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A FINITE-ELEMENT FORMULATION FOR SUBSONIC FLOWS AROUND COMPLEX CONFIGURATIONS

by

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ABSTRACT

The problem of potential steady subsonic flow around complex configurations is considered. This problem requires the solution of an integral equation relating the values of the potential on the surface of the body to the values of the normal derivative, which is known from the boundary conditions. The surface of the body is divided into small (hyperboloidal quadrilateral) surface elements, Σ_i , which are described in terms of the Cartesian components of the four corner points. The values of the potential (and its normal derivative) within each element is assumed to be constant and equal to its value at the centroid of the element. This yields a set of linear algebraic equations. The coefficients of the equation are given by source and doublet integrals over the surface elements, Σ_i. Closed form evaluations of the integrals are presented.

FOREWARD

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LIST OF SYMBOLS

$\vec{a}_i$	base vectors, Eqs. 1.13 and 2.8
a_{hk}	see Eq. 6.2
b_h	see Eq. 1.9
b_{hk}	see Eq. 1.10
c_p	pressure coefficient
$c_\ell = -\Delta c_p = c_{p,\ell} - c_{p,u}$	lifting pressure coefficient
c_{ki}	see Eq. 1.8
$I_D(\xi,\eta)$	see Eqs. 6.4 and 6.6
$I_S(\xi,\eta)$	see Eqs. 6.5 and 6.8
$J_W(\xi,\eta)$	see Eqs. 6.11 and 6.13
$\vec{n}$	normal to the surface Σ at P_1
N	number of elements
$\vec{P} \equiv (X,Y,Z)$	control point
$\vec{P}_{++}$, $\vec{P}_{+-}$, $\vec{P}_{-+}$, $\vec{P}_{--}$	see Eq. 2.4
$\vec{P}_c$, $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$	see Eq. 2.2
$\vec{P}_0$	see Eq. 2.13
$\vec{P}^{(k)}$	centroid of element σ_k
$\vec{q}$	see Eq. 2.12
$r = \left[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right]^{1/2}$	
r_h	see Eq. 2.19
U_∞	velocity of undisturbed flow
w_{ki}	see Eqs. 6.10 and 6.11
x,y,z	Cartesian coordinates
γ	vorticity
δ_{ki}	Kronecker delta
ξ,η	see Eq. 2.7

ρ_∞	density of undisturbed air
σ	surface surrounding body and wake
σ_k	surface element
φ	perturbation aerodynamic potential
φ_k	value of φ at $P^{(k)}$
Φ	aerodynamic potential
Ω	solid angle

SPECIAL SYMBOLS

∇	gradient operator in x,y,z coordinates

SUBSCRIPTS

1	Dummy variables
TE	trailing edge
*	evaluation at $P = P_*$

SECTION I

FORMULATION OF THE PROBLEM

1.1 Introduction

A general theory for compressible unsteady potential aerodynamic flow around lifting bodies having arbitrary shapes and motions is given in Refs. 1 and 2. Application to finite-thickness steady and oscillating wings in subsonic flow is given in Refs. 3, 4 and 5. Here, a general numerical formulation for complex configuration in steady subsonic flow is considered. By using the Prandtl-Glauert transformation, the incompressible flow is obtained.[1,2] Hence, for simplicity, only the incompressible flow is considered here.* In this case, the problem is governed by the Laplace equation with prescribed normal derivative on the body (exterior Newman problem for the Laplace equation) with an additional complication due to the presence of the wake (of unknown geometry). The method is described with the emphasis on the aerodynamic applications, but it is applicable to different physical problems as well (see Subsection 6.3).

The problem of the evaluation of the steady, incompressible potential aerodynamic flow around an aircraft of arbitrary configuration can be analyzed by solving the integral equation

$$\varphi_* = -\frac{1}{2\pi} \oiint \left[\frac{\partial \varphi}{\partial n} \frac{1}{r} - \varphi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right] d\sigma \tag{1.1}$$

where σ is a surface surrounding the aircraft and the wake.[2,5]

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*Subsonic oscillatory flow is considered in Appendix C.

For the moment, it will be assumed that the wake does not exist. The effect of the wake is considered in Section 5.

The value of $\frac{\partial \varphi}{\partial n}$ is obtained from the boundary condition (tangency condition)

$$\frac{\partial \Phi}{\partial n} = U_\infty \frac{\partial}{\partial n}(x + \varphi) = U_\infty \left(n_x + \frac{\partial \varphi}{\partial n}\right) = 0 \tag{1.2}$$

or

$$\frac{\partial \varphi}{\partial n} = - n_x = - \bar{n} \cdot \bar{i} \tag{1.3}$$

The integral equation can be studied by dividing the surface σ into N small finite elements σ_i to yield

$$\varphi_* = \frac{1}{2\pi} \sum_{k=1}^{N} \iint_{\sigma_k} \left[\bar{n} \cdot \bar{i} \frac{1}{r} + \varphi\, \bar{n} \cdot \bar{\nabla}\left(\frac{1}{r}\right)\right] d\sigma_k \tag{1.4}$$

Applying the mean value theorem one obtains

$$\varphi_* = \frac{1}{2\pi} \sum_{k=1}^{N} \iint_{\sigma_k} \left(\bar{n} \cdot \bar{i} \frac{1}{r}\right) d\sigma_k + \frac{1}{2\pi} \sum_{k=1}^{N} \varphi_k \iint_{\sigma_k} \bar{n} \cdot \bar{\nabla}\left(\frac{1}{r}\right) d\sigma_k \tag{1.5}$$

where φ_k is a suitable mean value of φ inside the element σ_k, which will be approximated by the value of φ at the centroid $P^{(K)}$ of the element, σ_K .

By satisfying Eq. (1.5) at the centroid, $P^{(h)}$, of the element σ_h , (h = 1, 2....N) yields

$$\varphi_h = \frac{1}{2\pi} \sum_{k=1}^{N} \iint_{\sigma_k} \bar{n} \cdot \bar{i} \frac{1}{r_h} d\sigma_k$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{N} \varphi_k \iint_{\sigma_k} \bar{n} \cdot \bar{\nabla}\left(\frac{1}{r_h}\right) d\sigma_k \qquad (h = 1, 2, 3, \cdots N) \tag{1.6}$$

where r_h is the distance of the centroid of the element σ_h from the dummy point of integration in the element σ_K .

Equation (1.6) is equivalent to*

$$\left[\delta_{hk} - C_{hk} \right] \left\{ \varphi_k \right\} = \left\{ b_h \right\} \tag{1.7}$$

where

$$C_{hk} = \frac{1}{2\pi} \iint_{\sigma_k} \bar{n} \cdot \bar{\nabla} \frac{1}{r_h} \, d\sigma_k \tag{1.8}$$

and

$$b_h = \sum_{K=1}^{N} b_{hk} \tag{1.9}$$

with

$$b_{hk} = \frac{1}{2\pi} \iint_{\sigma_k} \bar{n} \cdot \bar{i} \, \frac{1}{r_h} \, d\sigma_k \tag{1.10}$$

1.2 Surface Geometry

Let the geometry of the element σ_K be described by

$$\vec{P} = \vec{P} (\xi^1, \xi^2) \tag{1.11}$$

where ξ^1 and ξ^2 are the generalized curvilinear coordinate. Then the two base vectors $\vec{a}_i$ are given by (Fig. 1)

$$\vec{a}_i = \frac{\partial \vec{P}}{\partial \xi^i} \tag{1.12}$$

The unit normal to the surface is given by

$$\vec{n} = \frac{\vec{a}_1 \times \vec{a}_2}{| \vec{a}_1 \times \vec{a}_2 |} \tag{1.13}$$

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*The effect of the wake is not considered here (see Sections 5 and 6).

and is directed according to the right-hand rule (Fig. 1). The surface element $d\sigma$ is given by (Fig. 1)

$$d\sigma = \left| \vec{a}_1 d\xi^1 \times \vec{a}_2 d\xi^2 \right| = \left| \vec{a}_1 \times \vec{a}_2 \right| d\xi^1 d\xi^2 \tag{1.14}$$

1.3 Expressions for b_{hk} and c_{hk}

Combining Eqs. (1.10), (1.13) and (1.14) yields

$$b_{hk} = \frac{1}{2\pi} \iint_{\sigma_k} (\bar{a}_1 \times \bar{a}_2 \cdot \bar{i}) \frac{1}{r_h} d\xi^1 d\xi^2 \tag{1.15}$$

Similarly, combining Eqs. (1.8), (1.13) and (1.14) yields

$$C_{hk} = \frac{1}{2\pi} \iint_{\sigma_k} \bar{a}_1 \times \bar{a}_2 \cdot \bar{\nabla} \left(\frac{1}{r_h}\right) d\xi^1 d\xi^2$$

$$= -\frac{1}{2\pi} \iint_{\sigma_k} \frac{\bar{a}_1 \times \bar{a}_2 \cdot \bar{r}_h}{r_h^3} d\xi^1 d\xi^2 \tag{1.16}$$

where

$$\vec{r}_h = \begin{Bmatrix} x - x_h \\ y - y_h \\ z - z_h \end{Bmatrix} \tag{1.17}$$

In Section 2, these expressions are evaluated under the hypothesis that the surface element is a portion of a hyperboloid.

SECTION II

HYPERBOLOIDAL ELEMENT

2.1 Introduction

Consider the equations

$$x = x_c + x_1 \xi + x_2 \eta + x_3 \xi \eta$$

$$y = y_c + y_1 \xi + y_2 \eta + y_3 \xi \eta$$

$$z = z_c + z_1 \xi + z_2 \eta + z_3 \xi \eta \tag{2.1}$$

or, in vector notations

$$\bar{P} = \bar{P}_c + \bar{P}_1 \xi + \bar{P}_2 \eta + \bar{P}_3 \xi \eta \tag{2.2}$$

This represents a hyperboloid. The lines η = const and ξ = const are clearly straight lines. Consider the hyperboloidal element defined by the above equation with

$$\begin{aligned} -1 &\le \xi \le 1 \\ -1 &\le \eta \le 1 \end{aligned} \tag{2.3}$$

The centroid of the element is $\bar{p}_c$ ($\xi = \eta = 0$). The corner points of this element are

$$\begin{aligned}
\bar{P}_{++} &= \bar{P}_c + \bar{P}_1 + \bar{P}_2 + \bar{P}_3 & (\xi = +1,\ \eta = +1) \\
\bar{P}_{+-} &= \bar{P}_c + \bar{P}_1 - \bar{P}_2 - \bar{P}_3 & (\xi = +1,\ \eta = -1) \\
\bar{P}_{-+} &= \bar{P}_c - \bar{P}_1 + \bar{P}_2 - \bar{P}_3 & (\xi = -1,\ \eta = +1) \\
\bar{P}_{--} &= \bar{P}_c - \bar{P}_1 - \bar{P}_2 + \bar{P}_3 & (\xi = -1,\ \eta = -1)
\end{aligned} \tag{2.4}$$

The inverse relation is

$$\bar{p}_c = \frac{1}{4}\left(\bar{p}_{++} + \bar{p}_{+-} + \bar{p}_{-+} + \bar{p}_{--}\right)$$

$$\bar{p}_1 = \frac{1}{4}\left(\bar{p}_{++} + \bar{p}_{+-} - \bar{p}_{-+} - \bar{p}_{--}\right)$$

$$\bar{p}_2 = \frac{1}{4}\left(\bar{p}_{++} - \bar{p}_{+-} + \bar{p}_{-+} - \bar{p}_{--}\right)$$

$$\bar{p}_3 = \frac{1}{4}\left(\bar{p}_{++} - \bar{p}_{+-} - \bar{p}_{-+} + \bar{p}_{--}\right) \tag{2.5}$$

Note that the four boundaries of the element ($\xi = \pm 1$, $\eta = \pm 1$) are straight lines given by

$$\bar{p} = (\bar{p}_c + \bar{p}_1) + (\bar{p}_2 + \bar{p}_3)\eta \qquad -1 \le \eta \le 1$$

$$\bar{p} = (\bar{p}_c - \bar{p}_1) + (\bar{p}_2 - \bar{p}_3)\eta \qquad -1 \le \eta \le 1$$

$$\bar{p} = (\bar{p}_c + \bar{p}_2) + (\bar{p}_1 + \bar{p}_3)\xi \qquad -1 \le \xi \le 1$$

$$\bar{p} = (\bar{p}_c - \bar{p}_2) + (\bar{p}_1 - \bar{p}_3)\xi \qquad -1 \le \xi \le 1 \tag{2.6}$$

Next, assume that the surface of the aircraft is divided into curved quadrilateral elements with four corner points $\bar{p}_{++}$, $\bar{p}_{+-}$, $\bar{p}_{-+}$, $\bar{p}_{--}$. Then, as mentioned in Section I, these elements can be replaced by the hyperboloidal element (described above) which goes through the four corner points $\bar{p}_{++}$, $\bar{p}_{+-}$, $\bar{p}_{-+}$, $\bar{p}_{--}$ (see Fig. 2). It may be noted that the surface is continuous since adjacent elements have in common the straight line connecting the two common corner points. It may be noted also the $\bar{p}_C$ is the centroid of the hyperboloidal element σ_k and hence it will be indicated as

$$\bar{P}_c = \bar{P}^{(k)} \tag{2.6}$$

2.2 Geometry of Hyperboloid Element

The geometric quantities introduced in Section I can be written for the hyperboloid element described above. Letting

$$\xi^1 = \xi \, , \qquad \xi^2 = \eta \tag{2.7}$$

Equation (1.12) yields

$$\vec{a}_1 = \frac{\partial \vec{P}}{\partial \xi} = \vec{P}_1 + \vec{P}_3 \eta \tag{2.8-a}$$

$$\vec{a}_2 = \frac{\partial \vec{P}}{\partial \eta} = \vec{P}_2 + \vec{P}_3 \xi \tag{2.8-b}$$

This yields

$$\begin{aligned} \vec{a}_1 \times \vec{a}_2 &= (\bar{P}_1 \times \bar{P}_3 \eta) \times (\bar{P}_2 \times \bar{P}_3 \xi) \\ &= \bar{P}_1 \times \bar{P}_2 + \bar{P}_1 \times \bar{P}_3 \xi + \bar{P}_3 \times \bar{P}_2 \eta \end{aligned} \tag{2.9}$$

since $\bar{p}_3 \times \bar{p}_3 = 0$. In components notations

$$\begin{aligned} \bar{a}_1 \times \bar{a}_2 &= \mathrm{Det}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} + \mathrm{Det}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{bmatrix}\xi + \mathrm{Det}\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{bmatrix}\eta \\ &= \begin{Bmatrix} (y_1 z_2 - z_1 y_2) + (y_1 z_3 - z_1 y_3)\xi + (y_3 z_2 - z_3 y_2)\eta \\ (z_1 x_2 - x_1 z_2) + (z_1 x_3 - x_1 z_3)\xi + (z_3 x_2 - x_3 z_2)\eta \\ (x_1 y_2 - y_1 x_2) + (x_1 y_3 - y_1 x_3)\xi + (x_3 y_2 - y_3 x_2)\eta \end{Bmatrix} \end{aligned} \tag{2.10}$$

In particular, for the first component, one obtains

$$\bar{a}_1 \times \bar{a}_2 \cdot \bar{i} = (y_1 z_2 - z_1 y_2) + (y_1 z_3 - z_1 y_3)\xi + (y_3 z_2 - z_3 y_2)\eta \tag{2.11}$$

Note that, with present notations

$$\bar{q} = \bar{r}_h = \bar{p} - \bar{p}^{(h)} = \bar{p}^{(k)} - \bar{p}^{(h)} + \bar{p}_1\xi + \bar{p}_2\eta + \bar{p}_3\eta\xi$$

$$= \bar{p}_0 + \bar{p}_1\xi + \bar{p}_2\eta + \bar{p}_3\eta\xi \tag{2.12}$$

where

$$\bar{p}_0 = \bar{p}^{(k)} - \bar{p}^{(h)} \tag{2.13}$$

is the vector connecting the centroid $\bar{p}^{(h)}$ of the element σ_h, to the one, $\bar{p}^{(k)}$, of the element σ_k. Hence

$$\begin{aligned} &\bar{q} \cdot \bar{a}_1 \times \bar{a}_2 \\ = &(\bar{p}_1 \times \bar{p}_2 + \bar{p}_1 \times \bar{p}_3\xi + \bar{p}_3 \times \bar{p}_2\eta) \cdot (\bar{p}_0 + \bar{p}_1\xi + \bar{p}_2\eta + \bar{p}_3\xi\eta) \\ = &(\bar{p}_0 \cdot \bar{p}_1 \times \bar{p}_2) + (\bar{p}_0 \cdot \bar{p}_1 \times \bar{p}_3)\xi + (\bar{p}_0 \cdot \bar{p}_3 \times \bar{p}_2)\eta \\ &+ (\bar{p}_1 \cdot \bar{p}_3 \times \bar{p}_2)\xi\eta + (\bar{p}_2 \cdot \bar{p}_1 \times \bar{p}_3)\xi\eta + (\bar{p}_3 \cdot \bar{p}_1 \times \bar{p}_2)\xi\eta \end{aligned} \tag{2.14}$$

since

$$\bar{p}_1 \cdot \bar{p}_1 \times \bar{p}_2 = \bar{p}_1 \cdot \bar{p}_1 \times \bar{p}_3 = \bar{p}_2 \cdot \bar{p}_1 \times \bar{p}_2 = \bar{p}_2 \cdot \bar{p}_3 \times \bar{p}_3$$

$$= \bar{p}_3 \cdot \bar{p}_1 \times \bar{p}_3 = \bar{p}_3 \cdot \bar{p}_3 \times \bar{p}_2 = 0 \tag{2.15}$$

Noting that

$$\bar{p}_1 \cdot \bar{p}_2 \times \bar{p}_3 = \bar{p}_3 \cdot \bar{p}_1 \times \bar{p}_2 = -\bar{p}_1 \cdot \bar{p}_3 \times \bar{p}_2 = -\bar{p}_2 \cdot \bar{p}_1 \times \bar{p}_3 \tag{2.16}$$

