TM_{m_j' n_j'}$ modes

$$Y_{ij}^{TE,TM} = -\sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{\epsilon_{m_i} \epsilon_{m_j'}} \int_0^\infty \left[W_1(\beta)\, \xi_i^{TE}(\beta)\, \xi_j^{TM}(\beta)\, U_{ij}^{TE,TM}(\beta)\right] \beta \, d\beta \qquad (169)$$

where

$$U_{ij}^{TE,TM}(\beta) = 0 \qquad (R = 0;\ m_j' \neq m_i)$$

$$U_{ij}^{TE,TM}(\beta) = (\epsilon_{m_i} - 1) \sin m_i \phi_p \qquad (R = 0;\ m_j' = m_i)$$

$$U_{ij}^{TE,TM}(\beta) = (-1)^{m_j'} \left\{ J_{m_j'+m_i}(k_0 \beta R) \sin \left[(m_j' + m_i) \phi - m_j' \phi_p \right] - (-1)^{m_i} J_{m_j'-m_i}(k_0 \beta R) \sin \left[(m_j' - m_i) \phi - m_j' \phi_p \right] \right\} \qquad (R \neq 0)$$

For $TM_{m_i' n_i'}$ and $TE_{m_j n_j}$ modes

$$Y_{ij}^{TM,TE} = -\sqrt{\frac{\epsilon_0}{\mu_0}} \sqrt{\epsilon_{m_i'} \epsilon_{m_j}} \int_0^\infty \left[W_1(\beta)\, \xi_i^{TM}(\beta)\, \xi_j^{TE}(\beta)\, U_{ij}^{TM,TE}(\beta) \right] \beta \, d\beta \tag{170}$$

where

$$U_{ij}^{TM,TE}(\beta) = 0 \qquad (R = 0;\ m_j \neq m_i')$$

$$U_{ij}^{TM,TE}(\beta) = (\epsilon_{m_i'} - 1) \sin m_i' \phi_p \qquad (R = 0;\ m_j = m_i')$$

$$U_{ij}^{TM,TE}(\beta) = (-1)^{m_j} \left\{ J_{m_j+m_i'}(k_0 \beta R) \sin \left[(m_j + m_i') \phi - m_j \phi_p \right] - (-1)^{m_i'} J_{m_j-m_i'}(k_0 \beta R) \sin \left[(m_j - m_i') \phi - m_j \phi_p \right] \right\} \qquad (R \neq 0)$$

The integration on β in equations (167) to (170) must be numerically evaluated. A computer program has been written for the evaluation of these equations for the mutual admittance of two circular apertures radiating into a multilayered region of up to four layers, two of which may be inhomogeneous normal to the aperture plane. A listing of the computer program is included as appendix B.

DESCRIPTION OF EXPERIMENT

Hardware was constructed and an experiment was performed for the verification of the theoretical analysis. The verification was accomplished by comparing the measured and calculated TE_{11} mode mutual coupling between two circular waveguide-fed apertures for various combinations of frequency, spacing, and polarization. The hardware which was constructed and assembled for this purpose is shown in figure 4.

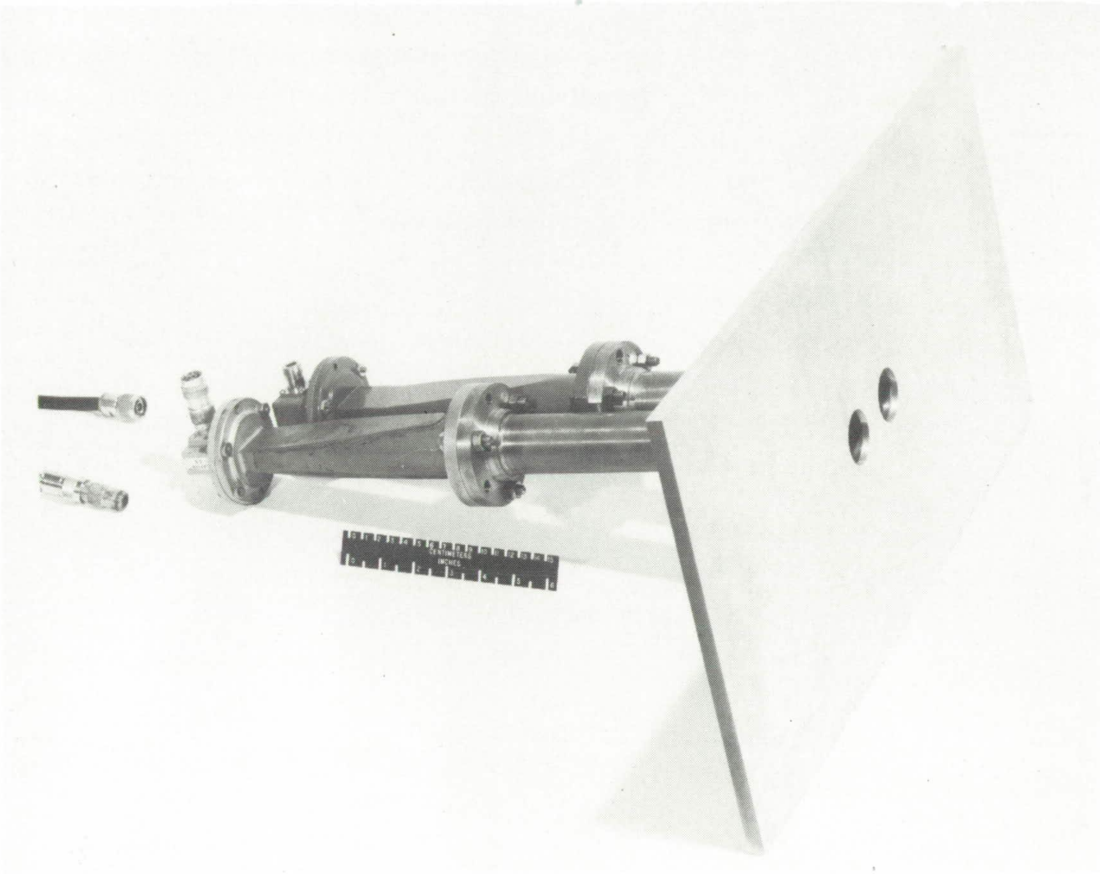

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Figure 4.- Experimental model for mutual coupling measurements.

The hardware in figure 4 consists of a 30.48 cm by 60.96 cm (12 in. by 24 in.) flat aluminum plate with two 3.81-cm-diameter (1.5-in.) circular waveguide-fed holes which are equally distant from the center of the rectangular plate. The circular waveguide sections are connected to standard coaxial adapters (RG 50/U rectangular to type N coaxial) by 25.4 cm (10 in.) circular to rectangular linearly tapered transitions. One of these transitions had been used in a previous experiment (ref. 130) and performed satisfactorily over the frequency range of interest. Since the other transition is dimensionally identical, it can be expected to give a similar performance.

Swivel flanges are used to connect the circular waveguide sections to the tapered transitions. This connection allows the polarization of each aperture to be changed by rotating the adapter and transition through the desired angle. The electric-field polarization of both apertures in figure 4 is vertical, as indicated by the position of the excitation probes of the coaxial to waveguide adapters.

The circular waveguide sections are flange mounted to the aluminum plate as illustrated in figure 5 by the unassembled cross-sectional view of the aluminum plate and one waveguide. The back side of the aluminum plate is recessed and the waveguide end extends out past the flange an equal amount in order to maintain accurate alinement of the waveguide with the circular hole. By mounting each waveguide to the aluminum plate in this manner, the same waveguide assembly can be used with a variety of flat plates with different hole spacings. The one shown in figure 4 is for a center-to-center spacing of 6.35 cm (2.5 in.). Other plates were constructed with center-to-center hole spacings of 8.89, 12.70, and 17.78 cm (3.5, 5.0, and 7.0 in.) but are not shown since they are identical to the one in figure 4 except for the different hole separations.

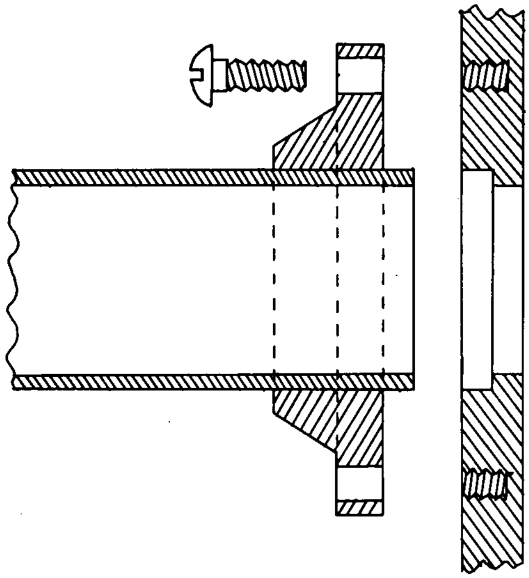

Figure 5.- Cross section of experimental model illustrating method of mounting circular waveguide to aluminum plate.

All parts for the experiment were purchased as commercial stock items, with the exception of the rectangular plates which were machined from stock aluminum.

The mutual coupling was measured by exciting one waveguide at the coaxial adapter and comparing the received signal level at the other coaxial adapter to a known reference. This measurement was accomplished by connecting coaxial cables to a signal generator and a receiver. The other ends of the cables (shown in fig. 4) were then connected together to obtain a reference signal level at the receiver. The mutual coupling was then measured by connecting the cable ends to the coaxial adapters and adjusting in-line calibrated attenuators until the received signal level was the same as the reference level. The net change in the calibrated attenuators is observed as the relative power coupled from the input terminal at one coaxial adapter to the output terminal at the other coaxial adapter through the open ends of the circular waveguides mounted to the flat aluminum plate.

The insertion loss due to the waveguide assembly was measured by the same method with the flat plate removed and the ends of the circular waveguides connected together as shown in figure 6. The insertion loss was measured at each frequency of interest and all the measured data presented in the section "Results and Discussion" have been corrected for the waveguide assembly insertion loss at each measurement frequency. The insertion loss correction was between 0.1 dB and 0.5 dB over the frequency range.

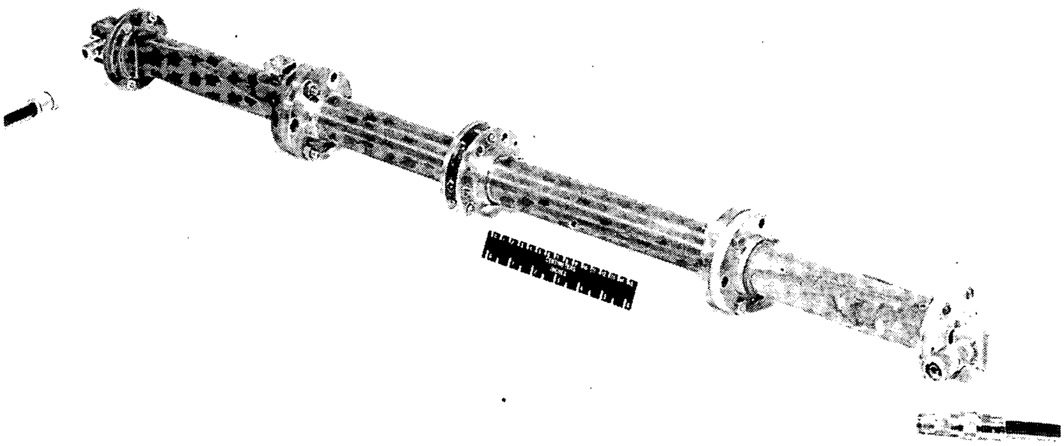

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Figure 6.- Experimental model for determination of waveguide assembly insertion loss.

RESULTS AND DISCUSSION

Comparison Between Measurements and Calculations

The data in this section are presented primarily for verification of the theoretical analysis. Data for various combinations of aperture spacings and polarizations are presented as a function of frequency in order to test all facets of the analysis.

The calculations in this and the following sections were obtained from the computer program listed in appendix B. The calculated mutual coupling values were taken from the appropriate off-diagonal terms of the scattering matrix.

Figures 7 and 8 show a comparison between calculations and measurements for the four different aperture separations. There is excellent agreement between the measured and calculated values in figure 7 for coupling in the E-plane; however, larger variations in the measured data as a function of frequency were observed for the H-plane coupling in figure 8. This oscillatory variation with frequency is typical of impedance measurements for antennas with a truncated ground plane. A slight variation can also be observed in figure 7; however, it is more pronounced in figure 8 because of a combination of two things. First, when the polarization is such that coupling occurs in the H-plane (fig. 8), the E-plane dimension of the ground plane is only 30.48 cm (12 in.) as compared with 60.96 cm (24 in.) for the E-plane coupling of figure 7. Past experience has shown that the dimension of the ground plane in the direction of the aperture electric-field vector has the most predominant diffraction effect. Second, the mutual coupling in the H-plane is naturally lower than in the E-plane and therefore the measurements become more sensitive to scattering from objects such as the edges of the ground plane. This later reasoning is verified by the data in figure 8 for the 6.35- and 8.89-cm (2.5- and 3.5-in.) spacings for which the mutual coupling is stronger and the scatter in the measured data is less. Since the 6.35-cm (2.5-in.) spacing yielded good agreement for both polarizations, the remaining data in this section are restricted to this spacing.

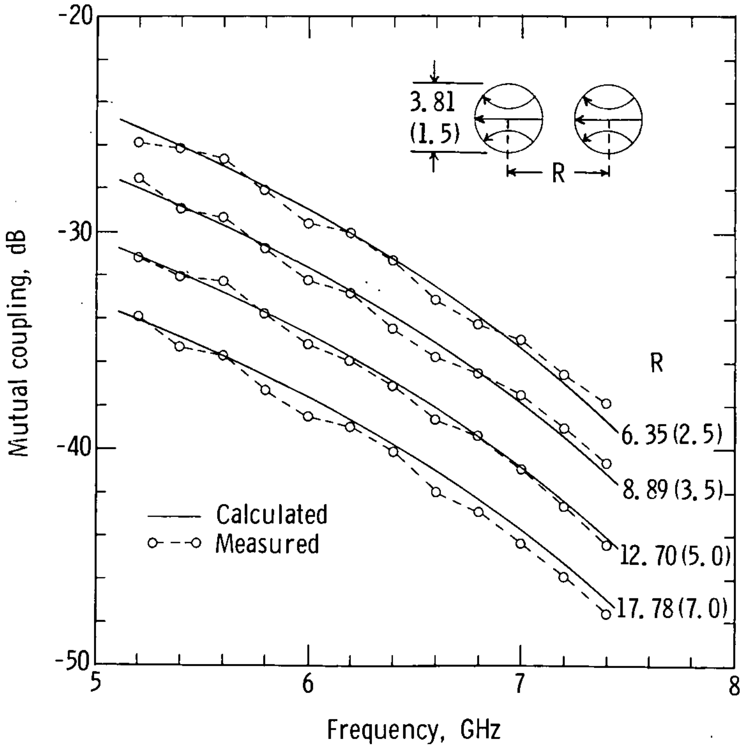

Figure 7.- TE_{11} mode mutual coupling between two circular waveguides radiating into free space (E-plane coupling). Dimensions are in centimeters (inches).

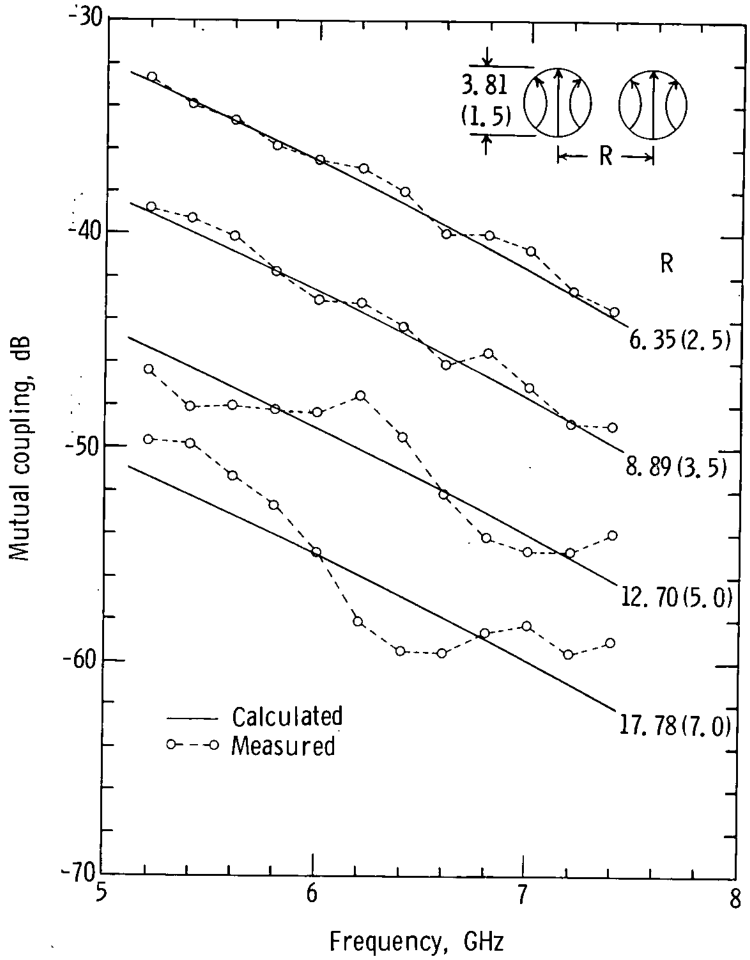

Figure 8.- TE_{11} mode mutual coupling between two circular waveguides radiating into free space (H-plane coupling). Dimensions are in centimeters (inches).

Figures 9 and 10 demonstrate the validity of the theoretical analysis for an arbitrary polarization of one aperture field with respect to the other. There is some scatter in the measured data due to ground plane edge effects; but, in general, the agreement is very good.

One thing to be observed by the data in figures 9 and 10 is that both the measured and calculated results indicate a trend toward complete isolation (Mutual coupling = $-\infty$ dB) for orthogonal polarization ($\phi_p = 90°$); however, this is not always true, as shown in figure 11. Here the principal electric fields are orthogonally polarized; however, both the measured and calculated results show an appreciable level of coupling between the aperture fields. These results indicate that, for certain geometries, the cross-polarized fields may have a significant influence upon the performance of a large array. Earlier analyses of infinite arrays (refs. 51 and 53) have shown some appreciable changes in the radiation characteristics of a phased array due to cross-polarized fields when the array beam is scanned at an angle far off the axis.

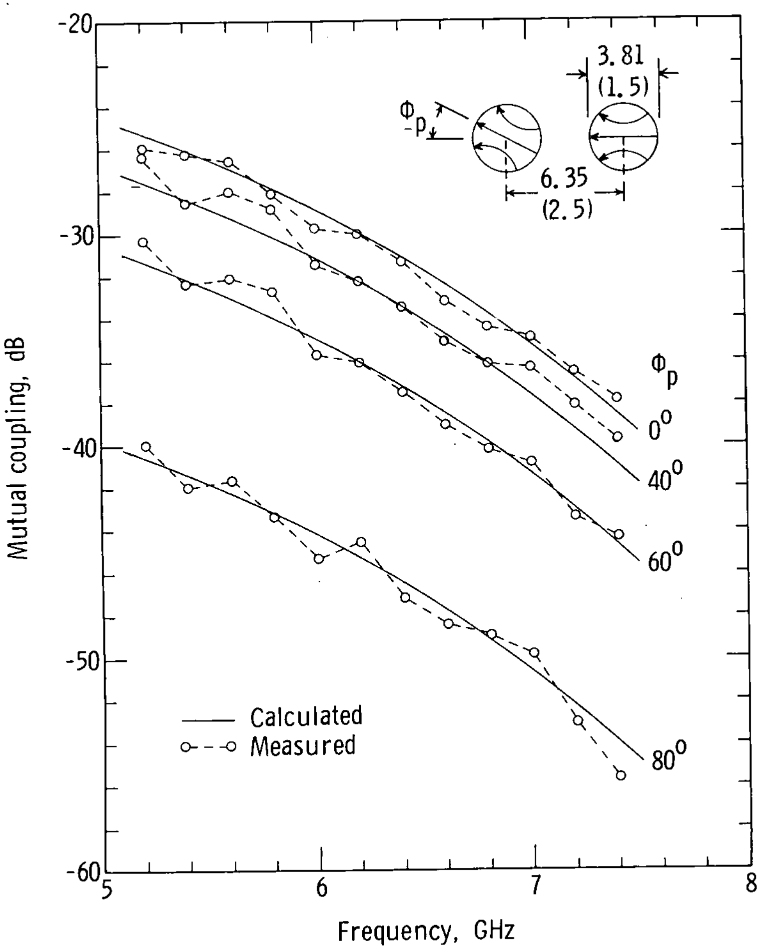

Figure 9.- TE_{11} mode mutual coupling with polarization as a parameter. Linear dimensions are given in centimeters (inches).

