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ANALYSIS OF LAMINATED, COMPOSITE, CIRCULAR CYLINDRICAL SHELLS WITH GENERAL BOUNDARY CONDITIONS

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16. Abstract

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ANALYSIS OF LAMINATED, COMPOSITE, CIRCULAR CYLINDRICAL SHELLS WITH GENERAL BOUNDARY CONDITIONS

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SUMMARY

This report develops (1) a refined approximate theory for the static and dynamic analyses of finite, laminated, composite, circular cylindrical shells with general boundary conditions; (2) an exact three-dimensional analysis of simply supported, laminated, composite, circular cylindrical shells; and (3) a thin-shell theory for laminated, composite, circular cylindrical shells.

In the refined approximate theory the displacements are assumed to be piecewise linear across the thickness and the effects of transverse shear deformations and transverse normal stress are included. A variational approach is followed to obtain the governing differential equations and boundary conditions. A general solution of the governing differential equations is also presented. The analysis of finite laminated shells with general boundary conditions involves satisfying the boundary conditions by making use of the appropriate part of the general solution. By using the refined approximate theory, the edge boundary conditions can be properly satisfied.

In the exact three-dimensional analysis of simply supported, laminated shells, each ply is treated as a homogeneous cylinder. The displacements are chosen to vary trigonometrically in the axial and circumferential directions. The three governing partial differential equations are reduced to three coupled ordinary differential equations with the radial coordinate as the independent variable. The three coupled ordinary differential equations are then solved by use of the Frobenius method to obtain the variation of displacements in the radial direction. By satisfying the interface and exterior surface conditions, a set of simultaneous algebraic equations is obtained. For free vibration, the determinant of the coefficient matrix is equated to zero, and the solution of this characteristic equation yields the frequencies. The analysis and results are applicable to wave propagation in infinite shells, since the simple boundary conditions simulate the axial nodes in infinite shells.

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The thin-shell theories for laminated, composite, circular cylindrical shells that are available in the literature contain some approximations in addition to the basic assumptions of thin-shell theory. The present thin-shell theory contains no additional assumptions.

The results obtained from the refined approximate theory and thin-shell theory are compared with the exact results for simply supported, laminated, composite circular cylindrical shells. The refined approximate theory is found to be very accurate even for thick shells with short nodal distances. In contrast, the thin-shell theory is found to be reasonably accurate only for thin shells with moderate nodal distances. Frequency calculations using the refined approximate theory and thin-shell theory took considerably less computer time than those using the exact three-dimensional theory.

INTRODUCTION

Structural application of laminated composites in which each ply is made up of different materials is ever on the increase. As a consequence, analysis of such structures is gaining importance. The present paper deals with the static and dynamic analyses of laminated, circular cylindrical shells made up of composite materials.

Analysis of laminated circular cylindrical shells has drawn considerable attention in recent times. If any of the plies is fiber reinforced or honeycomb, the normal procedure is to treat the ply as an equivalent homogeneous material having orthotropic or anisotropic properties. Dong (ref. 1), Bert, Baker, and Egle (ref. 2), and Stavsky and Loewy (ref. 3) have given differential equations for the dynamics of laminated cylindrical shells and solutions for free vibrations when the shell is simply supported. In these references the differential equations are based on different variations of thin-shell theory. The basic assumptions of thin-shell theory are (1) radial displacements are constant across the thickness, (2) axial and circumferential displacements vary linearly across the thickness and, (3) transverse shear deformations and transverse normal stress can be neglected. In references 1 to 3 some additional approximations are made.

Nelson, Dong, and Kalra (ref. 4) have analyzed the special problem of free vibrations of simply supported circular cylinders by following the Ritz technique. Using the theory of three-dimensional elasticity, Armenàkas (ref. 5) has analyzed the problem of wave propagation in two-layered circular cylindrical shells of infinite length and made of isotropic materials. For isotropic materials, the governing differential equations of elasticity can be easily solved in terms of displacement potentials; the variation of displacements in the radial direction, when determined, is in the form of Bessel functions. However, a similar approach is not possible if the material is orthotropic.

For laminated composite shells with general boundary conditions, the literature does not appear to offer any analysis which takes into account general variations of the displacements across the thickness. Such an analysis would automatically take into account the effects of transverse shear deformations and transverse normal stress. An analysis of this type is needed for the following reasons:

(1) The edge boundary conditions can be properly satisfied only if the variation of displacements across the thickness is unrestricted. The stresses and displacements close to the boundaries obtained from the thin-shell theory are highly inaccurate.

(2) For thick shells, especially composite shells, the thicknesswise variation of displacements is more general than the restricted distribution assumed in the thin-shell theory. Therefore, a general variation of displacements is necessary for a better estimation of frequencies of free vibrations, stresses, and displacements even in regions far from the edges.

An exact three-dimensional elastic analysis of laminated shells is computationally impractical except in the case of simply supported shells, the case that is derived herein. Therefore, an approximate theory, which takes into account general variations of displacements across the thickness, is developed for the static and dynamic analyses of laminated, composite, circular cylindrical shells with general boundary conditions. The approximate theory developed in this report is referred to as the "refined approximate theory." In the refined approximate theory the displacements are assumed to be piecewise linear across the thickness, and the three-dimensional problem is reduced to a two-dimensional problem in the circumferential and radial coordinates. Furthermore, a thin-shell theory is developed without making any approximations in addition to the basic assumptions inherent in thin-shell theory.

The accuracies of the refined approximate theory and the thin-shell theory are assessed by comparing results obtained by using them with the exact results for simply supported shells.

SYMBOLS

A_r, A_θ, A_z	arbitrary constants used in particular part of refined approximate analysis
a	outer radius of shell, $a^*(p+1)$
$a^*(i)$	inner radius of ith ply

$a(i) = \frac{a^*(i)}{a}$

$B_{11}(1,1), B_{12}(1,1), \ldots$	stiffnesses of shell used in thin-shell analysis
b	length of cylinder
$\bar{b}$	nodal distance or half-wavelength, b/n
$C_{11} \ldots C_{66}$	stiffness elastic constants
$c_{22} \ldots c_{66}$	modified elastic constants for thin-shell analysis
$D_\rho, D_{11}(1,1), D_{12}(1,1), \ldots$	stiffnesses of shell used in refined approximate analysis
d_r, d_θ, d_z	coefficients in power series for displacements in exact three-dimensional analysis
$F_{\rho r}, F_{\theta\theta}, F_{\theta z}, \ldots$	integrals of stresses in refined approximate analysis
$f_{\rho r}, f_{\theta\theta}, f_{\theta z}, \ldots$	integrals of stresses in thin-shell analysis
$G(i,k)$	arbitrary constants in expressions for displacements and stresses in exact three-dimensional analysis
$H(j,k)$	coefficient in power series expansion of displacements used in exact three-dimensional analysis
$h^*(i)$	thickness of ith ply

$h(i) = \frac{h^*(i)}{a}$

i, j, k, ℓ, β	indices
$\underset{\sim}{J}(\beta), \underset{\sim}{M}(\beta)$	matrices occurring in particular part of refined approximate analysis
m, n	integers occurring in trigonometric expansions of displacements and stresses in circumferential and axial directions, respectively

$N = \frac{n\pi a}{b}$ or $\frac{\pi a}{b}$

$\underset{\sim}{0}$ null matrix

p number of plies

p_m number of layers into which middle ply of a three-ply laminate is split in refined approximate analysis

$Q_{\theta\theta}, Q_{\theta z}, \ldots$ integrals of applied stresses on edges

$\left.\begin{matrix} q_{rr}(1), q_{r\theta}(1), \\ q_{rz}(1) \end{matrix}\right\}$ applied tractions on inner surface of shell in r, θ, and z directions, respectively

$\left.\begin{matrix} q_{rr}(2), q_{r\theta}(2), \\ q_{rz}(2) \end{matrix}\right\}$ applied tractions on outer surface of shell in r, θ, and z directions, respectively

$q_{\theta r}, q_{\theta\theta}, q_{\theta z}$ applied stresses on edges θ = Constant in r, θ, and z directions, respectively

$q_{zr}, q_{z\theta}, q_{zz}$ applied stresses on edges z = Constant in r, θ, and z directions, respectively

$q_{rrf}(1), q_{r\theta f}(1), q_{rzf}(1), \ldots$ Fourier coefficients of applied pressures

$R = \frac{r}{a}$

r radial coordinate

T kinetic energy

t time

$U_r(i)$ difference between radial displacements at inner and outer surfaces of ith ply used in refined approximate analysis

$U_\theta(i)$ difference between circumferential displacements at inner and outer surfaces of ith ply used in refined approximate analysis

$U_z(i)$	difference between axial displacements at inner and outer surfaces of ith ply used in refined approximate analysis
$U_{r0}, U_{\theta 0}, U_{z0}$	radial, circumferential, and axial displacements of inner surface in thin-shell analysis
u_r, u_θ, u_z	radial, circumferential, and axial displacements, respectively
V(1,k,i), V(2,k,i), . . .	coefficients of differential operators in governing differential equations of refined approximate analysis
W_s	strain energy
W_e	work done by applied loading
$(W_e)_{surfaces}$	work done by applied tractions on inner and outer surfaces of shell
$(W_e)_{edges}$	work done by applied stresses on edges
$Z = \frac{z}{a}$	
z	axial coordinate
$\alpha(k)$	indicial constants in Frobenius method of solution of differential equations used in exact three-dimensional analysis
$\left.\begin{matrix} \underset{\sim}{\Gamma}_\theta(\beta),\ \underset{\sim}{\Gamma}_z(\beta) \\ \underset{\sim}{\Xi}_\theta(\beta),\ \underset{\sim}{\Xi}_z(\beta) \\ \underset{\sim}{\Pi}_\theta(\beta),\ \underset{\sim}{\Pi}_z(\beta) \\ \underset{\sim}{\Upsilon}_\theta(\beta),\ \underset{\sim}{\Upsilon}_z(\beta) \\ \underset{\sim}{\Phi}_\theta(\beta),\ \underset{\sim}{\Phi}_z(\beta) \\ \underset{\sim}{\Psi}_\theta(\beta),\ \underset{\sim}{\Psi}_z(\beta) \end{matrix}\right\}$	matrices occurring in system of ordinary differential equations in complementary part of refined approximate analysis

$\delta(j,i)$ Kronecker delta

$\left.\begin{matrix}\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}, \\ \epsilon_{rz}, \epsilon_{r\theta}, \epsilon_{\theta z}\end{matrix}\right\}$ strains

$\zeta(j,k) = 1$ for $k \neq j$ and $\zeta(j,k) = \left[\dfrac{R - a(j)}{h(j)}\right]$ for $k = j$

$\left.\begin{matrix}\eta_r(i,\beta), \eta_\theta(i,\beta), \\ \eta_z(i,\beta)\end{matrix}\right\}$ functions of Z occurring in complementary part of refined approximate analysis

θ circumferential angular coordinate

λ nondimensional frequency parameter, equation (98)

$\left.\begin{matrix}\xi_r(i,\beta), \xi_\theta(i,\beta), \\ \xi_z(i,\beta)\end{matrix}\right\}$ functions of θ occurring in complementary part of refined approximate analysis

$\rho(i)$ mass density of ith layer

$\left.\begin{matrix}\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \\ \sigma_{rz}, \sigma_{r\theta}, \sigma_{\theta z}\end{matrix}\right\}$ stresses

$\tau(\beta)$ coefficients occurring in summation for coefficients $V(\beta,k,i)$

$\phi_r, \phi_\theta, \phi_z$ functions of radial coordinate occurring in double trigonometric expansion of displacements used in exact three-dimensional analysis

$\left.\begin{matrix}\chi_r(i,k), \chi_\theta(i,k), \\ \chi_z(i,k), \chi_{rr}(i,k), \\ \chi_{\theta\theta}(i,k), \chi_{zz}(i,k), \\ \chi_{rz}(i,k), \chi_{r\theta}(i,k), \\ \chi_{\theta z}(i,k)\end{matrix}\right\}$ functions which are power series in radial coordinate occurring in double trigonometric expansion of displacements and stresses used in exact three-dimensional analysis

Ω frequency of vibration

A tilde (~) under a symbol denotes a matrix. Primes denote differentiated quantities. Numbers in parentheses are indices.

STRESS-STRAIN RELATIONSHIP

Most of the materials commonly encountered in structural practice are either isotropic or orthotropic. But in some cases during analysis there arises a need to choose a coordinate system which is different from the axes of orthotropy. In such cases the stress-strain relationships exhibit anisotropy. As shown in reference 6, the anisotropic stress-strain relationship that might be encountered in the analysis of fiber-reinforced components is of the type

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \\ \sigma_{r\theta} \\ \sigma_{\theta z} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{rz} \\ \epsilon_{r\theta} \\ \epsilon_{\theta z} \end{bmatrix} \tag{1}$$

If the material is orthotropic, the elastic constants C_{16}, C_{26}, C_{36}, and C_{45} are zero.

STRAIN-DISPLACEMENT RELATIONS

The strain-displacement relations in polar coordinates are (from ref. 7)

$$\left.\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} & \epsilon_{rz} &= \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \\ \epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \epsilon_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\ \epsilon_{zz} &= \frac{\partial u_z}{\partial z} & \epsilon_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \end{aligned}\right\} \tag{2}$$

REFINED APPROXIMATE THEORY

Development of Theory

An approximate theory for the statics and dynamics of laminated, anisotropic, circular cylindrical shells (figs. 1 and 2) is developed. The displacements are assumed to be piecewise linear across the thickness; that is, the displacements of the jth ply are given by

$$\left.\begin{aligned} u_r(j) &= a \sum_{k=0}^{j} U_r(k)\, \zeta(j,k) \\ u_\theta(j) &= a \sum_{k=0}^{j} U_\theta(k)\, \zeta(j,k) \\ u_z(j) &= a \sum_{k=0}^{j} U_z(k)\, \zeta(j,k) \end{aligned}\right\} \tag{3}$$

where $j = 1, 2, \ldots, p$ and

$$\zeta(j,k) = \begin{cases} 1 & \text{for } k \neq j \\ \dfrac{R - a(j)}{h(j)} & \text{for } k = j \end{cases}$$

In equations (3), $U_r(k)$, $U_\theta(k)$, and $U_z(k)$ are functions of θ and z and are independent of r; $U_r(0)$, $U_\theta(0)$, and $U_z(0)$ are the displacements of the inner surface; $U_r(k)$, $U_\theta(k)$, and $U_z(k)$ are the differences between the displacements of the inner and outer surfaces of the kth ply at a given θ and z.

A variational approach is used to obtain the governing differential equations and boundary conditions. The variation in strain energy W_s, the kinetic energy T, and the work done by the applied forces W_e due to virtual displacements are calculated and substituted into the variational condition (known as Hamilton's principle)

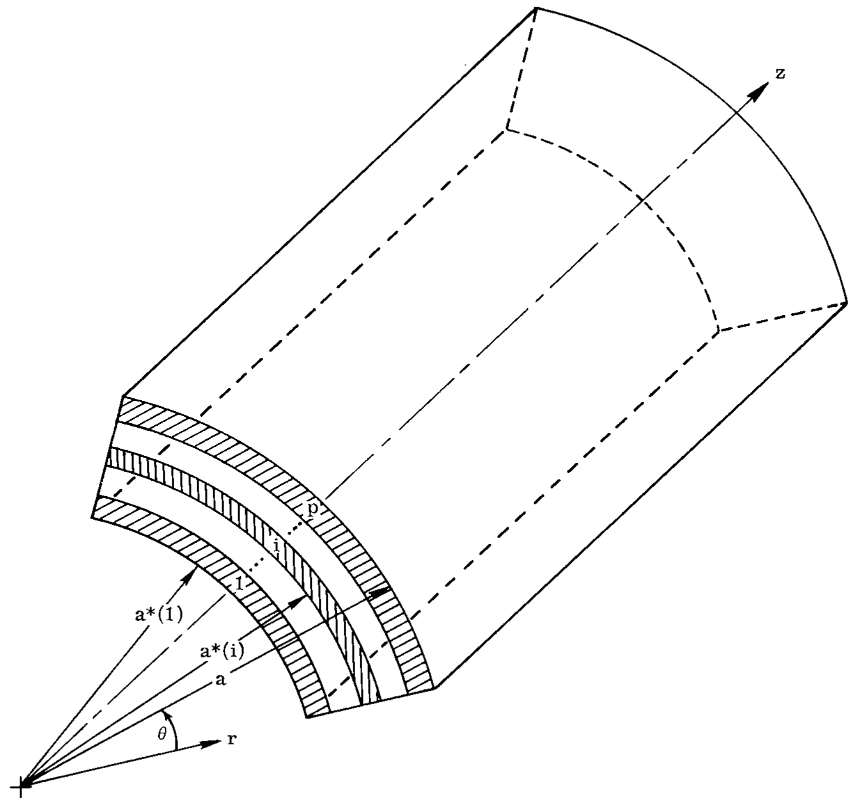


Figure 1.- Multi-ply cylindrical shell panel.

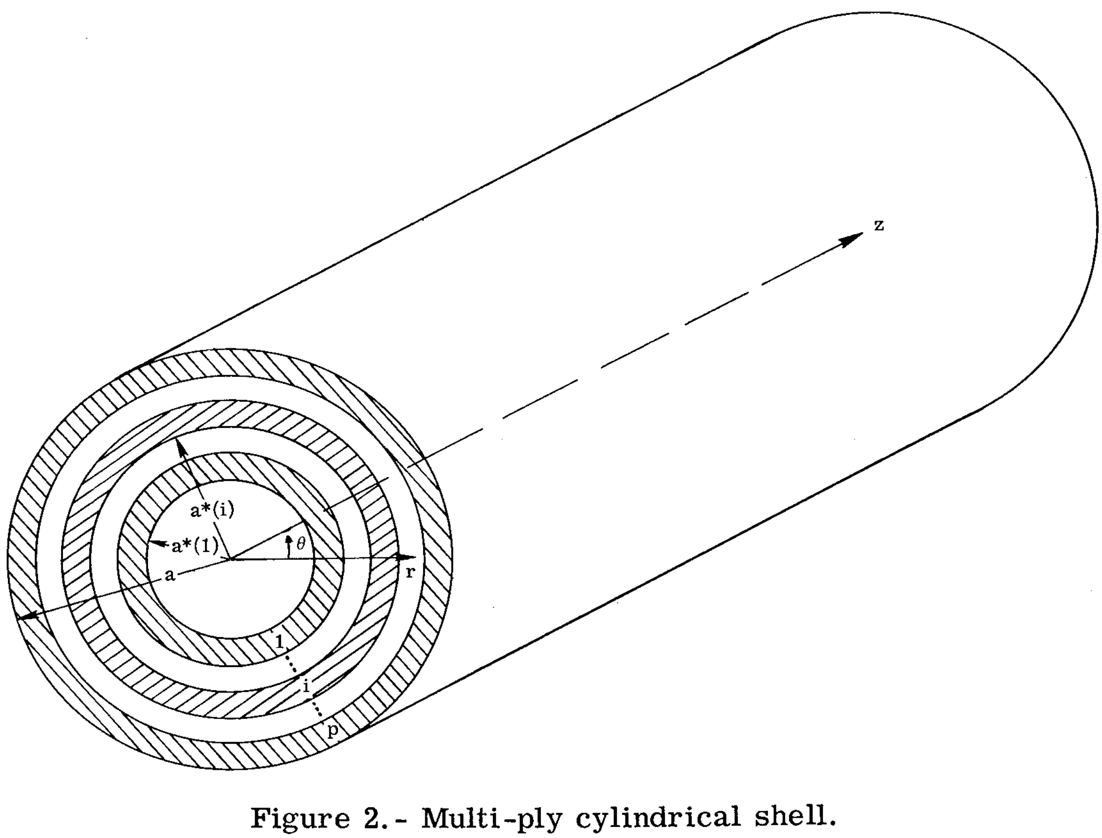


Figure 2.- Multi-ply cylindrical shell.

$$\int_{t_0}^{t_1} \left(\delta T + \delta W_e - \delta W_s\right) dt = 0 \tag{4}$$

where t_0 and t_1 are the initial and final times, respectively. The strain energy W_s is given by

$$W_s = \frac{a^3}{2} \sum_{j=1}^{p} \int_{\theta} \int_{z} \int_{a(j)}^{a(j+1)} \left[\sigma_{rr}(j)\ \epsilon_{rr}(j) + \sigma_{\theta\theta}(j)\ \epsilon_{\theta\theta}(j) + \sigma_{zz}(j)\ \epsilon_{zz}(j)\right.$$

$$\left. + \sigma_{rz}(j)\ \epsilon_{rz}(j) + \sigma_{r\theta}(j)\ \epsilon_{r\theta}(j) + \sigma_{\theta z}(j)\ \epsilon_{\theta z}(j)\right] R\ dR\ d\theta\ dZ \tag{5}$$

From equations (2) and (3) the strains can be expressed as

$$\left.\begin{aligned}
\epsilon_{rr}(j) &= \frac{U_r(j)}{h(j)} \\
\epsilon_{\theta\theta}(j) &= \frac{1}{R} \sum_{k=0}^{j} \left[\frac{\partial U_\theta(k)}{\partial \theta} + U_r(k)\right] \zeta(j,k) \\
\epsilon_{zz}(j) &= \sum_{k=0}^{j} \frac{\partial U_z(k)}{\partial Z} \zeta(j,k) \\
\epsilon_{rz}(j) &= \frac{U_z(j)}{h(j)} + \sum_{k=0}^{j} \frac{\partial U_r(k)}{\partial Z} \zeta(j,k) \\
\epsilon_{r\theta}(j) &= \frac{U_\theta(j)}{h(j)} + \frac{1}{R} \sum_{k=0}^{j} \left[\frac{\partial U_r(k)}{\partial \theta} - U_\theta(k)\right] \zeta(j,k) \\
\epsilon_{\theta z}(j) &= \sum_{k=0}^{j} \left[\frac{\partial U_\theta(k)}{\partial Z} + \frac{1}{R} \frac{\partial U_z(k)}{\partial \theta}\right] \zeta(j,k)
\end{aligned}\right\} \tag{6}$$

Substituting for the strains from equations (6) and integrating with respect to R give

$$W_s = \frac{a^3}{2} \int_\theta \int_z \sum_{j=1}^{p} \sum_{k=0}^{j} \left\{ F_{rr}(j,3,1)\, U_r(j)\, \delta(j,k) + F_{r\theta}(j,3,1)\, U_\theta(j)\, \delta(j,k) \right.$$

$$+ F_{rz}(j,3,1)\, U_z(j)\, \delta(j,k) + F_{\theta\theta}(j,1,\delta_1) \left[\frac{\partial U_\theta(k)}{\partial \theta} + U_r(k) \right]$$

$$+ F_{zz}(j,2,\delta_1) \frac{\partial U_z(k)}{\partial Z} + F_{r\theta}(j,1,\delta_1) \left[\frac{\partial U_r(k)}{\partial \theta} - U_\theta(k) \right]$$

$$+ F_{rz}(j,2,\delta_1) \frac{\partial U_r(k)}{\partial Z} + F_{\theta z}(j,2,\delta_1) \frac{\partial U_\theta(k)}{\partial Z}$$

$$\left. + F_{\theta z}(j,1,\delta_1) \frac{\partial U_z(k)}{\partial \theta} \right\} d\theta\, dZ \tag{7}$$

where $\delta_1 = 1 + \delta(j,k)$. The integrals F are integrals of stresses defined as

$$\left\{ F_\beta(j,1,\ell);\ F_\beta(j,2,\ell);\ F_\beta(j,3,\ell) \right\}$$

$$= \int_{a(j)}^{a(j+1)} \left\{ \sigma_\beta(j);\ R\sigma_\beta(j);\ \frac{R\sigma_\beta(j)}{h(j)} \right\} \left[\frac{R - a(j)}{h(j)} \right]^{\ell-1} dR \tag{8}$$

where

$$\beta = rr,\ \theta\theta,\ .\ .\ .,\ \theta z$$

Stresses can be written in terms of displacements by use of equations (1) and (6). By substituting for stresses in equation (8) and integrating, the integrals F can be expressed in terms of displacements. (See table I.)