Equation (2.14) reduces to

$$\bar{g} \cdot \bar{a}_1 \times \bar{a}_2 = (\bar{P}_0 \cdot \bar{P}_1 \times \bar{P}_2) + (\bar{P}_0 \cdot \bar{P}_1 \times \bar{P}_3)\xi$$
$$+ (\bar{P}_0 \cdot \bar{P}_3 \times \bar{P}_2)\eta - (\bar{P}_1 \cdot \bar{P}_2 \times \bar{P}_3)\xi\eta \tag{2.17}$$

In component notations, Eq. (2.17) reduces to

$$\bar{g} \cdot \bar{a}_1 \times \bar{a}_2 = \mathrm{Det}\begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} + \mathrm{Det}\begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{vmatrix}\xi$$

$$+ \mathrm{Det}\begin{vmatrix} x_0 & y_0 & z_0 \\ x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \end{vmatrix}\eta - \mathrm{Det}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\xi\eta$$

$$= \mathrm{Det}\begin{vmatrix} -\xi\eta & x_0 & y_0 & z_0 \\ \eta & x_1 & y_1 & z_1 \\ \xi & x_2 & y_2 & z_2 \\ -1 & x_3 & y_3 & z_3 \end{vmatrix} \tag{2.18}$$

Finally, according to Eq. (2.12)

$$r_h = |\bar{r}_h| = |\bar{g}| = |\bar{P}_0 + \bar{P}_1\xi + \bar{P}_2\eta + \bar{P}_3\xi\eta|$$
$$= \Big\{ \bar{P}_0 \cdot \bar{P}_0 + \bar{P}_1 \cdot \bar{P}_1\xi^2 + \bar{P}_2 \cdot \bar{P}_2\eta^2 + \bar{P}_3 \cdot \bar{P}_3\xi^2\eta^2$$
$$+ 2[\bar{P}_0 \cdot \bar{P}_1\xi + \bar{P}_0 \cdot \bar{P}_2\eta + (\bar{P}_0 \cdot \bar{P}_3 + \bar{P}_1 \cdot \bar{P}_2)\xi\eta + \bar{P}_1 \cdot \bar{P}_3\xi^2\eta + \bar{P}_2 \cdot \bar{P}_3\xi\eta^2]\Big\}^{\frac{1}{2}} \tag{2.19}$$

or, in components notations

$$r_h = \Big[(x_0 + x_1\xi + x_2\eta + x_3\xi\eta)^2 + (y_0 + y_1\xi + y_2\eta + y_3\xi\eta)^2$$
$$+ (z_0 + z_1\xi + z_2\eta + z_3\xi\eta)^2\Big]^{\frac{1}{2}} \tag{2.20}$$

2.3 Expression for b_{hk} and c_{hk}

By combining Eq. (1.15) with Eqs. (2.11) and (2.20), one obtains

$$b_{hk} = +\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\left[(y_1 z_2 - z_1 y_2) + (y_1 z_3 - z_1 y_3)\xi + (y_3 z_2 - z_3 y_2)\eta\right] \times$$

$$\times\left[(x_0 + x_1\xi + x_2\eta + x_3\xi\eta)^2 + (y_0 + y_1\xi + y_2\eta + y_3\xi\eta)^2 + (z_0 + z_1\xi + z_2\eta + z_3\xi\eta)^2\right]^{-\frac{1}{2}} d\xi\, d\eta \tag{2.21}$$

Similarly, combining Eq. (1.16) with Eqs. (2.18) and (2.20) yields,

$$C_{hk} = -\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} \mathrm{Det}\begin{vmatrix} -\xi\eta & x_0 & y_0 & z_0 \\ \eta & x_1 & y_1 & z_1 \\ \xi & x_2 & y_2 & z_2 \\ -1 & x_3 & y_3 & z_3 \end{vmatrix} \times$$

$$\times\left[(x_0 + x_1\xi + x_2\eta + x_3\xi\eta)^2 + (y_0 + y_1\xi + y_2\eta + y_3\xi\eta)^2 + (z_0 + z_1\xi + z_2\eta + z_3\xi\eta)^2\right]^{-\frac{3}{2}} d\xi\, d\eta \tag{2.22}$$

The integration of Eqs. (2.21) and (2.22) is discussed in Section 3 and 4, respectively.

SECTION 3

DOUBLET INTEGRAL

3.1 Integration with respect to ξ

Consider Eq. (2.22), which can be rewritten as

$$C_{hk} = -\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{m_0 - m_1\xi}{r_h^3}\,d\xi\,d\eta \tag{3.1}$$

where

$$m_0 = \mathrm{Det}\begin{vmatrix} 0 & x_0 & y_0 & z_0 \\ \eta & x_1 & y_1 & z_1 \\ 0 & x_2 & y_2 & z_2 \\ -1 & x_3 & y_3 & z_3 \end{vmatrix}$$

$$m_1 = \mathrm{Det}\begin{vmatrix} \eta & x_0 & y_0 & z_0 \\ 0 & x_1 & y_1 & z_1 \\ -1 & x_2 & y_2 & z_2 \\ 0 & x_3 & y_3 & z_3 \end{vmatrix} \tag{3.2}$$

and

$$r_h = |\bar{q}| = |\bar{Q}_0 + \xi\bar{Q}_1| = (Q_{00} + 2Q_{01}\xi + Q_{11}\xi^2)^{1/2} \tag{3.3}$$

with

$$Q_{ij} = \bar{Q}_i \cdot \bar{Q}_j \tag{3.4}$$

where

$$\begin{aligned} \bar{Q}_0 &= \bar{P}_0 + \eta\bar{P}_2 \\ \bar{Q}_1 &= \bar{P}_1 + \eta\bar{P}_3 \end{aligned} \tag{3.5}$$

The indefinite integral with respect to ξ in Eq. (3.1) is given by

$$i_D(\eta) = \int \frac{m_0 - m_1 \xi}{r_h^3} \, d\xi = \left[\frac{n_0 + n_1 \xi}{r_h} \right] \tag{3.6}$$

In order to obtain the relation between n_0, n_1 and m_0, m_1, consider the derivative of the expression in bracket in Eq. (3.6) and equate it to the integrand of Eq. (3.1). This yields

$$\begin{aligned}
\frac{\partial}{\partial \xi}\left[\frac{n_0 + n_1 \xi}{r_h}\right] &= \frac{n_1}{r_h} - (n_0 + n_1 \xi)\frac{1}{r_h^3}(\mathcal{Q}_{01} + \mathcal{Q}_{11}\xi) \\
&= \frac{1}{r_h^3}\left[n_1 (r_h^2 - \mathcal{Q}_{01}\xi - \mathcal{Q}_{11}\xi^2) - n_0(\mathcal{Q}_{01} + \mathcal{Q}_{11}\xi)\right] \\
&= \frac{1}{r_h^3}\left[n_1(\mathcal{Q}_{00} + \mathcal{Q}_{01}\xi) - n_0(\mathcal{Q}_{01} + \mathcal{Q}_{11}\xi)\right] \\
&= \frac{1}{r_h^3}(m_0 - m_1 \xi)
\end{aligned} \tag{3.7}$$

This implies

$$\begin{bmatrix} \mathcal{Q}_{00} & -\mathcal{Q}_{01} \\ -\mathcal{Q}_{01} & \mathcal{Q}_{11} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_0 \end{Bmatrix} = \begin{Bmatrix} m_0 \\ m_1 \end{Bmatrix} \tag{3.8}$$

or

$$\begin{Bmatrix} n_1 \\ n_0 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{01} \\ \mathcal{Q}_{01} & \mathcal{Q}_{00} \end{bmatrix} \begin{Bmatrix} m_0 \\ m_1 \end{Bmatrix} \tag{3.9}$$

Note that (see Eq. B.1)

$$D = \mathrm{Det}\begin{bmatrix} \mathcal{Q}_{00} & -\mathcal{Q}_{01} \\ -\mathcal{Q}_{01} & \mathcal{Q}_{11} \end{bmatrix} = \mathrm{Det}\begin{bmatrix} \bar{\mathcal{Q}}_0 \cdot \bar{\mathcal{Q}}_0 & \bar{\mathcal{Q}}_0 \cdot \bar{\mathcal{Q}}_1 \\ \bar{\mathcal{Q}}_1 \cdot \bar{\mathcal{Q}}_0 & \bar{\mathcal{Q}}_1 \cdot \bar{\mathcal{Q}}_1 \end{bmatrix}$$

$$= \left| \bar{\mathcal{Q}}_0 \times \bar{\mathcal{Q}}_1 \right|^2 \tag{3.10}$$

Finally, Eq. (3.9) yields

$$n_0 + n_1 \xi = \frac{1}{D}\left[(\mathcal{Q}_{01} m_0 + \mathcal{Q}_{00} m_1) + \xi (\mathcal{Q}_{11} m_0 + \mathcal{Q}_{01} m_1) \right]$$

$$= \frac{1}{D}\left[(\mathcal{Q}_{01} + \xi \mathcal{Q}_{11}) m_0 + (\mathcal{Q}_{00} + \xi \mathcal{Q}_{01}) m_1 \right]$$

$$= \frac{1}{D} (\bar{\mathcal{Q}}_0 + \xi \bar{\mathcal{Q}}_1) \cdot (\bar{\mathcal{Q}}_1 m_0 + \bar{\mathcal{Q}}_0 m_1)$$

$$= \frac{1}{D}\, \bar{q} \cdot \bar{Q} \tag{3.11}$$

since, according to Eqs. (2.12) and (3.5)

$$\bar{q} = \bar{P}_0 + \xi \bar{P}_1 + \eta \bar{P}_2 + \xi \eta \bar{P}_3$$

$$= \bar{\mathcal{Q}}_0 + \xi \bar{\mathcal{Q}}_1 \tag{3.12}$$

and, having defined

$$\bar{Q} = \bar{\mathcal{Q}}_1 m_0 + \bar{\mathcal{Q}}_0 m_1$$

$$= \mathrm{Det}\begin{bmatrix} \bar{\mathcal{Q}}_0 \eta & x_0 & y_0 & z_0 \\ \bar{\mathcal{Q}}_1 \eta & x_1 & y_1 & z_1 \\ -\bar{\mathcal{Q}}_0 & x_2 & y_2 & z_2 \\ -\bar{\mathcal{Q}}_1 & x_3 & y_3 & z_3 \end{bmatrix} = \mathrm{Det}\begin{bmatrix} 0 & x_0 + \eta x_2 & y_0 + \eta y_2 & z_0 + \eta z_2 \\ 0 & x_1 + \eta x_3 & y_1 + \eta y_3 & z_1 + \eta z_3 \\ -\bar{\mathcal{Q}}_0 & x_2 & y_2 & z_2 \\ -\bar{\mathcal{Q}}_1 & x_3 & y_3 & z_3 \end{bmatrix}$$

$$
= -\bar{\mathcal{Q}}_0 \begin{bmatrix} x_0+\eta x_2 & y_0+\eta y_2 & z_0+\eta z_2 \\ x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \end{bmatrix}
$$

$$
+\bar{\mathcal{Q}}_1 \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1+\eta x_3 & y_1+\eta y_3 & z_1+\eta z_3 \\ x_2 & y_2 & z_2 \end{bmatrix}
$$

$$
\equiv -\bar{\mathcal{Q}}_0 (\bar{\mathcal{Q}}_0 \cdot \bar{P}_1 \times \bar{P}_3) + \bar{\mathcal{Q}}_1 (\bar{P}_0 \cdot \bar{\mathcal{Q}}_1 \times \bar{P}_2)
$$

$$
\equiv -\bar{\mathcal{Q}}_0 (\bar{\mathcal{Q}}_0 \cdot \bar{P}_1 \times \bar{P}_3) - \bar{\mathcal{Q}}_1 (\bar{\mathcal{Q}}_1 \cdot \bar{P}_0 \times \bar{P}_2)
$$

$$
\equiv -(\bar{P}_0+\eta\bar{P}_2)\left[(\bar{P}_0+\eta\bar{P}_2)\cdot \bar{P}_1 \times \bar{P}_3\right]
$$

$$
-(\bar{P}_1+\eta\bar{P}_3)\left[(\bar{P}_1+\eta\bar{P}_3)\cdot \bar{P}_0 \times \bar{P}_2\right] \tag{3.13}
$$

Finally, combining Eqs. (3.6), (3.11) and (3.13) yields

$$
\lambda_D(\eta) = -\frac{(\mathcal{Q}_{00}+\xi\mathcal{Q}_{10})(\bar{\mathcal{Q}}_0\cdot\bar{P}_1\times\bar{P}_3)+(\mathcal{Q}_{10}+\xi\mathcal{Q}_{11})(\bar{\mathcal{Q}}_1\cdot\bar{P}_0\times\bar{P}_2)}{\mathcal{Q}_{00}\mathcal{Q}_{11}-\mathcal{Q}_{10}^2}\,\frac{1}{|\bar{\mathcal{Q}}_0+\xi\bar{\mathcal{Q}}_1|} \tag{3.14}
$$

an alternative form is (note that $\bar{a}_0 \cdot \bar{P}_1 \times \bar{P}_3 = \bar{a}_0 \cdot \bar{a}_1 \times \bar{P}_3$ and $\bar{a}_1 \cdot \bar{P}_0 \times \bar{P}_2 = \bar{a}_1 \cdot \bar{a}_0 \times \bar{P}_2$)

$$\lambda_D(\eta) = -\frac{\left[(a_{00}+\xi a_{01})\bar{P}_3 - (a_{10}+\xi a_{11})\bar{P}_2\right]}{a_{00}a_{11} - a_{10}^2} \cdot \frac{(\bar{a}_0 \times \bar{a}_1)}{|\bar{a}_0 + \xi \bar{a}_1|}$$

$$= -\frac{(\bar{a}_0 + \xi \bar{a}_1)}{|\bar{a}_0 + \xi \bar{a}_1|} \cdot \frac{(\bar{a}_0 \bar{P}_3 - \bar{a}_1 \bar{P}_2)\cdot(\bar{a}_0 \times \bar{a}_1)}{(\bar{a}_0 \times \bar{a}_1)\cdot(\bar{a}_0 \times \bar{a}_1)}$$

$$= \frac{-1}{|\bar{q}||\bar{a}_0 \times \bar{a}_1|^2}\Big[(\bar{q}\cdot\bar{a}_0)(\bar{P}_3\cdot\bar{a}_0\times\bar{a}_1) + \xi(\bar{q}\cdot\bar{a}_1)(\bar{P}_3\cdot\bar{a}_0\times\bar{a}_1) - (\bar{q}\cdot\bar{a}_1)(\bar{P}_2\cdot\bar{a}_0\times\bar{a}_1) - \xi(\bar{q}\cdot\bar{a}_1)(\bar{P}_3\cdot\bar{a}_0\times\bar{a}_1)\Big]$$

$$= \frac{-1}{|\bar{q}||\bar{q}\times\bar{a}_1|^2}\Big\{\bar{q}\cdot(\bar{a}_0+\xi\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{P}_3) - (\bar{q}\cdot\bar{a}_1)(\bar{P}_2+\xi\bar{P}_3)\cdot\bar{q}\times\bar{a}_1\Big\}$$

$$= \frac{1}{|\bar{q}||\bar{q}\times\bar{a}_1|^2}\Big[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{P}_3)\Big] \qquad (3.15)$$

3.2 Procedure to be Avoided

It may be noted that, according to Eqs. (3.1) and (3.14), C_{hk} is given by the sum of two integrals with respect to η is of the type

$$I_D = \int \frac{M_3(\eta)}{N_4(\eta)} \frac{1}{\sqrt{\alpha\eta^2+2\beta\eta+\gamma}}\, d\eta \tag{3.16}$$

where $M_3(\eta) = \sum_1^3 m_k \eta^k$ is a polynomial of third degree in η, while

$$N_4(\eta) = \sum_1^4 n_k \eta^k = \left|\bar{z}_0 \times \bar{z}_1\right| \geq 0 \tag{3.17}$$

is a nonnegative polynomial of fourth degree. If $N_4(\eta)$ were always positive, then the rational function could be replaced by a polinomial

$$\frac{M_3(\eta)}{N_4(\eta)} = C_0 + C_1(a\eta+b) + C_2(a\eta+b)^2 + \cdots \tag{3.18}$$

and the integral in Eq. (3.16) could be easily evaluated by using the recurrent formula (Eq. A.6)

$$\int \frac{(\alpha\eta+\beta)^n}{\sqrt{\alpha\eta^2+2\beta\eta+\gamma}}\, d\eta = \frac{1}{n}(\alpha\eta+\beta)^{n-1}\sqrt{\alpha\eta^2+2\beta\eta+\gamma} - \frac{n-1}{n}(\alpha\gamma-\beta^2)\int \frac{(\alpha\eta+\beta)^{n-2}}{\sqrt{\alpha\eta^2+2\beta\eta+\gamma}}\, d\eta \tag{3.19}$$

together with (Eqs. A.9 and A.10)

$$\int \frac{\alpha\eta+\beta}{\sqrt{\alpha\eta^2+2\beta\eta+\gamma}}\, d\eta = \sqrt{\alpha\eta^2+2\beta\eta+\gamma}$$

$$\int \frac{1}{\sqrt{\alpha\eta^2+2\beta\eta+\gamma}}\, d\eta = \frac{1}{\sqrt{\alpha}} \ln\left[(\alpha\eta+\beta) + \sqrt{\alpha}\sqrt{\alpha\eta^2+2\beta\eta+\gamma}\right] \tag{3.20}$$

However, if for a certain value, η_*, of η the denominator is equal to zero, Eq. (3.18) cannot be used. It may be noted that this implies at $\eta = \eta_*$, $|\bar{q}_0 \times \bar{q}_1| = 0$ or, that $\bar{q}_0 = \bar{p}_0 + \eta_* \bar{p}_2$ is parallel to $\bar{p}_1 + \eta_* \bar{p}_3$. As is evident for Fig. 2, this is the case if the point $\bar{P}_h$ belongs to the line $\eta = \eta_*$ (*). It may be noted that, as shown by Eq. (3.15), the numerator has a single root at $\eta = \eta_*$ while the denominator has a double root at $\eta = \eta_*$. In order to avoid this problem, it is convenient to follow the procedure described in the following subsection.

