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SHEAR-STRESS INTENSITY FACTORS FOR ELASTIC SHEETS WITH COVER PLATES

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SUMMARY

Shear-stress intensity factors are calculated for three problems concerning inextensible cover plates either bonded to or embedded in an elastic sheet which is under uniaxial tension. The stress intensity factors are small when the ratio of sheet thickness to cover-plate length is small and, as the ratio increases, rapidly approach their asymptotic values for infinite sheet thickness. In the problem of the embedded cover plate, the stress intensity factor also depends on the Poisson's ratio of the sheet material. The dependence on Poisson's ratio, however, is significant only when the ratio of sheet thickness to cover-plate length is small. Some possible implications of the present results for debonding of reinforced sheets under cyclic loading are briefly discussed.

INTRODUCTION

The bonding of dissimilar materials is increasingly used in aerospace construction. High-modulus composite materials, for example, are being bonded to selected surfaces of some conventional metal aircraft structural components to take advantage of the composite's higher ratio of strength (or stiffness) to weight and its usually superior fatigue resistance. This practice, however, can also have its disadvantages, one being the potentially severe stress concentrations which can occur near the edges of the interface between metal and composite. In conventional structures, stress concentrations often cause fatigue cracks; in bonded structures, they often cause progressive delamination and debonding. Both types of damage can occur at nominal stress levels far below the ultimate static strength of the structure.

Stress intensity analyses, on which the fracture mechanics discipline leans heavily, have been performed on a variety of bonded configurations weakened by cracks (see, for example, refs. 1 to 7). However, problems of singular stress fields in bonded materials not weakened by flaws have received considerably less attention. One class of such problems, collectively referred to as the cover-plate problem, is concerned with the analysis of the stress field in an elastic body with one or more thin plates bonded to its surfaces.

The elastic body is generally subjected to remote loads. The thin plates might be extensible or inextensible.

In reference 8, two such problems were studied; one related to a single cover plate on a half plane, and the other to a line inclusion in an infinite plane. The studies included the effects of various symmetric and antisymmetric loading conditions. Little attention, however, was given to the stress singularities which occur at the ends of the interface.

In reference 9, the general problem for an elastic half plane with an arbitrary number of cover plates was formulated, and the basic integral equation for periodic cover plates was obtained. The problem of inextensible cover plates was solved in closed form. Numerical solutions were obtained for the problems of one, two, and a periodic array of identical elastic cover plates. In all cases the contact shear stresses exhibited the inverse square root singularity at the ends of the interface and, hence, facilitated the presentation of results in terms of shear-stress intensity factors.

A potentially significant effect not investigated in reference 9 is that of finite thickness of the elastic body to which the cover plates are bonded. This effect is studied in the present paper through the analysis of three problems involving cover plates bonded to an elastic sheet under remote uniaxial tension: (1) a single cover plate, (2) identical cover plates on opposite surfaces, and (3) a thin plate embedded in the midplane of the elastic sheet. In each case, the plate, or plates, are assumed to be perfectly bonded, completely flexible in bending and, to isolate the thickness effect, inextensible. Results are presented in the form of curves of shear-stress intensity factor as a function of the ratio of sheet thickness to cover-plate length. The shear-stress intensity factors are normalized with respect to their limiting values from the corresponding solution for the infinitely thick sheet.

SYMBOLS

A,B,C,D	unknown coefficients in general solution to transformed biharmonic equation
A_m,A_n	series coefficients
a	half-length of cover plate
E	Young's modulus of elasticity
$f(\xi)$	unknown function in dual integral equations
G	modulus of shear rigidity

h	ratio of sheet thickness to cover-plate half-length for problem I and cover-plate length for problems II and III
K	shear-stress intensity factor
K_∞	shear-stress intensity factor for the infinitely thick sheet
k	arbitrary real positive number in solution of dual integral equations
$L_{m,n}$	coefficients in algebraic equations
m,n	integers
P_m,P_n	normalized series coefficients
u,v	dimensionless displacements
x,y	dimensionless rectangular Cartesian coordinates
$\epsilon_x,\epsilon_y,\gamma_{xy}$	elastic strains
μ	Poisson's ratio
ξ	transform variable
σ	dimensionless applied stress
$\sigma_x,\sigma_y,\sigma_{xy}$	dimensionless stresses
ϕ	dimensionless Airy stress function
∇^4	biharmonic operator, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$

Primes denote dimensional quantities. Bars denote transformed quantities.

ANALYSIS

The problems are analyzed under the plane stress assumptions. Of course, the results are easily convertible to the plane strain cases by appropriate redefinition of the elastic constants. The problems are identified in the following way (see fig. 1):

Problem I – Elastic sheet of thickness ah, under remote tension $E\sigma$, with one cover plate of length 2a.

Problem II – Elastic sheet of thickness 2ah, under remote tension $E\sigma$, with cover plates of length 2a symmetrically placed on opposite surfaces.

Problem III – Elastic sheet of thickness 2ah, under remote tension $E\sigma$, with one "cover" plate of length 2a embedded in its midplane.

The three problems are analyzed similarly. Therefore, the full details of only the first problem are presented here, the details of problems II and III being relegated to the appendix.