Taking the variation of the strain energy W_s, integrating by parts, and then changing the order of summation yield

$$
\begin{aligned}
\delta W_s = a^3 \sum_{k=0}^{p} \int_\theta \int_z \Bigg(& \left\{ \sum_{j=k}^{p} \left[\delta(j,k)\, F_{rr}(j,3,1) + F_{\theta\theta}(j,1,\delta_1) \right.\right. \\
& \left.\left. - \frac{\partial}{\partial\theta} F_{r\theta}(j,1,\delta_1) - \frac{\partial}{\partial Z} F_{rz}(j,2,\dot{\delta}_1) \right] \right\} \delta U_r(k) \\
& + \left\{ \sum_{j=k}^{p} \left[\delta(j,k) F_{r\theta}(j,3,1) - F_{r\theta}(j,1,\delta_1) - \frac{\partial}{\partial\theta} F_{\theta\theta}(j,1,\delta_1) \right.\right. \\
& \left.\left. - \frac{\partial}{\partial Z} F_{\theta z}(j,2,\delta_1) \right] \right\} \delta U_\theta(k) \\
& + \left\{ \sum_{j=k}^{p} \left[\delta(j,k)\, F_{rz}(j,3,1) - \frac{\partial}{\partial\theta} F_{\theta z}(j,1,\delta_1) - \frac{\partial}{\partial Z} F_{zz}(j,2,\delta_1) \right] \right\} \delta U_z(k) \Bigg) \, d\theta \, dZ \\
& + a^3 \sum_{k=0}^{p} \int_z \left\{ \left[\sum_{j=k}^{p} F_{r\theta}(j,1,\delta_1) \right] \delta U_r(k) \right. \\
& \left. + \left[\sum_{j=k}^{p} F_{\theta\theta}(j,1,\delta_1) \right] \delta U_\theta(k) + \left[\sum_{j=k}^{p} F_{\theta z}(j,1,\delta_1) \right] \delta U_z(k) \right\}_{\theta=\text{Constant}} dZ \\
& + a^3 \sum_{k=0}^{p} \int_\theta \left\{ \left[\sum_{j=k}^{p} F_{rz}(j,2,\delta_1) \right] \delta U_r(k) \right. \\
& \left. + \left[\sum_{j=k}^{p} F_{\theta z}(j,2,\delta_1) \right] \delta U_\theta(k) + \left[\sum_{j=k}^{p} F_{zz}(j,2,\delta_1) \right] \delta U_z(k) \right\}_{z=\text{Constant}} d\theta
\end{aligned}
\tag{9}
$$

The kinetic energy T is given by

$$T = \frac{a^3}{2} \sum_{j=1}^{p} \int_\theta \int_z \int_{a(j)}^{a(j+1)} \rho(j) \left\{ \left[\frac{\partial u_r(j)}{\partial t}\right]^2 + \left[\frac{\partial u_\theta(j)}{\partial t}\right]^2 + \left[\frac{\partial u_z(j)}{\partial t}\right]^2 \right\} R\, dR\, d\theta\, dZ \tag{10}$$

Taking the variation gives

$$\delta T = a^3 \sum_{j=1}^{p} \int_\theta \int_z \int_{a(j)}^{a(j+1)} \rho(j) \left[\frac{\partial u_r(j)}{\partial t} \frac{\partial \delta u_r(j)}{\partial t} + \frac{\partial u_\theta(j)}{\partial t} \frac{\partial \delta u_\theta(j)}{\partial t} + \frac{\partial u_z(j)}{\partial t} \frac{\partial \delta u_z(j)}{\partial t} \right] R\, dR\, d\theta\, dZ$$

After integration by parts,

$$\int_{t_0}^{t_1} \delta T\, dt = -a^3 \int_{t_0}^{t_1} \sum_{j=1}^{p} \int_\theta \int_z \int_{a(j)}^{a(j+1)} \rho(j) \left[\frac{\partial^2 u_r(j)}{\partial t^2} \delta u_r(j) + \frac{\partial^2 u_\theta(j)}{\partial t^2} \delta u_\theta(j) + \frac{\partial^2 u_z(j)}{\partial t^2} \delta u_z(j) \right] R\, dR\, d\theta\, dZ \tag{11}$$

Substituting equations (3) into equation (11) and integrating with respect to R give

$$\int_{t_0}^{t_1} \delta T\, dt = -a^5 \int_{t_0}^{t_1} \sum_{k=0}^{p} \int_\theta \int_z \left\{ \left[\sum_{j=k}^{p} F_{\rho r}(j, 2, \delta_1) \right] \delta U_r(k) + \left[\sum_{j=k}^{p} F_{\rho\theta}(j, 2, \delta_1) \right] \delta U_\theta(k) + \left[\sum_{j=k}^{p} F_{\rho z}(j, 2, \delta_1) \right] \delta U_z(k) \right\} d\theta\, dZ \tag{12}$$

(See table I for $F_{\rho r}$, $F_{\rho\theta}$, and $F_{\rho z}$.)

The total work done by applied loads W_e is the sum of the work done by applied tractions on the inner and outer surfaces of the shell $(W_e)_{surfaces}$ and the work done by applied stresses at the edges $(W_e)_{edges}$, that is,

$$W_e = (W_e)_{surfaces} + (W_e)_{edges} \tag{13}$$

Now if the shell is subjected to dynamic tractions $q_{rr}(1)$, $q_{r\theta}(1)$, and $q_{rz}(1)$ and $q_{rr}(2)$, $q_{r\theta}(2)$, and $q_{rz}(2)$ on the inner $(R = a(1))$ and outer surfaces $(R = 1)$ of the shell in the r, θ, and z directions, respectively, then the virtual work done by these applied tractions is

$$\delta(W_e)_{surfaces} = a^3 \int_\theta \int_z \sum_{\beta=r,\theta,z} \left[- q_{r\beta}(1)\; a(1)\; \delta U_\beta(0) + q_{r\beta}(2) \sum_{k=0}^{p} \delta U_\beta(k) \right] d\theta\; dZ$$

$$= a^3 \int_\theta \int_z \sum_{\beta=r,\theta,z} \sum_{k=0}^{p} \left[q_{r\beta}(2) - q_{r\beta}(1)\; \delta(k,0)\; a(1) \right] \delta U_\beta(k)\; d\theta\; dZ \tag{14}$$

If the edge z = Constant is subjected to stresses q_{zr}, $q_{z\theta}$, and q_{zz} and the edge θ = Constant is subjected to stresses $q_{\theta r}$, $q_{\theta\theta}$, and $q_{\theta z}$ in the r, θ, and z directions, respectively, the virtual work done by these stresses is

$$\delta(W_e)_{edges} = a^3 \int_\theta \sum_{j=1}^{p} \sum_{\beta=r,\theta,z} \int_{a(j)}^{a(j+1)} q_{z\beta} \left[\sum_{k=0}^{j} \delta U_\beta(k)\; \zeta(j,k) \right] R\; dR\; d\theta$$

$$+ a^3 \int_z \sum_{j=1}^{p} \sum_{\beta=r,\theta,z} \int_{a(j)}^{a(j+1)} q_{\theta\beta} \left[\sum_{k=0}^{j} \delta U_\beta(k)\; \zeta(j,k) \right] dR\; dZ$$

Integrating with respect to R and rearranging yield

$$\delta(W_e)_{edges} = a^3 \int_\theta \sum_{\beta=r,\theta,z} \sum_{k=0}^{p} \sum_{j=k}^{p} \delta U_\beta(k)\ Q_{z\beta}(j, \delta_1)\ d\theta$$

$$+ a^3 \int_Z \sum_{\beta=r,\theta,z} \sum_{k=0}^{p} \sum_{j=k}^{p} \delta U_\beta(k)\ Q_{\theta\beta}(j, \delta_1)\ dZ \tag{15}$$

where

$$Q_{z\beta}(j, \delta_1) = \int_{a(j)}^{a(j+1)} q_{z\beta}\, R \left[\frac{R - a(j)}{h(j)}\right]^{\delta_1 - 1} dR \tag{16a}$$

$$Q_{\theta\beta}(j, \delta_1) = \int_{a(j)}^{a(j+1)} q_{\theta\beta} \left[\frac{R - a(j)}{h(j)}\right]^{\delta_1 - 1} dR \tag{16b}$$

Equations (9), (12), (13), (14), and (15) are substituted into variational condition (4). Since the virtual displacements are arbitrary, the coefficients of $\delta U_r(k)$, $\delta U_\theta(k)$, and $\delta U_z(k)$ ($k = 0, 1, 2, \ldots, p$) can be equated to zero to yield the governing differential equations ($3p + 3$ in number) and boundary conditions ($3p + 3$ on a boundary).

The governing differential equations are

$$\sum_{j=k}^{p} \left[- a^2 F_{\rho r}(j, 2, \delta_1) - \delta(j,k)\ F_{rr}(j, 3, 1)\right.$$

$$\left. - F_{\theta\theta}(j, 1, \delta_1) + \frac{\partial}{\partial\theta} F_{r\theta}(j, 1, \delta_1) + \frac{\partial}{\partial Z} F_{rz}(j, 2, \delta_1)\right]$$

$$+ q_{rr}(2) - q_{rr}(1)\ \ \delta(k, 0)\ a(1) = 0 \tag{17}$$

$$\sum_{j=k}^{p} \left[- a^2 F_{\rho\theta}(j,2,\delta_1) - \delta(j,k)\, F_{r\theta}(j,3,1) + F_{r\theta}(j,1,\delta_1) \right.$$

$$\left. + \frac{\partial}{\partial\theta} F_{\theta\theta}(j,1,\delta_1) + \frac{\partial}{\partial Z} F_{\theta z}(j,2,\delta_1) \right]$$

$$+ q_{r\theta}(2) - q_{r\theta}(1)\ \delta(k,0)\ a(1) = 0 \tag{18}$$

$$\sum_{j=k}^{p} \left[- a^2 F_{\rho z}(j,2,\delta_1) - \delta(j,k)\, F_{rz}(j,3,1) + \frac{\partial}{\partial\theta} F_{\theta z}(j,1,\delta_1) \right.$$

$$\left. + \frac{\partial}{\partial Z} F_{zz}(j,2,\delta_1) \right] + q_{rz}(2) - q_{rz}(1)\ \delta(k,0)\ a(1) = 0 \tag{19}$$

for $k = 0, 1, 2, \ldots, p$.

The boundary conditions are as follows:

For θ = Constant,

$$\sum_{j=k}^{p} F_{r\theta}(j,1,\delta_1) - \sum_{j=k}^{p} Q_{\theta r}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_r(k) = 0 \tag{20a}$$

$$\sum_{j=k}^{p} F_{\theta\theta}(j,1,\delta_1) - \sum_{j=k}^{p} Q_{\theta\theta}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_\theta(k) = 0 \tag{20b}$$

$$\sum_{j=k}^{p} F_{\theta z}(j,1,\delta_1) - \sum_{j=k}^{p} Q_{\theta z}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_z(k) = 0 \tag{20c}$$

For z = Constant,

$$\sum_{j=k}^{p} F_{rz}(j,2,\delta_1) - \sum_{j=k}^{p} Q_{zr}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_r(k) = 0 \tag{21a}$$

$$\sum_{j=k}^{p} F_{\theta z}(j,2,\delta_1) - \sum_{j=k}^{p} Q_{z\theta}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_\theta(k) = 0 \tag{21b}$$

$$\sum_{j=k}^{p} F_{zz}(j,2,\delta_1) - \sum_{j=k}^{p} Q_{zz}(j,\delta_1) = 0 \quad \text{or} \quad \delta U_z(k) = 0 \tag{21c}$$

where $k = 0, 1, 2, \ldots, p$. It is noted that for cylindrical panels, the boundary conditions for both θ = Constant and z = Constant are satisfied, whereas for cylinders, only the boundary conditions for z = Constant are satisfied.

By substituting for the integrals F in terms of displacements, the governing differential equations (17) to (19) can be written as

$$\sum_{i=0}^{p} \left\{ \left[-V(1,k,i)\frac{\partial^2}{\partial t^2} + V(2,k,i)\frac{\partial^2}{\partial \theta^2} + V(3,k,i)\frac{\partial^2}{\partial Z^2} \right.\right.$$

$$\left. + V(4,k,i)\frac{\partial^2}{\partial\theta\,\partial Z} + V(5,k,i) \right] U_r(i)$$

$$+ \left[V(6,k,i)\frac{\partial}{\partial\theta} + V(7,k,i)\frac{\partial}{\partial Z} \right] U_\theta(i)$$

$$\left. + \left[V(8,k,i)\frac{\partial}{\partial\theta} + V(9,k,i)\frac{\partial}{\partial Z} \right] U_z(i) \right\} = q_{rr}(1)\,\delta(k,0)\,a(1) - q_{rr}(2) \tag{22}$$

$$
\sum_{i=0}^{p} \Bigg\{ \left[V(10,k,i) \frac{\partial}{\partial\theta} + V(11,k,i) \frac{\partial}{\partial Z} \right] U_r(i)
$$

$$
+ \left[- V(1,k,i) \frac{\partial^2}{\partial t^2} + V(12,k,i) \frac{\partial^2}{\partial\theta^2} + V(13,k,i) \frac{\partial^2}{\partial Z^2} \right.
$$

$$
\left. + V(14,k,i) \frac{\partial^2}{\partial\theta\ \partial Z} + V(15,k,i) \right] U_\theta(i)
$$

$$
+ \left[V(16,k,i) \frac{\partial^2}{\partial\theta^2} + V(17,k,i) \frac{\partial^2}{\partial Z^2} + V(18,k,i) \frac{\partial^2}{\partial\theta\ \partial Z} \right.
$$

$$
\left. + V(19,k,i) \right] U_z(i) \Bigg\} = q_{r\theta}(1)\ \delta(k,0)\ a(1) - q_{r\theta}(2) \tag{23}
$$

$$
\sum_{i=0}^{p} \Bigg\{ \left[V(20,k,i) \frac{\partial}{\partial\theta} + V(21,k,i) \frac{\partial}{\partial Z} \right] U_r(i)
$$

$$
+ \left[V(16,k,i) \frac{\partial^2}{\partial\theta^2} + V(17,k,i) \frac{\partial^2}{\partial Z^2} + V(18,k,i) \frac{\partial^2}{\partial\theta\ \partial Z} + V(22,k,i) \right] U_\theta(i)
$$

$$
+ \left[- V(1,k,i) \frac{\partial^2}{\partial t^2} + V(23,k,i) \frac{\partial^2}{\partial\theta^2} + V(24,k,i) \frac{\partial^2}{\partial Z^2} \right.
$$

$$
\left. + V(25,k,i) \frac{\partial^2}{\partial\theta\ \partial Z} + V(26,k,i) \right] U_z(i) \Bigg\} = q_{rz}(1)\ \delta(k,0)\ a(1) - q_{rz}(2) \tag{24}
$$

In equations (22) to (24) the V coefficients are pure functions of material properties and the inner and outer radii of the plies. (See table II.) They can also take into account variation of material properties within a ply.

Procedure for Analysis

The analysis of a cylinder or a cylindrical panel involves finding a solution of the governing differential equations (22) to (24) which satisfies the appropriate boundary conditions (eqs. (20) and (21)). The complete solution of the governing equations can be split into two parts: (1) a particular solution which takes care of the applied loading on the inner and outer surfaces and (2) a complementary solution which has arbitrary constants necessary for satisfying the edge boundary conditions. The procedure for obtaining the particular and complementary solutions is explained herein.

Particular solution. - In some cases, depending on the applied loading on the inner and outer surfaces of the cylinder, the particular solution can be obtained by inspection of the governing differential equations. However, the general procedure is as follows:

The applied loadings (amplitudes of loading in the case for forced vibrations) are expanded in a double Fourier series of the type

$$q_{rr}(\beta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[q_{rrf}(\beta,1) \cos m\theta \sin \frac{n\pi z}{b} + q_{rrf}(\beta,2) \cos m\theta \cos \frac{n\pi z}{b} + q_{rrf}(\beta,3) \sin m\theta \sin \frac{n\pi z}{b} + q_{rrf}(\beta,4) \sin m\theta \cos \frac{n\pi z}{b} \right] \tag{25a}$$

$$q_{r\theta}(\beta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[q_{r\theta f}(\beta,1) \sin m\theta \sin \frac{n\pi z}{b} + q_{r\theta f}(\beta,2) \sin m\theta \cos \frac{n\pi z}{b} + q_{r\theta f}(\beta,3) \cos m\theta \sin \frac{n\pi z}{b} + q_{r\theta f}(\beta,4) \cos m\theta \cos \frac{n\pi z}{b} \right] \tag{25b}$$

$$q_{rz}(\beta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[q_{rzf}(\beta,1) \cos m\theta \cos \frac{n\pi z}{b}$$

$$+ q_{rzf}(\beta,2) \cos m\theta \sin \frac{n\pi z}{b} + q_{rzf}(\beta,3) \sin m\theta \cos \frac{n\pi z}{b}$$

$$+ q_{rzf}(\beta,4) \sin m\theta \sin \frac{n\pi z}{b} \Big] \tag{25c}$$

where $\beta = 1, 2$. The Fourier load coefficients q_{rrf}, $q_{r\theta f}$, and q_{rzf} are functions of m and n. Depending on the loading, some of them might be zero.

The displacements $U_r(i)$, $U_\theta(i)$, and $U_z(i)$ are also chosen in double Fourier series as

$$U_r(i) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[A_r(i,1) \cos m\theta \sin \frac{n\pi z}{b}$$

$$+ A_r(i,2) \cos m\theta \cos \frac{n\pi z}{b} + A_r(i,3) \sin m\theta \sin \frac{n\pi z}{b}$$

$$+ A_r(i,4) \sin m\theta \cos \frac{n\pi z}{b} \Big] \tag{26a}$$

$$U_\theta(i) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big[A_\theta(i,1) \sin m\theta \sin \frac{n\pi z}{b}$$

$$+ A_\theta(i,2) \sin m\theta \cos \frac{n\pi z}{b} + A_\theta(i,3) \cos m\theta \sin \frac{n\pi z}{b}$$

$$+ A_\theta(i,4) \cos m\theta \cos \frac{n\pi z}{b} \Big] \tag{26b}$$

$$U_z(i) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[A_z(i,1) \cos m\theta \cos \frac{n\pi z}{b} \right.$$

$$+ A_z(i,2) \cos m\theta \sin \frac{n\pi z}{b} + A_z(i,3) \sin m\theta \cos \frac{n\pi z}{b}$$

$$\left. + A_z(i,4) \sin m\theta \sin \frac{n\pi z}{b} \right] \tag{26c}$$

where A_r, A_θ, and A_z are arbitrary constants. Equations (25) and (26) are substituted into the governing differential equations (22) to (24), and the various terms are grouped appropriately. Now, if the coefficients of $\cos m\theta \sin \frac{n\pi z}{b}$, $\cos m\theta \cos \frac{n\pi z}{b}$, $\sin m\theta \sin \frac{n\pi z}{b}$, and $\sin m\theta \cos \frac{n\pi z}{b}$ are equated to zero in each of the equations, a set of $4(3p + 3)$ simultaneous linear algebraic equations is obtained for each m and n combination. The equations can be written in the matrix form

$$\begin{bmatrix} \underset{\sim}{M}(1) & \underset{\sim}{0}_1 & \underset{\sim}{0}_1 & \underset{\sim}{J}(4) \\ \underset{\sim}{0}_1 & \underset{\sim}{M}(2) & \underset{\sim}{J}(3) & \underset{\sim}{0}_1 \\ \underset{\sim}{0}_1 & \underset{\sim}{J}(2) & \underset{\sim}{M}(3) & \underset{\sim}{0}_1 \\ \underset{\sim}{J}(1) & \underset{\sim}{0}_1 & \underset{\sim}{0}_1 & \underset{\sim}{M}(4) \end{bmatrix} \begin{bmatrix} \underset{\sim}{A}(1) \\ \underset{\sim}{A}(2) \\ \underset{\sim}{A}(3) \\ \underset{\sim}{A}(4) \end{bmatrix} = \begin{bmatrix} \underset{\sim}{K}(1) \\ \underset{\sim}{K}(2) \\ \underset{\sim}{K}(3) \\ \underset{\sim}{K}(4) \end{bmatrix} \tag{27}$$

where

$$\begin{aligned} \underset{\sim}{A}(\beta) = \{ & A_r(0,\beta), A_r(1,\beta), \ldots, A_r(p,\beta); \\ & A_\theta(0,\beta), A_\theta(1,\beta), \ldots, A_\theta(p,\beta); \\ & A_z(0,\beta), A_z(1,\beta), \ldots, A_z(p,\beta) \} \end{aligned} \tag{28}$$

$$\begin{aligned} \underset{\sim}{K}(\beta) = \{ & [q_{rrf}(1,\beta)\, a(1) - q_{rrf}(2,\beta)], -q_{rrf}(2,\beta), \ldots, -q_{rrf}(2,\beta); \\ & [q_{r\theta f}(1,\beta)\, a(1) - q_{r\theta f}(2,\beta)], -q_{r\theta f}(2,\beta), \ldots, -q_{r\theta f}(2,\beta); \\ & [q_{rzf}(1,\beta)\, a(1) - q_{rzf}(2,\beta)], -q_{rzf}(2,\beta), \ldots, -q_{rzf}(2,\beta) \} \end{aligned} \tag{29}$$

$\underset{\sim}{0}_1$ is a null square matrix of order $3p + 3$, and $\underset{\sim}{M}$ and $\underset{\sim}{J}$ are square matrices of order $3p + 3$. (See table III.)

If all the plies are orthotropic, the $\underset{\sim}{J}$ matrix becomes null, and correspondingly, equation (27) degenerates into four lower order equations,

$$\underset{\sim}{M}(\beta)\ \underset{\sim}{A}(\beta) = \underset{\sim}{K}(\beta) \tag{30}$$

where $\beta = 1,\ 2,\ 3,$ and 4.

Solution of the simultaneous algebraic equations (27) or (30) yields the values of A for each m and n combination. Summation of displacements and stresses with respect to m and n to desired accuracy completes the particular solution.

