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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

DEPARTMENT OF ENGINEERING SCIENCE AND MECHANICS

DYNAMICS OF A GRAVITY-GRADIENT STABILIZED FLEXIBLE SPACECRAFT

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Long Flexible Appendages)
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Abstract

This investigation is concerned with the dynamics of a gravity-gradient stabilized flexible satellite in the neighborhood of a deformed equilibrium configuration. First the equilibrium configuration is determined by solving a set of nonlinear differential equations. Then stability of motion about the deformed equilibrium is tested by means of the Liapunov direct method and the natural frequencies of oscillation of the complete structure calculated. The analysis is applicable to the RAE/B satellite.

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1. Introduction

With the advent of large spacecraft, flexibility has become an increasingly important factor in the system attitude stability. Early designs of spacecraft were based on rigid body analysis, according to which rotational motion is stable if it takes place about the axis of maximum or minimum moment of inertia and unstable if the body rotates about the axis of intermediate moment of inertia (see, for example, Ref. 1, Sec. 6.7). The erratic behavior of the Explorer I, a satellite stabilized about the axis of minimum moment of inertia, prompted a re-examination of the rigid body assumption. Indeed, Thomson and Reiter[2] were able to attribute the behavior of the Explorer I satellite to energy dissipation resulting from the vibration of flexible antennas. This conclusion was corroborated by Meirovitch.[3] References 2 and 3 used the so-called "energy sink" approach. Their main conclusion was that a flexible satellite cannot be stabilized about the axis of minimum moment of inertia, leaving as stability criterion what has come to be known as the "greatest moment of inertia" requirement.

For a number of years, no significant additional work on the stability of flexible spacecraft was performed. Some work on cable-connected space stations cannot be really considered pertinent. Some investigation that can be regarded as being related to flexible spacecraft is that by Hooker and Margulies,[4] who model a satellite as "a set of n rigid bodies interconnected by dissipative elastic joints," and forming so-called "topological trees."

The first serious attempt to treat rigorously the flexibility effects on the attitude stability of flexible satellites can be attributed to Meirovitch and Nelson.[5] Reference 5 investigated a satellite with elastic appendages by means of an infinitesimal analysis. It appears that Ref. 5 uses modal analysis for the first time in conjunction with the stability of flexible spacecraft. At the same time, Nelson and Meirovitch[6] used the Liapunov direct method to investigate the stability of a rigid satellite with elastically connected moving parts. The motion of a satellite consisting of two rigid bodies connected by an elastic structure was investigated by Robe and Kane[7] by means of an infinitesimal analysis. Simulating a spacecraft by a set of rigid masses interconnected by massless elastic members, Likins[8] derived the corresponding equations of motion, and indicated that a solution can be obtained by modal analysis. Reference 8, however, does not produce an algorithm for the solution of the equations. Thermal effects and solar radiation pressure were found by Etkin and Hughes[9] to cause the anomalous behavior of spinning satellites with long flexible antennas. The flexibility effects on the attitude motion of spacecraft were also investigated by Modi and Berenton[10] but the validity of their analysis is in doubt, as they restrict the satellite vibration to planar.

An interesting paper by Newton and Farrell[11] presents a method for evaluating the natural frequencies of a flexible gravity-gradient stabilized satellite. In the process, Reference 11 linearizes the equations of motion about the deformed equilibrium. As generalized coordinates, the investigators consider complete deformation patterns of the satellite. This procedure is not only unorthodox but also tends to limit the number of degrees of freedom of the simulation, not to mention the fact that one must guess in advance configuration patterns. Moreover, there is some question as to the evaluation of the equilibrium configuration. Nevertheless, the paper contains some interesting ideas. A paper by Likins and Wirsching[12] proposes to introduce the concept of "synthetic modes" in conjunction with a "hybrid" coordinate system, where the latter is defined as a set of coordinates consisting of rotational coordinates of the spacecraft as a whole and modal coordinates for the flexible appendages. This idea, however, was introduced earlier in Reference 5.

All preceding investigations have one thing in common, namely, they are all discretization schemes. Some use lumping of the distributed mass of the elastic members, a procedure referred to sometimes as spatial discretization, and others use series truncation in conjunction with modal analysis. In a first attempt to apply Liapunov's direct method to hybrid systems from the area of satellite dynamics, i.e., without using any discretization scheme, Meirovitch[13,14,15] studied the stability of spinning rigid bodies with elastic appendages. It should be pointed out that the term "hybrid" refers here to a system defined by sets of both ordinary and partial differential equations, a concept different from that used by Likins and Wirsching.[12] Several new ideas were introduced in Ref. 13, such as the use of the bounding properties of Rayleigh's quotient to eliminate spatial derivatives from the problem formulation and the use of testing density functions.

The ideas of Refs. 13-15 have been pursued by Meirovitch and Calico[16,17] for the case in which testing density functions cannot be defined readily. References 16 and 17 develop the so-called "method of integral coordinates," whereby certain integrals are identified and defined as generalized coordinates. Then, using the bounding properties of Rayleigh's quotient as well as certain Schwarz's inequalities for functions, a function κ bounding the Hamiltonian H from below is obtained, $\kappa \leq H$, so that κ can be used as a testing function in conjunction with Liapunov's direct method. The method of integral coordinates is basically a discretization scheme.

One problem that has received little attention in the technical literature is that of deformed equilibrium, which can be referred to mathematically as "nontrivial equilibrium." Such problems arise in the case of gravity-gradient or spin-stabilized satellites with very flexible appendages that are not aligned with the satellite's principal axes. Finding the equilibrium configuration can be quite a problem in itself, especially if the governing equations are nonlinear. Addressing himself to this problem, Flatley[18] obtained the nonlinear equilibrium configuration of the Radio Astronomy Explorer (RAE) satellite by means of an analogue computer. Deformed equilibrium has also been considered in Ref. 11, but the details are not clear and no plot of the deformed equilibrium is shown. In seeking stability statements for the RAE/B satellite, Meirovitch[19] obtained as a by-product the nonlinear deformed equilibrium, thus confirming the results of Ref. 18.

The present study is concerned with the stability of a hybrid dynamical system about nontrivial equilibrium. It contains many of the formulations and results of Ref. 19. Qualitative stability statements are obtained for the RAE/B satellite by both the Liapunov direct method and by an infinitesimal analysis. In connection with the infinitesimal analysis, the natural frequencies of oscillation about the nonlinear nontrivial equilibrium were obtained by a method developed by the first author of this report.[20] The method of Ref. 20 considers a state vector consisting of generalized coordinates and velocities, where the coordinates include both rotations and elastic deformations, and develops an eigenvalue problem in terms of real quantities alone. The stability statements of Ref. 19 and corresponding statements obtained from the solution of the eigenvalue problem agree completely.

2. Problem Formulation

We shall be concerned with the motion of a body consisting of n + 1 parts, of which one part is rigid and n parts are elastic. The domain of extension of the rigid part is denoted by D_0 and those of the elastic parts when in undeformed state by D_i $(i = 1,2,\ldots,n)$ (see Fig. 1). Correspondingly, the masses associated with the domains D_i are denoted by m_i $(i = 0,1,\ldots,n)$, so that the total mass is $m = \sum_{i=0}^{n} m_i$. The elastic domains are rigidly attached to D_0 and have common boundaries only with D_0.

The body m is assumed to move in a central-force gravitational field, with its mass center describing a given orbit about the center of force C.F., where the latter is assumed to be fixed in an inertial space.

In describing the motion of m it will prove convenient to identify a system of axes xyz (see Fig. 1) with the undeformed state. The origin c of xyz is taken to coincide with the mass center of m in the undeformed state and axes xyz themselves coincide with the principal axes of m in the same state. Note that the system xyz is embedded in the rigid part D_0 but is not necessarily a set of principal axes for that part. We shall assume here that the nature of the elastic motion is such that the mass center of the entire system remains at the origin of xyz. In measuring elastic deformations, we consider reference frames $x_i y_i z_i$ fixed relative to the elastic domains D_i $(i = 1,2,\ldots,n)$, where the direction of these axes is chosen parallel to that of the elastic deformations. The origin of axes $x_i y_i z_i$ is denoted by 0_i and in general it need not coincide with c.

Next let us denote the radius vector from the mass center c to a point in the domain D_i $(i = 0,1,\ldots,n)$ by $\underset{\sim}{h}_i + \underset{\sim}{r}_i$, where the point coincides with the position of an element of mass dm_i when the body is in undeformed state. The constant-magnitude vector $\underset{\sim}{h}_i$ denotes the radius vector from c to 0_i; clearly $\underset{\sim}{h}_0 = \underset{\sim}{0}$. On the other hand, $\underset{\sim}{r}_i$ is the radius vector from 0_i to the point in question, and its components represent the independent spatial variables associated with a point in the domain D_i. Denoting by $\underset{\sim}{i}_i$, $\underset{\sim}{j}_i$, and $\underset{\sim}{k}_i$ the unit vectors along axes x_i, y_i and z_i, respectively, we can write $\underset{\sim}{h}_i + \underset{\sim}{r}_i = (h_{xi} + x_i)\underset{\sim}{i}_i + (h_{yi} + y_i)\underset{\sim}{j}_i + (h_{zi} + z_i)\underset{\sim}{k}_i$ $(i = 0,1,\ldots,n)$.

In describing the elastic deformations, we can use the Lagrangian or the Eulerian approach. According to the Lagrangian approach the independent variables are those of the body in undeformed state, whereas in the Eulerian description of motion the independent variables are those of the body in deformed shape. For infinitesimally small deformations the two points of view coalesce, but for large deformations they do not. When it is necessary to calculate the stresses in a body undergoing large deformations, the Eulerian approach is more convenient. Although we shall be concerned with relatively large deformations, we have no interest in the internal stress distribution, and because of kinematical considerations we shall find it more convenient to use the Lagrangian approach. Hence, denoting by $\underset{\sim}{u}_i$ the elastic displacement vector of dm_i, and recognizing that the vector depends both on spatial position and time, we can write it in the form $\underset{\sim}{u}_i = u_i(x_i,y_i,z_i,t)\underset{\sim}{i}_i + v_i(x_i,y_i,z_i,t)\underset{\sim}{j}_i + w_i(x_i,y_i,z_i,t)\underset{\sim}{k}_i$, where u_i, v_i and w_i are displacement components measured along x_i, y_i and z_i, respectively. If $\underset{\sim}{R}_c$ is the radius vector from the center of force C.F. to the mass center c, then the position relative to the inertial space of a mass element dm_i in deformed state is given by $\underset{\sim}{R}_{di} = \underset{\sim}{R}_c + \underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i$. It should be noted that, by the definition of the mass center, $\sum_{i=0}^{n} \int_{m_i} (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i)dm_i = \underset{\sim}{0}$.

In view of the above discussion, the kinetic energy can be written as

$$T = \frac{1}{2}\int_{m_i} \dot{\underset{\sim}{R}}_{di} \cdot \dot{\underset{\sim}{R}}_{di}\, dm_i = \frac{1}{2} m\, \dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c + \frac{1}{2}\sum_{i=0}^{n}\int_{m_i} (\dot{\underset{\sim}{h}}_i + \dot{\underset{\sim}{r}}_i + \dot{\underset{\sim}{u}}_i) \cdot (\dot{\underset{\sim}{h}}_i + \dot{\underset{\sim}{r}}_i + \dot{\underset{\sim}{u}}_i)dm_i \tag{1}$$

where the first term on the right side of Eq. (1) is recognized as the kinetic energy of translation of the mass center c and the second one as the kinetic energy due to motion relative to c. Dots denote derivatives with respect to time. Denoting by $\underset{\sim}{\omega}$ the angular velocity of the set of axes xyz, hence also of the sets $x_iy_iz_i$ $(i = 1,2,\ldots,n)$, and recalling the expression for the time derivative of a vector expressed in terms of rotating coordinates, we obtain

$$\dot{\underset{\sim}{h}}_i + \dot{\underset{\sim}{r}}_i + \dot{\underset{\sim}{u}}_i = \dot{\underset{\sim}{u}}_i' + \underset{\sim}{\omega} \times (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i) \tag{2}$$

in which $\dot{\underset{\sim}{u}}_i' = \dot{u}_i\underset{\sim}{i}_i + \dot{v}_i\underset{\sim}{j}_i + \dot{w}\underset{\sim}{k}_i$ is the velocity of dm_i relative to c due to elastic effects alone. Introducing Eq. (2) into (1), we arrive at

$$T = \frac{1}{2} m \dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c + \frac{1}{2} \underset{\sim}{\omega} \cdot \underset{\sim}{J}_d \cdot \underset{\sim}{\omega} + \underset{\sim}{\omega} \cdot \sum_{i=1}^{n} \int_{m_i} (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i) \times \dot{\underset{\sim}{u}}_i' dm_i$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \int_{m_i} \dot{\underset{\sim}{u}}_i' \cdot \dot{\underset{\sim}{u}}_i' \, dm_i \tag{3}$$

where $\underset{\sim}{J}_d$ is the inertia dyadic of the body in deformed state about axes xyz.

Equation (3) is most conveniently expressed in matrix form. The matrix forms of the vectors $\dot{\underset{\sim}{R}}_c$ and $\underset{\sim}{\omega}$ are simply $\{\dot{R}_c\}$ and $\{\omega\}$, respectively. The inertia dyadic $\underset{\sim}{J}_d$ and the term on the right side of Eq. (3) require further attention. The inertia dyadic can be written as $\underset{\sim}{J}_d = \sum_{i=0}^{n} \underset{\sim}{J}_{di}^{(i)}$, where $\underset{\sim}{J}_{di}^{(i)}$ $(i = 0,1,\ldots,n)$ is the inertia dyadic associated with domain D_i when the corresponding mass is in deformed shape. The superscript i indicates that the dyadic is expressed in terms of the base $x_iy_iz_i$. This would require that we express $\underset{\sim}{\omega}$ in the same base. It is simpler, however, to express every $\underset{\sim}{J}_{di}$ in the base xyz instead. Denoting the vector $\underset{\sim}{r}_i$ by $\underset{\sim}{r}_i^{(0)}$ and $\underset{\sim}{r}_i^{(i)}$ when expressed in the base xyz and $x_iy_iz_i$, respectively, and by $\{r_i^{(0)}\}$ and $\{r_i^{(i)}\}$ the associated column matrices, the relation between the two can be written as $\{r_i^{(0)}\} = [\ell_i]^T\{r_i^{(i)}\}$, where $[\ell_i]$ is the matrix of direction cosines between axes $x_iy_iz_i$ and xyz. In a similar fashion, if we denote by $\underset{\sim}{J}_{di}^{(0)}$ and $\underset{\sim}{J}_{di}^{(i)}$ the inertia dyadics when expressed in the base xyz and $x_iy_iz_i$, respectively, and by $[J_{di}^{(0)}]$ and $[J_{di}^{(i)}]$ the associated inertia matrices, then the relation between the two can be shown to have the form $[J_{di}^{(0)}] = [\ell_i]^T\,[J_{di}^{(i)}][\ell_i]$. With the understanding that the inertia matrices imply the body in deformed shape, we can drop the subscript d. Moreover, we shall drop the superscript i when it agrees with the subscript. Hence, the inertia matrix for the entire body, expressed in the base xyz, takes the form $[J^{(0)}] = \sum_{i=0}^{n} [\ell_i]^T[J_i][\ell_i]$. We note that $[\ell_0] = [1]$, where $[1]$ is the unit matrix. A similar analysis can be performed with regard to the third term on the right side of Eq. (3). It follows that Eq. (3) can be written in the matrix form

$$T = \frac{1}{2} m \{\dot{R}_C\}^T\{\dot{R}_C\} + \frac{1}{2}\sum_{i=0}^{n} \{\omega\}^T [\ell_i]^T[J_i][\ell_i]\{\omega\} + \{\omega\}^T \sum_{i=1}^{n} \int_{m_i} [h_i^{(0)} + r_i^{(0)} + u_i^{(0)}][\ell_i]^T\{\dot{u}_i'\}dm_i + \frac{1}{2}\sum_{i=1}^{n}\int_{m_i} \{\dot{u}_i'\}^T\{\dot{u}_i'\}dm_i \tag{4}$$

where $[h_i^{(0)} + r_i^{(0)} + u_i^{(0)}]$ is a skew-symmetric matrix whose elements satisfy the relation $h_{imn}^{(0)} + r_{imn}^{(0)} + u_{imn}^{(0)} = \sum_{\ell=1}^{3} \epsilon_{nm\ell}\,(h_{i\ell}^{(0)} + r_{i\ell}^{(0)} + u_{i\ell}^{(0)})$, in which $\epsilon_{nm\ell}$ is the epsilon symbol (see Ref. 1, p. 109). Clearly, $\{\dot{u}_i'\}$ represents the matrix notation of $\dot{\underset{\sim}{u}}_i'$.

The potential energy results from two sources, namely, gravity and elastic deformations, denoted by V_G and V_E, respectively, so that $V = V_G + V_E$. From Ref. 15, we conclude that the gravitational potential energy can be written as

$$V_G = -\frac{Km}{R_C} - \frac{K}{2R_C^3}\sum_{i=0}^{n} tr\,([\ell_i]^T[J_i][\ell_i]) + \frac{3K}{2R_C^3}\sum_{i=0}^{n}\{\ell_a\}^T[\ell_i]^T[J_i][\ell_i]\{\ell_a\} \quad (5)$$

where tr denotes the trace of a matrix, and $\{\ell_a\}$ is the column matrix of direction cosines between the direction of the vector R_c and axes xyz.

The elastic potential energy, also known as strain energy, requires special attention, particularly in the case of large deformations. No general expression, such as for T and V_G, can be written for V_E. This is so because an explicit form requires the knowledge of the type of elastic members involved. For the moment, we shall be content to write

$$V_E = \sum_{i=1}^{n} V_{Ei} \qquad (6)$$

where V_{Ei} ($i = 1,2,\ldots,n$) is the elastic potential energy associated with the member occupying the domain D_i when the member is undeformed. We shall return to the elastic potential energy shortly.

At this point it appears desirable to determine the functional dependence of the kinetic and potential energy in order to derive general Lagrange's equations of motion. To this end, we must specify the nature of the elastic members. We shall be concerned with one-dimensional members capable of flexure in two orthogonal directions. Any axial displacements will be assumed to be a result of change of length caused by the transverse displacements and not because of axial flexibility. In essence, the members are cantilevered bars undergoing large transverse displacements (see Fig. 2). Although we shall use nonlinear theory for the elastic motion, this will be because geometric nonlinearities and not as a result of nonlinear stress-strain relations. The mass distribution is arbitrary, but some of the members carry tip masses.

Letting the radius vector $\underset{\sim}{r}_i$ be aligned with axis x_i when the bar is undeformed, we conclude from Fig. 2 that

$$\underset{\sim}{r}_i^{(i)} = \underset{\sim}{r}_i = x_i \underset{\sim}{i}_i \ , \ i = 1,2,\ldots,n \tag{7}$$

and

$$\underset{\sim}{u}_i^{(i)} = \underset{\sim}{u}_i(x_i,t) = v_i(x_i,t)\underset{\sim}{j}_i + w_i(x_i,t)\underset{\sim}{k}_i \ , \ i = 1,2,\ldots,n \tag{8}$$

In view of this, the elements of the inertia matrix for the rigid member can be written as

$$\begin{gathered} J_{011} = A_0 \ , \ J_{022} = B_0 \ , \ J_{033} = C_0 \\ J_{012} = J_{021} = J_{013} = J_{031} = J_{023} = J_{032} = 0 \end{gathered} \tag{9}$$

where A_0, B_0, C_0 are the principal moments of inertia of the rigid part, whereas these for member i are

$$J_{i11} = \int_0^{\ell_i} \rho_i[(h_{yi} + v_i)^2 + (h_{zi} + w_i)^2]dx_i + m_i[(h_{yi} + v_i)^2 + (h_{zi} + w_i)^2]\Big|_{x_i=\ell_i}$$

$$J_{i22} = \int_0^{\ell_i} \rho_i[(h_{xi} + x_i)^2 + (h_{zi} + w_i)^2]dx_i + m_i[(h_{xi} + x_i)^2 + (h_{zi} + w_i)^2]\Big|_{x_i=\ell_i}$$

$$J_{i33} = \int_0^{\ell_i} \rho_i[(h_{xi} + x_i)^2 + (h_{yi} + v_i)^2]dx_i + m_i[(h_{xi} + x_i)^2 + (h_{yi} + v_i)^2]\Big|_{x_i=\ell_i}$$

$$J_{i12} = J_{i21} = - \int_0^{\ell_i} \rho_i(h_{xi} + x_i)(h_{yi} + v_i)dx_i - m_i(h_{xi} + x_i)(h_{yi} + v_i)\Big|_{x_i=\ell_i}$$

$$J_{i13} = J_{i31} = - \int_0^{\ell_i} \rho_i(h_{xi} + x_i)(h_{zi} + w_i)dx_i - m_i(h_{xi} + x_i)(h_{zi} + w_i)\Big|_{x_i=\ell_i}$$

$$J_{i23} = J_{i32} = -\int_0^{\ell_i} \rho_i(h_{yi} + v_i)(h_{zi} + w_i)dx_i - m_i(h_{yi} + v_i)(h_{zi} + w_i)\Big|_{x_i=\ell_i}$$

$$i = 1,2,\ldots,n \qquad (10)$$

Note that ρ_i and m_i are mass densities and tip masses, respectively, and h_{xi}, h_{yi}, h_{zi} denote the coordinates of the points of attachment of the booms measured from the mass center along axes $x_i y_i z_i$ $(i = 1,2,\ldots,n)$. We shall not specify the mass densities at this point.

The desired equilibrium configuration is that of gravity-gradient stabilization. That implies that the mass center c moves in a circular orbit with the constant angular velocity $\underset{\sim}{\Omega}$ (see Fig. 3), and the set of axes xyz coincides with a set of orbital axes abc, where $\underline{a}$ coincides with the direction of the radius vector $\underset{\sim}{R}_c$, b is tangent to the orbit and in the direction of the motion, and c is normal to the motion. Note that the orbital axes abc rotate relative to an inertial space with angular velocity $\underset{\sim}{\Omega}$ about axis c. The orientation of axes xyz with respect to abc is given by three angles θ_j and $\{\omega\}$ depends on these angles and angular velocities $\dot{\theta}_j$ $(j = 1,2,3)$. Because the first term in the kinetic energy, Eq. (4), is constant for a circular orbit, it will be ignored in future discussions. Moreover, the last term depends on the elastic velocities, so that the functional dependence of T is

$$T = T(\theta_j, \dot{\theta}_j, v_i, \dot{v}_i, w_i, \dot{w}_i) \ , \ j = 1,2,3 \ ; \ i = 1,2,\ldots,n \qquad (11)$$

The gravitational potential energy V_G contains the matrix $\{\ell_a\}$, which is defined as the matrix of direction cosines between $\underset{\sim}{R}_c$ and xyz. Since xyz can be obtained from abc by means of the rotations θ_j $(j = 1,2,3)$, it follows that

$$V_G = V_G\ (\theta_j,\ v_i,\ w_i)\ ,\ j = 1,2,3\ ;\ i = 1,2,\ldots,n \tag{12}$$

It remains to establish the functional dependence of V_E. This requires some elaboration, particularly because of the geometric nonlinearities involved. First we wish to distinguish between the potential energy V_{EA} due to axial motion, and the potential energy V_{EB} due to flexure. Next let us consider Fig. 4 and denote by s_i the distance to any element of mass dm_i when measured along the deflected bar and by x_i when measured along the original direction of the undeflected bar. We shall assume that the <u>bar is inextensional</u>, so that these two distances remain the same, $s_i = x_i$. An element of length along the deflected bar can be obtained from

$$(ds_i)^2 = (dx_i + du_i)^2 + (dv_i)^2 + (dw_i)^2 \tag{13}$$

Assuming that du_i is one order of magnitude smaller than dv_i and dw_i, recalling that $ds_i = dx_i$, and rearranging Eq. (13), we arrive at

$$du_i = -\frac{1}{2}\left[\left(\frac{dv_i}{dx_i}\right)^2 + \left(\frac{dw_i}{dx_i}\right)^2\right]dx_i \quad ,\quad i = 1,2,\ldots,n \tag{14}$$

so that the axial displacement resulting from the transverse displacements is negative. Because for inextensional motion the axial force P_{xi} does not depend on the axial displacement, and, moreover, because a tensile force opposes the motion, we have

$$V_{EA} = -\sum_{i=1}^{n}\int_{\ell_i} P_{xi}\ du_i = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i} P_{xi}\ \left[\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial w_i}{\partial x_i}\right)^2\right]dx_i \tag{15}$$

where total derivatives have been replaced by partial derivatives in recognition of the fact that the displacements are functions not only of spatial position but also of time.