Under the plane stress assumptions, the following system of equations governs the three problems under consideration:

$$\nabla^4\phi'(x',y') = 0 \tag{1}$$

where

$$\left.\begin{aligned}\sigma_x' &= \frac{\partial^2\phi'}{\partial y'^2}\\ \sigma_y' &= \frac{\partial^2\phi'}{\partial x'^2}\\ \sigma_{xy}' &= -\frac{\partial^2\phi'}{\partial x'\,\partial y'}\end{aligned}\right\} \tag{2}$$

with the stress-strain relations

$$\left.\begin{aligned}\epsilon_x' &= \frac{1}{E}(\sigma_x' - \mu\sigma_y')\\ \epsilon_y' &= \frac{1}{E}(\sigma_y' - \mu\sigma_x')\\ \gamma_{xy}' &= \frac{\sigma_{xy}'}{G}\end{aligned}\right\} \tag{3}$$

and the strain-displacement relations

$$\left.\begin{aligned}
\epsilon_x' &= \frac{\partial u'}{\partial x'} \\
\epsilon_y' &= \frac{\partial v'}{\partial y'} \\
\gamma_{xy}' &= \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}
\end{aligned}\right\} \tag{4}$$

The following substitutions yield nondimensional variables:

$$\left.\begin{aligned}
&\phi' = Ea^2\phi && \epsilon_x' = \epsilon_x && u' = au \\
&\sigma_x' = E\sigma_x && \epsilon_y' = \epsilon_y && v' = av \\
&\sigma_y' = E\sigma_y && \gamma_{xy}' = \gamma_{xy} && x' = ax \\
&\sigma_{xy}' = E\sigma_{xy} && && y' = ay
\end{aligned}\right\} \tag{5}$$

Equations (1) to (4) then assume the following forms:

$$\nabla^4\phi(x,y) = 0 \tag{6}$$

where

$$\left.\begin{aligned}
\sigma_x &= \frac{\partial^2\phi}{\partial y^2} \\
\sigma_y &= \frac{\partial^2\phi}{\partial x^2} \\
\sigma_{xy} &= -\frac{\partial^2\phi}{\partial x\,\partial y}
\end{aligned}\right\} \tag{7}$$

with

$$\left.\begin{aligned}
\epsilon_x &= \sigma_x - \mu\sigma_y \\
\epsilon_y &= \sigma_y - \mu\sigma_x \\
\gamma_{xy} &= 2(1 + \mu)\sigma_{xy}
\end{aligned}\right\} \tag{8}$$

and

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \end{aligned}\right\} \tag{9}$$

In terms of the nondimensional coordinates, the cover plates are of length 2 and the elastic sheet is of thickness h in problem I and $2h$ in problems II and III.

It is convenient to decompose the problem into two subproblems. The first involves only the nominal stress field in the elastic sheet and its solution is elementary. The second involves the disturbance to the nominal stress field caused by the cover plate and its solution is of primary interest. With the nominal stresses removed by superposition, the only external loads in the disturbance problem are the contact shear stresses which prevent tangential displacement at the presumed location of each cover plate. The superposition procedure will be illustrated in detail in the analysis of problem I.

The configuration is well suited to the use of the Fourier transform. With the Fourier transform of a function defined by (see, for example, ref. 10)

$$\bar{f}(\xi,y) = \int_{-\infty}^{\infty} f(x,y)e^{-i\xi x}\,dx \tag{10}$$

and inversely

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\xi,y)e^{i\xi x}\,d\xi \tag{11}$$

its application to equation (6) yields

$$\left(\frac{d^2}{dy^2} - \xi^2\right)^2 \bar{\phi}(\xi,y) = 0 \tag{12}$$

it being assumed that stresses and displacements vanish as $|x| \to \infty$. The general solution to equation (12) is

$$\bar{\phi}(\xi,y) = (A + By)e^{-|\xi|y} + (C + Dy)e^{|\xi|y} \tag{13}$$

where A, B, C, and D are unknown functions of ξ. In terms of $\bar{\phi}$, the transformed stresses and displacements are

$$\left.\begin{aligned} \bar{\sigma}_x &= \frac{d^2\bar{\phi}}{dy^2} \\ \bar{\sigma}_y &= -\xi^2\bar{\phi} \\ \bar{\sigma}_{xy} &= -i\xi\frac{d\bar{\phi}}{dy} \end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned} \bar{u} &= \frac{1}{i\xi}\left(\frac{d^2\bar{\phi}}{dy^2} + \mu\xi^2\bar{\phi}\right) \\ \bar{v} &= \frac{1}{\xi^2}\left[\frac{d^3\bar{\phi}}{dy^3} - (2+\mu)\xi^2\frac{d\bar{\phi}}{dy}\right] \end{aligned}\right\} \tag{15}$$

Problem I

The problem of the remotely loaded elastic sheet with a single inextensible cover plate has the boundary conditions

$$\left.\begin{aligned} &\sigma_y(x,h) = 0 \\ &\sigma_{xy}(x,h) = 0 \\ &\sigma_y(x,0) = 0 \\ &u(x,0) = 0 \qquad (|x| < 1) \\ &\sigma_{xy}(x,0) = 0 \qquad (|x| > 1) \end{aligned}\right\} \tag{16}$$

and

$$\sigma_x \to \sigma; \quad \sigma_y, \sigma_{xy} \to 0 \qquad \text{as} \qquad |x| \to \infty$$

Subtraction of the nominal uniaxial stress field due to the remote loading yields the residual problem which has boundary conditions

$$\left.\begin{aligned}
&\sigma_y(x,h) = 0 \\
&\sigma_{xy}(x,h) = 0 \\
&\sigma_y(x,0) = 0 \\
&u(x,0) = -\sigma x && (|x| < 1) \\
&\sigma_{xy}(x,0) = 0 && (|x| > 1)
\end{aligned}\right\} \tag{17}$$

where

$$\sigma_x, \sigma_y, \sigma_{xy} \to 0 \qquad \text{as} \qquad |x| \to \infty$$

Hence, in the residual problem, the only external tractions on the sheet are the unknown shear stresses applied by the cover plate.