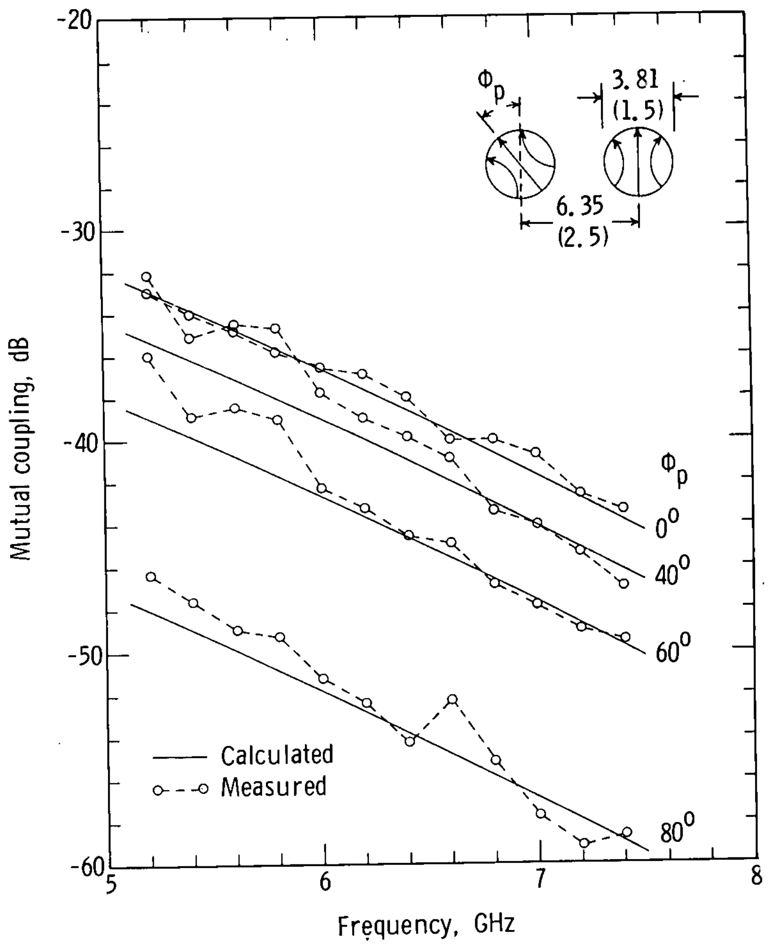

Figure 10.- TE_{11} mode mutual coupling with polarization as a parameter. Linear dimensions are given in centimeters (inches).

It should be noted that the excellent agreement between the measured and calculated results in figure 11 demonstrates the feasibility of using the present analysis to study cross-polarized effects in finite-size circular waveguide phased arrays. Such information is important in the design of arrays of circular polarized elements, in which case the axial ratio or polarization may vary drastically as a function of scan (ref. 51).

The data in figures 12 to 15 are presented to justify the theoretical analysis for dielectric-covered circular apertures. The dielectric constant (2.6) and loss tangent (0.006) used in the calculations are those measured by Von Hippel (ref. 131). The dielectric sheets used for the measurement were the same size as the ground plane and the thicknesses were such that only one surface wave mode can exist (ref. 132).

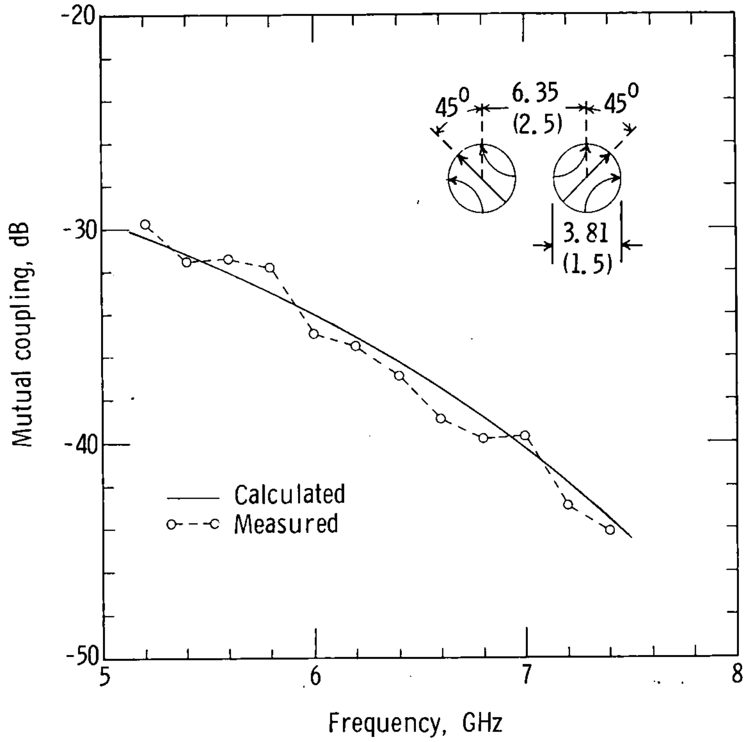

Figure 11.- TE_{11} mode free space mutual coupling between orthogonally polarized aperture fields. Linear dimensions are in centimeters (inches).

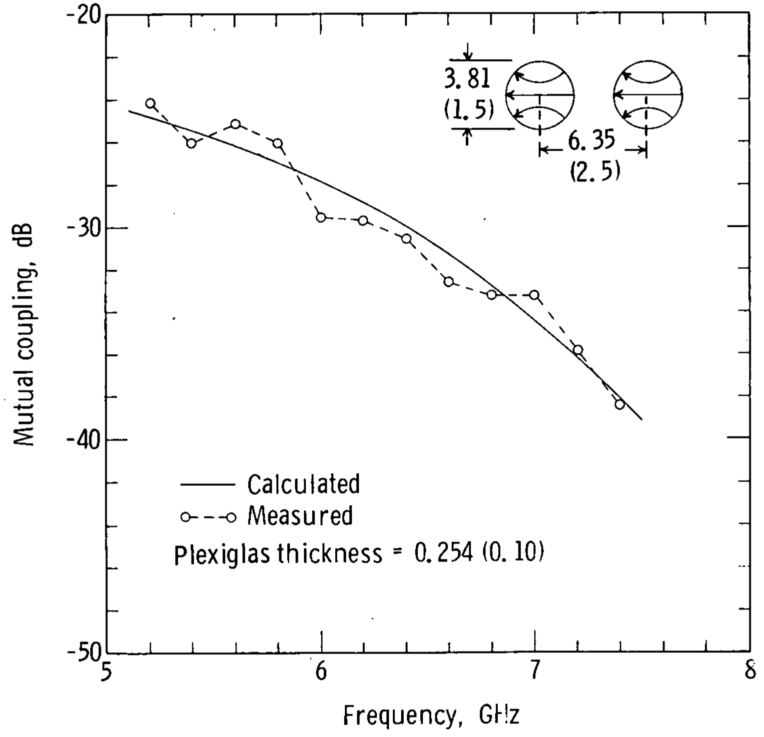

Figure 12.- TE_{11} mode mutual coupling between two dielectric-covered circular waveguides (dielectric constant, 2.6; loss tangent, 0.006; E-plane coupling). Linear dimensions are in centimeters (inches).

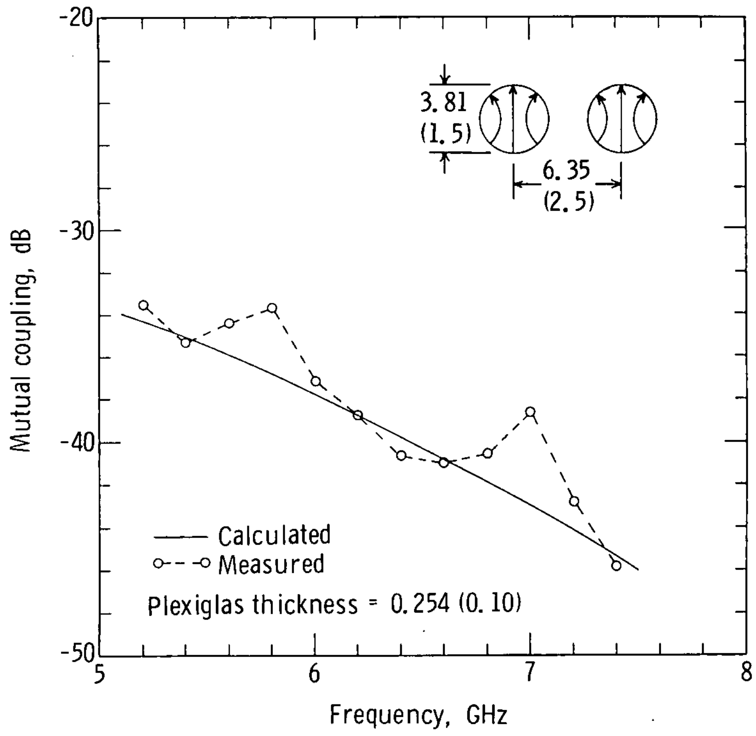

Figure 13.- TE_{11} mode mutual coupling between two dielectric-covered circular waveguides (dielectric constant, 2.6; loss tangent, 0.006; H-plane coupling). Linear dimensions are in centimeters (inches).

Very good agreement between measured and calculated values was obtained for coupling in the E-plane (figs. 12 and 14); however, large variations with frequency occurred in the measured data for coupling in the H-plane (figs. 13 and 15) because of the reflections of the surface wave from the ends of the dielectric sheet. Previous work (ref. 133) has shown that for thicknesses such that only one surface wave mode exists, the E-plane dimension of the finite dielectric sheet produces the largest perturbation in the measured data. This perturbation is also evident in the present data as observed in figures 12 and 14 where, as a result of the larger E-plane dimension, the oscillations are smaller and more closely spaced.

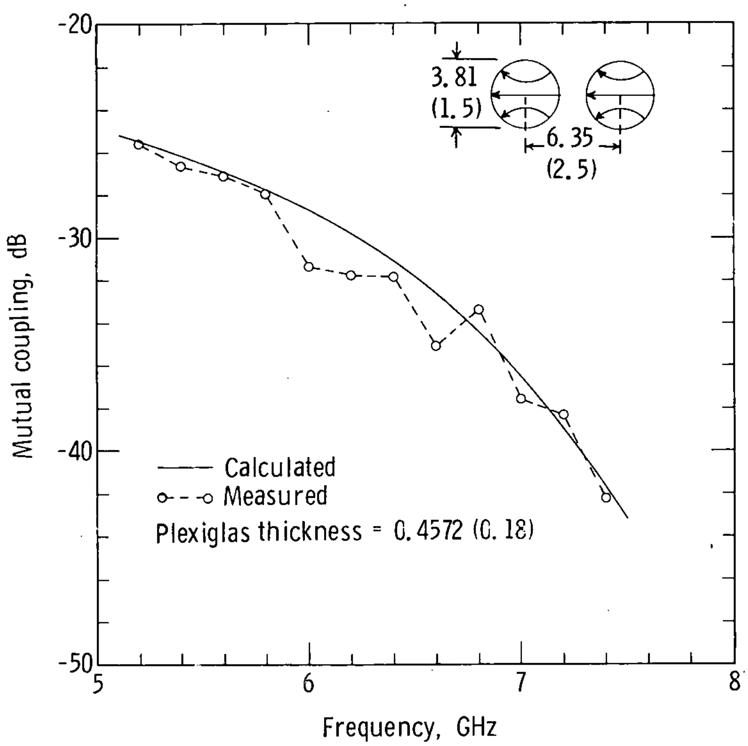

Figure 14.- TE_{11} mode mutual coupling between two dielectric-covered circular waveguides (dielectric constant, 2.6; loss tangent, 0.006; E-plane coupling). Linear dimensions are in centimeters (inches).

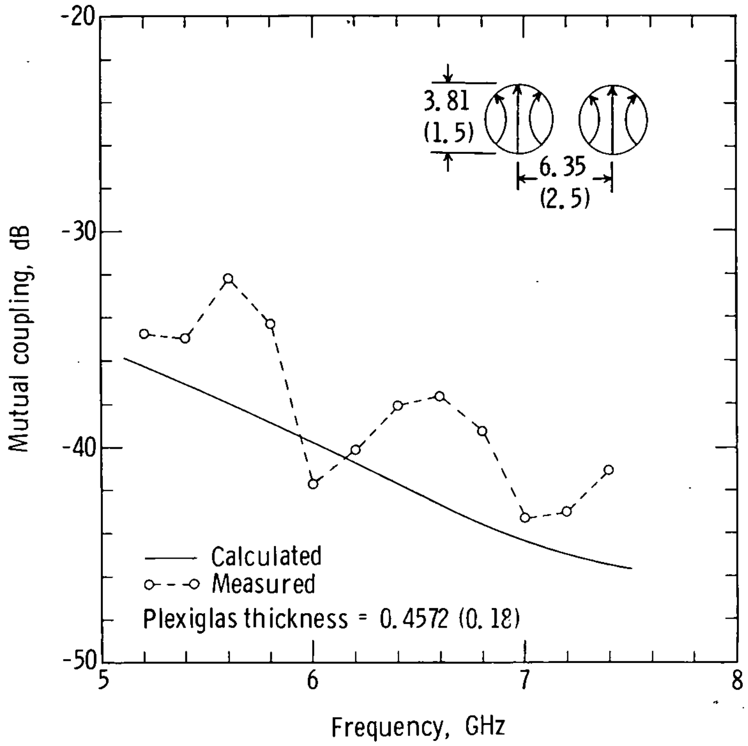

Figure 15.- TE_{11} mode mutual coupling between two dielectric-covered circular waveguides (dielectric constant, 2.6; loss tangent, 0.006; H-plane coupling). Linear dimensions are in centimeters (inches).

Although large variations in the measured data were observed in some instances, the primary differences between the measured and calculated results are due to the finiteness of the ground plane and the dielectric sheet, which was not considered in the theoretical model. However, the overall qualitative comparison between the measured and computed results has established the validity of the theoretical analysis.

Computed Higher Order Mode Effect

A brief parametric study was performed in order to determine the influence upon the mutual coupling due to higher order modes in the apertures. These data are summarized in table I.

The mutual coupling was first computed for two 0.75-wavelength-diameter circular waveguides in which the aperture field distributions were assumed to be that of the TE_{11} mode. Next the 4 by 4 complex scattering matrix was computed for two waveguides with two modes in each (TE_{11} plus one higher order mode). Then the appropriate value of the scattering matrix for the coupling between the TE_{11} modes (S_{13} in this case, was compared with that computed previously for only the TE_{11} mode assumption. These differences for both amplitude and phase are listed in table I for several of the next higher order modes.

TABLE I.- CHANGE IN FREE-SPACE MUTUAL COUPLING BETWEEN TE_{11} MODES DUE TO HIGHER ORDER MODES ASSUMED IN APERTURES OF 0.75-WAVELENGTH-DIAMETER CIRCULAR WAVEGUIDES

Polarization	Center-to-center spacing, wavelengths	Mutual coupling, TE_{11} mode only	Increase in numerical value of TE_{11} mode mutual coupling amplitude, dB and phase, deg												
			TE_{11} + TE_{21}	TE_{11} + TE_{01}	TE_{11} + TE_{31}	TE_{11} + TE_{12}	TE_{11} + TE_{13}	TE_{11} + TM_{01}	TE_{11} + TM_{11}	TE_{11} + TM_{21}	TE_{11} + TM_{31}	TE_{11} + TM_{12}	TE_{11} + TM_{13}	$TE_{11}+TE_{12}$ + $TM_{11}+TM_{12}$	TE_{11} + TE_{11} cross
	0.75	-25.0 dB 14.9 deg	0 .1	0 .1	-0.1 -.1	0 -1.8	0 -.4	0 0	0 22.0	0 0	0 0	-0.3 2.8	-0.1 .9	0 23.6	0 0
	1.25	-28.3 dB -172.0 deg	0 .1	0 .1	0 0	0 -1.4	0 -.4	0 0	-0.4 16.4	0 0	0 0	-0.2 1.9	-0.1 .6	-0.4 18.5	0 0
	2.00	-32.2 dB -84.0 deg	-0.1 0	-0.1 0	-0.1 0	0 -1.3	-0.1 -.3	0 0	-0.6 15.3	0 0	0 0	-0.2 1.7	-0.1 .6	-0.7 16.3	0 0
	0.75	-26.3 dB 117.7 deg	0 -.3	0 .1	0 -.3	0.3 2.3	0.1 .9	0 0	0.2 5.6	0 0	0 0	0 .6	0 .2	0.5 8.9	0 0
	1.25	-36.1 dB -77.7 deg	0 0	0 0	0 0	0 1.3	0 .2	0 0	-0.3 6.0	0 0	0 0	0 .5	0 .2	-0.4 7.6	0 0
	2.00	-44.5 dB 7.4 deg	-0.1 0	-0.1 0	-0.1 0	-0.2 .9	-0.1 .1	0 0	-0.6 7.0	0 0	0 0	-0.1 .5	0 .2	-0.8 8.1	0 0
	0.75	-29.7 dB 61.5 deg	0 -.2	0 .1	0 0	-0.3 .9	-0.1 .1	0 0	1.5 15.3	0 0	0 0	0.1 2.8	0 1.0	-0.5 19.1	0 -.2
	1.25	-33.9 dB -149.1 deg	0 0	0 0	0 0	-0.2 -1.3	-0.1 -0.4	0 0	0.2 15.1	0 0	0 0	-0.1 2.0	-0.1 .7	0 16.3	0 0
	2.00	-38.1 dB -70.3 deg	0 0	0 0	0 0	-0.1 -1.5	0 -.4	0 0	-0.3 14.6	0 0	0 0	-0.1 1.8	0 .6	-0.4 15.4	0 0
	0.75	-27.7 dB 159.4 deg	-0.1 0	0 0	0 -.1	0.4 -.6	0.1 .1	0 0	-1.1 14.6	0 0	0 0	-0.2 1.1	-0.1 .3	-0.8 15.6	0 .2
	1.25	-33.5 dB -13.6 deg	0 0	0 0	0 0	0.2 -.9	0 -.2	0 0	-0.9 14.8	0 0	0 0	-0.2 1.3	0 .4	-0.8 16.1	0 .1
	2.00	-38.0 dB 82.5 deg	0 0	0 0	0 0	0.1 -1.0	0 -.3	0 0	-0.8 14.9	0 0	0 0	-0.2 1.4	0 .4	-0.9 16.3	0 0

It is obvious that those modes, whose first index numbers are different, do not influence the mutual coupling calculations. Also, the only mode which has any noticeable effect is the TM_{11} and then primarily only in the phase. In a phase scanning array, this change in phase due to the presence of the TM_{11} mode may be of some significance and probably should receive more attention in future work.

The calculations, obtained by assuming that the aperture distributions contain the first four modes (S_{15} of the 8 by 8 matrix) whose first indices are the same, are also compared with the calculations obtained by assuming only the TE_{11} mode. It should be noted that the inclusion of additional modes other than the TE_{11} and TM_{11} has a negligible influence upon the mutual coupling.

Phased-Array Calculations

Data are presented in this section to illustrate the variation of the reflection coefficient as a function of scan for the elements of a finite planar array of circular apertures excited in the TE_{11} mode. The elements of the array will be in an equilateral triangular grid arrangement as indicated in figure 16. The dimensions were chosen to correspond to those of the infinite array analyzed by Amitay and Galindo (refs. 51 and 54). They employed a different time convention $(e^{-j\omega t})$ in their analysis; therefore, a change in sign for their infinite-array reflection coefficient phase calculations $(N = \infty)$ was necessary for a direct comparison with the finite-array results.

Data are presented for the two finite-array sizes indicated by the dashed circles inscribed on the array grid in figure 17. The elements in each finite array $(N = 37$ and $N = 183)$ are those whose centers lie either on or inside the dashed circle. Data are presented for the center element (C) for both array sizes and for two edge elements (A and B) of the larger array.