The flexure potential energy is due to the bending displacements v_i and w_i. We shall denote the bending moment associated with the displacement v_i by M_{zi} and the change in slope corresponding to an element of length in deformed state by $d\phi_{zi}$ because they both take place about the z_i-axis. Accordingly, the analogous quantities associated with flexure about y_i are denoted by M_{yi} and $d\phi_{yi}$, respectively. It follows that the flexure potential energy can be written as

$$V_{EB} = \frac{1}{2} \sum_{i=1}^{n} \int_{\ell_i} (M_{yi}\, d\phi_{yi} + M_{zi}\, d\phi_{zi}) \tag{16}$$

But the bending moments M_{yi} and M_{zi} can be written in terms of the associated flexural stiffness and radii of curvature, as follows

$$M_{yi} = \frac{EI_{yi}}{R_{yi}} \quad , \quad M_{zi} = \frac{EI_{zi}}{R_{zi}} \tag{17}$$

where, from Fig. 5, the radii of curvature have the form

$$R_{yi} = \frac{ds_{yi}}{d\phi_{yi}} \quad , \quad R_{zi} = \frac{ds_{zi}}{d\phi_{zi}} \tag{18}$$

in which

$$ds_{zi} = \left[1 + \left(\frac{dv_i}{dx_i}\right)^2\right]^{1/2} dx_i \quad , \quad ds_{yi} = \left[1 + \left(\frac{dw_i}{dx_i}\right)^2\right]^{1/2} dx_i \tag{19}$$

Moreover

$$\frac{dv_i}{dx_i} \cong \tan \phi_{zi} \quad , \quad \frac{dw_i}{dx_i} \cong \tan \phi_{yi} \tag{20}$$

From Eqs. (20), it follows that

$$d\phi_{zi} \cong d\left(\tan^{-1}\frac{dv_i}{dx_i}\right) = \frac{d\left(\dfrac{dv_i}{dx_i}\right)}{1+\left(\dfrac{dv_i}{dx_i}\right)^2} = \frac{\dfrac{d^2v_i}{dx_i{}^2}}{1+\left(\dfrac{dv_i}{dx_i}\right)^2}\,dx_i$$

$$d\phi_{yi} \cong d\left(\tan^{-1}\frac{dw_i}{dx_i}\right) \quad \frac{d\left(\dfrac{dw_i}{dx_i}\right)}{1+\left(\dfrac{dw_i}{dx_i}\right)^2} = \frac{\dfrac{d^2w_i}{dx_i{}^2}}{1+\left(\dfrac{dw_i}{dx_i}\right)^2}\,dx_i \tag{21}$$

Finally, introducing Eqs. (17) through (21) into Eq. (16), we obtain

$$V_{EB} = \frac{1}{2}\sum_{i=1}^{n}\int_{\ell_i}(M_{zi}\,d\phi_{zi} + M_{yi}\,d\phi_{yi}) = \frac{1}{2}\sum_{i=1}^{n}\int_{\ell_i}\left[EI_{zi}\frac{(d\phi_{zi})^2}{ds_{zi}}\right.$$

$$\left. + EI_{yi}\frac{(d\phi_{yi})^2}{ds_{yi}}\right] = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}\left\{EI_{zi}\frac{\left(\dfrac{d^2v_i}{dx_i{}^2}\right)^2}{\left[1+\left(\dfrac{dv_i}{dx_i}\right)^2\right]^{5/2}} + EI_{yi}\frac{\left(\dfrac{d^2w_i}{dx_i{}^2}\right)^2}{\left[1+\left(\dfrac{dw_i}{dx_i}\right)^2\right]^{5/2}}\right\}dx_i \tag{22}$$

Recalling that the flexural displacements depend also on time, and writing binomial expansions for the denominators in Eq. (22), we arrive at

$$V_{EB} \cong \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}\left\{EI_{zi}\left(\frac{\partial^2 v_i}{\partial x_i{}^2}\right)^2\left[1 - \frac{5}{2}\left(\frac{\partial v_i}{\partial x_i}\right)^2\right] + EI_{yi}\left(\frac{\partial^2 w_i}{\partial x_i{}^2}\right)^2\left[1\right.\right.$$

$$\left.\left. - \frac{5}{2}\left(\frac{\partial w_i}{\partial x_i}\right)^2\right]\right\}dx_i \tag{23}$$

where the terms involving $\partial v_i/\partial x_i$ and $\partial w_i/\partial x_i$ are recognized as the corrections due to the geometric nonlinear effect.

In view of the above, the potential energy has the general functional form

$$\begin{aligned} V_E &= V_{EA} + V_{EB} \\ &= V_E\,(v_i', v_i'', w_i', w_i'') \quad , \quad i = 1,2,\ldots,n \end{aligned} \tag{24}$$

where primes indicate differentiations with respect to x_i.

From Fig. 4, we conclude that we must still account for the distributed forces p_{yi} and p_{zi}. Regarding these forces as nonconservative, and assuming that they do not depend on the elastic deformations, we can account for their effect in the form of the nonconservative work

$$W_{nc} = \int_0^{\ell_i} (p_{yi}\, v_i + p_{zi}\, w_i)dx_i \tag{25}$$

so that the total work can be written as

$$W = W_c + W_{nc} = -V + W_{nc} \tag{26}$$

where the conservative work has been recognized as being equal to the negative of the potential energy.

The system differential equations of motion, and the appropriate boundary conditions, can be obtained from the extended Hamilton's principle (see Ref. 1, Sec. 2.7)

$$\int_{t_1}^{t_2} (\delta T + \overline{\delta W})dt = 0 \tag{27}$$

where all the virtual displacements must be set equal to zero at $t = t_1, t_2$. Introducing the Lagrangian $L = T - V$, Eq. (27) can be written as

$$\int_{t_1}^{t_2} (\delta L + \overline{\delta W}_{nc})dt = 0 \tag{28}$$

where the Lagrangian has the functional form

$$L = L(\theta_j, \dot{\theta}_j, v_i, \dot{v}_i, v_i', v_i'', w_i, \dot{w}_i, w_i', w_i'') \ , \ j = 1,2,3 \ ; \ i = 1,2,\ldots,n \tag{29}$$

It will prove convenient to separate the Lagrangian into that associated with the rigid domain D_0 and those associated with the elastic domains D_i. Hence, let the Lagrangian have the general functional form (see Ref. 19)

$$L(t) = L_0(t) + \sum_{i=1}^{n} \left\{ \int_0^{\ell_i} \hat{L}_i(x_i,t)dx_i + L_i(\ell_i,t) \right\} \tag{30}$$

where

$$L_0(t) = L_0[\theta_j(t), \dot{\theta}_j(t)], \qquad j = 1,2,3 \tag{31}$$

$$\hat{L}_i(x_i,t) = \hat{L}_i[\theta_j(t),\dot{\theta}_j(t),v_i(x_i,t), \dot{v}_i(x_i,t),v_i'(x_i,t),v_i''(x_i,t), w_i(x_i,t),\ldots,w_i''(x_i,t)]$$
$$i = 1,2,\ldots,n \tag{32}$$

$$L_i(\ell_i,t) = L_i[\theta_j(t),\dot{\theta}_j(t),v_i(\ell_i,t),\dot{v}_i(\ell_i,t),w_i(\ell_i,t),\dot{w}_i(\ell_i,t)]$$

in which L_0 is the Lagrangian corresponding to the system in undeformed state, $\hat{L}_i$ the Lagrangian density associated with any point of the elastic member i, and L_i the Lagrangian corresponding to the tip mass. Moreover,

ℓ_i represents the length of member i. From Eqs. (30), (31), and (32), we conclude that

$$\delta L = \sum_{j=1}^{3} \left(\frac{\partial L}{\partial \theta_j} \delta\theta_j + \frac{\partial L}{\partial \dot{\theta}_j} \delta\dot{\theta}_j \right) + \sum_{i=1}^{n} \left\{ \int_0^{\ell_i} \left(\frac{\partial \hat{L}_i}{\partial v_i} \delta v_i + \frac{\partial \hat{L}_i}{\partial \dot{v}_i} \delta \dot{v}_i \quad \frac{\partial \hat{L}_i}{\partial v_i'} \delta v_i' \right.\right.$$

$$\left. + \frac{\partial \hat{L}_i}{\partial v_i''} \delta v_i'' + \frac{\partial \hat{L}_i}{\partial w_i} \delta w_i + \frac{\partial \hat{L}_i}{\partial \dot{w}_i} \delta \dot{w}_i + \frac{\partial \hat{L}_i}{\partial w_i'} \delta w_i' + \frac{\partial \hat{L}_i}{\partial w_i''} \delta w_i'' \right) dx_i$$

$$+ \frac{\partial L_i}{\partial v_i(\ell_i,t)} \delta v_i(\ell_i,t) + \frac{\partial L_i}{\partial \dot{v}_i(\ell_i,t)} \delta \dot{v}_i(\ell_i,t) + \frac{\partial L_i}{\partial w_i(\ell_i,t)} \delta w_i(\ell_i,t)$$

$$\left. + \frac{\partial L_i}{\partial \dot{w}_i(\ell_i,t)} \delta \dot{w}_i(\ell_i,t) \right\} \tag{33}$$

In addition,

$$\delta W_{nc} = \sum_{i=1}^{n} \int_0^{\ell_i} (p_{yi} \delta v_i + p_{zi} \delta w_i) dx_i \tag{34}$$

Inserting Eqs. (33) and (34) into (32), and integrating by parts with respect to t, we arrive at <u>Lagrange's equations for the rotational motion</u>

$$\frac{\partial L}{\partial \theta_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_j} \right) = 0 \quad , \; j = 1,2,3 \tag{35}$$

Moreover, integrating by parts with respect to t and x_i, we obtain <u>Lagrange's equations for the transverse displacements</u>, and the associated boundary conditions, in the form

$$\frac{\partial \hat{L}_i}{\partial v_i} - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}_i}{\partial \dot{v}_i} \right) - \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{L}_i}{\partial v_i'} \right) + \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial \hat{L}_i}{\partial v_i''} \right) + p_{yi} = 0, \; 0 < x_i < \ell_i,$$

$$i = 1,2,\ldots,n \tag{36a}$$

and

$$\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right) + \frac{\partial L_i}{\partial v_i} - \frac{\partial}{\partial t}\left(\frac{\partial L_i}{\partial \dot{v}_i}\right)\right] \delta v_i = 0, \quad \frac{\partial \hat{L}_i}{\partial v_i''} \delta v_i = 0 \text{ at } x_i = \ell_i$$

$$i = 1,2,\ldots,n \tag{36b}$$

$$-\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right)\right]\delta v_i = 0, \quad -\frac{\partial \hat{L}_i}{\partial v_i''} \delta v_i' = 0 \text{ at } x_i = 0$$

Equations similar in structure to Eqs. (36) can be written for w_i by simply replacing v_i by w_i.

3. Nontrivial Equilibrium

Let us consider the case in which $p_{yi} = p_{zi} = 0$ and define an equilibrium configuration as a set of dependent variables θ_j, v_i, w_i constant in time and satisfying Lagrange's equations. Because these variables do not depend on time, they must satisfy the equations

$$\frac{\partial L}{\partial \theta_j} = 0, \quad j = 1,2,3 \tag{37}$$

and

$$\frac{\partial \hat{L}_i}{\partial v_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i'}\right) + \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right) = 0, \quad 0 < x_i < \ell_i, \quad i = 1,2,\ldots,n \tag{38a}$$

$$\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right) + \frac{\partial L_i}{\partial v_i}\right] \delta v_i = 0, \quad \frac{\partial \hat{L}_i}{\partial v_i''} \delta v_i' = 0 \text{ at } x_i = \ell_i$$

$$i = 1,2,\ldots,n \tag{38b}$$

$$-\left[\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right)\right]\delta v_i = 0, \quad -\frac{\partial \hat{L}_i}{\partial v_i''} \delta v_i' = 0 \text{ at } x_i = 0$$

as well as a set of equations similar to (38) for w_i. We shall denote the solutions of Eqs. (37) and (38), together with the set of equations for w_i, by θ_{j0}, $v_{i0}(x_i)$, $w_{i0}(x_i)$, where the first are constant and the latter functions of the spatial variables x_i alone.

4. Perturbations About Equilibrium. The Variational Equations of Motion

The interest lies in the stability of the system in the neighborhood of the nontrivial solutions θ_{j0}, $v_{i0}(x_i)$, $w_{i0}(x_i)$. We shall seek stability criteria by means of Liapunov's direct method, and, to this end, we let the solutions of Eqs. (35) and (36) and the companion equations to (36) have the form

$$\begin{gathered}\theta_j(t) = \theta_{j0} + \theta_{j1}(t),\ j = 1,2,3 \\ v_i(x_i,t) = v_{i0}(x_i) + v_{i1}(x_i,t),\ w_i(x_i,t) = w_{i0}(x_i) + w_{i1}(x_i,t), \\ i = 1,2,\ldots,n\end{gathered} \tag{39}$$

where $\theta_{j1}(t)$, $v_{i1}(x_i,t)$, $w_{i1}(x_i,t)$ are small perturbations. Inserting Eqs. (39) into Eq. (30), and expanding a Taylor's series about the nontrivial equilibrium, we obtain

$$\begin{aligned}L = {} & L(\theta_{j0},v_{i0},v'_{i0},v''_{i0},w_{i0},w'_{i0},w''_{i0}) + \sum_{j=1}^{3}\left(\frac{\partial L}{\partial\theta_{j0}}\theta_{j1} + \frac{\partial L}{\partial\dot\theta_{j0}}\dot\theta_{j1}\right) \\ & + \sum_{i=1}^{n}\left[\int_0^{\ell_i}\left(\frac{\partial\hat L_i}{\partial v_{i0}}v_{i1} + \frac{\partial\hat L_i}{\partial\dot v_{i0}}\dot v_{i1} + \frac{\partial\hat L_i}{\partial v'_{i0}}v'_{i1} + \frac{\partial\hat L_i}{\partial v''_{i0}}v''_{i1} + \frac{\partial\hat L_i}{\partial w_{i0}}w_{i1} + \ldots\right.\right. \\ & \left.\left. + \frac{\partial\hat L_i}{\partial w''_{i0}}w''_{i1}\right)dx_i + \left(\frac{\partial L_i}{\partial v_{i0}}v_{i1} + \ldots + \frac{\partial L_i}{\partial\dot w_{i0}}\dot w_{i1}\right)\Big|_{x_i = \ell_i}\right] \\ & + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial^2 L}{\partial\theta_{j0}\partial\theta_{k0}}\theta_{j1}\theta_{k1} + 2\frac{\partial^2 L}{\partial\theta_{j0}\partial\dot\theta_{k0}}\theta_{j1}\dot\theta_{k1} + \frac{\partial^2 L}{\partial\dot\theta_{j0}\partial\dot\theta_{k0}}\dot\theta_{j1}\dot\theta_{k1}\right)\end{aligned}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \left\{ \int_0^{\ell_i} \left[\frac{\partial^2 \hat{L}_i}{\partial v_{i0}^2} v_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial \dot{v}_{i0}^2} \dot{v}_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}'^2} v_{i1}'^2 + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}''^2} v_{i1}''^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}^2} w_{i1}^2 \right.\right.$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial \dot{w}_{i0}^2} \dot{w}_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}'^2} w_{i1}'^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}''^2} w_{i1}''^2 + 2 \sum_{j=1}^{3} \left(\frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial v_{i0}} \theta_{j1} v_{i1} \right.$$

$$\left. + \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \theta_{j1} \dot{v}_{i1} \quad \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial \dot{v}_{i0}} \dot{\theta}_{j1} \dot{v}_{i1} + \cdots + \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial \dot{w}_{i0}} \dot{\theta}_{j1} \dot{w}_{i1} \right)$$

$$+ 2 \left(\frac{\partial^2 \hat{L}_i}{\partial v_{i0} \partial w_{i0}} v_{i1} w_{i1} + \frac{\partial^2 \hat{L}_i}{\partial v_{i0} \partial \dot{w}_{i0}} v_{i1} \dot{w}_{i1} + \frac{\partial^2 \hat{L}_i}{\partial w_{i0} \partial \dot{v}_{i0}} w_{i1} \dot{v}_{i1} + \cdots \right.$$

$$\left.\left. + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}' \partial v_{i0}''} v_{i1}' v_{i1}'' + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}' \partial w_{i0}''} w_{i1}' w_{i1}'' \right) \right] dx_i + \left[\frac{\partial^2 L_i}{\partial v_{i0}^2} v_{i1}^2 + \frac{\partial^2 L_i}{\partial \dot{v}_{i0}^2} \dot{v}_{i1}^2 \right.$$

$$+ \frac{\partial^2 L_i}{\partial w_{i0}^2} w_{i1}^2 + \frac{\partial^2 L_i}{\partial \dot{w}_{i0}^2} \dot{w}_{i1}^2 + 2 \sum_{j=1}^{3} \left(\frac{\partial^2 L_i}{\partial \theta_{j0} \partial v_{i0}} \theta_{j1} v_{i1} \quad \frac{\partial^2 L_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \theta_{j1} \dot{v}_{i1} + \cdots \right.$$

$$\left. + \frac{\partial^2 L_i}{\partial \dot{\theta}_{j0} \partial \dot{w}_{i0}} \dot{\theta}_{j1} \dot{w}_{i1} \right) + 2 \left(\frac{\partial^2 L_i}{\partial v_{i0} \partial w_{i0}} v_{i1} w_{i1} + \frac{\partial^2 L_i}{\partial v_{i0} \partial \dot{w}_{i0}} v_{i1} \dot{w}_{i1} \right.$$

$$\left.\left.\left. + \frac{\partial^2 L_i}{\partial w_{i0} \partial \dot{v}_{i0}} w_{i1} \dot{v}_{i1} + \frac{\partial^2 L_i}{\partial \dot{v}_{i0} \partial \dot{w}_{i0}} \dot{v}_{i1} \dot{w}_{i1} \right) \right]_{x_i = \ell_i} \right\} + \cdots \tag{40}$$

where $\partial L/\partial \theta_{j0} = \partial L/\partial \theta_j \Big|_{\theta_j = \theta_{j0}, \ldots}$, etc. But the term $L(\theta_{j0}, v_{i0}, \ldots, w_{i0}'')$ is constant. Moreover, by virtue of Eqs. (37) and (38) and the companion equations for w_i, all the linear terms in the perturbed variables in expansion (40) reduce to

$$\sum_{j=1}^{3} \frac{\partial L}{\partial \dot{\theta}_{j0}} \dot{\theta}_{j1} + \sum_{i=1}^{n} \left[\int_0^{\ell_i} \left(\frac{\partial \hat{L}_i}{\partial \dot{v}_{i0}} \dot{v}_{i1} \quad \frac{\partial \hat{L}_i}{\partial \dot{w}_{i0}} \dot{w}_{i1} \right) dx_i + \left(\frac{\partial L_i}{\partial \dot{v}_{i0}} \dot{v}_{i1} \right.\right.$$

$$+ \frac{\partial L_i}{\partial \dot{w}_{i0}} \dot{w}_{i1}\Bigg)\Bigg|_{x_i = \ell_i}\Bigg] \tag{41}$$

which are all linear in the generalized velocities $\dot{\theta}_{j1}, \dot{v}_{i1}, \dot{w}_{i1}$. In view of this, if we retain terms through second order only, the Lagrangian becomes

$$L = T_{21} + T_{11} + T_{01} - V_1 \tag{42}$$

where

$$T_{21} = \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 L}{\partial \dot{\theta}_{j0} \partial \dot{\theta}_{k0}} \dot{\theta}_{j1} \dot{\theta}_{k1} + \frac{1}{2} \sum_{i=1}^{n} \Bigg\{ \int_0^{\ell_i} \Bigg[\frac{\partial^2 \hat{L}_i}{\partial \dot{v}_{i0}^2} \dot{v}_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial \dot{w}_{i0}^2} \dot{w}_{i1}^2$$

$$+ 2 \frac{\partial^2 \hat{L}_i}{\partial \dot{v}_{i0} \partial \dot{w}_{i0}} \dot{v}_{i1} \dot{w}_{i1} + 2 \sum_{j=1}^{3} \Bigg(\frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial \dot{v}_{i0}} \dot{\theta}_{j1} \dot{v}_{i1} + \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial \dot{w}_{i0}} \dot{\theta}_{j1} \dot{w}_{i1} \Bigg) \Bigg] dx_i$$

$$+ \Bigg[\frac{\partial^2 L_i}{\partial \dot{v}_{i0}^2} \dot{v}_{i1}^2 + \frac{\partial^2 L_i}{\partial \dot{w}_{i0}^2} \dot{w}_{i1}^2 + 2 \frac{\partial^2 L_i}{\partial \dot{v}_{i0} \partial \dot{w}_{i0}} \dot{v}_{i1} \dot{w}_{i1} + 2 \sum_{j=1}^{3} \Bigg(\frac{\partial^2 L_i}{\partial \dot{\theta}_{j0} \partial \dot{v}_{i0}} \dot{\theta}_{j1} \dot{v}_{i1}$$

$$+ \frac{\partial^2 L_i}{\partial \dot{\theta}_{j0} \partial \dot{w}_{i0}} \dot{\theta}_{j1} \dot{w}_{i1} \Bigg) \Bigg|_{x_i = \ell_i} \Bigg] \tag{43}$$

is quadratic in the generalized velocities,

$$T_{11} = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 L}{\partial \theta_{j0} \partial \dot{\theta}_{k0}} \theta_{j1} \dot{\theta}_{k1} + \sum_{i=1}^{n} \Bigg\{ \int_0^{\ell_i} \Bigg[\sum_{j=1}^{3} \Bigg(\frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \theta_{j1} \dot{v}_{i1}$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial v_{i0}} \dot{\theta}_{j1} v_{i1} + \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial \dot{w}_{i0}} \theta_{j1} \dot{w}_{i1} + \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{j0} \partial w_{i0}} \dot{\theta}_{j1} w_{i1} \Bigg)$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial v_{i0} \partial \dot{w}_{i0}} v_{i1}\dot{w}_{i1} + \frac{\partial^2 \hat{L}_i}{\partial \dot{v}_{i0} \partial w_{i0}} \dot{v}_{i1} w_{i1} \Bigg] dx_i + \Bigg[\sum_{j=1}^{3} \Bigg(\frac{\partial^2 L_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \theta_{j1} \dot{v}_{i1}$$

$$+ \frac{\partial^2 L_i}{\partial \dot{\theta}_{j0} \partial v_{i0}} \dot{\theta}_{j1} v_{i1} + \frac{\partial^2 L_i}{\partial \theta_{j0} \partial \dot{w}_{i0}} \theta_{j1} \dot{w}_{i1} + \frac{\partial^2 L_i}{\partial \dot{\theta}_{j0} \partial w_{i0}} \dot{\theta}_{j1} w_{i1} \Bigg)$$

$$+ \frac{\partial^2 L_i}{\partial v_{i0} \partial \dot{w}_{i0}} v_{i1}\dot{w}_{i1} + \frac{\partial^2 L_i}{\partial \dot{v}_{i0} \partial w_{i0}} \dot{v}_{i1} w_{i1} \Bigg]_{x_1 = \ell_1} \Bigg\} \tag{44}$$

is linear in the generalized velocities, and

$$T_{01} - V_1 = \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial^2 L}{\partial \theta_{j0} \partial \theta_{k0}} \theta_{j1} \theta_{k1} + \sum_{i=1}^{n} \Bigg\{ \int_0^{\ell_i} \Bigg[\frac{\partial^2 \hat{L}_i}{\partial v_{i0}^2} v_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}'^2} v_{i1}'^2$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial v_{i0}''^2} v_{i1}''^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}^2} w_{i1}^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}'^2} w_{i1}'^2 + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}''^2} w_{i1}''^2 + 2 \sum_{j=1}^{3} \Bigg(\frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial v_{i0}} \theta_{j1} v_{i1}$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial w_{i0}} \theta_{j1} w_{i1} \Bigg) + 2 \Bigg(\frac{\partial^2 \hat{L}_i}{\partial v_{i0} \partial w_{i0}} v_{i1} w_{i1} + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}' \partial v_{i0}''} v_{i1}' v_{i1}''$$

$$+ \frac{\partial^2 \hat{L}_i}{\partial w_{i0}' \partial w_{i0}''} w_{i1}' w_{i1}'' \Bigg) \Bigg] dx_i + \Bigg[\frac{\partial^2 L_i}{\partial v_{i0}^2} v_{i1}^2 + \frac{\partial^2 L_i}{\partial w_{i0}^2} w_{i1}^2 + 2 \sum_{j=1}^{3} \Bigg(\frac{\partial^2 L_i}{\partial \theta_{j0} \partial v_{i0}} \theta_{j1} v_{i1}$$

$$+ \frac{\partial^2 L_i}{\partial \theta_{j0} \partial w_{i0}} \theta_{j1} w_{i1} \Bigg) \Bigg]_{x_i = \ell_i} \Bigg\} \tag{45}$$

is free of generalized velocities.

In view of the above, the perturbed Lagrangian can be written in the general functional term

$$L = L(\theta_{j1}, \dot{\theta}_{j1}, v_{i1}, \dot{v}_{i1}, v_{i1}', \ldots, w_{i1}', w_{i1}''),\ j = 1,2,3;\ i = 1,2,\ldots,n \tag{46}$$

Consequently, the variational equations can be written in the form of the Lagrange equations, Eqs. (35) and (36), but with the subscripts j and i replaced by j1 and i1, respectively. Unlike Eqs. (35) and (36), the variational equations possess trivial equilibrium.