Complementary solution.- For obtaining the complete complementary solution, the displacements $U_r(i)$, $U_\theta(i)$, and $U_z(i)$ are assumed to be in the form

$$\left.\begin{aligned}
U_r(i) &= \sum_{n=0}^{\infty} \left[\xi_r(i,1) \sin \frac{n\pi z}{b} + \xi_r(i,2) \cos \frac{n\pi z}{b}\right] \\
&\quad + \sum_{m=0}^{\infty} \left[\eta_r(i,1) \sin m\theta + \eta_r(i,2) \cos m\theta\right] \\
U_\theta(i) &= \sum_{n=0}^{\infty} \left[\xi_\theta(i,1) \sin \frac{n\pi z}{b} + \xi_\theta(i,2) \cos \frac{n\pi z}{b}\right] \\
&\quad + \sum_{m=0}^{\infty} \left[\eta_\theta(i,1) \cos m\theta + \eta_\theta(i,2) \sin m\theta\right] \\
U_z(i) &= \sum_{n=0}^{\infty} \left[\xi_z(i,1) \cos \frac{n\pi z}{b} + \xi_z(i,2) \sin \frac{n\pi z}{b}\right] \\
&\quad + \sum_{m=0}^{\infty} \left[\eta_z(i,1) \sin m\theta + \eta_z(i,2) \cos m\theta\right]
\end{aligned}\right\} \tag{31}$$

The ξ are functions of θ whereas the η are functions of Z. Equations (31) are substituted into the differential equations (22) to (24), now with a zero right-hand side, and the various terms are properly grouped. If the coefficients of $\sin \frac{n\pi Z}{b}$, $\cos \frac{n\pi Z}{b}$, $\sin m\theta$, and $\cos m\theta$ are equated to zero in each of the equations, two sets of homogeneous simultaneous ordinary differential equations ($6p + 6$ in number) are obtained. One set has θ as the independent variable and ξ as the dependent variable. The second set has Z as the independent variable and η as the dependent variable. The equations can be written in the following form:

$$\begin{bmatrix} \underset{\sim}{\Psi}_\theta(1) & \underset{\sim}{\Pi}_\theta(2) \\ \underset{\sim}{\Pi}_\theta(1) & \underset{\sim}{\Psi}_\theta(2) \end{bmatrix} \frac{d^2}{d\theta^2} \begin{bmatrix} \underset{\sim}{\xi}(1) \\ \underset{\sim}{\xi}(2) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{\Gamma}_\theta(1) & \underset{\sim}{\Upsilon}_\theta(2) \\ \underset{\sim}{\Upsilon}_\theta(1) & \underset{\sim}{\Gamma}_\theta(2) \end{bmatrix} \frac{d}{d\theta} \begin{bmatrix} \underset{\sim}{\xi}(1) \\ \underset{\sim}{\xi}(2) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{\Phi}_\theta(1) & \underset{\sim}{\Xi}_\theta(2) \\ \underset{\sim}{\Xi}_\theta(1) & \underset{\sim}{\Phi}_\theta(2) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\xi}(1) \\ \underset{\sim}{\xi}(2) \end{bmatrix} = \underset{\sim}{0} \tag{32}$$

and

$$\begin{bmatrix} \underset{\sim}{\Psi}_Z(1) & \underset{\sim}{\Pi}_Z(2) \\ \underset{\sim}{\Pi}_Z(1) & \underset{\sim}{\Psi}_Z(2) \end{bmatrix} \frac{d^2}{dZ^2} \begin{bmatrix} \underset{\sim}{\eta}(1) \\ \underset{\sim}{\eta}(2) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{\Gamma}_Z(1) & \underset{\sim}{\Upsilon}_Z(2) \\ \underset{\sim}{\Upsilon}_Z(1) & \underset{\sim}{\Gamma}_Z(2) \end{bmatrix} \frac{d}{dZ} \begin{bmatrix} \underset{\sim}{\eta}(1) \\ \underset{\sim}{\eta}(2) \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{\Phi}_Z(1) & \underset{\sim}{\Xi}_Z(2) \\ \underset{\sim}{\Xi}_Z(1) & \underset{\sim}{\Phi}_Z(2) \end{bmatrix} \begin{bmatrix} \underset{\sim}{\eta}(1) \\ \underset{\sim}{\eta}(2) \end{bmatrix} = \underset{\sim}{0} \tag{33}$$

where

$$\underset{\sim}{\xi}(\beta) = \{\xi_r(0,\beta),\ \xi_r(1,\beta),\ \ldots,\ \xi_r(p,\beta);$$

$$\xi_\theta(0,\beta),\ \xi_\theta(1,\beta),\ \ldots,\ \xi_\theta(p,\beta);$$

$$\xi_z(0,\beta),\ \xi_z(1,\beta),\ \ldots,\ \xi_z(p,\beta)\} \tag{34}$$

and

$$\underset{\sim}{\eta}(\beta) = \{\eta_r(0,\beta),\ \eta_r(1,\beta),\ \ldots,\ \eta_r(p,\beta);$$

$$\eta_\theta(0,\beta),\ \eta_\theta(1,\beta),\ \ldots,\ \eta_\theta(p,\beta);$$

$$\eta_z(0,\beta),\ \eta_z(1,\beta),\ \ldots,\ \eta_z(p,\beta)\} \tag{35}$$

The matrices Ψ_θ, Π_θ, Γ_θ, Υ_θ, Φ_θ, Ξ_θ, Ψ_Z, Π_Z, Γ_Z, Υ_Z, Φ_Z, and Ξ_Z are square matrices of order $3p + 3$. (See table IV.) If all the plies are orthotropic, the matrices $\underset{\sim}{\Pi}_\theta$, $\underset{\sim}{\Upsilon}_\theta$, $\underset{\sim}{\Xi}_\theta$, $\underset{\sim}{\Pi}_Z$, $\underset{\sim}{\Upsilon}_Z$, and $\underset{\sim}{\Xi}_Z$ become null, and each of the equations (32) and (33) reduces to two equations of order $3p + 3$, that is,

$$\underset{\sim}{\Psi}_\theta(\beta)\frac{d^2}{d\theta^2}\underset{\sim}{\xi}(\beta) + \underset{\sim}{\Gamma}_\theta(\beta)\frac{d}{d\theta}\underset{\sim}{\xi}(\beta) + \underset{\sim}{\Phi}_\theta(\beta)\,\underset{\sim}{\xi}(\beta) = \underset{\sim}{0} \tag{36}$$

$$\underset{\sim}{\Psi}_Z(\beta)\frac{d^2}{dZ^2}\underset{\sim}{\eta}(\beta) + \underset{\sim}{\Gamma}_Z(\beta)\frac{d}{dZ}\underset{\sim}{\eta}(\beta) + \underset{\sim}{\Phi}_Z(\beta)\,\underset{\sim}{\eta}(\beta) = \underset{\sim}{0} \tag{37}$$

where $\beta = 1, 2$. Solution of the homogeneous simultaneous ordinary differential equations (32) and (33) or (36) and (37) yields expressions for ξ and η.

One of the procedures of solving homogeneous simultaneous ordinary differential equations in closed form is given in appendix A. The solution of the differential equations (32) and (33) can, in general, be written in the form

$$\left\{\xi_r(i,\beta);\ \xi_\theta(i,\beta);\ \xi_Z(i,\beta)\right\} = \sum_{\ell=1}^{12p+12}\left\{x_r(i,\beta,\ell);\ x_\theta(i,\beta,\ell);\ x_Z(i,\beta,\ell)\right\} X(\ell)\, e^{g(\ell)\theta} \tag{38}$$

and

$$\left\{\eta_r(i,\beta);\ \eta_\theta(i,\beta);\ \eta_z(i,\beta)\right\} = \sum_{\ell=1}^{12p+12} \left\{y_r(i,\beta,\ell);\ y_\theta(i,\beta,\ell);\ y_z(i,\beta,\ell)\right\} Y(\ell)\ e^{s(\ell)z} \tag{39}$$

where X and Y are arbitrary constants. The functions x_r, x_θ, x_z and y_r, y_θ, y_z could be either constants or functions of θ and z, respectively, depending on the multiplicity of roots $g(\ell)$ and $s(\ell)$, respectively. Equations (38) and (39) with equations (31) form the complementary solution.

Further remarks on analysis.- In the previous sections the procedure for obtaining the particular and complementary parts of the solution to the governing differential equations was described for a general case. In many cases, depending on the loading, material properties, and boundary conditions, it might be sufficient to consider only a part of the general solution. After the appropriate part of the general solution is chosen, the boundary conditions are satisfied either exactly or approximately, as the case may be, to obtain the values of the arbitrary constants of the complementary part. Convergence studies can be carried out with respect to the number of terms chosen in the complementary part.

Use of Refined Analysis with Simply Supported Cylinders

In order to assess its accuracy, the present approximate theory is applied to the analysis of simply supported, laminated orthotropic cylinders (fig. 2), and the results are compared with those obtained with the exact three-dimensional analysis.

The boundary conditions for a simply supported, laminated circular cylinder are on $z = 0$ and b

$$\sigma_{zz}(i) = u_r(i) = u_\theta(i) = 0 \tag{40}$$

The boundary conditions (40) are identically satisfied by the first and third sets in the particular solution (eqs. (26)). If the loading on the cylinder is such that it can be expanded in the form of the first set in equations (25), then it is sufficient to consider only the first set in equations (26). Because the particular solution itself satisfies the support boundary conditions, the complementary part is unnecessary. Since the material is orthotropic, from equation (30)

$$\underset{\sim}{M}(1)\ \underset{\sim}{A}(1) = \underset{\sim}{K}(1) \tag{41}$$

For static ($\Omega = 0$) or forced-vibration (Ω is known) problems, equation (41) is solved to obtain the values of A. The series for stresses and displacements is summed to the desired accuracy.

In free-vibration problems the right-hand side of equation (41) is zero; hence,

$$\underset{\sim}{M}(1)\ \underset{\sim}{A}(1) = \underset{\sim}{0} \tag{42}$$

The matrix $\underset{\sim}{M}(1)$ can be written as the sum of two matrices, $\underset{\sim}{\Delta}_1$ and $\underset{\sim}{\Delta}_2\Omega^2$, that is,

$$\underset{\sim}{M}(1) = \underset{\sim}{\Delta}_1 + \underset{\sim}{\Delta}_2\Omega^2 \tag{43}$$

Substituting equation (43) into equation (42) gives

$$\underset{\sim}{\Delta}_1\ \underset{\sim}{A}(1) + \Omega^2\underset{\sim}{\Delta}_2\ \underset{\sim}{A}(1) = \underset{\sim}{0} \tag{44}$$

Methods for numerical solution of characteristic matrix equations such as equation (44) are well known. Solution of equation (44) yields $3p + 3$ frequencies for each m and n combination.

EXACT THREE-DIMENSIONAL ANALYSIS OF SIMPLY SUPPORTED SHELLS

As mentioned previously, simply supported, orthotropic, laminated circular cylindrical shells (fig. 2) happen to be one of the very few cases for which exact three-dimensional analysis is possible. This is because the boundary conditions of a simply supported shell can be automatically satisfied by choosing the displacements and stresses in a double trigonometric series in θ and z coordinates. In the three-dimensional analysis, each ply of the laminated shell is treated as a homogeneous shell. The conditions of continuity and equilibrium at the interfaces are satisfied in addition to exterior surface conditions.

Governing Differential Equations

Consider a homogeneous orthotropic shell. The three-dimensional equations of equilibrium in a cylindrical polar coordinate system are (from ref. 7)

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} - \frac{\sigma_{\theta\theta}}{r} + \rho\Omega^2 u_r = 0 \tag{45a}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \rho\Omega^2 u_\theta = 0 \tag{45b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta z} + \frac{\partial}{\partial z}\sigma_{zz} + \rho\Omega^2 u_z = 0 \tag{45c}$$

By use of stress-strain and strain-displacement relationships (1) and (2), the governing differential equations of three-dimensional elasticity in terms of displacements become

$$\begin{aligned}
&\left(C_{11}\frac{\partial^2}{\partial r^2} + C_{11}\frac{1}{r}\frac{\partial}{\partial r} + C_{55}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - C_{22}\frac{1}{r^2} + \rho\Omega^2 + C_{44}\frac{\partial^2}{\partial z^2}\right)u_r \\
&+ \left[(C_{12} + C_{55})\frac{1}{r}\frac{\partial^2}{\partial r\,\partial \theta} - (C_{55} + C_{22})\frac{1}{r^2}\frac{\partial}{\partial \theta}\right]u_\theta \\
&+ \left[(C_{44} + C_{13})\frac{\partial^2}{\partial r\,\partial z} + (C_{13} - C_{23})\frac{1}{r}\frac{\partial}{\partial z}\right]u_z = 0
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\left[(C_{12} + C_{55})\frac{1}{r}\frac{\partial^2}{\partial r\,\partial \theta} + (C_{55} + C_{22})\frac{1}{r^2}\frac{\partial}{\partial \theta}\right]u_r \\
&+ \left(C_{55}\frac{\partial^2}{\partial r^2} + C_{22}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - C_{55}\frac{1}{r^2} + \rho\Omega^2 + C_{66}\frac{\partial^2}{\partial z^2} + C_{55}\frac{1}{r}\frac{\partial}{\partial r}\right)u_\theta \\
&+ \left[(C_{23} + C_{66})\frac{1}{r}\frac{\partial^2}{\partial \theta\,\partial z}\right]u_z = 0
\end{aligned} \tag{47}$$

$$\begin{aligned}
&\left[(C_{13} + C_{44})\frac{\partial^2}{\partial r\,\partial z} + (C_{44} + C_{23})\frac{1}{r}\frac{\partial}{\partial z}\right]u_r + \left[(C_{23} + C_{66})\frac{1}{r}\frac{\partial^2}{\partial \theta\,\partial z}\right]u_\theta \\
&+ \left(C_{44}\frac{\partial^2}{\partial r^2} + C_{44}\frac{1}{r}\frac{\partial}{\partial r} + C_{66}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \rho\Omega^2 + C_{33}\frac{\partial^2}{\partial z^2}\right)u_z = 0
\end{aligned} \tag{48}$$

Solution of Differential Equations

The displacements u_r, u_θ, and u_z are chosen to be in the form

$$\left.\begin{aligned}
u_r &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_r(r) \cos m\theta \sin \frac{n\pi z}{b} \\
u_\theta &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_\theta(r) \sin m\theta \sin \frac{n\pi z}{b} \\
u_z &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_z(r) \cos m\theta \cos \frac{n\pi z}{b}
\end{aligned}\right\} \tag{49}$$

where ϕ_r, ϕ_θ, and ϕ_z are pure functions of r. By substituting equations (49) in equations (46) to (48) and simplifying, a set of three homogeneous coupled ordinary differential equations in which the independent variable is r is obtained; the equations are

$$\begin{aligned}
&\left[C_{11} \frac{d^2}{dR^2} + C_{11} \frac{1}{R} \frac{d}{dR} - \frac{1}{R^2} (C_{55} m^2 + C_{22}) + (\rho \Omega^2 a^2 - C_{44} N^2)\right] \phi_r \\
&+ \left[(C_{55} + C_{12}) \frac{m}{R} \frac{d}{dR} - (C_{55} + C_{22}) \frac{m}{R^2}\right] \phi_\theta \\
&+ \left[- (C_{13} + C_{44}) N \frac{d}{dR} - (C_{13} - C_{23}) \frac{N}{R}\right] \phi_z = 0
\end{aligned} \tag{50}$$

$$\left[-(C_{55}+C_{12})\frac{m}{R}\frac{d}{dR}-(C_{55}+C_{22})\frac{m}{R^2}\right]\phi_r$$

$$+\left[C_{55}\frac{d^2}{dR^2}-(C_{22}m^2+C_{55})\frac{1}{R^2}+(\rho\Omega^2a^2-C_{66}N^2)+C_{55}\frac{1}{R}\frac{d}{dR}\right]\phi_\theta$$

$$+\left[(C_{23}+C_{66})\frac{mN}{R}\right]\phi_z=0 \tag{51}$$

$$\left[(C_{44}+C_{13})N\frac{d}{dR}+(C_{44}+C_{23})\frac{N}{R}\right]\phi_r+\left[(C_{66}+C_{23})\frac{mN}{R}\right]\phi_\theta$$

$$+\left[C_{44}\frac{d^2}{dR^2}+C_{44}\frac{1}{R}\frac{d}{dR}-C_{66}\frac{m^2}{R^2}+(\rho\Omega^2a^2-N^2C_{33})\right]\phi_z=0 \tag{52}$$

The Frobenius method is now used to solve equations (50) to (52). The functions ϕ_r, ϕ_θ, and ϕ_z are chosen in the form of a power series, that is,

$$\left\{\phi_r;\ \phi_\theta;\ \phi_z\right\}=\sum_{j=0}^{\infty}R^{\alpha+j}\left\{H_r(j);\ H_\theta(j);\ H_z(j)\right\} \tag{53}$$

and are substituted into equations (50) to (52). This substitution results in

$$\sum_{j=0}^{\infty}\left\{\left[C_{11}(j+\alpha)^2-(C_{55}m^2+C_{22})\right]H_r(j)\,R^{j+\alpha-2}+(\rho\Omega^2a^2-C_{44}N^2)\,H_r(j)\,R^{j+\alpha}\right.$$

$$+\left[(C_{55}+C_{12})\,m(j+\alpha)-(C_{55}+C_{22})\,m\right]R^{j+\alpha-2}\,H_\theta(j)$$

$$\left.+\left[-(C_{13}+C_{44})\,N(j+\alpha)-(C_{13}-C_{23})\,N\right]R^{j+\alpha-1}\,H_z(j)\right\}=0 \tag{54}$$

$$\sum_{j=0}^{\infty} \left\{ \left[-(C_{55} + C_{12})\, m(j+\alpha) - (C_{55} + C_{22})\, m \right] R^{j+\alpha-2}\, H_r(j) \right.$$

$$+ \left[C_{55}\,(j+\alpha)^2 - (C_{22}\, m^2 + C_{55}) \right] R^{j+\alpha-2}\, H_\theta(j)$$

$$+ (\rho\Omega^2 a^2 - C_{66}\, N^2)\, R^{j+\alpha}\, H_\theta(j)$$

$$\left. + \left[(C_{23} + C_{66})\, mN \right] R^{j+\alpha-1}\, H_z(j) \right\} = 0 \tag{55}$$

$$\sum_{j=0}^{\infty} \left\{ \left[(C_{44} + C_{13})\, N(j+\alpha) + (C_{44} + C_{23})\, N \right] R^{j+\alpha-1}\, H_r(j) \right.$$

$$+ \left[(C_{66} + C_{23})\, mN \right] R^{j+\alpha-1}\, H_\theta(j)$$

$$+ \left[C_{44}\,(j+\alpha)^2 - C_{66}\, m^2 \right] R^{j+\alpha-2}\, H_z(j)$$

$$\left. + (\rho\Omega^2 a^2 - N^2 C_{33})\, R^{j+\alpha}\, H_z(j) \right\} = 0 \tag{56}$$

If the coefficients of $R^{\alpha-2}$ are equated to zero in equations (54) to (56),

$$\begin{bmatrix} (C_{11}\alpha^2 - C_{55}m^2 - C_{22}) & \left[(C_{55}+C_{12})m\alpha - (C_{55}+C_{22})m\right] & 0 \\ \left[-(C_{55}+C_{12})m\alpha - (C_{55}+C_{22})m\right] & (C_{55}\alpha^2 - C_{22}m^2 - C_{55}) & 0 \\ 0 & 0 & (C_{44}\alpha^2 - C_{66}m^2) \end{bmatrix} \begin{bmatrix} H_r(0) \\ H_\theta(0) \\ H_z(0) \end{bmatrix} = \underset{\sim}{0} \tag{57}$$

For a nontrivial solution, the determinant of the coefficient matrix in equation (57) must be zero; as a result, the indicial equation

$$(\alpha^4 - 2v_1\alpha^2 + v_0)(C_{44}\alpha^2 - C_{66}m^2) = 0 \tag{58}$$

is obtained where

$$v_0 = \frac{C_{22}(m^2 - 1)^2}{C_{11}}$$

and

$$v_1 = \frac{C_{55}(C_{22} + C_{11}) + m^2(C_{11}C_{22} - 2C_{12}C_{55} - C_{12}^2)}{2C_{11}C_{55}}$$

Solution of equation (58) yields six roots for α:

$$\left.\begin{aligned}
\alpha(1) &= \sqrt{v_1 + \sqrt{v_1^2 - v_0}} \\
\alpha(2) &= -\alpha(1) \\
\alpha(3) &= \sqrt{v_1 - \sqrt{v_1^2 - v_0}} \\
\alpha(4) &= -\alpha(3) \\
\alpha(5) &= m\sqrt{C_{66}/C_{44}} \\
\alpha(6) &= -\alpha(5)
\end{aligned}\right\} \tag{59}$$

Solving equation (57) for the constants $H_r(0)$, $H_\theta(0)$, and $H_z(0)$, one obtains (index k is added to denote the root) the following:

For roots, 1, 2, 3, and 4,

$$\left.\begin{aligned} H_r(0,k) &= G(k) \\ H_\theta(0,k) &= \frac{C_{11}\alpha(k)^2 - C_{55}m^2 - C_{22}}{m(C_{55} + C_{22}) - m(C_{55} + C_{12})\alpha(k)} G(k) \\ H_z(0,k) &= 0 \end{aligned}\right\} \tag{60}$$

and for roots 5 and 6,

$$\left.\begin{aligned} H_r(0,k) &= H_\theta(0,k) = 0 \\ H_z(0,k) &= G(k) \end{aligned}\right\} \tag{61}$$

In equations (60) and (61), $G(k)$ for $k = 1, 2, \ldots, 6$ are arbitrary constants. By continuing the process of equating the coefficients of each power of R to zero (that is, $R^{\alpha(k)-1}$, $R^{\alpha(k)}$, $R^{\alpha(k)+1}$, etc.) and solving the resulting equations, the constants $H_r(j,k)$, $H_\theta(j,k)$, and $H_z(j,k)$, for $j = 1, 2, \ldots$ and $k = 1, 2, \ldots, 6$, can be expressed in terms of the arbitrary constant $G(k)$

$$\left\{H_r(j,k);\ H_\theta(j,k);\ H_z(j,k)\right\} = G(k)\ \left\{d_r(j,k);\ d_\theta(j,k);\ d_z(j,k)\right\} \tag{62}$$

where d_r, d_θ, and d_z are functions of $\alpha(k)$, j, m, N, Ω, and material properties. The functions d_r, d_θ, and d_z are obtained through recurrence relations (appendix B). From equations (53) and (62),

$$\left\{\phi_r;\ \phi_\theta;\ \phi_z\right\} = \sum_{k=1}^{6} G(k) \sum_{j=0}^{\infty} R^{\alpha(k)+j} \left\{d_r(j,k);\ d_\theta(j,k);\ d_z(j,k)\right\} \tag{63}$$

If $\alpha(k)$ happens to be a multiple root or an integer, the form of equation (63) changes slightly. The complete results are summarized in appendix B.

If equations (49) and (63) are combined, the displacements u_r, u_θ, and u_z can be formally written (index i is added to denote the ith ply of the laminate) as

$$\left.\begin{aligned}
u_r(i) &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_r(i,k)\, G(i,k) \\
u_\theta(i) &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sin m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_\theta(i,k)\, G(i,k) \\
u_z(i) &= a \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \cos \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_z(i,k)\, G(i,k)
\end{aligned}\right\} \tag{64}$$

Expressions for χ_r, χ_θ, and χ_z are given in appendix B. By making use of stress-strain and strain-displacement relations, stresses are obtained in the form

$$\left.\begin{aligned}
\sigma_{rr}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{rr}(i,k)\, G(i,k) \\
\sigma_{\theta\theta}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{\theta\theta}(i,k)\, G(i,k) \\
\sigma_{zz}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{zz}(i,k)\, G(i,k) \\
\sigma_{rz}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos m\theta \cos \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{rz}(i,k)\, G(i,k) \\
\sigma_{r\theta}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sin m\theta \sin \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{r\theta}(i,k)\, G(i,k) \\
\sigma_{\theta z}(i) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sin m\theta \cos \frac{n\pi z}{b} \sum_{k=1}^{6} \chi_{\theta z}(i,k)\, G(i,k)
\end{aligned}\right\} \tag{65}$$

Expressions for χ_{rr}, $\chi_{\theta\theta}$, . . . are given in appendix B.