The data are presented as a function of the differential phase shifts ψ_x and ψ_y between elements in the H-plane and E-plane directions, respectively. These phase shifts are related to the beam-pointing directional cosines T_x and T_y by (ref. 54)

$$\psi_x = 2\pi(0.714)\, T_x \tag{171}$$

$$\psi_y = 2\pi(0.714) \sin 60^\circ\, T_y \tag{172}$$

The finite-phased-array calculations are obtained by first determining the complex scattering matrix for the array. Then the complex amplitudes of the incident waveguide fields (a_{p_i}) are given a phase differential according to

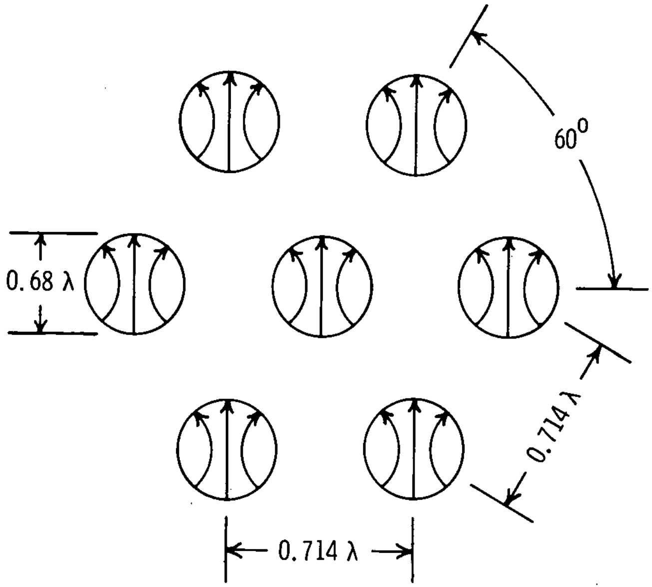

Figure 16.- Dimensions for equilateral triangular grid array of circular waveguide apertures excited in TE_{11} mode.

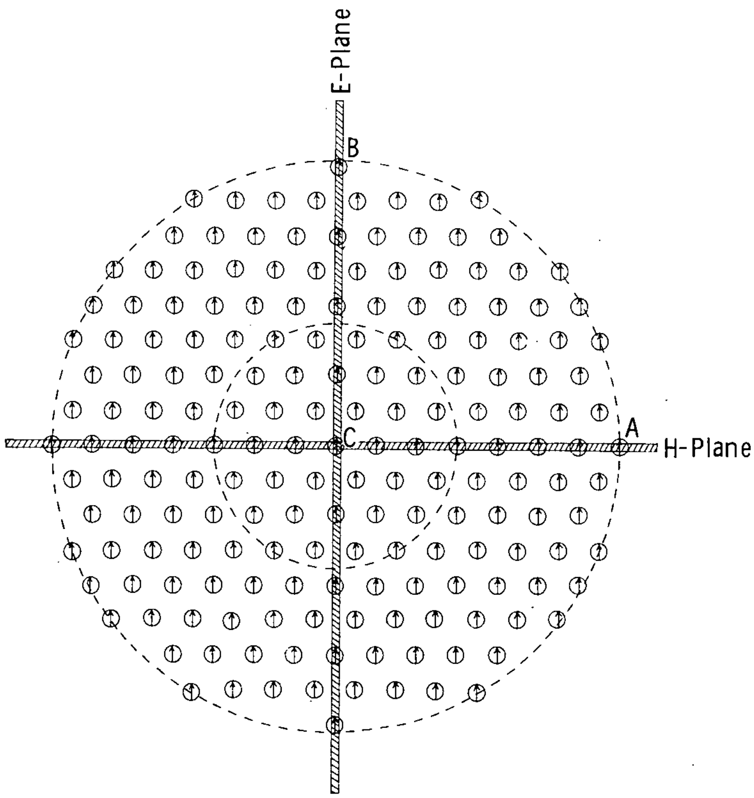

Figure 17.- Finite-size phased array of circular waveguide apertures in an equilateral triangular grid arrangement.

$$a_{p_i} = e^{-j\left(\psi_x x_i' + \psi_y y_i'\right)} \tag{173}$$

In order to scan the beam in the E-plane, ψ_x is set equal to zero and ψ_y is varied; and, in order to scan the beam in the H-plane, ψ_y is set equal to zero and ψ_x is varied. Simultaneous variation of ψ_x and ψ_y would scan the beam in some other direction as determined by the directional cosines T_x and T_y. At each value of ψ_x and ψ_y, the ratio of b_{p_i} (determined from the product of the scattering matrix [S] and the column matrix [a]) to a_{p_i} determines the amplitude and phase of the reflection coefficient of the ith aperture with all elements excited so as to point the beam in the direction specified by T_x and T_y.

The amplitude and phase of the reflection coefficient for the center element of the two finite-size arrays is presented in figures 18 and 19 together with the infinite-array

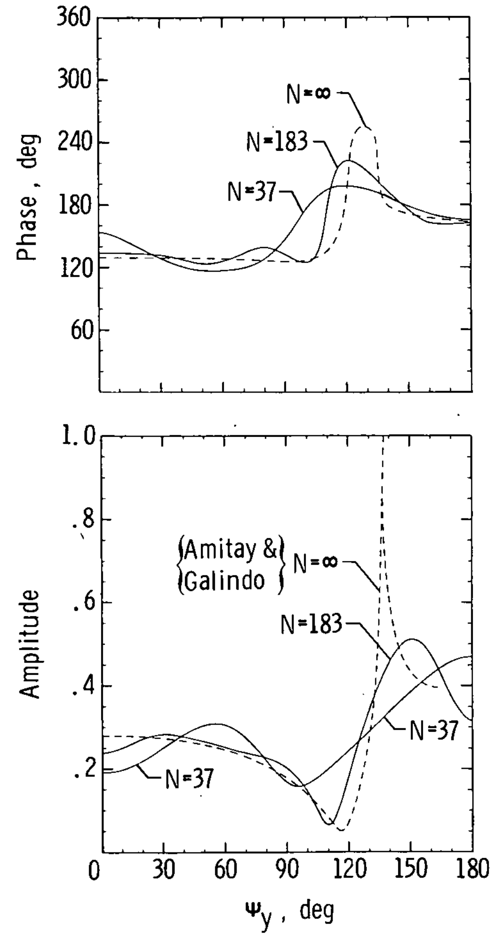

Figure 18.- TE_{11} mode reflection coefficient of center element of arrays in figure 17 as a function of E-plane scan.

calculations (shown dashed) of Amitay and Galindo (refs. 51 and 54). As the size of the array is increased, the reflection coefficient exhibits a resonance behavior corresponding to the "blind spot" of the infinite array. Although the reflection coefficient of the finite array never reaches unity, the qualitative agreement with the infinite array calculations tends to justify the present analysis as applied to finite arrays.

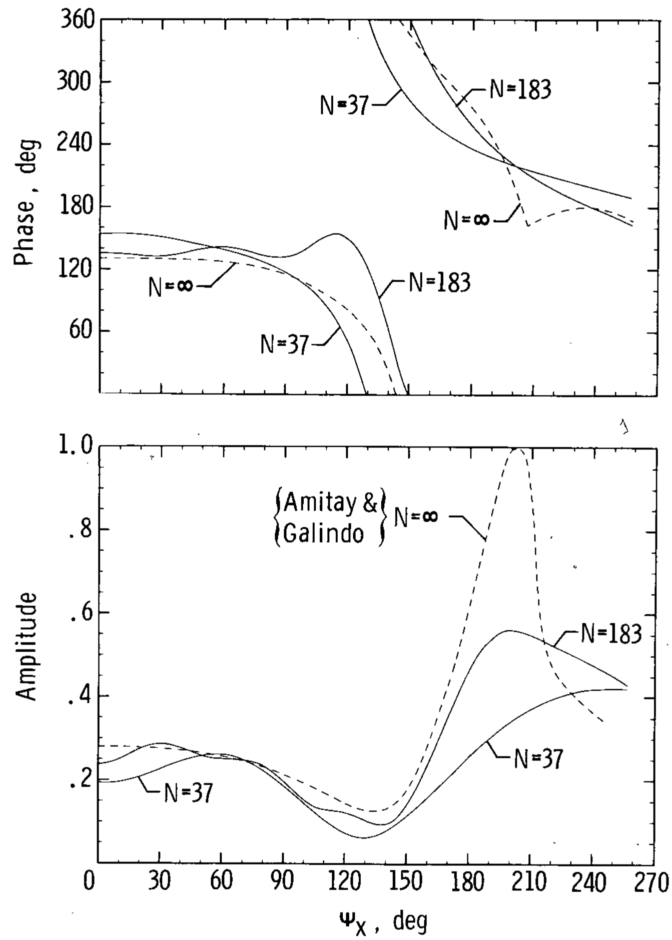

Figure 19.- TE_{11} mode reflection coefficient of center element of arrays in figure 17 as a function of H-plane scan.

Figures 20 and 21 show a comparison between the reflection coefficients of the center element and the edge elements of the larger array (N = 183). Notice that the reflection coefficient of the edge element exhibits a much sharper resonance when the array is scanned in one direction. When the array is scanned in the opposite direction, the edge element reflection coefficient is almost constant and very near the isolated element value (N = 1). This asymmetry with scan is a general characteristic of the edge elements in a large periodic array. (See ref. 98.)

The resonance phenomena which occurs for the edge element when the array is scanned in one direction can be attributed to the destructive interference between the direct radiation from the edge element and a leaky wave traveling in one direction on the periodic structure (ref. 30). In the case of the center element, leaky waves traveling in both directions can produce symmetrical element pattern interference nulls (ref. 30) or impedance resonances.

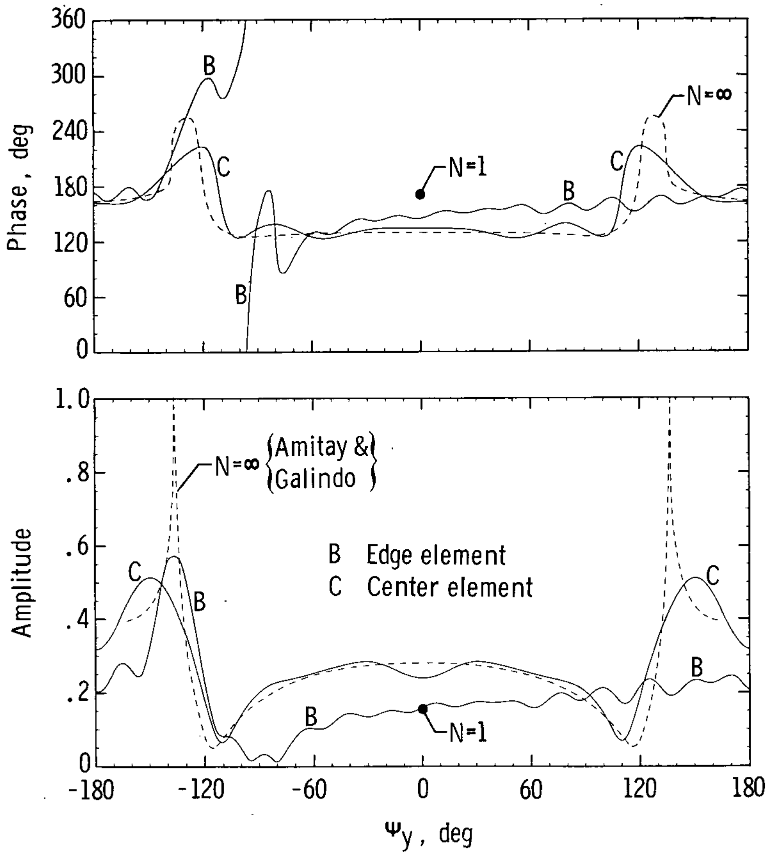

Figure 20.- TE_{11} mode reflection coefficient as a function of E-plane scan for center and edge elements of 183-element array.

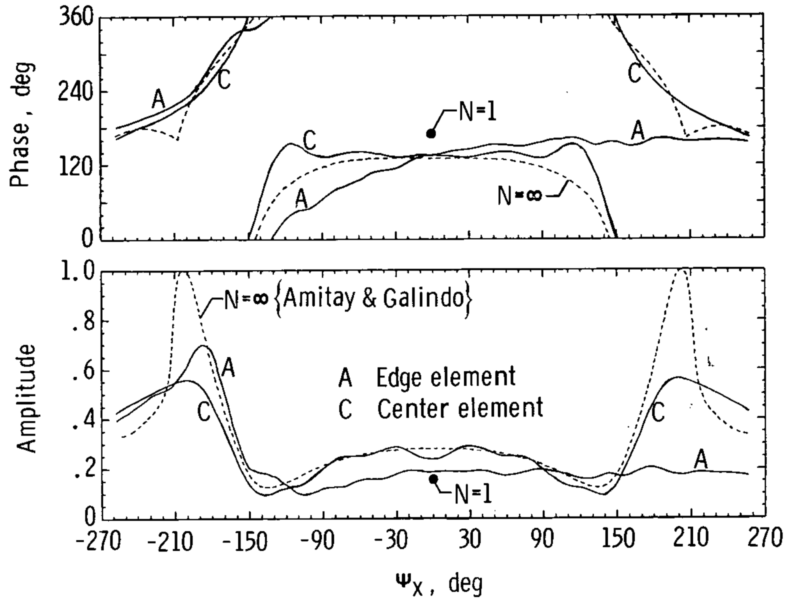

Figure 21.- TE_{11} mode reflection coefficient as a function of H-plane scan for center and edge elements of 183-element array.

The calculations in figures 22 to 24 are for the array geometry of figures 16 and 17 with a dielectric cover of 0.5-wavelength thickness and dielectric constant of 2.0. In order to avoid numerical difficulties, a dielectric loss tangent of 0.0001 was assumed for the finite-array calculations. Previous results for the self-admittance of a dielectric-covered rectangular slot (ref. 124) indicated that a loss tangent of 0.001 or less would yield 3 or 4 significant figure accuracy when compared with calculations for a lossless dielectric.

In figure 22, the calculations for the center element reflection coefficient of the larger finite array (N = 183) exhibit two peaks which appear to correspond to the resonances of the infinite array (ref. 54). In order to verify this result, the reflection coefficient amplitude of the edge element B is compared in figure 23 with the infinite-array calculations. When the array is scanned in one direction, the edge element "sees" a much larger periodic structure and the destructive interference mentioned earlier produces two sharper resonant peaks near the infinite-array "blind spots." This qualitative agreement between the dielectric-covered infinite-array and large finite-array calculations tends to establish further the validity of the present analysis.

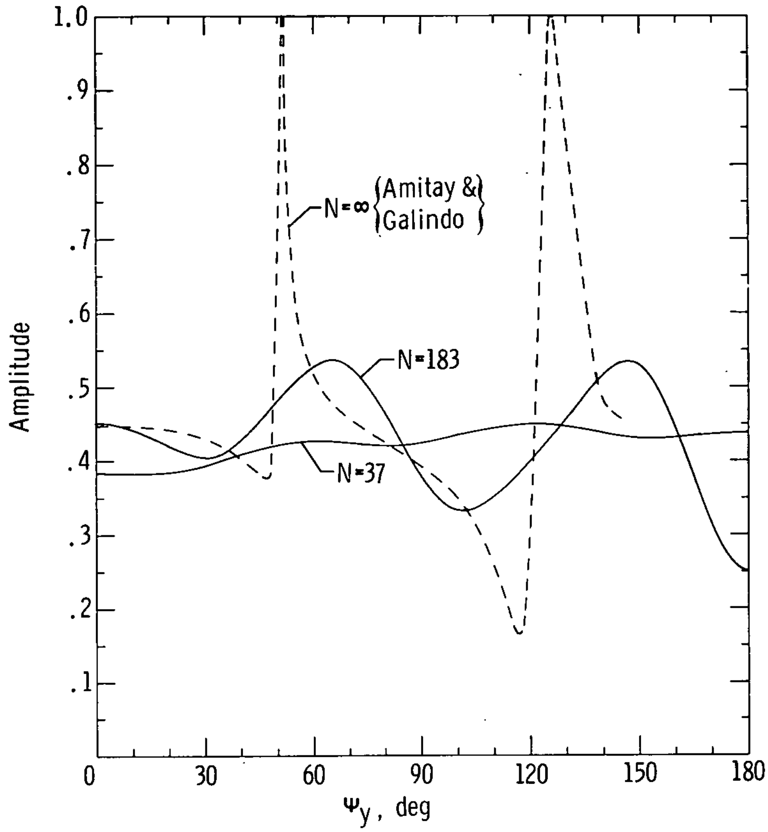

Figure 22.- TE_{11} mode reflection coefficient for the center element of arrays in figure 17 with a dielectric cover (dielectric constant, 2.0; loss tangent, 0.0001; dielectric thickness, 0.5λ; E-plane scan).

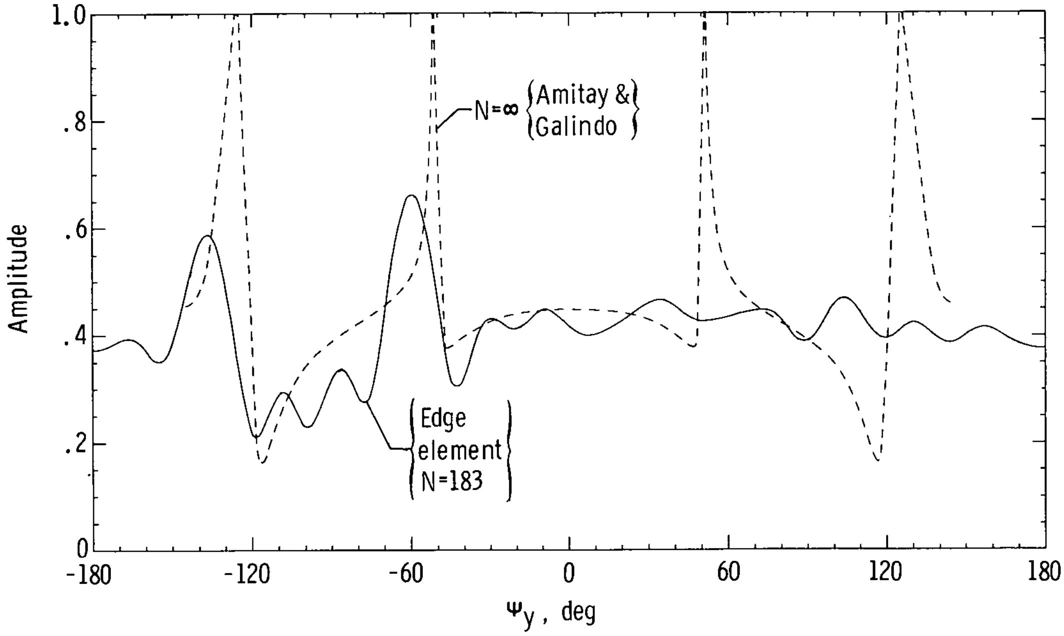

Figure 23.- TE_{11} mode reflection coefficient for edge element of 183-element array with dielectric cover (dielectric constant, 2,0; loss tangent, 0.0001; dielectric thickness, 0.5λ; E-plane scan).