5. Discretization by a Rayleigh-Ritz Approach

The variational equations discussed in the preceding section constitute a set of hybrid differential equations, in the sense that the equations for the rotational motion are ordinary differential equations and those for the elastic displacements are partial differential equations, where the latter are subject to given boundary conditions. It will prove convenient to transform the system into one consisting of ordinary differential equations alone. This can be done by using a discretization procedure based on the Rayleigh-Ritz approach. Indeed, let us introduce the notation

$$\theta_{j1}(t) = q_j(t), \; j = 1,2,3$$

$$\begin{aligned}
v_{11}(x_1,t) &= \sum_{j=4}^{p+3} \phi_j(x_1)q_j(t), & w_{11}(x_1,t) &= \sum_{j=p+4}^{2p+3} \psi_j(x_1)q_j(t) \\
v_{21}(x_2,t) &= \sum_{j=2p+4}^{3p+3} \phi_j(x_2)q_j(t), & w_{21}(x_2,t) &= \sum_{j=3p+4}^{4p+3} \psi_j(x_2)q_j(t) \\
&\text{-------------------} & &\text{-------------------} \\
v_{n1}(x_n,t) &= \sum_{j=2(n-1)p+4}^{(2n-1)p+3} \phi_j(x_n)q_j(t), & w_{n1}(x_n,t) &= \sum_{j=(2n-1)p+4}^{2np+3} \psi_j(x_n)q_j(t)
\end{aligned} \tag{47}$$

where $\phi_j(x_i)$ and $\psi_j(x_i)$ are admissible functions, taken as the eigenfunctions of the linearized system. With this notation, Eq. (43) can be written in the matrix form

$$T_{21} = \frac{1}{2}\{\dot{q}(t)\}^T[m]\{\dot{q}(t)\} \tag{48}$$

where [m] is a constant symmetric matrix having the elements

$$m_{jk} = \frac{\partial^2 L}{\partial\dot{\theta}_{j0}\partial\dot{\theta}_{k0}}, \; j,k = 1,2,3 \tag{49a}$$

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}} \phi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{v}_{i0}} \phi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 1,2,3;\; k = 2(i-1)p+4,\, 2(i-1)p+5,\ldots,(2i-1)p+3,$$

$$i = 1,2,\ldots,n \tag{49b}$$

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}} \psi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial\dot{\theta}_{j0}\partial\dot{w}_{i0}} \psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 1,2,3;\; k = (2i-1)p+4,\, (2i-1)p+5,\ldots,2ip+3,\; i = 1,2,\ldots,n \tag{49c}$$

$$m_{jk} = \int_0^{\ell i} \frac{\partial^2 \hat{L}_i}{\partial\dot{v}_{i0}^2} \phi_j(x_i)\phi_k(x_i)dx_i) + \left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}^2} \phi_j(x_i)\phi_k(x_i)\right]_{x_i = \ell_i}$$

$$j,k = 2(i-1)p+4,\, 2(i-1)p+5,\ldots,(2i-1)p+3,\; i = 1,2,\ldots,n \tag{49d}$$

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial\dot{v}_{i0}\partial\dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 2(i-1)p+4,\, 2(i-1)p+5,\ldots,(2i-1)p+3,$$

$$i = 1,2,\ldots,n \tag{49e}$$

$$k = (2i-1)p+4,\, (2i-1)p+5,\ldots,2ip+3,$$

$$m_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \dot{w}_{i0}^2} \psi_j(x_i)\psi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial \dot{w}_{i0}^2} \psi_j(x_i)\psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j,k = (2i-1)p+4,\ (2i-1)p+5, \ldots, 2ip+3,\ i = 1,2,\ldots,n \tag{49f}$$

On the other hand, Eq. (44) leads to the matrix form

$$T_{11} = \{q(t)\}^T[f]\{\dot{q}(t)\} \tag{50}$$

where [f] is a constant square matrix with the elements

$$f_{jk} = \frac{\partial^2 L}{\partial \theta_{j0} \partial \dot{\theta}_{k0}},\ j,k = 1,2,3 \tag{51a}$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \phi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial \theta_{j0} \partial \dot{v}_{i0}} \phi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 1,2,3;\ k = 2(i-1)p+4,\ 2(i-1)p+5, \ldots, (2i-1)p+3,\ i = 1,2,\ldots,n \tag{51b}$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial \dot{w}_{i0}} \psi_k(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial \theta_{j0} \partial \dot{w}_{i0}} \psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 1,2,3;\ k = (2i-1)p+4,\ (2i-1)p+5, \ldots, 2ip+3,\ i = 1,2,\ldots,n \tag{51c}$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{k0} \partial v_{i0}} \phi_j(x_i)dx_i + \left[\frac{\partial^2 L_i}{\partial \dot{\theta}_{k0} \partial v_{i0}} \phi_j(x_i)\right]_{x_i = \ell_i}$$

$$j = 2(i-1)p+4,\ 2(i-1)p+5, \ldots, (2i-1)p+3,\ i = 1,2,\ldots,n;\quad k = 1,2,3 \tag{51d}$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \dot{\theta}_{k0} \partial w_{i0}} \psi_j(x_i) dx_i + \left[\frac{\partial^2 L_i}{\partial \dot{\theta}_{k0} \partial w_{i0}} \psi_j(x_i)\right]_{x_i = \ell_i}$$

$$j = (2i-1)p+4,\ (2i-1)p+5, \ldots, 2ip+3,\ i = 1,2,\ldots,n;\ k = 1,2,3 \quad (51e)$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial v_{i0} \partial \dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i) dx_i + \left[\frac{\partial^2 L_i}{\partial v_{i0} \partial \dot{w}_{i0}} \phi_j(x_i)\psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 2(i-1)p+4,\ 2(i-1)p+5, \ldots, (2i-1)r+3,$$

$$i = 1,2,\ldots,n \quad (51f)$$

$$k = (2i-1)p+4,\ (2i-1)p+5, \ldots, 2ip+3,$$

$$f_{jk} = \int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \dot{v}_{i0} \partial w_{i0}} \phi_k(x_i)\psi_j(x_i) dx_i + \left[\frac{\partial^2 L_i}{\partial \dot{v}_{i0} \partial w_{i0}} \phi_k(x_i)\psi_j(x_i)\right]_{x_i = \ell_i}$$

$$j = (2i-1)p+4,\ (2i-1)p+5, \ldots, 2ip+3,$$

$$i = 1,2,\ldots,n \quad (51g)$$

$$k = 2(i-1)p+4,\ 2(i-1)p+5, \ldots, (2i-1)p+3,$$

Finally, from Eq. (45), we can write

$$T_{01} - V_1 = -\frac{1}{2}\{q(t)\}^T[k]\{q(t)\} \quad (52)$$

where [k] is a constant symmetric matrix with the elements

$$k_{jk} = -\frac{\partial^2 L}{\partial \theta_{j0} \partial \theta_{k0}}, \quad j,k = 1,2,3 \quad (53a)$$

$$k_{jk} = -\int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial \theta_{j0} \partial v_{i0}} \phi_k(x_i) dx_i - \left[\frac{\partial^2 L_i}{\partial \theta_{j0} \partial v_{i0}} \phi_k(x_i)\right]_{x_i = \ell_i}$$

$$j = 1,2,3;\ k = 2(i-1)p+4,\ 2(i-1)p+5, \ldots, (2i-1)p+3,$$

$$i = 1,2,\ldots,n \quad (53b)$$

$$k_{jk} = -\int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial\theta_{j0}\partial w_{i0}} \psi_k(x_i)dx_i - \left[\frac{\partial^2 L_i}{\partial\theta_{j0}\partial w_{i0}} \psi_k(x_i)\right]_{x_i = \ell_i}$$

$j = 1,2,3;\ k = (2i-1)p+4,\ (2i-1)p+5,\dots,2ip+3,\ i = 1,2,\dots,n$

(53c)

$$k_{jk} = -\int_0^{\ell_i} \left\{\frac{\partial^2 \hat{L}_i}{\partial v_{i0}^2} \phi_j(x_i)\phi_k(x_i) + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}'^2} \phi_j'(x_i)\phi_k'(x_i) + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}''^2} \phi_j''(x_i)\phi_k''(x_i)\right.$$

$$\left. + \frac{\partial^2 \hat{L}_i}{\partial v_{i0}'\partial v_{i0}''} \left(\phi_j'(x_i)\phi_k''(x_i) + \phi_j''(x_i)\phi_k'(x_i)\right)\right\}dx_i$$

$$-\left[\frac{\partial^2 L_i}{\partial v_{i0}^2} \phi_j(x_i)\phi_k(x_i)\right]_{x_i = \ell_i}$$

$j,k = 2(i-1)p+4,\ 2(i-1)p+5,\dots,(2i-1)p+3,\ i = 1,2,\dots,n$ (53d)

$$k_{jk} = -\int_0^{\ell_i} \frac{\partial^2 \hat{L}_i}{\partial v_{i0}\partial w_{i0}} \phi_j(x_i)\psi_k(x_i)dx_i - \left[\frac{\partial^2 L_i}{\partial v_{i0}\partial w_{i0}} \phi_j(x_i)\psi_k(x_i)\right]_{x_i = \ell_i}$$

$j = 2(i-1)p+4,\ 2(i-1)p+5,\dots,(2i-1)p+3,$

$i = 1,2,\dots,n$ (53e)

$k = (2i-1)p+4,\ (2i-1)p+5,\dots,2ip+3$

$$k_{jk} = -\int_0^{\ell_i} \left\{\frac{\partial^2 \hat{L}_i}{\partial w_{i0}^2} \psi_j(x_i)\psi_k(x_i) + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}'^2} \psi_j'(x_i)\psi_k'(x_i) + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}''^2} \psi_j''(x_i)\psi_k''(x_i)\right.$$

$$\left. + \frac{\partial^2 \hat{L}_i}{\partial w_{i0}'\partial w_{i0}''} \left[\psi_j'(x_i)\psi_k''(x_i) + \psi_j''(x_i)\psi_k'(x_i)\right]\right\}dx_i$$

$$-\left[\frac{\partial^2 L_i}{\partial w_{i0}^2}\psi_j(x_i)\psi_k(x_i)\right]_{x_i = \ell_i}$$

$$j,k = (2i-1)p+4,\ (2i-1)p+5,\ldots,2ip+3,\ i = 1,2,\ldots,n \tag{53f}$$

Introducing Eqs. (48), (50), and (52) into Eq. (42), we can write the Lagrangian in the matrix form

$$L = \frac{1}{2}\{\dot{q}\}^T[m]\{\dot{q}\} + \{q\}^T[f]\{\dot{q}\} - \frac{1}{2}\{q\}^T[k]\{q\} \tag{54}$$

Using the approach of Ref. 21 (see Sec. 3-4), we can write Lagrange's equations in the matrix form

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}}\right\} - \left\{\frac{\partial L}{\partial q}\right\} = \{0\} \tag{55}$$

Hence inserting Eq. (54) into (55), we obtain the equations of motion

$$[m]\{\ddot{q}\} + ([f]^T - [f])\{\dot{q}\} + [k]\{\dot{q}\} = \{0\} \tag{56}$$

so that, introducing the notation

$$[g] = [f]^T - [f] \tag{57}$$

where [g] is a skew-symmetric matrix, $[g]^T = -[g]$, we obtain

$$[m]\{\ddot{q}\} + [g]\{\dot{q}\} + [k]\{q\} = \{0\} \tag{58}$$

where [m] is identified as the inertia matrix, [g] is a "gyroscopic" matrix and [k] is a stiffness matrix which includes terms due to elastic, gravitational, and centrifugal effects.

6. Liapunov Stability Analysis

We shall seek criteria for the stability of motion in the neighborhood of the nontrivial equilibrium by means of the Liapunov direct method. This is equivalent to the problem of stability of the perturbed motion about the trivial solution. In terms of the discretized system, the perturbed motion is described by the vector $\{q(t)\}$, so that the interest lies in a stability analysis about the trivial equilibrium $\{q\} = \{0\}$.

It was shown in Ref. 15 that the Hamiltonian is a suitable Liapunov function for the type of problem at hand. Assuming that the system possesses a certain amount of internal damping, however small, the equilibrium is asymptotically stable if the Hamiltonian is positive definite. In terms of the perturbed variables, the Hamiltonian has the form

$$H = T_{21} - T_{01} + V_1 = \frac{1}{2}\{\dot{q}(t)\}^T[m]\{\dot{q}(t)\} + \frac{1}{2}\{q(t)\}^T[k]\{q(t)\} \tag{59}$$

But the function T_{21} is positive definite in the generalized velocities $\dot{q}_j(t)$ by definition. Hence, if the function

$$\kappa = \frac{1}{2}\{q(t)\}^T[k]\{q(t)\} \tag{60}$$

is positive definite in the generalized coordinates $q_j(t)$, then the Hamiltonian is a positive function in the generalized coordinates and velocities and the equilibrium is asymptotically stable. The function κ is positive definite if the matrix $[k]$ is positive definite. Whether $[k]$ is positive definite or not can be ascertained by means of Sylvester's criterion (Ref. 1, Sec. 6.7). The matrix $[k]$ will be referred to as a Hessian matrix.

7. Natural Frequencies of the Complete Structure

The Liapunov direct method provides qualitative information concerning the stability or lack of stability of an equilibrium configuration. Similar information can be extracted from the system of equations (58) via the eigenvalues. In addition, the eigenvalue problem yields results of a more quantitative nature in the form of the system natural frequencies and the normal modes for the complete structure, where the latter are defined later. It turns out that Eqs. (58) lead to an eigenvalue problem of a special nature. The nature of the eigenvalue problem can be conveniently discussed by converting the set of equations from second order to first order. Indeed, if the configuration vector $\{q(t)\}$ is of dimension N, then we can introduce the 2N-dimensional state vector $\{x(t)\}$ in the form

$$\{x(t)\} = \begin{Bmatrix} \{\dot{q}(t)\} \\ \{q(t)\} \end{Bmatrix} \tag{61}$$

No confusion should arise from denoting the state vector by $\{x(t)\}$, because the symbol x_i used to denote the position of a point in the elastic members represents a spatial coordinate independent of time and not a time-dependent generalized coordinate. Accordingly, if we introduce the $2N \times 2N$ matrices

$$[M] = \begin{bmatrix} [m] & [0] \\ [0] & [k] \end{bmatrix}, \quad [G] = \begin{bmatrix} [g] & [k] \\ -[k] & [0] \end{bmatrix} \tag{62}$$

then the set of N equations (58) can be transformed into a set of 2N first-order equations having the matrix form

$$[M]\{\dot{x}(t)\} + [G]\{x(t)\} = \{0\} \tag{63}$$

where [M] is symmetric and [G] is skew-symmetric,

$$[M] = [M]^T \ , \quad [G] = -[G]^T \tag{64}$$

because [m] and [k] are symmetric and [g] is skew-symmetric.

The matrix equation (63) is of the special form treated in Ref. 20, so that the eigenvalue problem can be solved by the method developed there. Hence, letting

$$\{x(t)\} = e^{\lambda t}\{x\} \tag{65}$$

where λ and $\{x\}$ are constant, we obtain the eigenvalue problem

$$\lambda[M]\{x\} + [G]\{x\} = \{0\} \tag{66}$$

It is shown in Ref. 20 that the solution of the eigenvalue problem (66) consists of 2N eigenvalues λ_r and eigenvectors $\{x\}_r$ $(r = 1,2,\ldots,2N)$, where the eigenvalues consist of pairs of pure imaginary complex conjugates, $\lambda_r = \pm\, i\omega_r$, and the eigenvectors also consist of pairs of associated complex conjugates $\{x\}_r$ and $\{x^*\}_r$ $(r = 1,2,\ldots,N)$. Moreover, the eigenvectors are orthogonal in a certain sense. Reference 20 provides an algorithm whereby the eigenvalue problem can be solved in terms of real quantities. The method will be used later in this work to solve the eigenvalue problem for a specific spacecraft.

8. Lagrange's Equations in Explicit Form

Lagrange's equations, Eqs. (35) and (36), are written in a general form. Before obtaining the nontrivial equilibrium and the corresponding variational equations, we must express them in a form in which the various coordinates appear explicitly. By virtue of the assumption that the satellite mass center moves in a circular orbit with orbital velocity Ω, we can replace K/R_c^3 by Ω^2 in Eq. (5). Moreover, the first terms in Eqs. (4) and (5) can be ignored because they are constant. In view of this, if we recall that the Lagrangian can be written as $L = T - V_G - V_{EA} - V_{EB}$, then we can substitute Eqs. (4), (5), (15), and (23) into L, and obtain

$$L(t) = \frac{1}{2}\{\omega\}^T[J^{(0)}]\{\omega\} + \{\omega\}^T\{K\} + T_E + \frac{1}{2}\Omega^2 tr[J^{(0)}]$$

$$- \frac{3}{2}\Omega^2\{\ell_a\}^T[J^{(0)}]\{\ell_a\} - \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i} [P_{xi}(v_i'^2 + w_i'^2)$$

$$+ EI_{zi}v_i''^2(1 - \frac{5}{2}v_i'^2) + EI_{yi}w_i''^2(1 - \frac{5}{2}w_i'^2)]dx_i \tag{67}$$

where P_{xi} is the axial force at any point of the slender rod, and

$$[J^{(0)}] = \sum_{i=0}^{n}[J_i^{(0)}] = \sum_{i=0}^{n}[\ell_i]^T[J_i][\ell_i] \tag{68a}$$

$$\{K\} = \sum_{i=1}^{n}\left\{\int_0^{\ell_i}\rho_i[h_i^{(0)} + r_i^{(0)} + u_i^{(0)}][\ell_i]^T\{\dot{u}_i\}dx_i + m_i[h_i^{(0)} + r_i^{(0)}\right.$$

$$\left. + u_i^{(0)}][\ell_i]^T\{\dot{u}_i\}\Big|_{x_i = \ell_i}\right\} \tag{68b}$$

$$T_E = \frac{1}{2}\sum_{i=1}^{n}\left\{\int_0^{\ell_i}\rho_i\{\dot{u}_i\}^T\{\dot{u}_i\}dx_i + m_i\{\dot{u}_i\}^T\{\dot{u}_i\}\Big|_{x_i = \ell_i}\right\} \tag{68c}$$

in which $[J^{(0)}]$ is the inertia matrix of the body in deformed state in terms of the reference system xyz, $\{K\}$ is an angular momentum matrix due to the elastic velocities, and T_E is the kinetic energy due to the elastic velocities. The elements of $[J_i]$ are given by Eqs. (9) and (10). Introducing the notation

$$\hat{J}_{i11} = \rho_i[(h_{yi}+v_i)^2+(h_{zi}+w_i)^2], \quad J_{i11}(\ell_i) = m_i[(h_{yi}+v_i)^2+(h_{zi}+w_i)^2]\Big|_{x_i=\ell_i}$$

$$\hat{J}_{i22} = \rho_i[(h_{xi}+x_i)^2+(h_{zi}+w_i)^2], \quad J_{i22}(\ell_i) = m_i[(h_{xi}+x_i)^2+(h_{zi}+w_i)^2]\Big|_{x_i=\ell_i}$$

$$\hat{J}_{i33} = \rho_i[(h_{xi}+x_i)^2+(h_{yi}+v_i)^2], \quad J_{i33}(\ell_i) = m_i[(h_{xi}+x_i)^2+(h_{yi}+v_i)^2]\Big|_{x_i=\ell_i}$$

$$\hat{J}_{i12} = \hat{J}_{i21} = -\rho_i(h_{xi}+x_i)(h_{yi}+v_i), \quad J_{i12}(\ell_i) = J_{i21}(\ell_i) = -m_i(h_{xi}+x_i)(h_{yi}+v_i)\Big|_{x_i=\ell_i}$$

$$\hat{J}_{i13} = \hat{J}_{i31} = -\rho_i(h_{xi}+x_i)(h_{zi}+w_i), \quad J_{i13}(\ell_i) = J_{i31}(\ell_i) = -m_i(h_{xi}+x_i)(h_{zi}+w_i)\Big|_{x_i=\ell_i}$$

$$\hat{J}_{i23} = \hat{J}_{i32} = -\rho_i(h_{yi}+v_i)(h_{zi}+w_i), \quad J_{i23}(\ell_i) = J_{i32}(\ell_i) = -m_i(h_{yi}+v_i)(h_{zi}+w_i)\Big|_{x_i=\ell_i} \qquad i = 1,2,\ldots,n \tag{69}$$

we can write

$$[J_i] = \int_0^{\ell_i} [\hat{J}_i]dx_i + [J_i(\ell_i)] \tag{70}$$

In a similar way, from the second of Eqs. (68), we have

$$\{K\} = \sum_{i=1}^{n} \left(\int_0^{\ell_i} \{\hat{K}_i\} dx_i + \{K_i(\ell_i)\} \right) \tag{71}$$

In view of the above, the Lagrangian densities can be written as follows

$$\hat{L}_i(x_i,t) = \frac{1}{2}\{\omega\}^T[\hat{J}_i^{(0)}]\{\omega\} + \{\omega\}^T\{\hat{K}_i\} + \frac{1}{2}\rho_i\{\dot{u}_i\}^T\{\dot{u}_i\} + \frac{1}{2}\Omega^2 tr[\hat{J}_i^{(0)}]$$

$$- \frac{3}{2}\Omega^2\{\ell_a\}^T[\hat{J}_i^{(0)}]\{\ell_a\} - \frac{1}{2}P_{xi}(v_i'^2 + w_i'^2) - \frac{1}{2}EI_{zi}\, v_i''^2(1 - \frac{5}{2}v_i'^2)$$

$$- \frac{1}{2}EI_{yi}\, w_i''^2(1 - \frac{5}{2}w_i'^2), \quad i = 1,2,\ldots,n \tag{72}$$

whereas the parts of the Lagrangian associated with the discrete masses are

$$L_i(\ell_i,t) = \frac{1}{2}\{\omega\}^T[J_i^{(0)}(\ell_i)]\{\omega\} + \{\omega\}^T\{K_i(\ell_i)\} + \frac{1}{2}m_i\{\dot{u}_i\}^T\{\dot{u}_i\}\Big|_{x_i=\ell_i}$$

$$+ \frac{1}{2}\Omega^2 tr[J_i^{(0)}(\ell_i)] - \frac{3}{2}\Omega^2\{\ell_a\}^T[J_i^{(0)}(\ell_i)]\{\ell_a\} \tag{73}$$

From the context it should be obvious when brackets and braces denote matrices in Eqs. (67) through (73) and when they do not.

Substituting Eq. (67) into Lagrange's equations for the rotational motion, Eqs. (65), we obtain

$$\left(\frac{\partial}{\partial\theta_j}\{\omega\}^T\right)[J^{(0)}]\{\omega\} + \left(\frac{\partial}{\partial\theta_j}\{\omega\}^T\right)\{K\} - 3\Omega^2\left(\frac{\partial}{\partial\theta_j}\{\ell_a\}^T\right)[J^{(0)}]\{\ell_a\}$$

$$- \frac{\partial}{\partial t}\left(\frac{\partial}{\partial\dot{\theta}_i}\{\omega\}^T\right)([J^{(0)}]\{\omega\} + \{K\}) = 0 \quad j = 1,2,3 \tag{74}$$

Moreover, Lagrange's equations for the transverse displacements v_i are

$$\frac{1}{2}\{\omega\}^T\left(\frac{\partial}{\partial v_i}[\hat{J}_i^{(0)}]\right)\{\omega\}+\{\omega\}^T\left(\frac{\partial}{\partial v_i}\{\hat{K}_i\}\right)+\frac{1}{2}\Omega^2\,\mathrm{tr}\left(\frac{\partial}{\partial v_i}[\hat{J}_i^{(0)}]\right)$$

$$-\frac{3}{2}\Omega^2\{\ell_a\}^T\left(\frac{\partial}{\partial v_i}[\hat{J}_i^{(0)}]\right)\{\ell_a\}-\frac{\partial}{\partial t}\left(\{\omega\}^T\frac{\partial}{\partial \dot{v}_i}\{\hat{K}\}+\rho_i\,\dot{v}_i\right)$$

$$-\frac{\partial}{\partial x_i}(-P_{xi}v_i'+\frac{5}{2}EI_{zi}\,v_i'v_i''^2)+\frac{\partial^2}{\partial x_i^2}[-EI_{zi}\,v_i''\,(1$$

$$-\frac{5}{2}v_i'^2)]+p_{yi}=0\quad,\quad 0<x_i<\ell_i\ ,\quad i=1,2,\ldots,n \tag{75a}$$

which are subject to the boundary conditions

$$-(P_{xi}-\frac{5}{2}EI_{zi}\,v_i''^2)\,v_i'+\frac{\partial}{\partial x_i}[EI_{zi}\,v_i''\,(1-\frac{5}{2}v_i'^2)]$$

$$+\frac{1}{2}\Omega^2(\{\omega\}^T\frac{\partial}{\partial v_i}[J_i^{(0)}]\{\omega\}+\mathrm{tr}\,\frac{\partial}{\partial v_i}[J^{(0)}]-3\{\ell_a\}^T\frac{\partial}{\partial v_i}[J^{(0)}]\{\ell_a\})$$

$$\left.\begin{aligned}&-\frac{\partial}{\partial t}(\{\omega\}^T\frac{\partial}{\partial \dot{v}_i}\{K\}+\rho_i\,\dot{v}_i)=0\\ &EI_{zi}\,v_i''\,(1-\frac{5}{2}v_i'^2)=0\end{aligned}\right\}\ \text{at}\ x_i=\ell_i,\ i=1,2,\ldots,n \tag{75b}$$

$$v_i=0,\ v_i'=0\ \text{at}\ x_i=0,\ i=1,2,\ldots,n \tag{75c}$$

Similarly, Lagrange's equations for the displacements w_i are

$$\frac{1}{2}\{\omega\}^T\left(\frac{\partial}{\partial w_i}[\hat{J}_i^{(0)}]\right)\{\omega\}+\{\omega\}^T\left(\frac{\partial}{\partial w_i}\{\hat{K}_i\}\right)+\frac{1}{2}\Omega^2\,\mathrm{tr}\left(\frac{\partial}{\partial w_i}[\hat{J}_i^{(0)}]\right)$$

$$- \frac{3}{2} \Omega^2 \{\ell_a\}^T \left[\frac{\partial}{\partial w_i} [\hat{J}_i^{(0)}]\right] \{\ell_a\} - \frac{\partial}{\partial t} \left[\{\omega\} \frac{\partial}{\partial \dot{w}_i} \{\hat{K}\} + \rho_i \dot{w}_i\right]$$

$$- \frac{\partial}{\partial x_i} (-P_{xi} w_i' + \frac{5}{2} EI_{yi} w_i' w_i''^2) + \frac{\partial^2}{\partial x_i^2} [-EI_{yi} w_i (1$$

$$- \frac{5}{2} w_i'^2)] + p_{zi} = 0 , \quad 0 < x_i < \ell_i, \quad i = 1,2,\dots,n \tag{76a}$$

which are subject to the boundary conditions

$$\left.\begin{aligned} &-(P_{xi} - \frac{5}{2} EI_{zi} w_i''^2) w_i' + \frac{\partial}{\partial x_i} [EI_{yi} w_i''(1 - \frac{5}{2} w_i'^2)] \\ &\quad + \frac{1}{2} \Omega^2 \left(\{\omega\}^T \frac{\partial}{\partial w_i} [J_i^{(0)}] \{\omega\} + tr \frac{\partial}{\partial w_i} [J^{(0)}] -3 \{\ell_a\}^T \frac{\partial}{\partial w_i} [J^{(0)}]\{\ell_a\}\right. \\ &\quad - \frac{\partial}{\partial t} \left[\{\omega\}^T \frac{\partial}{\partial \dot{w}_i} \{K\} + \rho_i \dot{w}_i\right] = 0 \\ &EI_{yi} w_i''(1 - \frac{5}{2} w_i'^2) = 0 \end{aligned}\right\} \text{ at } x_i = \ell_i, \; i = 1,2,\dots,n \tag{76b}$$

$$w_i = 0, \; w_i' = 0 \text{ at } x_i = 0, \; i = 1,2,\dots,n \tag{76c}$$

9. Equilibrium Equations in Explicit Form

For a gravity-gradient stabilized satellite, the angles $\theta_j (j = 1,2,3)$ are measured relative to an orbiting system of axes. The orbit being circular, with the orbital angular velocity being equal to Ω, the orbital axes rotate relative to an inertial space with angular velocity Ω about an axis normal to the orbital plane. This axis is denoted by c (see complete definition later). Hence, the angular velocity matrix $\{\omega\}$ can be written as

$$\{\omega\} = \Omega\{\ell_c\} + \{\omega\}_r \tag{77}$$

where $\{\ell_c\} = \{\ell_c(\theta_j)\}$ is the matrix of direction cosines between axis c and the reference system xyz, and $\{\omega\}_r = \{\omega(\theta_j,\dot{\theta}_j)\}_r$ is a matrix whose elements are the angular velocity components of system xyz relative to the orbital axes. They are linear combinations of the velocities $\dot{\theta}_j$ $(j = 1,2,3)$.