Boundary Conditions

The displacements and stresses (eqs. (64) and (65)) satisfy the boundary conditions (eq. (40)) identically. The conditions that are to be satisfied are

For the inner surface $R = a(1)$,

$$\sigma_{rr}(1) = q_{rr}(1), \quad \sigma_{r\theta}(1) = q_{r\theta}(1), \quad \sigma_{rz}(1) = q_{rz}(1) \tag{66}$$

For the outer surface $R = 1$,

$$\sigma_{rr}(p) = q_{rr}(2), \quad \sigma_{r\theta}(p) = q_{r\theta}(2), \quad \sigma_{rz}(p) = q_{rz}(2) \tag{67}$$

For the interfaces $R = a(\ell)$ where $\ell = 2, 3, \ldots, p$,

$$\sigma_{rr}(\ell - 1) - \sigma_{rr}(\ell) = \sigma_{r\theta}(\ell - 1) - \sigma_{r\theta}(\ell) = \sigma_{rz}(\ell - 1) - \sigma_{rz}(\ell) = 0 \tag{68a}$$

$$u_r(\ell - 1) - u_r(\ell) = u_\theta(\ell - 1) - u_\theta(\ell) = u_z(\ell - 1) - u_z(\ell) = 0 \tag{68b}$$

Let the applied loadings be expandable in double Fourier series as in the first set of equation (25). When conditions (66) to (68) are satisfied, a set of $6p$ simultaneous algebraic equations is obtained for each m and n combination. The equations can be written in the matrix form as

$$\underset{\sim}{P}\underset{\sim}{G} = \underset{\sim}{\gamma} \tag{69}$$

where

$$\begin{aligned}\underset{\sim}{G} = \{&G(1,1), G(1,2), \ldots, G(1,6);\\ &G(2,1), G(2,2), \ldots, G(2,6);\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &G(p,1), G(p,2), \ldots, G(p,6)\}\end{aligned} \tag{70}$$

$$\begin{aligned}\underset{\sim}{\gamma} = \{&q_{rrf}(1,1), q_{r\theta f}(1,1), q_{rzf}(1,1); \underset{\sim}{0}_3;\\ &q_{rrf}(2,1), q_{r\theta f}(2,1), q_{rzf}(2,1)\}\end{aligned} \tag{71}$$

($\underset{\sim}{0}_3$ is a null column matrix of order $6p - 6$) and

$$
\underset{\sim}{P} = \begin{bmatrix}
\underset{\sim}{\mu}(a(1),1) & \underset{\sim}{0}_1 & \underset{\sim}{0}_1 & \cdot & \cdot\ \cdot & \cdot & \cdot & \underset{\sim}{0}_1 \\
\underset{\sim}{L}(a(2),1) & -\underset{\sim}{L}(a(2),2) & \underset{\sim}{0}_2 & \cdot & \cdot\ \cdot & \cdot & \cdot & \underset{\sim}{0}_2 \\
\underset{\sim}{0}_2 & \underset{\sim}{L}(a(3),2) & -\underset{\sim}{L}(a(3),3) & \cdot & \cdot\ \cdot & \cdot & \cdot & \underset{\sim}{0}_2 \\
\underset{\sim}{0}_2 & \underset{\sim}{0}_2 & \underset{\sim}{L}(a(4),3) & -\underset{\sim}{L}(a(4),4) & \cdot\ \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \underset{\sim}{L}(a(p-1),p-2) & -\underset{\sim}{L}(a(p-1),p-1) & \underset{\sim}{0}_2 \\
\underset{\sim}{0}_2 & \underset{\sim}{0}_2 & \cdot & \cdot & \cdot\ \cdot & \underset{\sim}{0}_2 & \underset{\sim}{L}(a(p),p-1) & -\underset{\sim}{L}(a(p),p) \\
\underset{\sim}{0}_1 & \underset{\sim}{0}_1 & \cdot & \cdot & \cdot\ \cdot & \cdot & \underset{\sim}{0}_1 & \underset{\sim}{\mu}(1,p)
\end{bmatrix} \tag{72}
$$

In equation (72),

$$
\underset{\sim}{\mu}(\nu,i) = \begin{bmatrix}
\chi_{rr}(i,1), & \chi_{rr}(i,2), & \ldots, & \chi_{rr}(i,6) \\
\chi_{r\theta}(i,1), & \chi_{r\theta}(i,2), & \ldots, & \chi_{r\theta}(i,6) \\
\chi_{rz}(i,1), & \chi_{rz}(i,2), & \ldots, & \chi_{rz}(i,6)
\end{bmatrix}_{R=\nu} \tag{73}
$$

$$
\underset{\sim}{L}(\nu,i) = \begin{bmatrix}
 & \underset{\sim}{\mu}(\nu,i) & & \\
\hline
\chi_{r}(i,1), & \chi_{r}(i,2), & \ldots, & \chi_{r}(i,6) \\
\chi_{\theta}(i,1), & \chi_{\theta}(i,2), & \ldots, & \chi_{\theta}(i,6) \\
\chi_{z}(i,1), & \chi_{z}(i,2), & \ldots, & \chi_{z}(i,6)
\end{bmatrix}_{R=\nu} \tag{74}
$$

and $\underset{\sim}{0}_1$ and $\underset{\sim}{0}_2$ are null matrices of order (3×6) and (6×6).

In static ($\Omega = 0$) and forced-vibration (Ω is known) problems, equation (69) can be solved to obtain the constants G, and evaluation of equations (64) and (65) to the desired accuracy yields the displacements and stresses.

In a free-vibration or wave-propagation problem, the right-hand side of equation (69) is zero. For a nontrivial solution,

$$\det \underset{\sim}{P} = 0 \tag{75}$$

The solution of this characteristic equation yields the frequencies of free vibration. For given circumferential and axial wave numbers m and n, there is an infinite spectrum of natural frequencies, each corresponding to a different thickness or radial mode.

When either m or n equals zero, the characteristic determinant (of order 6p) degenerates into a product of two determinants of orders 4p and 2p.

For $m = 0$, the determinant of order 4p corresponds to axisymmetric motion $\left(\frac{\partial}{\partial \theta} = 0, \quad u_\theta = 0\right)$, and the determinant of order 2p corresponds to torsional vibration $\left(\frac{\partial}{\partial \theta} = 0, \quad u_r = u_z = 0\right)$. Furthermore, in the axisymmetric motion, if the nodal distance $\left(\bar{b} = b/n\right)$ is infinite, the radial and axial displacements become uncoupled, and the determinant of order 4p degenerates into a product of two determinants, each of order 2p. One corresponds to radial motion and the other to axial motion.

For $n = 0$, the determinant of order 4p corresponds to plane strain motion $\left(\frac{\partial}{\partial z} = 0, u_z = 0\right)$, and the determinant of order 2p corresponds to thickness shear in the axial direction $\left(\frac{\partial}{\partial z} = 0, \quad u_r = u_\theta = 0\right)$.

Procedure for Numerical Evaluation

Static and forced-vibration problems.- If the geometric and material properties and the loading are known, the steps to evaluate stresses and displacements are as follows:

(1) Express the loading in double Fourier series and obtain the Fourier load coefficients.

(2) Compute the necessary χ functions for radial coordinates $R = a(1), \ldots, a(p), 1$.

(3) Solve equation (69) to obtain the constants $G(i, k)$, for $k = 1, 2, \ldots, 6$ and $i = 1, 2, \ldots, p$.

(4) Evaluate equations (64) and (65) to desired accuracy by summing the series with respect to m and n to obtain the displacements and stresses at points of interest.

Free vibration. - If m, n, and the geometric and material properties are known, the steps to obtain the frequencies of vibration Ω are as follows:

(1) Assume a starting value for the frequency.
(2) Compute the necessary χ functions for $R = a(1), \ldots, a(p), 1$.
(3) Calculate the determinant $|\underset{\sim}{P}|$.
(4) Adjust the frequency by a suitable amount and repeat steps (2) and (3) until the sign of the determinant for two successive values of frequency is different.
(5) Linearly interpolate the frequency for which the determinant is zero, and by using this as the initial value and employing the well-known regula falsi technique, refine the approximate frequency to the desired accuracy.

Some remarks on computation. - In all the problems – static, forced vibration, or free vibration – it is necessary to evaluate the χ functions at the required radii, not only for solving equation (69), but also for calculating stresses and displacements at the required locations. In free vibrations the χ functions will have to be evaluated for each trial frequency. The χ functions, which are power series in the radial coordinate R, are slowly convergent. For the numerical results presented in this paper, the number of terms summed for good accuracy ranged between 80 and 140 depending on thickness, thinner cylinders requiring more terms. The numerical evaluation was time consuming. In free-vibration problems, in order to save computer time, frequencies were evaluated first by using the refined approximate theory, and these frequencies were then used as the starting values for solving the characteristic equation (eq. (75)). In contrast to the exact three-dimensional analysis, frequency calculations using the refined approximate theory and thin-shell theory took very little time.

THIN-SHELL THEORY

In this section the Flügge type analysis for homogeneous isotropic shells (ref. 8) is extended to anisotropic laminated circular cylindrical shells. The displacement distribution consistent with the basic assumptions of thin-shell theory is

$$\left.\begin{aligned} u_r &= aU_{r0} \\ u_\theta &= a\left\{\frac{R}{a(1)}\,U_{\theta 0} - \left[\frac{R - a(1)}{a(1)}\right]\frac{\partial U_{r0}}{\partial \theta}\right\} \\ u_z &= a\left\{U_{z0} - [R - a(1)]\,\frac{\partial U_{r0}}{\partial Z}\right\} \end{aligned}\right\} \tag{76}$$

where U_{r0}, $U_{\theta 0}$, and U_{z0} are the displacements at the inner surface of the shell. The strains are

$$
\left.
\begin{aligned}
\epsilon_{\theta\theta} &= \frac{1}{a(1)} \frac{\partial U_{\theta 0}}{\partial \theta} - \left[\frac{R - a(1)}{Ra(1)}\right] \frac{\partial^2 U_{r0}}{\partial \theta^2} + \frac{U_{r0}}{R} \\
\epsilon_{zz} &= \frac{\partial U_{z0}}{\partial Z} - [R - a(1)] \frac{\partial^2 U_{r0}}{\partial Z^2} \\
\epsilon_{\theta z} &= \frac{R}{a(1)} \frac{\partial U_{\theta 0}}{\partial Z} + \frac{1}{R} \frac{\partial U_{z0}}{\partial \theta} - \left[\frac{R}{a(1)} - \frac{a(1)}{R}\right] \frac{\partial^2 U_{r0}}{\partial \theta\, \partial Z}
\end{aligned}
\right\} \tag{77}
$$

The modified relationships between stresses $\sigma_{\theta\theta}$, σ_{zz}, and $\sigma_{\theta z}$ and strains $\epsilon_{\theta\theta}$, ϵ_{zz}, and $\epsilon_{\theta z}$ for the jth ply are obtained as follows: Equation (1) is solved for strains ϵ_{rr}, $\epsilon_{\theta\theta}$, ϵ_{zz}, and $\epsilon_{\theta z}$,

$$
\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \epsilon_{\theta z} \end{bmatrix}_j
= \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{36} \\ C_{16} & C_{26} & C_{36} & C_{66} \end{bmatrix}_j^{-1}
\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \end{bmatrix}_j
$$

$$
= \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{16} \\ E_{12} & E_{22} & E_{23} & E_{26} \\ E_{13} & E_{23} & E_{33} & E_{36} \\ E_{16} & E_{26} & E_{36} & E_{66} \end{bmatrix}_j
\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \end{bmatrix}_j \tag{78}
$$

Now, solving for stresses $\sigma_{\theta\theta}$, σ_{ZZ}, and $\sigma_{\theta Z}$ in terms of $\epsilon_{\theta\theta}$, ϵ_{ZZ}, and $\epsilon_{\theta Z}$ and making use of the last three equations of equation (78) yield

$$\begin{bmatrix} \sigma_{\theta\theta} \\ \sigma_{ZZ} \\ \sigma_{\theta Z} \end{bmatrix}_j = \begin{bmatrix} E_{22} & E_{23} & E_{26} \\ E_{23} & E_{33} & E_{36} \\ E_{26} & E_{36} & E_{66} \end{bmatrix}_j^{-1} \begin{bmatrix} \epsilon_{\theta\theta} \\ \epsilon_{ZZ} \\ \epsilon_{\theta Z} \end{bmatrix}_j$$

$$= \begin{bmatrix} c_{22} & c_{23} & c_{26} \\ c_{23} & c_{33} & c_{36} \\ c_{26} & c_{36} & c_{66} \end{bmatrix}_j \begin{bmatrix} \epsilon_{\theta\theta} \\ \epsilon_{ZZ} \\ \epsilon_{\theta Z} \end{bmatrix}_j \tag{79}$$

A procedure identical to that for the refined approximate theory is now followed to derive the governing differential equations and boundary conditions. The strain energy W_s is given by

$$W_s = \frac{a^3}{2} \int_\theta \int_Z \sum_{j=1}^{p} \int_{a(j)}^{a(j+1)} \Big[\sigma_{\theta\theta}(j)\ \epsilon_{\theta\theta}(j) + \sigma_{ZZ}(j)\ \epsilon_{ZZ}(j)$$

$$+ \sigma_{\theta Z}(j)\ \epsilon_{\theta Z}(j)\Big]\ R\ dR\ d\theta\ dZ \tag{80}$$

Substituting for the strains from equations (77) yields

$$W_s = \frac{a^3}{2} \int_\theta \int_Z \Bigg[f_{\theta\theta}(1)\ \frac{\partial U_{\theta 0}}{\partial \theta} - f_{\theta\theta}(2)\ \frac{\partial^2 U_{r0}}{\partial \theta^2} + f_{\theta\theta}(3)\ U_{r0}$$

$$+ f_{ZZ}(1)\ \frac{\partial U_{Z0}}{\partial Z} - f_{ZZ}(2)\ \frac{\partial^2 U_{r0}}{\partial Z^2} + f_{\theta Z}(1)\ \frac{\partial U_{\theta 0}}{\partial Z}$$

$$+ f_{\theta Z}(2)\ \frac{\partial U_{Z0}}{\partial \theta} - f_{\theta Z}(3)\ \frac{\partial^2 U_{r0}}{\partial \theta\ \partial Z} \Bigg] d\theta\ dZ \tag{81}$$

The stress integrals f can be written in terms of displacements. (See table V.) Taking the virtual variation of strain energy and then integrating by parts give

$$\delta W_s = a^3 \int_\theta \int_Z \left\{ \left[f_{\theta\theta}(3) - \frac{\partial^2 f_{\theta\theta}(2)}{\partial\theta^2} - \frac{\partial^2 f_{ZZ}(2)}{\partial Z^2} - \frac{\partial^2 f_{\theta Z}(3)}{\partial\theta\, \partial Z} \right] \delta U_{r0} \right.$$

$$+ \left[-\frac{\partial f_{\theta Z}(1)}{\partial Z} - \frac{\partial f_{\theta\theta}(1)}{\partial\theta} \right] \delta U_{\theta 0}$$

$$\left. + \left[-\frac{\partial f_{ZZ}(1)}{\partial Z} - \frac{\partial f_{\theta Z}(2)}{\partial\theta} \right] \delta U_{Z0} \right\} d\theta\, dZ$$

$$+ a^3 \int_Z \left\{ \left[\frac{\partial f_{\theta\theta}(2)}{\partial\theta} + \frac{\partial f_{\theta Z}(3)}{\partial Z} \right] \delta U_{r0} - f_{\theta\theta}(2)\, \delta \frac{\partial U_{r0}}{\partial\theta} \right.$$

$$\left. + f_{\theta\theta}(1)\, \delta U_{\theta 0} + f_{\theta Z}(2)\, \delta U_{Z0} \right\} dZ$$

$$+ a^3 \int_\theta \left\{ \left[\frac{\partial f_{\theta Z}(3)}{\partial\theta} + \frac{\partial f_{ZZ}(2)}{\partial Z} \right] \delta U_{r0} - f_{ZZ}(2)\, \delta \frac{\partial U_{r0}}{\partial Z} \right.$$

$$\left. + f_{\theta Z}(1)\, \delta U_{\theta 0} + f_{ZZ}(1)\, \delta U_{Z0} \right\} d\theta$$

$$- a^3 \left[f_{\theta Z}(3)\, \delta U_{r0} \right]_{\text{at corners}} \tag{82}$$

The virtual variation of kinetic energy, including the rotary inertia terms, is

$$\int_{t_0}^{t_1} \delta T\ dt = a^5 \int_{t_0}^{t_1} \left(- \int_\theta \int_Z \left\{ \left[f_{\rho r}(1) + \frac{\partial f_{\rho r}(2)}{\partial \theta} + \frac{\partial f_{\rho r}(3)}{\partial Z} \right] \delta U_{r0} \right. \right.$$

$$\left. + f_{\rho\theta}(1)\ \delta U_{\theta 0} + f_{\rho z}(1)\ \delta U_{z0} \right\} d\theta\ dZ$$

$$\left. + \int_Z f_{\rho r}(2)\ \delta U_{r0}\ dZ + \int_\theta f_{\rho r}(3)\ \delta U_{r0}\ d\theta \right) dt \qquad (83)$$

where $f_{\rho r}$, $f_{\rho\theta}$, and $f_{\rho z}$ are given in table V. The virtual work due to applied tractions on the inner and outer surfaces is

$$\delta(W_e)_{surfaces} = a^3 \int_\theta \int_Z \left(\left\{ q_{rr}(2) - a(1)\ q_{rr}(1) + \frac{1 - a(1)}{a(1)} \frac{\partial q_{r\theta}(2)}{\partial \theta} \right. \right.$$

$$\left. + \left[1 - a(1)\right] \frac{\partial q_{rz}(2)}{\partial Z} \right\} \delta U_{r0}$$

$$+ \left[\frac{q_{r\theta}(2)}{a(1)} - a(1)\ q_{r\theta}(1) \right] \delta U_{\theta 0}$$

$$\left. + \left[q_{rz}(2) - a(1)\ q_{rz}(1) \right] \delta U_{z0} \right) d\theta\ dZ$$

$$- a^3 \int_Z \left[\frac{1 - a(1)}{a(1)} \right] q_{r\theta}(2)\ \delta U_{r0}\ dZ$$

$$- a^3 \int_\theta \left[1 - a(1)\right]\ q_{rz}(2)\ \delta U_{r0}\ d\theta \qquad (84)$$

The virtual work due to applied stresses on the edges is

$$\delta(W_e)_{edges} = a^3 \int_\theta \left[\left(Q_{zr} + \frac{\partial \overline{Q}_{z\theta}}{\partial \theta}\right) \delta U_{r0} - \overline{Q}_{zz}\, \delta \frac{\partial U_{r0}}{\partial Z} + Q_{z\theta}\, \delta U_{\theta 0} + Q_{zz}\, \delta U_{z0}\right] d\theta$$

$$+ a^3 \int_Z \left[\left(Q_{\theta r} + \frac{\partial \overline{Q}_{\theta z}}{\partial Z}\right) \delta U_{r0} - \overline{Q}_{\theta\theta}\, \delta \frac{\partial U_{r0}}{\partial \theta} + Q_{\theta\theta}\, \delta U_{\theta 0} + Q_{\theta z}\, \delta U_{z0}\right] dZ$$

$$- a^3 \left[(\overline{Q}_{z\theta} + \overline{Q}_{\theta z})\, \delta U_{r0}\right]_{\text{at corners}} \tag{85}$$

where

$$\left\{Q_{zr};\, Q_{z\theta};\, Q_{zz};\, \overline{Q}_{z\theta};\, \overline{Q}_{zz};\, Q_{\theta r};\, Q_{\theta z};\, Q_{\theta\theta};\, \overline{Q}_{\theta z};\, \overline{Q}_{\theta\theta}\right\}$$

$$= \sum_{j=1}^{p} \int_{a(j)}^{a(j+1)} \left\{q_{zr} R;\; q_{z\theta} \frac{R^2}{a(1)};\; q_{zz} R;\; q_{z\theta} \frac{R\,[R - a(1)]}{a(1)};\; q_{zz}\,[R - a(1)]\,R;\; q_{\theta r};\right.$$

$$\left. q_{\theta z};\; q_{\theta\theta} \frac{R}{a(1)};\; q_{\theta z}\,[r - a(1)];\; q_{\theta\theta} \frac{R - a(1)}{a(1)}\right\} dR \tag{86}$$

Equations (82) to (85) and (13) are substituted into equation (4), and the various terms are properly grouped. Since the virtual displacements are arbitrary, their coefficients are equated to zero to obtain three differential equations and four boundary conditions. The three differential equations are (after substituting for the integrals f in terms of displacement)

$$
\begin{aligned}
&\left\{ - B_{\rho}(9,1) \frac{\partial^2}{\partial t^2} + B_{\rho}(7,3) \frac{\partial^4}{\partial t^2 \partial \theta^2} + B_{\rho}(9,3) \frac{\partial^4}{\partial t^2 \partial Z^2} - B_{22}(1,3) \frac{\partial^4}{\partial \theta^4} \right. \\
&+ 2B_{22}(2,2) \frac{\partial^2}{\partial \theta^2} - \left[2B_{23}(5,3) + B_{66}(12,3) \right] \frac{\partial^4}{\partial \theta^2 \partial Z^2} - B_{22}(3,1) \\
&+ 2B_{23}(6,2) \frac{\partial^2}{\partial Z^2} - B_{33}(9,3) \frac{\partial^4}{\partial Z^4} - 2B_{26}(18,3) \frac{\partial^4}{\partial \theta^3 \partial Z} \\
&\left. + 2B_{26}(13,2) \frac{\partial^2}{\partial \theta\, \partial Z} - 2B_{36}(16,3) \frac{\partial^4}{\partial \theta\, \partial Z^3} \right\} U_{r0} \\
&+ \left\{ - B_{\rho}(10,2) \frac{\partial^3}{\partial t^2 \partial \theta} + B_{22}(4,2) \frac{\partial^3}{\partial \theta^3} - B_{22}(5,1) \frac{\partial}{\partial \theta} \right. \\
&+ \left[B_{66}(14,2) + B_{23}(8,2) \right] \frac{\partial^3}{\partial \theta\, \partial Z^2} + \left[B_{26}(7,2) + B_{26}(17,2) \right] \frac{\partial^3}{\partial \theta^2 \partial Z} \\
&\left. - B_{26}(8,1) \frac{\partial}{\partial Z} + B_{36}(15,2) \frac{\partial^3}{\partial Z^3} \right\} U_{\theta 0} \\
&+ \left\{ - B_{\rho}(9,2) \frac{\partial^3}{\partial t^2 \partial Z} + \left[B_{66}(13,2) + B_{23}(5,2) \right] \frac{\partial^3}{\partial \theta^2 \partial Z} \right. \\
&- B_{23}(6,1) \frac{\partial}{\partial Z} + B_{33}(9,2) \frac{\partial^3}{\partial Z^3} + B_{26}(2,2) \frac{\partial^3}{\partial \theta^3} \\
&\left. - B_{26}(3,1) \frac{\partial}{\partial \theta} + \left[B_{36}(6,2) + B_{36}(16,2) \right] \frac{\partial^3}{\partial \theta\, \partial Z^2} \right\} U_{z0} \\
&= a(1)\, q_{rr}(1) - q_{rr}(2) - \left[\frac{1 - a(1)}{a(1)} \right] \frac{\partial q_{r\theta}(2)}{\partial \theta} - \left[1 - a(1) \right] \frac{\partial q_{rz}(2)}{\partial Z} \qquad (87
\end{aligned}
$$

$$\left\{B_\rho(10,2)\ \frac{\partial^3}{\partial t^2 \partial\theta} - \left[B_{66}(14,2) + B_{23}(8,2)\right]\ \frac{\partial^3}{\partial\theta\ \partial Z^2}\right.$$

$$- B_{22}(4,2)\ \frac{\partial^3}{\partial\theta^3} + B_{22}(5,1)\ \frac{\partial}{\partial\theta} - \left[B_{26}(7,2) + B_{26}(17,2)\right]\ \frac{\partial^3}{\partial\theta^2 \partial Z}$$

$$\left. + B_{26}(8,1)\ \frac{\partial}{\partial Z} - B_{36}(15,2)\ \frac{\partial^3}{\partial Z^3}\right\} U_{r0}$$

$$+ \left\{-B_\rho(11,1)\ \frac{\partial^2}{\partial t^2} + B_{22}(7,1)\ \frac{\partial^2}{\partial\theta^2} + B_{66}(11,1)\ \frac{\partial^2}{\partial Z^2} + 2B_{26}(10,1)\ \frac{\partial^2}{\partial\theta\ \partial Z}\right\} U_{\theta 0}$$

$$+ \left\{\left[B_{66}(8,1) + B_{23}(8,1)\right] \frac{\partial^2}{\partial\theta\ \partial Z} + B_{26}(5,1)\ \frac{\partial^2}{\partial\theta^2} + B_{36}(15,1)\ \frac{\partial^2}{\partial Z^2}\right\} U_{z0}$$

$$= a(1)\ q_{r\theta}(1) - \frac{q_{r\theta}(2)}{a(1)} \tag{88}$$

$$\left\{B_\rho(9,2)\ \frac{\partial^3}{\partial t^2 \partial Z} - \left[B_{23}(5,2) + B_{66}(13,2)\right]\ \frac{\partial^3}{\partial\theta^2 \partial Z} + B_{23}(6,1)\ \frac{\partial}{\partial Z}\right.$$

$$- B_{33}(9,2)\ \frac{\partial^3}{\partial Z^3} - B_{36}(16,2) \frac{\partial^3}{\partial\theta\ \partial Z^2} - B_{26}(2,2)\ \frac{\partial^3}{\partial\theta^3} + B_{26}(3,1)\ \frac{\partial}{\partial\theta}$$

$$\left. - B_{36}(6,2)\ \frac{\partial^3}{\partial Z^2 \partial\theta}\right\} U_{r0}$$

$$+ \left\{\left[B_{23}(8,1) + B_{66}(8,1)\right] \frac{\partial^2}{\partial\theta\ \partial Z} + B_{36}(15,1)\ \frac{\partial^2}{\partial Z^2} + B_{26}(5,1)\ \frac{\partial^2}{\partial\theta^2}\right\} U_{\theta 0}$$

$$+ \left\{- B_\rho(9,1)\ \frac{\partial^2}{\partial t^2} + B_{33}(9,1)\ \frac{\partial^2}{\partial Z^2} + B_{66}(3,1)\ \frac{\partial^2}{\partial\theta^2} + 2B_{36}(6,1) \frac{\partial^2}{\partial\theta\ \partial Z}\right\} U_{z0}$$

$$= a(1)\ q_{rz}(1) - q_{rz}(2) \tag{89}$$

The quantities B_{22}, B_{23}, . . . are defined in table V. The boundary conditions are as follows:

