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ORDINARY ELECTROMAGNETIC MODE INSTABILITY*

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ABSTRACT

The instability of the ordinary electromagnetic mode propagating perpendicular to an external magnetic field is studied for a single-species plasma with ring velocity distribution. The marginal instability boundaries for both the purely growing mode and the propagating growing modes are calculated from the instability criteria. The dispersion characteristics for various sets of plasma parameters are also given. The typical growth rates are of the order of the cyclotron frequency and enhanced by increasing $\beta_{\parallel}$.

I. INTRODUCTION

The instability of the ordinary electromagnetic mode propagating perpendicular to an external magnetic field has received considerable attention. Hamasaki (1968a, b) and Davidson and Wu (1970) investigated a stationary plasma with anisotropic temperatures and showed that for temperature anisotropy $T_\perp/T_\parallel < 1$ and sufficiently large $\beta_\parallel > 2$, where $\beta_\parallel$ is the ratio of parallel kinetic pressure to magnetic pressure, purely growing ordinary electromagnetic modes exist for propagation perpendicular to the external magnetic field. Counter-streaming plasmas with and without temperature anisotropy have also been previously investigated by many authors for the ordinary mode instability (Gaffey, Thompson and Liu, 1972, 1973; Lee and Armstrong, 1971; Bornatici and Lee, 1970; Tzoar and Yang, 1970). In this paper we wish to present a comprehensive picture of the ordinary electromagnetic mode for a single-species plasma. From the study of the dispersion relation derived from the linearized Vlasov-Maxwell equations, it is found that the instability can occur for $\beta_\parallel > 1$ and is dependent on the values of $\beta_\perp$ and α, where $\beta_\perp$ is the ratio of perpendicular kinetic pressure to magnetic pressure and α is the ratio $v_{o\perp}/c^2$, with c as the velocity of light. $v_{o\perp}$ is the average perpendicular particle velocity. Attention is called to the prediction of propagating instability with the real part of the complex frequency, ω_r, being non-zero. A detailed analysis is made of the steady state conditions of the plasma to determine the stability boundary. The solution of the very complicated dispersion relation is done numerically and compared with the analytical solutions that are obtained in certain limiting cases. The justification of the instability boundaries predicted by the instability criteria which require the instability to be absolute is also made through the numerical solutions of the dispersion relation. The growth rates are found to be typically of the order of the cyclotron frequency.

The electromagnetic instability is important in high-β plasmas and is of interest in a broad range of astrophysical and laboratory applications. Recently the experimental observation of the electric field fluctuations made by the IMP-6 spacecraft in the outermagnetosphere was reported by Gurnett and Shaw (1973). They observed strong electromagnetic radiation of harmonically related bands in the frequency spectrum with electric field fluctuation parallel to the external magnetic field. They suggested that it is due to an ordinary electromagnetic plasma instability and this suggestion is justified by our work.

The basic plasma model that is investigated consists of an equal number of electrons and ions immersed in a uniform external magnetic field and free of any electric field. Ions are considered to form a uniform positive background. In section II the dispersion relation is obtained. Various limiting forms of the dispersion relation are discussed and general stability considerations are given. Section III presents the analysis of instability. Three conditions for instability are given and the instability boundaries are calculated. In Section IV the numerical solutions of the dispersion relation of the various parametric dependences are discussed. Finally the conclusions of this study are summarized in Section V.

II. DISPERSION RELATION

For an infinite collisionless single-species plasma immersed in a uniform, static magnetic field $\vec{B}_o$ we take the equilibrium distribution function to be an even function of the component of velocity parallel to $\vec{B}_o$ so that the ordinary mode decouples from the extraordinary modes for wave vector $\vec{k} \perp \vec{B}_o$ ($\vec{k} = k\hat{e}_x$ and $\vec{B}_o = B_o\hat{e}_z$). The ordinary mode is a purely electromagnetic wave with the electric field $\vec{E} \parallel \vec{B}_o$.