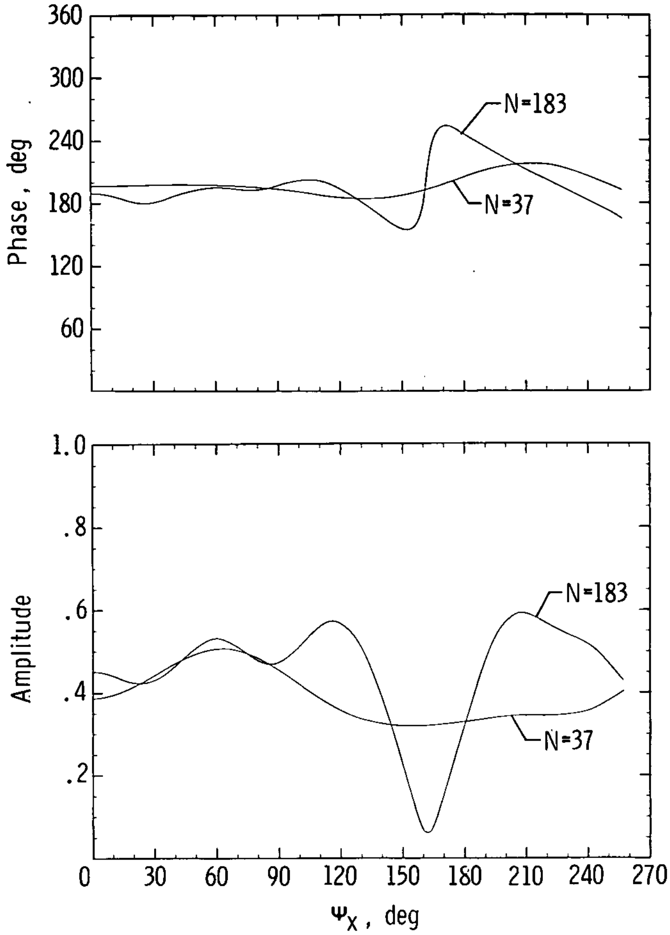

Figure 24.- TE_{11} mode reflection coefficient for the center element of arrays in figure 17 with a dielectric cover (dielectric constant, 2.0; loss tangent, 0.0001; dielectric thickness, 0.5λ; H-plane scan).

The reflection coefficient of the center element of the two dielectric-covered finite arrays is presented in figure 24 as a function of the H-plane scan parameter. The infinite-array calculations were not available for comparison. The finite-array calculations for this case are included since a different type of resonance occurs which probably warrants further investigation. It appears to be related to the phased-array impedance-matching techniques by the dielectric loading of the aperture plane (refs. 78 to 80) and indicates that the present analysis may also be of some benefit in the impedance matching of finite arrays. Minor modifications (ref. 134) of the present analysis could also be used to study the impedance properties of finite arrays with dielectric plugs.

CONCLUDING REMARKS

A variational expression has been derived for the self and mutual admittances of waveguide-fed apertures radiating into a multilayered region which may contain inhomogeneous layers. The general expression has been evaluated for circular apertures excited in the TE_{mn} and TM_{mn} waveguide modes and a computer program written which can include up to four external layers, two of which may be inhomogeneous normal to the aperture plane.

Good agreement was obtained between measured and calculated values for the transverse electric (TE_{11}) mode mutual coupling of two circular waveguides radiating into free space and one dielectric layer. A comparison was made between measured and calculated results for several combinations of frequency, polarization, and spacing. Very good agreement was obtained in all cases, except where the diffractions from the edges of the 30.48 cm by 60.96 cm (12 in. by 24 in.) ground plane produced large scatter in the measured data.

By performing a parametric study, it was determined that the only significant effect of higher order modes is due to the transverse magnetic (TM_{11}) mode and then primarily in the phase of the coupling coefficient.

A comparison was also made between the reflection coefficient of an infinite array and the reflection coefficients of several elements of two finite arrays. It was shown that the center element of a 183-element array (approximately 10 wavelengths wide) had similar impedance characteristics to that of the infinite array. Although total reflection did not occur in the finite array, a definite peak in the reflection coefficient, corresponding to the "blind spot" of the infinite array, was observed.

The validity of the theoretical model has been established by a comparison with measurements on two circular waveguide-fed apertures and with calculations on an infinite periodic array.

This work allows the determination of mutual coupling between the elements of a finite array of circular apertures including any number of higher order modes and an arbitrary polarization for each aperture field.

The analysis presented here can also be applied to other aperture shapes for which the Fourier transforms of the aperture electric fields can be determined.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., July 3, 1973.

APPENDIX A

VECTOR POTENTIALS AND WAVE EQUATIONS

The definition of the vector potentials ($\vec{A}$ and $\vec{F}$) in their relationship to the electromagnetic fields ($\vec{E}$ and $\vec{H}$) and the wave equations, as used in this analysis, are given here for reference.

It is assumed throughout the paper that the electromagnetic fields contain a harmonic time variation of the form $e^{j\omega t}$; then, Maxwell's equations for a charge-free region are

$$\vec{\nabla} \times \vec{H} - j\omega\epsilon \vec{E} = 0 \tag{A1}$$

$$\vec{\nabla} \times \vec{E} + j\omega\mu \vec{H} = 0 \tag{A2}$$

$$\vec{\nabla} \cdot \mu \vec{H} = 0 \tag{A3}$$

$$\vec{\nabla} \cdot \epsilon \vec{E} = 0 \tag{A4}$$

where the permittivity ϵ and permeability μ may be complex and also be a function of the x_i, y_i, and z_i coordinate variables.

From equation (A3), the vector $\mu\vec{H}$ can be defined as the curl of another vector $\vec{A}$, that is,

$$\vec{H} = \frac{1}{\mu} \vec{\nabla} \times \vec{A} \tag{A5}$$

Substituting equation (A5) into equation (A2) gives

$$\vec{\nabla} \times (\vec{E} + j\omega\vec{A}) = 0 \tag{A6}$$

Or since a curl-free vector is the gradient of a scalar

$$\vec{E} + j\omega\vec{A} = -\vec{\nabla}\Psi \tag{A7}$$

the electric field is given by

$$\vec{E} = -\vec{\nabla}\Psi - j\omega\vec{A} \tag{A8}$$

Similarly, by defining another vector $\vec{F}$ which satisfies equation (A4) so that

$$\vec{E} = -\frac{1}{\epsilon} \vec{\nabla} \times \vec{F} \tag{A9}$$

then from equation (A1)

$$\vec{H} = -\vec{\nabla}\Phi - j\omega\vec{F} \tag{A10}$$

By superimposing the results due to the assumed magnetic vector and scalar potentials and the electric vector and scalar potentials,

$$\vec{E} = -\frac{1}{\epsilon}\vec{\nabla} \times \vec{F} - \vec{\nabla}\Psi - j\omega\vec{A} \tag{A11}$$

$$\vec{H} = \frac{1}{\mu}\vec{\nabla} \times \vec{A} - \vec{\nabla}\Phi - j\omega\vec{F} \tag{A12}$$

describes the electromagnetic fields in terms of a set of arbitrary vector and scalar functions.

Since these functions are arbitrary, $\vec{A}$ is chosen as

$$\vec{A} = A\,\hat{z}_i \tag{A13}$$

where A is a function of x_i, y_i, z_i. Then with Φ and $\vec{F}$ temporarily set to zero and by using the vector identities

$$\vec{\nabla} \times \left[\frac{1}{\mu}\vec{\nabla} \times \vec{A}\right] \equiv \vec{\nabla}\left(\frac{1}{\mu}\right) \times [\vec{\nabla} \times \vec{A}] + \frac{1}{\mu}[\vec{\nabla} \times \vec{\nabla} \times \vec{A}] \tag{A14}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{A15}$$

equation (A1) becomes

$$\left[\frac{\partial}{\partial z_i}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial x_i} + \frac{1}{\mu}\frac{\partial^2 A}{\partial x_i \partial z_i} + j\omega\epsilon\frac{\partial \Psi}{\partial x_i}\right]\hat{x}_i + \left[\frac{\partial}{\partial z_i}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial y_i} + \frac{1}{\mu}\frac{\partial^2 A}{\partial y_i \partial z_i} + j\omega\epsilon\frac{\partial \Psi}{\partial y_i}\right]\hat{y}_i$$

$$+\left[-\frac{\partial}{\partial x_i}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial x_i} - \frac{\partial}{\partial y_i}\left(\frac{1}{\mu}\right)\frac{\partial A}{\partial y_i} + \frac{1}{\mu}\frac{\partial^2 A}{\partial z_i^2} + j\omega\epsilon\frac{\partial \Psi}{\partial z_i} - \omega^2\epsilon A - \frac{1}{\mu}\nabla^2 A\right]\hat{z}_i = 0 \tag{A16}$$

Equating either the $\hat{x}_i$ or $\hat{y}_i$ components to zero gives

$$\Psi = -\frac{1}{j\omega\epsilon}\frac{\partial}{\partial z_i}\left(\frac{A}{\mu}\right) \tag{A17}$$

Then the $\hat{z}_i$ components yield

$$\frac{\partial}{\partial x_i}\left(\frac{1}{\mu}\frac{\partial A}{\partial x_i}\right) + \frac{\partial}{\partial y_i}\left(\frac{1}{\mu}\frac{\partial A}{\partial y_i}\right) + \frac{\partial^2}{\partial z_i^2}\left(\frac{A}{\mu}\right) - \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial z_i}\frac{\partial}{\partial z_i}\left(\frac{A}{\mu}\right) + k_0^2\left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)\left(\frac{A}{\mu}\right) = 0 \tag{A18}$$

And if the medium is assumed to be homogeneous in the x_i and y_i directions, equation (A18) yields the wave equation

$$\nabla^2\left[\frac{A(x_i, y_i, z_i)}{\mu(z_i)}\right] - \frac{1}{\epsilon(z_i)}\frac{\partial\epsilon(z_i)}{\partial z_i}\frac{\partial}{\partial z_i}\left[\frac{A(x_i, y_i, z_i)}{\mu(z_i)}\right] + k_0^2\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\epsilon_0}\left[\frac{A(x_i, y_i, z_i)}{\mu(z_i)}\right] = 0 \tag{A19}$$

Likewise, if it is assumed that $\Psi = 0$ and $\vec{A} = 0$ momentarily and

$$\vec{F} = F\,\hat{z}_i \tag{A20}$$

then from equation (A2)

$$\Phi = -\frac{1}{j\omega\mu}\frac{\partial}{\partial z_i}\left(\frac{F}{\epsilon}\right) \tag{A21}$$

and, for $\mu = \mu(z_i)$ and $\epsilon = \epsilon(z_i)$,

$$\nabla^2\left[\frac{F(x_i, y_i, z_i)}{\epsilon(z_i)}\right] - \frac{1}{\mu(z_i)}\frac{\partial\mu(z_i)}{\partial z_i}\frac{\partial}{\partial z_i}\left[\frac{F(x_i, y_i, z_i)}{\epsilon(z_i)}\right] + k_0^2\frac{\mu(z_i)\,\epsilon(z_i)}{\mu_0\epsilon_0}\left[\frac{F(x_i, y_i, z_i)}{\epsilon(z_i)}\right] = 0 \tag{A22}$$

Therefore, by superposition, the electric and magnetic fields can be derived from a set of z_i directed vector potentials which satisfy the differential equations (A19) and (A22) subject to the appropriate boundary conditions on the electric and magnetic fields.

If bidimensional Fourier transforms are assumed so that

$$A(k_x, k_y, z_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{A(x_i, y_i, z_i)}{\mu(z_i)} e^{-jk_x x_i} e^{-jk_y y_i} dx_i \, dy_i \tag{A23}$$

$$F(k_x, k_y, z_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F(x_i, y_i, z_i)}{\epsilon(z_i)} e^{-jk_x x_i} e^{-jk_y y_i} dx_i \, dy_i \tag{A24}$$

$$\vec{E}(k_x, k_y, z_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{E}(x_i, y_i, z_i) e^{-jk_x x_i} e^{-jk_y y_i} dx_i \, dy_i \tag{A25}$$

$$\vec{H}(k_x, k_y, z_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{H}(x_i, y_i, z_i) e^{-jk_x x_i} e^{-jk_y y_i} dx_i \, dy_i \tag{A26}$$

and inversely

$$\frac{A(x_i, y_i, z_i)}{\mu(z_i)} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k_x, k_y, z_i) e^{jk_x x_i} e^{jk_y y_i} dk_x \, dk_y \tag{A27}$$

$$\frac{F(x_i, y_i, z_i)}{\epsilon(z_i)} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y, z_i) e^{jk_x x_i} e^{jk_y y_i} dk_x \, dk_y \tag{A28}$$

$$\vec{E}(x_i, y_i, z_i) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{E}(k_x, k_y, z_i) e^{jk_x x_i} e^{jk_y y_i} dk_x \, dk_y \tag{A29}$$

$$\vec{H}(x_i, y_i, z_i) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{H}(k_x, k_y, z_i) e^{jk_x x_i} e^{jk_y y_i} dk_x \, dk_y \tag{A30}$$

then the transverse components of the transformed electric and magnetic fields become

$$E_{x_i}(k_x, k_y, z_i) = - \frac{k_x}{\omega\epsilon(z_i)} A'(k_x, k_y, z_i) + jk_y F(k_x, k_y, z_i) \tag{A31}$$

$$E_{y_i}(k_x, k_y, z_i) = - \frac{k_y}{\omega\epsilon(z_i)} A'(k_x, k_y, z_i) - jk_x F(k_x, k_y, z_i) \tag{A32}$$

$$H_{x_i}(k_x, k_y, z_i) = - jk_y A(k_x, k_y, z_i) - \frac{k_x}{\omega\mu(z_i)} F'(k_x, k_y, z_i) \tag{A33}$$

$$H_{y_i}(k_x, k_y, z_i) = jk_x A(k_x, k_y, z_i) - \frac{k_y}{\omega\mu(z_i)} F'(k_x, k_y, z_i) \tag{A34}$$

where the primes on A and F denote differentiation with respect to z_i, and where the transformed potential functions now satisfy the differential equations

$$\frac{d^2}{dz_i^2} A(k_x, k_y, z_i) - \frac{1}{\epsilon(z_i)} \frac{d\epsilon(z_i)}{dz_i} \frac{d}{dz_i} A(k_x, k_y, z_i)$$

$$+ k_0^2 \left[\frac{\mu(z_i)\ \epsilon(z_i)}{\mu_0 \epsilon_0} - \frac{k_x^2 + k_y^2}{k_0^2} \right] A(k_x, k_y, z_i) = 0 \tag{A35}$$

$$\frac{d^2}{dz_i^2} F(k_x, k_y, z_i) - \frac{1}{\mu(z_i)} \frac{d\mu(z_i)}{dz_i} \frac{d}{dz_i} F(k_x, k_y, z_i)$$

$$+ k_0^2 \left[\frac{\mu(z_i)\ \epsilon(z_i)}{\mu_0 \epsilon_0} - \frac{k_x^2 + k_y^2}{k_0^2} \right] F(k_x, k_y, z_i) = 0 \tag{A36}$$

At a boundary $(z_i = d_p)$ between two media $(\mu_p(z_i)\ \epsilon_p(z_i)$ and $\mu_{p+1}(z_i)\ \epsilon_{p+1}(z_i))$, continuity of the transverse electric and magnetic fields or their transforms shows that the transformed potentials for the two regions must satisfy the boundary conditions

$$A_p(k_x, k_y, d_p) = A_{p+1}(k_x, k_y, d_p) \tag{A37}$$

$$F_p(k_x, k_y, d_p) = F_{p+1}(k_x, k_y, d_p) \tag{A38}$$

$$\left[\frac{d}{dz_i} A_p(k_x, k_y, z_i)\right]_{z_i=d_p} = \frac{\epsilon_p(d_p)}{\epsilon_{p+1}(d_p)} \left[\frac{d}{dz_i} A_{p+1}(k_x, k_y, z_i)\right]_{z_i=d_p} \tag{A39}$$

$$\left[\frac{d}{dz_i} F_p(k_x, k_y, z_i)\right]_{z_i=d_p} = \frac{\mu_p(d_p)}{\mu_{p+1}(d_p)} \left[\frac{d}{dz_i} F_{p+1}(k_x, k_y, z_i)\right]_{z_i=d_p} \tag{A40}$$

APPENDIX B

COMPUTER PROGRAM FOR THE CALCULATION OF THE SCATTERING MATRIX OF A PLANAR ARRAY OF CIRCULAR WAVEGUIDES RADIATING INTO EITHER FREE SPACE OR FOUR DIELECTRIC LAYERS

The program listed here computes the complex mutual admittances between the modal fields of all the apertures in the array. These admittance values are used to form a complex square matrix which is operated on with the appropriate matrix algebra and inversion to obtain the complex scattering matrix for the array.

The basic program for the mutual admittance calculation is a modification of a previous computer program (ref. 135) for the calculation of the self admittance of a TE_{11} mode excited circular aperture. The computer program in its present state is limited to a maximum of four external layers over the apertures; however, the third and fourth layers may be inhomogeneous normal to the aperture plane.

The wave equations (eqs. (58) and (59) with $\mu(z_i)$ = Constant) for the third and fourth layers are solved numerically by using a spline routine for curve fitting through discrete points of the dielectric or plasma profile. The numerical integration of equations (167) to (170) is performed by a Runge-Kutta method containing a variable increment which is continuously subdivided until a specified accuracy is achieved over each integration step. More detailed discussion of the numerical techniques used in the original computer program are given in reference 135.

The present program accepts as input the following parameters:

NUMMODE — total number of waveguide modes assumed in each aperture (each aperture field distribution is assumed to be a superposition of the same waveguide modes)

NUMHOLE — total number of circular apertures in the array

NUMTE — total number of transverse electric waveguide modes assumed in each aperture field distribution

NUMTM — total number of transverse magnetic waveguide modes assumed in each aperture field distribution

(Note that NUMMODE = NUMTE + NUMTM)

MIJ(I), NIJ(I) — indices of $TE_{m_i n_i}$ modes, I = 1 to NUMTE (if NUMTE = 0, omit)

MMIJ(I), NNIJ(I)	indices of $TM_{m_i' n_i'}$ modes, I = 1 to NUMTM (if NUMTM = 0, omit)
AIJ(I)	radius (a_i) of each aperture, I = 1 to NUMHOLE
XI(I), YI(I)	x,y coordinates of center of each aperture (x_i', y_i'), I = 1 to NUMHOLE
PHIJP(I)	angular rotation (ϕ_i') of x_i axis with respect to x axis for each aperture, I = 1 to NUMHOLE
F	frequency
ZI, Z2, Z3, Z4	distances (d_1, d_2, d_3, d_4) from aperture plane to outer surfaces of layers 1, 2, 3, 4, respectively (For radiation into free space, set Z4 = 0.0.)
CONVERT	a conversion factor to change all input dimensions to centimeters (that is, for input dimensions in inches, set CONVERT = 2.54)
ER	relative dielectric constant of material completely filling all waveguides (ER = 1.0 for air-filled guides)

For radiation into free space (Z4 = 0.0), no additional input data is needed; however, for Z4 > 0.0, the relative values of Z1, Z2, Z3, and Z4 are compared to determine which external layers are to be considered in the calculations and appropriate parameters are read in as follows:

V1, W1	real and imaginary parts of complex relative dielectric constant of layer nearest to aperture plane (layer 1) (if Z3 = 0.0, and Z4 > 0.0, omit)
V2, W2	real and imaginary parts of complex relative dielectric constant of layer 2 (if Z3 = 0.0 and Z4 > 0.0, omit)
NP3	number of points used in approximation of the inhomogeneous profile for the dielectric constant of layer 3 (if Z3 = 0.0 and Z4 > 0.0, omit; or if Z2 = Z3 = Z4, omit)
ZD(I)	distance from aperture plane to discrete points in layer 3 dielectric profile (if Z3 = 0.0 and Z4 > 0.0, or if Z2 = Z3 = Z4, omit)
V3(I), W3(I)	real and imaginary parts of relative dielectric constant at discrete points ZD(I) in layer 3 inhomogeneous profile (if Z3 = 0.0 and Z4 > 0.0, or if Z2 = Z3 = Z4, omit)

NP4	number of points used in approximation of the inhomogeneous profile for the electron density and collision frequency for the plasma of layer 4 (if Z3 = Z4, omit)
ZND(I)	distance from aperture plane to discrete points in layer 4 plasma profile (if Z3 = Z4, omit)
NE(I), NU(I)	electron density and collision frequency at discrete points, ZND(I), in layer 3 inhomogeneous plasma profile (if Z3 = Z4, omit)

The values of ZD(I) and ZND(I) must be monotonically increasing. The first value of ZD(I) must equal Z2, the last value of ZD(I) and the first value of ZND(I) must equal Z3, and the last value of ZND(I) must equal Z4. Any deviation from these values will cause errors to occur in the calculations.