In terms of the transformed stress function, equations (17) become

$$\left.\begin{aligned}
&\bar{\phi}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\bar{\phi}(\xi,0) = 0 \\
&\sigma x = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{d^2\bar{\phi}}{dy^2}(\xi,0)\, e^{i\xi x}\, \frac{d\xi}{\xi} && (|x| < 1) \\
&\text{and} \\
&0 = \int_{-\infty}^{\infty} \xi \frac{d\bar{\phi}}{dy}(\xi,0)\, e^{i\xi x}\, d\xi && (|x| > 1)
\end{aligned}\right\} \tag{18}$$

Substitution of the general solution (13) into the first three of equations (18) gives

$$\left.\begin{aligned}
&A = -C = \frac{2\xi h^2 D}{\Delta} \\
&B = \frac{D}{\Delta}\left(e^{2\xi h} - 2\xi h - 1\right)
\end{aligned}\right\} \tag{19}$$

where

$$\Delta = e^{-2\xi h} + 2\xi h - 1$$

Using equations (19), the remaining (mixed) boundary condition becomes

$$-\sigma x = \frac{2i}{\pi} = \int_{-\infty}^{\infty} \frac{D}{\Delta}(\sinh 2\xi h - 2\xi h)\, e^{i\xi x}\, d\xi \qquad (|x| < 1)$$

$$0 = \int_{-\infty}^{\infty} \frac{\xi D}{\Delta}(\cosh 2\xi h - 2\xi^2 h^2 - 1)\, e^{i\xi x}\, d\xi \qquad (|x| > 1)$$

Symmetry considerations reduce these to the following, to be solved for D:

$$\left.\begin{aligned} \frac{\pi\sigma x}{4} &= \int_{-\infty}^{\infty} \frac{D}{\Delta}(\sinh 2\xi h - 2\xi h) \sin \xi x\, d\xi \qquad (|x| < 1) \\ 0 &= \int_{0}^{\infty} \frac{\xi D}{\Delta}(\cosh 2\xi h - 2\xi^2 h^2 - 1) \sin \xi x\, d\xi \qquad (|x| > 1) \end{aligned}\right\} \quad (20)$$

With the identity

$$\sin \xi x = \left(\frac{\pi \xi x}{2}\right)^{1/2} J_{1/2}(\xi x)$$

and the substitution

$$f(\xi) = \frac{\xi^{3/2} D}{\Delta}(\cosh 2\xi h - 2\xi^2 h^2 - 1) \qquad (21)$$

the dual integral equations become

$$\left.\begin{aligned} \frac{\sigma}{2}\left(\frac{\pi x}{2}\right)^{1/2} &= \int_{0}^{\infty} f(\xi) \left\{\frac{\sinh 2\xi h - 2\xi h}{\cosh 2\xi h - 2\xi^2 h^2 - 1}\right\} J_{1/2}(\xi x)\, d\xi \qquad (|x| < 1) \\ 0 &= \int_{0}^{\infty} f(\xi) J_{1/2}(\xi x)\, d\xi \qquad (|x| > 1) \end{aligned}\right\} \quad (22)$$

Following a method presented in reference 11, it is assumed that

$$f(\xi) = \xi^{1-k} \sum_{m=0}^{\infty} A_m J_{\frac{1}{2}+2m+k}(\xi) \qquad (23)$$

in which k is required only to be real and positive, and the A_m's are to be determined. This form for $f(\xi)$ satisfies the second of equations (22) automatically. Further application of the method of reference 11 yields the following infinite set of simultaneous linear algebraic equations to be solved for the A_m's:

$$\left.\begin{aligned} & A_0 + \sum_{m=0}^{\infty} L_{m,0}A_m = \frac{\sigma\pi}{2}\frac{2^{-k}}{\Gamma\left(k+\frac{1}{2}\right)} \\ & A_n + \sum_{m=0}^{\infty} L_{m,n}A_m = 0 \qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{24}$$

where

$$L_{m,n} = (4n + 2k + 1)\int_0^{\infty}\left\{\xi^{1-2k}\left(\frac{\sinh 2\xi h - 2\xi h}{\cosh 2\xi h - 2\xi^2h^2 - 1} - 1\right)\right\}J_{2m+k+\frac{1}{2}}(\xi)J_{2n+k+\frac{1}{2}}(\xi)\frac{d\xi}{\xi} \tag{25}$$

With the choice $k = 1/2$ and the substitution $A_m = \frac{\sigma\pi}{4}P_m$, equations (24) and (25) become

$$\left.\begin{aligned} & P_0 + \sum_{m=0}^{\infty} L_{m,0}P_m = 1 \\ & P_n + \sum_{m=0}^{\infty} L_{m,n}P_m = 0 \qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{26}$$

and

$$L_{m,n} = 2(2n + 1)\int_0^{\infty}\left\{\frac{\sinh 2\xi h - 2\xi h}{\cosh 2\xi h - 2\xi^2h^2 - 1} - 1\right\}J_{2m+1}(\xi)\,J_{2n+1}(\xi)\,\frac{d\xi}{\xi} \tag{27}$$

Of particular interest is the shear stress along the sheet—cover-plate interface. It is given by

$$\sigma_{xy}(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\sigma}_{xy}(\xi,0)e^{i\xi x}\,d\xi$$

or, in terms of $\bar{\phi}$,

$$\sigma_{xy}(x,0) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\xi\frac{d\bar{\phi}}{dy}(\xi,0)e^{i\xi x}\,d\xi \tag{28}$$