The electric field in the $(\vec{k},\omega)$ space for the ordinary mode is (e.g., Baldwin, Bernstein and Weenink, 1969)

$$\vec{E}(k,\omega) = \frac{\vec{N}(k,\omega)}{D(k,\omega)} \tag{1}$$

where the numerical function is

$$\vec{N}(k,\omega) = i\omega\vec{e} + ic\vec{k} \times \vec{b} + \hat{e}_z \frac{\omega\omega_p^2}{e\omega_c} \int d\vec{v}\, v_\parallel\, g(k, v_\perp, v_\parallel)$$

$$\times \sum_{n=-\infty}^{\infty} \frac{J_n^2(\rho)}{(\omega + n\omega_c)}$$

$$- \hat{e}_z \frac{\omega_p^2 b}{mc} \int d\vec{v}\, v_\parallel \left(v_\parallel \frac{\partial f_o}{\partial v_\perp} - v_\perp \frac{\partial f_o}{\partial v_\parallel} \right) \sum_{n=-\infty}^{\infty} \frac{nJ_n(\rho) \frac{d}{d\rho} J_n(\rho)}{(\omega + n\omega_c)} \tag{2}$$

and the dispersion function is

$$D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 + \sum_{n=-\infty}^{\infty} \frac{A_n(k)n\omega_c}{\omega - n\omega_c} \tag{3}$$

$$A_n(k) = 2\pi\, \omega_p^2 \int J_n^2(\rho)\, v_{\parallel}^2 \frac{\partial f_o}{\partial v_{\perp}}\, dv_{\parallel}\, dv_{\perp}$$

with $\omega_p^2 = 4\pi n_o e^2/m$; $\omega_c = eB_o/mc$; $\rho = kv_{\perp}/\omega_c$; J_n is the nth order Bessel function of the first kind; f_o is the equilibrium distribution function; and $\vec{e}$, $\vec{b}$, g are defined by

$$(\vec{e},\ \vec{b},\ g) = \int \frac{d\vec{x}}{(2\pi)^3}\, e^{-\vec{k}\cdot\vec{x}}\, [\vec{E}\ (t = 0),\ \vec{B}\ (t = 0),\ f^{(1)}\ (t = 0)]\ .$$

The singularities at $\omega = n\omega_c$ in $N(k,\omega)$ are matched by the singularities at $\omega = n\omega_c$ in $D(k,\omega)$. Therefore all of the singularities in $E(k,\omega)$ come from the zeros of the dispersion relation $D(k,\omega) = 0$. If we solve for ω (k real), exponential growth and damping are implied for $\omega_i > 0$ and $\omega_i < 0$, respectively, while ω_i is the imaginary part of ω.

In general the infinite sum in Eq. (3) converges rapidly for arguments of the Bessel functions less than order unity because

$$J_n(\rho) \approx \frac{1}{|n|!}\left(\frac{\rho}{2}\right)^{|n|} \qquad \text{for } \rho \ll 1 \quad . \tag{4}$$

However for ρ larger than order unity the series converges very slowly because

$$J_n(\rho) \approx \sqrt{\frac{2}{\pi\rho}} \cos\left(\rho - \frac{\pi}{4} - \frac{n\pi}{2}\right) \qquad \text{for } \rho \gg 1 \quad . \tag{5}$$

Therefore a large number of terms must be kept in order to obtain accurate results. This extensive calculation can be avoided by using both the integral representation of the Bessel functions and the Sommerfeld-Watson transformation technique (see Appendix A). The dispersion function then reduces to

$$D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 + 2\pi\omega_p^2 \int_{-\infty}^{\infty} v_{\parallel}^2\, dv_{\parallel} \int_0^{\infty} d\rho\, \frac{\rho f_o}{\sin \Omega\pi}$$

$$\cdot \int_o^{\pi} d\tau\, \sin \Omega\tau\, \sin \tau\, J_o\left(2\rho \cos \frac{\tau}{2}\right) \tag{6}$$