The output of the computer program is as follows:

YC(II,JJ)	elements of complex admittance matrix
PRMT(2)	upper limit of numerical integration (maximum value is set at 50.0)
IHOLE	ith aperture
JHOLE	jth aperture
IMODE	pth mode in ith aperture
JMODE	qth mode in jth aperture
YMN(I)	characteristic admittance of pth mode in ith aperture
S(I, J)	elements of complex scattering matrix

```
      PROGRAM CIRWG(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C**********************************************************************
C       MI,NI,MJ,NJ  ARE THE SUBSCRIPTS OF THE APERTURE EXCITATION MODES
C          AI = RADIUS OF I-TH APERTURE
C          AJ = RADIUS OF J-TH APERTURE
C           R = CENTER TO CENTER SPACING
C         PHI = ANGULAR ROTATION OF R WITH RESPECT TO XI-AXIS (DEGREES)
C      PHIJPP = ANGULAR ROTATION OF XJ-AXIS WITH RESPECT TO XI-AXIS (DEG.)
C**********************************************************************
      DIMENSION AUX(8,4),DERY(4),PRMT(5),Y(4)
      DIMENSION ZD(50),ZND(50),NE(50),NU(50)
C               DIMENSIONS FOR COMMON VARIABLES
      DIMENSION V3(50),V31(50),V32(50),V33(50),
     A          V4(50),V41(50),V42(50),V43(50),
     B          W4(50),W41(50),W42(50),W43(50),
     C          W3(50),W31(50),W32(50),W33(50),
     D          Z(50),ZN(50),XMNP(8,3),XMN(8,3)
C               COMMON - DIMENSIONED VARIABLES
      COMMON V3,V31,V32,V33,W3,W31,W32,W33,Z,XMNP,XMN,
     A       V4,V41,V42,V43,W4,W41,W42,W43,ZN,
C               COMMON - UNDIMENSIONED VARIABLES
     B       BSQ,CCA,CCB,D2,KIND,L3,L4,MOST,TMI,TMJ,
     C       NEW,NP3,NP4,RKERR,RK3,RK4,TERMA,TERMB,TERMC,TERMD,
     D          V1,     V2,      V3XI3,V4XI4,
     E          W1,W1SQ,W2,W2SQ,W3XI3,W4XI4,XI1,XI2,XI3,XI4,
     F MI,MJ,NI,NJ,MIP,MJP,AKZEROI,AKZEROJ,RKZERO,FACTORI,FACTORJ,
     G FACTSQI,FACTSQJ,COSP,COSM,SINP,SINM,COSPHIJ,PHIJPP
      LOGICAL TMI,TMJ
      REAL KZERO,NE,NU,NED,NUD,ISOLATE
      COMPLEX CCA,CCB,COEFF,GAMMA,CON,YC,YTE,YTM,YMNZERO
      COMPLEX A,B,DETERM,YAP
      DIMENSION YC(25,25),A(25,25),B(25,25),IPIVOT(25),INDEX(25,2)
      DIMENSION AIJ(25),XI(25),YI(25),PHIJP(25),MIJ(10),NIJ(10)
      DIMENSION MMIJ(10),NNIJ(10)
      EXTERNAL FINDC
C           ESTABLISH CONSTANTS
      XMNP(1,1)=3.832$XMNP(1,2)=7.016$XMNP(1,3)=10.173$XMNP(2,1)=1.84118
      XMNP(2,2)=5.331$XMNP(2,3)=8.536$XMNP(3,1)=3.054$XMNP(3,2)=6.706
      XMNP(3,3)=9.969$XMNP(4,1)=4.201$XMNP(4,2)=8.015$XMNP(5,1)=5.317
      XMNP(5,2)=9.282$XMNP(6,1)=6.416$XMNP(6,2)=10.52$XMNP(7,1)=7.501
      XMNP(8,1)=8.578
      XMN(1,1)=2.405$XMN(2,1)=3.832$XMN(3,1)=5.136$XMN(1,2)=5.520
      XMN(4,1)=6.380$XMN(2,2)=7.016$XMN(5,1)=7.588$XMN(3,2)=8.417
      XMN(1,3)=8.654$XMN(6,1)=8.771$XMN(4,2)=9.761$XMN(7,1)=9.936
      XMN(2,3)=10.173$XMN(5,2)=11.065$XMN(8,1)=11.086
      PI=2.0*ASIN(1.0)
      TWOPI=2.0*PI
      FORK=TWOPI/(3.*1.E10)
      FOMEGA=(TWOPI*8970.)**2
      CON=CMPLX(0.0,1.0)
C           START READING INPUT
C**********************************************************************
C         NUMMODE = NUMBER OF MODES PER APERTURE
C         NUMHOLE = NUMBER OF APERTURES
C         NUMTE = NUMBER OF TE MODES
C         NUMTM = NUMBER OF TM MODES
C**********************************************************************
    1 READ(5,11)NUMHOLE,NUMMODE,NUMTE,NUMTM
   11 FORMAT(4I5)
```

```
      IF(EOF,5)900,2
    2 M=NUMHOLE*NUMMODE
      IF(M.GT.25)3,4
    3 WRITE(6,80)
   80 FORMAT(1H1*M EXCEEDS DIMENSION OF YC*)
      STOP
    4 IF(NUMTE.GT.10)GO TO 5
      IF(NUMTM.GT.10)GO TO 5
      IF(NUMTE.GT.NUMMODE)GO TO 900
      IF(NUMTM.GT.NUMMODE)GO TO 900
      IF(NUMMODE.GT.20)5,6
    5 WRITE(6,81)
   81 FORMAT(1H1*NUMMODE EXCEEDS DIMENSION OF MIJ AND NIJ*)
      STOP
    6 IF(NUMHOLE.GT.25)7,12
    7 WRITE(6,84)
   84 FORMAT(1H1*NUMHOLE EXCEEDS DIMENSION OF AIJ*)
      STOP
C*********************************************************************
C        MIJ(I),NIJ(I) = INDICES OF I-TH MODE TE-MN
C        MMIJ(I),NNIJ(I) = INDICES OF I-TH TM-MN MODE
C*********************************************************************
   12 IF(NUMTE.EQ.0)GO TO 17
      READ(5,14)((MIJ(I),NIJ(I)),I=1,NUMTE)
   17 IF(NUMTM.EQ.0)GO TO 10
      READ(5,14)((MMIJ(I),NNIJ(I)),I=1,NUMTM)
   10 CONTINUE
   14 FORMAT(20(2I1,2X))
      IF(NUMTE.EQ.0)GO TO 55
      DO 50 I=1,NUMTE
      IF(MIJ(I).GT.7.OR.NIJ(I).GT.3) GO TO 601
   50 CONTINUE
   55 IF(NUMTM.EQ.0)GO TO 53
      DO 52 I=1,NUMTM
      IF(MMIJ(I).GT.7.OR.NNIJ(I).GT.3)GO TO 602
   52 CONTINUE
   53 CONTINUE
C*********************************************************************
C     AIJ(I) = RADIUS OF I-TH APERTURE
C     XI(I) AND YI(I) = X,Y COORDINATES OF CENTER OF I-TH APERTURE
C     PHIJP(I) = ANGULAR ROTATION OF XI-AXIS WITH RESPECT TO X-AXIS
C                (DEGREES COUNTER-CLOCKWISE).
C*********************************************************************
      READ(5,15) (AIJ(I),I=1,NUMHOLE)
      READ(5,15) (XI(I),I=1,NUMHOLE)
      READ(5,15) (YI(I),I=1,NUMHOLE)
      READ(5,15) (PHIJP(I),I=1,NUMHOLE)
   15 FORMAT(8F10.2)
      SIZE=1.0
      POL=0.0
      IF(NUMHOLE.GT.1)8,9
    8 DO 60 I=2,NUMHOLE
      J=I-1
      IF(AIJ(I).NE.AIJ(J)) SIZE=0.0
   60 CONTINUE
      DO 51 I=2,NUMHOLE
      J=I-1
      IF(PHIJP(I).NE.PHIJP(J)) POL=1.0
   51 CONTINUE
    9 CONTINUE
```

```
C***********************************************************************
C     F = FREQUENCY ( CYCLES PER SECOND)
C     Z1,Z2,Z3,Z4 = DISTANCES FROM APERTURE TO OUTER SURFACE OF LAYERS
C                   1,2,3,4 RESPECTIVELY.
C     CONVERT = CONVERSION FACTOR FOR CONVERTING INPUT DIMENSION TO
C               CENTIMETERS (IF INPUT IN CENTIMETERS, LEAVE BLANK).
C     ER = RELATIVE DIELECTRIC CONSTANT OF MATERIAL FILLING ALL WAVEGUIDES
C***********************************************************************
C        FOR ALL INPUT DIMENSIONS IN WAVELENGTHS
C        SET F=3.0E10 AND CONVERT=1.0
C
C        FOR FREE SPACE,SET Z4=0.0
C
      READ(5,16)F,Z1,Z2,Z3,Z4,CONVERT,ER
      IF(ER.EQ.0.0)ER=1.0
   16 FORMAT(E10.2,7F10.2)
      WRITE(6,41)
   41 FORMAT(1H1)
      RKERR=0.0001
      EPSLN=0.00001
C        RKERR,EPSLN = ERROR TOLERANCES FOR RUNGE-KUTTA INTEGRATION
C                      AND SPLINE CURVE-FIT ROUTINE.
      DIMEN=1.0
      IF(CONVERT.NE.0.0)DIMEN=CONVERT
      WRITE(6,105)
      IF(Z4.NE.0.0)GO TO 19
      WRITE(6,18)F,DIMEN
   18 FORMAT(3X*MUTUAL COUPLING OF CIRCULAR APERTURES RADIATING INTO FRE
     1E SPACE*//5X*F = *E12.5/5X*DIMEN = *F10.6/)
      KIND=4
      V1=V2=1.0
      W1=W2=0.0
      GO TO 21
   19 CONTINUE
      WRITE(6,20)F,Z1,Z2,Z3,Z4,DIMEN
   20 FORMAT(3X*MUTUAL COUPLING OF CIRCULAR APERTURES RADIATING INTO A M
     1ULTILAYERED DIELECTRIC UNDER A NONHOMOGENEOUS PLASMA LAYER.*//5X*
     2INPUTS*//2X*F = *E12.5/1X*Z1  = *F10.6/1X*Z2 = *F10.6/1X*Z3 = *F10.
     36/1X*Z4 = *F10.6//1X*DIMEN = *F10.6)
      WRITE(6,105)
   21 CONTINUE
      WRITE(6,85)ER
   85 FORMAT(1X*ER = *F10.6)
      DO 70 I=1,NUMHOLE
      WRITE(6,92)I,AIJ(I),I,XI(I),I,YI(I),I,PHIJP(I)
   92 FORMAT(1X*AIJ(*I2*)=*F8.5,5X*XI(*I2*)=*F8.5,3X*YI(*I2*)=*F8.5,5X*P
     1HIJP(*I2*)=*F8.3* DEG.*)
   70 CONTINUE
   93 FORMAT(1X*MODE*I2* = TE-*2I1)
      WRITE(6,105)
      DO 72 I=1,NUMMODE
      IF(I.GT.NUMTE)GO TO 71
      WRITE(6,93)I,MIJ(I),NIJ(I)
      GO TO 72
   71 IDEM=I-NUMTE
      WRITE(6,91)I,MMIJ(IDEM),NNIJ(IDEM)
   91 FORMAT(1X*MODE*I2* = TM-*2I1)
   72 CONTINUE
      WRITE(6,105)
      IF(Z4.EQ.0.0)GO TO 100
```

```
C          FIND INPUT DATA CASE
      KIND=1
      IF(Z4-Z3)26,47,33
   26 WRITE(6,27)
   27 FORMAT(10X,16HINPUT DATA ERROR)
      GO TO 900
   33 IF(Z3)26,40,82
   40 KIND=3
      GO TO 109
   47 IF(Z3-Z2)26,54,61
   54 KIND=4
      GO TO 82
   61 KIND=2
C          GET OTHER INPUT NEEDED BASED ON INPUT DATA CASE
C***********************************************************************
C     V1,W1 = REAL AND IMAGINARY PARTS OF DIELECTRIC CONSTANT OF LAYER 1.
C     V2,W2 = REAL AND IMAGINARY PARTS OF DIELECTRIC CONSTANT OF LAYER 2.
C***********************************************************************
   82 READ(5,13)V1,W1,V2,W2
      WRITE(6,105)
      WRITE(6,83)V1,W1,V2,W2
   83 FORMAT(3X*V1 = *F8.5,5X*W1 = *F8.5/3X*V2 = *F8.5,5X*W2 = *F8.5/)
      IF(KIND.EQ.4) GO TO 100
C***********************************************************************
C          NP3 = NUMBER OF POINTS FOR LAYER 3 DIELECTRIC PROFILE
C***********************************************************************
      READ(5,110)NP3
      IF(NP3)89,89,96
   89 WRITE(6,90)
   90 FORMAT(10X,42HERROR IN NUMBER OF POINTS FOR V3 W3 TABLES/)
      GO TO 900
   96 IF(NP3-50)102,102,89
C***********************************************************************
C     V3(I),W(3) = REAL AND IMAGINARY PARTS OF DIELECTRIC CONSTANT AT
C                  POINTS ZD(I) INSIDE LAYER 3.
C***********************************************************************
  102 READ(5,15) (ZD(I),I=1,NP3)
      READ(5,15) (V3(I),I=1,NP3)
      READ(5,15) (W3(I),I=1,NP3)
      WRITE(6,103)
  103 FORMAT(14X,1HZ,15X,5HV3(Z),12X,5HW3(Z)/)
      WRITE(6,104)(ZD(I),V3(I),W3(I),I=1,NP3)
  104 FORMAT(5X,3F17.5)
      WRITE(6,105)
  105 FORMAT(1H )
      IF(KIND.EQ.2) GO TO 100
C***********************************************************************
C          NP4 = NUMBER OF POINTS FOR LAYER 4 PLASMA PROFILE
C***********************************************************************
  109 READ(5,110)NP4
  110 FORMAT(I5)
      IF(NP4)116,116,123
  116 WRITE(6,117)
  117 FORMAT(10X,42HERROR IN NUMBER OF POINTS FOR NE NU TABLES/)
      GO TO 900
  123 IF(NP4-50)130,130,116
C***********************************************************************
C     NE(I),NU(I) = ELECTRON DENSITY (ELECTRONS PER CUBIC CENTIMETER)AND
C                   ELECTRON COLLISION FREQUENCY (PER SECOND) AT POINTS
C                   ZND(I) INSIDE LAYER 4.
```

```
C***************************************************************************
  130 READ(5,15)  (ZND(I),I=1,NP4)
      READ(5,13)  (NE(I),I=1,NP4)
      READ(5,13)  (NU(I),I=1,NP4)
   13 FORMAT(8E10.2)
      WRITE(6,138)
  138 FORMAT(14X,1HZ,12X,5HNE(Z),9X,5HNU(Z)/)
      WRITE(6,139)(ZND(I),NE(I),NU(I),I=1,NP4)
  139 FORMAT(8X,   3E14.5)
      WRITE(6,105)
  100 CONTINUE
      WRITE(6,94)
      DO 1000  IHOLE=1,NUMHOLE
      DO 1000  JHOLE=1,NUMHOLE
      DO 1000  IMODE=1,NUMMODE
      DO 1000  JMODE=1,NUMMODE
      TMI=.FALSE.
      TMJ=.FALSE.
      IF(IMODE.GT.NUMTE)TMI=.TRUE.
      IF(JMODE.GT.NUMTE)TMJ=.TRUE.
      PHI=0.0
      II=(IHOLE-1)*NUMMODE+IMODE
      JJ=(JHOLE-1)*NUMMODE+JMODE
      IF(IHOLE.EQ.JHOLE)500,503
  500 IF(SIZE.EQ.1.0)501,503
  501 IF(IHOLE.GT.1)502,503
  502 YC(II,JJ)=YC(IMODE,JMODE)
      GO TO 1000
  503 CONTINUE
   94 FORMAT(1H0*----------------------------------------------------*/)
      AI=AIJ(IHOLE)*DIMEN
      AJ=AIJ(JHOLE)*DIMEN
      IF(TMI)504,505
  504 IDEM=IMODE-NUMTE
      PHPI=PHIJP(IHOLE)*PI/180.
      MI=MMIJ(IDEM)
      NI=NNIJ(IDEM)
      GO TO 506
  505 MI=MIJ(IMODE)
      NI=NIJ(IMODE)
      PHPI=PHIJP(IHOLE)*PI/180.
  506 IF(TMJ)507,508
  507 IDEM=JMODE-NUMTE
      PHPJ=PHIJP(JHOLE)*PI/180.
      MJ=MMIJ(IDEM)
      NJ=NNIJ(IDEM)
      GO TO 509
  508 MJ=MIJ(JMODE)
      NJ=NIJ(JMODE)
      PHPJ=PHIJP(JHOLE)*PI/180.
  509 CONTINUE
      IF(TMI.AND.(MJ.EQ.0).AND.(.NOT.TMJ))GO TO 99999
      IF(TMJ.AND.(MI.EQ.0).AND.(.NOT.TMI))GO TO 99999
      PHIJPP=PHPJ-PHPI
      MIP=MI+1
      MJP=MJ+1
      XJI=DIMEN*(XI(JHOLE)-XI(IHOLE))
      YJI=DIMEN*(YI(JHOLE)-YI(IHOLE))
      R=SQRT(XJI*XJI+YJI*YJI)
      IF(IHOLE.EQ.JHOLE)R=0.0
```

```
      IF(ABS(PHIJPP).LT.1.0E-04)PHIJPP=0.0
      IF(TMI.AND..NOT.TMJ)PHIJPP=PHIJPP-PI/2.
      IF(TMJ.AND..NOT.TMI)PHIJPP=PHIJPP-PI/2.
      IF(R.EQ.0.0). GO TO 703
      IF(ABS(XJI).LT.1.0E-06)700,701
  700 PHI=0.5*PI
      IF(YJI.LT.0.0)PHI=PHI+PI
      PHI=PHI-PHPI
      GO TO 702.
  701 PHI=ATAN2(YJI,XJI)-PHPI
  702 ARG=(MJ+MI)*PHI-MJ*PHIJPP
      COSP=COS(ARG)
      SINP=SIN(ARG)
      ARG=(MJ-MI)*PHI-MJ*PHIJPP
      COSM=COS(ARG)
      SINM=SIN(ARG)
  703 COSPHIJ=COS(PHIJPP)
      KZERO=FORK*F
      AKZEROI=AI*KZERO
      AKZEROJ=AJ*KZERO
      RKZERO=R*KZERO
      IF(TMI)704,705
  704 FACTORI=XMN(MIP,NI)/AKZEROI
      GO TO 706
  705 FACTORI=XMNP(MIP,NI)/AKZEROI
  706 IF(TMJ)707,708
  707 FACTORJ=XMN(MJP,NJ)/AKZEROJ
      GO TO 709