3.3 Integration with Respect to η

Consider Eq. (3.16) as mentioned above, $N_4(\eta)$ is a non-negative polynomial of fourth degree. Hence, it can always be decomposed as

$$N_4(\eta) = n_4 (\eta - \eta_1)(\eta - \eta_2)(\eta - \eta_3)(\eta - \eta_4)$$

where η_i are the roots of the polynomial N_4, and n_4 is the coefficient of η^4, namely $|\bar{p}_2 \times \bar{p}_3|^2$. Then the rational function M_3/N_4 can be separated into the sum of four terms

$$\frac{M_3(\eta)}{N_4(\eta)} = \frac{c_1}{\eta - \eta_1} + \frac{c_2}{\eta - \eta_2} + \frac{c_3}{\eta - \eta_3} + \frac{c_4}{\eta - \eta_4} \tag{3.21}$$

where C_i are constants. Hence, the evaluation of c_{hk} is

(*) It may be noted that the problem could be avoided by integrating with respect to η first.

reduced to the evaluation of integrals of the type

$$I_D = \int \frac{c_i}{\eta - \eta_i} \frac{d\eta}{\sqrt{\alpha\eta^2 + 2\beta\eta + \gamma}}$$

$$= -\frac{c_k}{r_i} \ln \frac{r_i \gamma + r_i^2 + (\beta + \gamma\eta_i)(\eta - \eta_i)}{\eta - \eta_i} \tag{3.22}$$

with

$$r_i = \alpha\eta_i^2 + 2\beta\eta_i + \gamma$$

3.4 Particular Cases

A few special cases need a special treatment or can be evaluated in a simpler manner. They are considered in the following Subsections.

3.4.1 $N_4 = 0$ in the integration interval

As mentioned above, N_4 is a nonnegative polinomial. Hence, if $N_4(\eta_1) = 0$ with η_1 real and within the interval (-1, 1), then N_4 has at least a double root at η_1. In this case, the decomposition in fractions is different and Eq. (3.21) must be replaced by

$$\frac{M_3(\eta)}{N_4(\eta)} = \frac{c_1}{\eta - \eta_1} + \frac{c_2}{(\eta - \eta_1)^2} + \frac{c_3}{\eta - \eta_3} + \frac{c_4}{\eta - \eta_4} \tag{3.23}$$

However, as mentioned above, when this occurs, $|\bar{\mathfrak{z}}_0 \times \bar{\mathfrak{z}}_1| = 0$ and the numerator is also equal to zero (single root). This implies that C_2 in Eq. (3.23) is equal to zero. Hence, only integrals of the type given in Eq. (3.22) are involved.

3.4.2 $N_4 \equiv 0$

The case $N_4 = 0$ is also possible. This implies that $\bar{\mathcal{Q}}_0$ is always parallel to $\bar{\mathcal{Q}}_1$. This, on the other hand, implies that all the lines ξ = const converge into the point $\bar{p}^{(h)}$. This means that the element is a planar element with two edges passing through the point $\bar{p}^{(h)}$, which lies outside the element. In this case, the coefficient C_{hk} is identically equal to zero, as evident from Eq. (1.16).

3.4.3 Trapezoidal planar element

If two edges of the element are parallel (trapezoidal planar element) the integration can be performed in a simple fashion. Choose ξ and η such that the parallel edges are the edges $\eta = \pm 1$ or

$$\bar{q} = (\bar{p}_0 \pm \bar{p}_2) + \xi\,(\bar{p}_1 \pm \bar{p}_3) \tag{3.24}$$

In order for the edge to be parallel, the vector $\bar{p}_1$ and $\bar{p}_3$ must be parallel, namely

$$\begin{aligned} \bar{p}_1 &= \chi\,\bar{u} \\ \bar{p}_3 &= \psi\,\bar{u} \end{aligned} \tag{3.25}$$

(where $\bar{u} = \dfrac{\bar{p}_1}{|\bar{p}_1|}$ is a unit vector) and

$$\bar{a}_1 = \bar{\mathcal{Q}}_1 \equiv (\bar{p}_1 + \eta\,\bar{p}_3) = (\chi + \eta\psi)\,\bar{u} \tag{3.26}$$

Substituting these into Eq. (3.15) gives (note that $\bar{q}\cdot\bar{a}_1\times\bar{p}_3 = 0$)

$$\lambda_D = -\frac{\bar{q}\cdot\bar{u}}{|\bar{q}|}\;\frac{\bar{p}_0\cdot\bar{u}\times\bar{p}_2}{|\bar{\mathcal{Q}}_0\times\bar{u}|^2} \tag{3.27}$$

Integrating with respect to η , one obtains

$$I_D = -\frac{1}{2\pi}\int i_D\, d\eta = -\frac{\bar{P}_0 \cdot \bar{u} \times \bar{P}_2}{2\pi}\int \frac{\bar{q}\cdot\bar{u}}{|\bar{q}|}\frac{1}{|\bar{q}\times\bar{u}|^2}d\eta \tag{3.28}$$

Noting that (see Eq. B.1)

$$|\bar{q}|^2 = |\bar{q}\times\bar{u}|^2 + |\bar{q}\cdot\bar{u}|^2 \tag{3.29}$$

one obtains

$$I_D = -\frac{1}{2\pi}\frac{f}{|f|}J \tag{3.30}$$

with (Eq. A.1)

$$J = |f|\int \frac{a+b\eta}{c+2d\eta+e\eta^2}\frac{1}{\sqrt{(a+b\eta)^2+(c+2d\eta+e\eta^2)}}d\eta \tag{3.31}$$

$$= \frac{|f|}{\sqrt{ec-d^2}}\tan^{-1}\left[\frac{(ae-bd)\eta+(ad-bc)}{\sqrt{ec-d^2}}\frac{1}{r_h}\right]$$

where it has been set

$$\bar{q}\cdot\bar{u} = (\bar{P}_0+\xi\bar{P}_1)\cdot\bar{u} + \eta(\bar{P}_2+\xi\bar{P}_3)\cdot\bar{u}$$

$$= a + b\eta$$

$$|\bar{q}\times\bar{u}|^2 = |\bar{q}|^2 - (\bar{q}\cdot\bar{u})^2$$

$$= |P_0+\xi P_1|^2 + 2(P_0+\xi P_1)(P_2+\xi P_2)\eta + |P_2+\xi P_3|^2\eta^2 - (a+b\eta)^2 \tag{3.32}$$

$$= c + 2d\eta + e\eta^2$$

with

$$a = \bar{C}_0 \cdot \bar{u}$$

$$b = \bar{C}_2 \cdot \bar{u}$$

$$\alpha = \bar{C}_0 \cdot \bar{C}_0$$

$$\beta = \bar{C}_0 \cdot \bar{C}_2$$

$$\gamma = \bar{C}_2 \cdot \bar{C}_2$$

$$c = \alpha - a^2 = |\bar{C}_0 \times \bar{u}|^2 \equiv |\bar{P}_0 \times \bar{u}|^2$$

$$d = \beta - ab = (\bar{C}_0 \times \bar{u}) \cdot (\bar{C}_2 \times \bar{u}) = (\bar{P}_0 \times \bar{u}) \cdot (\bar{P}_2 \times \bar{u})$$

$$e = \gamma - b^2 = |\bar{C}_2 \times \bar{u}|^2 = |\bar{P}_2 \times \bar{u}|^2$$

$$f = \bar{q} \cdot \bar{u} \times \bar{P}_2 \tag{3.33}$$

where

$$\bar{C}_0 = \bar{P}_0 + \xi \bar{P}_1$$

$$\bar{C}_2 = \bar{P}_2 + \xi \bar{P}_3 \tag{3.34}$$

The results obtained above can be rewritten in a more compact form by noting that (see Eq. B.1)

$$\begin{aligned} ec - d^2 &= |\bar{P}_0 \times \bar{u}|^2 |\bar{P}_2 \times \bar{u}|^2 - \left[(\bar{P}_0 \times \bar{u}) \cdot (\bar{P}_2 \times \bar{u})\right]^2 \\ &= |(\bar{P}_0 \times \bar{u}) \times (\bar{P}_2 \times \bar{u})|^2 = |\bar{P}_0^N \times \bar{P}_2|^2 \\ &= \left(\bar{P}_0^N \times \bar{P}_2^N \cdot \bar{u}\right)^2 = (\bar{P}_2 \times \bar{P}_2 \cdot \bar{u})^2 = f^2 \end{aligned} \tag{3.35}$$

where the superscript N indicates the part normal to $\bar{u}$:

$$\bar{P}_i^N = \bar{P}_i - P_i^T \bar{u} \qquad \left(P_i^T = \bar{P}_i \cdot \bar{u}\right) \tag{3.36}$$

Also, it may be noted that

$$\frac{1}{r|f|}\left[(ae - bd)\eta + (ad - bc)\right] = \left[a(e\eta + d) - b(d\eta + c)\right] \frac{1}{r_h |f|}$$

$$
\begin{aligned}
&= \left\{ a\left[(\gamma - b^2)\eta + (\beta - ab)\right] - b\left[(\beta - ab)\eta + (\alpha - a^2)\right]\right\} \frac{1}{r_h |f|} \\
&= \left[a\,(\gamma\eta + \beta) - b\,(\beta\eta + \alpha)\right] \frac{1}{r_h |f|} \\
&= \left[(\bar{c}_0 \cdot \bar{u})(\bar{c}_0 + \eta \bar{c}_2)\cdot \bar{c}_2 - (\bar{c}_2 \cdot \bar{u})(\bar{c}_0 + \eta \bar{c}_2)\cdot \bar{c}_0 \right] \frac{1}{r_h |f|} \\
&= \left[(\bar{c}_0 \cdot \bar{u})(\bar{q} \cdot \bar{c}_2) - (\bar{c}_2 \cdot \bar{u})(\bar{q} \cdot \bar{c}_0)\right] \frac{1}{r_h |f|} \\
&= - \frac{(\bar{c}_0 \times \bar{c}_2)\cdot(\bar{q} \times \bar{u})}{r_h |\bar{q} \cdot \bar{u} \times \bar{P}_2|} = - \frac{(\bar{q} \times \bar{c}_2)\cdot(\bar{q} \times \bar{u})}{r_h |\bar{q} \cdot \bar{u} \times \bar{c}_2|} \\
&= - \frac{(\bar{q} \times \bar{c}_2)\cdot(\bar{q} \times \bar{a}_1)}{r_h |\bar{q} \cdot \bar{a}_1 \times \bar{c}_2|} = - \frac{(\bar{q} \times \bar{a}_1)\cdot(\bar{q} \times \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}}\, |\bar{q} \cdot \bar{a}_1 \times \bar{a}_2|}
\end{aligned}
\tag{3.37}
$$

Combining Eqs. (3.30), (3.31), (3.35) and (3.37) yields

$$
I_D = - \frac{1}{2\pi}\, \mathrm{sign}\,(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)\; J \tag{3.38}
$$

with

$$
J = + \tan^{-1} \frac{-(\bar{q} \times \bar{a}_1)\cdot(\bar{q} \times \bar{a}_2)}{|\bar{q}|\; |\bar{q} \cdot \bar{a}_1 \times \bar{a}_2|} \tag{3.39}
$$

3.5 Hyperboloidal Element

In this section, it will be shown by differentiation that the result obtained above is valid for any hyperboloidal element. Note first that

$$\frac{\partial \bar{q}}{\partial \eta} = \bar{a}_2$$

$$\frac{\partial \bar{a}_1}{\partial \eta} = \bar{P}_3$$

$$\frac{\partial \bar{a}_2}{\partial \eta} = 0 \tag{3.40}$$

Next consider (± 1 indicates the sign of $\bar{q} \cdot \bar{a}_1 \times \bar{a}_2$)

$$\frac{\partial J}{\partial \eta} = \frac{\partial}{\partial \eta} \tan^{-1} \frac{-(\bar{q} \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}}\ (\pm\ \bar{q} \cdot \bar{a}_1 \times \bar{a}_2)}$$

$$= \frac{\mp 1}{1 + \left[\dfrac{(\bar{q} \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}}\ (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)}\right]^2} \times$$

$$\times \left\{ \left[(\bar{a}_2 \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) + (\bar{q} \times \bar{P}_3) \cdot (\bar{q} \times \bar{a}_2) + (\bar{q} \times \bar{a}_1) \cdot (\bar{a}_2 \times \bar{a}_2) \right] \frac{1}{|\bar{q}|\ (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)} \right.$$

$$\left. + \frac{(\bar{q} \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2)}{-(\bar{q} \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)^2} \left[\frac{\bar{a}_2 \cdot \bar{q}}{\sqrt{\bar{q} \cdot \bar{q}}}\ \bar{q} \cdot \bar{a}_1 \times \bar{a}_2 + \sqrt{\bar{q} \cdot \bar{q}} \left(\bar{a}_2 \cdot \bar{a}_1 \times \bar{a}_2 + \bar{q} \cdot \bar{P}_3 \times \bar{a}_2 \right) \right] \right\}$$

$$= \mp \frac{r^2(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2}{r^2(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+\left[(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\right]^2}\frac{1}{r^3(\bar{q}\cdot\bar{a}_1\times a_2)^2}$$

$$\times\Big\{\left[(\bar{a}_2\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)+(\bar{q}\times\bar{P}_3)\cdot(\bar{q}\times\bar{a}_2)\right](\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\left[(\bar{a}_2\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+\bar{q}\cdot\bar{q}\,(\bar{q}\cdot\bar{P}_3\times\bar{a}_2)\right]\Big\}$$

$$= \frac{\mp 1}{r^2(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+\left[(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\right]^2}\ \frac{1}{r}$$

$$\times\Big\{\left[(\bar{a}_2\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{q})-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{a}_2\cdot q)\right](\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$+\left[(\bar{q}\times\bar{P}_3)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{P}_3\times\bar{a}_2)\right]\bar{q}\cdot\bar{q}\Big\}$$

$$(3.41)$$

Next note that (see Eq. B.1)

$$\bar{q}\cdot\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+\left[(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\right]^2 \equiv |\bar{q}\times\bar{a}_1|^2\,|\bar{q}\times\bar{a}_2|^2 \qquad (3.42)$$

For

$$|\bar{q}\times\bar{a}_1|^2\,|\bar{q}\times\bar{a}_2|^2-\left\{(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+\left[(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)\right]^2\right\}$$

$$=\left[(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_1)-(\bar{q}\cdot\bar{a}_1)^2\right]\left[(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{a}_2)-(\bar{q}\cdot\bar{a}_2)^2\right]$$

$$-\bar{q}\cdot\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2-\left[(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_2)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2)\right]^2$$

$$=(\bar{q}\cdot\bar{q})^2(\bar{a}_1\cdot\bar{a}_1)(\bar{a}_2\cdot\bar{a}_2)+\cancel{(\bar{q}\cdot\bar{a}_1)^2(\bar{q}\cdot\bar{a}_2)^2}-(\bar{q}\cdot\bar{q})\left[(\bar{a}_1\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2)^2+(\bar{a}_2\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)^2\right]$$

$$-(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2-(\bar{q}\cdot\bar{q})^2(\bar{a}_1\cdot\bar{a}_2)^2-\cancel{(\bar{q}\cdot\bar{a}_1)^2(\bar{q}\cdot\bar{a}_2)^2}+2(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2)$$

$$=(\bar{q}\cdot\bar{q})\Big\{(\bar{q}\cdot\bar{q})\left[(\bar{a}_1\cdot\bar{a}_1)(\bar{a}_2\cdot\bar{a}_2)-(\bar{a}_1\cdot\bar{a}_2)^2\right]-(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2$$

$$-\left[(\bar{a}_1\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2)^2+(\bar{a}_2\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)^2-2(\bar{a}_1\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_2)\right]\Big\}$$

$$=(\bar{q}\cdot\bar{q})\Big\{(\bar{q}\cdot\bar{q})(\bar{a}_1\times\bar{a}_2)\cdot(\bar{a}_1\times\bar{a}_2)-\bar{q}\cdot(\bar{a}_1\times\bar{a}_2)\,\bar{q}\cdot(\bar{a}_1\times\bar{a}_2)$$

$$-\left|\bar{a}_1(\bar{q}\cdot\bar{a}_2)-\bar{a}_2(\bar{q}\cdot\bar{a}_1)\right|^2\Big\}$$

$$=(\bar{q}\cdot\bar{q})\Big\{|\bar{q}\times(\bar{a}_1\times\bar{a}_2)|^2-|\bar{a}_1(\bar{q}\cdot\bar{a}_2)-\bar{a}_2(\bar{q}\cdot\bar{a}_1)|^2\Big\}$$

$$=0$$

(3.43)

since

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}\,(\bar{A}\cdot\bar{C}) - \bar{C}\,(\bar{A}\cdot\bar{B}) \tag{3.44}$$

It is worth noting that, as shown above,

$$(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 = |\bar{q}|^2\,|\bar{a}_1\times\bar{a}_2|^2 - \left|\bar{a}_1(\bar{q}\cdot\bar{a}_2) - \bar{a}_2(\bar{q}\cdot\bar{a}_1)\right|^2 \tag{3.45}$$