For θ = Constant,

$$\left[\frac{\partial f_{\theta\theta}(2)}{\partial\theta} + \frac{\partial f_{\theta z}(3)}{\partial Z} - f_{\rho r}(2) + q_{r\theta}(2)\ \frac{1 - a(1)}{a(1)}\right] = Q_{\theta r} + \frac{\partial \overline{Q}_{\theta z}}{\partial Z} \quad \text{or} \quad \delta U_{r0} = 0 \tag{90a}$$

$$f_{\theta\theta}(2) = \overline{Q}_{\theta\theta} \quad \text{or} \quad \delta\left(\frac{\partial U_{r0}}{\partial\theta}\right) = 0 \tag{90b}$$

$$f_{\theta\theta}(1) = Q_{\theta\theta} \quad \text{or} \quad \delta U_{\theta 0} = 0 \tag{90c}$$

$$f_{\theta z}(2) = Q_{\theta z} \quad \text{or} \quad \delta U_{z0} = 0 \tag{90d}$$

For z = Constant,

$$\left\{\frac{\partial f_{\theta z}(3)}{\partial\theta} + \frac{\partial f_{zz}(2)}{\partial Z} - f_{\rho r}(3) + q_{rz}(2)\left[1 - a(1)\right]\right\} = Q_{zr} + \frac{\partial \overline{Q}_{z\theta}}{\partial\theta} \quad \text{or} \quad \delta U_{r0} = 0 \tag{91a}$$

$$f_{zz}(2) = \overline{Q}_{zz} \quad \text{or} \quad \delta\left(\frac{\partial U_{r0}}{\partial Z}\right) = 0 \tag{91b}$$

$$f_{\theta z}(1) = Q_{z\theta} \quad \text{or} \quad \delta U_{\theta 0} = 0 \tag{91c}$$

$$f_{zz}(1) = Q_{zz} \quad \text{or} \quad \delta U_{z0} = 0 \tag{91d}$$

and at corners,

$$f_{\theta z}(3) = \overline{Q}_{z\theta} + \overline{Q}_{\theta z} \quad \text{or} \quad \delta U_{r0} = 0 \tag{92}$$

It is noted here that the thin-shell theory has only three differential equations in contrast to the $3p + 3$ equations of the refined approximate theory. Also, the number of boundary conditions that can be specified at any location on the boundary is 4 in thin-shell theory compared with $3p + 3$ in the refined approximate theory.

Procedure for Analysis

The analysis of a circular cylindrical shell now involves finding a solution of the governing differential equations (eqs. (87) to (89)) which satisfies the appropriate boundary conditions (eqs. (90) and (91)). The complete solution of the governing differential equations can be split into a particular part and a complementary part. The procedures for obtaining the particular and complementary parts are similar to those for the refined approximate analysis.

Use of Thin-Shell Theory With Simply Supported Cylinders

In order to assess the accuracy of the thin-shell theory, it is applied to the analysis of harmonic vibrations of simply supported, laminated orthotropic cylinders (fig. 2), and the results are compared with those obtained by using an exact three-dimensional analysis. The displacements U_{r0}, $U_{\theta 0}$, and U_{z0} are assumed to be of the form

$$\left.\begin{aligned} U_{r0} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_r \cos m\theta \sin \frac{n\pi z}{b} \\ U_{\theta 0} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_\theta \sin m\theta \sin \frac{n\pi z}{b} \\ U_{z0} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_z \cos m\theta \cos \frac{n\pi z}{b} \end{aligned}\right\} \tag{93}$$

If the applied loading can be expressed in the form of the first set in equation (25), then by substituting equation (93) along with the first set in equation (25) into equations (87) to (89), a set of three simultaneous algebraic equations is obtained for each combination of m and n. These equations can be written in the form

$$\underset{\sim}{P}_1 \underset{\sim}{S} + \Omega^2 \underset{\sim}{P}_2 \underset{\sim}{S} = \underset{\sim}{\omega} \tag{94}$$

where

$$\underset{\sim}{S} = \left\{S_r;\ S_\theta;\ S_z\right\} \tag{95}$$

$$\underset{\sim}{\omega} = \left\{\left(a(1)\, q_{rrf}(1) - q_{rrf}(2) - \left[\frac{1 - a(1)}{a(1)}\right] m q_{r\theta f}(2) + [1 - a(1)]\; N q_{rzf}(2)\right);\right.$$

$$\left.\left(a(1)\, q_{r\theta f}(1) - \frac{q_{r\theta f}(2)}{a(1)}\right); \left(a(1)\, q_{rzf}(1) - q_{rzf}(2)\right)\right\} \tag{96}$$

and $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ are square matrices of order 3.

The matrices $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ are of the form

$$\underset{\sim}{P}_1 = \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ \kappa_2 & \kappa_4 & \kappa_5 \\ \kappa_3 & \kappa_5 & \kappa_6 \end{bmatrix}$$

$$\underset{\sim}{P}_2 = \begin{bmatrix} \kappa_7 & \kappa_8 & \kappa_9 \\ \kappa_8 & \kappa_{10} & 0 \\ \kappa_9 & 0 & \kappa_{11} \end{bmatrix}$$

where

$$\kappa_1 = -B_{22}(1,3)\, m^4 - 2B_{22}(2,2)\, m^2 - \left[2B_{23}(5,3) + B_{66}(12,3)\right] m^2N^2 - B_{22}(3,1)$$

$$- 2\, B_{23}(6,2)\, N^2 - B_{33}(9,3)\, N^4$$

$$\kappa_2 = -B_{22}(4,2)\, m^3 - B_{22}(5,1)\, m - \left[B_{66}(14,2) + B_{23}(8,2)\right] mN^2$$

$$\kappa_3 = \left[B_{66}(13,2) + B_{23}(5,2)\right] m^2N + B_{23}(6,1)\, N + B_{33}(9,2)\, N^3$$

$$\kappa_4 = -B_{66}(11,1)\, N^2 - B_{22}(7,1)\, m^2$$

$$\kappa_5 = \left[B_{66}(8,1) + B_{23}(8,1)\right] mN$$

$$\kappa_6 = -B_{33}(9,1)\,N^2 - B_{66}(3,1)\,m^2$$

$$\kappa_7 = B_\rho(9,1) + B_\rho(7,3)\,m^2 + B_\rho(9,3)\,N^2$$

$$\kappa_8 = B_\rho(10,2)\,m$$

$$\kappa_9 = -B_\rho(9,2)\,N$$

$$\kappa_{10} = B_\rho(11,1)$$

$$\kappa_{11} = B_\rho(9,1)$$

See table V for B relations.

In the case of static or forced-vibration problems, equation (94) can be solved to obtain the constants S_r, S_θ, and S_z. The series for stresses and displacements are summed to the desired accuracy.

In the case of free vibrations, equation (94) becomes

$$\underset{\sim}{P}_1 \underset{\sim}{S} + \Omega^2 \underset{\sim}{P}_2 \underset{\sim}{S} = \underset{\sim}{0} \tag{97}$$

This characteristic matrix equation is solved to obtain the frequencies of free vibration. Since the order of $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_2$ matrices is 3, the number of frequencies is also 3 for a given m and n combination.

NUMERICAL RESULTS AND DISCUSSION

In this section the numerical results for the free vibrations of simply supported, laminated, orthotropic cylinders are presented for two typical three-ply cylinders. Results from the present refined approximate theory and the thin-shell theory are compared with results from the exact three-dimensional analysis. Since the simple-support conditions simulate the conditions at nodes in wave propagation in infinite cylinders, the present results are applicable to the problem of wave propagation. In wave-propagation problems, the nodal distance $\bar{b}$ is equal to one-half the wavelength, and the wave velocity is equal to $\bar{b}\Omega/\pi$.

The frequencies are presented in terms of a dimensionless frequency parameter λ, defined as

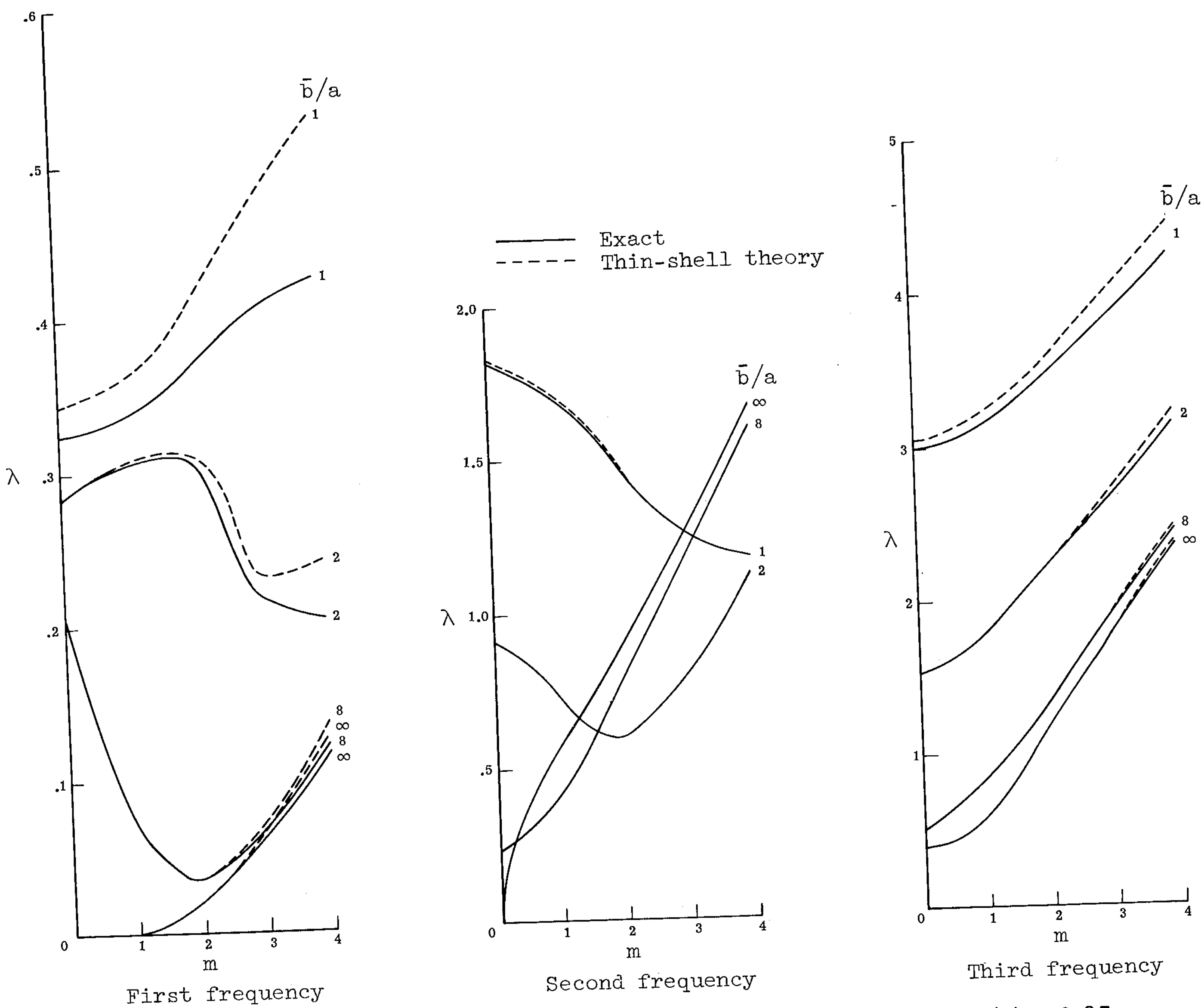
$$\lambda = \Omega a \left\{ \frac{\sum_{j=1}^{p} \rho(j) \left[a(j+1)^2 - a(j)^2\right]}{\sum_{j=1}^{p} C_{33}(j) \left[a(j+1)^2 - a(j)^2\right]} \right\}^{1/2} \tag{98}$$

The first three frequencies of free vibration are presented in figure 3 and table VI for a thin cylinder (total thickness is 5 percent of outer radius) and a thick cylinder (total thickness is 20 percent of outer radius). The material and geometric properties are given in table VI. In both cylinders the middle ply is thicker and also of lower elastic moduli than the other two plies. In figure 3 the frequency parameter is plotted against the circumferential wave number m for various ratios of nodal distance to outer radius. In figures 4 and 5 the displacement distributions across the thickness are plotted for the thick cylinder.

On attempting to classify the first three radial or thickness modes, it was found that in most cases not one of the three displacements was clearly dominating. In this respect the laminated composite cylinders differ from homogeneous isotropic cylinders, where the three thickness modes can be distinguished reasonably well as modes associated with either large radial displacements, large axial displacements, or large circumferential displacements. In the numerical results presented, for $m = 0$, the first mode was axisymmetric, the second mode was torsional, and the third mode was axisymmetric. In contrast, when $m = 0$ in homogeneous isotropic cylinders, the first mode is generally torsional.

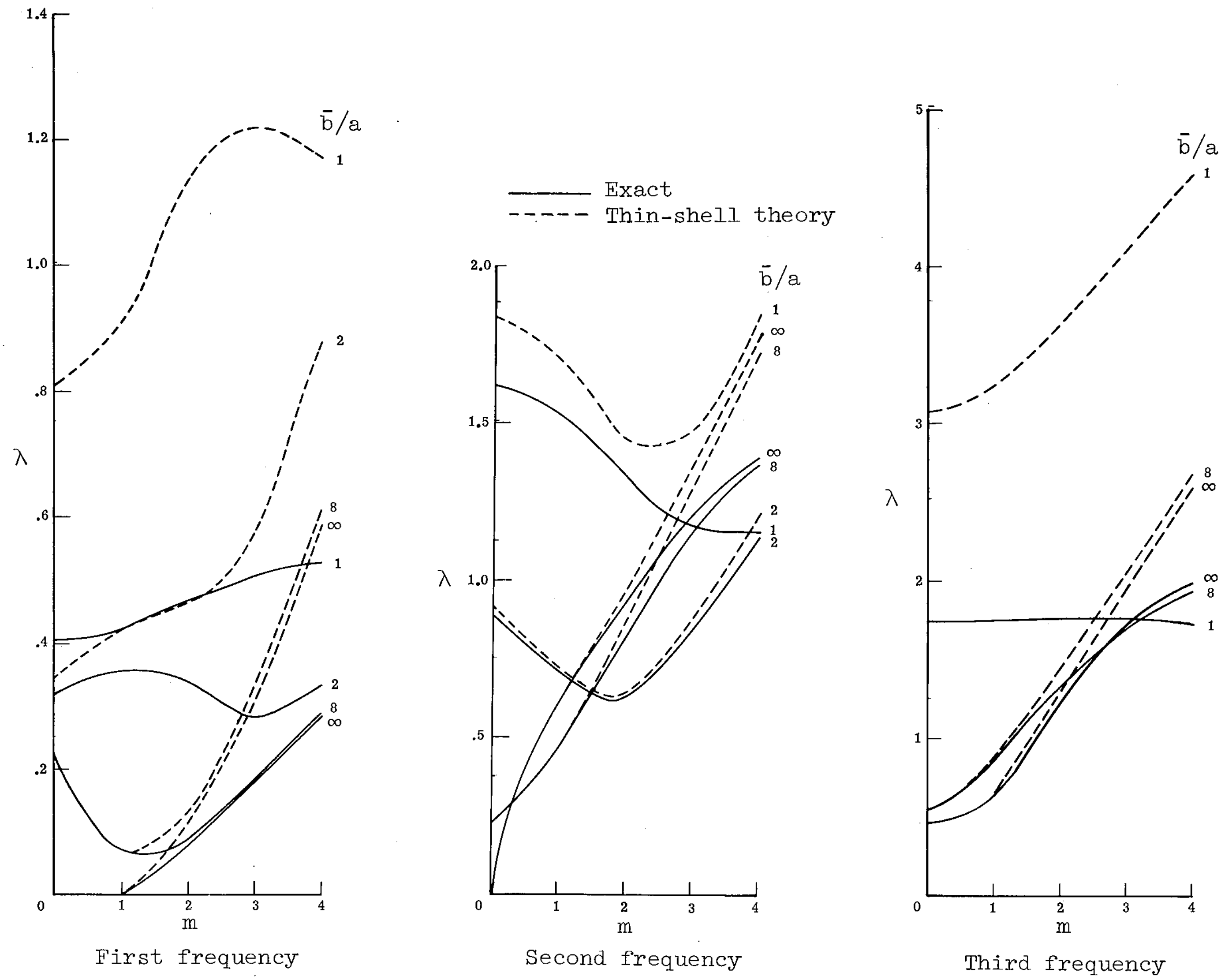
As in homogeneous isotropic cylinders, as the nodal distance approaches infinity, the first and second frequencies vanish when $m = 0$, and only the first frequency vanishes when $m = 1$. For $m \geqq 2$, all the frequencies approach nonzero finite values as the nodal distance approaches infinity. The frequencies are not always monotonic functions of circun ferential wave number m; the nature of variation of frequencies with m depends on nod distance and material properties. (See fig. 3 and table VI.)

In axisymmetric modes (fig. 4), even when the cylinder is thick and the nodal distance is short, the true u_r distribution across the thickness is nearly piecewise linear in the first two modes. Also, the true u_z distribution is piecewise linear in the first mode and becomes nonlinear in the second mode. In the torsional mode the true u_θ distributio is nonlinear.



(a) Thin cylinder: Total thickness, 5 percent of outer radius; a(1) = 0.95; a(2) = 0.955; a(3) = 0.995.

Figure 3.- Frequency parameter λ as a function of circumferential wave number m for various nodal distances. (See table VI for material properties.)



(b) Thick cylinder: Total thickness, 20 percent of outer radius; a(1) = 0.8; a(2) = 0.82; a(3) = 0.98.

Figure 3.- Concluded.

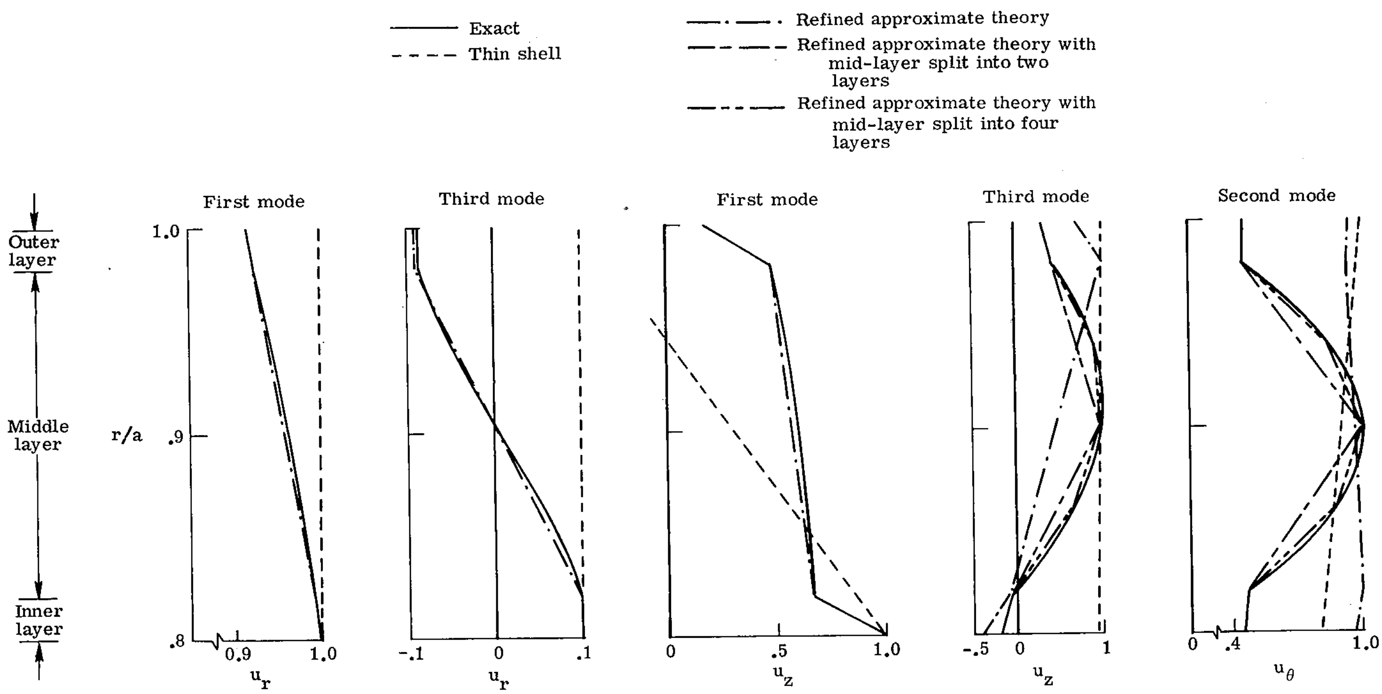


Figure 4.- Displacement distribution across thickness in a thick three-ply cylinder. $m = 0$; $\bar{b}/a = 1$. (First and third modes are axisymmetric, $u_\theta = 0$. Second mode is torsional ($u_r = u_z = 0$). Displacements have been normalized with respect to their maximum values.)

When $m \geqq 1$ (fig. 5), the true displacement distributions across the thickness are nearly piecewise linear for the first mode even when the cylinder is thick and the nodal distance is short. But for the second and third modes, especially when the cylinder is thick, the true displacement distribution across the thickness within each ply is nonlinear.