The equilibrium equations can be obtained by deleting from Eqs. (74) - (76) all the terms involving derivatives with respect to time. This implies that we can replace $\{\omega\}$ by $\Omega\{\ell_c\}$ in these equations. Hence, the nontrivial equilibrium must satisfy the general equations for the rotational motion

$$\{\ell_c\}^T[J^{(0)}] \frac{\partial}{\partial\theta_j} \{\ell_c\} - 3\{\ell_a\}^T[J^{(0)}] \frac{\partial}{\partial\theta_j} \{\ell_a\} = 0, \; j = 1,2,3 \tag{78}$$

as well as the boundary-value problems defined by the differential equations

$$\frac{1}{2}\Omega^2\left(\{\ell_c\}^T \frac{\partial}{\partial v_i} [\hat{J}_i^{(0)}]\{\ell_c\} + \mathrm{tr}\frac{\partial}{\partial v_i}[\hat{J}_i^{(0)}] - 3\{\ell_a\}^T \frac{\partial}{\partial v_i}[\hat{J}_i^{(0)}]\{\ell_a\}\right)$$

$$+ \frac{\partial}{\partial x_i}[(P_{xi} - \frac{5}{2} EI_{zi}\, v_i''^2)v_i'] - \frac{\partial^2}{\partial x_i^2}[EI_{zi}v_i''(1 - \frac{5}{2} v_i'^2)] = 0,$$

$$0 < x_i < \ell_i, \; i = 1,2,\ldots,n \tag{79a}$$

and the boundary conditions

$$-(P_{xi} - \frac{5}{2} EI_{zi}v_i''^2)v_i' + \frac{\partial}{\partial x_i}[EI_{zi}v_i''(1 - \frac{5}{2} v_i'^2)] + \frac{1}{2}\Omega^2\left(\{\ell_c\}^T \frac{\partial}{\partial v_i}[J_i^{(0)}]\{\ell_c\}\right.$$

$$\left.\begin{aligned} &+ \mathrm{tr}\,\frac{\partial}{\partial v_i}\,[J^{(0)}] - 3\{\ell_a\}^T \frac{\partial}{\partial v_i}\,[J_i^{(0)}]\{\ell_a\}\Bigg| = 0 \\ &- EI_{zi} v_i''(1 - \frac{5}{2}\,v_i'^2) = 0 \end{aligned}\right\} \text{at } x_i = \ell_i,\ i = 1,2,\ldots,n \tag{79b}$$

$$v_i = 0,\ v_i' = 0 \text{ at } x_i = 0,\ i = 1,2,\ldots,n \tag{79c}$$

Moreover, it must satisfy a set of equations similar in structure to Eqs. (79), but with v_i replaced by w_i.

10. The Variational Equations for the Discretized System

The variational equations for the discretized system were obtained earlier in the form (58), where the matrices [m], [g], and [k] are defined by Eqs. (49), (51), (53), and (57). Although the equations just mentioned have the advantage of revealing the symmetry of [m] and [k] and the skew-symmetry of [g], the formulas for deriving the elements of the matrices are not the most suitable from a computational point of view. Indeed, we wish to present a procedure whereby the actual derivation of the variational equations is performed by a digital computer.

Consistent with earlier notation, we shall denote quantities associated with equilibrium by the subscript 0 and perturbed quantities by the subscript 1. With this in mind, we can write the Lagrangian in the form

$$L = L_0 + L_1 \tag{80}$$

$$\begin{aligned} L_0 = \frac{1}{2}\,\{\omega\}_0^T[J^{(0)}]_0\{\omega\}_0 + \{\omega\}_0^T\{K\}_0 + T_{E0} + \frac{1}{2}\,\Omega^2\ \mathrm{tr}[J^{(0)}]_0 \\ - \frac{3}{2}\,\Omega^2\ \{\ell_a\}_0^T[J^{(0)}]_0\{\ell_a\}_0 - V_{E0} \end{aligned} \tag{81}$$

and

$$L_1 = \{\omega\}_1^T[J^{(0)}]_0\{\omega\}_0 + \frac{1}{2}\{\omega\}_1^T[J^{(0)}]_0\{\omega\}_1 + \frac{1}{2}\{\omega\}_0^T[J^{(0)}]_1\{\omega\}_0$$

$$+ \{\omega\}_1^T[J^{(0)}]_1\{\omega\}_0 + \{\omega\}_1^T\{K\}_0 + \{\omega\}_1^T\{K\}_1 + \{\omega\}_0^T\{K\}_1 + T_{E1}$$

$$+ \frac{1}{2}\Omega^2\,\mathrm{tr}[J^{(0)}]_1 - 3\Omega^2\{\ell_a\}_1^T[J^{(0)}]_0\{\ell_a\}_0 - \frac{3}{2}\Omega^2\{\ell_a\}_1^T[J^{(0)}]_0\{\ell_a\}_1$$

$$- \frac{3}{2}\Omega^2\{\ell_a\}_0^T[J^{(0)}]_1\{\ell_a\}_0 - 3\Omega^2\{\ell_a\}_1^T[J^{(0)}]_1\{\ell_a\}_0 - V_{E1} \qquad (82)$$

in which

$$\{\omega\} = \{\omega\}_0 + \{\omega\}_1 = \Omega\{\ell_c\}_0 + \sum_{i=1}^{3}\left[\left(\frac{\partial}{\partial\theta_{i0}}\{\omega\}\right)q_i + \left(\frac{\partial}{\partial\dot{\theta}_{i0}}\{\omega\}\right)\dot{q}_i\right]$$

$$+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\left(\frac{\partial^2}{\partial\theta_{i0}\partial\theta_{j0}}\{\omega\}\right)q_iq_j + 2\left(\frac{\partial^2}{\partial\theta_{j0}\partial\dot{\theta}_{i0}}\{\omega\}\right)q_j\dot{q}_i\right] \qquad (83)$$

$$\{\ell_a\} = \{\ell_a\}_0 + \{\ell_a\}_1 = \{\ell_a\}_0 + \sum_{i=1}^{3}\left(\frac{\partial}{\partial\theta_{i0}}\{\ell_a\}\right)q_i$$

$$+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2}{\partial\theta_{i0}\partial\theta_{j0}}\{\ell_a\}\right)q_iq_j \qquad (84)$$

$$[J^{(0)}] = \sum_{i=0}^{n}[\ell_i]^T[J_i][\ell_i] = \sum_{i=0}^{n}[\ell_i]^T\left(\int_0^{\ell_i}[\hat{J}_i]dx_i + [J_i(\ell_i)]\right)[\ell_i]$$

$$= [J^{(0)}]_0 + [J^{(0)}]_1 = [J^{(0)}]_0 + \sum_{i=0}^{n}[\ell_i]^T\left(\int_0^{\ell_i}\left[\left[\frac{\partial\hat{J}_i}{\partial v_{i0}}\right]v_{i1}\right.\right.$$

$$+ \left[\frac{\partial \hat{J}_i}{\partial w_{i0}}\right] w_{i1} + \frac{1}{2}\left[\frac{\partial^2 \hat{J}_i}{\partial v_{i0}^2}\right] v_{i1}^2 + \left[\frac{\partial^2 \hat{J}_i}{\partial v_{i0}\partial w_{i0}}\right] v_{i1} w_{i1} + \frac{1}{2}\left[\frac{\partial^2 \hat{J}_i}{\partial w_{i0}^2}\right] w_{i1}^2 \Bigg] dx_i$$

$$+ \left[\frac{\partial J_i(\ell_i)}{\partial v_{i0}}\right] v_{i1}(\ell_i) + \left[\frac{\partial J_i(\ell_i)}{\partial w_{i0}}\right] w_{i1}(\ell_i) + \frac{1}{2}\left[\frac{\partial^2 J_i(\ell_i)}{\partial v_{i0}^2}\right] v_{i1}^2(\ell_i)$$

$$+ \left[\frac{\partial^2 J_i(\ell_i)}{\partial v_{i0}\partial w_{i0}}\right] v_{i1}(\ell_i) w_{i1}(\ell_i) + \frac{1}{2}\left[\frac{\partial^2 J_i(\ell_i)}{\partial w_{i0}^2}\right] w_{i1}^2(\ell_i) \Bigg\} [\ell_i] \qquad (85)$$

$$\{K\} = \sum_{i=1}^{n} \left\{ \int_0^{\ell_i} \{\hat{K}_i\} dx_i + \{K_i(\ell_i)\} \right\}$$

$$\{K\}_0 = 0$$

$$\{K\}_1 = \sum_{i=1}^{n} \int_0^{\ell_i} \Bigg[\left\{\frac{\partial \hat{K}_i}{\partial v_{i0}}\right\} v_{i1} + \left\{\frac{\partial \hat{K}_i}{\partial w_{i0}}\right\} w_{i1} + \left\{\frac{\partial \hat{K}_i}{\partial v'_{i0}}\right\} v'_{i1} + \left\{\frac{\partial \hat{K}_i}{\partial w'_{i0}}\right\} w'_{i1}$$

$$+ \left\{\frac{\partial^2 \hat{K}_i}{\partial v_{i0}\partial w'_{i0}}\right\} v_{i1} w'_{i1} + \left\{\frac{\partial^2 \hat{K}_i}{\partial v'_{i0}\partial w_{i0}}\right\} v'_{i1} w_{i1} \Bigg] dx_i + \left\{\frac{\partial K_i(\ell_i)}{\partial v_{i0}}\right\} v_{i1}(\ell_i)$$

$$+ \left\{\frac{\partial K_i(\ell_i)}{\partial w_{i0}}\right\} w_{i1}(\ell_i) + \left\{\frac{\partial K_i(\ell_i)}{\partial v'_{i0}}\right\} v'_{i1}(\ell_i) + \left\{\frac{\partial K_i(\ell_i)}{\partial w'_{i0}}\right\} w'_{i1}(\ell_i)$$

$$+ \left\{\frac{\partial^2 K_i(\ell_i)}{\partial v_{i0}\partial w'_{i0}}\right\} v_{i1}(\ell_i) w'_{i1}(\ell_i) + \left\{\frac{\partial^2 K_i(\ell_i)}{\partial v'_{i0}\partial w_{i0}}\right\} v'_{i1}(\ell_i) w_{i1}(\ell_i) \qquad (86)$$

Note that Eqs. (83) - (86) represent Taylor's series expansions of $\{\omega\}$, $\{\ell_a\}$, $[J^{(0)}]$, and $\{K\}$ about equilibrium. Moreover, we have

$$T_{E0} = 0 \tag{87}$$

and

$$T_{E1} = \frac{1}{2}\sum_{i=1}^{n}\left[\int_0^{\ell_i} \rho_i\{\dot{u}_i\}^T\{\dot{u}_i\}dx_i + m_i\{\dot{u}_i\}^T\{\dot{u}_i\}\Big|_{x_i = \ell_i}\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n}\int_0^{\ell_i} \rho_i(\dot{v}_{i1}^2 + \dot{w}_{i1}^2)dx_i + m_i(\dot{v}_{i1}^2 + \dot{w}_{i1}^2)\Big|_{x_i = \ell_i}\right] \tag{88}$$

as well as

$$V_{E0} = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}\left\{P_{xi}\left[\left(\frac{\partial v_{i0}}{\partial x_i}\right)^2 + \left(\frac{\partial w_{i0}}{\partial x_i}\right)^2\right] + EI_{zi}\left(\frac{\partial^2 v_{i0}}{\partial x_i^2}\right)^2\left[1 - \frac{5}{2}\left(\frac{\partial v_{i0}}{\partial x_i}\right)^2\right]\right.$$

$$\left. + EI_{yi}\left(\frac{\partial^2 w_{i0}}{\partial x_i{}^2}\right)^2\left[1 - \frac{5}{2}\left(\frac{\partial w_{i0}}{\partial x_i}\right)^2\right]\right\}dx_i \tag{89}$$

and

$$V_{E1} = \frac{1}{2}\sum_{i=1}^{n}\int_0^{\ell_i}\left\{P_{xi}\left[2\frac{\partial v_{i0}}{\partial x_i}\frac{\partial v_{i1}}{\partial x_i} + 2\frac{\partial w_{i0}}{\partial x_i}\frac{\partial w_{i1}}{\partial x_i} + \left(\frac{\partial v_{i1}}{\partial x_i}\right)^2 + \left(\frac{\partial w_{i1}}{\partial x_i}\right)^2\right]\right.$$

$$- 5\,EI_{zi}\left(\frac{\partial v_{i0}}{\partial x_i}\right)\left(\frac{\partial^2 v_{i0}}{\partial x_i^2}\right)^2\frac{\partial v_{i1}}{\partial x_i} - \frac{5}{2}EI_{zi}\left(\frac{\partial^2 v_{i0}}{\partial x_i^2}\right)^2\left(\frac{\partial v_{i1}}{\partial x_i}\right)^2$$

$$+ 2\,EI_{zi}\left[1 - \frac{5}{2}\left(\frac{\partial v_{i0}}{\partial x_i}\right)^2\right]\left(\frac{\partial^2 v_{i0}}{\partial x_i^2}\right)\frac{\partial^2 v_{i1}}{\partial x_i^2} + EI_{zi}\left[1 - \frac{5}{2}\left(\frac{\partial v_{i0}}{\partial x_i}\right)^2\right]\left(\frac{\partial^2 v_{i1}}{\partial x_i^2}\right)^2$$

$$- 10\, EI_{zi} \frac{\partial v_{i0}}{\partial x_i} \frac{\partial^2 v_{i0}}{\partial x_i^2} \left(\frac{\partial v_{i1}}{\partial x_i}\right)\left(\frac{\partial^2 v_{i1}}{\partial x_i^2}\right)$$

$$- 5\, EI_{yi} \left(\frac{\partial w_{i0}}{\partial x_i}\right)\left(\frac{\partial^2 w_{i0}}{\partial x_i^2}\right)^2 \frac{\partial w_{i1}}{\partial x_i} - \frac{5}{2} EI_{yi} \left(\frac{\partial^2 w_{i0}}{\partial x_i^2}\right)^2 \left(\frac{\partial w_{i1}}{\partial x_i}\right)^2$$

$$+ 2\, EI_{yi} \left[1 - \frac{5}{2}\left(\frac{\partial w_{i0}}{\partial x_i}\right)^2\right]\left(\frac{\partial^2 w_{i0}}{\partial x_i^2}\right) \frac{\partial^2 w_{i1}}{\partial x_i^2} + EI_{yi} \left[1 - \frac{5}{2}\left(\frac{\partial w_{i0}}{\partial x_i}\right)^2\right]\left(\frac{\partial^2 w_{i1}}{\partial x_i^2}\right)^2$$

$$\left. - 10\, EI_{yi} \frac{\partial w_{i0}}{\partial x_i} \frac{\partial^2 w_{i0}}{\partial x_i^2} \left(\frac{\partial w_{i1}}{\partial x_i}\right)\left(\frac{\partial^2 w_{i1}}{\partial x_i^2}\right)\right\} dx_i \tag{90}$$

To obtain the variational equations in terms of the discrete coordinates $q_j(t)$ $(j = 1,2,\ldots,2np+3)$, we must insert the modal expansions (47) into L_1 and perform the indicated integrations over the spatial variables x_i $(i = 1,2,\ldots,n)$. Because the resulting expressions are very lengthy, we shall not write them explicity, but proceed with the derivation of Lagrange's equations instead. To this end, it will prove convenient to denote constant terms by the subscript $_c$ and terms that are linear in the generalized coordinates $q_j(t)$ and generalized velocities $\dot{q}_j(t)$ by the subscript ℓ. This enables us to write

$$\frac{\partial L_1}{\partial \dot{q}_j} = \left(\frac{\partial}{\partial \dot{q}_j} \{\omega\}_1^T\right)_\ell \left([J^{(0)}]_0\{\omega\}_0\right) + \left(\frac{\partial}{\partial \dot{q}_j} \{\omega\}_1^T\right)_c \left[[J^{(0)}](\{\omega\}_1)_\ell \right.$$

$$\left. + ([J^{(0)}]_1)_\ell\{\omega\}_0 + (\{K\}_1)_\ell\right] \quad j = 1,2,3 \tag{91a}$$

$$\frac{\partial L_1}{\partial \dot{q}_j} = \left(\{\omega\}_1^T\right)_\ell \left(\frac{\partial}{\partial \dot{q}_j}\{K\}_1\right)_c + \{\omega\}_0^T \left(\frac{\partial}{\partial \dot{q}_j}\{K\}_1\right)_\ell + \left(\frac{\partial}{\partial \dot{q}_j} T_{E1}\right)_\ell$$

$$j = 4,5,\ldots,2np+3 \qquad (91b)$$

$$\frac{\partial L_1}{\partial q_j} = \left(\frac{\partial}{\partial q_j}\{\omega\}_1^T\right)_\ell \left([J^{(0)}]_0 \{\omega\}_0\right) + \left(\frac{\partial}{\partial q_j}\{\omega\}_1^T\right)_c \left[[J^{(0)}]_0 \left(\{\omega\}_1\right)_\ell\right.$$

$$\left. + \left([J^{(0)}]_1\right)_\ell \{\omega\}_0 + \left(\{K\}_1\right)_\ell\right] - 3\,\Omega^2 \left(\frac{\partial}{\partial q_j}\{\ell_a\}_1^T\right)_\ell \left([J^{(0)}]_0 \{\ell_a\}_0\right)$$

$$- 3\,\Omega^2 \left(\frac{\partial}{\partial q_j}\{\ell\ \}_1^T\right)_c \left[[J^{(0)}] \left(\{\ell_a\}_1\right)_\ell + \left([J^{(0)}]\right)_\ell \{\ell_a\}_0\right]$$

$$j = 1,2,3 \qquad (91c)$$

$$\frac{\partial L_1}{\partial q_j} = \{\omega\}_0^T \left[\frac{1}{2}\left(\frac{\partial}{\partial q_j}[J^{(0)}]_1\right)_\ell \{\omega\}_0 + \left(\frac{\partial}{\partial q_j}\{K\}_1\right)_\ell\right]$$

$$+ \left(\{\omega\}_1^T\right)_\ell \left[\left(\frac{\partial}{\partial q_j}[J^{(0)}]_1\right)_c \{\omega\}_0 + \left(\frac{\partial}{\partial q_j}\{K\}_1\right)_c\right]$$

$$+ \frac{1}{2}\Omega^2 \frac{\partial}{\partial q_j}\left(\mathrm{tr}\,[J^{(0)}]_1\right)_\ell - \frac{3}{2}\Omega^2 \{\ell_a\}_0^T \left(\frac{\partial}{\partial q_j}[J^{(0)}]_1\right)_\ell \{\ell_a\}_0$$

$$- 3\,\Omega^2 \left(\{\ell\ \}_1^T\right)_\ell \left(\frac{\partial}{\partial q_j}[J^{(0)}]_1\right)_c \{\ell_a\}_0 + \left(\frac{\partial}{\partial q_j} V_{E1}\right)_\ell$$

$$j = 4,5,\ldots 2np+3 \qquad (91d)$$

which enables us to write Lagrange's equations for the perturbed motion in the compact form

$$\left(\frac{\partial}{\partial q_j}\{\omega\}_1^T\right)_\ell \left(\left[J^{(0)}\right]_0 \{\omega\}_0\right) + \left(\frac{\partial}{\partial q_j}\{\omega\}_1^T\right)_c \left[\left[J^{(0)}\right]_0 \left(\{\omega\}_1\right)_\ell\right.$$

$$\left. + \left(\left[J^{(0)}\right]_1\right)_\ell \{\omega\}_0 + \left(\{K\}_1\right)_\ell\right] - 3\,\Omega^2 \left(\frac{\partial}{\partial q_j}\{\ell_a\}_1^T\right)_\ell \left(\left[J^{(0)}\right]_0 \{\ell_a\}_0\right)$$

$$- 3\,\Omega^2 \left(\frac{\partial}{\partial q_j}\{\ell_a\}_1^T\right)_c \left[\left[J^{(0)}\right]_0 \left(\{\ell_a\}_1\right)_\ell + \left(\left[J^{(0)}\right]_1\right)_\ell \{\ell_a\}_0\right]$$

$$- \frac{d}{dt}\left\langle \left(\frac{\partial}{\partial \dot{q}_j}\{\omega\}_1^T\right)_\ell \left(\left[J^{(0)}\right]_0 \{\omega\}_0\right) + \left(\frac{\partial}{\partial \dot{q}_j}\{\omega\}_1^T\right)_c \left[\left[J^{(0)}\right]_0 \left(\{\omega\}_1\right)_\ell\right.\right.$$

$$\left.\left. + \left(\left[J^{(0)}\right]_1\right)_\ell \{\omega\}_0 + \left(\{K\}_1\right)_\ell\right]\right\rangle = 0 \ , \ j = 1,2,3 \tag{92a}$$

$$\{\omega\}_0^T \left\langle \frac{1}{2}\left(\frac{\partial}{\partial q_j}\left[J^{(0)}\right]_1\right)_\ell \{\omega\}_0 + \left(\frac{\partial}{\partial q_j}\{K\}_1\right)_\ell\right\rangle$$

$$+ \left(\{\omega\}_1^T\right)_\ell \left\langle \left(\frac{\partial}{\partial q_j}\left[J^{(0)}\right]_1\right)_c \{\omega\}_0 + \left(\frac{\partial}{\partial q_j}\{K\}_1\right)_c\right\rangle$$

$$+ \frac{1}{2}\Omega^2 \frac{\partial}{\partial q_j}\left(\mathrm{tr}\left[J^{(0)}\right]_1\right)_\ell - \frac{3}{2}\Omega^2 \{\ell_a\}_0^T \left(\frac{\partial}{\partial q_j}\left[J^{(0)}\right]_1\right)_\ell \{\ell_a\}_0$$

$$- 3\,\Omega^2 \left(\{\ell_a\}_1^T\right)_\ell \left(\frac{\partial}{\partial q_j}\left[J^{(0)}\right]_1\right)_c \{\ell_a\}_0 + \left(\frac{\partial}{\partial q_j} V_{E1}\right)_\ell$$

$$- \frac{d}{dt}\left\langle \left(\{\omega\}_1^T\right)_\ell \left(\frac{\partial}{\partial \dot{q}_j}\{K\}_1\right)_c + \{\omega\}_0^T \left(\frac{\partial}{\partial \dot{q}_j}\{K\}_1\right)_\ell + \left(\frac{\partial}{\partial \dot{q}_j} T_{E1}\right)_\ell\right\rangle = 0$$

$$j = 4,5,\ldots,2np+3 \tag{92b}$$

As mentioned already, the advantage of Lagrange's equations (92) over those derived in Sec. 4 is that Eqs. (92) permit automatic derivation by means of a digital computer.

Before specializing the equations to a particular satellite, let us derive an expression for the axial force P_{xi} in terms of matrix notation. The axial force P_{xi} is due to centrifugal and differential gravity effects. Introducing the modified potential energy density associated with member i

$$\hat{V}_{i\,mod} = -\frac{1}{2}\Omega^2(\{\ell_c\}^T[\hat{J}_i^{(0)}]\{\ell_c\} + tr[\hat{J}_i^{(0)}] - 3\{\ell_a\}^T[\hat{J}_i^{(0)}]\{\ell_a\})$$

$$-\frac{1}{2}\Omega^2(\{\ell_c\}^T[J_i^{(0)}(x_i)]\{\ell_c\} + tr[J_i^{(0)}(x_i)]$$

$$-3\{\ell_a\}^T[J_i^{(0)}(x_i)]\{\ell_a\})\delta(x_i - \ell_i) \tag{93}$$

where the terms inside parentheses and multiplying $\delta(x_i-\ell_i)$ are due to the tip masses, the axial force density can be written in the form

$$P_{xi}(x_i) = -\frac{\partial\hat{V}_{i\,mod}}{\partial x_i}$$

$$= \frac{1}{2}\Omega^2(\{\ell_c\}^T[\hat{J}_i^{(0)}]'\{\ell_c\} + tr[\hat{J}_i^{(0)}]' - 3\{\ell_a\}^T[\hat{J}_i^{(0)}]'\{\ell_a\}) \tag{94}$$

in which we introduced the notation

$$[\hat{J}_i^{(0)}]' = \frac{\partial}{\partial x_i}[\hat{J}_i^{(0)}] + \frac{\partial}{\partial x_i}[\hat{J}_i^{(0)}(x_i)]\delta(x_i-\ell_i) \tag{95}$$

Observing from Eq. (67) that P_{xi} is multiplied by $(v_i'^2 + w_i'^2)$, we ignore any transverse terms in $[\hat{J}_i^{(0)}]'$, so that using the first of Eqs. (68) and Eqs. (69) we obtain the approximation

$$[\hat{J}_i^{(0)}]' \simeq 2(h_{xi} + x_i) \left\langle \rho_i + m_i\delta(x_i - \ell_i) \right\rangle [\ell_i]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\ell_i] \tag{96}$$

Inserting Eqs. (95) and (96) into Eq. (94), we can write the axial force P_{xi} at any point x_i in the form of the integral

$$P_{xi} = \int_{x_i}^{\ell} \hat{P}_{\xi i}(\xi_i)d\xi_i = \Omega^2(\{\ell_c\}^T[\int_{x_i}^{\ell_i} J_i^{(0)'}(\xi_i)d\xi_i]\{\ell_c\}$$

$$+ \mathrm{tr}[\int_{x_i}^{\ell_i} \hat{J}_i^{(0)'}(\xi_i)d\xi_i] - 3\{\ell_a\}^T[\int_{x_i}^{\ell_i} \hat{J}_i^{(0)'}(\xi_i)d\xi_i]\{\ell_a\}) \tag{97}$$

where, assuming that ρ_i = const, we have

$$[\int_{x_i}^{\ell_i} \hat{J}_i^{(0)'}(\xi_i)d\xi_i] = \Big\langle \rho_i[(h_{xi}+\ell_i)^2 - (h_{xi}+x_i)^2]$$

$$+ 2m_i(h_{xi}+\ell_i)\Big\rangle [\ell_i]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\ell_i] \tag{98}$$

It follows that the desired expression has the form

$$P_{xi} = \Omega^2 \Big\langle \frac{1}{2} \rho_i[(h_{xi}+\ell_i)^2 - (h_{xi}+x_i)^2]$$

$$+ m_i(h_{xi}+\ell_i)\Big\rangle \left(\{\ell_c\}^T[\ell_i]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\ell_i]\{\ell_c\} - \mathrm{tr}[\ell_i]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\ell_i] \right.$$

$$- 3\{\ell_a\}^T[\ell_i]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\ell_i]\{\ell_a\} \Bigg\} \tag{99}$$