where $\Omega = \omega/\omega_c$. Clearly this expression is defined for all Ω except possibly at $\Omega = n$ and hence is the proper analytic continuation of Eq. (3). This form is particularly convenient for computational purposes. Before solving the dispersion relation exactly, it is important to examine some interesting features of the dispersion relation. In the limit of a cold plasma system, the dispersion relation becomes $D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 = 0$, which shows how the presence of the plasma alters the vacuum propagation of an electromagnetic wave. In particular, ω is independent of B_o. In the cutoff limit $(k \to 0)$, the dispersion relation has roots at the plasma frequency and all harmonics of the cyclotron frequency except $n = 0$. In the resonance limit $(k \to \infty)$, the dispersion relation has roots at $\omega = \pm (c^2k^2 + \omega_p^2)^{1/2}$ and $\omega = n\omega_c$ $(n = \pm 1, \pm 2, \ldots)$. In the limit of very strong B_o, $\omega_c \to \infty$ and hence $\rho \to 0$, the dispersion relation has roots at $\omega = \pm (c^2k^2 + \omega_p^2)^{1/2}$, and $\omega = n\omega_c$ $(n = \pm 1, \pm 2, \ldots)$. This implies that, from comparing the result with cold plasma limit, the very large B_o plasma behaves just like a cold plasma except for the intrinsic cyclotron harmonic behavior.

Now in order to get some idea on the relation between the equilibrium distribution and instability of the system, we let $\omega^2 = (\omega_r^2 - \omega_i^2) + i(2\omega_r\omega_i)$. Then separate the dispersion function (3) into its real and imaginary parts, we have

$$\mathrm{Re}[D(k,\omega)] = P_r - c^2k^2 - \omega_p^2 + \sum_{n=1}^{\infty} \frac{B_n(k)\, n^2\omega_c^2 \left(P_r - n^2\omega_c^2\right)}{\left[\left(P_r - n^2\omega_c^2\right)^2 + P_i^2\right]} = 0 \tag{7}$$

$$\mathrm{Im}[D(k,\omega)] = P_i \left[1 - \sum_{n=1}^{\infty} \frac{B_n(k)\, n^2 \omega_c^2}{\left[\left(P_r - n^2 \omega_c^2 \right)^2 + P_i^2 \right]} \right] = 0 \tag{8}$$

where $P_r = \omega_r^2 - \omega_i^2$, $P_i = 2\omega_r \omega_i$, and $B_n(k) = 2A_n(k)$.

Now assume that $B_n(k) \le 0$ for all $n \ge 1$, then Eq. (8) implies that either $\omega_r = 0$ or $\omega_i = 0$ and there is no propagating instability. If there are more than one particle species in the plasma, the above conclusion is still correct so long as each velocity distribution satisfies the condition $B_n(k) \le 0$ for all $n \ge 1$. One class of distribution functions that satisfies this condition is defined from Eq. (3) by $\partial f_o/\partial v_\perp \le 0$ for all $v_\perp > 0$. Hence a necessary condition for propagating instabilities is that the inequality $\partial f_o/\partial v_\perp > 0$ must be satisfied for some range of $v_\perp$.

Instabilities associated with the class of distributions with $\partial f_o/\partial v_\perp \le 0$ have been investigated by many authors (Hamasaki, 1968a, b; Davidson and Wu, 1970; Tzoar and Yang, 1970; Bornatici and Lee, 1970; Lee and Armstrong, 1971; Gaffey, Thompson and Liu, 1972, 1973). Our interest is to investigate the type of distribution with $\partial f_o/\partial v_\perp > 0$ for some range of $v_\perp$. We will take the ring distribution which occurs naturally in the earth magnetosphere when high energy charged particles, streaming from the sun, are trapped by the earth's magnetic field at the bow shock (Kennel and Petschek, 1966).