Subsitituting equations (19) into equation (13), then using equations (21) and (23) (remembering that $k = 1/2$ and $A_m = \frac{\sigma\pi}{4} P_m$), equation (28) becomes, after appropriate symmetry considerations,

$$\sigma_{xy}(x,0) = \frac{\sigma}{2} \sum_{m=0}^{\infty} P_m \int_0^{\infty} J_{2m+1}(\xi)\sin \xi x \, d\xi \tag{29}$$

Evaluation of the integral (see, for example, ref. 12) yields

$$\sigma_{xy}(x,0) = \begin{cases} \dfrac{\sigma}{2} \displaystyle\sum_{m=0}^{\infty} P_m \dfrac{\sin\left[(2m+1)\sin^{-1}x\right]}{\sqrt{1-x^2}} & (|x| < 1) \\ 0 & (|x| > 1) \end{cases} \tag{30}$$

In order to study the effect of sheet thickness on stress intensity, it is convenient to examine the ratio of the interface shear stresses for the sheet and the half-space. The interface shear stress for the half-space can be obtained from reference 9. With some changes in notation, it can be written as

$$\sigma_{xy}(x,0) = \begin{cases} \dfrac{\sigma}{2} \dfrac{x}{\sqrt{1-x^2}} & (|x| < 1) \\ 0 & (|x| > 1) \end{cases} \tag{31}$$

Forming the ratio of equations (30) to (31) (for $|x| < 1$) and proceeding to the limit as $|x| \to 1$ yields the ratio of shear-stress intensity factors

$$\frac{K}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m \tag{32}$$

The P_m's are obtained by solving a suitably truncated set of equations (26), after calculation of the necessary coefficients from equation (27) by numerical integration. The number of equations (26) required for accurate determination of $\frac{K}{K_\infty}$ depends on the value of h. When h is large the coefficient matrix in equations (26) is diagonally dominant; hence, few equations are required. As h is reduced, the required number of equations increases. For example, three equations were sufficient for $h = 5$, while 14 were required for $h = 0.1$. This was true for all three problems.

Problem II

In this problem, two identical cover plates are attached to the opposite surfaces of the elastic sheet, which is now of dimensionless thickness $2h$. (See fig. 1(b).) After subtraction of the nominal uniaxial stress field, the problem has the following boundary conditions:

$$\sigma_y(x,0) = \sigma_y(x,2h) = 0$$

$$u(x,0) = u(x,2h) = -\sigma x \qquad (|x| < 1)$$

$$\sigma_{xy}(x,0) = \sigma_{xy}(x,2h) = 0 \qquad (|x| > 1)$$

Because of symmetry about the midplane of the sheet, only the lower half of the sheet ($0 \leqq y \leqq h$) need be considered. Incorporation of the additional boundary conditions arising from symmetry considerations yields the boundary conditions

$$\left.\begin{aligned}
&\sigma_{xy}(x,h) = 0 \\
&v(x,h) = 0 \\
&\sigma_y(x,0) = 0 \\
&u(x,0) = -\sigma x \qquad (|x| < 1) \\
&\sigma_{xy}(x,0) = 0 \qquad (|x| > 1)
\end{aligned}\right\} \tag{33}$$

By the analytical procedure employed in problem I, the interface shear stress is found (see appendix) again to have the form

$$\sigma_{xy}(x,0) = \left\{\begin{array}{ll}
\dfrac{\sigma}{2} \displaystyle\sum_{m=0}^{\infty} P_m \frac{\sin\left[(2m+1)\sin^{-1}x\right]}{\sqrt{1-x^2}} & (|x| < 1) \\
0 & (|x| > 1)
\end{array}\right\} \tag{34}$$

The P_m's again are obtained from equations (26), which now have coefficients given by

$$L_{m,n} = 2(2n + 1) \int_0^\infty \left\{ \frac{\cosh 2\xi h + 1}{\sinh 2\xi h + 2\xi h} - 1 \right\} J_{2m+1}(\xi) J_{2n+1}(\xi) \frac{d\xi}{\xi} \tag{35}$$

The ratio of shear-stress intensity factors has the same form as in problem I:

$$\frac{K}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

Of course, the P_m's are generally different from those of problem I.

Problem III

In this problem the inextensible zero-thickness plate is embedded in the midplane of the sheet. (See fig. 1(c).) After subtraction of the nominal uniaxial stress field, the boundary conditions along the free surfaces are

$$\sigma_{xy}(x,\pm h) = \sigma_y(x,\pm h) = 0$$

and, at the interface,

$$u(x,0) = -\sigma x \qquad (|x| < 1)$$

With symmetry taken into account, and with attention restricted to the upper half of the sheet, the complete boundary conditions are

$$\left.\begin{aligned}
&\sigma_y(x,h) = 0 \\
&\sigma_{xy}(x,h) = 0 \\
&v(x,0) = 0 \\
&u(x,0) = -\sigma x \qquad && (|x| < 1) \\
&\sigma_{xy}(x,0) = 0 \qquad && (|x| > 1)
\end{aligned}\right\} \tag{36}$$

After an analysis similar to those of problems I and II (see appendix for details), the ratio of shear-stress intensity factors again has the form

$$\frac{K}{K_\infty} \sum_{m=0}^{\infty} (-1)^m P_m$$

Again, the P_m's are different from those in the preceding solutions. The P_m's are calculated from equations (26), whose coefficients are now obtained by evaluating

$$L_{m,n} = 2(2n + 1) \int_0^\infty \left\{ \frac{\cosh 2\xi h + 2\left(\frac{1 + \mu}{3 - \mu}\right)\xi^2 h^2 + \frac{5 - 2\mu + \mu^2}{(1 + \mu)(3 - \mu)}}{\sinh 2\xi h + 2\xi h} - 1 \right\} J_{2m+1}(\xi) J_{2n+1}(\xi) \frac{d\xi}{\xi} \tag{37}$$

Note that, in contrast to problems I and II, the values of $L_{m,n}$ (hence, the ratios of the stress intensity factors) depend on Poisson's ratio.