The thin-shell theory overestimates frequencies (fig. 3 and table VI). Table VII summarizes the influence of thickness, wave number m, and nodal distance on the accuracy of frequencies obtained from thin-shell theory. In general, the error in the second frequency is less than the error in the first and third frequencies. The thickness-wise displacement distributions predicted by thin-shell theory are highly erroneous (figs. 4 and 5).

The first frequency obtained from the refined approximate theory is very close to the exact value, even for thick cylinders, for all values of m, nodal distance, and material properties (table VI). The second and third frequencies are also reasonably close to the exact values. For the thick cylinder, the maximum errors in the second and third frequencies obtained from the refined approximate theory are 13 percent and 6 percent, respectively. In comparison, the corresponding maximum errors in the second and third frequencies obtained by thin-shell theory are 60 percent and 165 percent. The accuracy

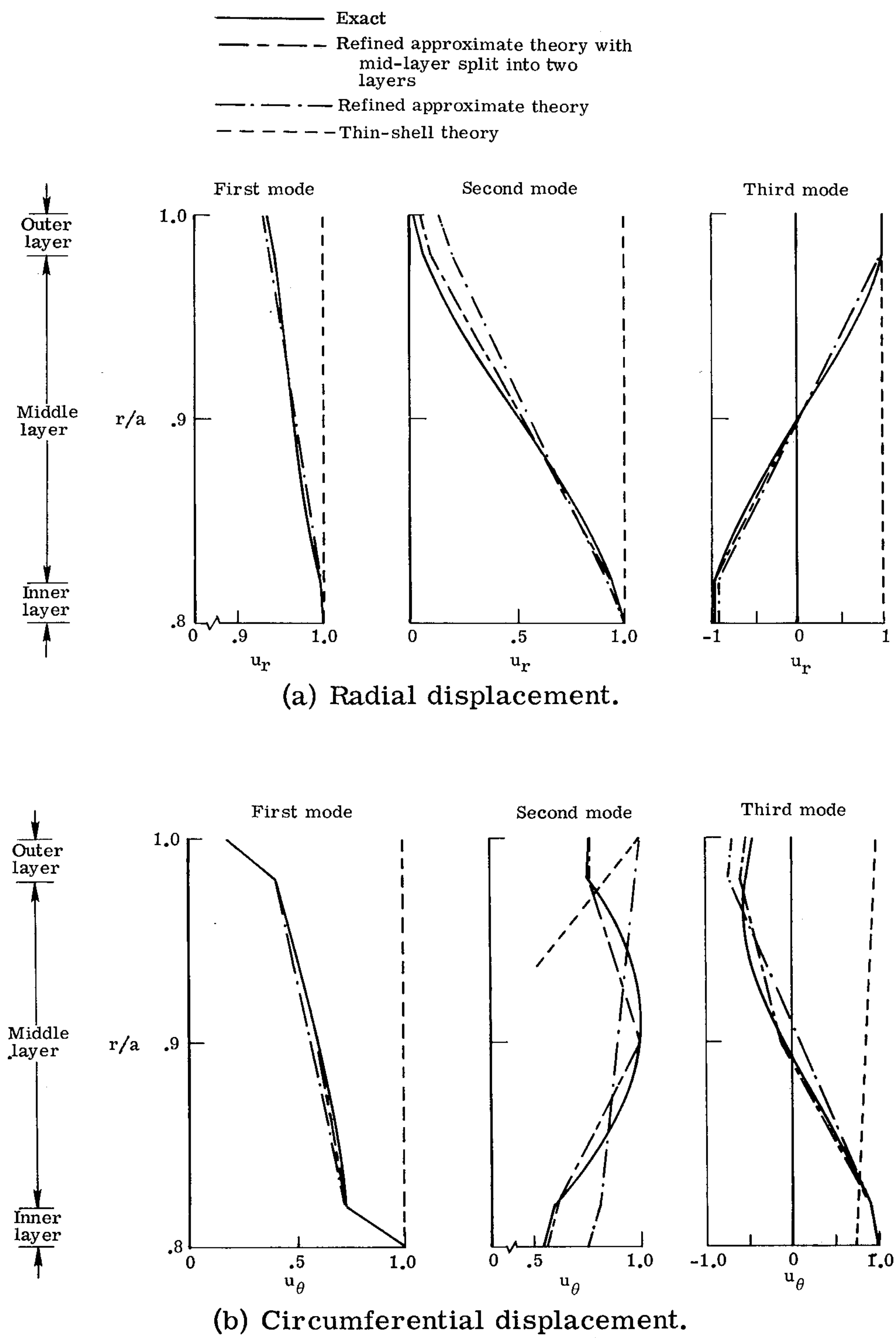
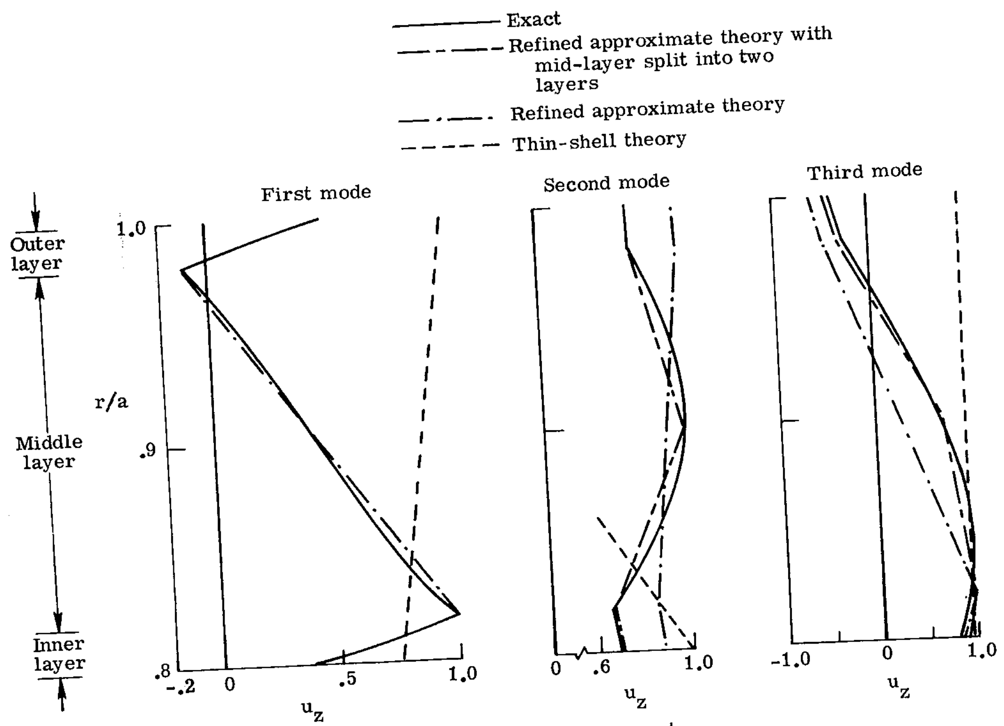


Figure 5.- Displacement distribution across thickness in a thick three-ply cylinder. $m = 4$; $\overline{b}/a = 1$. (See table VI for geometric and material properties. Displacements have been normalized with respect to their maximum values.)



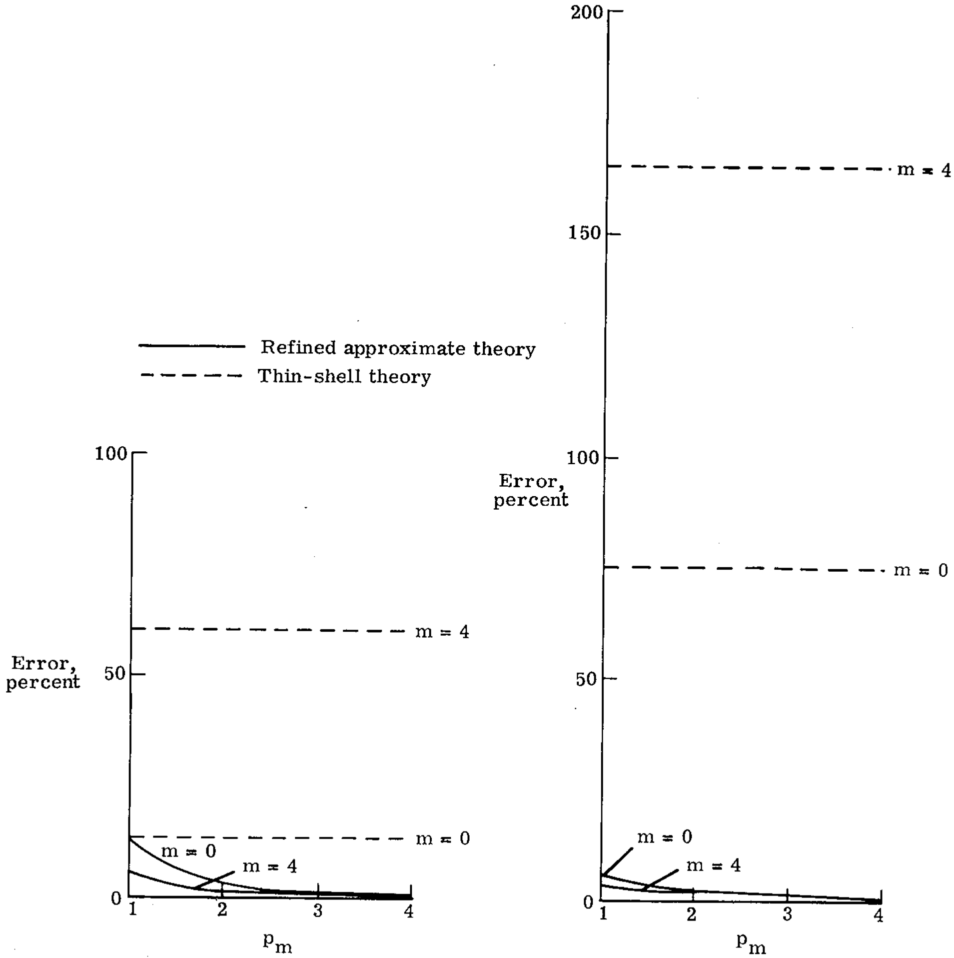
(c) Axial displacement.

Figure 5.- Concluded.

of the frequencies obtained by the refined approximate theory can be further improved by splitting the thicker ply into a number of thinner plies. The second and third frequency values obtained by splitting the middle ply into two plies of equal thickness are given in table VI(b). The frequencies obtained from the refined approximate theory can be brought as close to exact values as desired by increasing p_m, the number of plies into which the middle ply is split (fig. 6). The improvement in the accuracy with increase in p_m is rapid at first but slows down subsequently. The amount of computation increases slightly with increase in p_m.

The number of frequencies predicted by the refined approximate theory for a given m and nodal distance is $3p + 3$, where p is the total number of plies including the artificial splitting of the actual plies. The higher order frequencies (not shown in table VI) from the refined approximate theory correspond to the higher order frequencies of the exact three-dimensional analysis. The refined approximate theory might identify some modes which are not identified by thin-shell theory but are important in response and impact problems.

The refined approximate theory yields a piecewise linear representation of the exact distribution (figs. 4 and 5). Thus, for cases in which the exact displacement distribution is nearly piecewise linear, the approximate and exact distributions are close. In cases where the exact distribution is nonlinear within each ply, the representation of the true distribution by the approximate distribution can be improved by artificially splitting the thicker plies into thinner plies (figs. 4 and 5).



(a) Second frequency. (b) Third frequency.

Figure 6.- Percent error in the frequency parameter as a function of the number of layers p_m into which middle ply is split. $\overline{b}/a = 1$; thick cylinder: Total thickness, 20 percent of outer radius; a(1) = 0.8; a(2) = 0.82; a(3) = 0.98. (See table VI for material properties.)

CONCLUDING REMARKS

A refined approximate theory for the static and dynamic analysis of finite laminated composite shells was developed. The analysis was reduced to a two-dimensional problem in the axial and circumferential coordinates by assuming piecewise linearity of displacements across the thickness. The governing differential equations and the boundary conditions were derived by using a variational approach. General solutions of the governing differential equations were developed in trigonometric series form. Analysis of finite laminated shells with general boundary conditions now involved satisfying the boundary conditions by making use of the appropriate part of the general solution. This theory allows 3 + (3 × Number of plies) boundary conditions on a boundary in contrast to only four boundary conditions of thin-shell theory. Thus, the refined approximate theory allows a proper satisfaction of the boundary conditions and, in turn, the results close to boundaries should be obtainable to a high degree of accuracy.

An exact three-dimensional analysis of simply supported shells was also developed, and the results from it for free vibrations were used for assessing the accuracy of the refined approximate theory and that of thin-shell theory. The thin-shell theory developed in this report does not contain any approximations beyond the basic assumptions inherent in thin-shell theory.

The refined approximate theory was found to be accurate even for thick shells with short nodal distances and high circumferential wave numbers. In contrast, thin-shell theory was found to be accurate for moderately thin shells (thickness about 5 percent of the radius) only in a narrow range of nodal distances and circumferential wave numbers.

The accuracy of the results obtained from the refined approximate theory can be further improved, if needed, by splitting the actual plies into a number of thinner plies. The accuracy can be studied through improvement of the results as the number of plies into which the thicker plies are split increases. The refined approximate theory can be applied to a variety of static and dynamic problems of laminated, composite, circular cylindrical shells. The theory can be extended to arbitrary shells.

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National Aeronautics and Space Administration,
Hampton, Va., August 20, 1973.

APPENDIX A

SOLUTION OF HOMOGENEOUS SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

A procedure for solving a system of homogeneous simultaneous ordinary differential equations of the type of equations (32) and (33) is developed herein. The system of second-order differential equations is converted to a system of first-order equations, which can then be solved by following standard techniques. Consider a system of α equations,

$$\underset{\sim}{A} \frac{d^2}{dy^2} \underset{\sim}{\Phi} + \underset{\sim}{B} \frac{d}{dy} \underset{\sim}{\Phi} + \underset{\sim}{C}\underset{\sim}{\Phi} = \underset{\sim}{0} \tag{A1}$$

where y is the independent variable, $\underset{\sim}{\Phi}$ is the column matrix of dependent variables, and $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ are the coefficient square matrices of order α. Now by choosing

$$\underset{\sim}{\phi} = \frac{d}{dy} \underset{\sim}{\Phi} \tag{A2}$$

equation (A1) can be written in the form

$$\underset{\sim}{A} \frac{d}{dy} \underset{\sim}{\phi} + \underset{\sim}{B}\underset{\sim}{\phi} + \underset{\sim}{C}\underset{\sim}{\Phi} = \underset{\sim}{0} \tag{A3}$$

Equation (A3) can be written after some manipulation as

$$-\underset{\sim}{A}^{-1}\underset{\sim}{B}\underset{\sim}{\phi} - \underset{\sim}{A}^{-1}\underset{\sim}{C}\underset{\sim}{\Phi} = \frac{d}{dy} \underset{\sim}{\phi} \tag{A4}$$

Combining equations (A2) and (A4) yields

$$\underset{\sim}{D} \begin{bmatrix} \underset{\sim}{\Phi} \\ \hline \underset{\sim}{\phi} \end{bmatrix} = \frac{d}{dy} \begin{bmatrix} \underset{\sim}{\Phi} \\ \hline \underset{\sim}{\phi} \end{bmatrix} \tag{A5}$$

where

$$\underset{\sim}{D} = \left[\begin{array}{c:c} \underset{\sim}{0}_1 & \underset{\sim}{I} \\ \hdashline -\underset{\sim}{A}^{-1}\underset{\sim}{C} & -\underset{\sim}{A}^{-1}\underset{\sim}{B} \end{array} \right] \tag{A6}$$

In equation (A6) $\underset{\sim}{I}$ is a unit matrix of order α, and $\underset{\sim}{0}_1$ is a null square matrix of order α. Solution of equation (A5) can be assumed to be of the form

$$\{\underset{\sim}{\Phi}; \underset{\sim}{\phi}\} = \{\underset{\sim}{\Psi}; \underset{\sim}{\psi}\} e^{\mu y} \tag{A7}$$

Substituting equation (A7) into equation (A5) and simplifying give

$$\underset{\sim}{D} \begin{bmatrix} \underset{\sim}{\Psi} \\ \hline \underset{\sim}{\psi} \end{bmatrix} = \mu \begin{bmatrix} \underset{\sim}{\Psi} \\ \hline \underset{\sim}{\psi} \end{bmatrix} \tag{A8}$$

Solution of characteristic matrix equations such as equation (A8) is well known. Since the order of $\underset{\sim}{D}$ is 2α, there will be 2α values of μ. Thus, the solution of equation (A1) can be finally written in the form

$$\underset{\sim}{\Phi} = \sum_{\ell=1}^{2\alpha} \underset{\sim}{\Xi}(\ell)\, T(\ell)\, e^{\mu(\ell)y} \tag{A9}$$

where $\underset{\sim}{\Xi}(\ell)$ is the eigenvector corresponding to root $\mu(\ell)$, and $T(\ell)$ is the arbitrary constant.

In cases where multiple roots occur, the special solutions corresponding to such roots can be obtained by following the usual procedure.

APPENDIX B

χ FUNCTIONS OF EQUATIONS (64) AND (65)

In this appendix, expressions for the χ functions occurring in equations (64) and (65) are given. The form of the solution given by equation (63) is true if the roots are nonzero, are nonmultiple, and do not differ by an integer. When any of these conditions are not satisfied, as in the cases discussed later, the form of solution is different from that of equation (63), and special procedures must be followed to obtain the solution.

Ordinary Case

When equations (63) and (64) are compared, it is seen that the displacement functions χ_r, χ_θ, and χ_z are

$$\left\{\chi_r(k);\ \chi_\theta(k);\ \chi_z(k)\right\} = \sum_{j=0}^{\infty} R^{\alpha(k)+j} \left\{d_r(j,k);\ d_\theta(j,k);\ d_z(j,k)\right\} \tag{B1}$$

The functions $\chi_{rr}(k)$, $\chi_{\theta\theta}(k)$, . . . can be written in the form

$$\chi_{rr}(k) = \sum_{j=0}^{\infty} R^{\alpha(k)+j-1} \left[\psi_1 d_r(j,k) + \psi_2 d_\theta(j,k) + \psi_3 d_z(j,k)\right] \tag{B2}$$

where

	ψ_1	ψ_2	ψ_3
$\chi_{rr}(k)$	$C_{11}\left[\alpha(k) + j\right] + C_{12}$	$C_{12}m$	$-C_{13}NR$
$\chi_{\theta\theta}(k)$	$C_{12}\left[\alpha(k) + j\right] + C_{22}$	$C_{22}m$	$-C_{23}NR$
$\chi_{zz}(k)$	$C_{13}\left[\alpha(k) + j\right] + C_{23}$	$C_{23}m$	$-C_{33}NR$
$\chi_{rz}(k)$	$C_{44}NR$	0	$C_{44}\left[\alpha(k) + j\right]$
$\chi_{r\theta}(k)$	$-C_{55}m$	$C_{55}\left[\alpha(k) + j - 1\right]$	0
$\chi_{\theta z}(k)$	0	$C_{66}NR$	$-C_{66}m$

and d_r, d_θ, and d_z are given in table VIII.

Special Case: $m = 1$

When $m = 1$, roots $\alpha(3)$ and $\alpha(4)$ are zero, and the original solutions, given in equations (B1) and (B2), corresponding to these two indicial constants become identical. For the root $\alpha(3)$, the original solution is retained, but for the indicial constant $\alpha(4)$, a special solution, which is linearly independent of the solution for $\alpha(3)$, is found. The special solution is obtained by differentiating the original solutions (eqs. (B1) and (B2)) with respect to α, then substituting $k = 4$ and taking the limit $\alpha(4) \rightarrow 0$. The solution thus obtained is given as follows (In the following equation, the index k is retained instead of replacing it with 4 since the same equation will be used later with a different value for k):

With $k = 4$,

$$\left\{\chi_r(k);\ \chi_\theta(k);\ \chi_z(k)\right\} = \sum_{j=0}^{\infty} R^j \left\{\left[d_r(j,k) \log_e R + d_r'(j,k)\right];\right.$$

$$\left.\left[d_\theta(j,k) \log_e R + d_\theta'(j,k)\right];\ \left[d_z(j,k) \log_e R + d_z'(j,k)\right]\right\} \tag{B3}$$

Functions $\chi_{rr}(k)$, $\chi_{\theta\theta}(k)$, . . . can be written in the form

$$\chi_{rr}(k) = \sum_{j=0}^{\infty} R^{j-1} \left[\psi_1\left\{d_r(j,k) \log_e R + d_r'(j,k)\right\} + \psi_2\left\{d_\theta(j,k) \log_e R\right.\right.$$

$$\left. + d_\theta'(j,k)\right\} + \psi_3\left\{d_z(j,k) \log_e R + d_z'(j,k)\right\}$$

$$\left. + \psi_4 d_r(j,k) + \psi_5 d_\theta(j,k) + \psi_6 d_z(j,k)\right] \tag{B4}$$

where

	ψ_4	ψ_5	ψ_6
$\chi_{rr}(k)$	C_{11}	0	0
$\chi_{\theta\theta}(k)$	C_{12}	0	0
$\chi_{zz}(k)$	C_{13}	0	0
$\chi_{rz}(k)$	0	0	C_{44}
$\chi_{r\theta}(k)$	0	C_{55}	0
$\chi_{\theta z}(k)$	0	0	0

In equations (B3) and (B4) the primes denote that the value has been differentiated with respect to α and then the limit has been taken as $\alpha \rightarrow 0$. The functions d_r', d_θ', and d_z' are given in table VIII.

Special Case: $m = 0$

When $m = 0$, $\alpha(3) = 1$, $\alpha(4) = -1$, and $\alpha(5) = \alpha(6) = 0$. For $m = 0$, that is $\frac{\partial}{\partial\theta} = 0$, displacements u_r and u_z are not coupled with u_θ; also, the differential equations (50) and (52) are not coupled with equation (51). The motion that is associated with u_r and u_z is axisymmetric, and the associated indicial constants are $\alpha(1)$, $\alpha(2)$, $\alpha(5)$, and $\alpha(6)$. The motion associated with u_θ is torsional, and the corresponding indicial constants are $\alpha(3)$ and $\alpha(4)$.

Axisymmetric motion.- Among the four associated indicial constants, $\alpha(5)$ and $\alpha(6)$ are zero. Therefore, the original solutions, given in equations (B1) and (B2), corresponding to these two indicial constants, become identical. For $\alpha(5)$ the original solution (eqs. (B1) and (B2)) is retained, and a special solution corresponding to $\alpha(6)$ is found. This solution is obtained by differentiating the original solution with respect to α and then substituting $k = 6$ and $\alpha(k) \rightarrow 0$. The solution is the same as given in equations (B3) and (B4), now with $k = 6$.