11. The RAE/B Satellite. General Formulation.

a. Equations of motion

Next let us specialize the equations to the case of a satellite consisting of a rigid core with six flexible booms, as shown in Fig. 6. First, we wish to determine the matrices $[\ell_i]$ of the direction cosines between axes $x_i y_i z_i$ and xyz. From Fig. 6, it is easy to verify that

$$[\ell_1] = \begin{bmatrix} c\alpha & s\alpha & 0 \\ -s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\ell_2] = \begin{bmatrix} -c\alpha & s\alpha & 0 \\ -s\alpha & -c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\ell_3] = \begin{bmatrix} -c\alpha & -s\alpha & 0 \\ s\alpha & -c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\ell_4] = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{100}$$

$$[\ell_5] = \begin{bmatrix} 0 & s\beta & c\beta \\ 1 & 0 & 0 \\ 0 & c\beta & -s\beta \end{bmatrix}, \quad [\ell_6] = \begin{bmatrix} 0 & -s\beta & -c\beta \\ 1 & 0 & 0 \\ 0 & -c\beta & s\beta \end{bmatrix}$$

where $s\alpha = \sin\alpha$, $c\alpha = \cos\alpha$, $s\beta = \sin\beta$, and $c\beta = \cos\beta$. Moreover, to write the angular velocity matrix $\{\omega\}$ in explicit form, we must specify the rotations θ_j $(j = 1,2,3)$. Assuming that system xyz is obtained from system abc by means of the rotations θ_2 about y, $-\theta_1$ about x, and θ_3 about z, and recalling that axes abc rotate about c with the constant angular velocity Ω, matrix $\{\omega\}$ can be shown to have the expression

$$\{\omega\} = \Omega \begin{Bmatrix} -(s\theta_2\ c\theta_3 + s\theta_1\ c\theta_2\ s\theta_3) \\ s\theta_2\ s\theta_3 - s\theta_1\ c\theta_2\ c\theta_3 \\ c\theta_1\ c\theta_2 \end{Bmatrix} + \begin{bmatrix} -c\theta_3 & c\theta_1\ s\theta_3 & 0 \\ s\theta_3 & c\theta_1\ c\theta_3 & 0 \\ 0 & s\theta_1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \tag{101}$$

where $s\theta_1 = \sin\theta_1$, $c\theta_1 = \cos\theta_1$, etc. Because the direction of the radius vector R_c coincides with that of axis a at all times, the direction matrix $\{\ell_a\}$ can be written as

$$\{\ell_a\} = \begin{Bmatrix} c\theta_2\ c\theta_3 - s\theta_1\ s\theta_2\ s\theta_3 \\ -(c\theta_2\ s\theta_3 + s\theta_1\ s\theta_2\ c\theta_3) \\ c\theta_1\ s\theta_2 \end{Bmatrix} \tag{102}$$

It will prove convenient to rewrite matrices $\{\omega\}$ and $\{\ell_a\}$ as follows

$$\{\omega\} = [\theta]_3\ [\theta^*]_1\ \{\dot{\theta}\} + \{\dot{\theta}\}_3 + \Omega\ [\theta]_3[\theta]_1\{\theta\}_2 \tag{103}$$

where

$$[\theta]_3 = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad [\theta^*]_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & c\theta_1 & 0 \\ 0 & s\theta_1 & 0 \end{bmatrix},\quad \{\dot{\theta}\}_3 = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{Bmatrix}$$

$$[\theta]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_1 & -s\theta_1 \\ 0 & s\theta_1 & c\theta_1 \end{bmatrix},\quad \{\theta\}_2 = \begin{Bmatrix} -s\theta_2 \\ 0 \\ c\theta_2 \end{Bmatrix} \tag{104}$$

Introducing Eq. (103) into (4), and recalling Eq. (68), the kinetic energy becomes

$$
\begin{aligned}
T &= \frac{1}{2}\{\omega\}^T [J^{(0)}] \{\omega\} + \{\omega\}^T \{K\} + T_E \\
&= \frac{1}{2}\{\dot\theta\}^T [\theta^*]_1^T [\theta]_3^T [J^{(0)}] [\theta]_3 [\theta^*]_1 \{\dot\theta\} + \{\dot\theta\}^T [\theta^*]_1^T [\theta]_3^T [J^{(0)}] \{\dot\theta\}_3 \\
&\quad + \frac{1}{2}\{\dot\theta\}_3^T [J^{(0)}] \{\dot\theta\}_3 + \{\dot\theta\}^T [\theta^*]_1 [\theta]_3^T \{K\} + \{\dot\theta\}_3^T \{K\} + T_E \\
&\quad + \Omega \{\theta\}_2^T [\theta]_1^T [\theta]_3^T [J^{(0)}] [\theta]_3 [\theta^*]_1 \{\dot\theta\} + \Omega \{\theta\}_2^T [\theta]_1^T [\theta]_3^T [J^{(0)}] \{\dot\theta\}_3 \\
&\quad + \Omega \{\theta\}_2^T [\theta]_1^T [\theta]_3^T \{K\} + \frac{1}{2}\Omega^2 \{\theta\}_2^T [\theta]_1^T [\theta]_3^T [J^{(0)}] [\theta]_3 [\theta]_1 \{\theta\}_2 \\
&= \frac{1}{2}\{\dot\theta\}^T [\theta^*]_1^T [J^*] [\theta^*]_1 \{\dot\theta\} + \{\dot\theta\}^T [\theta^*]_1^T [\theta]_3^T [J] \{\dot\theta\}_3 \\
&\quad + \{\dot\theta\}^T [\theta^*]_1^T \{K^*\} + \frac{1}{2} J_{33} \dot\theta_3^2 + K_3 \dot\theta_3 + T_E \\
&\quad + \Omega \{\theta\}_2^T [\theta]_1^T [J^*] [\theta^*]_1 \{\dot\theta\} + \Omega \{\theta\}_2^T [\theta]_1^T [\theta]_3^T [J] \{\dot\theta\}_3 \\
&\quad + \Omega \{\theta\}_2^T [\theta]_1^T \{K^*\} + \frac{1}{2}\Omega^2 \{\theta\}_2^T [\theta]_1^T [J^*] [\theta]_1 \{\theta\}_2 \qquad (105)
\end{aligned}
$$

where $[J^*] = [\theta]_3^T [J^{(0)}] [\theta_3]$ and $\{K^*\} = [\theta]_3^T \{K\}$. Moreover, inserting Eq. (103) into (5), and recognizing that $\{\ell_a\}^T = -(\{\theta\}_2')^T [\theta]_1^T \{\theta\}_3^T$, we obtain the gravitational potential energy in the form

$$
\begin{aligned}
V_G &= -\frac{1}{2}\Omega^2 \operatorname{tr}[J^{(0)}] + \frac{3}{2}\Omega^2 (-\{\theta\}_2')^T [\theta]_1^T [\theta]_3^T [J^{(0)}] [\theta]_3 [\theta]_1 (-\{\theta\}_2') \\
&= -\frac{1}{2}\Omega^2 \operatorname{tr}[J^{(0)}] + \frac{3}{2}\Omega^2 (\{\theta\}_2')^T [\theta]_1^T [J^*] [\theta]_1 \{\theta\}_2' \qquad (106)
\end{aligned}
$$

where primes indicate differentiation with respect to θ_2. Expanding the matrix involved in T and V_G, and recalling that $L = T - V_G - V_{EA} - V_{EB}$, the Lagrangian L can be written in the form

$$
L = \frac{1}{2}[J_{11}^* \dot\theta_1^2 + (J_{22}^* c^2\theta_1 + J_{33}^* s^2\theta_1 + 2 J_{23}^* s\theta_1 c\theta_1)\dot\theta_2^2
$$

$$- 2\,(J_{12}^{*}c\theta_1 + J_{13}^{*}s\theta_1)\,\dot\theta_1\dot\theta_2 - 2J_{13}^{*}\,\dot\theta_1\dot\theta_3 + J_{33}^{*}\,\dot\theta_3^2$$

$$+ 2\,(J_{23}^{*}c\theta_1 + J_{33}^{*}s\theta_1)\,\dot\theta_2\dot\theta_3]$$

$$- K_1^{*}\dot\theta_1 + (K_2^{*}c\theta_1 + K_3^{*}s\theta_1)\,\dot\theta_2 + K_3^{*}\,\dot\theta_3 + \frac{1}{2}\sum_{i=1}^{6}\int_{m_i}(\dot u_i^2 + \dot v_i^2 + \dot w_i^2)dm_i$$

$$+ \Omega\,\{(J_{11}^{*}s\theta_2 + J_{12}^{*}\,s\theta_1c\theta_2 - J_{13}^{*}\,c\theta_1c\theta_2)\dot\theta_1 + [-J_{12}^{*}\,c\theta_1\,s\theta_2$$

$$- J_{22}^{*}\,s\theta_1c\theta_1c\theta_2 + J_{33}^{*}\,s\theta_1c\theta_1c\theta_2 - J_{13}^{*}\,s\theta_1s\theta_2 + J_{23}^{*}\,c\theta_2\,(c^2\theta_1$$

$$- s^2\theta_1)]\dot\theta_2 + (J_{33}^{*}\,c\theta_1c\theta_2 - J_{13}^{*}\,s\theta_2 - J_{23}^{*}\,s\theta_1c\theta_2)\dot\theta_3 - K_1^{*}\,s\theta_2$$

$$- K_2^{*}\,s\theta_1c\theta_2 + K_3^{*}\,c\theta_1c\theta_2\} + \Omega^2\,[\frac{1}{2}\,(J_{11} + J_{22} + J_{33}) + \frac{1}{2}\,J_{11}^{*}\,(s^2\theta_2$$

$$- 3c^2\theta_2) + \frac{1}{2}\,J_{22}^{*}\,s^2\theta_1\,(c^2\theta_2 - 3s^2\theta_2) + 4\,J_{12}^{*}\,s\theta_1s\theta_2c\theta_2$$

$$+ \frac{1}{2}\,J_{33}^{*}\,c^2\theta_1\,(c^2\theta_2 - 3s^2\theta_2) - 4J_{13}^{*}\,c\theta_1s\theta_2\,c\theta_2 + J_{23}^{*}\,s\theta_1c\theta_1\,(3s^2\theta_2$$

$$- c^2\theta_2)] - \frac{1}{2}\sum_{i=1}^{6}\int_0^{\ell_i} P_{xi}\left[\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial w_i}{\partial x_i}\right)^2\right]dx_i$$

$$- \frac{1}{2}\sum_{i=1}^{6}\int_0^{\ell_i}\left\{EI_{zi}\left(\frac{\partial^2 v_i}{\partial x_i^2}\right)^2\left[1 - \frac{5}{2}\left(\frac{\partial v_i}{\partial x_i}\right)^2\right] + EI_{yi}\left(\frac{\partial^2 w_i}{\partial x_i^2}\right)^2\left[1\right.\right.$$

$$\left.\left.- \frac{5}{2}\left(\frac{\partial w_i}{\partial x_i}\right)^2\right]\right\}dx_i \qquad (107)$$

To obtain Lagrange's equations for the rotational motion, we introduce Eq. (107) into Eqs. (35), and obtain

$$[(J_{33}^{*} - J_{22}^{*})\,s\theta_1c\theta_1 + J_{23}^{*}\,(c^2\theta_1 - s^2\theta_1)]\dot\theta_2^2 + (J_{12}^{*}\,s\theta_1 - J_{13}^{*}\,c\theta_1)\dot\theta_1\dot\theta_2$$

$$+ (-J_{23}^* s\theta_1 + J_{33}^* c\theta_1)\dot{\theta}_2\dot{\theta}_3 - (K_2^* s\theta_1 - K_3^* c\theta_1)\dot{\theta}_2$$

$$+ \Omega \{(J_{12}^* c\theta_1 c\theta_2 + J_{13}^* s\theta_1 c\theta_2)\dot{\theta}_1 + [J_{12}^* s\theta_1 s\theta_2 - J_{13}^* c\theta_1 s\theta_2$$

$$+ (J_{33}^* - J_{22}^*)(c^2\theta_1 - s^2\theta_1)c\theta_2 - 4\, J_{23}^* c\theta_1 s\theta_1 c\theta_2]\dot{\theta}_2$$

$$- (J_{33}^* s\theta_1 c\theta_2 + J_{23}^* c\theta_1 c\theta_2)\dot{\theta}_3 - K_2^* c\theta_1 c\theta_2 - K_3^* s\theta_1\, c\theta_2\}$$

$$+ \Omega^2\, [4\, J_{12}^* c\theta_1 s\theta_2 c\theta_2 + (J_{22}^* - J_{33}^*)\, s\theta_1 c\theta_1\, (c^2\theta_2 - 3s^2\theta_2)$$

$$+ 4\, J_{13}^* s\theta_1 s\theta_2 c\theta_2 + J_{23}^* (c^2\theta_1 - s^2\theta_1)(3s^2\theta_2 - c^2\theta_2)]$$

$$- \frac{d}{dt}\, [J_{11}^*\, \dot{\theta}_1 - (J_{12}^* c\theta_1 + J_{13}^* s\theta_1)\dot{\theta}_2 - J_{13}^*\, \dot{\theta}_3 - K_1^*$$

$$+ \Omega\, (J_{11}^* s\theta_2 + J_{12}^* s\theta_1 c\theta_2 - J_{13}^* c\theta_1 c\theta_2)] = 0 \qquad (108a)$$

$$\Omega\, \{[J_{11}^* c\theta_2 - J_{12}^* s\theta_1 s\theta_2 + J_{13}^* c\theta_1 s\theta_2]\dot{\theta}_1 + [-J_{12}^* c\theta_1 c\theta_2 + J_{22}^* s\theta_1 s\theta_2 c\theta_1$$

$$- J_{33}^* c\theta_1 s\theta_1 s\theta_2 - J_{13}^* s\theta_1 c\theta_2 - J_{23}^* s\theta_2(c^2\theta_1 - s^2\theta_1)]\dot{\theta}_2 - (J_{33}^* c\theta_1 s\theta_2$$

$$+ J_{13}^* c\theta_2 - J_{23}^* s\theta_1 s\theta_2)\dot{\theta}_3 - K_1^* c\theta_2 + K_2^* s\theta_1 s\theta_2 - K_3^* c\theta_1 s\theta_2\}$$

$$+ \Omega^2\, [4\, J_{11}^* s\theta_2\, c\theta_2 + 4\, J_{12}^* s\theta_1(c^2\theta_2 - s^2\theta_2) - 4\, J_{22}^*\, s^2\theta_1 s\theta_2 c\theta_2$$

$$- 4\, J_{13}^* c\theta_1(c^2\theta_2 - s^2\theta_2) + 8\, J_{23}^* s\theta_1 c\theta_1 s\theta_2 c\theta_2 - 4\, J_{33}^*\, c^2\theta_1 s\theta_2 c\theta_2]$$

$$- \frac{d}{dt}\, \{[J_{22}^*\, c^2\theta_1 + J_{33}^*\, s^2\theta_1 + 2\, J_{23}^* s\theta_1 c\theta_1]\dot{\theta}_2 - (J_{12}^* c\theta_1 + J_{13}^* s\theta_1)\dot{\theta}_1$$

$$+ (J_{23}^* c\theta_1 + J_{33}^* s\theta_1)\dot{\theta}_3 + K_2^* c\theta_1 + K_3^* s\theta_1 + \Omega\, [-J_{12}^* c\theta_1 s\theta_2$$

$$- J_{22}^* s\theta_1 c\theta_1 c\theta_2 - J_{13}^* s\theta_1 s\theta_2 + J_{23}^* (c^2\theta_1 - s^2\theta_1)c\theta_2$$

$$+ J_{33}^* s\theta_1 c\theta_1 c\theta_2]\} = 0 \qquad (108b)$$

$$- J_{12}^{*}\,\dot{\theta}_1^2 + (J_{12}^{*}\,c^2\theta_1 + J_{13}^{*}\,s\theta_1 c\theta_1)\dot{\theta}_2^2 + [(J_{22}^{*} - J_{11}^{*})c\theta_1 + J_{23}^{*}\,s\theta_1]\dot{\theta}_1\dot{\theta}_2$$

$$+ J_{23}^{*}\,\dot{\theta}_1\dot{\theta}_3 + J_{13}^{*}\,c\theta_1\dot{\theta}_2\dot{\theta}_3 + K_2^{*}\,\dot{\theta}_1 + K_1^{*}\,c\theta_1\dot{\theta}_2$$

$$+ \Omega\,\{[-2\,J_{12}^{*}\,s\theta_2 + (J_{11}^{*} - J_{22}^{*})\,s\theta_1 c\theta_2 + J_{23}^{*}\,c\theta_1 c\theta_2]\dot{\theta}_1$$

$$+ [(J_{22}^{*} - J_{11}^{*})\,c\theta_1 s\theta_2 - 2J_{12}^{*}\,s\theta_1 c\theta_1 c\theta_2 + J_{23}^{*}\,s\theta_1 s\theta_2$$

$$+ J_{13}^{*}\,(c^2\theta_1 - s^2\theta_1)c\theta_2]\dot{\theta}_2 + (J_{23}^{*}\,s\theta_2 - J_{13}^{*}\,s\theta_1 c\theta_2)\dot{\theta}_3 + K_2^{*}\,s\theta_2$$

$$- K_1^{*}\,s\theta_1 c\theta_2\} + \Omega^2\,[-J_{12}^{*}\,(s^2\theta_2 - 3c^2\theta_2) + 4(J_{11}^{*} - J_{22}^{*})s\theta_1 s\theta_2 c\theta_2$$

$$+ J_{12}^{*}\,s^2\theta_1\,(c^2\theta_2 - 3s^2\theta_2) + 4J_{23}^{*}\,c\theta_1 s\theta_2 c\theta_2$$

$$- J_{13}^{*}\,s\theta_1 c\theta_1(c^2\theta_2 - 3s^2\theta_2)] - \frac{d}{dt}\,[-J_{13}^{*}\,\dot{\theta}_1 + (J_{13}^{*}\,c\theta_1 + J_{33}^{*}\,s\theta_1)\dot{\theta}_2$$

$$+ J_{33}^{*}\,\dot{\theta}_3 + K_3^{*} + \Omega\,(-J_{13}^{*}\,s\theta_2 - J_{23}^{*}\,s\theta_1 c\theta_2 + J_{33}^{*}\,c\theta_1 c\theta_2)] = 0 \qquad (108c)$$

Considering Eq. (36) in conjunction with Eqs. (69), (99), (100), and (102), and letting $i = 1$, we obtain the differential equation for v_1

$$\rho_1\,\{(h_{x1} + x_1)\,[(\omega_1 c\alpha + \omega_2 s\alpha)(\omega_1 s\alpha - \omega_2 c\alpha) - 3\Omega^2\,(\ell_{a1}c\alpha + \ell_{a2}s\alpha)(\ell_{a1}s\alpha$$

$$- \ell_{a2}c\alpha)] + (h_{y1} + v_1)\,[(\omega_1 c\alpha + \omega_2 s\alpha)^2 + \omega_3^2 + 2\Omega^2 - 3\Omega^2\,\Big\langle(\ell_{a1}c\alpha$$

$$+ \ell_{a2}s\alpha)^2 + \ell_{a3}^2\Big\rangle] + (h_{z1} + w_1)[\omega_3(\omega_1 s\alpha - \omega_2 c\alpha) - 3\Omega^2\,\ell_{a3}\,(\ell_{a1}s\alpha$$

$$- \ell_{a2}c\alpha)] + \dot{w}_1\,(c\alpha\omega_1 + s\alpha\omega_2)\} - \frac{\partial}{\partial t}\,\rho_1\,[-(w_1 + h_{z1})(\omega_1 c\alpha + \omega_2 s\alpha)$$

$$+ (h_{x1} + x_1)\,\omega_3 + \dot{v}_1]$$

$$- \frac{\partial}{\partial x_1}\Big\langle -\,\Omega^2\,\{(\ell_{c1}s\alpha - \ell_{c2}c\alpha)^2 + \ell_{c3}^2 + 2 - 3[(\ell_{a1}s\alpha - \ell_{a2}c\alpha)^2$$

$$+ \ell_{a3}^2]\} \times \{\frac{1}{2} \rho_1[(h_{x1} + \ell_1)^2 - (h_{x1} + x_1)^2] + m_1 (h_{x1} + \ell_1)\} v_1'$$

$$+ \frac{5}{2} EI_{z1} v_1' v_1''^2\Big\rangle + \frac{\partial^2}{\partial x_1^2} [-EI_{z1} v_1'' (1 - \frac{5}{2} v_1'^2)] + p_{y1} = 0 ,$$

$$0 < x_1 < \ell_1 \tag{109a}$$

which is subject to the boundary conditions

$$v_1(0) = 0 \quad , \quad v_1'(0) = 0 \tag{109b}$$

$$m_1 \{(h_{x1} + \ell_1) [(\omega_1 c\alpha + \omega_2 s\alpha)(\omega_1 s\alpha - \omega_2 c\alpha) - 3 \Omega^2 (\ell_{a1} c\alpha$$

$$+ \ell_{a2} s\alpha)(\ell_{a1} s\alpha - \ell_{a2} c\alpha)] + (h_{y1} + v_1) [(\omega_1 c\alpha + \omega_2 s\alpha)^2$$

$$+ \omega_3^2 + 2 \Omega^2 - 3 \Omega^2 \Big\langle (\ell_{a1} c\alpha + \ell_{a2} s\alpha)^2 + \ell_{a3}^2 \Big\rangle] + (h_{z1} + w_1) [\omega_3(\omega_1 s\alpha$$

$$- \omega_2 c\alpha) - 3 \Omega^2 \ell_{a3} (\ell_{a1} s\alpha - \ell_{a2} c\alpha)] + \dot{w}_1 (c\alpha\omega_1 + s\alpha\omega_2)$$

$$- \frac{\partial}{\partial t} [-(w_1 + h_{z1})(\omega_1 c\alpha + \omega_2 s\alpha) + (h_{x1} + \ell_1) \omega_3 + \dot{v}_1]$$

$$- \Omega^2 \Big\langle (\ell_{c1} s\alpha - \ell_{c2} c\alpha)^2 + \ell_{c3}^2 + 2 - 3[(\ell_{a1} s\alpha - \ell_{a2} c\alpha)^2 + \ell_{a3}^2] \Big\rangle (h_{x1}$$

$$+ \ell_1) v_1'\} + \frac{5}{2} EI_{z1} v_1' v_1''^2$$

$$\left.\begin{aligned} &+ \frac{\partial}{\partial x_1} [EI_{z1} v_1'' (1 - \frac{5}{2} v_1'^2)]\Big|_{x_1 = \ell_1} = 0 \\ &EI_{z1} v_1'' (1 - \frac{5}{2} v_1'^2)\Big|_{x_1 = \ell_1} = 0 \end{aligned}\right\} \tag{109c}$$

Similarly, the differential equation for w_1 is

$$\rho_1 \{(h_{x1} + x_1) [-\omega_3(\omega_1 c\alpha + \omega_2 s\alpha) + 3\Omega^2 \ell_{a3}(\ell_{a1} c\alpha + \ell_{a2} s\alpha)]$$

$$+ (h_{y1} + v_1) [\omega_3(\omega_1 s\alpha - \omega_2 c\alpha) - 3\Omega^2 \ell_{a3} (\ell_{a1} s\alpha - \ell_{a2} c\alpha)]$$

$$+ (h_{z1} + w_1) [\omega_1^2 + \omega_2^2 + 2\Omega^2 - 3\Omega^2 (\ell_{a1}^2 + \ell_{a2}^2)]$$

$$- \dot{v}_1 (\omega_1 c\alpha + \omega_2 s\alpha)\} - \frac{\partial}{\partial t} \rho_1 [(h_{x1} + x_1)(\omega_1 s\alpha - \omega_2 c\alpha)$$

$$+ (h_{y1} + v_1)(\omega_1 c\alpha + \omega_2 s\alpha) + \dot{w}_1] - \frac{\partial}{\partial x_1} \Big\langle - \Omega^2 \{(\ell_{c1} s\alpha - \ell_{c2} c\alpha)^2$$

$$+ \ell_{c3}^2 + 2 - 3[(\ell_{a1} s\alpha - \ell_{a2} c\alpha)^2 + \ell_{a3}^2]\} \times \{\frac{1}{2} \rho_1 [(h_{x1} + \ell_1)^2$$

$$- (h_{x1} + x_1)^2] + m_1 (h_{x1} + \ell_1)\} w_1' + \frac{5}{2} EI_{y1} w_1' w_1''^2 \Big\rangle$$

$$+ \frac{\partial^2}{\partial x_1^2} [-EI_{y1} w_1'' (1 - \frac{5}{2} w_1'^2)] + p_{z1} = 0 \quad , \quad 0 < x_1 < \ell_1 \tag{110a}$$

where w_1 must satisfy the boundary conditions

$$w_1(0) = 0 \quad , \quad w_1'(0) = 0 \tag{110b}$$

$$m_1 \{(h_{x1} + \ell_1) [-\omega_3(\omega_1 c\alpha + \omega_2 s\alpha) + 3\Omega^2 \ell_{a3} (\ell_{a1} c\alpha + \ell_{a2} s\alpha)]$$

$$+ (h_{y1} + v_1) [\omega_3(\omega_1 s\alpha - \omega_2 c\alpha) - 3\Omega^2 \ell_{a3} (\ell_{a1} s\alpha - \ell_{a2} c\alpha)]$$

$$+ (h_{z1} + w_1) [\omega_1^2 + \omega_2^2 + 2\Omega^2 - 3\Omega^2 (\ell_{a1}^2 + \ell_{a2}^2)]$$

$$- \dot{v}_1 (\omega_1 c\alpha + \omega_2 s\alpha) - \frac{\partial}{\partial t} [(h_{x1} + \ell_1)(\omega_1 s\alpha - \omega_2 c\alpha) + (h_{y1} + v_1)(\omega_1 c\alpha$$

$$+ \omega_2 s\alpha) + \dot{w}_1] - \Omega^2 \Big\langle (\ell_{c1} s\alpha - \ell_{c2} c\alpha)^2 + \ell_{c3}^2 + 2 - 3[(\ell_{a1} s\alpha$$

$$- \ell_{a2} c\alpha)^2 + \ell_{a3}^2] \Big\rangle (h_{x1} + \ell_1) w_1'\} + \frac{5}{2} EI_{y1} w_1' w_1''^2$$

$$\left.\begin{aligned} &+ \frac{\partial}{\partial x_1} [EI_{y1} w_1'' (1 - \frac{5}{2} w_1'^2)] \Big|_{x_1 = \ell_1} = 0 \\ &EI_{y1} w_1'' (1 - \frac{5}{2} w_1'^2) \Big|_{x_1 = \ell_1} = 0 \end{aligned}\right\} \tag{110c}$$

In the above

$$\omega_1 = -\Omega(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3) - c\theta_3\dot{\theta}_1 + c\theta_1 s\theta_3\dot{\theta}_2$$

$$\omega_2 = \Omega(s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3) + s\theta_3\dot{\theta}_1 + c\theta_1 c\theta_3\dot{\theta}_2 \tag{111a}$$

$$\omega_3 = \Omega c\theta_1 c\theta_2 + s\theta_1\dot{\theta}_2 + \dot{\theta}_3$$

$$\ell_{a1} = c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3$$

$$\ell_{a2} = -(c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3) \tag{111b}$$

$$\ell_{a3} = c\theta_1 s\theta_2$$

$$\ell_{c1} = -(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3)$$

$$\ell_{c2} = s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3 \tag{111c}$$

$$\ell_{c3} = c\theta_1 c\theta_2$$

The equations of motion and boundary conditions associated with the booms 2, 3, and 4 are obtained from Eqs. (109) and (110) by replacing α by $\pi-\alpha$, $\pi+\alpha$, and $2\pi-\alpha$, respectively, and, of course, changing the subscripts of v_1, and w_1 accordingly.