RESULTS AND DISCUSSION

Normalized interface shear-stress intensity factors are listed in table 1 for various values of h from 0.1 to 10. The results are also plotted in figure 2 for values from 0.1 to 5.

Similar trends are noted in all three solutions. Stress intensity factors are low when h is small, and they initially increase rapidly as h increases. The initial rate of increase is especially rapid in problem II, where the two cover plates combine to cause a considerably more severe state of shear stress, especially at lower values of h.

In contrast, the initial rate of increase of $\frac{K}{K_\infty}$ with h is substantially lower in problem I. This lower rate of increase could be interpreted as a consequence of the choice of h, rather than 2h, for the sheet thickness; however, replotting the solution to problem I with h replaced by 2h would still yield a $\frac{K}{K_\infty}$ curve below those for the other two solutions. A comparison of the $\frac{K}{K_\infty}$ curves for problems I and II, then, suggests that when metal structures are reinforced with composite overlays, an overlay on one surface should cause a less severe state of shear stress in the sheet than comparable overlays on both surfaces. This assumes, of course, that an overlay on one surface satisfies all other design requirements, for example, minimum bending stiffness.

The trends of all three solutions may have implications for the design of composite-reinforced metal structures, since they indicate that longer cover plates (or overlays) cause less severe states of shear stress. On the assumption that debond initiation under cyclic loading is related to shear-stress intensity, the results suggest that debond initiation should be delayed by longer overlays (smaller h).

In problem III, that of the sheet with the embedded "cover" plate, the stress intensity factors depend on the Poisson's ratio of the sheet material. As can be seen in table 1 and figure 2, the dependence is essentially negligible for $h > 1$. The stress intensity factors for different Poisson's ratios, however, begin to differ significantly as h falls below 1, indicating the potential importance of Poisson's ratio in composite-reinforced metal structures.

CONDLUDING REMARKS

Interface shear-stress intensity factors have been calculated for three problems involving inextensible cover plates either perfectly bonded to or embedded in elastic sheets under uniaxial tension. In all three problems the stress intensity factors are small when the ratio of sheet thickness to cover-plate length is small. With increases in the ratio they rapidly approach their asymptotic values for infinite sheet thickness. This trend is less pronounced in the single-cover-plate problem, leading to the observation that a single cover plate might be less likely to debond under cyclic loading than a pair of comparable cover plates on the opposite surfaces of the sheet.

The stress intensity factor for the problem of an embedded cover plate depends on the Poisson's ratio of the sheet material.

Limitations of the present solution method preclude accurate and reasonably economical calculations for h much smaller than 0.1. The present results are not applicable to the problem of the behavior of the stress intensity factors for vanishingly small ratios of sheet thickness to cover-plate length. This appears to be a potentially fruitful area for additional research, since the reinforcement of metal structures with composite overlays usually involves very small ratios of sheet thickness to cover-plate (composite overlay) length.

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APPENDIX

ANALYSIS DETAILS OF PROBLEMS II AND III

Problem II – The configuration is shown in figure 1(b). After subtraction of the nominal uniaxial stress field and restriction of attention to the upper half of the sheet, the boundary conditions are

$$\left.\begin{aligned}
&\sigma_{xy}(x,h) = 0 & & \\
&v(x,h) = 0 & & \\
&\sigma_y(x,0) = 0 & & \\
&u(x,0) = -\sigma x & &(|x| < 1) \\
&\sigma_{xy}(x,0) = 0 & &(|x| > 1)
\end{aligned}\right\} \tag{A1}$$

In terms of the transformed stress function, these become

$$\left.\begin{aligned}
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 & & \\
&\frac{d^3\bar{\phi}}{dy^3}(\xi,h) = 0 & & \\
&\bar{\phi}(\xi,0) = 0 & & \\
&-\sigma x = \frac{i}{\pi}\int_0^\infty \frac{d^2\bar{\phi}}{dy^2}(\xi,0)\sin \xi x \frac{d\xi}{\xi} & &(|x| < 1) \\
&0 = \int_0^\infty \xi \frac{d\bar{\phi}}{dy}(\xi,0)\sin \xi x \, d\xi & &(|x| > 1)
\end{aligned}\right\} \tag{A2}$$

Substitution of the general solution (13) into the first three of equations (A2) and solving for three of the constants of integration in terms of the fourth gives

$$\left.\begin{aligned} A &= -C = \frac{2hD}{1 + e^{-2\xi h}} \\ B &= -De^{2\xi h} \end{aligned}\right\} \tag{A3}$$

By virtue of equations (A3), the remaining boundary condition becomes

$$\left.\begin{aligned} \frac{-\sigma\pi x}{2} &= \int_0^\infty D\left(1 + e^{2\xi h}\right) \sin \xi x \, d\xi && (|x| < 1) \\ 0 &= \int_0^\infty \xi D \left\{\frac{\sinh 2\xi h + 2\xi h}{1 + e^{-2\xi h}}\right\} \sin \xi x \, d\xi && (|x| > 1) \end{aligned}\right\} \tag{A4}$$