Torsional motion.- The two indicial constants corresponding to torsional motion are $\alpha(3)$ and $\alpha(4)$, which are equal to +1 and -1. Since the indicial constants differ by an integer, it would be impossible to obtain the solution corresponding to the indicial constant of -1 by use of equations (B1) and (B2). Therefore a special solution corresponding to root $\alpha(4)$ must be obtained. Instead of trying to obtain a special solution starting from the general solution, in this case it is much easier to return to the appropriate differential equation and solve it directly. The differential equation for torsional vibration is (from eq. (51))

$$\left[C_{55}\frac{d^2}{dR^2} - \frac{C_{55}}{R^2} + (\rho\Omega^2 a^2 - C_{66}N^2) + C_{55}\frac{1}{R}\frac{d}{dR}\right]\phi_\theta = 0 \tag{B5}$$

Rearranging gives

$$\left[\hat{R}^2\frac{d^2}{d\hat{R}^2} + \hat{R}\frac{d}{d\hat{R}} + (\hat{R}^2 v - 1)\right]\phi_\theta = 0 \tag{B6}$$

where

$$\left.\begin{aligned} &\hat{R} = \overline{\omega}R \\ &\overline{\omega} = \sqrt{|\omega^2|} \\ &v = 1,\ -1 \qquad\qquad (\omega^2 > 0,\ \omega^2 < 0) \end{aligned}\right\} \tag{B7}$$

and

$$\omega^2 = \frac{\rho\Omega^2 a^2}{C_{55}} - \frac{C_{66}}{C_{55}}N^2 \tag{B8}$$

Equation (B6) is a Bessel equation when $\omega^2 > 0$ and a modified Bessel equation when $\omega^2 < 0$. The two solutions of equation (B6) are $\chi_\theta(3)$ and $\chi_\theta(4)$, given by

$$\chi_\theta(3) = \begin{cases} J_1(\hat{R}) & (\omega^2 > 0) \\ I_1(\hat{R}) & (\omega^2 < 0) \end{cases} \tag{B9}$$

$$\chi_\theta(4) = \begin{cases} Y_1(\hat{R}) & (\omega^2 > 0) \\ K_1(\hat{R}) & (\omega^2 < 0) \end{cases} \tag{B10}$$

where J_1 and Y_1 are Bessel functions of first and second kind of order 1, and I_1 and K_1 are modified Bessel functions of first and second kind of order 1.

Among the other χ functions, χ_r, χ_z, χ_{rr}, $\chi_{\theta\theta}$, χ_{zz}, and χ_{rz} are zero, and

$$\left.\begin{aligned} \chi_{r\theta}(3) &= C_{55}\,\varpi\left[\frac{dJ_1(\hat{R})}{d\hat{R}} - \frac{J_1(\hat{R})}{\hat{R}}\right] \\ \chi_{r\theta}(4) &= C_{55}\,\varpi\left[\frac{dY_1(\hat{R})}{d\hat{R}} - \frac{Y_1(\hat{R})}{\hat{R}}\right] \\ \chi_{\theta z}(3) &= C_{66}NJ_1(\hat{R}) \\ \chi_{\theta z}(4) &= C_{66}NY_1(\hat{R}) \end{aligned}\right\} \tag{B11}$$

When $\omega^2 < 0$ in equations (B11), J is replaced by I and Y by K.

Special Case: $m\sqrt{C_{66}/C_{44}}$ = Integer

When $m\sqrt{C_{66}/C_{44}}$ is an integer, the indicial constants $\alpha(5)$ and $\alpha(6)$ differ by an even integer. In this case it would be impossible to calculate the solution corresponding to $\alpha(6)$ by using equations (B1) and (B2). The special solution is obtained by multiplying the original solution (eqs. (B1) and (B2)) by $\alpha - \alpha(6)$, differentiating with respect to α, and then substituting $\alpha = \alpha(6)$. Now,

$$\begin{aligned} \left\{\chi_r(6);\ \chi_\theta(6);\ \chi_z(6)\right\} &= \sum_{j=0}^{\beta-1} R^{\alpha(6)+j}\left\{d_r(j,6);\ d_\theta(j,6);\ d_z(j,6)\right\} \\ &+ \sum_{j=\beta}^{\infty} R^{\alpha(6)+j}\left\{\left[\hat{d}_r(j,6)\log_e R + \hat{d}'_r(j,6)\right];\right. \\ &\left.\left[\hat{d}_\theta(j,6)\log_e R + \hat{d}'_\theta(j,6)\right];\ \left[\hat{d}_z(j,6)\log_e R + \hat{d}'_z(j,6)\right]\right\} \end{aligned} \tag{B12}$$

where $\beta = \chi(5) - \chi(6) = 2m\sqrt{C_{66}/C_{44}}$. $\chi_{rr}(6)$, $\chi_{r\theta}(6)$, . . . can be written in the form

$$\chi_{rr}(6) = \sum_{j=0}^{\beta-1} R^{\alpha(6)+j-1} \left[\psi_1 d_r(j,6) + \psi_2 d_\theta(j,6) + \psi_3 d_z(j,6)\right]$$

$$+ \sum_{j=\beta}^{\infty} R^{\alpha(6)+j-1} \left\{\psi_4 \hat{d}_r(j,6) + \psi_5 \hat{d}_\theta(j,6) + \psi_6 \hat{d}_z(j,6)\right.$$

$$+ \psi_1 \left[\hat{d}_r(j,6) \log_e R + \hat{d}'_r(j,6)\right]$$

$$\left. + \psi_2 \left[\hat{d}_\theta(j,6) \log_e R + \hat{d}'_\theta(j,6)\right] + \psi_3 \left[\hat{d}_z(j,6) \log_e R + \hat{d}'_z(j,6)\right]\right\} \qquad \text{(B13)}$$

In equations (B12) and (B13),

$$\hat{d}_r(j,6) = \lim_{\alpha \to \alpha(6)} \left\{\left[\alpha - \alpha(6)\right] d_r(j,6)\right\}$$

$$\hat{d}'_r(j,6) = \frac{\partial}{\partial \alpha} \left\{\left[\alpha - \alpha(6)\right] d_r(j,6)\right\} \quad \text{at} \quad \alpha = \alpha(6)$$

and $\hat{d}_r$, $\hat{d}'_r$, . . . are given in table VIII.

Special Case: Isotropic Material

When the material is isotropic, all the indicial constants $\alpha(1)$, $\alpha(2)$, . . ., $\alpha(6)$ are integers. Special solutions can be obtained starting from the general solution given by equations (B1) and (B2). But, as mentioned in the Introduction, for the isotropic case the solution for the governing differential equations (eqs. (46) to (48)) can be obtained very easily by following the displacement potential technique (see refs. 5 and 9).

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TABLE I.- EQUATIONS FOR F

[The $F_{rr}(j,3,1)$, $F_{r\theta}(j,3,1)$, . . . can be expressed in the form $F_{rr}(j,3,1) = \sum_{i=0}^{j} \left[\kappa_r(i)\,U_r(i) + \kappa_\theta(i)\,U_\theta(i) + \kappa_z(i)\,U_z(i)\right]$.]

	$\kappa_r(i)$	$\kappa_\theta(i)$	$\kappa_z(i)$
$F_{rr}(j,3,1)$	$\delta(j,i)\,D_{11}(j,7,1) + D_{12}(j,5,\delta_4)$	$D_{12}(j,5,\delta_4)\frac{\partial}{\partial\theta} + D_{16}(j,4,\delta_4)\frac{\partial}{\partial Z}$	$D_{16}(j,5,\delta_4)\frac{\partial}{\partial\theta} + D_{13}(j,4,\delta_4)\frac{\partial}{\partial Z}$
$F_{r\theta}(j,3,1)$	$D_{55}(j,5,\delta_4)\frac{\partial}{\partial\theta} + D_{45}(j,4,\delta_4)\frac{\partial}{\partial Z}$	$\delta(j,i)\,D_{55}(j,7,1) - D_{55}(j,5,\delta_4)$	$\delta(j,i)\,D_{45}(j,7,1)$
$F_{rz}(j,3,1)$	$D_{45}(j,5,\delta_4)\frac{\partial}{\partial\theta} + D_{44}(j,4,\delta_4)\frac{\partial}{\partial Z}$	$\delta(j,i)\,D_{45}(j,7,1) - D_{45}(j,5,\delta_4)$	$\delta(j,i)\,D_{44}(j,7,1)$
$F_{r\theta}(j,1,\delta_1)$	$D_{55}(j,3,\delta_5)\frac{\partial}{\partial\theta} + D_{45}(j,1,\delta_5)\frac{\partial}{\partial Z}$	$\delta(j,i)\,D_{55}(j,5,\delta_1) - D_{55}(j,3,\delta_5)$	$\delta(j,i)\,D_{45}(j,5,\delta_1)$
$F_{\theta\theta}(j,1,\delta_1)$	$\delta(j,i)\,D_{12}(j,5,\delta_1) + D_{22}(j,3,\delta_5)$	$D_{22}(j,3,\delta_5)\frac{\partial}{\partial\theta} + D_{26}(j,1,\delta_5)\frac{\partial}{\partial Z}$	$D_{26}(j,3,\delta_5)\frac{\partial}{\partial\theta} + D_{23}(j,1,\delta_5)\frac{\partial}{\partial Z}$
$F_{\theta z}(j,1,\delta_1)$	$\delta(j,i)\,D_{16}(j,5,\delta_1) + D_{26}(j,3,\delta_5)$	$D_{26}(j,3,\delta_5)\frac{\partial}{\partial\theta} + D_{66}(j,1,\delta_5)\frac{\partial}{\partial Z}$	$D_{66}(j,3,\delta_5)\frac{\partial}{\partial\theta} + D_{36}(j,1,\delta_5)\frac{\partial}{\partial Z}$
$F_{rz}(j,2,\delta_1)$	$D_{45}(j,1,\delta_5)\frac{\partial}{\partial\theta} + D_{44}(j,2,\delta_5)\frac{\partial}{\partial Z}$	$\delta(j,i)\,D_{45}(j,4,\delta_1) - D_{45}(j,1,\delta_5)$	$\delta(j,i)\,D_{44}(j,4,\delta_1)$
$F_{\theta z}(j,2,\delta_1)$	$\delta(j,i)\,D_{16}(j,4,\delta_1) + D_{26}(j,1,\delta_5)$	$D_{26}(j,1,\delta_5)\frac{\partial}{\partial\theta} + D_{66}(j,2,\delta_5)\frac{\partial}{\partial Z}$	$D_{66}(j,1,\delta_5)\frac{\partial}{\partial\theta} + D_{36}(j,2,\delta_5)\frac{\partial}{\partial Z}$
$F_{zz}(j,2,\delta_1)$	$\delta(j,i)\,D_{13}(j,4,\delta_1) + D_{23}(j,1,\delta_5)$	$D_{23}(j,1,\delta_5)\frac{\partial}{\partial\theta} + D_{36}(j,2,\delta_5)\frac{\partial}{\partial Z}$	$D_{36}(j,1,\delta_5)\frac{\partial}{\partial\theta} + D_{33}(j,2,\delta_5)\frac{\partial}{\partial Z}$
$F_{\rho r}(j,2,\delta_1)$	$D_\rho(j,2,\delta_5)\frac{\partial^2}{\partial t^2}$	0	0
$F_{\rho\theta}(j,2,\delta_1)$	0	$D_\rho(j,2,\delta_5)\frac{\partial^2}{\partial t^2}$	0
$F_{\rho z}(j,2,\delta_1)$	0	0	$D_\rho(j,2,\delta_5)\frac{\partial^2}{\partial t^2}$

Note: $\delta_4 = 1 + \delta(j,i)$; $\delta_5 = \delta_1 + \delta(j,i)$

The stiffnesses D are pure functions of the material properties and the inner and outer radii of the individual plies; that is,

$$\left\{D_\beta(j,1,\ell), \ldots, D_\beta(j,7,\ell)\right\} = \int_{a(j)}^{a(j+1)} \left\{1;\ R;\ \frac{1}{R};\ \frac{R}{h(j)};\ \frac{1}{h(j)};\ \frac{1}{Rh(j)};\ \frac{R}{h(j)^2}\right\} C_\beta(j) \left[\frac{R - a(j)}{h(j)}\right]^{\ell-1} dR$$

where $\beta = 11, 12, \ldots, 66, \rho$. When $\beta = \rho$, $C_\beta = \rho$. If the material properties are constant across a ply,

$$D_\beta(j,1,\ell) = \frac{C_\beta(j)\,h(j)}{\ell} \qquad D_\beta(j,3,2) = C_\beta(j)\left\{1 - \frac{a(j)}{h(j)}\log_e\left[1 + \frac{h(j)}{a(j)}\right]\right\} \qquad D_\beta(j,5,\ell) = \frac{D_\beta(j,1,\ell)}{h(j)}$$

$$D_\beta(j,2,\ell) = C_\beta(j)\,h(j)\left[\frac{h(j)}{\ell + 1} + \frac{a(j)}{\ell}\right] \qquad D_\beta(j,3,3) = C_\beta(j)\left\{\frac{1}{2} - \frac{a(j)}{h(j)} + \frac{a(j)^2}{h(j)^2}\log_e\left[1 + \frac{h(j)}{a(j)}\right]\right\} \qquad D_\beta(j,6,\ell) = \frac{D_\beta(j,3,\ell)}{h(j)}$$

$$D_\beta(j,3,1) = C_\beta(j)\log_e\left[1 + \frac{h(j)}{a(j)}\right] \qquad D_\beta(j,4,\ell) = \frac{D_\beta(j,2,\ell)}{h(j)} \qquad D_\beta(j,7,\ell) = \frac{D_\beta(j,2,\ell)}{h(j)^2}$$

TABLE II.- COEFFICIENTS V

$$\left[\begin{array}{l} V(\beta,k,i) = \sum_{j=\bar{k}}^{p} \Delta(j,i)\,\tau(\beta); \\ \bar{k} = 1 \quad \text{for} \quad k = 0, \quad \bar{k} = k \quad \text{for} \quad k > 0; \\ \Delta(j,i) = 0 \quad \text{for} \quad i > j, \quad \Delta(j,i) = 1 \quad \text{for} \quad i \leqq j \end{array}\right]$$

β	$\tau(\beta)$
1	$D_\rho(j,2,\delta_5)\,a^2$
2	$D_{55}(j,3,\delta_5)$
3	$D_{44}(j,2,\delta_5)$
4	$2D_{45}(j,1,\delta_5)$
5	$-\delta(j,k)\,\delta(j,i)\,D_{11}(j,7,1) - \delta(j,k)\,D_{12}(j,5,\delta_4) - \delta(j,i)\,D_{12}(j,5,\delta_1) - D_{22}(j,3,\delta_5)$
6	$-\delta(j,k)\,D_{12}(j,5,\delta_4) - D_{22}(j,3,\delta_5) - D_{55}(j,3,\delta_5) + \delta(j,i)\,D_{55}(j,5,\delta_1)$
7	$-\delta(j,k)\,D_{16}(j,4,\delta_4) - D_{45}(j,1,\delta_5) + \delta(j,i)\,D_{45}(j,4,\delta_1) - D_{26}(j,1,\delta_5)$
8	$-\delta(j,k)\,D_{16}(j,5,\delta_4) + \delta(j,i)\,D_{45}(j,5,\delta_1) - D_{26}(j,3,\delta_5)$
9	$-\delta(j,k)\,D_{13}(j,4,\delta_4) - D_{23}(j,1,\delta_5) + \delta(j,i)\,D_{44}(j,4,\delta_1)$
10	$-\delta(j,k)\,D_{55}(j,5,\delta_4) + D_{55}(j,3,\delta_5) + \delta(j,i)\,D_{12}(j,5,\delta_1) + D_{22}(j,3,\delta_5)$
11	$D_{45}(j,1,\delta_5) + \delta(j,i)\,D_{16}(j,4,\delta_1) - \delta(j,k)\,D_{45}(j,4,\delta_4) + D_{26}(j,1,\delta_5)$
12	$D_{22}(j,3,\delta_5)$
13	$D_{66}(j,2,\delta_5)$
14	$2D_{26}(j,1,\delta_5)$
15	$\delta(j,k)\,D_{55}(j,5,\delta_4) - \delta(j,k)\,\delta(j,i)\,D_{55}(j,7,1) - D_{55}(j,3,\delta_5) + \delta(j,i)\,D_{55}(j,5,\delta_1)$
16	$D_{26}(j,3,\delta_5)$
17	$D_{36}(j,2,\delta_5)$
18	$D_{23}(j,1,\delta_5) + D_{66}(j,1,\delta_5)$
19	$-\delta(j,k)\,\delta(j,i)\,D_{45}(j,7,1) + \delta(j,i)\,D_{45}(j,5,\delta_1)$
20	$D_{26}(j,3,\delta_5) - \delta(j,k)\,D_{45}(j,5,\delta_4) + \delta(j,i)\,D_{16}(j,5,\delta_1)$
21	$\delta(j,i)\,D_{13}(j,4,\delta_1) - \delta(j,k)\,D_{44}(j,4,\delta_4) + D_{23}(j,1,\delta_5)$
22	$-\delta(j,k)\,\delta(j,i)\,D_{45}(j,7,1) + \delta(j,k)\,D_{45}(j,5,\delta_4)$
23	$D_{66}(j,3,\delta_5)$
24	$D_{33}(j,2,\delta_5)$
25	$2D_{36}(j,1,\delta_5)$
26	$-\delta(j,k)\,\delta(j,i)\,D_{44}(j,7,1)$

Notes: $\delta_1 = 1 + \delta(j,k)$, $\delta_5 = 1 + \delta(j,i) + \delta(j,k)$, $\delta_4 = 1 + \delta(j,i)$, and the stiffnesses D are given in table I.

TABLE III.- MATRICES $\underset{\sim}{M}(\beta)$ AND $\underset{\sim}{J}(\beta)$

[Matrices $\underset{\sim}{M}(\beta)$ and $\underset{\sim}{J}(\beta)$ can be written in terms of nine submatrices, in the form

$$\underset{\sim}{M}(\beta) = \begin{bmatrix} \underset{\sim}{\Psi}_1(\beta) & \underset{\sim}{\Psi}_2(\beta) & \underset{\sim}{\Psi}_3(\beta) \\ \underset{\sim}{\Psi}_4(\beta) & \underset{\sim}{\Psi}_5(\beta) & \underset{\sim}{\Psi}_6(\beta) \\ \underset{\sim}{\Psi}_7(\beta) & \underset{\sim}{\Psi}_8(\beta) & \underset{\sim}{\Psi}_9(\beta) \end{bmatrix}$$

General terms of matrices $\Psi_1(\beta), \ldots, \Psi_9(\beta)$ are given below in which k refers to row and i to column.]

	$\underset{\sim}{M}(\beta)$	$\underset{\sim}{J}(\beta)$
$\underset{\sim}{\Psi}_1(\beta)$	$V(1,k,i)\,\Omega^2 - V(2,k,i)\,m^2 - V(3,k,i)\,N^2 + V(5,k,i)$	$-V(4,k,i)\,mN\delta_2(\beta)\,\delta_3(\beta)$
$\underset{\sim}{\Psi}_2(\beta)$	$V(6,k,i)\,m\delta_2(\beta)$	$V(7,k,i)\,N\delta_3(\beta)$
$\underset{\sim}{\Psi}_3(\beta)$	$-V(9,k,i)\,N\delta_3(\beta)$	$-V(8,k,i)\,m\delta_2(\beta)$
$\underset{\sim}{\Psi}_4(\beta)$	$-V(10,k,i)\,m\delta_2(\beta)$	$V(11,k,i)\,N\delta_3(\beta)$
$\underset{\sim}{\Psi}_5(\beta)$	$V(1,k,i)\,\Omega^2 - V(12,k,i)\,m^2 - V(13,k,i)\,N^2 + V(15,k,i)$	$V(14,k,i)\,mN\delta_2(\beta)\,\delta_3(\beta)$
$\underset{\sim}{\Psi}_6(\beta)$	$V(18,k,i)\,mN\delta_2(\beta)\,\delta_3(\beta)$	$-V(16,k,i)\,m^2 - V(17,k,i)\,N^2 + V(19,k,i)$
$\underset{\sim}{\Psi}_7(\beta)$	$V(21,k,i)\,N\delta_3(\beta)$	$-V(20,k,i)\,m\delta_2(\beta)$
$\underset{\sim}{\Psi}_8(\beta)$	$V(18,k,i)\,mN\delta_2(\beta)\,\delta_3(\beta)$	$-V(16,k,i)\,m^2 - V(17,k,i)\,N^2 + V(22,k,i)$
$\underset{\sim}{\Psi}_9(\beta)$	$V(1,k,i)\,\Omega^2 - V(23,k,i)\,m^2 - V(24,k,i)\,N^2 + V(26,k,i)$	$V(25,k,i)\,mN\delta_2(\beta)\,\delta_3(\beta)$

Note: $\delta_2(1) = \delta_2(2) = 1$, $\delta_2(3) = \delta_2(4) = -1$, $\delta_3(1) = \delta_3(3) = 1$, $\delta_3(2) = \delta_3(4) = -1$.
The coefficients V are given in table II.

TABLE IV.- MATRICES $\underset{\sim}{\Psi}_\theta(\beta), \underset{\sim}{\Pi}_\theta(\beta), \ldots$ OCCURRING IN EQUATIONS (32) AND (33)

Matrices $\underset{\sim}{\Psi}_\theta(\beta), \underset{\sim}{\Pi}_\theta(\beta), \ldots$ can be written in terms of nine submatrices, each of order $p + 1$, in the form

$$\underset{\sim}{\Psi}_\theta(\beta) = \begin{bmatrix} \underset{\sim}{\Theta}_1(\beta) & \underset{\sim}{\Theta}_2(\beta) & \underset{\sim}{\Theta}_3(\beta) \\ \underset{\sim}{\Theta}_4(\beta) & \underset{\sim}{\Theta}_5(\beta) & \underset{\sim}{\Theta}_6(\beta) \\ \underset{\sim}{\Theta}_7(\beta) & \underset{\sim}{\Theta}_8(\beta) & \underset{\sim}{\Theta}_9(\beta) \end{bmatrix}$$

General terms of matrices $\underset{\sim}{\Theta}_1(\beta), \underset{\sim}{\Theta}_2(\beta), \ldots$ are given in which k refers to row and i to column.

	$\underset{\sim}{\Theta}_1(\beta)$	$\underset{\sim}{\Theta}_2(\beta)$	$\underset{\sim}{\Theta}_3(\beta)$	$\underset{\sim}{\Theta}_4(\beta)$	$\underset{\sim}{\Theta}_5(\beta)$	$\underset{\sim}{\Theta}_6(\beta)$	$\underset{\sim}{\Theta}_7(\beta)$	$\underset{\sim}{\Theta}_8(\beta)$	$\underset{\sim}{\Theta}_9(\beta)$
$\underset{\sim}{\Psi}_\theta(\beta)$	V(2,k,i)	0	0	0	V(12,k,i)	0	0	0	V(23,k,i)
$\underset{\sim}{\Pi}_\theta(\beta)$	0	0	0	0	0	V(16,k,i)	0	V(16,k,i)	0
$\underset{\sim}{\Gamma}_\theta(\beta)$	0	V(6,k,i)	0	V(10,k,i)	0	$-V(18,k,i)\,N\delta_6(\beta)$	0	$V(18,k,i)\,N\delta_6(\beta)$	0
$\underset{\sim}{\Upsilon}_\theta(\beta)$	$V(4,k,i)\,N\delta_6(\beta)$	0	V(8,k,i)	0	$V(14,k,i)\,N\delta_6(\beta)$	0	V(20,k,i)	0	$-V(25,k,i)\,N\delta_6(\beta)$
$\underset{\sim}{\Phi}_\theta(\beta)$	$V(1,k,i)\,\Omega^2 - V(3,k,i)\,N^2 + V(5,k,i)$	0	$-V(9,k,i)\,N\delta_6(\beta)$	0	$V(1,k,i)\,\Omega^2 - V(13,k,i)\,N^2 + V(15,k,i)$	0	$V(21,k,i)\,N\delta_6(\beta)$	0	$V(1,k,i)\,\Omega^2 - V(24,k,i)\,N^2 + V(26,k,i)$
$\underset{\sim}{\Xi}_\theta(\beta)$	0	$V(7,k,i)\,N\delta_6(\beta)$	0	$V(11,k,i)\,N\delta_6(\beta)$	0	$-V(17,k,i)\,N^2 + V(19,k,i)$	0	$-V(17,k,i)\,N^2 + V(22,k,i)$	0
$\underset{\sim}{\Psi}_z(\beta)$	V(3,k,i)	0	0	0	V(13,k,i)	0	0	0	V(24,k,i)
$\underset{\sim}{\Pi}_z(\beta)$	0	0	0	0	0	V(17,k,i)	0	V(17,k,i)	0
$\underset{\sim}{\Gamma}_z(\beta)$	0	0	V(9,k,i)	0	0	$V(18,k,i)\,m\delta_6(\beta)$	V(21,k,i)	$-V(18,k,i)\,m\delta_6(\beta)$	0
$\underset{\sim}{\Upsilon}_z(\beta)$	$V(4,k,i)\,m\delta_6(\beta)$	V(7,k,i)	0	V(11,k,i)	$-V(14,k,i)\,m\delta_6(\beta)$	0	0	0	$V(25,k,i)\,m\delta_6(\beta)$
$\underset{\sim}{\Phi}_z(\beta)$	$V(1,k,i)\,\Omega^2 - V(2,k,i)\,m^2 + V(5,k,i)$	$-V(6,k,i)\,m\delta_6(\beta)$	0	$V(10,k,i)\,m\delta_6(\beta)$	$V(1,k,i)\,\Omega^2 - V(12,k,i)\,m^2 + V(15,k,i)$	0	0	0	$V(1,k,i)\,\Omega^2 - V(23,k,i)\,m^2 + V(26,k,i)$
$\underset{\sim}{\Xi}_z(\beta)$	0	0	$V(8,k,i)\,m\delta_6(\beta)$	0	0	$-V(16,k,i)\,m^2 + V(19,k,i)$	$V(20,k,i)\,m\delta_6(\beta)$	$-V(16,k,i)\,m^2 + V(22,k,i)$	0

Notes: $\delta_6(1) = 1$; $\delta_6(2) = -1$.