  708 FACTORJ=XMNP(MJP,NJ)/AKZEROJ
  709 CONTINUE
      FACTSQI=FACTORI**2
      FACTSQJ=FACTORJ**2
      XI1=Z1*KZERO*DIMEN
      XI2=Z2*KZERO*DIMEN
      XI3=Z3*KZERO*DIMEN
      XI4=Z4*KZERO*DIMEN
      D2=XI2-XI1
      RK3=RKERR
      RK4=RKERR
      IF(R.EQ.0.0.AND.MI.NE.MJ) GO TO 99999
      IF(KIND.EQ.3) GO TO 220
      W2SQ=W2*W2
      CCA=CMPLX(V1,-W1)/CMPLX(V2,-W2)
      IF(KIND.NE.4) GO TO 210
      TERMC=V2
      TERMD=-W2
      GO TO 149
  210 CONTINUE
      DO 107 I=1,NP3
      Z(I)=ZD(I)*DIMEN
  107 Z(I)=Z(I)*KZERO
C           SET UP ARRAYS FOR SPLINE INTERPOLATION
      CALL SPLRED(NP3,EPSLN,Z,V3,V31,V32,V33)
      CALL SPLRED(NP3,EPSLN,Z,W3,W31,W32,W33)
      L3=NP3
      CALL SPLD2(NP3,L3,XI3,Z,V3,W3,V31,W31,V32,W32,V33,W33,V3XI3,
     A           W3XI3,DUMMY,DUMMY)
      IF(KIND.EQ.2)GO TO 146
  220 CONTINUE
C           ESTABLISH V4 AND W4
```

```
      OMEGA=TWOPI*F
      OMEGSQ=OMEGA*OMEGA
      DO 137 I=1,NP4
      ZN(I)=ZND(I)*DIMEN
      OPSQ=FOMEGA*NE(I)
      DENOM=OMEGSQ+NU(I)**2
      V4(I)=1.-OPSQ/DENOM
      W4(I)=NU(I)*OPSQ/(OMEGA*DENOM)
  137 CONTINUE
      DO 140 I = 1,NP4
  140 ZN(I)=ZN(I)*KZERO
      CALL SPLRED(NP4,EPSLN,ZN,V4,V41,V42,V43)
      CALL SPLRED(NP4,EPSLN,ZN,W4,W41,W42,W43)
      L4=NP4
      CALL SPLD2(NP4,L4,XI4,ZN,V4,W4,V41,W41,V42,W42,V43,W43,V4XI4,
     A             W4XI4,DUMMY,DUMMY)
      CALL SPLD2(NP4,L4,XI3,ZN,V4,W4,V41,W41,V42,W42,V43,W43,V4XI3,
     A             W4XI3,DUMMY,DUMMY)
C         SET UP INITIAL CONDITIONS FOR BASE RUNGE-KUTTA INTEGRATION
      IF(KIND.NE.1)GO TO 148
      DENOM=V4XI3**2+W4XI3**2
      TERMA=(V3XI3*V4XI3+W3XI3*W4XI3)/DENOM
      TERMB=(V3XI3*W4XI3-W3XI3*V4XI3)/DENOM
  146 L3=1
      CALL SPLD2(NP3,L3,XI2,Z,V3,W3,V31,W31,V32,W32,V33,W33,V3XI2,
     A             W3XI2,DUMMY,DUMMY)
      DENOM=V3XI2**2+W3XI2**2
      TERMC=(V2*V3XI2+W2*W3XI2)/DENOM
      TERMD=(V2*W3XI2-W2*V3XI2)/DENOM
      GO TO 149
  148 V1=V4XI3
      W1=W4XI3
  149 W1SQ=W1*W1
      CCB=CMPLX(-V1,W1)
      PRMT(1)=0.
      NPRM=0
      PRMT(2)=0.01
      PRMT(3)=(PRMT(2)-PRMT(1))/5.
      Y1TEST=Y2TEST=Y3TEST=Y4TEST=0.0
      PRMT(4)=RKERR
  150 PRMT(5)=0.
      NEW=0
      MOST=0
      DO 151 I=1,4
      Y(I)=0.
      DERY(I)=.25
  151 CONTINUE
      CALL LRKS1(PRMT,Y,DERY,4,FINDC,AUX)
      IF(PRMT(5))1,165,158
  158 IF(NPRM.GE.2)GO TO 161
      IF(PRMT(1).NE.0.0)GO TO 162
      WRITE(6,157)PRMT(1),PRMT(2)
      PRMT(1)=0.001
      PRMT(3)=(PRMT(2)-PRMT(1))/5.
      PRMT(4)=RKERR
      NPRM=0
      GO TO 150
  162 WRITE(6,157)PRMT(1),PRMT(2)
      NPRM=NPRM+1
      PRMT(3)=PRMT(3)/5.
```

```
      GO TO 150
  161 PRMT(4)=PRMT(4)*10.
      WRITE(6,159)PRMT(4)
  159 FORMAT(10X,42HERROR TOLERANCE FOR LRKS1 INTEGRATION HAS ,/
     A       10X,18HBEEN INCREASED TO ,E12.5)
      WRITE(6,157)PRMT(1),PRMT(2)
  157 FORMAT(10X*PRMT(1)=*F6.3,3X*PRMT(2)=*F6.3)
      GO TO 150
C
C          INTEGRATION COMPLETE, CALCULATE FINAL ANSWERS
C          Y(1) = REAL PART OF YTE INTEGRAL
C          Y(2) = REAL PART OF YTM INTEGRAL
C          Y(3) = IMAGINARY PART OF YTE INTEGRAL
C          Y(4) = IMAGINARY PART OF YTM INTEGRAL
C
  165 CONTINUE
      IF(PRMT(2).LT.6.0)166,171
  166 Y1TEST=Y1TEST+Y(1)
      Y2TEST=Y2TEST+Y(2)
      Y3TEST=Y3TEST+Y(3)
      Y4TEST=Y4TEST+Y(4)
      PRMT(1)=PRMT(2)
      NPRM=0
      IF(ABS(PRMT(2)-1.0).LE.1.E-05)PRMT(2)=1.0
      PRMT(1)=PRMT(2)
      IF(ABS(PRMT(1)-1.0).LE.1.E-05)PRMT(1)=1.00001
      PRDEL=0.24
      IF(PRMT(2).GE.0.25.AND.PRMT(2).LT.2.0)PRDEL=0.25
      IF(PRMT(2).GE.2.0.AND.PRMT(2).LT.4.0)PRDEL=0.5
      IF(PRMT(2).GE.4.0)PRDEL=1.0
      PRMT(2)=PRMT(2)+PRDEL
      IF(ABS(PRMT(2)-1.0).LE.1.E-05)PRMT(2)=0.99999
      PRMT(3)=(PRMT(2)-PRMT(1))/5.
      PRMT(4)=RKERR
      GO TO 150
  171 IF(PRMT(2).GE.50.0)GO TO 175
      IF(ABS(Y1TEST).LT.1.E-200)GO TO 172
      IF(ABS(Y(1)/Y1TEST)-1.E-04)172,166,166
  172 IF(ABS(Y2TEST).LT.1.E-200)GO TO 173
      IF(ABS(Y(2)/Y2TEST)-1.E-04)173,166,166
  173 IF(ABS(Y3TEST).LT.1.E-200)GO TO 174
      IF(ABS(Y(3)/Y3TEST)-1.E-04)174,166,166
  174 IF(ABS(Y4TEST).LT.1.E-200)GO TO 175
      IF(ABS(Y(4)/Y4TEST)-1.E-04)175,166,166
  175 Y(1)=Y(1)+Y1TEST
      Y(2)=Y(2)+Y2TEST
      Y(3)=Y(3)+Y3TEST
      Y(4)=Y(4)+Y4TEST
      EMI=2.0
      EMJ=2.0
      IF(MI.EQ.0)EMI=1.0
      IF(MJ.EQ.0)EMJ=1.0
      MPLUS=MJ+MI
      SQRTXI=SQRT(XMNP(MIP,NI)**2-MI**2)
      SQRTXJ=SQRT(XMNP(MJP,NJ)**2-MJ**2)
      IF(TMI)SQRTXI=1.0
      IF(TMJ)SQRTXJ=1.0
      MIP=MI+1
      MJP=MJ+1
      CY=-SQRT(EMI*EMJ)/(120.*PI*SQRTXI*SQRTXJ)
```

```
      COEFF=CMPLX(0.0,CY)
 8899 CONTINUE
      YTE=COEFF*CMPLX(Y(1),Y(3))
      YTM=COEFF*CMPLX(Y(2),Y(4))
      IF(R.EQ.0.0)8891,8892
 8891 IF(TMI.AND.TMJ)8801,8802
 8801 VB=0.0
      UB=-(2./EMI)*COS(FLOAT(MI)*PHIJPP)
      GO TO 8810
 8802 IF(TMI)8803,8804
 8803 VB=0.0
      UB=(EMI-1.0)*SIN(FLOAT(MI)*PHIJPP)
      GO TO 8810
 8804 IF(TMJ)8805,8806
 8805 VB=0.0
      UB=(EMI-1.0)*SIN(FLOAT(MI)*PHIJPP)
      GO TO 8810
 8806 VB=(2./EMI)*COS(FLOAT(MI)*PHIJPP)
      UB=-(EMI-1.0)*COS(FLOAT(MI)*PHIJPP)
 8810 YTE=YTE*VB
      YTM=YTM*UB
 8892 YC(II,JJ)=YTE+YTM
      WRITE(6,97)II,JJ,YC(II,JJ),PRMT(2),IHOLE,IMODE,JHOLE,JMODE
   97 FORMAT(1X*Y(*I2*,*I2*) = *E12.5* +J(*E12.5*)*3X*PRMT(2)=*F6.3,3X*I
     1HOLE=*I2,2X*IMODE=*I2,3X*JHOLE=*I2,2X*JMODE=*I2/)
      PHIJPP=PHIJPP*180./PI
      R=R/DIMEN
      GO TO 1000
99999 YC(II,JJ)=CMPLX(0.0,0.0)
 1000 CONTINUE
      DO 1001  IHOLE=1,NUMHOLE
      DO 1001  JHOLE=1,NUMHOLE
      DO 1001  IMODE=1,NUMMODE
      DO 1001  JMODE=1,NUMMODE
      TMI=.FALSE.
      IF(IMODE.GT.NUMTE)TMI=.TRUE.
      II=(IHOLE-1)*NUMMODE+IMODE
      JJ=(JHOLE-1)*NUMMODE+JMODE
      AKZEROI=AIJ(IHOLE)*KZERO*DIMEN
      IF(TMI)1003,1002
 1002 NI=NIJ(IMODE)
      MIP=MIJ(IMODE)+1
      FACTSQI=(XMNP(MIP,NI)/AKZEROI)**2
      FTSQ=ER-FACTSQI
      IF(ABS(FTSQ).LT.1.0E-200)FTSQ=0.0
      IF(FTSQ.GE.0.0)YMNZERO=(1.0,0.0)*SQRT(FTSQ)
      IF(FTSQ.LT.0.0)YMNZERO=-CON*SQRT(-FTSQ)
      GO TO 1004
 1003 IDEM=IMODE-NUMTE
      NI=NNIJ(IDEM)
      MIP=MMIJ(IDEM)+1
      FACTSQI=(XMN(MIP,NI)/AKZEROI)**2
      FTSQ=ER-FACTSQI
      CONVERT=FTSQ
      IF(ABS(FTSQ).LT.1.0E-290)CONVERT=1.0E-290
      IF(FTSQ.GE.0.0)YMNZERO=(1.0,0.0)/SQRT(CONVERT)
      IF(FTSQ.LT.0.0)YMNZERO=(-1.,0.)/(CON*SQRT(-CONVERT))
 1004 CONTINUE
      A(II,JJ)=YC(II,JJ)
      B(II,JJ)=-YC(II,JJ)
```

```
      IF(II.EQ.JJ)A(II,JJ)=YMNZERO/(120.*PI)+YC(II,JJ)
      IF(II.EQ.JJ)B(II,JJ)=YMNZERO/(120.*PI)-YC(II,JJ)
      YC(II,JJ)=B(II,JJ)
      YMNZERO=YMNZERO/(120.*PI)
      IF(II.EQ.JJ)WRITE(6,98)II,YMNZERO,IHOLE,IMODE
   98 FORMAT(1X*YMN(*I2*) = *E12.5* +J(*E12.5*)*3X*IHOLE=*I2,3X*IMODE=*I
     12)
 1001 CONTINUE
      WRITE(6,94)
      WRITE(6,813)
  813 FORMAT(10X*SCATTERING MATRIX*)
      MAX=25
      CALL CXINV(A,M,B,M,DETERM,IPIVOT,INDEX,MAX,ISCALE)
      DO 802 I=1,M
      DO 802 J=1,M
      B(I,J)=CMPLX(0.,0.)
      DO 830 K=1,M
      B(I,J)=B(I,J)+YC(I,K)*A(K,J)
  830 CONTINUE
  802 CONTINUE
      WRITE(6,105)
      DO 805 I=1,M
      DO 805 J=1,M
      XDX=CABS(B(I,J))
      IF(XDX.LT.5.E-16)GO TO 810
      ISOLATE=20.*ALOG10(XDX)
      XXXX1=AIMAG(B(I,J))
      XXXX2=REAL(B(I,J))
      PHASE=(180./PI)*ATAN2(XXXX1,XXXX2)
      WRITE(6,803)I,J,B(I,J),ISOLATE,PHASE
  803 FORMAT(1X*S(*I2*,*I2*) = *E12.5* +J(*E12.5*)*3X,F9.4* DB*3X,F8.3*
     1DEG.*)
      GO TO 805
  810 WRITE(6,806)I,J,B(I,J)
  806 FORMAT(1X*S(*I2*,*I2*) = *E12.5* +J(*E12.5*)*3X*BELOW -300 DB*)
  805 CONTINUE
      WRITE(6,95)
   95 FORMAT(1H *+++++++++++++++++++++++++++++++++++++++++++++++++++++++++
     1++++++++++++++*/)
      GO TO 1
  601 WRITE(6,88)
   88 FORMAT(1H0*MODE SUBSCRIPTS ARE OUT OF RANGE OF XMNP ARRAY*)
  900 WRITE(6,901)
  901 FORMAT(1H1,10X,10HEND OF JOB )
      STOP
  602 WRITE(6,99)
   99 FORMAT(1X*MODE SUBSCRIPTS FOR TM MODES OUT OF RANGE OF XMN ARRAY*)
      STOP
      END
```

```
      SUBROUTINE FINDC(BETA,Y,DERY)
      DIMENSION BESSEL(21),DERIV(8),DERY(1),EXTRA(8,8),HOLD(8),
     A          INDEX(100),PARM(5),RUST(8),SAVE(500),Y(1)
C             DIMENSIONS FOR COMMON VARIABLES
      DIMENSION V3(50),V31(50),V32(50),V33(50),
     A          V4(50),V41(50),V42(50),V43(50),
     B          W4(50),W41(50),W42(50),W43(50),
     C          W3(50),W31(50),W32(50),W33(50),
     D          Z(50),ZN(50),XMNP(8,3),XMN(8,3)
C             COMMON - DIMENSIONED VARIABLES
      COMMON V3,V31,V32,V33,W3,W31,W32,W33,Z,XMNP,XMN,
     A        V4,V41,V42,V43,W4,W41,W42,W43,ZN,
C             COMMON - UNDIMENSIONED VARIABLES
     B        BSQ,CCA,CCB,D2,KIND,L3,L4,MOST,TMI,TMJ,
     C        NEW,NP3,NP4,RKERR,RK3,RK4,TERMA,TERMB,TERMC,TERMD,
     D           V1,      V2,      V3XI3,V4XI4,
     E           W1,W1SQ,W2,W2SQ,W3XI3,W4XI4,XI1,XI2,XI3,XI4,
     F MI,MJ,NI,NJ,MIP,MJP,AKZEROI,AKZEROJ,RKZERO,FACTORI,FACTORJ,
     G FACTSQI,FACTSQJ,COSP,COSM,SINP,SINM,COSPHIJ,PHIJPP
      LOGICAL TMI,TMJ
      COMPLEX CARGO,CCA,CCB,CCON,COSINE,
     A                 F1B0,F1PB0,F1PXI1,F1XI1,F2PXI2,F2XI2,
     B                 G1B0,G1PB0,G1PXI1,G1XI1,G2PXI2,G2XI2,K1,K2,SINE
      EXTERNAL LAYER3,LAYER4
      IF(NEW)7,105,7
C          FIND OUT IF THIS BETA IS IN SAVE TABLE
    7 IF(BETA-SAVE(LAST))14,84,35
C          BETA IS LESS THAN LAST TABLE VALUE USED
   14 LAST=LAST-1
      IF(BETA-SAVE(LAST))21,84,28
   21 IF(LAST-1)22,22,14
   22 WRITE(6,23)
   23 FORMAT(10X,8HERROR 23)
      GO TO 900
   28 NEXT=LAST
      LAST=LAST+1
      GO TO 63
C          BETA IS GREATER THAN LAST TABLE VALUE USED
   35 IF(MOST-LAST)42,42,49
   42 NEED=1
      GO TO 109
   49 LAST=LAST+1
      IF(BETA-SAVE(LAST))56,84,35
   56 NEXT=LAST-1
C          AT THIS POINT WE KNOW THAT BETA LIES BETWEEN
C          SAVE(NEXT) AND SAVE(LAST)
   63 IF(ABS((BETA-SAVE(NEXT))/BETA)-1.E-6)70,70,77
   70 LAST=NEXT
      GO TO 84
   77 IF(ABS((BETA-SAVE(LAST))/BETA)-1.E-6)84,84,81
   81 NEED=0
      GO TO 109
C          GET INTEGRAND VALUES FROM SAVE TABLE
   84 NOW=INDEX(LAST)
      DO 91 I=1,4
      DERY(I)=SAVE(NOW)
      NOW=NOW+1
   91 CONTINUE
      RETURN
C          BETA IS ZERO
```

```
  105 IF(BETA.NE.0.0)GO TO 109
      DO 107 I=1,4
      DERY(I)=0.
  107 CONTINUE
      GO TO 403
C          CALCULATE INTEGRANDS
C
C                RUST ARRAY DEFINED
C          RUST(1)=R(XI)
C          RUST(2)=S(XI)
C          RUST(3)=T(XI)
C          RUST(4)=U(XI)
C          RUST(5)=R-(XI)
C          RUST(6)=S-(XI)
C          RUST(7)=T-(XI)
C          RUST(8)=U-(XI)
  109 BSQ=BETA*BETA
      ROOT=SQRT(ABS(BSQ-1.))
      GO TO(154,133,154,112),KIND
C
C          CASE 4 - XI4=XI3=XI2
C
  112 RUST(1)=1.
      RUST(2)=0.
      RUST(3)=1.
      RUST(4)=0.
      IF(BETA-1.)119,119,126
  119 RUST(5)=0.
      RUST(6)=-ROOT
      RUST(7)=0.
      RUST(8)=-ROOT
      GO TO 273
  126 RUST(5)=-ROOT
      RUST(6)=0.
      RUST(7)=-ROOT
      RUST(8)=0.
      GO TO 273
C
C          CASE 2 - XI4=XI3
C
  133 ASSIGN 135 TO JAIL
      PARM(1)=XI3
      PARM(2)=XI2
      PARM(3)=-.5
  135 RUST(1)=1.
      RUST(2)=0.
      RUST(3)=1.
      RUST(4)=0.
      IF(BETA-1.)140,140,147
  140 RUST(5)=0.
      RUST(6)=-ROOT
      RUST(7)=-W3XI3*ROOT
      RUST(8)=-V3XI3*ROOT
      GO TO 249
  147 RUST(5)=-ROOT
      RUST(6)=0.
      RUST(7)=-V3XI3*ROOT
      RUST(8)= W3XI3*ROOT
      GO TO 249
C
```

```
C          CASE 1 OR 3 - SET INITIAL VALUES FOR INTEGRATION
C          FROM XI4 TO XI3
C
  154 PARM(1)=XI4
      PARM(2)=XI3
      PARM(3)=-.5
  161 PARM(4)=RK4
      PARM(5)=0.
      DO 168 I=1,8
      DERIV(I)=.125
  168 CONTINUE
      RUST(1)=1.
      RUST(2)=0.
      RUST(3)=1.
      RUST(4)=0.
      IF(BETA-1.)175,175,182
  175 RUST(5)=0.
      RUST(6)=-ROOT
      RUST(7)=-W4XI4*ROOT
      RUST(8)=-V4XI4*ROOT
      GO TO 189
  182 RUST(5)=-ROOT
      RUST(6)=0.
      RUST(7)=-V4XI4*ROOT
      RUST(8)= W4XI4*ROOT
C          INTEGRATE FROM XI4 TO XI3
  189 CALL LRKS2(PARM,RUST,DERIV,8,LAYER4,EXTRA)
      IF(PARM(5))900,203,196
  196 RK4=RK4*10.
      WRITE(6,197)RK4
  197 FORMAT(10X,39HERROR TOLERANCE FOR LAYER 4 INTEGRATION,/
     A        10X,21HHAS BEEN INCREASED TO,  E13.5/)
      GO TO 161
C
  203 IF(KIND-3)217,210,217
C
C          CASE 3 - XI3=0 - SET LAYER 1 FUNCTIONS AND PROCEED
C                           TO FINAL CALCULATIONS
  210 F1B0=CMPLX(RUST(1),RUST(2))
      G1B0=CMPLX(RUST(3),RUST(4))
      F1PB0=CMPLX(RUST(5),RUST(6))