Next note that (see Eq. B.1)

$$\Big[(\bar{a}_2\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{q}) - (\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{a}_2\cdot\bar{q})\Big](\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$+\Big[(\bar{q}\times\bar{p}_3)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\Big](\bar{q}\cdot\bar{q})$$

$$= \Big\{\Big[\cancel{(\bar{a}_2\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_2)} - (\bar{a}_2\cdot\bar{a}_2)(\bar{a}_1\cdot\bar{q})\Big](\bar{q}\cdot\bar{q})$$

$$-\Big[\cancel{(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_2)} - (\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)\Big](\bar{a}_2\cdot\bar{q})\Big\}(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$+\Big\{\Big[(\bar{q}\cdot\bar{q})(\bar{p}_3\cdot\bar{a}_2) - (\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{p}_3)\Big](\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$-\Big[(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_2) - (\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1)\Big](\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\Big\}(\bar{q}\cdot\bar{q})$$

$$= -\Big[(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{a}_2) - (\bar{q}\cdot\bar{a}_2)^2\Big](\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$+\Big\{(\bar{q}\cdot\bar{q})\Big[(\bar{a}_2\cdot\bar{p}_3)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{a}_1\cdot\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\Big]$$

$$-(\bar{q}\cdot\bar{a}_2)\Big[(\bar{q}\cdot\bar{p}_3)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{p}_3\times\bar{a}_2)\Big]\Big\}\bar{q}\cdot\bar{q}$$

$$= -|\bar{q}\times\bar{a}_2|^2(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)$$

$$+|\bar{q}\times\bar{a}_2|^2(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3) \qquad (3.46)$$

since (note the change of the order in the triple product)

$$(\bar{q}\cdot\bar{q})\left[(\bar{a}_2\cdot\bar{p}_3)(\bar{q}\times\bar{a}_2\cdot\bar{a}_1)-(\bar{a}_1\cdot\bar{a}_2)(\bar{q}\times\bar{a}_2\cdot\bar{p}_3)\right]$$

$$-(\bar{q}\cdot\bar{a}_2)\left[(\bar{q}\cdot\bar{p}_3)(\bar{q}\times\bar{a}_2\cdot\bar{a}_1)-(\bar{q}\cdot\bar{a}_1)(\bar{q}\times\bar{a}_2\cdot\bar{p}_3)\right]$$

$$=(\bar{q}\cdot\bar{q})\left[\bar{a}_2\times(\bar{q}\times\bar{a}_2)\right]\cdot(\bar{p}_3\times\bar{a}_1)$$

$$-(\bar{q}\cdot\bar{a}_2)\left[\bar{q}\times(\bar{q}\times\bar{a}_2)\right]\cdot(\bar{p}_3\times\bar{a}_1)$$

$$=(\bar{q}\cdot\bar{q})\left[(\bar{a}_2\cdot\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_1)-\cancel{(\bar{a}_2\cdot\bar{q})(\bar{a}_2\cdot\bar{p}_3\times\bar{a}_1)}\right]$$

$$-(\bar{q}\cdot\bar{a}_2)\left[(\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{p}_3\times\bar{a}_1)-\cancel{(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{p}_3\times\bar{a}_1)}\right]$$

$$=\left[(\bar{q}\cdot\bar{q})(\bar{a}_2\cdot\bar{a}_2)-(\bar{q}\cdot\bar{a}_2)^2\right]\bar{q}\cdot\bar{p}_3\times\bar{a}_1$$

$$=|\bar{q}\times\bar{a}_2|^2\,\bar{q}\cdot\bar{p}_3\times\bar{a}_1 \qquad (3.47)$$

Finally, by combining Eqs. (3.38), (3.39), (3.41), (3.42) and (3.46) one obtains

$$\frac{\partial}{\partial \eta} \tan^{-1} \frac{-(\bar{q} \times \bar{a}_1)\cdot(\bar{q} \times \bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\;(\pm\bar{q}\cdot\bar{a}_1 \times \bar{a}_2)}$$

$$= \frac{1}{r} \frac{\pm 1}{|\bar{q}\times\bar{a}_1|^2\,|\bar{q}\times\bar{a}_2|^2} |\bar{q}\times\bar{a}_2|^2 \left[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{P}_3)\right]$$

$$= \frac{\pm 1}{r} \frac{(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{P}_3)}{|\bar{q}\times a_1|^2} \tag{3.48}$$

and, according to Eq. (3.38) and (3.15)

$$\frac{\partial I_D}{\partial \eta} = \mp\frac{1}{2\pi}\frac{\partial J}{\partial \eta} = \frac{-1}{2\pi}\,\frac{1}{r}\,\frac{(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2) - (\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{P}_3)}{|\bar{q}\times a_1|^2}$$

$$= -\frac{1}{2\pi}\lambda_D \tag{3.49}$$

in agreement with Eq. (3.28). This completes the proof that Eq. (3.38) is valid for any hyperboloidal element. However, for the sake of completeness, the derivative with respect to ξ is also performed (note that $\frac{\partial \bar{a}_1}{\partial \xi} = 0$, $\frac{\partial \bar{a}_2}{\partial \xi} = \bar{P}_3$ and $\frac{\partial}{\partial \xi}(\bar{q}\times\bar{a}_1) = 0$)

$$2\pi \frac{\partial^2 I_D}{\partial \xi \partial \eta} = \mp \frac{\partial^2}{\partial \xi \partial \eta} \tan^{-1} \frac{-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\;(\pm\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}$$

$$= \frac{\partial}{\partial \xi}\left(\frac{(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{P}_3) - (\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}{\sqrt{\bar{q}\cdot\bar{q}}\;|\bar{q}\times\bar{a}_1|^2}\right)$$

$$= \frac{1}{|\bar{q}\times\bar{a}_1|^2}\left\{-\frac{\bar{q}\cdot\bar{a}_1}{(\bar{q}\cdot\bar{q})^{3/2}}\left[\cancel{(\bar{q}\cdot\bar{q})(\bar{q}\cdot}\bar{a}_1\times\bar{P}_3) - (\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\right] + \right.$$

$$+\frac{1}{\sqrt{\bar{q}\cdot\bar{q}}}\Big[2\,\cancel{(\bar{a}_1\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)}+(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_1\times\bar{p}_3)$$

$$-(\bar{a}_1\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-(\bar{q}\cdot\bar{a}_1)(\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2)-\cancel{(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1\times\bar{p}_3)}\Big]\Big\}$$

$$=\frac{1}{r^3\,|\bar{q}\times\bar{a}_1|^2}\Big[(\bar{q}\cdot\bar{a}_1)(\bar{q}\cdot\bar{a}_1)-(\bar{q}\cdot\bar{q})(\bar{a}_1\cdot\bar{a}_1)\Big]\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2$$

$$=-\frac{1}{r_3}\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2 \tag{3.50}$$

in agreement with Eq. (1.16)

SECTION 4

SOURCE INTEGRAL

1. Introduction

Consider the evaluation of b_{hk}. According to Eq. (2.21), b_{hk} can be evaluated as

$$b_{hk} = \frac{1}{2\pi}\int_{-1}^{1}\left[\, i_s \,\right]_{\xi=-1}^{\xi=1} d\eta \tag{4.1}$$

where

$$i_s = \int \frac{\hat{a}\,\xi + \hat{b}}{\sqrt{\hat{\alpha}\,\xi^2 + 2\hat{\beta}\,\xi + \hat{\gamma}}}\, d\xi + \hat{c}\,(\eta) \tag{4.2}$$

with

$$\begin{aligned}
\hat{a} &= (y_1 z_3 - z_1 y_3) = \bar{P}_1 \times \bar{P}_3 \cdot \bar{i}_x \\
\hat{b} &= (y_1 z_2 - z_1 y_2) + \eta\,(y_3 z_2 - z_3 y_2) = (\bar{P}_1 + \eta \bar{P}_3) \times \bar{P}_2 \cdot \bar{i}_x \\
\hat{\alpha} &= \bar{\mathcal{Q}}_1 \cdot \bar{\mathcal{Q}}_1 = |\,\bar{P}_1 + \eta \bar{P}_3\,|^2 \\
\hat{\beta} &= \bar{\mathcal{Q}}_0 \cdot \bar{\mathcal{Q}}_1 = (\bar{P}_1 + \eta \bar{P}_3) \cdot (\bar{P}_0 + \eta \bar{P}_2) \\
\hat{\gamma} &= \bar{\mathcal{Q}}_0 \cdot \bar{\mathcal{Q}}_0 = (\bar{P}_0 + \eta \bar{P}_2) \cdot (\bar{P}_0 + \eta \bar{P}_2)
\end{aligned} \tag{4.3}$$

Note that Eq. (4.2) yields (Eqs. A.9 and A.10)

$$\begin{aligned}
i_s &= \frac{\hat{a}}{\hat{\alpha}} \int \frac{\hat{\alpha}\,\xi + \hat{\beta}}{\sqrt{\hat{\alpha}\,\xi^2 + 2\hat{\beta}\,\xi + \hat{\gamma}}}\, d\xi + \frac{\hat{b}\hat{\alpha} - \hat{a}\hat{\beta}}{\hat{\alpha}} \int \frac{1}{\sqrt{\hat{\alpha}\,\xi^2 + 2\hat{\beta}\,\xi + \hat{\gamma}}}\, d\xi + \hat{c}(\eta) \\
&= \frac{\hat{a}}{\hat{\alpha}} \sqrt{\hat{\alpha}\,\xi^2 + 2\hat{\beta}\,\xi + \hat{\gamma}} + \frac{\hat{b}\hat{\alpha} - \hat{a}\hat{\beta}}{\hat{\alpha}\sqrt{\hat{\alpha}}} \ln\left(\sqrt{\hat{\alpha}}\, r + \hat{\alpha}\,\xi + \hat{\beta}\right) + \hat{c}(\eta)
\end{aligned} \tag{4.4}$$

The integral obtained combining Eqs. (4.1) and (4.4) can be easily evaluated numerically. Note that b is a linear function of η while $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ quadratic function of η. However, for elements with two parallel edges, the integral can be evaluated analytically, as shown in the next Subsection.

4.2 Trapezoidal Planar Element

Consider an element with two parallel edges (trapezoidal planar element). Choose ξ and η such that the parallel edges correspond to $\eta = \pm 1$. For this case

$$\begin{aligned} \bar{P}_1 &= \chi\, \bar{u} \\ \bar{P}_3 &= \psi\, \bar{u} \end{aligned} \tag{4.5}$$

(where $\bar{u} = \dfrac{\bar{P}_1}{|\bar{P}_1|}$) and

$$\bar{\mathcal{Q}}_1 = (\bar{P}_1 + \eta \bar{P}_3) = (\chi + \eta\psi)\,\bar{u} \tag{4.6}$$

These imply

$$\begin{aligned} \hat{a} &= 0 \\ \hat{b} &= (\chi + \eta\psi)(\bar{u} \times \bar{P}_2 \cdot \bar{i}_x) = (\chi + \eta\psi)\, B \\ \hat{\alpha} &= (\chi + \eta\psi)^2 \\ \hat{\beta} &= (\chi + \eta\psi)(\bar{u} \cdot \bar{\mathcal{Q}}_0) \\ \hat{\gamma} &= \bar{\mathcal{Q}}_0 \cdot \bar{\mathcal{Q}}_0 \end{aligned} \tag{4.7}$$

and Eq. (4.4) reduces to

$$\begin{aligned} \lambda_s &= \frac{\hat{b}}{\sqrt{\hat{\alpha}}} \ln\left(\sqrt{\hat{\alpha}}\, r + \hat{\alpha}\xi + \hat{\beta}\right) + \hat{c}(\eta) \\ &= B\frac{\chi + \eta\psi}{|\chi + \eta\psi|} \ln\left[|\chi+\eta\psi|\, r + (\chi+\eta\psi)^2 \xi + (\chi+\eta\psi)(\bar{u}\cdot\bar{\mathcal{Q}}_0)\right] \\ &\quad + \hat{c}(\eta) \end{aligned} \tag{4.8}$$

Note that

$$|P_3| < |P_1| \tag{4.9}$$

otherwise the boundaries $\xi = \pm 1$ would be crossing. This implies

$$\chi + \eta \psi > 0 \tag{4.10}$$

and hence, Eq. (4.8) reduces to

$$\begin{aligned} \lambda_s &= B \ln [r + (\chi + \eta\psi)\xi + \bar{u} \cdot \bar{\mathcal{Q}}_0] + C_1(\eta) \\ &= B \ln [r + (\bar{\mathcal{Q}}_0 + \xi \bar{\mathcal{Q}}_1) \cdot \bar{u}] + C_1(\eta) \\ &= B \ln (|\bar{q}| + \bar{q} \cdot \bar{u}) + C_1(\eta) \end{aligned} \tag{4.11}$$

with

$$C_1(\eta) = \hat{C}(\eta) + B \ln (\chi + \eta \psi) \tag{4.12}$$

Consider

$$I_s = \frac{1}{2\pi} \int \lambda_s \, d\eta = \frac{B}{2\pi} \int \ln (|\bar{q}| + \bar{q} \cdot \bar{u}) d\eta + C(\eta) \tag{4.13}$$

where

$$C(\eta) = \frac{1}{2\pi} \int C_1(\eta) d\eta + C \tag{4.14}$$

Integrating by parts yields

$$\frac{2\pi}{B} I_s = (\eta - \eta_*) \ln (|\bar{q}| + \bar{q} \cdot \bar{u}) - \int (\eta - \eta_*) \frac{1}{|\bar{q}| + \bar{q} \cdot \bar{u}} \left(\frac{\bar{q} \cdot (\bar{P}_2 + \xi \bar{P}_3)}{|\bar{q}|} + (\bar{P}_2 + \xi \bar{P}_3) \cdot \bar{u} \right) d\eta$$

$$= (\eta - \eta_*) \ln (|\bar{q}| + \bar{q} \cdot \bar{u}) - \int \frac{(\eta - \eta_*)(\bar{P}_2 + \xi \bar{P}_3) \cdot (\bar{q} + |\bar{q}| \bar{u})}{|\bar{q}| + \bar{q} \cdot \bar{u}} \frac{d\eta}{|\bar{q}|} + C(\eta)$$

(4.15)

where η_* is a suitable constant. The choice for η_* is discussed later on (see Eq. 4.30). Note that it is possible to write

$$(\eta - \eta_*)(\bar{P}_2 + \xi \bar{P}_3) = \bar{q} - \bar{q}_* \tag{4.16}$$

with

$$\bar{q}_* = \bar{P}_0 + \xi \bar{P}_1 + \eta_* (\bar{P}_2 + \xi \bar{P}_3) \tag{4.17}$$

By using Eq. (4.16), Eq. (4.15) reduces to

$$I_{s1} = \frac{2\pi}{B} I_s - (\eta - \eta_*) \ln (|\bar{q}| + \bar{q}\cdot\bar{u})$$

$$= -\int \frac{\bar{q}\cdot(\bar{q} + |\bar{q}|\,\bar{u})}{|\bar{q}| + \bar{q}\cdot\bar{u}} \frac{1}{|\bar{q}|} d\eta + \int \bar{q}_* \cdot \frac{\bar{q} + |\bar{q}|\,\bar{u}}{|\bar{q}| + \bar{q}\cdot\bar{u}} \frac{d\eta}{|\bar{q}|} + C(\eta)$$

$$= \int \bar{q}_* \frac{\bar{q} + |\bar{q}|\,\bar{u}}{|\bar{q}| + \bar{q}\cdot\bar{u}} \frac{1}{|\bar{q}|} d\eta + C_2(\eta) \tag{4.18}$$

with

$$C_2(\eta) = C(\eta) - \int \bar{q} \cdot \frac{(\bar{q} + |\bar{q}|\,\bar{u})}{|\bar{q}| + \bar{q}\cdot\bar{u}} \frac{1}{|\bar{q}|} d\eta = C(\eta) - \eta \tag{4.19}$$

Next note that

$$I_{S1} = \int \bar{q}_* \cdot \frac{(\bar{q} + |\bar{q}|\,\bar{u})}{|\bar{q}| + \bar{q}\cdot\bar{u}} \frac{|\bar{q}| - (\bar{q}\cdot\bar{u})}{|\bar{q}| - (\bar{q}\cdot\bar{u})} \frac{1}{|\bar{q}|} d\eta + C_2(\eta)$$

$$= \int \frac{\bar{q}_*\cdot(\bar{q} + |\bar{q}|\bar{u})}{|\bar{q}|^2 - (\bar{q}\cdot\bar{u})^2} \frac{|\bar{q}| - \bar{q}\cdot\bar{u}}{|\bar{q}|} d\eta + C_2(\eta)$$

$$= \int \frac{\bar{q}_*\cdot\bar{q} - (\bar{q}_*\cdot u)(\bar{q}\cdot\bar{u})}{|\bar{q}|^2 - (\bar{q}\cdot\bar{u})^2} d\eta + \int \frac{(\bar{q}_*\cdot\bar{u})|\bar{q}|^2 - (\bar{q}_*\cdot\bar{q})(\bar{q}\cdot\bar{u})}{|\bar{q}|^2 - (\bar{q}\cdot\bar{u})^2} \frac{d\eta}{|\bar{q}|} + C_2(\eta) \tag{4.20}$$

Applying the general formula (see Appendix B)

$$(\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) = (\bar{A} \cdot \bar{C})(\bar{B} \cdot \bar{D}) - (\bar{A} \cdot \bar{D})(\bar{B} \cdot \bar{C}) \tag{4.21}$$

one obtains (note that $\bar{u} \cdot \bar{u} = 1$)

$$|\bar{q}|^2 - |\bar{q} \cdot \bar{u}|^2 = |\bar{q} \times \bar{u}|^2 = |\bar{Q}_0 \times \bar{u}|^2$$

$$\bar{q}_* \cdot \bar{q} - (\bar{q}_* \cdot \bar{u})(\bar{q} \cdot \bar{u}) = (\bar{q}_* \times \bar{u}) \cdot (\bar{q} \times \bar{u}) = [(\bar{P}_0 + \eta_* \bar{P}_2) \times \bar{u}] \cdot (\bar{Q}_0 \times \bar{u})$$

$$(\bar{q}_* \cdot \bar{u})|\bar{q}|^2 - (\bar{q}_* \cdot \bar{q})(\bar{q} \cdot \bar{u}) = (\bar{q}_* \times \bar{q})(\bar{u} \times \bar{q}) = (\bar{q}_* \times \bar{q}) \cdot (\bar{u} \times \bar{Q}_0)$$

(4.22)

since

$$\bar{q}_* \times \bar{u} = \bar{P}_0 + \xi \bar{P}_1 + \eta_* (\bar{P}_2 + \xi \bar{P}_3) \times \bar{u} = (\bar{P}_0 + \eta_* \bar{P}_2) \times \bar{u} \tag{4.23}$$

$$\bar{q} \times \bar{u} = (\bar{Q}_0 + \xi \bar{Q}_1) \times \bar{u} = \bar{Q}_0 \times \bar{u} \tag{4.24}$$

Combining Eqs. (4.20) and (4.22) yields

$$I_{s1} = \int \frac{[(\bar{P}_0 + \eta_* \bar{P}_2) \times \bar{u}] \cdot (\bar{Q}_0 \times \bar{u})}{|\bar{Q}_0 \times \bar{u}|^2} d\eta + \int \frac{(\bar{q} \times \bar{q}_*) \cdot (\bar{q} \times \bar{u})}{|\bar{q} \times \bar{u}|^2} \frac{d\eta}{|\bar{q}|} + C_2(\eta)$$

$$= \int \frac{(\bar{q} \times \bar{q}_*) \cdot (\bar{q} \times \bar{u})}{|\bar{q} \times \bar{u}|^2} \frac{d\eta}{|\bar{q}|} + C_3(\eta)$$

(4.25)

where

$$C_3(\eta) = C_2(\eta) + \int \frac{[(\bar{P}_0 + \eta_* \bar{P}_2) \times \bar{u}] \cdot (\bar{Q}_0 \times \bar{u})}{|\bar{Q}_0 \times \bar{u}|^2} d\eta \tag{4.26}$$

since the integrand in Eq. (4.26) is independent of ξ .