The coefficients V are given in table II.

TABLE V.- EQUATIONS FOR f

$f_{\theta\theta}(1), f_{\theta\theta}(2), \ldots$ can be expressed in the form

$$f_{\theta\theta}(1) = \left[\upsilon_r(1) - \upsilon_r(2)\frac{\partial^2}{\partial\theta^2} - \upsilon_r(3)\frac{\partial^2}{\partial Z^2} - \upsilon_r(4)\frac{\partial^2}{\partial\theta\,\partial Z}\right]U_{r0}$$

$$+\left[\upsilon_\theta(1)\frac{\partial}{\partial\theta} + \upsilon_\theta(2)\frac{\partial}{\partial Z}\right]U_{\theta 0} + \left[\upsilon_z(1)\frac{\partial}{\partial\theta} + \upsilon_z(2)\frac{\partial}{\partial Z}\right]U_{z0}$$

	$\upsilon_r(1)$	$\upsilon_r(2)$	$\upsilon_r(3)$	$\upsilon_r(4)$	$\upsilon_\theta(1)$	$\upsilon_\theta(2)$	$\upsilon_z(1)$	$\upsilon_z(2)$
$f_{\theta\theta}(1)$	$B_{22}(5,1)$	$B_{22}(4,2)$	$B_{23}(8,2)$	$B_{26}(17,2)$	$B_{22}(7,1)$	$B_{26}(10,1)$	$B_{26}(5,1)$	$B_{23}(8,1)$
$f_{\theta\theta}(2)$	$B_{22}(2,2)$	$B_{22}(1,3)$	$B_{23}(5,3)$	$B_{26}(18,3)$	$B_{22}(4,2)$	$B_{26}(7,2)$	$B_{26}(2,2)$	$B_{23}(5,2)$
$f_{\theta\theta}(3)$	$B_{22}(3,1)$	$B_{22}(2,2)$	$B_{23}(6,2)$	$B_{26}(13,2)$	$B_{22}(5,1)$	$B_{26}(8,1)$	$B_{26}(3,1)$	$B_{23}(6,1)$
$f_{zz}(1)$	$B_{23}(6,1)$	$B_{23}(5,2)$	$B_{33}(9,2)$	$B_{36}(16,2)$	$B_{23}(8,1)$	$B_{36}(15,1)$	$B_{36}(6,1)$	$B_{33}(9,1)$
$f_{zz}(2)$	$B_{23}(6,2)$	$B_{23}(5,3)$	$B_{33}(9,3)$	$B_{36}(16,3)$	$B_{23}(8,2)$	$B_{36}(15,2)$	$B_{36}(6,2)$	$B_{33}(9,2)$
$f_{\theta z}(1)$	$B_{26}(8,1)$	$B_{26}(7,2)$	$B_{36}(15,2)$	$B_{66}(14,2)$	$B_{26}(10,1)$	$B_{66}(11,1)$	$B_{66}(8,1)$	$B_{36}(15,1)$
$f_{\theta z}(2)$	$B_{26}(3,1)$	$B_{26}(2,2)$	$B_{36}(6,2)$	$B_{66}(13,2)$	$B_{26}(5,1)$	$B_{66}(8,1)$	$B_{66}(3,1)$	$B_{36}(6,1)$
$f_{\theta z}(3)$	$B_{26}(13,2)$	$B_{26}(18,3)$	$B_{36}(16,3)$	$B_{66}(12,3)$	$B_{26}(17,2)$	$B_{66}(14,2)$	$B_{66}(13,2)$	$B_{36}(16,2)$

$$f_{\rho r}(1) = B_\rho(9,1)\frac{\partial^2}{\partial t^2}U_{r0}$$

$$f_{\rho r}(2) = -B_\rho(7,3)\frac{\partial^3}{\partial t^2\,\partial\theta}U_{r0} + B_\rho(10,2)\frac{\partial^2}{\partial t^2}U_{\theta 0}$$

$$f_{\rho r}(3) = -B_\rho(9,3)\frac{\partial^3}{\partial t^2\,\partial Z}U_{r0} + B_\rho(9,2)\frac{\partial^2}{\partial t^2}U_{z0}$$

$$f_{\rho\theta}(1) = -B_\rho(10,2)\frac{\partial^3}{\partial t^2\,\partial\theta}U_{r0} + B_\rho(11,1)\frac{\partial^2}{\partial t^2}U_{\theta 0}$$

$$f_{\rho z}(1) = -B_\rho(9,2)\frac{\partial^3}{\partial t^2\,\partial Z}U_{r0} + B_\rho(9,1)\frac{\partial^2}{\partial t^2}U_{z0}$$

The B terms are pure functions of the material properties and the inner and outer radii of the individual plies.

$$\left\{B_\beta(1,\ell); B_\beta(2,\ell); \ldots; B_\beta(18,\ell)\right\}$$

$$= \sum_{j=1}^{p}\int_{a(j)}^{a(j+1)}\left\{\frac{1}{Ra(1)^2}; \frac{1}{Ra(1)}; \frac{1}{R}; \frac{1}{a(1)^2}; \frac{1}{a(1)}; 1; \frac{R}{a(1)^2}; \frac{R}{a(1)}; R; \frac{R^2}{a(1)^2}; \frac{R^3}{a(1)^2};\right.$$

$$\left.\frac{\left[R + a(1)\right]^2}{Ra(1)^2}; \frac{\left[R + a(1)\right]}{Ra(1)}; \frac{R\left[R + a(1)\right]}{a(1)^2}; \frac{R^2}{a(1)}; \frac{\left[R + a(1)\right]}{a(1)}; \frac{\left[R + a(1)\right]}{a(1)^2}; \frac{\left[R + a(1)\right]}{Ra(1)^2}\right\} c_\beta(j)\left[R - a(1)\right]^{\ell - 1}dR$$

$\beta = 22, \ldots, 66, \rho$; when $\beta = \rho$, $c_\beta = \rho$.

TABLE VI.- FREQUENCY PARAMETER λ FOR THREE-PLY CYLINDERS

$$\left[\begin{array}{l} C_{11}(i): C_{12}(i): C_{13}(i): C_{22}(i): C_{23}(i): C_{33}(i): C_{44}(i): C_{55}(i): C_{66}(i) \\ \qquad = 0.08: 0.05: 0.07: 0.19: 0.32: 1: 0.04: 0.03: 0.34 \qquad (i = 1,2,3) \\ C_{33}(1)/C_{33}(2) = C_{33}(3)/C_{33}(2) = 20 \end{array}\right]$$

(a) Thin cylinder; total thickness, 5 percent of outer radius; a(1) = 0.95; a(2) = 0.955; and a(3) = 0.995

$\frac{b}{a}$	m	First frequency			Second frequency			Third frequency		
		Exact	Thin shell	Refined laminate	Exact	Thin shell	Refined laminate	Exact	Thin shell	Refined laminate
1	0	0.32461	0.34393	0.32475	1.8186	1.8323	1.8321	3.0037	3.0582	3.0531
	1	.33631	.36091	.33646	1.7031	1.7141	1.7139	3.1438	3.2082	3.2024
	2	.36737	.40749	.36753	1.4689	1.4757	1.4756	3.4615	3.5521	3.5440
	3	.40447	.47064	.40467	1.2612	1.2657	1.2654	3.8451	3.9760	3.9636
	4	.42507	.52990	.42533	1.1822	1.1875	1.1857	4.2502	4.4368	4.4173
2	0	.28282	.28342	.28286	.91444	.91614	.91612	1.5441	1.5508	1.5503
	1	.30591	.30722	.30596	.73851	.73938	.73936	1.7807	1.7917	1.7909
	2	.30838	.31225	.30846	.60334	.60419	.60379	2.2071	2.2292	2.2275
	3	.21959	.23314	.21974	.79860	.80118	.79965	2.6748	2.7153	2.7122
	4	.20414	.24179	.20444	1.1413	1.1467	1.1445	3.1559	3.2241	3.2187
8	0	.20999	.20995	.21000	.22901	.22904	.22904	.51742	.51743	.51757
	1	.07054	.07054	.07054	.41947	.41954	.41957	.84022	.84128	.84126
	2	.03594	.03679	.03602	.78831	.78949	.78928	1.3512	1.3559	1.3556
	3	.06940	.07419	.06966	1.1918	1.1964	1.1956	1.8985	1.9113	1.9107
	4	.12237	.13560	.12283	1.5963	1.6080	1.6058	2.4564	2.4841	2.4828
∞	0	0	0	0	0	0	0	.40893	.40875	.40897
	1	0	0	0	.57734	.57756	.57764	.59782	.59819	.59818
	2	.02316	.02375	.02328	.91106	.91294	.91266	1.1935	1.1964	1.1963
	3	.06372	.06712	.06401	1.2851	1.2910	1.2900	1.7847	1.7946	1.7943
	4	.11781	.12857	.11829	1.6695	1.6833	1.6808	2.3694	2.3927	2.3921

(b) Thick cylinder; total thickness, 20 percent of outer radius; a(1) = 0.8; a(2) = 0.82; a(3) = 0.98

$\frac{b}{a}$	m	First frequency			Second frequency				Third frequency			
		Exact	Thin shell	Refined laminate; $p_m = 1$	Exact	Thin shell	Refined laminate		Exact	Thin shell	Refined laminate	
							$p_m = 1$	$p_m = 2$			$p_m = 1$	$p_m = 2$
1	0	0.40438	0.81033	0.40466	1.6205	1.8399	1.8312	1.6637	1.7475	3.0636	1.8508	1.7901
	1	.42140	.90411	.42173	1.5294	1.7059	1.6938	1.5648	1.7530	3.2348	1.8543	1.7947
	2	.46495	1.1402	.46544	1.3354	1.4500	1.4331	1.3580	1.7633	3.6180	1.8620	1.8037
	3	.50904	1.2187	.50984	1.1742	1.4813	1.2375	1.1894	1.7626	4.0843	1.8508	1.8022
	4	.52631	1.1676	.52774	1.1540	1.8484	1.2171	1.1695	1.7309	4.5893	1.7982	1.7670
2	0	.31807	.34461	.31837	.89129	.91996	.91833	.89790	1.4316	1.5573	1.5388	1.4580
	1	.35573	.41939	.35608	.71061	.72587	.72295	.71368	1.4242	1.8267	1.4445	1.4441
	2	.33947	.46341	.33990	.62782	.68829	.63563	.62979	1.3608	2.3061	1.3801	1.3789
	3	.28099	.57619	.28145	.85051	.98070	.87356	.85655	1.5134	2.8370	1.5451	1.5381
	4	.33070	.87921	.33178	1.1398	1.3556	1.2125	1.1597	1.7260	3.3934	1.7896	1.7581
8	0	.21844	.21804	.21861	.22955	.22999	.22996	.22965	.54260	.54266	.54543	.54332
	1	.06638	.06642	.06639	.46207	.46637	.46439	.46267	.86049	.88568	.88059	.86560
	2	.08773	.13212	.08787	.82383	.86418	.84514	.82944	1.3288	1.4482	1.4065	1.3499
	3	.18459	.33591	.18496	1.1547	1.2943	1.2350	1.1765	1.6937	2.0573	1.7601	1.7266
	4	.28616	.61270	.28705	1.3692	1.7296	1.5441	1.4177	1.9282	2.6843	1.9793	1.9503
∞	0	0	0	0	0	0	0	0	.44779	.44467	.44865	.44802
	1	0	0	0	.60992	.61894	.61637	.61160	.64047	.65076	.64808	.64238
	2	.08004	.11167	.08019	.91989	.97362	.95316	.92872	1.2280	1.3015	1.2853	1.2418
	3	.18123	.31182	.18159	1.2068	1.3758	1.3065	1.2338	1.7339	1.9523	1.8120	1.7733
	4	.28393	.58693	.28481	1.3884	1.7937	1.5749	1.4405	1.9938	2.6030	2.0327	2.0090

TABLE VII.- ACCURACY OF FREQUENCIES OBTAINED BY USING THIN-SHELL THEORY

Thickness of laminate		Nodal distance $\frac{\bar{b}}{a}$ / Radius			Circumferential wave number, m	Accuracy		
Thin	Thick	Long	Moderate	Short		Reasonably accurate	Inaccurate	Highly inaccurate
X		X	X		$\leqq 2$	X		
X		X	X		>2		X	
X				X	All		X	
	X	X			$\leqq 1$	X		
	X	X			>1		X	
	X		X	X	All			X

Notes:

1. Long: $\bar{b}/a > 8$
   Moderate: $2 < \bar{b}/a < 8$
   Short: $\bar{b}/a < 2$
2. Reasonably accurate: error less than 5 percent
   Inaccurate: error 5 percent to 25 percent
   Highly inaccurate: error greater than 25 percent
3. Thickness up to 5 percent of radius

TABLE VIII.- RELATIONSHIPS FOR d_r, d_θ, AND d_z AND FOR $e_0, e_1, \ldots$

(a) Relations for d_r, d_θ, and d_z

	$j = 0$ for –		$j > 0$ for –	
	$k = 1,2,3,4$	$k = 5,6$	j is even and $k = 1,2,3,4$ or j is odd and $k = 5,6$	j is even and $k = 5,6$ or j is odd and $k = 1,2,3,4$
$d_r(j,k)$	1	0	$(e_2e_6 - e_4e_5)/e_7$	0
$d_\theta(j,k)$	e_0	0	$(e_3e_5 - e_1e_6)/e_7$	0
$d_z(j,k)$	0	1	0	e_8/e_9
$d'_r(j,k)$	0	0	$\left[(e'_2e_6 + e_2e'_6 - e'_4e_5 - e_4e'_5)e_7 - (e_2e_6 - e_4e_5)e'_7\right]/{e_7}^2$	0
$d'_\theta(j,k)$	e'_0	0	$\left[(e'_3e_5 + e_3e'_5 - e'_1e_6 - e_1e'_6)e_7 - (e_3e_5 - e_1e_6)e'_7\right]/{e_7}^2$	0
$d'_z(j,k)$	0	0	0	$(e'_8e_9 - e_8e'_9)/{e_9}^2$
$\hat{d}_r(j,k)$			$(e_2\hat{e}_6 - e_4\hat{e}_5)/e_7$	0
$\hat{d}_\theta(j,k)$			$(e_3\hat{e}_5 - e_1\hat{e}_6)/e_7$	0
$\hat{d}_z(j,k)$			0	$\hat{e}_8/\hat{e}_9$
$\hat{d}'_r(j,k)$			$\left[(e_2\hat{e}'_6 + e'_2\hat{e}_6 - e_4\hat{e}'_5 - e'_4\hat{e}_5)e_7 - (e_2\hat{e}_6 - e_4\hat{e}_5)e'_7\right]/{e_7}^2$	0
$\hat{d}'_\theta(j,k)$			$\left[(e_3\hat{e}'_5 + e'_3\hat{e}_5 - e_1\hat{e}'_6 - e'_1\hat{e}_6)e_7 - (e_3\hat{e}_5 - e_1\hat{e}_6)e'_7\right]/{e_7}^2$	0
$\hat{d}'_z(j,k)$			0	$(\hat{e}'_8\hat{e}_9 - \hat{e}_8\hat{e}'_9)/{\hat{e}_9}^2$

TABLE VIII.- RELATIONSHIPS FOR d_r, d_θ, AND d_z AND FOR $e_0, e_1, \ldots$ – Continued

(b) Relations for $e_0, e_1, \ldots$

	Relation
e_0	$\dfrac{C_{11}\alpha(k)^2 - C_{55}m^2 - C_{22}}{m(C_{55} + C_{22}) - m(C_{55} + C_{12})\alpha(k)}$
e_1	$C_{11}\left[\alpha(k) + j\right]^2 - C_{55}m^2 - C_{22}$
e_2	$m\left\{(C_{55} + C_{12})\left[\alpha(k) + j\right] - C_{55} - C_{22}\right\}$
e_3	$m\left\{-(C_{55} + C_{12})\left[\alpha(k) + j\right] - C_{55} - C_{22}\right\}$
e_4	$C_{55}\left[\alpha(k) + j\right]^2 - C_{22}m^2 - C_{55}$
e_5	$\left(\rho\Omega^2a^2 - C_{44}N^2\right)d_r(j-2, k) + Nd_z(j-1, k)\left\{-(C_{13} + C_{44})\left[\alpha(k) + j - 1\right] - C_{13} + C_{23}\right\}$
e_6	$\left(\rho\Omega^2a^2 - C_{66}N^2\right)d_\theta(j-2, k) + mNd_z(j-1, k)(C_{23} + C_{66})$
e_7	$e_1e_4 - e_2e_3$
e_8	$\left\{(C_{44} + C_{13})\left[\alpha(k) + j - 1\right] + C_{44} + C_{23}\right\}Nd_r(j-1, k) + (C_{66} + C_{23})mNd_\theta(j-1, k) + \left(\rho\Omega^2a^2 - C_{33}N^2\right)d_z(j-2, k)$
e_9	$C_{66}m^2 - C_{44}\left[\alpha(k) + j\right]^2$

Note: $d_r(-1,k)$, $d_\theta(-1,k)$, and $d_z(-1,k)$, when they occur, are zero.

(c) Relations for $e'_0, e'_1, \ldots$

	Relation
e'_0	$-\dfrac{C_{55} + C_{12}}{C_{55} + C_{22}}$
e'_1	$2C_{11}\left[\alpha(k) + j\right]$
e'_2	$m(C_{55} + C_{12})$
e'_3	$-m(C_{55} + C_{12})$
e'_4	$2C_{55}\left[\alpha(k) + j\right]$
e'_5	$\left(\rho\Omega^2a^2 - C_{44}N^2\right)d'_r(j-2, k) - N(C_{13} + C_{44})d_z(j-1, k) + Nd'_z(j-1, k)\left\{-(C_{13} + C_{44})\left[\alpha(k) + j - 1\right] - C_{13} + C_{23}\right\}$
e'_6	$\left(\rho\Omega^2a^2 - C_{66}N^2\right)d'_\theta(j-2, k) + (C_{23} + C_{66})mNd'_z(j-1, k)$
e'_7	$e'_1e_4 + e_1e'_4 - e'_2e_3 - e_2e'_3$
e'_8	$\left(\rho\Omega^2a^2 - C_{33}N^2\right)d'_z(j-2, k) + (C_{44} + C_{13})Nd_r(j-1, k) + \left\{(C_{44} + C_{13})\left[\alpha(k) + j - 1\right] + C_{44} + C_{23}\right\}Nd'_r(j-1, k) + (C_{66} + C_{23})mNd'_\theta(j-1, k)$
e'_9	$-2C_{44}\left[\alpha(k) + j\right]$

Note: $d'_r(-1,k)$, $d'_\theta(-1,k)$, and $d'_z(-1,k)$, when they occur, are zero.

TABLE VIII.- RELATIONSHIPS FOR d_r, d_θ, AND d_z AND FOR $e_0, e_1, \ldots$ – Concluded

(d) Values for $\hat{e}_5, \hat{e}_5', \ldots$

	$j = \beta$ (j is even)	$j = \beta + 1$ (j is odd)	$j \geqq \beta + 2$
$\hat{e}_8$	e_8		
$\hat{e}_9$	$C_{44}[3\alpha(6) - \alpha]$ at $\alpha = \alpha(6)$; thus, $\hat{e}_9 = 2C_{44}\alpha(6)$		
$\hat{e}_8'$	e_8'		
$\hat{e}_9'$	$-C_{44}$		
$\hat{e}_5$		$N\hat{d}_z(\beta,6)\left\{-(C_{13} + C_{44})[\alpha(6) + j - 1] - C_{13} + C_{23}\right\}$	
$\hat{e}_6$		$mN\hat{d}_z(\beta,6)(C_{23} + C_{66})$	
$\hat{e}_5'$		$\left(\rho\Omega^2a^2 - C_{44}N^2\right)d_r(\beta-1, 6) - N\hat{d}_z(\beta,6)(C_{13} + C_{44}) + N\hat{d}_z'(\beta,6)\left\{-(C_{13} + C_{44})[\alpha(6) + j - 1] - C_{13} + C_{23}\right\}$	
$\hat{e}_6'$		$\left(\rho\Omega^2a^2 - C_{66}N^2\right)d_\theta(\beta-1,6) + mN\hat{d}_z'(\beta,6)(C_{23} + C_{66})$	
$\hat{e}_8$			$\left(\rho\Omega^2a^2 - C_{33}N^2\right)\hat{d}_z(j-2, 6) + \left\{(C_{44} + C_{13})[\alpha(6) + j - 1] + C_{44} + C_{23}\right\}N\hat{d}_r(j-1, 6) + (C_{66} + C_{23})mN\hat{d}_\theta(j-1, 6)$
$\hat{e}_9$			e_9
$\hat{e}_5$			$\left(\rho\Omega^2a^2 - C_{44}N^2\right)\hat{d}_r(j-2, 6) + N\left\{-(C_{13} + C_{44})[\alpha(6) + j - 1] - C_{13} + C_{23}\right\}\hat{d}_z(j-1, 6)$
$\hat{e}_6$			$\left(\rho\Omega^2a^2 - C_{66}N^2\right)\hat{d}_\theta(j-2, 6) + mN(C_{23} + C_{66})\hat{d}_z(j-1, 6)$
$\hat{e}_8'$			$\left(\rho\Omega^2a^2 - C_{33}N^2\right)\hat{d}_z'(j-2, 6) + (C_{44} + C_{13})N\hat{d}_r(j-1, 6) + \left\{(C_{44} + C_{13})[\alpha(6) + j - 1] + C_{44} + C_{23}\right\}N\hat{d}_r'(j-1, 6) + (C_{66} + C_{23})mN\hat{d}_\theta'(j-1, 6)$
$\hat{e}_9'$			$e_9' = -2C_{44}[\alpha(6) + j]$
$\hat{e}_5'$			$\left(\rho\Omega^2a^2 - C_{44}N^2\right)\hat{d}_r'(j-2, 6) - (C_{13} + C_{44})N\hat{d}_z(j-1, 6) + N\left\{-(C_{13} + C_{44})[\alpha(6) + j - 1] - C_{13} + C_{23}\right\}\hat{d}_z'(j-1, 6)$
$\hat{e}_6'$			$\left(\rho\Omega^2a^2 - C_{66}N^2\right)\hat{d}_\theta'(j-2, 6) + mN(C_{23} + C_{66})\hat{d}_z'(j-1, 6)$

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