With the identity

$$\sin \xi x = \left(\frac{\pi\xi x}{2}\right)^{1/2} J_{1/2}(\xi x)$$

and the substitution

$$f(\xi) = \xi^{3/2} D \left\{\frac{\sinh 2\xi h + 2\xi h}{1 + e^{-2\xi h}}\right\} \tag{A5}$$

equations (A4) become

$$\left.\begin{aligned} -\frac{\sigma}{2}\left(\frac{\pi x}{2}\right)^{1/2} &= \int_0^\infty \left\{\frac{\cosh 2\xi h + 1}{\sinh 2\xi h + 2\xi h}\right\} f(\xi) J_{1/2}(\xi x) \frac{d\xi}{\xi} && (|x| < 1) \\ 0 &= \int_0^\infty f(\xi) J_{1/2}(\xi x) \, d\xi && (|x| > 1) \end{aligned}\right\} \tag{A6}$$

As in problem I, $f(\xi)$ is assumed to have the form given by equation (23), which automatically satisfies the second of equations (A6) for k real and positive. With the choice $k = 1/2$, further application of the method of reference 11 to the remaining equation gives

$$A_0 + \sum_{m=0}^{\infty} L_{m,0} A_m = -\frac{\sigma\pi}{4}$$

$$A_n + \sum_{m=0}^{\infty} L_{m,n} A_m = 0 \qquad (n = 1, 2, 3, \ldots)$$

which, with the substitution $A_m = \frac{-\sigma\pi}{4} P_m$, become

$$\left.\begin{aligned} &P_0 + \sum_{m=0}^{\infty} L_{m,0} P_m = 1 \\ &P_n + \sum_{m=0}^{\infty} L_{m,n} P_m = 0 \qquad\qquad (n = 1, 2, 3, \ldots) \end{aligned}\right\} \tag{A7}$$

where

$$L_{m,n} = 2(2n + 1) \int_0^{\infty} \left\{ \frac{\cosh 2\xi h + 1}{\sinh 2\xi h + 2\xi h} - 1 \right\} J_{2m+1}(\xi) J_{2n+1}(\xi) \frac{d\xi}{\xi} \tag{A8}$$

A truncated set of equations (A7) is solved after specification of h and calculation of the coefficients from equation (A8) by numerical integration.

The transformed shear stress along the line $y = 0$ is

$$\bar{\sigma}_{xy}(\xi,0) = -\frac{i\sigma\pi}{2} \sum_{m=0}^{\infty} P_m J_{2m+1}(\xi)$$

Inverse transformation gives

$$\sigma_{xy}(x,0) = \left\{ \begin{array}{ll} \dfrac{\sigma}{2} \displaystyle\sum_{m=0}^{\infty} P_m \dfrac{\sin\left[(2m + 1)\sin^{-1}x\right]}{\sqrt{1 - x^2}} & (|x| < 1) \\ 0 & (|x| > 1) \end{array} \right\} \tag{A9}$$

The ratio of shear-stress intensity factors for the sheet and the similarly loaded half-space again has the form

$$\frac{K}{K_{\infty}} = \sum_{m=0}^{\infty} (-1)^m P_m$$

where the P_m's are, in general, different from those of problem I.

Problem III – The configuration is shown in figure 1(c). The inextensible "cover" plate is embedded in the middle surface of the elastic sheet. After subtraction of the nominal uniaxial stress field and restriction of attention to the upper half of the sheet, the boundary conditions are

$$\left.\begin{aligned}
&\sigma_y(x,h) = 0 \\
&\sigma_{xy}(x,h) = 0 \\
&v(x,0) = 0 \\
&u(x,0) = -\sigma x && (|x| < 1) \\
&\sigma_{xy}(x,0) = 0 && (|x| > 1)
\end{aligned}\right\} \quad \text{(A10)}$$

In terms of the transformed stress function, these become

$$\left.\begin{aligned}
&\bar{\phi}(\xi,h) = 0 \\
&\frac{d\bar{\phi}}{dy}(\xi,h) = 0 \\
&\frac{d^3\bar{\phi}}{dy^3}(\xi,0) - (2+\mu)\xi^2 \frac{d\bar{\phi}}{dy}(\xi,0) = 0 \\
&-\sigma x = \frac{i}{\pi}\int_0^\infty \left\{ \frac{d^2\bar{\phi}}{dy^2}(\xi,0) + \mu\ \xi^2\bar{\phi}(\xi,0)\right\} \sin \xi x \frac{d\xi}{\xi} && (|x| < 1) \\
&0 = \int_0^\infty \xi \frac{d\bar{\phi}}{dy}(\xi,0) \sin \xi x\, d\xi && (|x| > 1)
\end{aligned}\right\} \quad \text{(A11)}$$

Substitution of the general solution (13) into the first three of equations (A11) and solving for three of the constants of integration in terms of the fourth yields

$$\left.\begin{aligned}
A &= \frac{D}{\Delta}\left\{e^{2\xi h} + 2\alpha\xi^2h^2 + 2\xi h - 1\right\} \\
B &= -\frac{\xi D}{\Delta}\left\{2 + \alpha + 2\alpha\xi h + \alpha e^{2\xi h}\right\} \\
C &= \frac{D}{\Delta}\left\{2\alpha\xi^2h^2 - 2\xi h - 1 - e^{-2\xi h}\right\}
\end{aligned}\right\} \quad \text{(A12)}$$

where

$$\Delta = \xi\left\{2 + \alpha - 2\alpha\xi h + \alpha e^{-2\xi h}\right\}$$

and

$$\alpha = \frac{1 + \mu}{1 - \mu}$$

By use of equations (A12), the remaining boundary condition becomes

$$\left.\begin{aligned}
\frac{\sigma\pi x}{2(1 + \mu)(3 - \mu)} &= \int_0^\infty \frac{\xi D}{(1 - \mu)\Delta}\left\{\cosh 2\xi h + 2\left(\frac{1 + \mu}{3 - \mu}\right)\xi^2 h^2 + \frac{5 - 2\mu + \mu^2}{(1 + \mu)(3 - \mu)}\right\}\sin \xi x \, d\xi \qquad (|x| < 1) \\
0 &= \int_0^\infty \frac{\xi^2 D}{(1 - \mu)\Delta}\left\{\sinh 2\xi h + 2\xi h\right\}\sin \xi x \, d\xi \qquad (|x| > 1)
\end{aligned}\right\} \quad \text{(A13)}$$