Finally, note that it is possible to write

$$\bar{q}_* = (\bar{p}_0 + \xi \bar{p}_1) + \eta_* (\bar{p}_2 + \xi \bar{p}_3) = \bar{q}_*^{\,N} + q_*^{\,T} \bar{u} \tag{4.27}$$

where $\bar{q}_*^{\,N}$ is normal to both $\bar{u}$ and $\bar{p}_2 + \xi \bar{p}_3$ or

$$\bar{q}_*^{\,N} \cdot \bar{u} = 0 \quad , \quad \bar{q}_*^{\,N} \cdot (\bar{p}_2 + \xi \bar{p}_3) = 0 \tag{4.28}$$

Combining Eqs. (4.17), (4.27), and (4.28) yields

$$[(\bar{p}_0 + \xi \bar{p}_1) + \eta_* (\bar{p}_2 + \xi \bar{p}_3) - q_*^{\,T} \bar{u}] \cdot \bar{u} = 0$$

$$[(\bar{p}_0 + \xi \bar{p}_1) + \eta_* (\bar{p}_2 + \xi \bar{p}_3) - q_*^{\,T} \bar{u}] \cdot (\bar{p}_2 + \xi \bar{p}_3) = 0 \tag{4.29}$$

By using Eq. (3.33) and noting that $\bar{u} \cdot \bar{u} = 1$, Eq. (4.29) may be rewritten as

$$a + \eta_* b - q_*^{\,T} = 0$$

$$\beta + \eta_* \gamma - q_*^{\,T} b = 0$$

that is, a system of two equations for the two unknowns η_* and $q_*^{\,T}$, which has the solution

$$\eta_* = - \frac{\beta - ab}{\gamma - b^2} = - \frac{d}{e}$$

$$q_*^{\,T} = \frac{a\gamma - b\beta}{\gamma - b^2} = \frac{a\gamma - b\beta}{e} \tag{4.30}$$

Finally, combining Eqs. (4.25) and (4.27) yields

$$I_{s1} = \int \frac{(\bar{q} \times \bar{q}_*^N)\cdot(\bar{q}\times\bar{u})}{|\bar{q}\times\bar{u}|^2} \frac{d\eta}{|\bar{q}|} + q_*^T \int \frac{1}{|\bar{q}|} d\eta + C_3(\eta)$$

$$= \int \frac{\bar{q}_*^N\cdot\bar{u}\,|\bar{q}|^2 - (\bar{q}_*^N\cdot\bar{q})(\bar{q}\cdot\bar{u})}{|\bar{q}\times\bar{u}|^2} \frac{d\eta}{|\bar{q}|} + q_*^T \int \frac{1}{|\bar{q}|} d\eta + C_3(\eta)$$

(4.31)

or

$$I_{s1} = -\bar{q}_*^N\cdot\bar{P}_0 \int \frac{\bar{q}\cdot\bar{u}}{|\bar{q}\times\bar{u}|^2}\frac{d\eta}{|\bar{q}|} + q_*^T \int \frac{1}{|\bar{q}|} d\eta + C_3(\eta) \quad (4.32)$$

since $\bar{q}_*^N\cdot\bar{u} = 0$ (Eq. 4.28) and

$$\bar{q}_*^N\cdot\bar{q} = \bar{q}_*^N\cdot[\bar{P}_0 + \xi\bar{P}_1 + \eta(\bar{P}_2 + \xi\bar{P}_3)] = \bar{q}_*^N\cdot\bar{P}_0 \quad (4.33)$$

For $\bar{q}_*^N$ is normal to $\bar{u}$ and $\bar{P}_2 + \xi\bar{P}_3$ (Eq. 4.28). Note that (see Eq. A.10)

$$\int \frac{1}{|\bar{q}|} d\eta = \int (\alpha + 2\beta\eta + \gamma)^{-\frac{1}{2}} d\eta$$

$$= \frac{1}{\sqrt{\gamma}} \ln(\sqrt{\gamma}\,|\bar{q}| + \gamma\eta + \beta) \quad (4.34)$$

with α, β and γ given by Eq. (3.33), while the first integral in Eq. (4.33) is proportional to I_D given by Eq. (3.28).

Finally combining Eqs. (4.18), (4.32), (4.34), one obtains

$$I_S(\xi,\eta) = \frac{B}{2\pi}\Big[(\eta-\eta_*)\ln(|\bar{q}| + \bar{q}\cdot\bar{u}) + \frac{q_*^T}{\sqrt{\gamma}}\ln(\sqrt{\gamma}\,|\bar{q}| + \gamma\eta + \beta)$$

$$- \bar{q}_*^N\cdot\bar{P}_0 \frac{1}{|f|} J\Big] \quad (4.35)$$

with J given by Eq. (3.39). The results obtained above may be rewritten in a more expressive form. Note that

$$\eta - \eta_* = \eta + \frac{d}{e} = \frac{e\eta + d}{e} = \frac{(\bar{c}_2 \times \bar{u} \cdot \bar{c}_2 \times \bar{u})\eta + (\bar{c}_0 \times \bar{u}) \cdot (\bar{c}_2 \times \bar{u})}{(\bar{c}_2 \times \bar{u}) \cdot (\bar{c}_2 \times \bar{u})}$$

$$= \frac{(\bar{q} \times \bar{u}) \cdot (\bar{c}_2 \times \bar{u})}{|\bar{c}_2 \times \bar{u}|^2} = \frac{(\bar{q} \times \bar{a}_1) \cdot (\bar{a}_2 \times \bar{a}_1)}{|\bar{a}_1 \times \bar{a}_2|^2} \tag{4.36}$$

$$q^{*T} = \frac{a\gamma - b\beta}{e} = \frac{(\bar{c}_0 \cdot \bar{u})(\bar{c}_2 \cdot \bar{c}_2) - (\bar{c}_2 \cdot \bar{u})(\bar{c}_0 \cdot \bar{c}_2)}{|\bar{c}_2 \times \bar{u}|^2}$$

$$= \frac{(\bar{c}_0 \times \bar{c}_2) \cdot (\bar{u} \times \bar{c}_2)}{|\bar{c}_2 \times \bar{u}|^2} = \frac{(\bar{q} \times \bar{a}_2) \cdot (\bar{a}_1 \times \bar{a}_1)\, |a_1|}{|\bar{a}_1 \times \bar{a}_2|} \tag{4.37}$$

and finally

$$\frac{\bar{q}_*^N \cdot \bar{P}_0}{|f|} = \frac{\bar{q}_*^N \cdot \bar{q}_*^N}{|f|} = \frac{|\bar{P}_0 \cdot \bar{n}|^2}{|\bar{P}_0 \cdot \bar{u} \times \bar{P}_2|}$$

$$= \frac{|\bar{P}_0 \cdot \bar{n}|^2}{|\bar{P}_0 \cdot \bar{n}|\,|\bar{u} \times \bar{P}_2|} = \frac{|\bar{P}_0 \cdot \bar{n}|}{|\bar{u} \times \bar{P}_2|} = \frac{|a_1|\,|\bar{q} \cdot \bar{a}_1 \times \bar{a}_2|}{|\bar{a}_1 \times \bar{a}_2|^2} \tag{4.38}$$

Combining Eqs. (4.35) through (4.38), adding for convenience the function of η, $\ln\sqrt{\bar{a}_1 \cdot \bar{a}_1}$, and noting that (Eq. 4.7)

$$B = \bar{u} \times \bar{P}_2 \cdot i_x \tag{4.39}$$

yields

$$I_s = \frac{B}{2\pi}\Big\{(\eta-\eta_*)\big[\ln(r+\bar{q}\cdot\bar{u}) + \ln\sqrt{\bar{a}_1\cdot\bar{a}_1}\,\big] +$$

$$+\frac{q_*^T}{\sqrt{\gamma}}\ln(\sqrt{\gamma}\;r+\bar{q}\cdot\bar{c}_2) - \frac{\bar{q}_*^N\cdot\bar{P}_0}{|f|}\,J\Big\}$$

$$= \frac{1}{2\pi}\frac{(\bar{a}_1\times\bar{a}_2\cdot\bar{i}_x)}{|\bar{a}_1\times\bar{a}_2|^2}\Big\{(\bar{q}\times\bar{a}_1)\cdot(\bar{a}_2\times\bar{a}_1)\frac{1}{\sqrt{\bar{a}_1\cdot\bar{a}_1}}\ln(\sqrt{\bar{a}_1\cdot\bar{a}_1}\sqrt{\bar{q}\cdot\bar{q}}+\bar{q}\cdot\bar{a}_1)$$

$$+(\bar{q}\times\bar{a}_2)\cdot(\bar{a}_1\times\bar{a}_2)\frac{1}{\sqrt{\bar{a}_2\cdot\bar{a}_2}}\ln(\sqrt{\bar{a}_2\cdot\bar{a}_2}\;\sqrt{\bar{q}\cdot\bar{q}}+\bar{q}\cdot\bar{a}_2)$$

$$-|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|\;J\Big\}$$

(4.40)

4.3 Quadrilateral Planar Element

In this section, it will be shown, by differentiation, that the results obtained above are valid for any quadrilateral planar element. For this element, the normal

$$\bar{n} = \frac{\bar{a}_1\times\bar{a}_2}{|\bar{a}_1\times\bar{a}_2|} \qquad (4.41)$$

is independent of ξ and η and Eq. (4.40) reduces to

$$I_s = \frac{1}{2\pi}\,\bar{n}\cdot\bar{\lambda}\Big\{-\bar{q}\times\bar{a}_1\cdot\bar{n}\,\frac{1}{|\bar{a}_1|}\,\ln\left(|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1\right)$$

$$+\,\bar{q}\times\bar{a}_2\cdot\bar{n}\,\frac{1}{|\bar{a}_2|}\,\ln\left(|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_2\right)$$

$$-\,|\bar{q}\cdot\bar{n}|\,J\Big\} \tag{4.42}$$

Note that $\dfrac{\partial\bar{a}_1}{\partial\xi} = 0$

$$\frac{\partial}{\partial\xi}\left(\bar{q}\times\bar{a}_1\cdot\bar{n}\right) = \bar{a}_1\times\bar{a}_1\cdot\bar{n} = 0 \tag{4.43}$$

and

$$\frac{\partial}{\partial\xi}\ln\left(|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1\right)$$
$$= \frac{1}{|\bar{a}_1||\bar{q}| + \bar{q}\cdot\bar{a}_1}\left(\frac{\bar{q}\cdot\bar{a}_1}{\sqrt{\bar{q}\cdot\bar{q}}}\,|\bar{a}_1| + \bar{a}_1\cdot\bar{a}_1\right] = \frac{\sqrt{\bar{a}_1\cdot\bar{a}_1}}{\sqrt{\bar{q}\cdot\bar{q}}} \tag{4.44}$$

Hence

$$\frac{\partial^2}{\partial\xi\,\partial\eta}\left[\bar{q}\times\bar{a}_1\cdot\bar{n}\,\frac{1}{|a_1|}\,\ln\left(|a_1||\bar{q}| + \bar{q}\cdot\bar{a}_1\right)\right]$$

$$= \frac{\partial}{\partial\eta}\left(\frac{\bar{q}\times\bar{a}_1\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}}\right) = \frac{\bar{a}_2\times\bar{a}_1\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} + \frac{\bar{q}\times\bar{P}_3\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}} - \bar{q}\times\bar{a}_1\cdot\bar{n}\,\frac{\bar{q}\cdot\bar{a}_2}{r^3} \tag{4.45}$$

Similarly

$$\frac{\partial^2}{\partial\xi\partial\eta}\left[\bar{q}\times\bar{a}_2\cdot\bar{n}\,\frac{1}{|\bar{a}_2|}\,\ln\left(|\bar{a}_2||\bar{q}|+\bar{q}\cdot\bar{a}_2\right)\right]$$

$$=\frac{\partial}{\partial\xi}\left(\frac{\bar{q}\times\bar{a}_2\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}}\right)=\frac{\bar{a}_1\times\bar{a}_2\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}}+\frac{\bar{q}\times\bar{P}_3\cdot\bar{n}}{\sqrt{\bar{q}\cdot\bar{q}}}-\bar{q}\times\bar{a}_2\cdot\bar{n}\,\frac{\bar{q}\cdot\bar{a}_1}{r^3}$$

(4.46)

Furthermore, noting that

$$\frac{\partial}{\partial\xi}(\bar{q}\cdot\bar{n})=\bar{a}_1\cdot\bar{n}=0$$

$$\frac{\partial}{\partial\eta}(\bar{q}\cdot\bar{n})=\bar{a}_2\cdot\bar{n}=0 \qquad (4.47)$$

and using Eq. (3.50) yields

$$\frac{\partial^2}{\partial\xi\partial\eta}\left(|\bar{q}\cdot\bar{n}|\,J\right)=\mp 2\pi\,\frac{\partial^2}{\partial\xi\partial\eta}\left[\pm(\bar{q}\cdot\bar{n})\,I_D\right]$$

$$=\bar{q}\cdot\bar{n}\;\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{r^3}$$

(4.48)

Finally, combining Eqs. (4.42), (4.45), (4.46) and (4.48) yields

$$\frac{2\pi}{\bar{n}\cdot\bar{\lambda}_x}\,\frac{\partial^2 I_s}{\partial\xi\partial\eta}=\frac{|\bar{a}_1\times\bar{a}_2|}{r}+(\bar{q}\cdot\bar{a}_1\times\bar{n})\,\frac{\bar{q}\cdot\bar{a}_2}{r^3}+$$

$$+\frac{|\bar{a}_1\times\bar{a}_2|}{r}-(\bar{q}\cdot\bar{a}_2\times\bar{n})\,\frac{\bar{q}\cdot\bar{a}_1}{r^3}-\frac{(\bar{q}\cdot\bar{n})\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{r^3}$$

$$= 2\,\frac{|\bar{a}_1 \times \bar{a}_2|}{r} + \frac{1}{r^3}\Big[\big(-\bar{q}\times\bar{n}\cdot\bar{a}_1\big)\big(\bar{q}\cdot\bar{a}_2\big) + \big(\bar{q}\times\bar{n}\cdot\bar{a}_2\big)\big(\bar{q}\cdot\bar{a}_1\big) - \big(\bar{q}\cdot\bar{n}\big)\big(\bar{q}\cdot\bar{a}_1\times\bar{a}_2\big)\Big]$$

$$= 2\,\frac{|\bar{a}_1 \times \bar{a}_2|}{r} + \frac{1}{r^3}\Big[-\big(\bar{q}\times\bar{n}\big)\times\bar{q}\cdot\big(\bar{a}_1\times\bar{a}_2\big) - \big(\bar{q}\cdot\bar{n}\big)\big(\bar{q}\cdot\bar{a}_1\times\bar{a}_2\big)\Big]$$

$$= 2\,\frac{|\bar{a}_1 \times \bar{a}_2|}{r} + \frac{1}{r^3}\Big[\cancel{\big(\bar{q}\cdot\bar{n}\big)\big(\bar{q}\cdot\bar{a}_1\times\bar{a}_2\big)} - \big(\bar{q}\cdot\bar{q}\big)\big(\bar{n}\cdot\bar{a}_1\times\bar{a}_2\big) - \cancel{\big(\bar{q}\cdot\bar{n}\big)\big(\bar{q}\cdot\bar{a}_1\times\bar{a}_2\big)}\Big]$$

$$= 2\,\frac{|\bar{a}_1 \times \bar{a}_2|}{r} - \frac{1}{r}\,|\bar{n}\cdot\bar{a}_1\times\bar{a}_2| = \frac{|\bar{a}_1 \times \bar{a}_2|}{r} \tag{4.49}$$

in agreement with Eq. (1.15).

4.4 Comments

It may be noted that triangular elements are the limiting case of quadrilateral planar elements. Hence, the above formulation is exact for triangular elements. On the other hand, if the derivatives of $\bar{n}$ with respect to ξ and η are negligible, then Eq. (4.42) can still be used. It may be noted that the error introduced by assuming $\frac{\partial \bar{n}}{\partial \xi} = \frac{\partial \bar{n}}{\partial \eta} = 0$ within the element is of the same order of magnitude as the one introduced by using constant-potential elements. Smaller elements are thus required where $\bar{n}$ is varying rapidly, that is at the leading edge and tip, where, incidentally φ is also varying rapidly.

SECTION 5

THE WAKE

5.1 Dynamics of the Wake

As mentioned in Section 1, the surface σ in Eq. 1.1, surrounds the body and the wake. The effect of the wake, disregarded in Section 1, yields an additional term in Eq. (1.4), given by[2,5]

$$I_w = \frac{1}{2\pi} \iint_{\sigma} \Delta\varphi \, \bar{n}_u \cdot \bar{\nabla} \frac{1}{r_h} \, d\sigma \tag{5.1}$$

with

$$\Delta\varphi = \varphi_u - \varphi_\ell \tag{5.2}$$

This represents a distribution of doublets with intensity $\Delta\varphi$. The geometry of the wake is not known. An iterative procedure can be used to solve the problem: consider the surface of the wake divided into small elements. Assume initially that the wake is composed of straight vortex lines (see next subsection); then find the values of $\varphi_\wedge$ and then evaluate the velocity at the corner of the elements. Find a new location for the corner of the element such that the elements approximate the stream-surface emanating from the trailing edge and repeat the procedure mentioned above. Needless to say, convergence of the iterative procedure should be verified, numerically, if not theoretically.