Following the same procedure as that used to obtain Eqs. (109) and (110), the equation of motion and boundary conditions for v_5 are

$$\rho_5 \{(h_{x5} + x_5)\,[-\omega_1(\omega_2 s\beta + \omega_3 c\beta) + 3\Omega^2\ell_{a1}(\ell_{a2}s\beta + \ell_{a3}c\beta)]$$
$$+ (h_{y5} + v_5)[\omega_2^2 + \omega_3^2 + 2\Omega^2 - 3\Omega^2\,(\ell_{a2}^2 + \ell_{a3}^2)] + (h_{z5} + w_5)[\omega_1(\omega_3 s\beta$$

$$- \omega_2 c\beta) - 3\Omega^2 \ell_{a1} (\ell_{a3} s\beta - \ell_{a2} c\beta)] + (\omega_3 c\beta + \omega_2 s\beta)\dot{w}_5\}$$

$$- \frac{\partial}{\partial t} \rho_5 [(h_{x5} + x_5)(\omega_2 c\beta - \omega_3 s\beta) - (h_{z5} + w_5)(\omega_2 s\beta + \omega_3 c\beta) + \dot{v}_5]$$

$$- \frac{\partial}{\partial x_5} \Big\langle -\Omega^2 \{\ell_{c1}^2 + (\ell_{c2} c\beta - \ell_{c3} s\beta)^2 + 2 - 3[\ell_{a1}^2 + (\ell_{a2} c\beta - \ell_{a3} s\beta)^2]\} \times$$

$$\{\frac{1}{2} \rho_5 [(h_{x5} + \ell_5)^2 - (h_{x5} + x_5)^2]\} v_5' + \frac{5}{2} EI_{z5} v_5' v_5''^2 \Big\rangle$$

$$+ \frac{\partial^2}{\partial x_5^2} [-EI_{z5} v_5'' (1 - \frac{5}{2} v_5'^2)] + p_{y5} = 0 \quad , \quad 0 < x_5 < \ell_5 \tag{112a}$$

$$v_5(0) = 0 \quad , \quad v_5'(0) = 0 \tag{112b}$$

$$\left.\begin{aligned} &\frac{5}{2} EI_{z5} v_5' v_5''^2 + \frac{\partial}{\partial x_5} [EI_{z5} v_5'' (1 - \frac{5}{2} v_5'^2)]\Big|_{x_5 = \ell_5} = 0 \\ &EI_{z5} v_5'' (1 - \frac{5}{2} v_5'^2)\Big|_{x_5 = \ell_5} = 0 \end{aligned}\right\} \tag{112c}$$

and those for w_5 are

$$\rho_5 \{(h_{x5} + x_5) [(\omega_3 s\beta - \omega_2 c\beta)(\omega_2 s\beta + \omega_3 c\beta) - 3\Omega^2 (\ell_{a3} s\beta - \ell_{a2} c\beta)(\ell_{a2} s\beta$$

$$+ \ell_{a3} c\beta)] - (h_{y5} + v_5)[\omega_1 (\omega_2 c\beta - \omega_3 s\beta) - 3\Omega^2 \ell_{a1} (\ell_{a2} c\beta - \ell_{a3} s\beta)$$

$$+ (h_{z5} + w_5) \Big\langle \omega_1^2 + (\omega_2 s\beta + \omega_3 c\beta)^2 + 2\Omega^2 - 3\Omega^2 [\ell_{a1}^2 + (\ell_{a2} s\beta$$

$$+ \ell_{a3} c\beta)^2] \Big\rangle - \dot{v}_5 (\omega_2 s\beta + \omega_3 c\beta)\}$$

$$- \frac{\partial}{\partial t} \rho_5 [-(h_{x5} + x_5)\omega_1 + (h_{y5} + v_5)(\omega_3 c\beta + \omega_2 s\beta)]$$

$$- \frac{\partial}{\partial x_5} \Big\langle -\Omega^2 \{\ell_{c1}^2 + (\ell_{c2} c\beta - \ell_{c3} s\beta)^2 + 2 - 3[\ell_{a1}^2 + (\ell_{a2} c\beta - \ell_{a3} s\beta)^2]\} \times$$

$$\{\frac{1}{2} \rho_5 [(h_{x5} + \ell_5)^2 - (h_{x5} + x_5)^2]\} w_5' + \frac{5}{2} EI_{y5} w_5' w_5''^2 \Big\rangle$$

$$+\frac{\partial^2}{\partial x_5^2}\left[-EI_{y5}w_5''\left(1-\frac{5}{2}w_5'^2\right)\right]+p_{z5}=0 \quad , \quad 0 < x_5 < \ell_5 \tag{113a}$$

$$w_5(0)=0 \quad , \quad w_5'(0)=0 \tag{113b}$$

$$\left.\begin{aligned}&\frac{5}{2}EI_{y5}\,w_5'w_5''^2+\frac{\partial}{\partial x_5}\left[EI_{y5}w_5''\left(1-\frac{5}{2}w_5'^2\right)\right]\Big|_{x_5=\ell_5}=0\\&EI_{y5}w_5''\left(1-\frac{5}{2}w_5'^2\right)\Big|_{x_5=\ell_5}=0\end{aligned}\right\} \tag{113c}$$

The equations for boom 6 are obtained from Eqs. (112) and (113) by an appropriate change in subscript, and by replacing β by $\pi+\beta$.

b. Perturbation solution of the equilibrium problem.

The first problem in attempting a solution of the equations of motion, Eqs. (108), (109), (112), and (113), is to identify the equilibrium configurations. To this end, we must let all the velocities and accelerations equal to zero in these equations. This leaves us three transcendental equations for the rotations θ_j ($j = 1,2,3$) and twelve nonlinear differential equations for the elastic displacements v_i, w_i ($i = 1,2,\ldots,6$).

We shall consider the solution of the nonlinear equilibrium problem in the form

$$v_{i0}(x_i)=v_{i00}(x_i)+v_{i01}(x_i),\ w_{i0}(x_i)=w_{i00}(x_i)+w_{i01}(x_i), \qquad i=1,2,\ldots,6 \tag{114}$$

where the third subscripts on the right side of Eqs. (114) indicate the solution of the linearized problem if the subscript is zero and relatively small perturbations if the subscript is one. It follows that the inertia matrix of the deformed body can be written as

$$[J^{(0)}] = [J^{(0)}]_0 + [J^{(0)}]_1 = \sum_{i=0}^{6} [\ell_i]^T[J_i]_0[\ell_i] + \sum_{i=1}^{6} [\ell_i]^T[J_i]_1[\ell_i] \quad (115)$$

where $[J^{(0)}]_0$ is the inertia matrix as if the body was entirely rigid, in which

$$\begin{aligned} &J_{0110} = A_0,\ J_{0220} = B_0,\ J_{0330} = C_0 \\ &J_{0120} = J_{0210} = J_{0130} = J_{0310} = J_{0230} = J_{0320} = 0 \end{aligned} \quad (116)$$

are the moments of inertia of the rigid hub, and

$$\begin{aligned} J_{i110} &= \int_0^{\ell_i} \rho_i(h_{yi}^2+h_{zi}^2)dx_i+m_i(h_{yi}^2+h_{zi}^2) \\ J_{i220} &= \int_0^{\ell_i} \rho_i[(h_{xi}+x_i)^2+h_{zi}^2]dx_i+m_i[(h_{xi}+\ell_i)^2+h_{zi}^2] \\ J_{i330} &= \int_0^{\ell_i} \rho_i[(h_{xi}+x_i)^2+h_{yi}^2]dx_i+m_i[(h_{xi}+\ell_i)^2+h_{yi}^2] \qquad i = 1,2,\ldots,6 \\ J_{i120} &= J_{i210} = -\int_0^{\ell_i} \rho_i(h_{xi}+x_i)h_{yi}dx_i-m_i(h_{xi}+\ell_i)h_{yi} \\ J_{i130} &= J_{i310} = -\int_0^{\ell_i} \rho_i(h_{xi}+x_i)h_{zi}dx_i-m_i(h_{xi}+\ell_i)h_{zi} \\ J_{i230} &= J_{i320} = -\int_0^{\ell_i} \rho_i h_{yi}h_{zi}dx_i-m_i h_{yi}h_{zi} \end{aligned} \quad (117)$$

are the moments of inertia of the appendages when in undeformed state, expressed in terms of local coordinates. Moreover, $[J^{(0)}]_1$ is the change in the inertia matrix due to first order elastic displacements, which has the elements

$$J_{i111} = \int_0^{\ell_i} \rho_i (2h_{yi}v_{i00}+v_{i00}^2+2h_{zi}w_{i00}+w_{i00}^2)dx_i$$
$$+ m_i(2h_{yi}v_{i00}+v_{i00}^2+2h_{zi}w_{i00}+w_{i00}^2)\Big|_{x_i = \ell_i}$$

$$J_{i221} = \int_0^{\ell_i} \rho_i (2h_{zi}w_{i00}+w_{i00}^2)dx_i+m_i(2h_{zi}w_{i00}+w_{i00}^2)\Big|_{x_i = \ell_i}$$

$$J_{i331} = \int_0^{\ell_i} \rho_i (2h_{yi}v_{i00}+v_{i00}^2)dx_i+m_i(2h_{yi}v_{i00}+v_{i00}^2)\Big|_{x_i = \ell_i}$$

$$J_{i121} = J_{i211} = -\int_0^{\ell_i} \rho_i (h_{xi}+x_i)v_{i00}dx_i - m_i(h_{xi}+x_i)v_{i00}\Big|_{x_i = \ell_i}$$

$$J_{i131} = J_{i311} = -\int_0^{\ell_i} \rho_i (h_{xi}+x_i)w_{i00}dx_i - m_i(h_{xi}+x_i)w_{i00}\Big|_{x_i = \ell_i}$$

$$J_{i231} = J_{i321} = -\int_0^{\ell_i} \rho_i (h_{yi}w_{i00}+h_{zi}v_{i00}+v_{i00}w_{i00})dx_i$$

$$-m_i(h_{yi}w_{i00}+h_{zi}v_{i00}+v_{i00}w_{i00})\Big|_{x_i = \ell_i}$$

$$i = 1,2,\ldots,6 \qquad (118)$$

To linearize the algebraic equations for the angles θ_j ($j = 1,2,3$) we would have to assume that the angles are small. This, however, is not always true for an arbitrary satellite, so that linearization cannot be justified. Fortunately, it is not difficult to solve the nonlinear algebraic equations for the angles θ_j ($j = 1,2,3$) by means of Newton-Raphson method for the moments of inertia given. As a first iteration, we insert the moments of inertia of the satellite regarded as rigid into the tree transcendental equations for θ_j $(j = 1,2,3)$, and obtain some preliminary values for these angles. Hence, letting all terms in Eqs. (108) involving time derivatives equal to zero, we obtain

$$4J^*_{120}c\theta_1 s\theta_2 c\theta_2 + (J^*_{220} - J^*_{330})s\theta_1 c\theta_1 \ (c^2\theta_1 - 3s^2\theta_2) + 4J^*_{130}s\theta_1 s\theta_2 c\theta_2$$
$$+ J^*_{230}(c^2\theta_1 - s^2\theta_1)(3s^2\theta_2 - c^2\theta_1) = 0 \tag{119a}$$

$$4J^*_{110}s\theta_2 c\theta_2 + 4J^*_{120}s\theta_1(c^2\theta_2 - s^2\theta_2) - 4J^*_{220}s^2\theta_1 s\theta_2 c\theta_2 - 4J^*_{130}c\theta_1(c^2\theta_2$$
$$- s^2\theta_2) + 8J^*_{230}s\theta_1 c\theta_1 s\theta_2 c\theta_2 - 4J^*_{330}c^2\theta_1 s\theta_2 c\theta_2 = 0 \tag{119b}$$

$$-J^*_{120}(s^2\theta_2 - 3c^2\theta_2) + 4(J^*_{110} - J^*_{220})s\theta_1 s\theta_2 c\theta_2 + J^*_{120}s^2\theta_1(c^2\theta_2 - 3s^2\theta_2)$$
$$+ 4J^*_{230}c\theta_1 s\theta_2 c\theta_2 - J^*_{130}s\theta_1 c\theta_1(c^2\theta_2 - 3s^2\theta_2) = 0 \tag{119c}$$

where $[J^*]_0 = [\theta]_3^T \, [J^{(0)}]_0 \, [\theta]_3$ in which $[\theta]_3$ is given by the first of Eqs. (104). Regarding the angles θ_j $(j = 1,2,3)$ as known constants, we can linearize Eqs. (109), (110), (112), and (113) with respect to v_i, w_i and their derivatives, and solve for the perturbed elastic displacements. Hence, inserting Eqs. (114) into (109), we obtain the equations for v_{100} in the form

$$\rho_1 \ \{(h_{x1} + x_1) \ [(\omega_{10}c\alpha + \omega_{20}s\alpha)(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2(\ell_{a10}c\alpha$$
$$+ \ell_{a20}s\alpha)(\ell_{a10}s\alpha - \ell_{a20}c\alpha)] + (h_{y1} + v_{100}) \ [(\omega_{10}c\alpha + \omega_{20}s\alpha)^2 + \omega_{30}^2$$
$$+ 2\Omega^2 - 3\Omega^2 \left\langle(\ell_{a10}c\alpha + \ell_{a20}s\alpha)^2 + \ell_{a30}^2\right\rangle] + (h_{z1} + w_{100})[\omega_{30}(\omega_{10}s\alpha$$

$$- \omega_{20}c\alpha) - 3\Omega^2 \ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]\} - v''_{100}\, \rho_1\, (h_{x1}$$

$$+ x_1)\, \Omega^2 \{(\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell^2_{c30} + 2 - 3[(\ell_{a10}s\alpha - \ell_{a20}c\alpha)^2$$

$$+ \ell^2_{a30}]\} + v''_{100}\, \Omega^2 \{\tfrac{1}{2}\rho_1\, [(h_{x1} + \ell_1)^2 - (h_{x1} + x_1)^2]$$

$$+ m_1(h_{x1} + \ell_1)\}\, \{(\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell^2_{c30} + 2 - 3[(\ell_{a10}s\alpha$$

$$- \ell_{a20}c\alpha)^2 + \ell^2_{a30}]\} - EI\, v''''_{100} \tag{120a}$$

where v_{100} is subject to the boundary conditions

$$v_{100}(0) = 0 \quad , \quad v'_{100}(0) = 0 \tag{120b}$$

$$m_1 \{(h_{x1} + \ell_1)\, [(\omega_{10}c\alpha + \omega_{20}s\alpha)(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\, (\ell_{a10}c\alpha$$

$$+ \ell_{a20}s\alpha)(\ell_{a10}s\alpha - \ell_{a20}c\alpha)] + (h_{y1} + v_{100})[(\omega_{10}c\alpha + \omega_{20}s\alpha)^2$$

$$+ \omega^2_{30} + 2\Omega^2 - 3\Omega^2 \left\langle(\ell_{a10}c\alpha + \ell_{a20}s\alpha)^2 + \ell^2_{a30}\right\rangle\,] + (h_{z1}$$

$$+ w_{100})\, [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\, \ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$\left.\begin{aligned} &- v'_{100}\, \Omega^2\, (h_{x1} + \ell_1) \left\langle(\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell^2_{c30} + 2 - 3[(\ell_{a10}s\alpha \right. \\ &\left. - \ell_{a20}c\alpha)^2 + \ell^2_{a30}\right\rangle\} + EI\, v'''_{100}\Big|_{x_1 = \ell_1} = 0 \\ &\qquad\qquad EI\, v''_{100}\Big|_{x_1 = \ell_1} = 0 \end{aligned}\right\} \tag{120c}$$

The quantities ω_{j0}, ℓ_{aj0} and ℓ_{cj0} $(j = 1,2,3)$ appearing in Eqs. (120) are to be calculated by using θ_j $(j = 1,2,3)$ as given by Eqs. (119). Note that now primes designate total derivatives with respect to x_1 because v_{100} depends on x alone and not on t. Similarly, for w_{100} we have

$$\rho_1 \{(h_{x1} + x_1)[-\omega_{30}(\omega_{10}c\alpha + \omega_{20}s\alpha) + 3\Omega^2\ell_{a30}(\ell_{a10}c\alpha + \ell_{a20}s\alpha)]$$

$$+ (h_{y1} + v_{100}) [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$+ (h_{z1} + w_{100}) [(\omega_{10}^2 + \omega_{20}^2) + 2\Omega^2 - 3\Omega^2(\ell_{a10}^2 + \ell_{a20}^2)]\}$$

$$- \rho_1 w'_{100} (h_{x1} + x_1) \Omega^2 \{(\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha$$

$$- \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\} + w''_{100} \Omega^2 \{\frac{1}{2} \rho_1 [(h_{x1} + \ell_1)^2 - (h_{x1} + x_1)^2]$$

$$+ m_1(h_{x1} + \ell_1)\} \{(\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha$$

$$- \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\} - EI_{y1} w''''_{100} = 0 \qquad (121a)$$

where w_{100} is subject to the boundary conditions

$$w_{100}(0) = 0 \quad , \quad w'_{100}(0) = 0 \qquad (121b)$$

$$m_1 \{(h_{x1} + \ell_1) [-\omega_{30}(\omega_{10}c\alpha + \omega_{20}s\alpha) + 3\Omega^2\ell_{a3}(\ell_{a10}c\alpha + \ell_{a20}s\alpha)]$$

$$+ (h_{y1} + v_{100}) [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\ell_{a3}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$+ (h_{z1} + w_{100}) [\omega_{10}^2 + \omega_{20}^2 + 2\Omega^2 - 3\Omega^2 (\ell_{a10}^2 + \ell_{a20}^2)]$$

$$- w'_{100} \Omega^2 (h_{x1} + \ell_1) \left\langle (\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha\right.$$

$$\left.\left.\left. - \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\right\rangle \right\} + EI\, w'''_{100}\right|_{x_1 = \ell_1} = 0$$

$$\left. EI\, w''_{100}\right|_{x_1 = \ell_1} = 0 \qquad (121c)$$

The differential equations and boundary conditions for v_{i00} and w_{i00} ($i = 2,3,4$) are obtained from Eqs. (120) and (121) by replacing the subscripts of v_{100} and w_{100} by the appropriate ones and the angle α by $\pi-\alpha$, $\pi+\alpha$, and $2\pi-\alpha$, respectively. On the other hand, the differential equations and boundary conditions for v_{500} and w_{500} are obtained from Eqs. (112) and (113) in the form

$$\rho_5 \{(h_{x5} + x_5) [-\omega_{10}(\omega_{20}s\beta + \omega_{30}c\beta) + 3\Omega^2\ell_{a10}(\ell_{a20}s\beta + \ell_{a30}c\beta)]$$
$$+ (h_{y5} + v_{500}) [\omega_{20}^2 + \omega_{30}^2 + 2\Omega^2 - 3\Omega^2 (\ell_{a20}^2 + \ell_{a30}^2)]$$
$$+ (h_{z5} + w_{500}) [\omega_{10}(\omega_{30}s\beta - \omega_{20}c\beta) - 3\Omega^2\ell_{a10}(\ell_{a30}s\beta - \ell_{a20}c\beta)]\}$$
$$- v_{500}' \rho 5\ \Omega^2 (h_{x5} + x_5) \{\ell_{c10}^2 + (\ell_{c20}c\beta - \ell_{c30}s\beta)^2$$
$$+ 2 - 3[\ell_{a10}^2 + (\ell_{a20}c\beta - \ell_{a30}s\beta)^2]\} + v_{500}'' \Omega^2 \{\frac{1}{2} \rho_5 [(h_{x5}$$
$$+ \ell_5)^2 - (h_{x5} + x_5)^2]\} \quad \{\ell_{c10}^2 + (\ell_{c20}c\beta - \ell_{c30}s\beta)^2 + 2$$
$$- 3[\ell_{a10}^2 + (\ell_{a20}c\beta - \ell_{a30}s\beta)^2]\} - EI_{z5} v_{500}'''' = 0 \qquad (122a)$$

where v_{500} satisfies the boundary conditions

$$v_{500}(0) = 0 \quad , \quad v_{500}'(0) = 0 \qquad (122b)$$

$$EI_{z5} v_{500}'''\Big|_{x_5 = \ell_5} = 0 \quad , \quad EI_{z5} v_{500}''\Big|_{x_5 = \ell_5} = 0 \qquad (122c)$$

as well as

$$\rho_5 \{(h_{x5} + x_5) [(\omega_{30}s\beta - \omega_{20}c\beta)(\omega_{20}s\beta + \omega_{30}c\beta) - 3\Omega^2(\ell_{a30}s\beta$$

$$- \ell_{a20}c\beta)(\ell_{a20}s\beta + \ell_{a30}c\beta)] - (h_{y5} + v_{500}) [\omega_{10}(\omega_{20}c\beta - \omega_{30}s\beta)$$

$$- 3\Omega^2 \ell_{a10}(\ell_{a20}c\beta - \ell_{a30}s\beta) + (h_{z5} + w_{500}) \{\omega_{10}^2 + (\omega_{20}s\beta + \omega_{30}c\beta)^2$$

$$+ 2\Omega^2 - 3\Omega^2 [\ell_{a10}^2 + (\ell_{a20}s\beta + \ell_{a30}c\beta)^2]\} - w'_{500} \rho_5 \Omega^2 (h_{x5}$$

$$+ x_5) \{\ell_{c10}^2 + (\ell_{c20}c\beta - \ell_{c30}s\beta)^2 + 2 - 3[\ell_{a10}^2 + (\ell_{a20}c\beta - \ell_{a30}s\beta)^2]\}$$

$$+ w''_{500} \Omega^2 \{\frac{1}{2} \rho_5 [(h_{x5} + \ell_5)^2 - (h_{x5} + x_5)^2]\} \{\ell_{c10}^2 + (\ell_{c20}c\beta$$

$$- \ell_{c30}s\beta)^2 + 2 - 3[\ell_{a10}^2 + (\ell_{a20}c\beta - \ell_{a30}s\beta)^2]\} - EI_{z5} w''''_{500} = 0 \quad (123a)$$

$$w_{500}(0) = 0 \quad , \quad w'_{500}(0) = 0 \quad (123b)$$

$$EI_{y5} w'''_{500}\Big|_{x_5 = \ell_5} = 0 \quad , \quad EI_{y5} w''_{500}\Big|_{x_5 = \ell_5} = 0 \quad (123c)$$

The differential equations and boundary conditions for v_{600} and w_{600} are obtained by replacing in Eqs. (122) and (123) v_{500} and w_{500} by v_{600} and w_{600} and β by $\pi+\beta$, respectively.