With the identity

$$\sin \xi x = \left(\frac{\pi \xi x}{2}\right)^{1/2} J_{1/2}(\xi x)$$

and the substitution

$$f(\xi) = \frac{\xi^{5/2} D}{(1 - \mu)\Delta}(\sinh 2\xi h + 2\xi h)$$

equations (A13) become

$$\left.\begin{aligned}
\frac{\sigma}{(1 + \mu)(3 - \mu)}\left(\frac{\pi x}{2}\right)^{1/2} &= \int_0^\infty \left\{\frac{\cosh 2\xi h + 2\left(\frac{1 + \mu}{3 - \mu}\right)\xi^2 h^2 + \frac{5 - 2\mu + \mu^2}{(1 + \mu)(3 - \mu)}}{\sinh 2\xi h + 2\xi h}\right\} f(\xi) J_{1/2}(\xi x)\frac{d\xi}{\xi} \qquad (|x| < 1) \\
0 &= \int_0^\infty f(\xi) J_{1/2}(\xi x) \, d\xi \qquad (|x| > 1)
\end{aligned}\right\} \quad \text{(A14)}$$

Again, $f(\xi)$ is assumed to have the form shown in equation (23). Hence, the second of equations (A14) is satisfied and, with the choice $k = 1/2$, the first of equations (A14) becomes, after further application of the method of reference 11,

$$A_0 + \sum_{m=0}^{\infty} L_{m,0} A_m = \frac{\sigma\pi}{2(1 + \mu)(3 - \mu)}$$

$$A_n + \sum_{m=0}^{\infty} L_{m,n} A_m = 0 \qquad (n = 1, 2, 3, \ldots)$$

With the substitution $A_m = \frac{\sigma\pi}{2(1 + \mu)(3 - \mu)} P_m$, these become

$$P_0 + \sum_{m=0}^{\infty} L_{m,0} P_m = 1$$

$$P_n + \sum_{m=0}^{\infty} L_{m,n} P_m = 0 \qquad (n = 1, 2, 3, \ldots)$$

where

$$L_{m,n} = 2(2n + 1) \int_0^{\infty} \left\{ \frac{\cosh 2\xi h + 2\left(\frac{1 + \mu}{3 - \mu}\right)\xi^2 h^2 + \frac{5 - 2\mu + \mu^2}{(1 + \mu)(3 - \mu)}}{\sinh 2\xi h + 2\xi h} \right\} J_{2m+1}(\xi)\, J_{2n+1}(\xi)\, \frac{d\xi}{\xi}$$

In this problem, both μ and h must be specified for the calculation of the $L_{m,n}$'s (hence, the P_m's).

The transformed shear stress along $y = 0$ is given by

$$\bar{\sigma}_{xy}(\xi,0) = \frac{2\sigma}{(1 + \mu)(3 - \mu)} \sum_{m=0}^{\infty} P_m J_{2m+1}(\xi)$$

Inverse transformation gives

$$\sigma_{xy}(x,0) = \begin{cases} \dfrac{2\sigma}{(1 + \mu)(3 - \mu)} \displaystyle\sum_{m=0}^{\infty} P_m \frac{\sin\left[(2m + 1)\sin^{-1}x\right]}{\sqrt{1 - x^2}} & (|x| < 1) \\ 0 & (|x| > 1) \end{cases} \qquad (A15)$$

Again, the ratio shear-stress intensity factors for the sheet and the similarly loaded infinite body is found to be

$$\frac{K}{K_\infty} = \sum_{m=0}^{\infty} (-1)^m P_m$$

in which the P_m's are generally different from those of problems I and II. An additional distinction is their dependence on Poisson's ratio.

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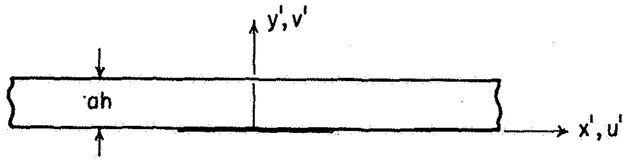
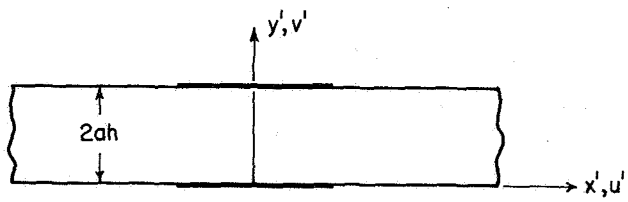
TABLE 1. - SHEAR-STRESS INTENSITY FACTORS

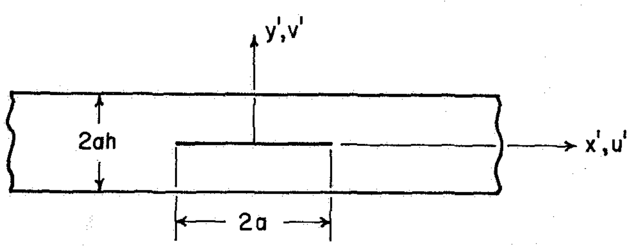
h	$\frac{K}{K_\infty}$ for -			
	Problem I	Problem II	Problem III	
			$\mu = 0$	$\mu = 0.5$
0.1	0.091	0.279	0.219	0.257
.2	.160	.464	.368	.426
.4	.273	.726	.584	.652
.6	.368	.882	.724	.776
.8	.451	.963	.811	.842
1.0	.524	1.00	.864	.879
1.5	.667	1.02	.929	.928
2	.763	1.02	.957	.951
3	.870	1.01	.979	.975
4	.920	1.01	.988	.985
5	.946	1.00	.992	.990
10	.986	1.00	.998	.997



(a) Problem I – sheet with one cover plate.