Finally, the values of $\Delta\varphi$ at the centroids of the elements can be obtained as follows. Consider the Bernoulli theorem for potential flow

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \int \frac{dp}{\rho} = \text{const} \tag{5.3}$$

Since no pressure difference can exist across the wake, then

$$\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2\right)_u - \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2\right)_\ell = 0 \tag{5.4}$$

or

$$\frac{\partial}{\partial t}(\phi_u - \phi_\ell) + \frac{1}{2}\left[(\bar{\nabla}\phi_u \cdot \bar{\nabla}\phi_u) - (\bar{\nabla}\phi_\ell \cdot \bar{\nabla}\phi_\ell)\right] = 0 \tag{5.5}$$

This can be rewritten as

$$\frac{\partial}{\partial t}(\phi_u - \phi_\ell) + \frac{1}{2}\left[\bar{\nabla}(\phi_u + \phi_\ell) \cdot \bar{\nabla}(\phi_u - \phi_\ell)\right] = 0 \tag{5.6}$$

or

$$\frac{D}{Dt}\Delta\Phi = 0 \tag{5.7}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{Q}_m \cdot \bar{\nabla} \tag{5.8}$$

is the total time derivative obtained by following a particle having the mean velocity, ,

$$\bar{Q}_m = \frac{\nabla \phi_u + \nabla \phi_\ell}{2} \tag{5.9}$$

For steady state flow, Eq. (5.7) reduces to

$$\bar{Q}_m \cdot \bar{\nabla}(\phi_u - \phi_\ell) = 0 \tag{5.10}$$

(in agreement with the result obtained by Mangler and Smith[7]) or

$$\Delta \Phi = \phi_u - \phi_\ell = \text{const} \tag{5.11}$$

along a streamline. Hence it is convenient to use elements with edges approximately coincident with stream lines. Then the value of $\Delta\Phi$ is the same along the strip obtained by continuing all the elements between two streamlines. This yields a simpler set of equations since only the value of $\Delta\Phi$ at the trailing edge (rather than the values of $\Delta\varphi$ at the centroid of each element) would be involved.

It may be noted that the above derivation is exact in the sense that no small perturbation hypothesis has been made. It may also be interesting to interpret these results in terms of velocity: the vortices are parallel to the streamlines, the total vorticity between two streamlines (equal to the difference in $\Delta\varphi$ between the streamlines) is constant, while the intensity of the vortices decreases if the vortex lines diverge. Note that the vorticity is given by

$$\gamma = \frac{d}{ds}\left(\phi_u - \phi_\ell\right) \tag{5.12}$$

where s is the arclength in the direction normal to the lines $\Delta\Phi$ = const.

It may be worth noting that (Ref. 1)

$$\iint_\sigma \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\sigma = \Omega \tag{5.13}$$

where Ω is the solid angle of the surface as seen from the control point. Hence, the important factor is the contour of the strip (rather than the shape of the strip) since the solid angle depends only upon the contour. Note that the doublet integral is exact for any hyperboloidal element.

Furthermore, it should be noted that after a few spanlengths, the wake-sheet rolls up into two vortices. Hence, after a few spanlengths, the strips can be replaced by two concentrated vortices.

Finally, it should be mentioned that, as shown by Mangler and Smith[7],"the vortex sheet shed from the trailing edge of a lifting wing with non-zero angle, in inviscid subsonic flow, leaves the trailing edge tangentially to the upper or lower surface, or exceptionally, in an intermediate direction. The exceptional intermediate direction is possible in three circumstances only: either there is no mean flow or no shed verticity at the point" or both. If the shed verticity is positive (negative) and the mean flow is outboard (inboard), the sheet is tangential to the upper surface, otherwise is tangential to the lower surface. Note that is has been implicitly assumed that the flow leaves the aircraft at a sharp trailing edge. Shedding of vortices from the body requires the use of viscous flow equations (Ref. 1) and is not considered here.

5.2 Simplified Treatment of the Wake

A simplified treatment of the wake (used in Refs. 1-5) consists of assuming that the wake is composed of straight vortex-lines emanating from the trailing edge and parallel to the x-axis (direction of the flow). For this case, the surface of the wake is divided into infinitely long elements, $\hat{\sigma}_{\ell}$, with two edges parallel to the x-axis. These elements are the continuation of the elements of the wing having an edge in contact with the trailing edge (Fig. 4).

Hence, by assuming that the value of $\Delta\varphi_{TE}$ can be approximated by the value at the centroid of the element $\hat{\sigma}_{\ell}$ the contribution I_w (see Eq. 5.1) is given by

$$\Sigma\, \omega_{ik}\, \varphi_k \tag{5.14}$$

with

$$\omega_{ik} = I_w = \pm \frac{1}{2\pi} \iint_{\hat{\sigma}_{\ell}} \bar{n}_u \cdot \bar{\nabla}\left(\frac{1}{r_h}\right) d\hat{\sigma}_{\ell} \tag{5.15}$$

for the elements with an edge in contact with the trailing edge, and

$$\omega_{ik} = 0 \tag{5.16}$$

for the others.

In order to evaluate the integral in Eq. (5.15), it is convenient to consider that the element $\hat{\sigma}_{\ell}$ is the limit of the parallelepipedal element obtained by truncating the element $\hat{\sigma}_{\ell}$ at the finite distance (Fig. 5). The limit is obtained by letting

$$\bar{P}_1 = \chi\, \bar{u} = \chi\, \bar{i} \tag{5.17}$$

go to infinity; note that $\bar{u} = \bar{i}$ since two edges are parallel to the x-axis. Note that (see Fig. 5)

$$\bar{P}_0 - \bar{P}_1 = \frac{\bar{P}_+ + \bar{P}_-}{2} = \bar{P}_m$$
$$\bar{P}_1 = \chi\, \bar{i} = \bar{Q}_1$$

$$\bar{P}_2 = \frac{\bar{P}_+ - \bar{P}_-}{2} = \bar{P}_d = \bar{a}_2$$

$$\bar{P}_3 = 0 \tag{5.19}$$

It is convenient to separate the contribution from the trailing edge ($\xi = -1$) and the edge that goes to infinity ($\xi = 1$):

$$I_W = \mp \frac{S}{2\pi} \Big[J_W(1,\eta) - J_W(-1,\eta) \Big]_{\eta=-1}^{\eta=1} \tag{5.20}$$

where (note that $\bar{q} = \bar{P}_m + (1+\xi)\chi + \eta \bar{P}_d$)

$$S = \mathrm{sign}(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) = \mathrm{sign}(\bar{P}_{md} \cdot \bar{\lambda} \times \bar{P}_d) \tag{5.21}$$

with

$$\bar{P}_{md} = \bar{P}_m + \eta \bar{P}_d \tag{5.22}$$

while (note that $\bar{q}(\xi=1) = \bar{P}_{md} + 2\chi\bar{\lambda}$)

$$J_W(1,\eta) = \lim_{\chi\to\infty} \tan^{-1}\left(\frac{-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{|\bar{q}|\,|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|} \right)_{\xi=1}$$

$$= \lim_{\chi\to\infty} \tan^{-1}\left(\frac{-(\bar{q}\times\bar{\lambda})\cdot(\bar{q}\times\bar{P}_d)}{|\bar{q}|\,(\bar{q}\cdot\bar{\lambda}\times\bar{P}_d)} \right)_{\xi=1}$$

$$= \lim_{\chi\to\infty} \tan^{-1} \frac{-\left[(\bar{P}_m+\eta\bar{P}_d)\times\bar{\lambda}\right]\cdot\left[(P_m+2\chi\bar{\lambda})\times\bar{P}_d\right]}{\left[(\bar{P}_m+\eta\bar{P}_d+2\chi\bar{\lambda})\cdot(\bar{P}_m+\eta\bar{P}_d+2\chi\bar{\lambda})\right]^{1/2}|\bar{P}_m\cdot\bar{\lambda}\times\bar{P}_d|}$$

$$= \tan^{-1} \frac{-(\bar{P}_{md}\times\bar{\lambda})\cdot(\bar{\lambda}\times\bar{P}_d)}{|\bar{P}_{md}\cdot\bar{\lambda}\times\bar{P}_d|}$$

$$= \tan^{-1} \frac{\bar{P}_{md}\cdot\bar{P}_m - (\bar{P}_{md}\cdot\bar{\lambda})(\bar{P}_d\cdot\bar{\lambda})}{|\bar{P}_{md}\cdot\bar{\lambda}\times\bar{P}_d|}$$

(5.23)

and similarly (note that $\bar{q}(\xi=-1) = \bar{P}_m + \eta\,\bar{P}_d = \bar{P}_{md}$)

$$J_w(-1,\eta) = \tan^{-1}\left[-\frac{(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{|\bar{q}|\;|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|}\right]_{\xi=-1}$$

$$= \tan^{-1}\left[-\frac{(\bar{q}\times\bar{i})\cdot(\bar{q}\times\bar{P}_d)}{|\bar{q}|\;|\bar{q}\cdot\bar{i}\times\bar{P}_d|}\right]_{\xi=-1}$$

$$= \tan^{-1}\left[\frac{-(\bar{P}_{md}\times\bar{i})\cdot(\bar{P}_{md}\times\bar{P}_d)}{|\bar{P}_{md}|\;|\bar{P}_{md}\cdot\bar{i}\times\bar{P}_d|}\right]_{\xi=-1}$$

$$= \tan^{-1}\left[\frac{-(\bar{P}_{md}\cdot\bar{P}_{md})(\bar{i}\cdot\bar{P}_d)-(\bar{P}_{md}\cdot\bar{P}_d)(\bar{P}_{md}\cdot\bar{i})}{|\bar{P}_{md}|\;|\bar{P}_{md}\cdot\bar{i}\times\bar{P}_d|}\right]_{\xi=-1} \tag{5.24}$$

SECTION 6

COMMENTS AND SUMMARY

6.1 Comments

As shown in the preceding sections, the use of hyperboloidal elements is quite cumbersome unless two edges of the element have two parallel edges. In this case, the element reduces to the trapezoidal planar element considered in Ref. 1 (in particular triangular).

Hence, it is convenient to divide the surface into triangular elements (which are a particular case of trapezoidal elements. For flat surfaces (the surface of a building for instance) quadrilateral elements are more convenient.

It should be noted that the doublet integral is equal to the solid angle (multiplied by $\frac{-1}{2\pi}$) of the surface σ_K as seen by the control point $\bar{p}^{(h)}$. Hence, the correct shape of the solid angle depends only upon the contour of the surface σ_K but not upon its actual shape. This implies also that it is important to use triangular elements, rather than tangent plane approximations to the hyperboloidal elements, which would yield discontinuities in the surface and hence, the total solid angle would be changed. A measure of the accuracy of the method in evaluating the coefficients C_{hk} (Ref. 1) is that the sum of the coefficients should be equal to -1 (solid angle multiplied by $\frac{-1}{2\pi}$). Note that if the two hyperboloidal elements in Fig. 3a are replaced by the two hyperboloidal elements (in the limit triangular elements) in

Fig. 3b, the centroids of the elements move very slightly. This is convenient for the evaluation of the pressure coefficient $C_{p_x} = -2\frac{\partial \varphi}{\partial x}$ by central difference (since centroids of the elements lie on the same wing section if the elements are bounded by wing sections). On the other hand, hyperboloidal elements can be used if the derivatives of $\bar{n}$ with respect to ξ and η are negligible (Section 4.4) in the source integral (the doublet integral is exact for any hyperboloidal element).

6.2 Summary

Assume that the surface has been divided into quadrilateral elements. The elements are described by the vectors $\bar{P}_c, \bar{P}_1, \bar{P}_2, \bar{P}_3$, from which one obtains

$$\bar{q} = (\bar{P}_c + \xi\bar{P}_1 + \eta\bar{P}_2 + \xi\eta\bar{P}_3)^{(K)} - P_c^{(i)}$$

$$\bar{a}_1 = (\bar{P}_1 + \eta\bar{P}_3)^{(K)} \quad , \qquad \bar{a}_2 = (\bar{P}_2 + \xi\bar{P}_3)^{(K)}$$

Then the approximate solution of Eq. (1.1) is obtained by solving the linear system of equations

$$[a_{iK}]\{\varphi_K\} = \{b_i\} \tag{6.1}$$

with

$$a_{iK} = \delta_{iK} - C_{iK} - w_{iK} \tag{6.2}$$

$$b_i = \Sigma\, b_{iK} \tag{6.3}$$

where w_{ik} represents the contribution of the wake discussed in Section 5, while C_{ik} and b_{ik} were derived in Sections 3 and 4.

For convenience, the results are summarized here. The coefficients C_{ik} and b_{ik} are given by

$$C_{ik} = I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \tag{6.4}$$

$$b_{ik} = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_D(-1,-1) \tag{6.5}$$

In Eq. (6.4), I_D is given by

$$I_D(\xi,\eta) = -\frac{1}{2\pi}\,\mathrm{sign}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)\,J \tag{6.6}$$

with

$$J = \tan^{-1}\frac{-(\bar{q}\times\bar{a}_1)\cdot(\bar{q}\times\bar{a}_2)}{|\bar{q}|\;|\bar{q}\cdot\bar{a}_1\times\bar{a}_2|} \tag{6.7}$$

On the other hand, in Eq. (6.5), I_S is given by

$$I_S(\xi,\eta) = -\bar{n}\cdot\bar{i}\;\hat{I}_S(\xi,\eta) \tag{6.8}$$

with

$$\begin{aligned}\hat{I}_S(\xi,\eta) = -\frac{1}{2\pi}\Big\{ &-(\bar{q}\times\bar{a}_1\cdot\bar{n})\frac{1}{\sqrt{\bar{a}_1\cdot\bar{a}_1}}\ln\left(\sqrt{(\bar{a}_1\cdot\bar{a}_1)(\bar{q}\cdot\bar{q})}+\bar{q}\cdot\bar{a}_1\right)\\ &+(\bar{q}\times\bar{a}_2)\cdot\bar{n}\;\frac{1}{\sqrt{\bar{a}_2\cdot\bar{a}_2}}\ln\left(\sqrt{(\bar{a}_2\cdot\bar{a}_2)(\bar{q}\cdot\bar{q})}+\bar{q}\cdot\bar{a}_2\right)\\ &-|\bar{q}\cdot\bar{n}|\;J\Big\}\end{aligned} \tag{6.9}$$

Equation (6.9) is exact for quadrilateral planar elements and may be used for hyperboloidal elements if the derivatives $\frac{\partial\bar{n}}{\partial\xi}$, $\frac{\partial\bar{n}}{\partial\eta}$ are negligible : this is possible if small elements are used where the surface curvature is high (leading edge and tip).

Finally, the coefficients w_{ik} in Eq. (6.2) are given by

$$W_{ik} = 0 \tag{6.10}$$

for the elements without any edge on the trailing edge. For the elements in contact with the trailing edge, assuming that the wake is composed for straight vortex lines, the coefficients w_{ik} are given by

$$W_{ik} = \mp \frac{1}{2} \operatorname{sign}(\bar{P}_m \cdot \bar{i} \times \bar{P}_d)\left(J_w(1,1) - J_w(1,-1) - J_w(-1,1) + J_w(-1,-1)\right) \tag{6.11}$$

with

$$J_w(1,\eta) = \tan^{-1} \frac{-(\bar{P}_{md} \times \bar{i}) \cdot (\bar{i} \times \bar{P}_d)}{|\bar{P}_{md} \cdot \bar{i} \times \bar{P}_d|} \tag{6.12}$$

and

$$J_w(-1,\eta) = \tan^{-1} \frac{-(\bar{P}_{md} \times \bar{i}) \cdot (\bar{P}_{md} \times \bar{P}_d)}{|\bar{P}_{md}|\,|\bar{P}_{md} \cdot \bar{i} \times \bar{P}_d|} \tag{6.13}$$

where $\bar{i}$ is the unit vector in the flow direction,

$$\bar{P}_m = \frac{\bar{P}_+ + \bar{P}_-}{2} \tag{6.14}$$

$$\bar{P}_d = \frac{\bar{P}_+ - \bar{P}_-}{2} \tag{6.15}$$

$$\bar{P}_{md} = \bar{P}_m + \eta\,\bar{P}_d \tag{6.16}$$

where $\bar{p}_+$ and $\bar{p}_-$ are the location of the trailing-edge corner points $\bar{P}'_{TE}$, $\bar{P}''_{TE}$ of the elements with respect to the control point $\bar{p}^{(K)}$

$$\bar{P}_+ = \bar{P}'_{TE} - \bar{P}^{(K)} \qquad , \qquad \bar{P}_- = \bar{P}''_{TE} - \bar{P}^{(K)} \tag{6.17}$$

6.3 Exterior Neuman Problem

The results obtained above can be applied for the solution of the Laplace equation outside the surface σ , with prescribed normal derivative on σ and usual regularity conditions at infinity. In this case, Eq. (1.6) is replaced by

$$\varphi_h = \Sigma \left(\frac{\partial \varphi}{\partial n}\right)_K \iint_{\sigma_K} \frac{1}{r_h} \, d\sigma_K$$

$$+ \Sigma \, \varphi_K \iint_{\sigma_K} \bar{n} \cdot \bar{\nabla}\left(\frac{1}{r_h}\right) d\sigma_K \qquad h = 1, 2, \cdots N \tag{6.18}$$

and correspondingly, Eq. (6.1) is replaced by

$$[\, a_{iK} \,] \{ \varphi_K \} = [\, \hat{b}_{iK} \,] \left\{ \left(\frac{\partial \varphi}{\partial n}\right)_K \right\} \tag{6.19}$$

with a_{ik} given by Eq. (6.2) (with $w_{ik} = 0$ if there is no wake) and

$$\hat{b}_{iK} = \hat{I}_s(1,1) - \hat{I}_s(1,-1) - \hat{I}_s(-1,1) + \hat{I}_s(-1,-1) \tag{6.20}$$

with $\hat{I}_s(\xi, \eta)$ given by Eq. (6.9). Clearly, if the boundary condition is given by Eq. (1.3), Eq. (6.19) reduces to Eq. (6.1).