On the other hand, the boundary-value problem for the perturbation v_{101} is defined by the differential equation

$$\rho_1 v_{101} \{(\omega_{10}c\alpha + \omega_{20}s\alpha)^2 + \omega_{30} + 2\Omega^2 - 3\Omega^2 [(\ell_{a10}c\alpha + \ell_{a20}s\alpha)^2 + \ell_{a30}^2]\}$$

$$+ \rho_1 w_{101} [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2 \ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$+ v'_{101} \{-\rho_1 \Omega^2 (h_{x1} + x_1) \Big\langle (\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell_{c30}^2 + 2$$

and that for the perturbation w_{101} by

$$\rho_1 v_{101}\, [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2 \ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$+ \rho_1 w_{101}\ \ [\omega_{10}^2 + \omega_{20}^2\ + 2\Omega^2 - 3\Omega^2\ (\ell_{a10}^2 + \ell_{a20}^2)]$$

$$+ w'_{101}\ \{-\rho_1\Omega^2\ (h_{x1} + x_1)\ \Big\langle(\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell_{c30}^2 + 2$$

$$- 3[(\ell_{a10}s\alpha - \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\Big\rangle\ \ + EI_{y1}\ [10\ w''_{100}w'''_{100}$$

$$+ 5\ w''''_{100}w'_{100}]\} + w''_{101}\ \{\Omega^2\Big\langle\frac{1}{2}\rho_1\ [(h_{x1} + \ell_1)^2 - (h_{x1}\ + x_1)^2]$$

$$+ m_1(h_{x1} + \ell_1)\Big\rangle\ \Big\langle(\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha$$

$$- \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\Big\rangle\ + EI_{y1}\ [\frac{15}{2}\ (w''_{100})^2 + 10\ w'_{100}w'''_{100}]\}$$

$$+ EI_{y1}w'''_{101}\ \ (10\ w'_{100}w''_{100})$$

$$- EI_{y1}\ w''''_{101}\ \ [1 - \frac{5}{2}\ (w'_{100})^2] = -\ EI_{y1}\ [\frac{5}{2}\ (w''_{100})^3 + 10\ w'_{100}w''_{100}w'''_{100}$$

$$+ \frac{5}{2}\ w''''_{100}(w'_{100})^2] \tag{125a}$$

$$w_{101}(0) = 0\quad ,\quad w'_{101}(0) = 0 \tag{125b}$$

$$m_1 v_{101}\ [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\ell_{a30}(\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$

$$+ m_1 w_{101}\ \ [\ \omega_{10}^2 + \omega_{20}^2\ \ + 2\Omega^2 - 3\Omega^2\ (\ell_{a10}^2 + \ell_{a20}^2)]$$

$$+ w'_{101}\ \{m\Omega^2\ (h_{x1} + \ell_1)\ \Big\langle\ (\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha$$

$$- \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\Big\rangle - \frac{5}{2}\ EI_{y1}\ [2w'_{100}w'''_{100} + (w''_{100})^2]$$

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$$
\left.
\begin{aligned}
&- EI_{y1}\, w''_{101}\, (5w'_{100}w''_{100}) \\
&+ EI_{y1}\, w'''_{101}\, [1 - \tfrac{5}{2}\, (w'_{100})^2] = \tfrac{5}{2}\, EI_{y1}\, [(w'_{100})^2\, w'''_{100} \\
&\qquad + 2w'_{100}\, (w''_{100})^2]\Big|_{x_5 = \ell_5} \\
- EI_{y1}\, &[(1 - \tfrac{5}{2}\, (w'_{10})^2]\, w''_{101} \\
&\qquad - 5w'_{100}w''_{100}w'_{101}] = - \tfrac{5}{2}\, EI_{y1}(w'_{100})^2\, w''_{100}\Big|_{x_5 = \ell_5}
\end{aligned}
\right\} \qquad (125c)
$$

with companion equations for v_{i01} and w_{i01} $(i = 2,3,4)$. In a like manner

$$
\begin{aligned}
&\rho_5 v_{501}\, [\omega_{20}^2 + \omega_{30}^2 + 2\Omega^2 - 3\Omega^2\, (\ell_{a20}^2 + \ell_{a30}^2)] \\
&\quad + \rho_5 w_{501}\, [\omega_{10}(\omega_{30} s\beta - \omega_{20} c\beta) - 3\Omega^2 \ell_{a10}(\ell_{a30} s\beta - \ell_{a20} c\beta)] \\
&\quad + v'_{501}\, \{-\rho_5\Omega^2\, (h_{x5} + x_5)\, \Big\langle \ell_{c10}^2 + (\ell_{c20} c\beta - \ell_{c30} s\beta)^2 + 2 \\
&\quad - 3[\ell_{a10}^2 + (\ell_{a20} c\beta - \ell_{a30} s\beta)^2] \Big\rangle + EI_{z5}\, [10\, v''_{500} v'''_{500} + 5\, v''''_{500} v'_{500}]\} \\
&\quad + v''_{501}\, \{\Omega^2 \Big\langle \tfrac{1}{2}\, \rho_5\, [(h_{x5} + \ell_5)^2 - (h_{x5} + x_5)^2] \Big\rangle\, \Big\langle \ell_{c10}^2 + (\ell_{c20} c\beta \\
&\quad - \ell_{c30} s\beta)^2 + 2 - 3[\ell_{a10}^2 + (\ell_{a20} c\beta - \ell_{a30} s\beta)^2] \Big\rangle + EI_{z5}\, [\tfrac{15}{2}\, (v''_{500})^2 \\
&\quad + 10\, v'_{500} v'''_{500}]\} + EI_{z5}\, v'''_{501}\, (10\, v'_{500} v''_{500}) \\
&\quad - EI_{z5}\, v''''_{501}\, [1 - \tfrac{5}{2}\, (v'_{500})^2] = - EI_{z5}\, [\tfrac{5}{2}\, (v''_{500})^3 \\
&\quad + 10\, v'_{500} v''_{500} v'''_{500} + \tfrac{5}{2}\, v''''_{500}\, (v'_{500})^2]
\end{aligned} \qquad (126a)
$$

$$
v_{501}(0) = 0 \quad , \quad v'_{501}(0) = 0 \qquad (126b)
$$

$$
\left.
\begin{aligned}
& EI_{z5} \{-\tfrac{5}{2} [2 \, v'_{500} v'''_{500} + (v''_{500})^2] \, v'_{501} - 5 \, v'_{500} v''_{500} v'_{501} \\
& \quad + [1 - \tfrac{5}{2} (v'_{500})^2] \, v'''_{501}\} \\
& \quad = \tfrac{5}{2} EI_{z5} \, [(v'_{500})^2 \, v'''_{500} + 2 \, v'_{500} \, (v''_{500})^2] \\
& - EI_{z5} \, \{[(1 - \tfrac{5}{2} (v'_{500})^2] \, v''_{501} \\
& \qquad - 5 \, v'_{500} v''_{500} v'_{501}\} = - \tfrac{5}{2} EI_{z5} \, (v'_{500})^2 \, v''_{500}
\end{aligned}
\right\} \text{ at } x_5 = \ell_5 \qquad (126c)
$$

and

$$
\begin{aligned}
& \rho_5 v_{501} \, [\omega_{10}(\omega_{20} c\beta - \omega_{30} s\beta) - 3\Omega^2 \ell_{a10}(\ell_{a20} c\beta - \ell_{a30} s\beta)] \\
& \quad + \rho_5 w_{501} \left\langle \omega_{10}^2 + (\omega_{20} s\beta + \omega_{30} c\beta)^2 + 2\Omega^2 - 3\Omega^2 \, [\ell_{a10}^2 + (\ell_{a20} s\beta \right. \\
& \quad \left. + \ell_{a30} c\beta)^2] \right\rangle + w'_{501} \, \{- \rho_5 \Omega^2 \, (h_{x5} + x_5) \left\langle \ell_{c10}^2 + (\ell_{c20} c\beta \right. \\
& \quad \left. - \ell_{c30} s\beta)^2 + 2 - 3[\ell_{a10}^2 + (\ell_{a20} c\beta - \ell_{a30} s\beta)^2] \right\rangle \\
& \quad + EI_{y5} \, [10 \, w''_{500} w'''_{500} + 5 \, w''''_{500} w'_{500}]\} \\
& \quad + w''_{501} \, \{\Omega^2 \left\langle \tfrac{1}{2} \rho_5 \, [(h_{x5} + \ell_5)^2 - (h_{x5} + x_5)^2] \right\rangle \left\langle \ell_{c10}^2 + (\ell_{c20} c\beta \right. \\
& \quad \left. - \ell_{c30} s\beta)^2 + 2 - 3[\ell_{a10}^2 + (\ell_{a20} c\beta - \ell_{a30} s\beta)^2] \right\rangle + EI_{y5} \, [\tfrac{15}{2} \, (w''_{500})^2 \\
& \quad + 10 \, w'_{500} w'''_{500}]\} + EI_{y5} \, w'''_{501} \, (10 \, w'_{500} w''_{500}) \\
& \quad - EI_{y5} \, w''''_{501} \, [1 - \tfrac{5}{2} (w'_{500})^2] = - EI_{y5} \, [\tfrac{5}{2} \, (w''_{500})^3 \\
& \quad + 10 \, w'_{500} w''_{500} w'''_{500} + \tfrac{5}{2} \, w''''_{500} \, (w'_{500})^2]
\end{aligned}
\qquad (127a)
$$

$$
w_{501}(0) = 0 \quad , \quad w'_{501}(0) = 0 \qquad (127b)
$$

$$
\left.\begin{aligned}
&EI_{y5}\{-\frac{5}{2}[2\,w'_{500}w'''_{500}+(w'_{500})^2]\,w'_{501}-5\,w'_{500}w''_{500}w'_{501}\\
&\qquad+[1-\frac{5}{2}(w'_{500})^2]\,w'''_{501}\}\\
&\quad=\frac{5}{2}EI_{y5}\,[(w'_{500})^2\,w'''_{500}+2\,w'_{500}\,(w''_{500})^2]\\
&-EI_{y5}\,\{[(1-\frac{5}{2}(w'_{500})^2]\,w''_{501}\\
&\qquad\quad-5\,w'_{500}w''_{500}w'_{501}\}=-\frac{5}{2}EI_{y5}\,(w'_{500})^2\,w''_{500}
\end{aligned}\right\}\text{ at } x_5=\ell_5 \qquad (127c)
$$

It should be pointed out that this particular perturbation scheme enables us to solve first Eqs. (119) for the first approximation rotations θ_{j00} ($j = 1,2,3$) independently of the elastic displacements. The rotations are then introduced into Eqs. (120) - (123), yielding the first approximation for the elastic displacements v_{i00} and w_{i00} ($i = 1,2,\ldots,6$) independently of one another. Inserting the first approximation v_{i00} and w_{i00} into Eqs. (124) - (127), we can obtain the corrections v_{i01} and w_{i01} to the elastic displacements. The sums of these solutions yield v_{i0} and w_{i0} ($i = 1,2,\ldots,6$) according to Eqs. (114). Then, inserting v_{i0} and w_{i0} ($i = 1,2,\ldots,6$) back into Eqs. (119), we obtain the angles θ_{j0} ($j = 1,2,3$). In the vast majority of cases, this approximation is sufficient. If not, having the new angles, we can iterate once more to improve the elastic displacements v_{i0} and w_{i0}, as well as the angles θ_{j0}.

c. Liapunov stability analysis and the eigenvalue problem.

The values θ_{j0}, v_{i0}, and w_{i0} obtained above, together with sets of admissible functions $\phi_j(x_i)$ and $\psi_j(x_i)$, are subsequently introduced into Eqs. (49), (51), and (53), to obtain the coefficients m_{jk}, f_{jk}, and

k_{jk}. The coefficients m_{jk} and k_{jk} yield the symmetric matrices [m] and [k], whereas using Eq. (57) the coefficients f_{jk} yield the skew symmetric matrix [g].

From Sec. 6, if [k] represents a positive definite matrix, then the nontrivial equilibrium is asympototically stable. On the other hand, to obtain the natural frequencies, we must solve the eigenvalue problem in the form (66). However, before the nontrivial equilibrium can be determined, the system stability tested, and the natural frequencies calculated, it is desirable to use specific values for the system parameters. This is done in the next section.

d. The shortening of the projections effect

As indicated in Sec. 2, the booms are assumed to be inextensional, so that there is no longitudinal vibration. However, because of the transverse displacements, there is a shortening of the projection on the nominal axis of any element of length of the boom. In fact, from Eq. (14), the change in length of projection of any element of length dx_i is

$$du_i = -\frac{1}{2}\left[\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial w_i}{\partial x_i}\right)^2\right]dx_i \quad , \quad i = 1,2,\ldots,6 \tag{128}$$

We shall treat this shortening as a perturbation of the spatial coordinate x_i, so that we can write

$$x_i = x_{i0} + x_{i1} \quad , \quad 0 \leq x_{i0} \leq \ell_i \quad , \quad i = 1,2,\ldots,6 \tag{129}$$

where x_{i0} are the original spatial coordinates and x_{i1} are the perturbations. From Eqs. (128), however, we conclude that the shortening is a second-order effect. Hence, it will not affect Eqs. (120) - (123) except that x_i are to be regarded in these equations as x_{i0} $(i = 1,2,\ldots,6)$. This enables us to solve for v_{i00} and w_{i00} and write the shortening of the projections in the form

$$x_{i1} = \int_0^{x_{i0}} du_i = -\frac{1}{2}\int_0^{x_{i0}} \left[\left(\frac{\partial v_{i00}}{\partial \xi_i}\right)^2 + \left(\frac{\partial w_{i00}}{\partial \xi_i}\right)^2\right] d\xi_i \,, \qquad i = 1,2,\ldots,6 \tag{130}$$

where ξ_i is a dummy variable. On the other hand, the perturbation equations, Eqs. (124) - (127), must be modified to account for the shortening effect. For example, the boundary-value problem for v_{i01} becomes

$$\rho_1 v_{101} \{(\omega_{10}c\alpha + \omega_{20}s\alpha)^2 + \omega_{30}^2 + 2\Omega^2 - 3\Omega^2[(\ell_{a10}c\alpha + \ell_{a20}s\alpha)^2 + \ell_{a30}^2]\}$$
$$+ \rho_1 w_{101} [\omega_{30}(\omega_{10}s\alpha - \omega_{20}c\alpha) - 3\Omega^2\ell_{a30} (\ell_{a10}s\alpha - \ell_{a20}c\alpha)]$$
$$+ v'_{101} \{- \rho_1\Omega^2 (h_{x1} + x_{10}) \Big\langle(\ell_{c10}s\alpha - \ell_{c20}c\alpha) + \ell_{c30}^2 + 2$$
$$- 3[(\ell_{a10}s\alpha - \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\Big\rangle + EI_{z1} (10\, v''_{100}v'''_{100} + 5\, v''''_{100}v'_{100})\}$$
$$+ v''_{101} \{\Omega^2\Big\langle\frac{1}{2}\rho_1 [(h_{x1} + \ell_{10})^2 - (h_{x1} + x_{10})^2] + m_1(h_{x1}$$
$$+ \ell_{10})\Big\rangle \Big\langle(\ell_{c10}s\alpha - \ell_{c20}c\alpha)^2 + \ell_{c30}^2 + 2 - 3[(\ell_{a10}s\alpha - \ell_{a20}c\alpha)^2 + \ell_{a30}^2]\Big\rangle$$
$$+ EI_{z1} [\frac{15}{2} (v''_{100})^2 + 10\, v'_{100}v'''_{100}]\} + EI_{z1} \{v'''_{101} (10\, v'_{100}v''_{100})$$
$$- v''''_{101} [1 - \frac{5}{2} (v'_{100})^2]\} = -EI_{z1} [\frac{5}{2} (v''_{100})^2 + 10\, v'_{100}v''_{100}v'''_{100}$$
$$+ \frac{5}{2} v''''_{100} (v'_{100})^2] + \frac{1}{2}\rho_1 [(\omega_{10}c\alpha + \omega_{20}s\alpha)(\omega_{10}s\alpha - \omega_{20}c\alpha)$$

$$- 3\Omega^2 (\ell_{a10}c\alpha + \ell_{a20}s\alpha)(\ell_{a10}s\alpha - \ell_{a20}c\alpha) \int_0^{x_{10}} \left[\left(\frac{\partial v_{100}}{\partial \xi_1}\right)^2 + \left(\frac{\partial w_{100}}{\partial \xi_1}\right)^2\right] d\xi_1 \quad , \quad 0 \le x_{10} \le \ell_{10} \tag{131}$$

subject to boundary conditions (125b) and (125c), where in the latter ℓ_1 must be replaced by the shortened length ℓ_{10}.

12. The RAE/B Satellite. Numerical Results.

The general formulation of Sec. 11 has been used to obtain the nontrivial equilibrium configuration of the RAE/B satellite, to test the stability of equilibrium, and to calculate the natural frequencies of oscillation about the nontrivial equilibrium. The system parameters are as follows:

$A_0 = 87.74$ slug ft^2, $B_0 = 83.74$ slug ft^2, $C_0 = 18$ slug ft^2

$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 4.348 \times 10^{-4}$ slug ft^{-1} $\rho_5 = \rho_6 = 4.596 \times 10^{-4}$ slug ft^{-1}

$m_1 = m_2 = m_3 = m_4 = 2.40 \times 10^{-3}$ slug, $m_5 = m_6 = 0$

$\ell_1 = \ell_2 = \ell_3 = \ell_4 = 600$ ft, $\ell_5 = \ell_6 = 315$ ft

$EI_{y1} = EI_{z1} = EI_{y2} = \ldots = EI_{z4} = 15.278$ lb ft^2, $\alpha = 30°$

$EI_{y5} = EI_{z5} = EI_{y6} = EI_{z6} = 13.889$ lb ft^2, $\beta = 25°$

$h_{x1} = h_{x4} = 0.973$ ft, $h_{x2} = h_{x3} = 0.878$ ft, $h_{x5} = h_{x6} = 0$

$h_{y1} = -h_{y4} = 0.705$ ft, $h_{y2} = -h_{y3} = -0.760$ ft, $h_{y5} = h_{y6} = -1.800$ ft

$h_{z1} = h_{z2} = h_{z3} = h_{z4} = h_{z5} = h_{z6} = 0$

$\Omega = 4.653 \times 10^{-4}$ rad sec^{-1}

We shall present the results of the analyses in the order listed above.

a. Nontrivial equilibrium

Inserting the above data into Eqs. (116) and (117), as indicated in Eq. (115), and solving Eqs. (119), we obtain

$$\theta_{100} = 0.13537 \text{ rad} = 7.756 \text{ deg.}$$

$$\theta_{200} = -5.63789 \times 10^{-8} \text{ rad} = -3.2302 \times 10^{-6} \text{ deg.}$$

$$\theta_{300} = 1.37374 \times 10^{-6} \text{ rad} = 7.87096 \times 10^{-5} \text{ deg.}$$

We note that θ_{100} is caused largely by the damper booms. The fact that the rods are not attached at the satellite mass center turns out to have an insignificant effect on θ_{j00} $(j = 1,2,3)$.

To evaluate the elastic displacements $v_{i00}(x_i)$ and $w_{i00}(x_i)$ $(i = 1,2,\ldots,6)$. We assume the solution of Eqs. (120) - (123) in the form

$$v_{i00}(x_i) = \sum_{r=1}^{p} a_{ri0}\phi_r(x_i)$$

$$i = 1,2,\ldots,6 \qquad (132)$$

$$w_{i00}(x_i) = \sum_{r=1}^{p} b_{ri0}\phi_r(x_i)$$

where

$$\phi_r(x_i) = A_r[(\cos \beta_r\ell_i + \cosh \beta_r\ell_i)(\sin \beta_r x_i - \sinh \beta_r x_i)$$

$$- (\sin \beta_r\ell_i + \sinh \beta_r\ell_i)(\cos \beta_r x_i - \cosh \beta_r x_i)] \qquad (133)$$

are eigenfunctions corresponding to a bar in bending with the end $x_i = 0$ fixed and having a mass m_i attached at the end $x_i = \ell_i$. The eigenvalues $\beta_r\ell_i$ are solutions of the characteristic equation

$$(1 + \cos \beta_r \ell_i \cosh \beta_r \ell_i) = \beta_r \ell_i \frac{m_i}{\rho_i \ell_i} (\sin \beta_r \ell_i \cosh \beta_r \ell_i$$
$$- \cos \beta_r \ell_i \sinh \beta_r \ell_i) \tag{134}$$

Moreover, the amplitudes A_r are such that the eigenfunctions $\phi_r(x_i)$ are orthonormal, i.e., they satisfy the relation

$$\int_0^{\ell_i} \rho_i \phi_r(x_i) \phi_s(x_i) dx_i + m_i \phi_r(\ell_i) \phi_s(\ell_i) = \delta_{rs} \tag{135}$$

where δ_{rs} is the Kronecher delta. Limiting the series in (132) to two terms, p = 2, the first two roots of Eq. (134) and the amplitudes A_r corresponding to i = 1,2,3,4 are

$$\begin{aligned} \beta_1 \ell_i &= 1.85813 \qquad & A_1 &= 0.47696 \quad \text{slug}^{-1/2} \\ \beta_2 \ell_i &= 4.65310 \qquad & A_2 &= 0.03789 \quad \text{slug}^{-1/2} \end{aligned} \tag{136}$$

In addition, the coefficients a_{ri0}, b_{ri0} (i = 1,2,3,4) are

Table I.

i	a_{1i0}	a_{2i0}	b_{1i0}	b_{2i0}
1	-0.13656×10^2	-0.98055×10^{-1}	0.55184	0.46385×10^{-2}
2	0.13652×10^2	0.97981×10^{-1}	0.55188	0.46385×10^{-2}
3	-0.13652×10^2	-0.97980×10^{-1}	-0.55187	-0.46384×10^{-2}
4	0.13656×10^2	0.98054×10^{-1}	-0.55184	-0.46385×10^{-2}

The first two roots of Eq. (134) with $m_i = 0$, and the amplitudes A_1 and A_2 corresponding to i = 5,6 are

$$\beta_1 \ell_i = 1.87511 \qquad A_1 = 0.63510 \quad \text{slug}^{-1/2}$$
$$\beta_2 \ell_i = 4.69414 \qquad A_2 = 0.04899 \quad \text{slug}^{-1/2} \tag{137}$$

whereas the coefficients a_{ri} and b_{ri} are

Table II

i	a_{1i0}	a_{2i0}	b_{1i0}	b_{2i0}
5	-0.93855×10^{-2}	-0.12900×10^{-3}	0.17756	0.70688×10^{-3}
6	-0.93840×10^{-2}	-0.12900×10^{-3}	0.17756	0.70688×10^{-3}

It will prove of interest to list the elastic displacements of the end points, as calculated by means of the linearized equations. These displacements are

$$\begin{aligned}
v_{100}(\ell_1) &= -52.205 \text{ ft}, & w_{100}(\ell_1) &= 2.1071 \text{ ft}, \\
v_{200}(\ell_2) &= 52.192 \text{ ft}, & w_{200}(\ell_2) &= 2.1072 \text{ ft}, \\
v_{300}(\ell_3) &= -52.191 \text{ ft}, & w_{300}(\ell_3) &= -2.1072 \text{ ft}, \\
v_{400}(\ell_4) &= 52.204 \text{ ft}, & w_{400}(\ell_4) &= -2.1071 \text{ ft}, \\
v_{500}(\ell_5) &= -4.8655 \times 10^{-2} \text{ ft}, & w_{500}(\ell_5) &= 0.92961 \text{ ft}, \\
v_{600}(\ell_6) &= -4.8648 \times 10^{-2} \text{ ft}, & w_{600}(\ell_6) &= 0.92960 \text{ ft}
\end{aligned}$$

The above values of $v_{i00}(x_i)$ and $w_{i00}(x_i)$ $(i = 1,2,\ldots,6)$ enable us to solve Eqs. (119) for the angles θ_{j0} $(j = 1,2,3)$ and Eq. (131) and the companion ones for the perturbations $v_{i01}(x_i)$, $w_{i01}(x_i)$ $(i = 1,2,\ldots,6)$. The resulting angles are

$\theta_{10} = 0.19695$ rad $= 11.2846$ deg.

$\theta_{20} = -6.54250 \times 10^{-8}$ rad $= -3.74858 \times 10^{-6}$ deg.

$\theta_{30} = 1.37608 \times 10^{-6}$ rad $= 7.88438 \times 10^{-5}$ deg.

Instead of listing the perturbations v_{i01} and w_{i01}, we shall list the complete solutions v_{i0} and w_{i0} in the form of the series

$$v_{i0}(x_i) = \sum_{r=1}^{2} a_{ri}\phi_r(x_i)$$

$$i = 1,2,\ldots,6 \qquad (138)$$

$$w_{i0}(x_i) = \sum_{r=1}^{2} b_{ri}\phi_r(x_i)$$

where $\phi_{ri}(x_i)$ are still given by Eqs. (133), in which the eigenvalues $\beta_r\ell_i$ and amplitudes A_r $(r = 1,2)$ are given by (136) and (137). The final results are tabulated as follows

Table III

i	a_{1i}	a_{2i}	b_{1i}	b_{2i}
1	-0.13803×10^{2}	-0.86379×10^{-1}	0.54865	0.46378×10^{-2}
2	0.13800×10^{2}	0.86313×10^{-1}	0.54869	0.46378×10^{-2}
3	-0.13800×10^{2}	-0.86313×10^{-1}	-0.54869	-0.46378×10^{-2}
4	0.13803×10^{2}	0.86379×10^{-1}	-0.54865	-0.46378×10^{-2}
5	-0.93855×10^{-2}	-0.12900×10^{-3}	0.17756	0.70670×10^{-3}
6	-0.93840×10^{-2}	-0.12900×10^{-3}	0.17756	0.70670×10^{-3}

Moreover, the final end displacements are

$$v_{10}(\ell_1) = -52.816 \text{ ft}, \qquad w_{10}(\ell_1) = 2.0948 \text{ ft},$$

$$v_{20}(\ell_2) = 52.803 \text{ ft}, \qquad w_{20}(\ell_2) = 2.0950 \text{ ft},$$

$$v_{30}(\ell_3) = -52.803 \text{ ft}, \qquad w_{30}(\ell_3) = -2.0950 \text{ ft},$$

$$v_{40}(\ell_4) = 52.816 \text{ ft}, \qquad w_{40}(\ell_4) = -2.0948 \text{ ft},$$

$$v_{50}(\ell_5) = -4.8655 \times 10^{-2} \text{ ft}, \qquad w_{50}(\ell_5) = 0.92962 \text{ ft},$$

$$v_{60}(\ell_6) = -4.8648 \times 10^{-2} \text{ ft}, \qquad w_{60}(\ell_6) = 0.92962 \text{ ft},$$

and we note that the nonlinear effect is virtually zero for booms 5 and 6. The nontrivial equilibrium is depicted in Fig. 7, where only the radial booms are shown because the displacements of the damper booms are insignificant.

b. Liapunov stability analysis

A stability analysis using κ, as given by Eq. (60), as a testing function has been carried out. Essentially, the analysis reduced to testing the matrix [k] for positive definiteness, where the elements of [k] are given by Eqs. (53). The numerical values of the elements for the particular configuration at hand are listed in the next subsection. The matrix was found to be positive definite, so that the equilibrium is asymptotically stable.

c. Eigenvalue problem

Using Eqs. (49), (51), (53), and (57), in conjunction with the above data, we obtain the elements

Table IV

	3.42090×10^{4}	0	-6.31727×10^{-3}	4.78807×10^{-1}	-7.67484×10
$m_{1,j}$	-4.78845×10^{-1}	-7.67515×10	4.78840×10^{-1}	7.67512×10	-4.78808×10^{-1}
	7.67480×10	8.49246×10^{-5}	6.81765×10	-8.49247×10^{-5}	6.81765×10
	0	1.12564×10^{5}	2.20835×10^{4}	3.44739×10	-1.57077×10^{2}
$m_{2,j}$	3.44664×10	1.57032×10^{2}	3.44664×10	1.57032×10^{2}	3.44739×10
	-1.57077×10^{2}	5.48304×10	-3.23228×10^{-1}	-5.48305×10	3.23042×10^{-1}
	-6.31727×10^{-3}	2.20835×10^{4}	1.31824×10^{5}	1.77103×10^{2}	0
$m_{3,j}$	1.77064×10^{2}	0	1.77064×10^{2}	0	1.77130×10^{2}
	0	-2.89735×10	-4.94522×10^{-1}	2.89734×10	4.94522×10^{-1}
	4.78807×10^{-1}	3.44739×10	1.77103×10^{2}	1	0
$m_{4,j}$	0	0	0	0	0
	0	0	0	0	0
	-7.67484×10	-1.57707×10^{2}	0	0	1
$m_{5,j}$	0	0	0	0	0
	0	0	0	0	0

	-4.78845×10^{-1}	3.44664×10	1.77064×10^{2}	0	0
$m_{6,j}$	1	0	0	0	0
	0	0	0	0	0
	-7.67515×10	1.57032×10^{2}	0	0	0
$m_{7,j}$	0	1	0	0	0
	0	0	0	0	0
	4.78840×10^{-1}	3.44664×10	1.77064×10^{2}	0	0
$m_{8,j}$	0	0	1	0	0
	0	0	0	0	0
	7.67512×10	1.57032×10^{2}	0	0	0
$m_{9,j}$	0	0	0	1	0
	0	0	0	0	0
	-4.78808×10^{-1}	3.44739×10	1.77103×10^{2}	0	0
$m_{10,j}$	0	0	0	0	1
	0	0	0	0	0

	7.67480×10	-1.57077×10^{2}	0	0	0
$m_{11,j}$	0	0	0	0	0
	1	0	0	0	0
	8.49246×10^{-5}	5.48304×10	-2.89735×10	0	0
$m_{12,j}$	0	0	0	0	0
	0	1	0	0	0
	6.81765×10	-3.23228×10^{-1}	-4.94522×10^{-1}	0	0
$m_{13,j}$	0	0	0	0	0
	0	0	1	0	0
	-8.49247×10^{-5}	-5.48305×10	2.89734×10	0	0
$m_{14,j}$	0	0	0	0	0
	0	0	0	1	0
	6.81765×10	3.23042×10^{-1}	4.94522×10^{-1}	0	0
$m_{15,j}$	0	0	0	0	0
	0	0	0	0	1