      G1PB0=CMPLX(RUST(7),RUST(8))
      GO TO 300
C
C          CASE 1 - ALL LAYERS - SET INITIAL CONDITIONS
C
  217 PARM(1)=XI3
      PARM(2)=XI2
      PARM(3)=-.5
      DO 224 I=1,8
      HOLD(I)=RUST(I)
  224 CONTINUE
      ASSIGN 231 TO JAIL
      GO TO 245
  231 DO 238 I=1,6
      RUST(I)=HOLD(I)
  238 CONTINUE
  245 RUST(7)=TERMA*HOLD(7)-TERMB*HOLD(8)
      RUST(8)=TERMA*HOLD(8)+TERMB*HOLD(7)
  249 DO 252 I=1,8
```

```
      DERIV(I)=.125
  252 CONTINUE
      PARM(4)=RK3
      PARM(5)=0.
C          INTEGRATE FROM XI3 TO XI2
      CALL LRKS2(PARM,RUST,DERIV,8,LAYER3,EXTRA)
      IF(PARM(5))900,273,259
  259 RK3=RK3*10.
      WRITE(6,260)RK3
  260 FORMAT(10X,39HERROR TOLERANCE FOR LAYER 3 INTEGRATION,/
     A         10X,21HHAS BEEN INCREASED TO,  E13.5/)
      GO TO JAIL,(135,231)
C     DO NOT PASS GO
C     DO NOT COLLECT $200
  273 IF(XI4.EQ.0.0)GO TO 300
      DO 280  I=7,8
      HOLD(I)=RUST(I)
  280 CONTINUE
      RUST(7)=TERMC*HOLD(7)-TERMD*HOLD(8)
      RUST(8)=TERMC*HOLD(8)+TERMD*HOLD(7)
C          ALGEBRA FROM XI2 TO XI1
      F2XI2=CMPLX(RUST(1),RUST(2))
      G2XI2=CMPLX(RUST(3),RUST(4))
      F2PXI2=CMPLX(RUST(5),RUST(6))
      G2PXI2=CMPLX(RUST(7),RUST(8))
      VMBSQ=V2-BSQ
      IF(W2)282,282,286
  282 IF(VMBSQ)284,283,285
  283 F1XI1=F2XI2-F2PXI2*D2
      G1XI1=G2XI2-G2PXI2*D2
      F1PXI1=F2PXI2
      G1PXI1=CCA*G2PXI2
      GO TO 291
  284 P2=0.0
      Q2=SQRT(-VMBSQ)
      GO TO 287
  285 P2=SQRT(VMBSQ)
      Q2=0.0
      GO TO 287
  286 ROOT=SQRT(VMBSQ*VMBSQ+W2SQ)
      P2=SQRT(.5*(ROOT+VMBSQ))
      Q2=SQRT(.5*(ROOT-VMBSQ))
  287 K2=CMPLX(P2,-Q2)
      CARGO=D2*K2
      SINE=CSIN(CARGO)
      COSINE=CCOS(CARGO)
      CCON=SINE/K2
      F1XI1=F2XI2*COSINE-F2PXI2*CCON
      G1XI1=G2XI2*COSINE-G2PXI2*CCON
      F1PXI1=K2*F2XI2*SINE+F2PXI2*COSINE
      G1PXI1=K2*CCA*G2XI2*SINE+CCA*G2PXI2*COSINE
C          ALGEBRA FROM XI1 TO 0.0
  291 VMBSQ=V1-BSQ
      IF(W1)292,292,296
  292 IF(VMBSQ)294,293,295
  293 F1B0=F1XI1-F1PXI1*XI1
      G1B0=G1XI1-G1PXI1*XI1
      F1PB0=F1PXI1
      G1PB0=G1PXI1
      GO TO 300
```

```
  294 P1=0.0
      Q1=SQRT(-VMBSQ)
      GO TO 297
  295 P1=SQRT(VMBSQ)
      Q1=0.0
      GO TO 297
  296 ROOT=SQRT(VMBSQ*VMBSQ+W1SQ)
      P1=SQRT(.5*(ROOT+VMBSQ))
      Q1=SQRT(.5*(ROOT-VMBSQ))
  297 K1=CMPLX(P1,-Q1)
      CARGO=XI1*K1
      SINE=CSIN(CARGO)
      COSINE=CCOS(CARGO)
      CCON=SINE/K1
      F1BO=F1XI1*COSINE-F1PXI1*CCON
      G1BO=G1XI1*COSINE-G1PXI1*CCON
      CCON=K1*SINE
      F1PBO=F1XI1*CCON+F1PXI1*COSINE
      G1PBO=G1XI1*CCON+G1PXI1*COSINE
C
C          FINAL INTEGRAND CALCULATIONS
C
C          GET BESSEL FUNCTIONS NEEDED
C               BESSEL(1)=J0(ARG)
C               BESSEL(2)=J1(ARG)
C
  300 CONTINUE
      IF(RKZERO.EQ.0.0)GO TO 301
      MPLUS=MJ+MI
      MMINUS=MJ-MI
      IF(MMINUS.LT.0)MMINUS=MI-MJ
      ARG=RKZERO*BETA
      MMP=MPLUS+1
      CALL BESJS(ARG,BESSEL,MMP)
      BESP=BESSEL(MMP)
      MMP=MMINUS+1
      CALL BESJS(ARG,BESSEL,MMP)
      BESM=BESSEL(MMP)
      IF(MI.GT.MJ)BESM=(-1.0)*MMINUS*BESM
      IF(TMI.AND.TMJ)311,312
  311 VBETA=0.0
      UBETA=-(-1.0)**MJ*(BESP*COSP+(-1.0)**MI*BESM*COSM)
      GO TO 315
  312 IF(TMI.OR.TMJ)313,314
  313 VBETA=0.0
      UBETA=(-1.0)**MJ*(BESP*SINP-(-1.0)**MI*BESM*SINM)
      GO TO 315
  314 VBETA=(-1.0)**MJ*(BESP*COSP+(-1.0)**MI*BESM*COSM)
      UBETA=(-1.0)**MJ*(BESP*COSP-(-1.0)**MI*BESM*COSM)
  315 CONTINUE
      GO TO 302
  301 UBETA=1.0
      VBETA=1.0
  302 IF(TMI)1310,1320
 1310 MMP=MI+1
      MMPP=MMP+1
      IF(ABS(FACTSQI-BSQ)-1.0E-06)1311,1312,1312
 1311 ARG=XMN(MIP,NI)
      CALL BESJS(ARG,BESSEL,MMPP)
      XIBETA=BETA*BESSEL(MMPP)/(2.*ARG)
```

```
      YIBETA=0.0
      GO TO 350
 1312 ARG=AKZEROI*BETA
      CALL BESJS(ARG,BESSEL,MMP)
      XIBETA=BETA*BESSEL(MMP)/(FACTSQI-BSQ)
      YIBETA=0.0
      GO TO 350
 1320 MMP=MI+1
      IF(ABS(FACTSQI-BSQ)-1.0E-06)1321,1322,1322
 1321 ARG=XMNP(MIP,NI)
      CALL BESJS(ARG,BESSEL,MMP)
      YIBETA=BESSEL(MMP)*(ARG*ARG-MI*MI)/(2.*FACTORI)
      GO TO 1323
 1322 MMPP=MMP+1
      ARG=AKZEROI*BETA
      CALL BESJS(ARG,BESSEL,MMPP)
      YIBETA=XMNP(MIP,NI)*FACTORI*(MI*BESSEL(MMP)/ARG-BESSEL(MMPP))/(FAC
     1TSQI-BSQ)
 1323 XIBETA=MI*BESSEL(MMP)/BETA
  350 IF(TMJ)1330,1340
 1330 MMP=MJ+1
      MMPP=MMP+1
      IF(ABS(FACTSQJ-BSQ)-1.0E-06)1331,1332,1332
 1331 ARG=XMN(MJP,NJ)
      CALL BESJS(ARG,BESSEL,MMPP)
      XJBETA=BETA*BESSEL(MMPP)/(2.*ARG)
      YJBETA=0.0
      GO TO 360
 1332 ARG=AKZEROJ*BETA
      CALL BESJS(ARG,BESSEL,MMP)
      XJBETA=BETA*BESSEL(MMP)/(FACTSQJ-BSQ)
      YJBETA=0.0
      GO TO 360
 1340 MMP=MJ+1
      IF(ABS(FACTSQJ-BSQ)-1.0E-06)1341,1342,1342
 1341 ARG=XMNP(MJP,NJ)
      CALL BESJS(ARG,BESSEL,MMP)
      YJBETA=BESSEL(MMP)*(ARG*ARG-MJ*MJ)/(2.*FACTORJ)
      GO TO 1343
 1342 MMPP=MMP+1
      ARG=AKZEROJ*BETA
      CALL BESJS(ARG,BESSEL,MMPP)
      YJBETA=XMNP(MJP,NJ)*FACTORJ*(MJ*BESSEL(MMP)/ARG-BESSEL(MMPP))/(FAC
     1TSQJ-BSQ)
 1343 XJBETA=MJ*BESSEL(MMP)/BETA
  360 CONTINUE
      IF(TMI.OR.TMJ)VBETA=0.0
      FACTRM=XIBETA*XJBETA*UBETA*BETA
      FACTRE=-YIBETA*YJBETA*VBETA*BETA
      IF(XI4.EQ.0.0)GO TO 3015
C           SEPARATE REAL AND IMAGINARY PARTS
      NTYPE=0
 3010 IF(ABS(REAL(F1B0)).GT.1.E120)GO TO 3011
      IF(ABS(AIMAG(F1B0)).GT.1.E120)GO TO 3011
      IF(ABS(REAL(F1PB0)).GT.1.E120)GO TO 3011
      IF(ABS(AIMAG(F1PB0)).GT.1.E120)GO TO 3011
      NTYPE=0
      GO TO 3012
 3011 NTYPE=NTYPE+1
      IF(NTYPE.GT.5)GO TO 900
```

```
      F1B0=F1B0*(1.E-120,0.)
      F1PB0=F1PB0*(1.E-120,0.)
      GO TO 3010
 3012 IF(ABS(REAL(G1B0)).GT.1.E120)GO TO 3013
      IF(ABS(AIMAG(G1B0)).GT.1.E120)GO TO 3013
      IF(ABS(REAL(G1PB0)).GT.1.E120)GO TO 3013
      IF(ABS(AIMAG(G1PB0)).GT.1.E120)GO TO 3013
      NTYPE=0
      GO TO 3014
 3013 NTYPE=NTYPE+1
      IF(NTYPE.GT.5)GO TO 900
      G1B0=G1B0*(1.E-120,0.)
      G1PB0=G1PB0*(1.E-120,0.)
      GO TO 3012
 3014 CONTINUE
      CCON=F1PB0/F1B0
      DERY(1)=FACTRE*REAL(CCON)
      DERY(3)=FACTRE*AIMAG(CCON)
      CCON=CCB*G1B0/G1PB0
      DERY(2)=FACTRM*REAL(CCON)
      DERY(4)=FACTRM*AIMAG(CCON)
      IF(X14.NE.0.0)GO TO 3016
 3015 DERY(1)=FACTRE*RUST(5)
      DERY(3)=FACTRE*RUST(6)
      DERY(2)=0.0
      DERY(4)=-FACTRM/ROOT
      IF(BETA.LE.1.0)GO TO 3016
      DERY(2)=-DERY(4)
      DERY(4)=0.0
 3016 CONTINUE
C
C          SAVE INTEGRANDS
C
      IF(NEW)449,403,400
  400 IF(NEED)414,414,407
  403 NEW=1
C          MOVE TO TOP OF BETA SAVE TABLE
  407 MOST=MOST+1
      LAST=MOST
      GO TO 428
C          MAKE SPACE IN MIDDLE OF BETA SAVE TABLE
  414 MOST=MOST+1
      NA=LAST+1
      MOVE=MOST+NA
      DO 421 N=NA,MOST
      M=MOVE-N
      SAVE(M)=SAVE(M-1)
      INDEX(M)=INDEX(M-1)
  421 CONTINUE
C          SAVE BETA AND POINTER
  428 SAVE(LAST)=BETA
      INDEX(LAST)=101+4*(MOST-1)
      NOW=INDEX(LAST)
C          SAVE INTEGRANDS
      DO 435 I=1,4
      SAVE(NOW)=DERY(I)
      NOW=NOW+1
  435 CONTINUE
C          CHECK FOR TABLE FULL
      IF(100-MOST)442,442,800
```

```
  442 NEW=-1
      GO TO 800
C          BETA SAVE TABLE IS FULL
  449 KEEP=INDEX(1)
      IF(NFED)456,456,470
C          PUSH DOWN SAVE TABLE FROM SAVE(NEXT)
  456 LIMIT=NEXT-1
      DO 463 I=1,LIMIT
      SAVE(I)=SAVE(I+1)
      INDEX(I)=INDEX(I+1)
  463 CONTINUE
      LAST=NEXT
      GO TO 484
C          PUSH DOWN ENTIRE BETA SAVE TABLE
  470 DO 477 I=1,99
      SAVE(I)=SAVE(I+1)
      INDEX(I)=INDEX(I+1)
  477 CONTINUE
      LAST=MOST
C          SAVE BETA AND POINTER
  484 SAVE(LAST)=BETA
      INDEX(LAST)=KEEP
C          SAVE INTEGRANDS
      DO 491 I=1,4
      SAVE(KEEP)=DERY(I)
      KEEP=KEEP+1
  491 CONTINUE
  800 RETURN
  900 WRITE(6,901)
      IF(NTYPE.NE.0)WRITE(6,904)F1B0,F1PB0,G1B0,G1PB0
  904 FORMAT(1X*F1B0=*2E13.5/1X*F1PB0=*2E13.5/1X*G1B0=*2E13.5/1X*G1PB0=*
     12E13.5/),
  901 FORMAT(1H1,10X,10HEND OF JOB )
      STOP
  902 WRITE(6,903)
  903 FORMAT(1X*MPLUS EXCEEDES DIMENSION OF ARRAY BESSEL*)
      STOP
      END
```

```
      SUBROUTINE SPLRED(N,EPSLN,X,Y,DELY,S2,S3)
      DIMENSION X(1),Y(1),DELY(1),S2(1),S3(1),
     A          H(100),H2(100),B(100),DELSQY(100),C(100)
      N1=N-1
      DO 7 I=1,N1
      H(I)=X(I+1)-X(I)
    7 DELY(I)=(Y(I)-Y(I+1))/H(I)
      DO 14 I=2,N1
      H2(I)=H(I-1)+H(I)
      B(I)=.5*H(I-1)/H2(I)
      DELSQY(I)=(DELY(I)-DELY(I-1))/H2(I)
      S2(I)=DELSQY(I)+DELSQY(I)
   14 C(I)=S2(I)+DELSQY(I)
      S2(1)=0.
      S2(N)=0.
      OMEGA=1.071797
   21 ETA=0.
      DO 35 I=2,N1
      W=(C(I)-B(I)*S2(I-1)-(.5-B(I))*S2(I+1)-S2(I))*OMEGA
      IF(ABS(W)-ETA)35,35,28
   28 ETA=ABS(W)
   35 S2(I)=S2(I)+W
      IF(ETA-EPSLN)42,21,21
   42 DO 49 I=1,N1
   49 S3(I)=(S2(I+1)-S2(I))/H(I)
      RETURN
      END
```

```
      SUBROUTINE SPLD2(N,M,T,X,Y,Z,DELY,DELZ,S2,T2,S3,T3,SS,TT,SS1,TT1)
      DIMENSION X(1),Y(1),DELY(1),DELZ(1),S2(1),T2(1),S3(1),T3(1),Z(1)
      DATA SIXTH/.16666666666667/
    7 I=M
      IF(M-1)77,21,14
   14 IF(M-N)21,21,77
   21 IF(T-X(1))63,28,35
   28 I=1
      GO TO 105
   35 IF(T-X(N))42,91,73
   42 IF(T-X(I))56,105,49
   49 I=I+1
      IF(T-X(I))98,105,49
   56 I=I-1
      IF(T-X(I))56,105,105
   63 IF(T-X(1)+1.E-6)65,64,64
   64 T=X(1)
      GO TO 28
   65 WRITE(6,70) T
   70 FORMAT(10X,10HARGUMENT =,E14.6,22HOUT OF RANGE IN SPLD2 )
      WRITE(6,71)(X(L),L=1,N)
   71 FORMAT(/10X,24HRANGE OF ARGUMENT VALUES/(E20.6))
      STOP
   73 IF(T-X(N)-1.E-6)75,75,65
   75 T=X(N)
      GO TO 91
   77 WRITE(6,84)
   84 FORMAT(10X,23HM OUT OF RANGE IN SPLD2)
      STOP
   91 I=N
   98 I=I-1
  105 HT1=T-X(I)
      HT2=T-X(I+1)
      PROD=HT1*HT2
      SS2=S2(I)+HT1*S3(I)
      TT2=T2(I)+HT1*T3(I)
      DELSQS=(S2(I)+S2(I+1)+SS2)*SIXTH
      DELSQT=(T2(I)+T2(I+1)+TT2)*SIXTH
      SS=Y(I)+HT1*DELY(I)+PROD*DELSQS
      TT=Z(I)+HT1*DELZ(I)+PROD*DELSQT
      H12=HT1+HT2
      PRCON=PROD*SIXTH
      SS1=DELY(I)+H12*DELSQS+PRCON*S3(I)
      TT1=DELZ(I)+H12*DELSQT+PRCON*T3(I)
      M=I
      RETURN
      END
```

```
      SUBROUTINE LAYER3(XI,RUST,DERIV)
      DIMENSION RUST(1),DERIV(1)
C                DIMENSIONS FOR COMMON VARIABLES
      DIMENSION V3(50),V31(50),V32(50),V33(50),
     A          V4(50),V41(50),V42(50),V43(50),
     B          W4(50),W41(50),W42(50),W43(50),
     C          W3(50),W31(50),W32(50),W33(50),
     D          Z(50),ZN(50),XMNP(8,3),XMN(8,3)
C                COMMON - DIMENSIONED VARIABLES
      COMMON V3,V31,V32,V33,W3,W31,W32,W33,Z,XMNP,XMN,
     A       V4,V41,V42,V43,W4,W41,W42,W43,ZN,
C                COMMON - UNDIMENSIONED VARIABLES
     B       BSQ,CCA,CCB,D2,KIND,L3,L4,MOST,TMI,TMJ,
     C       NEW,NP3,NP4,RKERR,RK3,RK4,TERMA,TERMB,TERMC,TERMD,
     D          V1,     V2,      V3XI3,V4XI4,
     E          W1,W1SQ,W2,W2SQ,W3XI3,W4XI4,XI1,XI2,XI3,XI4,
     F MI,MJ,NI,NJ,MIP,MJP,AKZEROI,AKZEROJ,RKZERO,FACTORI,FACTORJ,
     G FACTSQI,FACTSQJ,COSP,COSM,SINP,SINM,COSPHIJ,PHIJPP
      LOGICAL TMI,TMJ
      COMPLEX CCA,CCB
      CALL SPLD2(NP3,L3,XI,Z,V3,W3,V31,W31,V32,W32,V33,W33,V3XI,
     A           W3XI,V3PXI,W3PXI)
      DERIV(1)=RUST(5)
      DERIV(2)=RUST(6)
      DERIV(3)=RUST(7)
      DERIV(4)=RUST(8)
      BSQMV=BSQ-V3XI
      DERIV(5)=BSQMV*RUST(1)-W3XI*RUST(2)
      DERIV(6)=BSQMV*RUST(2)+W3XI*RUST(1)
      DENOM=V3XI*V3XI+W3XI*W3XI
      FIRST=V3XI/DENOM
      SECOND=V3PXI*RUST(7)+W3PXI*RUST(8)
      THIRD=W3XI/DENOM
      FOURTH=V3PXI*RUST(8)-W3PXI*RUST(7)
      DERIV(7)=FIRST*SECOND-THIRD*FOURTH+BSQMV*RUST(3)-W3XI*RUST(4)
      DERIV(8)=FIRST*FOURTH+THIRD*SECOND+BSQMV*RUST(4)+W3XI*RUST(3)
      RETURN
      END
```

```
      SUBROUTINE LAYER4(XI,RUST,DERIV)
      DIMENSION RUST(1),DERIV(1)
C                DIMENSIONS FOR COMMON VARIABLES
      DIMENSION V3(50),V31(50),V32(50),V33(50),
     A          V4(50),V41(50),V42(50),V43(50),
     B          W4(50),W41(50),W42(50),W43(50),
     C          W3(50),W31(50),W32(50),W33(50),
     D          Z(50),ZN(50),XMNP(8,3),XMN(8,3)
C                COMMON - DIMENSIONED VARIABLES
      COMMON V3,V31,V32,V33,W3,W31,W32,W33,Z,XMNP,XMN,
     A       V4,V41,V42,V43,W4,W41,W42,W43,ZN,
C                COMMON - UNDIMENSIONED VARIABLES
     B       BSQ,CCA,CCB,D2,KIND,L3,L4,MOST,TMI,TMJ,
     C       NEW,NP3,NP4,RKERR,RK3,RK4,TERMA,TERMB,TERMC,TERMD,
     D          V1,      V2,      V3XI3,V4XI4,
     E          W1,W1SQ,W2,W2SQ,W3XI3,W4XI4,XI1,XI2,XI3,XI4,
     F MI,MJ,NI,NJ,MIP,MJP,AKZEROI,AKZEROJ,RKZERO,FACTORI,FACTORJ,