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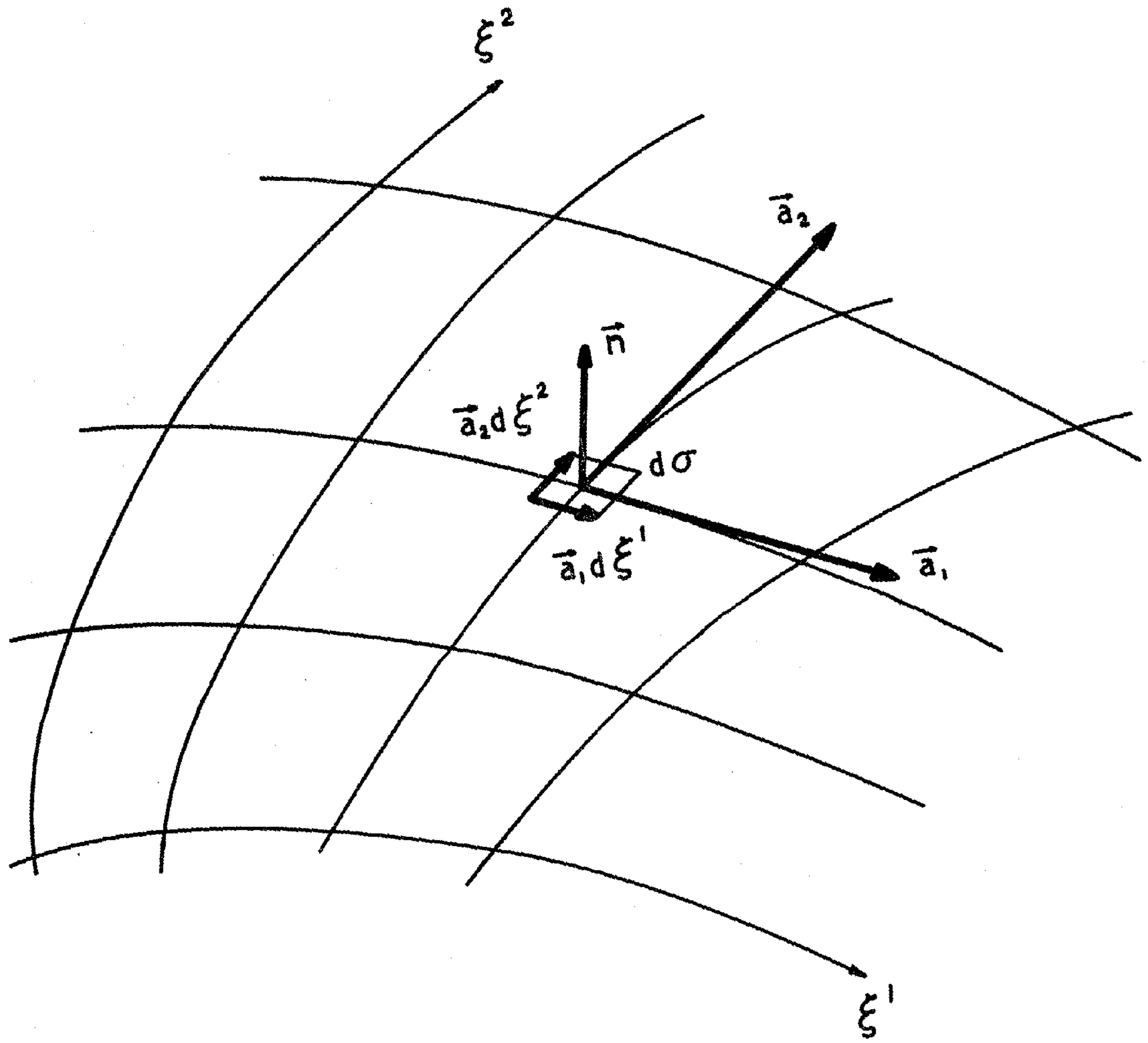


Fig. 1 Surface geometry

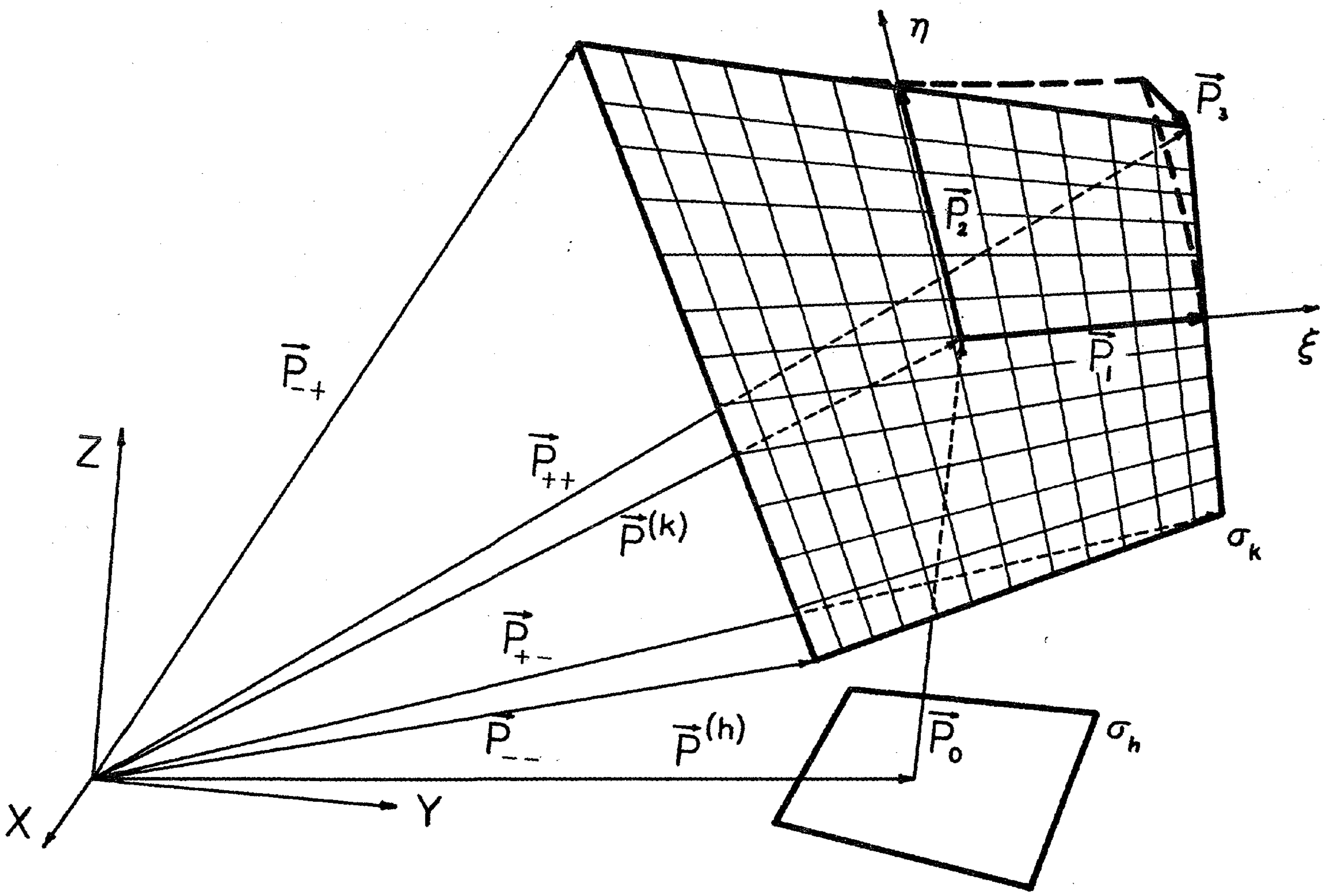
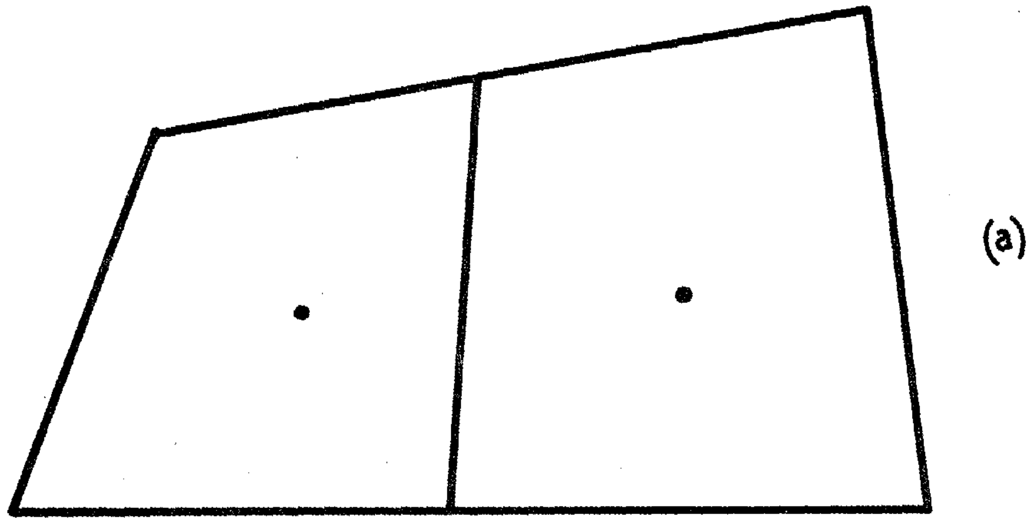


Fig. 2 Geometry of hyperboloidal element



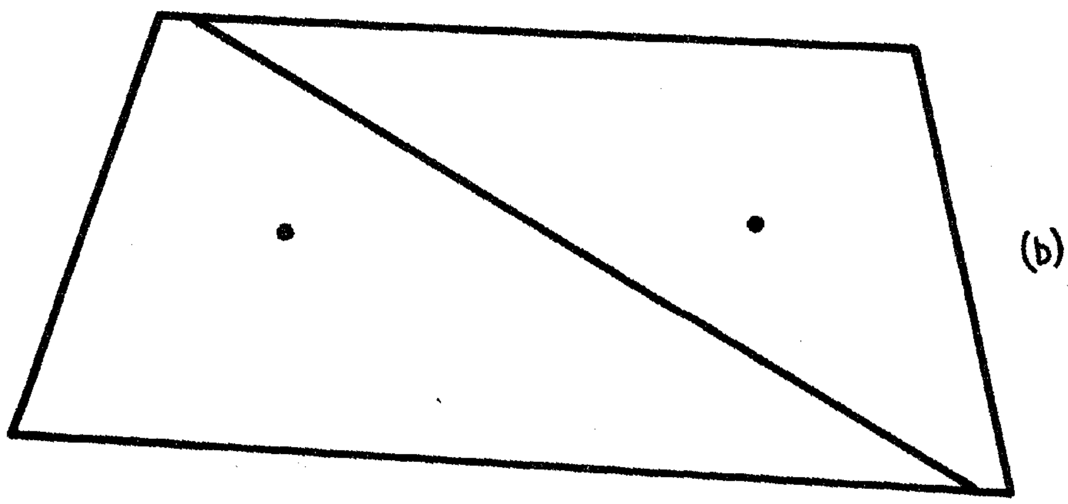


Fig. 3 Rectangular and triangular elements

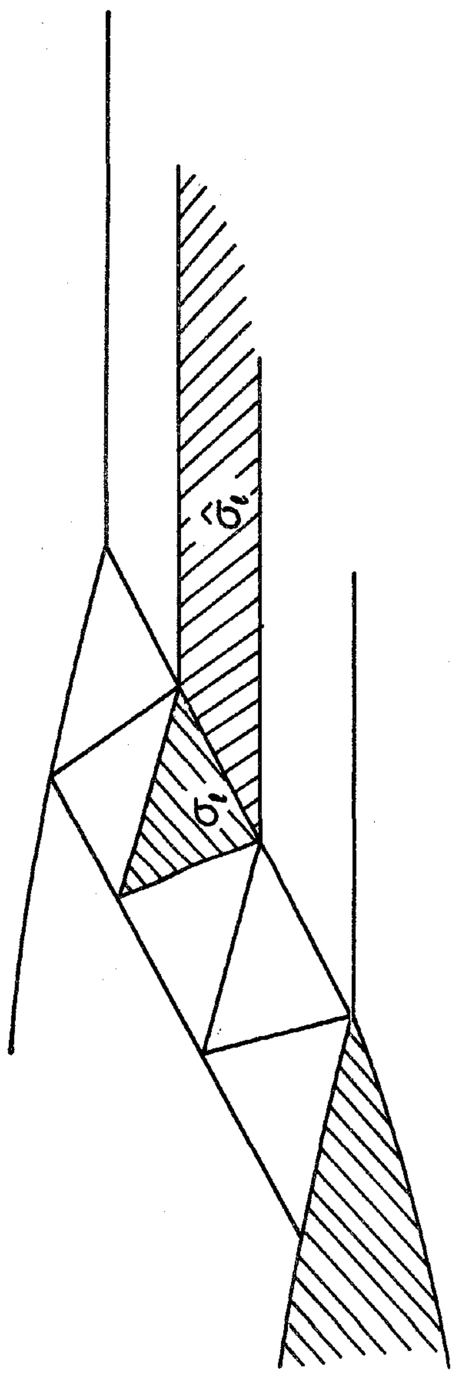


Fig. 4 Wake Geometry

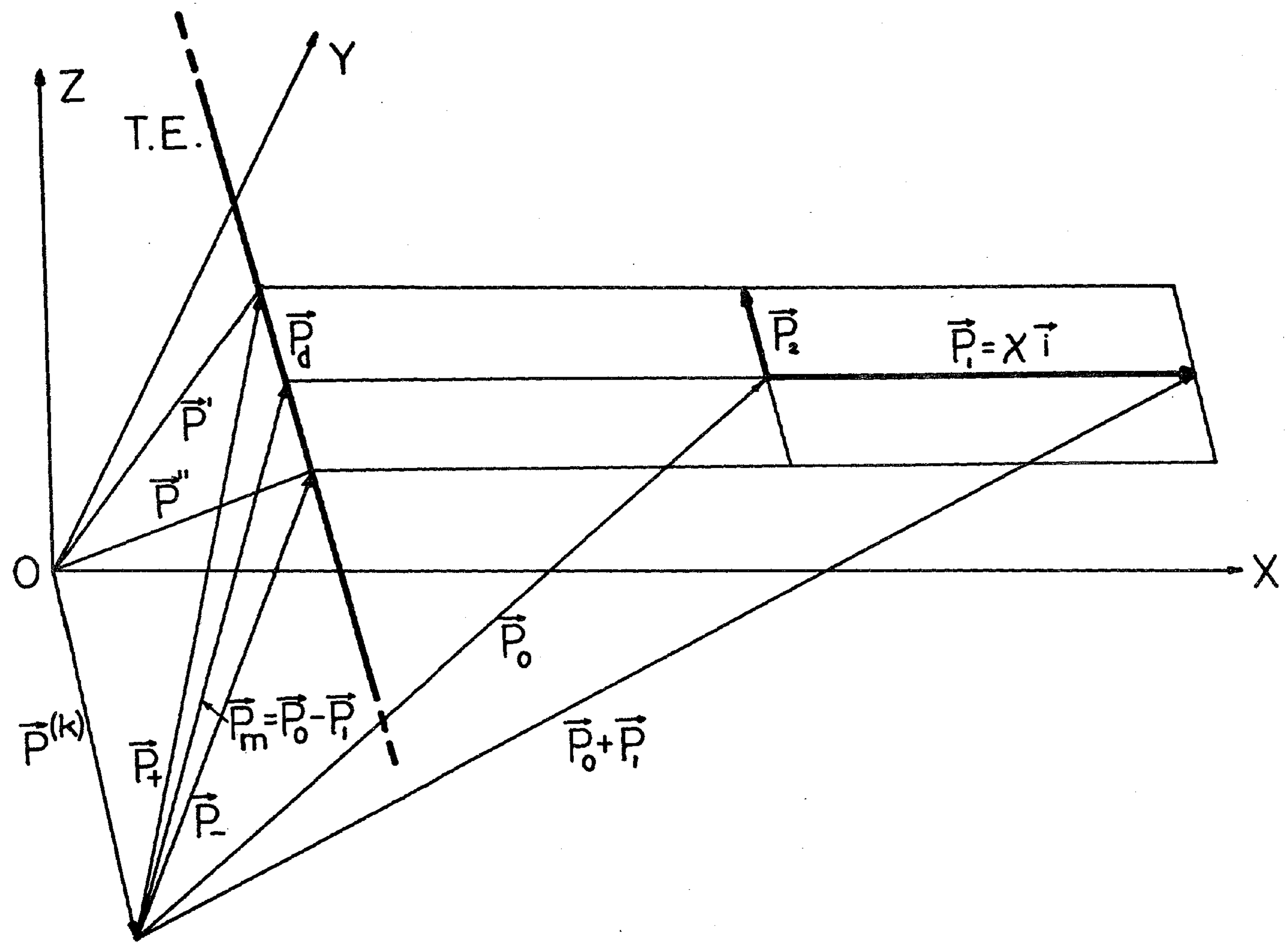


Fig. 5 Wake element

APPENDIX A

USEFUL INDEFINITE INTEGRALS

A.1 Introduction

For the sake of completeness, some indefinite integrals used in this report are verified in this Appendix.