Table V

$g_{1,j}$	0	6.59282	1.29341	6.74937×10^{-3}	-3.36689×10^{-11}
	6.74965×10^{-3}	-3.36715×10^{-11}	6.74958×10^{-3}	3.36711×10^{-11}	6.74931×10^{-3}
	3.36689×10^{-11}	5.09923×10^{-2}	-6.36069×10^{-8}	-5.09924×10^{-2}	-6.36071×10^{-8}
$g_{2,j}$	-6.59282	0	-4.23315×10^{-7}	1.14630×10^{-2}	2.64825×10^{-3}
	-1.14635×10^{-2}	2.64836×10^{-3}	1.14634×10^{-2}	-2.64834×10^{-3}	-1.14630×10^{-2}
	-2.64824×10^{-3}	-6.88069×10^{-8}	-4.10887×10^{-2}	0.88070×10^{-8}	-4.10889×10^{-2}
$g_{3,j}$	-1.29341	4.23315×10^{-7}	0	-1.24289×10^{-2}	1.40029×10^{-2}
	1.24054×10^{-2}	1.40035×10^{-2}	-1.24053×10^{-2}	-1.40034×10^{-2}	1.24288×10^{-2}
	-1.40028×10^{-2}	-4.97588×10^{-4}	2.17121×10^{-2}	-4.97587×10^{-4}	2.17121×10^{-2}
$g_{4,j}$	-6.74937×10^{-3}	-1.14630×10^{-2}	1.24289×10^{-2}	0	9.12262×10^{-5}
	0	0	0	0	0
	0	0	0	0	0
$g_{5,j}$	3.36689×10^{-11}	-2.64825×10^{-3}	-1.40029×10^{-2}	-9.12262×10^{-5}	0
	0	0	0	0	0
	0	0	0	0	0

	-6.74965×10^{-3}	1.14635×10^{-2}	-1.24054×10^{-2}	0	0
$g_{6,j}$	0	9.12263×10^{-5}	0	0	0
	0	0	0	0	0
	3.36715×10^{-11}	-2.64836×10^{-3}	-1.40035×10^{-2}	0	0
$g_{7,j}$	-9.12263×10^{-5}	0	0	0	0
	0	0	0	0	0
	-6.74958×10^{-3}	-1.14634×10^{-2}	1.24053×10^{-2}	0	0
$g_{8,j}$	0	0	0	-9.12258×10^{-5}	0
	0	0	0	0	0
	-3.36711×10^{-11}	2.64834×10^{-3}	1.40034×10^{-2}	0	0
$g_{9,j}$	0	0	9.12258×10^{-5}	0	0
	0	0	0	0	0
	-6.74931×10^{-3}	1.14630×10^{-2}	-1.24288×10^{-2}	0	0
$g_{10,j}$	0	0	0	0	0
	-9.12259×10^{-5}	0	0	0	0

	-3.36689×10^{-11}	2.64824×10^{-3}	1.40028×10^{-2}	0	0
$g_{11,j}$	0	0	0	0	9.12259×10^{-5}
	0	0	0	0	0
	-5.09923×10^{-2}	6.88069×10^{-8}	4.97588×10^{-4}	0	0
$g_{12,j}$	0	0	0	0	0
	0	0	-7.49379×10^{-4}	0	0
	6.36069×10^{-8}	4.10887×10^{-2}	-2.17121×10^{-2}	0	0
$g_{13,j}$	0	0	0	0	0
	0	7.49379×10^{-4}	0	0	0
	5.09924×10^{-2}	-6.88070×10^{-8}	4.97587×10^{-4}	0	0
$g_{14,j}$	0	0	0	0	0
	0	0	0	0	7.49380×10^{-4}
	6.36071×10^{-8}	4.10889×10^{-2}	-2.17121×10^{-2}	0	0
$g_{15,j}$	0	0	0	0	0
	0	0	0	-7.49380×10^{-4}	0

Table VI

$k_{1,j}$	4.33117×10^{-3}	0	-2.98546×10^{-9}	5.43385×10^{-6}	-1.53634×10^{-5}
	-5.43408×10^{-6}	-1.53641×10^{-5}	5.43405×10^{-6}	1.53640×10^{-5}	-5.43384×10^{-6}
	1.53634×10^{-5}	-1.59271×10^{-11}	-4.36481×10^{-6}	1.592715×10^{-11}	-4.36488×10^{-6}
$k_{2,j}$	0	8.50947×10^{-2}	1.66943×10^{-2}	1.72626×10^{-5}	-1.35856×10^{-4}
	1.72556×10^{-5}	1.35817×10^{-4}	1.72558×10^{-5}	1.35817×10^{-4}	1.72628×10^{-5}
	-1.35856×10^{-4}	-4.74228×10^{-5}	-2.79445×10^{-7}	4.74229×10^{-5}	2.79513×10^{-7}
$k_{3,j}$	-2.98546×10^{-9}	1.66943×10^{-2}	5.21464×10^{-2}	6.59442×10^{-5}	-6.66325×10^{-6}
	6.59170×10^{-5}	6.66133×10^{-6}	6.59174×10^{-5}	6.66134×10^{-6}	6.59447×10^{-5}
	-6.66325×10^{-6}	1.64685×10^{-5}	-3.34538×10^{-7}	-1.64684×10^{-5}	3.34444×10^{-7}
$k_{4,j}$	5.43385×10^{-6}	1.72626×10^{-5}	6.59442×10^{-5}	3.49900×10^{-6}	-3.60230×10^{-8}
	0	0	0	0	0
	0	0	0	0	0
$k_{5,j}$	-1.53634×10^{-5}	-1.35856×10^{-4}	-6.66325×10^{-6}	-3.60230×10^{-8}	4.02153×10^{-6}
	0	0	0	0	0
	0	0	0	0	0

	-5.43408×10^{-6}	1.72556×10^{-5}	6.59170×10^{-5}	0	0
$k_{6,j}$	3.49896×10^{-6}	3.60229×10^{-8}	0	0	0
	0	0	0	0	0
	-1.53641×10^{-5}	1.35817×10^{-4}	6.66133×10^{-6}	0	0
$k_{7,j}$	3.60229×10^{-8}	4.02153×10^{-6}	0	0	0
	0	0	0	0	0
	5.43405×10^{-6}	1.72558×10^{-5}	6.59174×10^{-5}	0	0
$k_{8,j}$	0	0	3.49897×10^{-6}	3.60230×10^{-8}	0
	0	0	0	0	0
	1.53640×10^{-5}	1.35817×10^{-4}	6.66134×10^{-6}	0	0
$k_{9,j}$	0	0	3.60230×10^{-8}	4.02153×10^{-6}	0
	0	0	0	0	0
	-5.43384×10^{-6}	1.72628×10^{-5}	6.59447×10^{-5}	0	0
$k_{10,j}$	0	0	0	0	3.49900×10^{-6}
	-3.60230×10^{-8}	0	0	0	0

	1.53634×10^{-5}	-1.35856×10^{-4}	-6.66325×10^{-6}	0	0
$k_{11,j}$	0	0	0	0	-3.60230×10^{-8}
	4.02153×10^{-6}	0	0	0	0
	-1.59271×10^{-11}	-4.74228×10^{-5}	1.64685×10^{-5}	0	0
$k_{12,j}$	0	0	0	0	0
	0	3.71129×10^{-5}	8.10039×10^{-13}	0	0
	-4.36481×10^{-6}	-2.79445×10^{-7}	-3.34538×10^{-7}	0	0
$k_{13,j}$	0	0	0	0	0
	0	8.10039×10^{-13}	3.88946×10^{-6}	0	0
	1.59271×10^{-11}	4.74229×10^{-5}	-1.64684×10^{-5}	0	0
$k_{14,j}$	0	0	0	0	0
	0	0	0	3.71129×10^{-5}	-8.10040×10^{-13}
	-4.36488×10^{-6}	2.79513×10^{-7}	3.34444×10^{-7}	0	0
$k_{15,j}$	0	0	0	0	0
	0	0	0	-8.10040×10^{-13}	3.88945×10^{-6}

Note that the elastic displacements were represented by one mode each.

Using the formulation of Section 7, we obtain the following natural frequencies

Table VII

	1.514583×10^{-2}	-1.514583×10^{-2}	9.554415×10^{-3}
	-9.554415×10^{-3}	9.900792×10^{-3}	-9.900792×10^{-3}
	6.143208×10^{-2}	-6.143208×10^{-3}	4.319115×10^{-3}
	-4.319115×10^{-3}	2.020401×10^{-3}	-2.020401×10^{-3}
	2.020401×10^{-3}	-2.020401×10^{-3}	1.975175×10^{-3}
ω_i - rad sec^{-1}	-1.975175×10^{-3}	1.955743×10^{-3}	-1.955743×10^{-3}
	1.871461×10^{-3}	-1.871461×10^{-3}	1.856504×10^{-3}
	-1.856504×10^{-3}	1.856504×10^{-3}	-1.856504×10^{-3}
	8.708815×10^{-4}	-8.708815×10^{-4}	6.155318×10^{-4}
	-6.155318×10^{-4}	3.439481×10^{-4}	-3.439481×10^{-4}

d. Parametric study

The stability analysis was carried one step farther by varying the angle α. It was found that the system was asymptotically stable for $\alpha = 50°$, but became unstable for $\alpha = 51°$. The results can be easily explained by the fact that in the absence of damper booms and for completely rigid radial booms the system becomes unstable around $\alpha = 45°$. The gravitational and centrifugal effects tend to deform the flexible booms in a manner that the moments of inertia about the local vertical and about an axis tangent to the orbit are the same for an angle α such that $50° < \alpha < 51°$. It should be mentioned that instability in both cases can be traced to angle θ_3, which tends to become large when the moment of inertia about the local vertical becomes larger than that about the axis tangent to the orbit, as at this point the "least moment of inertia" criterion is violated.

The same parametric study was undertaken with respect to the natural frequencies. In terms of natural frequencies, instability occurs when at least one natural frequency (we recall that in our case the natural frequencies occur in pairs) reduces to zero. Here again the system becomes unstable for $50° < \alpha < 51°$, thus corroborating the results obtained by the Liapunov stability analysis.

13. Summary and Conclusions

Two new theories for studying the motion characteristics of a rotating system with flexible parts about undeformed equilibrium have been developed. The first is qualitative and the second quantitative. Specifically, the first represents a stability theory and the second a method for obtaining the system natural frequencies.

The stability theory is based on the Liapunov direct method and makes use of modal analysis to represent elastic displacements. The novelty of the formulation lies in the fact that for the first time a nontrivial equilibrium is considered in conjunction with the Liapunov direct method for a stability analysis of spinning flexible bodies capable of large deformations.

The stability analysis can be divided into two major parts: the evaluation of the nontrivial equilibrium and the stability analysis itself.

When the body is capable of large deformations, nonlinear algebraic and differential equations must be solved for the rotational and elastic displacements, respectively, where these displacements define the equilibrium configurations of the system. Because the problem is one of stability about nontrivial equilibrium, it is necessary to expand the Liapunov function about that equilibrium. Assuming small displacements from equilibrium, the problem reduces to the evaluation of a Hessian matrix at the nontrivial equilibrium and testing the matrix for sign definiteness by means of Sylvester's criterion. It should be pointed out that the size of the Hessian matrix depends on the number of eigenfunctions used to represent the elastic displacements.

The method for obtaining the natural frequencies of the system makes use of the variational equations about the nontrivial equilibrium. Then the set of second-order differential equations is converted into a set of twice the number of first-order differential equations. The associated eigenvalue problem yields the system natural frequencies.

The two methods are quite general in scope, and can be used for testing stability and calculating the natural frequencies of a large variety of hybrid systems. As an application, the theory has been used to test the stability of the RAE/B satellite. First, the nonlinear equations have been solved for the nontrivial equilibrium configuration, and then this configuration has been used to evaluate the associated Hessian matrix. The satellite was found to be stable. Then one of the systems parameters has been varied to predict at which point the equilibrium becomes unstable. The results are in line with the expectations. In addition, the system natural frequencies for oscillation about the deformed equilibrium were calculated. The parametric study used in conjunction with the Liapunov stability analysis was used to examine how the frequencies are affected. The study resulted in the same instability statement.

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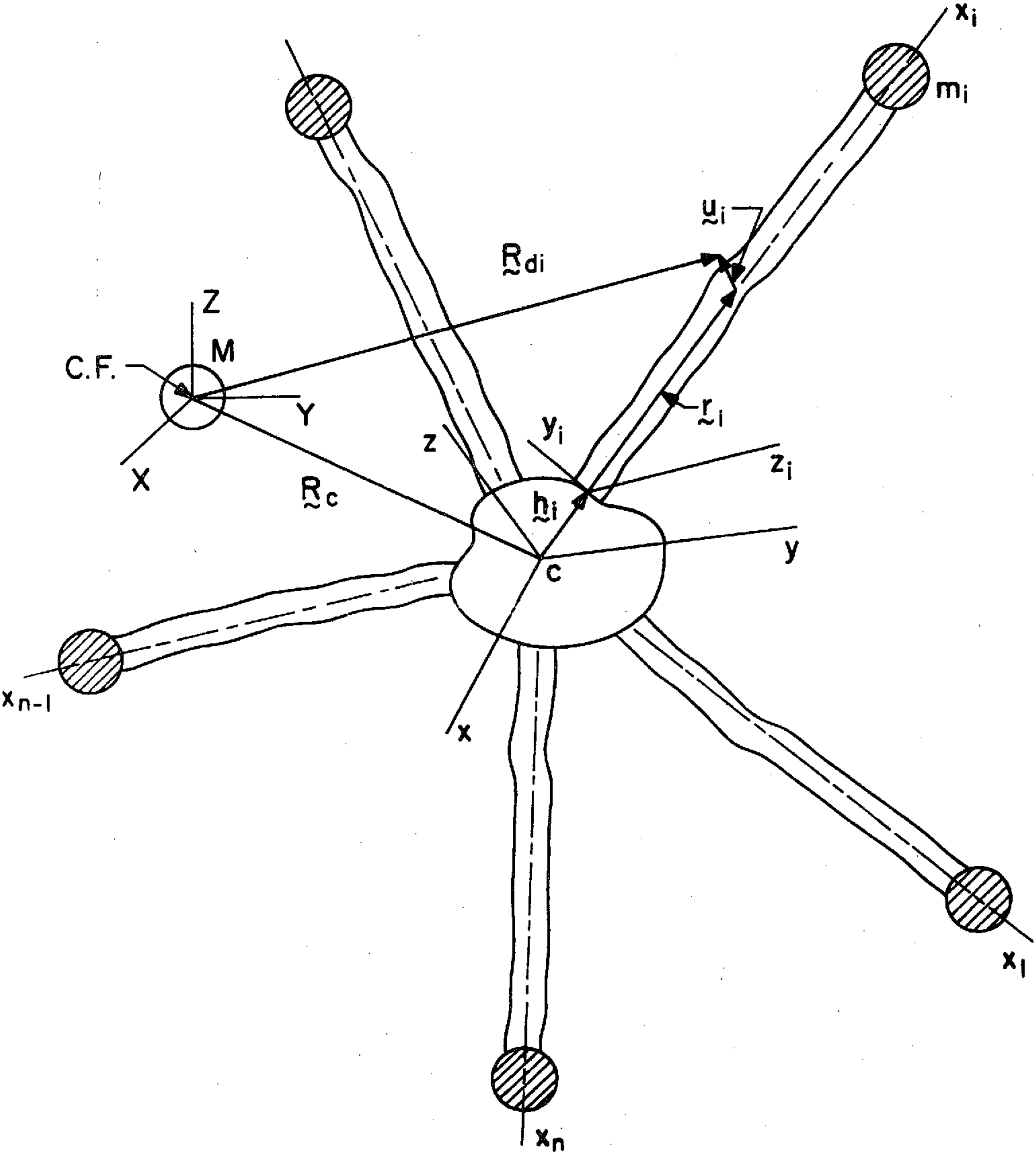

FIGURE 1 - GENERAL MATHEMATICAL MODEL

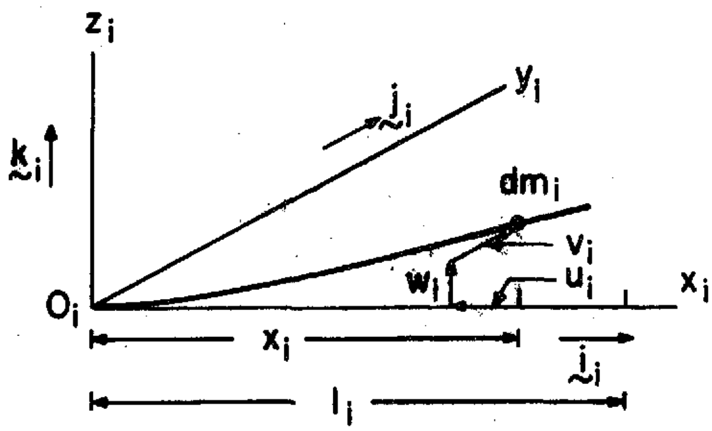

FIGURE 2 - ELASTIC DISPLACEMENTS

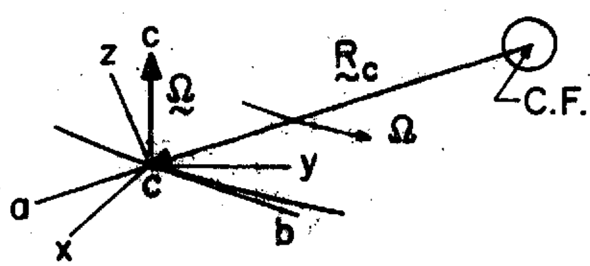

FIGURE 3 - ORBITAL AXES AND BODY AXES

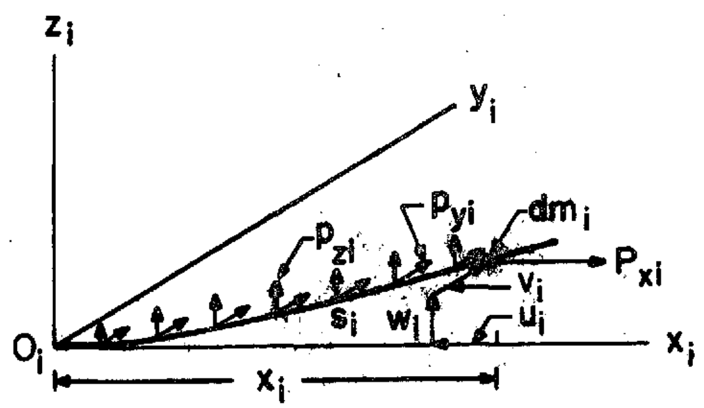

FIGURE 4 - FORCES ON ELASTIC BOOM

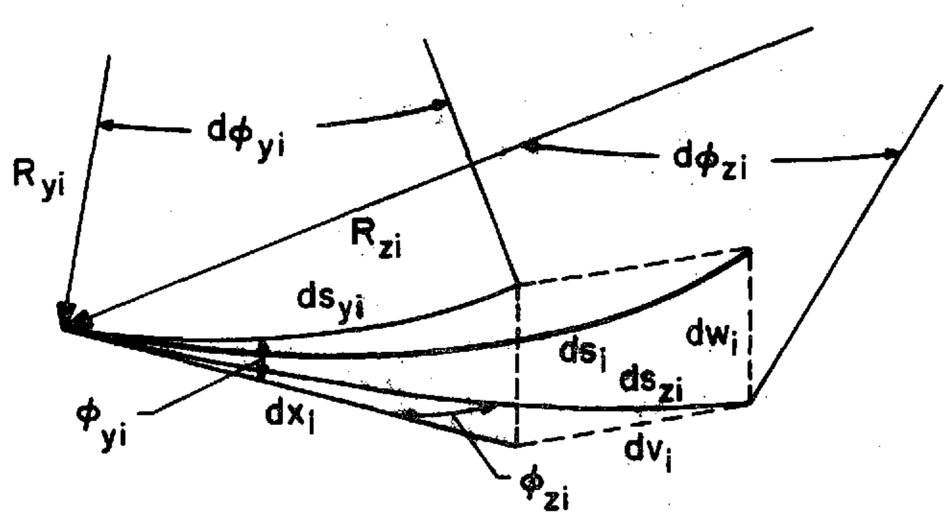

FIGURE 5 - DEFORMATION OF ELASTIC BOOM

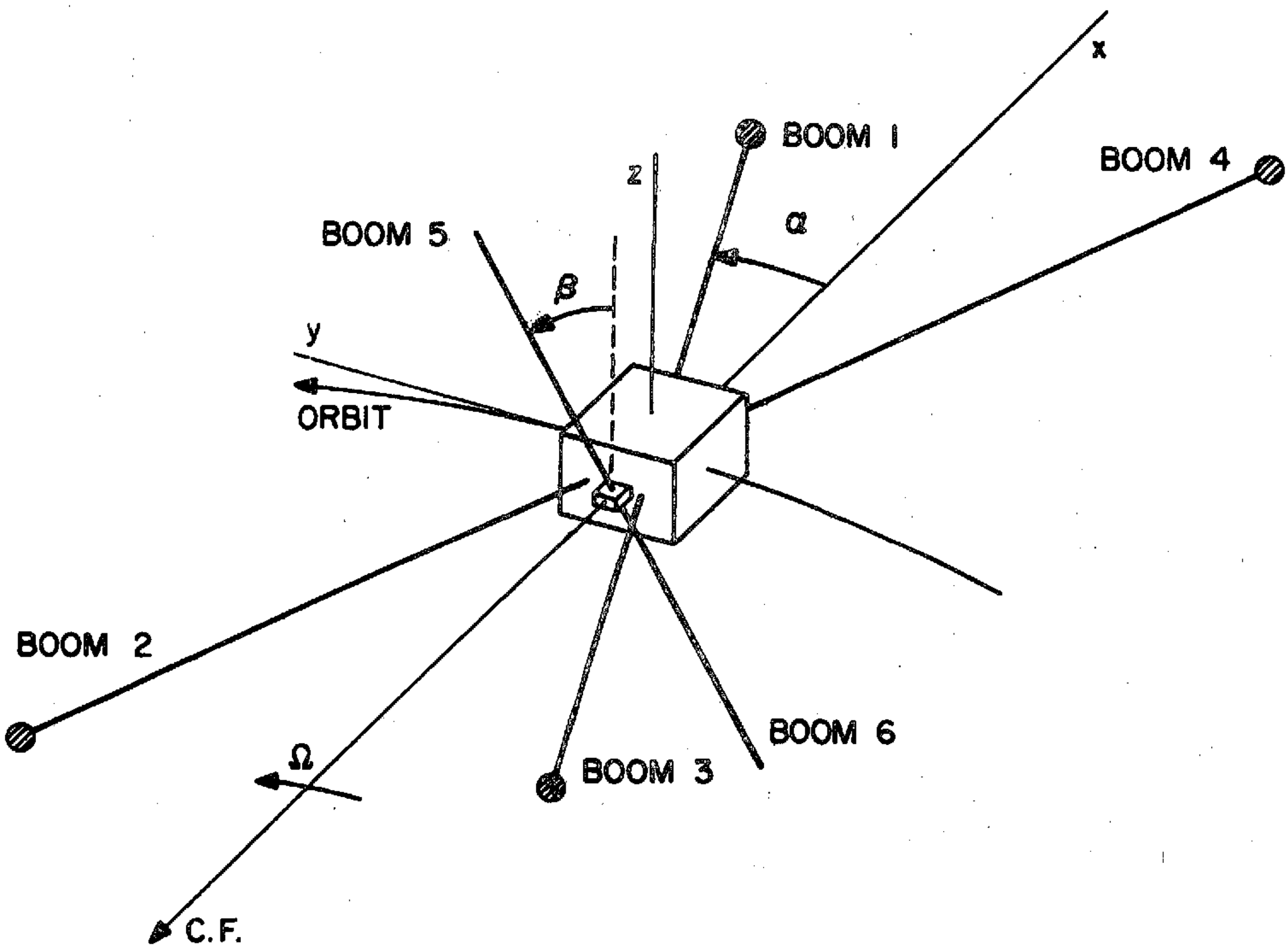

FIGURE 6 – RADIO ASTRONOMY EXPLORER – LUNAR (RAE/B) SATELLITE

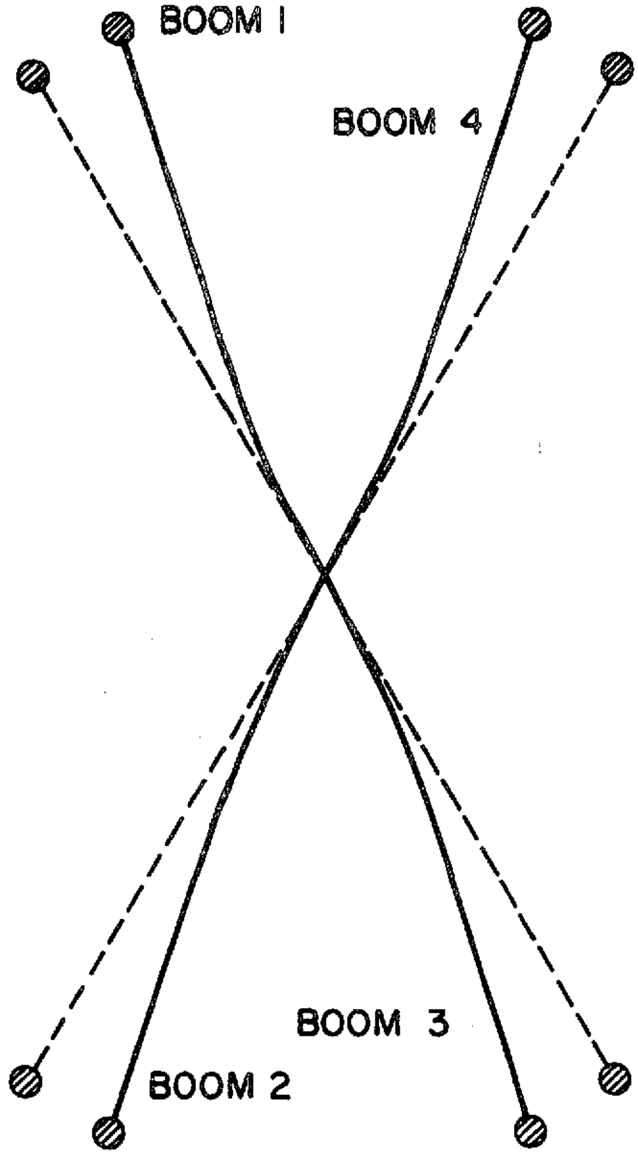

FIGURE 7 – NOTRIVIAL EQUILIBRIUM CONFIGURATION