     G FACTSQI,FACTSQJ,COSP,COSM,SINP,SINM,COSPHIJ,PHIJPP
      LOGICAL TMI,TMJ
      COMPLEX CCA,CCB
      CALL SPLD2(NP4,L4,XI,ZN,V4,W4,V41,W41,V42,W42,V43,W43,V4XI,
     A           W4XI,V4PXI,W4PXI)
      DERIV(1)=RUST(5)
      DERIV(2)=RUST(6)
      DERIV(3)=RUST(7)
      DERIV(4)=RUST(8)
      BSQMV=BSQ-V4XI
      DERIV(5)=BSQMV*RUST(1)-W4XI*RUST(2)
      DERIV(6)=BSQMV*RUST(2)+W4XI*RUST(1)
      DENOM=V4XI*V4XI+W4XI*W4XI
      FIRST=V4XI/DENOM
      SECOND=V4PXI*RUST(7)+W4PXI*RUST(8)
      THIRD=W4XI/DENOM
      FOURTH=V4PXI*RUST(8)-W4PXI*RUST(7)
      DERIV(7)=FIRST*SECOND-THIRD*FOURTH+BSQMV*RUST(3)-W4XI*RUST(4)
      DERIV(8)=FIRST*FOURTH+THIRD*SECOND+BSQMV*RUST(4)+W4XI*RUST(3)
      RETURN
      END
```

```
      SUBROUTINE BESJS(XX,BJ,MT)
      DIMENSION BJ(1),B(130)
C           ROUTINE FINDS BESSEL J OF X FOR ORDERS ZERO THROUGH MT
C           AND LOADS THEM INTO BJ(1) THROUGH BJ(MT+1).
      X=ABS(XX)
      BJ(1) = 1.0
      N = MT + 1
      IF(X.GE.80.)10,20
   10 PI=3.141592653589793
      DO 11 I=1,N
   11 BJ(I)=SQRT(2.0/(PI*X))*COS(X-0.25*PI-0.5*PI*(I-1))
      GO TO 220
   20 CONTINUE
      DO 5 I = 2,N
    5 BJ(I) = .0
      IF(X-15.)32,32,34
   32 NTEST = 20.+10.*X-X**2/3
      GO TO 36
   34 NTEST = 90.+X/2.
   36 IF(MT-NTEST)40,38,38
   38 N = NTEST - 1
      GO TO 45
   40 N = MT
   45 BPREV = .0
      N1 = N+1
      F = 2./X
      D = 1.0E-6
C           COMPUTE STARTING VALUE OF M
      IF(X-5.)50,60,60
   50 MA = X + 6.
      GO TO 70
   60 MA = 1.4*X + 60./X
   70 MB = N+IFIX(X)/4+2
      MZERO = MAX0(MA,MB)
C           SET UPPER LIMIT OF M
      MMAX = NTEST
      DO 190 M = MZERO,MMAX,3
      FM1 = 1.0E-28
      FM = .0
      ALPHA = .0
      IF(M-(M/2)*2)120,110,120
  110 JT = -1
      GO TO 130
  120 JT = 1
  130 M2 = M-2
      DO 160 K = 1,M2
      MK = M-K
      B(MK) = F*FLOAT(MK)*FM1-FM
```

```
C          OVERFLOW TEST
      IF(B(MK)-1.0E68)140,220,220
  140 FM = FM1
      FM1 = B(MK)
      JT = -JT
      S = 1+JT
  160 ALPHA = ALPHA+B(MK)*S
      B(1) = F*FM1-FM
      ALPHA = ALPHA+B(1)
      BTEST = B(N1)
      BTEST = BTEST/ALPHA
      IF(ABS(BTEST-BPREV)-ABS(D*BTEST))200,200,190
  190 BPREV = BTEST
  200 DO 210 I = 1,N1
  210 BJ(I) = B(I)/ALPHA
  220 IF(XX.LT.0.0)GO TO 230
      RETURN
  230 N=MT+1
      DO 231 I=1,N
      NN=I-1
  231 BJ(I)=BJ(I)*(-1.0)**NN
      RETURN
      END
```

```
      SUBROUTINE LRKS1(PRMT,Y,DERY,NDIM,FCT,AUX)
      DIMENSION Y(1),DERY(1),AUX(8,2),A(4),B(4),C(4),PRMT(1)
   50 FORMAT(52H MORE THAN 15 BISECTIONS NEEDED IN LRKS1 INTEGRATION)
   51 FORMAT(47H INITIAL INCREMENT IS ZERO ON LRKS1 INTEGRATION )
   52 FORMAT(54H INITIAL INCREMENT HAS WRONG SIGN IN LRKS1 INTEGRATION)
      DO 1 I=1,NDIM
    1 AUX(8,I) = .066666666666667*DERY(I)
      X=PRMT(1)
      XEND=PRMT(2)
      H=PRMT(3)
      CALL FCT(X,Y,DERY)
C
C     ERROR TEST
      IF(H*(XEND-X))38,37,2
C
C     PREPARATIONS FOR RUNGE-KUTTA METHOD
    2 A(1)=.5
      A(2) = .292893218813452
      A(3) = 1.707106781186555
      A(4) = .166666666666667
      B(1)=2.
      B(2)=1.
      B(3)=1.
      B(4)=2.
      C(1)=.5
      C(2) = A(2)
      C(3) = A(3)
      C(4)=.5
C
C     PREPARATIONS OF FIRST RUNGE-KUTTA STEP
      DO 3 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
      AUX(3,I)=0.
    3 AUX(6,I)=0.
      IREC=0
      H=H+H
      IHLF=-1
      ISTEP=0
      IEND=0
C
C
C     START OF A RUNGE-KUTTA STEP
    4 IF((X+H-XEND)*H)7,6,5
    5 H=XEND-X
    6 IEND=1
    7 ITEST = 0
    9 ISTEP=ISTEP+1
C
C
C     START OF INNERMOST RUNGE-KUTTA LOOP
      J=1
   10 AJ=A(J)
      BJ=B(J)
      CJ=C(J)
      DO 11 I=1,NDIM
      R1=H*DERY(I)
      R2=AJ*(R1-BJ*AUX(6,I))
      Y(I)=Y(I)+R2
      R2=R2+R2+R2
```

```
   11 AUX(6,I)=AUX(6,I)+R2-CJ*R1
      IF(J-4)12,15,15
   12 J=J+1
      IF(J-3)13,14,13
   13 X=X+.5*H
   14 CALL FCT(X,Y,DERY)
      GOTO 10
C     END OF INNERMOST RUNGE-KUTTA LOOP
C
C
C     TEST OF ACCURACY
   15 IF(ITEST)16,16,20
C
C     IN CASE ITEST=0 THERE IS NO POSSIBILITY FOR TESTING OF ACCURACY
   16 DO 17 I=1,NDIM
   17 AUX(4,I)=Y(I)
      ITEST=1
      ISTEP=ISTEP+ISTEP-2
   18 IHLF=IHLF+1
      X=X-H
      H=.5*H
      DO 19 I=1,NDIM
      Y(I)=AUX(1,I)
      DERY(I)=AUX(2,I)
   19 AUX(6,I)=AUX(3,I)
      GOTO 9
C
C     IN CASE ITEST=1 TESTING OF ACCURACY IS POSSIBLE
   20 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)21,23,21
   21 CALL FCT(X,Y,DERY)
      DO 22 I=1,NDIM
      AUX(5,I)=Y(I)
   22 AUX(7,I)=DERY(I)
      GOTO 9
C
C     COMPUTATION OF TEST VALUE DELT
   23 DELT=0.
      DO 24 I=1,NDIM
      IF (Y(I)) 242, 241, 242
  241 DELT = DELT + AUX(8,I) *  ABS (AUX(4,I))
      GO TO 24
  242 DELT = DELT + AUX(8,I) *  ABS((AUX(4,I) - Y(I)) / Y(I))
   24 CONTINUE
      IF(DELT-PRMT(4))28,28,25
C
C     ERROR IS TOO GREAT
   25 IF(IHLF-15)26,36,36
   26 DO 27 I=1,NDIM
   27 AUX(4,I)=AUX(5,I)
      ISTEP=ISTEP+ISTEP-4
      X=X-H
      IEND=0
      GOTO 18
C
C     RESULT VALUES ARE GOOD
   28 CALL FCT(X,Y,DERY)
      DO 29 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
```

```
      AUX(3,I)=AUX(6,I)
      Y(I)=AUX(5,I)
   29 DERY(I)=AUX(7,I)
   30 DO 31 I=1,NDIM
      Y(I)=AUX(1,I)
   31 DERY(I)=AUX(2,I)
      IREC=IHLF
      IF(IEND)32,32,40
C
C     INCREMENT GETS DOUBLED
   32 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      IF(IHLF)4,33,33
   33 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)4,34,4
   34 IF(DELT-.02*PRMT(4))35,35,4
   35 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      GOTO 4
C
C
C     RETURNS TO CALLING PROGRAM
   36 WRITE(6,50)
      PRMT(5)=1.0
      GO TO 40
   37 WRITE(6,51)
      GO TO 39
   38 WRITE(6,52)
   39 PRMT(5)=-1.0
   40 RETURN
      END
```

```
      SUBROUTINE LRKS2(PRMT,Y,DERY,NDIM,FCT,AUX)
      DIMENSION Y(1),DERY(1),AUX(8,1),A(4),B(4),C(4),PRMT(1)
   50 FORMAT(52H MORE THAN 15 BISECTIONS NEEDED IN LRKS2 INTEGRATION)
   51 FORMAT(47H INITIAL INCREMENT IS ZERO ON LRKS2 INTEGRATION )
   52 FORMAT(54H INITIAL INCREMENT HAS WRONG SIGN IN LRKS2 INTEGRATION)
      DO 1 I=1,NDIM
    1 AUX(8,I) = .0666666666666667*DERY(I)
      X=PRMT(1)
      XEND=PRMT(2)
      H=PRMT(3)
      CALL FCT(X,Y,DERY)
C
C     ERROR TEST
      IF(H*(XEND-X))38,37,2
C
C     PREPARATIONS FOR RUNGE-KUTTA METHOD
    2 A(1)=.5
      A(2) = .292893218813452
      A(3) = 1.707106781186655
      A(4) = .1666666666666667
      B(1)=2.
      B(2)=1.
      B(3)=1.
      B(4)=2.
      C(1)=.5
      C(2) = A(2)
      C(3) = A(3)
      C(4)=.5
C
C     PREPARATIONS OF FIRST RUNGE-KUTTA STEP
      DO 3 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
      AUX(3,I)=0.
    3 AUX(6,I)=0.
      IREC=0
      H=H+H
      IHLF=-1
      ISTEP=0
      IEND=0
C
C
C     START OF A RUNGE-KUTTA STEP
    4 IF((X+H-XEND)*H)7,6,5
    5 H=XEND-X
    6 IEND=1
    7 ITEST = 0
    9 ISTEP=ISTEP+1
C
C
C     START OF INNERMOST RUNGE-KUTTA LOOP
      J=1
   10 AJ=A(J)
      BJ=B(J)
      CJ=C(J)
      DO 11 I=1,NDIM
      R1=H*DERY(I)
      R2=AJ*(R1-BJ*AUX(6,I))
      Y(I)=Y(I)+R2
      R2=R2+R2+R2
```

```
   11 AUX(6,I)=AUX(6,I)+R2-CJ*R1
      IF(J-4)12,15,15
   12 J=J+1
      IF(J-3)13,14,13
   13 X=X+.5*H
   14 CALL FCT(X,Y,DERY)
      GOTO 10
C     END OF INNERMOST RUNGE-KUTTA LOOP
C
C
C     TEST OF ACCURACY
   15 IF(ITEST)16,16,20
C
C     IN CASE ITEST=0 THERE IS NO POSSIBILITY FOR TESTING OF ACCURACY
   16 DO 17 I=1,NDIM
   17 AUX(4,I)=Y(I)
      ITEST=1
      ISTEP=ISTEP+ISTEP-2
   18 IHLF=IHLF+1
      X=X-H
      H=.5*H
      DO 19 I=1,NDIM
      Y(I)=AUX(1,I)
      DERY(I)=AUX(2,I)
   19 AUX(6,I)=AUX(3,I)
      GOTO 9
C
C     IN CASE ITEST=1 TESTING OF ACCURACY IS POSSIBLE
   20 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)21,23,21
   21 CALL FCT(X,Y,DERY)
      DO 22 I=1,NDIM
      AUX(5,I)=Y(I)
   22 AUX(7,I)=DERY(I)
      GOTO 9
C
C     COMPUTATION OF TEST VALUE DELT
   23 DELT=0.
      DO 24 I=1,NDIM
      IF (Y(I)) 242, 241, 242
  241 DELT = DELT + AUX(8,I) *  ABS (AUX(4,I))
      GO TO 24
  242 DELT = DELT + AUX(8,I) *  ABS((AUX(4,I) - Y(I)) / Y(I))
   24 CONTINUE
C
      IF(DELT-PRMT(4))28,28,25
C     ERROR IS TOO GREAT
   25 IF(IHLF-15)26,36,36
   26 DO 27 I=1,NDIM
   27 AUX(4,I)=AUX(5,I)
      ISTEP=ISTEP+ISTEP-4
      X=X-H
      IEND=0
      GOTO 18
C
C     RESULT VALUES ARE GOOD
   28 CALL FCT(X,Y,DERY)
      DO 29 I=1,NDIM
      AUX(1,I)=Y(I)
      AUX(2,I)=DERY(I)
```

```
      AUX(3,I)=AUX(6,I)
      Y(I)=AUX(5,I)
   29 DERY(I)=AUX(7,I)
   30 DO 31 I=1,NDIM
      Y(I)=AUX(1,I)
   31 DERY(I)=AUX(2,I)
      IREC=IHLF
      IF(IEND)32,32,40
C
C     INCREMENT GETS DOUBLED
   32 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      IF(IHLF)4,33,33
   33 IMOD=ISTEP/2
      IF(ISTEP-IMOD-IMOD)4,34,4
   34 IF(DELT-.02*PRMT(4))35,35,4
   35 IHLF=IHLF-1
      ISTEP=ISTEP/2
      H=H+H
      GOTO 4
C
C
C     RETURNS TO CALLING PROGRAM
   36 WRITE(6,50)
      PRMT(5)=1.0
      GO TO 40
   37 WRITE(6,51)
      GO TO 39
   38 WRITE(6,52)
   39 PRMT(5)=-1.0
   40 RETURN
      END
```

```
      SUBROUTINE CXINV(A,N,B,M,DET,IPIV,INDX,MAX,ISCALE)
C
C        COMPLEX MATRIX INVERSION WITH SOLUTION OF LINEAR EQUATIONS
C
C        CAVM = CABS(A(MAX)),   CAVA = CABS(A(I,J))
C        CADM = CABS(DETERM),   CAPV = CABS(PIVOT)
C
      COMPLEX A(MAX,N), B(MAX,M), SWAP, DET, PIV, PIVI, C0, C1
      DIMENSION IPIV(N), INDX(MAX,2)
C
C        CONSTANTS, INITIALIZATION
C
      C0 = (0.0,0.0)
      C1 = (1.0,0.0)
      ISCALE = 0
      RL = 10.0**100
      RS = 1.0/RL
      DET = C1
      CADM = 1.0
      DO 20 J=1,N
   20 IPIV(J) = 0
      DO 500 I=1,N
C
C        SEARCH FOR PIVOT ELEMENT
C
      CAVM = 0.0
      DO 105 J=1,N
      IF (IPIV(J) .EQ. 1) GO TO 105
      DO 100 K=1,N
      IF (IPIV(K) - 1) 50,100,750
   50 CONTINUE
      CAVA = CABS(A(J,K))
      IF (CAVM .GE. CAVA) GO TO 100
      IROW = J
      ICOL = K
      CAVM = CAVA
  100 CONTINUE
  105 CONTINUE
      IF (CAVM .EQ. 0.0) GO TO 720
      IPIV(ICOL) = IPIV(ICOL) + 1
C
C        INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
      IF (IROW .EQ. ICOL) GO TO 230
      DET = -DET
      DO 200 L=1,N
      SWAP = A(IROW,L)
      A(IROW,L) = A(ICOL,L)
      A(ICOL,L) = SWAP
  200 CONTINUE
      IF (M .LE. 0) GO TO 230
      DO 220 L=1,M
      SWAP = B(IROW,L)
      B(IROW,L) = B(ICOL,L)
      B(ICOL,L) = SWAP
  220 CONTINUE
  230 CONTINUE
      INDX(I,1) = IROW
      INDX(I,2) = ICOL
      PIV = A(ICOL,ICOL)
```

```
      CAPV = CABS(PIV)
      IF (CAPV .EQ. 0.0) GO TO 720
C
C        SCALE DETERMINANT
C
      PIVI = PIV
      CADM = CABS(DET)
      IF (CADM  .LT. RL ) GO TO 260
      DET = DET/RL
      CADM = CABS(DET)
      ISCALE = ISCALE + 1
      IF (CADM .LT. RL ) GO TO 290
      DET = DET/RL
      ISCALE = ISCALE + 1
      GO TO 290
  260 CONTINUE
      IF (CADM .GT. RS) GO TO 290
      DET = DET*RL
      CADM = CABS(DET)
      ISCALE = ISCALE - 1
      IF (CADM .GT. RS) GO TO 290
      DET = DET*RL
      ISCALE = ISCALE - 1
  290 CONTINUE
      CAPV = CABS(PIVI)
      IF (CAPV .LT. RL) GO TO 320
      PIVI = PIVI/RL
      CAPV = CABS(PIVI)
      ISCALE = ISCALE + 1
      IF (CAPV .LT. RL) GO TO 340
      PIVI = PIVI/RL
      ISCALE = ISCALE + 1
      GO TO 340
  320 CONTINUE
      IF (CAPV .GT. RS) GO TO 340
      PIVI = PIVI*RL
      CAPV = CABS(PIVI)
      ISCALE = ISCALE - 1
      IF (CAPV .GT. RS) GO TO 340
      PIVI = PIVI*RL
      ISCALE = ISCALE - 1
  340 CONTINUE
      DET = DET * PIVI
C
C        DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
      A(ICOL,ICOL) = C1
      DO 350 L=1,N
  350 A(ICOL,L) = A(ICOL,L)/PIV
      IF (M .LE. 0) GO TO 380
      DO 370 L=1,M
  370 B(ICOL,L) = B(ICOL,L)/PIV
C
C        REDUCE NON-PIVOT ROWS
C
  380 CONTINUE
      DO 500 L1=1,N
      IF (L1 .EQ. ICOL) GO TO 500
      SWAP = A(L1,ICOL)
      A(L1,ICOL) = C0
```

```
      DO 400 L=1,N
  400 A(L1,L) = A(L1,L) - A(ICOL,L)*SWAP
      IF (M .LE. 0) GO TO 500
      DO 450 L=1,M
  450 B(L1,L) = B(L1,L) - B(ICOL,L)*SWAP
  500 CONTINUE
C
C         INTERCHANGE COLUMNS
C
      DO 700 I=1,N
      L = N+1-I
      IF (INDX(L,1) .EQ. INDX(L,2))GO TO 700
      IROW = INDX(L,1)
      ICOL = INDX(L,2)
      DO 690 K=1,N
      SWAP = A(K,IROW)
      A(K,IROW) = A(K,ICOL)
      A(K,ICOL) = SWAP
  690 CONTINUE
  700 CONTINUE
      GO TO 750
  720 DET = C0
    . ISCALE = 0
  750 RETURN
      END
```

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