A.2 Doublet Integral

For the evaluation of the doublet integral, I_D, the use of the following integral is required

$$
\begin{aligned}
J &= \int \frac{a+b\eta}{c+2d\eta+e\eta^2}\,\frac{1}{r}\,d\eta \\
&= \frac{1}{\sqrt{ec-d^2}}\tan^{-1}\frac{(ea-bd)\eta+(ad-bc)}{\sqrt{ec-d^2}}\,\frac{1}{r}
\end{aligned}
\tag{A.1}
$$

with

$$
\begin{aligned}
r &= \left[(a+b\eta)^2+(c+2d\eta+e\eta^2)\right]^{\frac{1}{2}} \\
&= \left[(a^2+c)+2(ab+d)\eta+(b^2+e)\eta^2\right]^{\frac{1}{2}}
\end{aligned}
\tag{A.2}
$$

Differentiating Eq. (A.1), one obtains

$$
\begin{aligned}
&\frac{1}{\sqrt{ec-d^2}}\,\frac{d}{d\eta}\tan^{-1}\left[\frac{(ae-bd)\eta+(ad-bc)}{\sqrt{ec-d^2}}\,\frac{1}{r}\right] \\
&= \frac{1}{\sqrt{ec-d^2}}\,\frac{1}{1+\left[\dfrac{(ae-bd)\eta+(ad-bc)}{\sqrt{ec-d^2}\,r}\right]^2}\,\frac{(ae-bd)r^2+\left[(ae-bd)\eta+(ad-bc)\right]\left[(b^2+e)\eta+(ab+d)\right]}{\sqrt{ec-d^2}\,r^3} \\
&= \frac{(ae-bd)\left[r^2-(b^2+e)\eta^2-(ab+d)\eta\right]-(ad-bc)\left[(b^2+e)\eta+(ab+d)\right]}{(ec-d^2)r^2+\left[(ac-bd)\eta+(ad-bc)\right]^2}\,\frac{1}{r} \\
&= \frac{a+b\eta}{c+2d\eta+e\eta^2}\,\frac{1}{r}
\end{aligned}
\tag{A.3}
$$

since
$$(ae-bd)\left[r^2-(b+e)\eta^2-(ab+d)\eta\right]-(ad-bc)\left[(b^2+e)\eta+(ab+d)\right]$$

$$=(ae-bd)\left[(ab+d)\eta+(a^2+e)\right]-(ad-bc)\left[(b^2+e)\eta+(ab+d)\right]$$

$$=\left[(ae-bd)(ab+d)-(ad-bc)(b^2+e)\right]\eta+\left[(ae-bd)(a^2+c)-(ad-bc)(ab+d)\right]$$

$$=\left[a^2be+ade-ab^2d-bd^2-a^2b^2d-ade+b^3c+bce\right]\eta$$
$$+\left[a^3e+ace-a^3bd-bcd-a^2bd-ad^2+ab^2c+bcd\right]$$

$$=(b\eta+a)(a^2e-d^2+b^2c+ce-2abd) \qquad \text{(A.4)}$$

and
$$(ce-d^2)r^2+\left[(ae-bd)\eta+(ad-bc)\right]^2$$

$$=\left[(ce-d^2)(b^2+e)+(ae-bd)^2\right]\eta^2+\left[(ce-d^2)(ab+d)+\right.$$
$$\left.(ae-bd)(ad-bc)\right]2\eta+\left[(ce-d^2)(a^2+c)+(ad-bc)^2\right]$$

$$=\left[b^2ce+ce^2-b^2d^2-d^2e+a^2e^2-2abde+b^2d^2\right]\eta^2$$
$$+\left[abce+cde-abd^2-d^3+a^2de-abce-abd^2+b^2cd\right]2\eta$$
$$+\left[a^2ce+c^2e-a^2d^2-cd^2+a^2d^2-2abcd+b^2c^2\right]$$

$$=(e\eta^2+2d\eta+c)(a^2e-d^2+b^2c+ce-2abd) \qquad \text{(A.5)}$$

A.3 Recurrent Formula

The recurrent formula
$$\int(\beta+\gamma\eta)^n\frac{1}{r}\,d\eta=(\beta+\gamma\eta)^{n-1}r-\frac{n-1}{n}(\alpha\gamma-\beta^2)\int(\beta+\gamma\eta)^{n-2}\frac{1}{r}\,d\eta \qquad \text{(A.6)}$$

with $$r = \sqrt{\alpha + 2\beta\eta + \gamma\eta^2} \tag{A.7}$$

is easily verified by differentiation

$$\frac{\partial}{\partial\eta}(\beta+\gamma\eta)^{n-1}\sqrt{\alpha+2\beta\eta+\gamma\eta^2}$$

$$= \gamma(n-1)(\beta+\gamma\eta)^{n-2}\sqrt{\alpha+2\beta\eta+\gamma\eta^2} + (\beta+\gamma\eta)^{n-1}\frac{\beta+\gamma\eta}{\sqrt{\alpha+2\beta\eta+\gamma\eta^2}}$$

$$= \frac{(\beta+\gamma\eta)^{n-2}}{\sqrt{\alpha+2\beta\eta+\gamma\eta^2}}\left[\gamma(n-1)(\alpha+2\beta\eta+\gamma\eta^2)+(\beta+\gamma\eta)^2\right]$$

$$= \frac{(\beta+\gamma\eta)^{n-2}}{\sqrt{\alpha+2\beta\eta+\gamma\eta^2}}\left[n(\beta+\gamma\eta)^2+(n-1)(\alpha\gamma-\beta^2)\right]$$

$$= n(\beta+\gamma\eta)^n\frac{1}{r} + (n-1)(\alpha\gamma-\beta^2)(\beta+\gamma\eta)^{n-2}\frac{1}{r} \tag{A.8}$$

In particular, for n = 1, one obtains

$$\int\frac{\beta+\gamma\eta}{\sqrt{\alpha+2\beta\eta+\gamma\eta^2}}\,d\eta = \sqrt{\alpha+2\beta\eta+\gamma\eta^2} \tag{A.9}$$

Note that for n = 0, Eq. (A.6) is not valid and is replaced by

$$\int\frac{1}{r}\,d\eta = \frac{1}{\sqrt{\gamma}}\ln(\sqrt{\gamma}\,r+\gamma\eta+\beta) \tag{A.10}$$

For $$\frac{1}{\sqrt{\gamma}}\frac{d}{d\eta}\ln(\sqrt{\gamma}\,r+\gamma\eta+\beta)$$

$$= \frac{1}{\sqrt{\gamma}}\frac{1}{\sqrt{\gamma}\,r+\gamma\eta+\beta}\left[\sqrt{\gamma}\,\frac{\gamma\eta+\beta}{r}+\gamma\right]$$

$$= \frac{1}{r} \tag{A.11}$$

APPENDIX B

A VECTOR CALCULUS FORMULA

A convenient formula used extensively in this report is

$$(\bar{A}\cdot\bar{B})(\bar{C}\cdot\bar{D}) - (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D}) = (\bar{A}\times\bar{D})\cdot(\bar{B}\times\bar{C}) \tag{B.1}$$

which is easily verified in the following:

$$\begin{aligned}
&(\bar{A}\cdot\bar{B})(\bar{C}\cdot\bar{D}) - (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D}) \\
&= (A_x B_x + A_y B_y + A_z B_z)(C_x D_x + C_y D_y + C_z D_z) \\
&\quad - (A_x C_x + A_y C_y + A_z C_z)(B_x D_x + B_y D_y + B_z D_z) \\
&= A_x \cancel{B_x C_x} D_x + A_y \cancel{B_y C_y} D_y + A_z \cancel{B_z C_z} D_z + A_x B_x C_y D_y \\
&\quad + A_x B_x C_z D_z + A_y B_y C_x D_x + A_z B_z C_x D_x + A_y B_y C_z D_z \\
&\quad + A_z B_z C_y D_y - \{ A_x \cancel{C_x B_x} D_x + A_y \cancel{C_y B_y} D_y + A_z \cancel{C_z B_z} D_z \\
&\quad + A_x C_x B_y D_y + A_x C_x B_z D_z + A_y C_y B_x D_x + A_z C_z B_x D_x \\
&\quad + A_y C_y B_z D_z + A_z C_z B_y D_y \}
\end{aligned} \tag{B.2}$$

while

$$\begin{aligned}
&(\bar{A}\times\bar{D})\cdot(\bar{B}\times\bar{C}) \\
&= \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_x & A_y & A_z \\ D_x & D_y & D_z \end{pmatrix} \cdot \begin{pmatrix} \bar{i} & \bar{j} & \bar{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix} =
\end{aligned}$$

$$
\begin{aligned}
= \; & (A_y D_z - A_z D_y)(B_y C_z - B_z C_y) \\
& + (A_z D_x - A_x D_z)(B_z C_x - B_x C_z) \\
& + (A_x D_y - A_y D_x)(B_x C_y - B_y C_x) \\
= \; & A_y B_y C_z D_z - A_y B_z C_y D_z - (A_z B_y C_z D_y - A_z B_z C_y D_y) \\
& + A_z B_z C_x D_x + A_x B_x C_z D_z - (A_z C_z B_x D_x + A_x C_x B_z D_z) \\
& + A_x B_x C_y D_y + A_y B_y C_x D_x - (A_y C_y B_x D_x + A_x C_x B_y D_y)
\end{aligned}
$$

APPENDIX C

SUBSONIC OSCILLATORY FLOW

C.1 Integral Equation

In this Appendix, it is shown how the results obtained in the main body of this report can be extended to subsonic oscillatory flow. Introducing the variables

$$X = \frac{x}{\beta \ell} \qquad Y = \frac{y}{\ell} \qquad Z = \frac{z}{\ell} \qquad T = \frac{\beta a_\infty t}{\ell} \qquad \Omega = \frac{\omega \ell}{\beta a_\infty} \tag{C.1}$$

and the complex potential $\hat{\phi}$ such that

$$\Phi(x, y, z) = U_\infty \ell \left[X + \hat{\phi}(X, Y, Z) e^{i\Omega(T + MX)} \right] \tag{C.2}$$

the integral equation for the subsonic oscillatory flow is given by

$$2\pi \hat{\phi} = -\oint\!\!\oint_\Sigma \left[\frac{\partial \hat{\phi}}{\partial N} \frac{e^{-i\Omega R}}{R} + \hat{\phi} \frac{\partial}{\partial N} \left(\frac{e^{-i\Omega R}}{R} \right) \right] d\Sigma \tag{C.3}$$

where Σ surrounds body and wake.

C.2 Boundary Condition

The boundary condition is given by

$$\nabla_{xyz} S \cdot \nabla_{xyz} \varphi = -\frac{\partial S}{\partial t} - U_\infty \frac{\partial S}{\partial x} \tag{C.4}$$

or

$$\nabla_{XYZ} S \cdot \nabla_{XYZ} \phi = \frac{\beta}{M} \frac{\partial S}{\partial T} + \frac{1}{\beta} \frac{\partial S}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial S}{\partial X} \frac{\partial \phi}{\partial X} = 0 \tag{C.5}$$

where φ and ϕ are such that

$$\Phi = U_\infty x + \varphi = U_\infty \ell (X + \phi) \tag{C.6}$$

Next, assume that the motion of the surface consists of small harmonic oscillations around a rest configuraiton, that is

$$S = S_0(X,Y,Z) + \tilde{S}(X,Y,Z)\, e^{i\Omega T} \tag{C.7}$$

Then, setting

$$\phi = \phi_0(X,Y,Z) + \tilde{\phi}(X,Y,Z)\, e^{i\Omega T} \tag{C.8}$$

one obtains

$$\begin{aligned}
&\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi_0 + \left(\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_0\right) e^{i\Omega T} + \\
&\left(\nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \tilde{\phi}\right) e^{i2\Omega T} + \frac{\beta}{M}\, i\Omega \tilde{S}\, e^{i\Omega T} + \\
&+ \frac{1}{\beta}\left(\frac{\partial S_0}{\partial X} + \frac{\partial \tilde{S}}{\partial X} e^{i\Omega T}\right) \\
&+ \frac{M^2}{\beta^2}\left[\frac{\partial S_0}{\partial X} \frac{\partial \phi_0}{\partial X} + \left(\frac{\partial S_0}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_0}{\partial X}\right) e^{i\Omega T} + \right. \\
&\left. + \frac{\partial \tilde{S}}{\partial T} \frac{\partial \tilde{\phi}}{\partial T}\, e^{i2\Omega T}\right] = 0
\end{aligned} \tag{C.9}$$

Assuming

$$S_0 = O(1) \tag{C.10}$$

$$\frac{\partial S_0}{\partial X} = O(\varepsilon) \tag{C.11}$$

with

$$\nabla_{XYZ} S_0 = O(1) \tag{C.12}$$

and

$$\tilde{S} = O(\varepsilon^2) \tag{C.13}$$

$$\Omega = O(1) \tag{C.14}$$

$$\frac{\partial \tilde{S}}{\partial X} = O(1) \tag{C.15}$$

it is easy to see that

$$\phi_0 = O(\varepsilon) \tag{C.16}$$

$$\tilde{\phi} = O(\varepsilon^2) \tag{C.17}$$

Neglecting the terms which contain $e^{i2\Omega T}$ (of order ε^4) and separating the steady from the oscillatory terms, one obtains

$$\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi_0 + \frac{1}{\beta}\frac{\partial S_0}{\partial X} + \frac{M^2}{\beta^2}\frac{\partial S_0}{\partial X}\frac{\partial \phi_0}{\partial X} = 0 \tag{C.18}$$

$$\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_0 + \frac{\beta}{M} i\Omega \tilde{S} + \frac{1}{\beta}\frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{\beta^2}\left(\frac{\partial S_0}{\partial X}\frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \phi_0}{\partial X}\right) = 0 \tag{C.19}$$

Introducing $\hat{\phi}$ such that

$$\tilde{\phi} = \hat{\phi}\, e^{i\Omega M X} \tag{C.20}$$

Equation (C.19) reduces to

$$\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \hat{\phi}\, e^{i\Omega M X} + i\Omega M \frac{\partial S_0}{\partial X}\hat{\phi}\, e^{i\Omega M X} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_0 + \frac{\beta}{M} i\Omega \tilde{S} + \frac{1}{\beta}\frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{\beta^2}\left[\frac{\partial S_0}{\partial X}\left(\frac{\partial \hat{\phi}}{\partial X} + i\Omega M \hat{\phi}\right) e^{i\Omega M X} + \frac{\partial \tilde{S}}{\partial X}\frac{\partial \phi_0}{\partial X}\right] = 0 \tag{C.21}$$

Finally, neglecting terms of order ε^2 in Eq. (C.18) and terms of order ε^3 in Eq. (C.19), one obtains

$$\nabla_{xyz} S_o \cdot \nabla_{xyz} \phi_o = -\frac{1}{\beta} \frac{\partial S_o}{\partial X} \tag{C.22}$$

$$\nabla_{xyz} S_o \cdot \nabla_{xyz} \hat{\phi} = -\left(i \frac{\beta \Omega}{M} \tilde{S} + \frac{1}{\beta} \frac{\partial \tilde{S}}{\partial X} \right) \tag{C.23}$$

In particular for

$$S = \pm \frac{1}{\ell} \left[z - z_o(x,y) - \tilde{z}(x,y)\, e^{i\omega t} \right]$$

(where the upper[lower] sign holds on the upper [lower] surface), one obtains

$$S_o = \pm \frac{1}{\ell} \left[z - z_o(x,y) \right] \tag{C.24}$$

$$\tilde{S} = \mp \frac{1}{\ell} \tilde{z}(x,y) \tag{C.25}$$

$$\frac{1}{|\nabla_{xyz} S_o|} = |N_z| = \pm N_z \tag{C.26}$$

and

$$\frac{\partial \hat{\phi}}{\partial N} \equiv \frac{\nabla_{xyz} S_o \cdot \nabla_{xyz} \hat{\phi}}{|\nabla_{xyz} S_o|} = -N_z \left(i k \tilde{z} + \frac{\partial \tilde{z}}{\partial x} \right) \tag{C.27}$$

where

$$k = \frac{\beta \Omega}{M} = \frac{\omega \ell}{U_\infty} \tag{C.28}$$

Equation (C.27) gives the value of $\partial\hat{\phi}/\partial N$ to be used in Eq. (C.3).

C.3 Pressure Coefficient

The pressure coefficient is given by the linearized Bernoulli theorem as

$$C_p = -\frac{2}{U_\infty^2}\left(\frac{\partial \varphi}{\partial t} + U_\infty \frac{\partial \varphi}{\partial x}\right)$$

$$= -2\left(\frac{\beta}{M}\frac{\partial \phi}{\partial T} + \frac{1}{\beta}\frac{\partial \phi}{\partial X}\right) \qquad \text{(C.29)}$$

For oscillatory flow, setting

$$\phi = \tilde{\phi}\, e^{i\Omega T} = \hat{\phi}\, e^{i\Omega(T + MX)} \qquad \text{(C.30)}$$

$$C_p = \tilde{C}_p\, e^{i\Omega T} \qquad \text{(C.31)}$$

one obtains

$$\tilde{C}_p = -2\left(\frac{\beta}{M}\, i\Omega\, \tilde{\phi} + \frac{1}{\beta}\frac{\partial \tilde{\phi}}{\partial X}\right)$$

$$= -2\left[i\Omega\left(\frac{\beta}{M} + \frac{M}{\beta}\right)\hat{\phi} + \frac{1}{\beta}\frac{\partial \hat{\phi}}{\partial X}\right] e^{i\Omega MX}$$

$$= -\frac{2}{\beta}\left[i\frac{\Omega}{M}\hat{\phi} + \frac{\partial \hat{\phi}}{\partial X}\right] e^{i\Omega MX}$$

$$= -\frac{2}{\beta}\left[e^{-i\Omega X/M}\frac{\partial}{\partial X}\left(\hat{\phi}\, e^{i\Omega X/M}\right)\right] e^{i\Omega MX}$$

$$= -\frac{2}{\beta}\, e^{-i\Omega \frac{\beta^2}{M} X}\frac{\partial}{\partial X}\left(\hat{\phi}\, e^{i\Omega X/M}\right)$$

$$= -\frac{2}{\beta}\, e^{-ik\beta X}\frac{\partial}{\partial X}\left(\hat{\phi} e^{ikX/\beta}\right) \qquad \text{(C.32)}$$