(b) Problem II – sheet with two cover plates.



(c) Problem III – sheet with embedded cover plate.

Figure 1.- Configurations and coordinate systems.

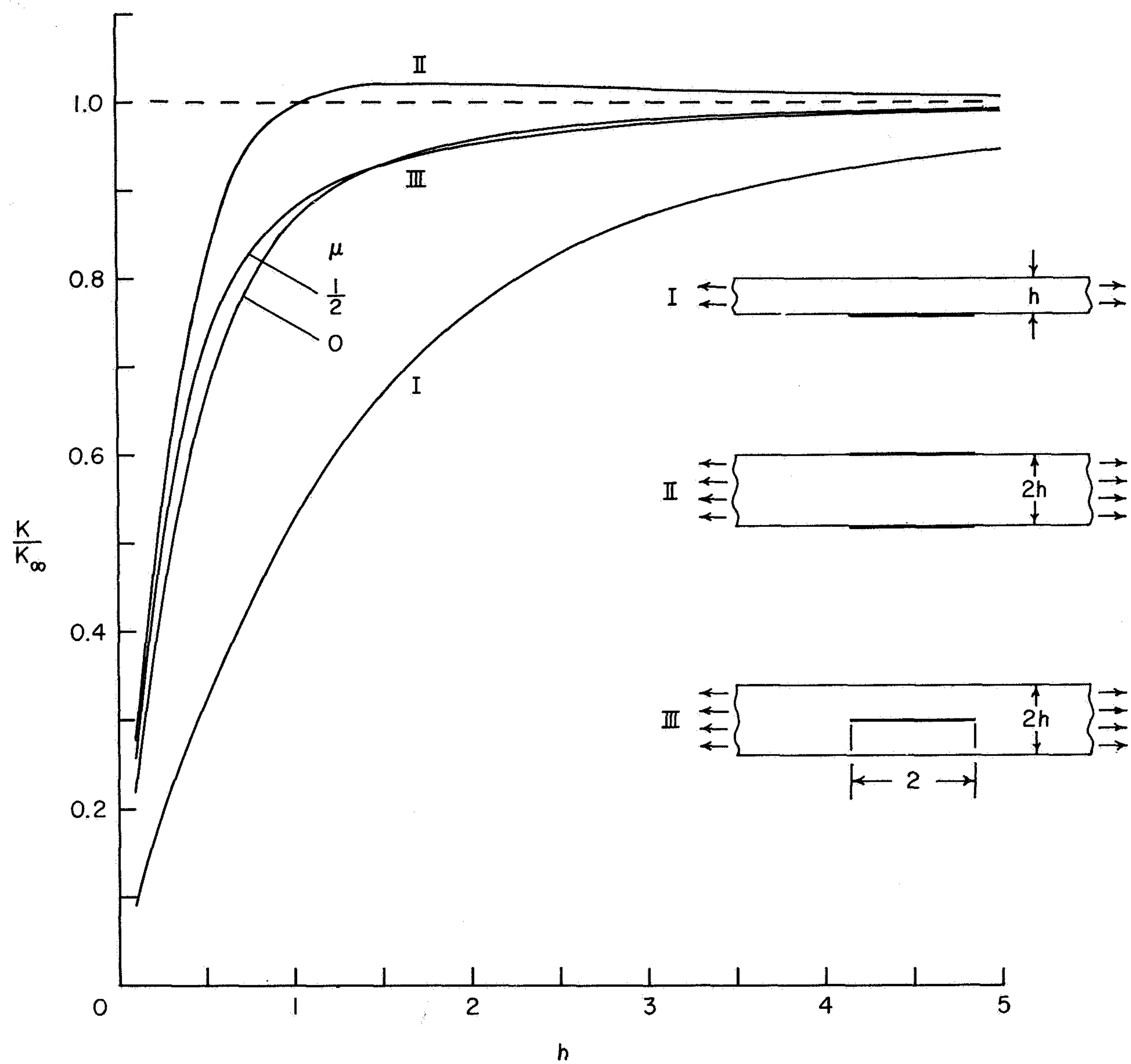


Figure 2.- Shear-stress intensity factors as functions of the ratio of sheet thickness to cover-plate length.

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16. Abstract

Shear-stress intensity factors are calculated for three problems concerning inextensible cover plates either bonded to or embedded in an elastic sheet which is under uniaxial tension. The stress intensity factors are small when the ratio of sheet thickness to cover-plate length is small and, as the ratio increases, rapidly approach their asymptotic values for infinite sheet thickness. In the problem of the embedded cover plate, the stress intensity factor also depends on the Poisson's ratio of the sheet material. The dependence on Poisson's ratio, however, is significant only when the ratio of sheet thickness to cover-plate length is small. Some possible implications of the present results for debonding of reinforced sheets under cyclic loading are briefly discussed.

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