III. INSTABILITY ANALYSIS

The equilibrium distribution is taken to be the ring distribution

$$f_o = \frac{\delta(v_\perp - v_{o\perp})}{2\pi v_{o\perp}} \frac{1}{\sqrt{2\pi}\, v_{o\parallel}} \exp\left(\frac{-v_\parallel^2}{2\, v_{o\parallel}^2}\right) \quad . \tag{9}$$

The distribution function is normalized so that

$$\int f_o\, d\vec{v} = 1 \quad . \tag{10}$$

The dispersion relation can be written from Eqs. (3) and (6) as

$$D(k,\omega) = \frac{\omega_c^2}{\alpha}\left[\alpha\Omega^2 - \beta_\perp - \rho_o^2 - \beta_\parallel \rho_o \sum_{n=-\infty}^{\infty} \frac{n \frac{\partial}{\partial \rho_o} J_n^2(\rho_o)}{\Omega - n}\right]$$

$$= \frac{\omega_c^2}{\alpha}\left[\alpha\Omega^2 - \beta_\perp - \rho_o^2 + \frac{\beta_\parallel \rho_o^2}{\sin \Omega\pi} \int_0^\pi d\tau \sin\Omega\tau \sin\tau\, J_o\left(2\rho_o \cos\frac{\tau}{2}\right)\right]$$

$$= 0 \tag{11}$$

where

$$\rho_o = \frac{kv_{o\perp}}{\omega_c} \quad ,$$

$$\Omega = \frac{\omega}{\omega_c} \quad ,$$

$$\alpha = \frac{v_{o\perp}^2}{c^2} \quad ,$$

$$\beta_{\parallel} = \frac{\left(\omega_p^2/\omega_c^2\right)}{\left(c^2/v_{o\parallel}^2\right)} = \frac{\left(n_o m v_{o\parallel}^2/2\right)}{\left(B_o^2/8_{\parallel}\right)} \quad ,$$

and

$$\beta_{\perp} = \frac{\left(\omega_p^2/\omega_c^2\right)}{\left(c^2/v_{o\perp}^2\right)} = \frac{\left(n_o m v_{o\perp}^2/2\right)}{\left(B_o^2/8\pi\right)} \quad .$$

If we consider an equilibrium distribution of the counter-streaming type, say,

$$f_o = \frac{\delta(v_{\perp} - v_{o\perp})}{2\pi\, v_{o\perp}} \frac{1}{2\sqrt{2\pi}\, v_{o\parallel}} \left[\exp \frac{-(v_{\parallel} - u)^2}{2v_{o\parallel}^2} + \exp \frac{-(v_{\parallel} + u)^2}{2v_{o\parallel}^2}\right] \tag{12}$$

the dispersion relation will have the same form as Eq. (11) with $\beta_\parallel$ replaced by $\beta_\parallel' = [\omega_p^2/\omega_c^2]/[c^2/(v_{o\parallel}^2 + u^2)]$. Distribution functions of the form (12) produce no current along the magnetic field. Thus the effect of streaming is only to increase the value of $\beta_\parallel$.

In order to determine the curves of marginal instability in the parameter space $\alpha - \beta_\parallel - \beta_\perp$, we study the threshold conditions for the onset of complex solutions of the dispersion relation with $\omega_i > 0$. The necessary condition (Briggs, 1964) for an onset of a pair of complex conjugate solutions of $D(k,\omega) = 0$ at $\Omega = \tilde{\Omega}$, $\rho_o = \tilde{\rho}_o$ is that $(\tilde{\Omega}, \tilde{\rho}_o)$ be a double root of Ω for $\rho_o = \tilde{\rho}_o$, i.e.,

$$\left.\frac{\partial D}{\partial \Omega}\right|_{\substack{\Omega=\tilde{\Omega}\\ \rho_o=\tilde{\rho}_o}} = 0 \quad . \tag{13}$$

The pinching-pole criterion (Briggs, 1964; Derfler, 1967) which is necessary for absolute instability is that $D(k,\omega) = 0$ has a double root ρ_o at $\Omega = \tilde{\Omega}$, $\rho_o = \tilde{\rho}_o$, i.e.,

$$\left.\frac{\partial D}{\partial \rho_o}\right|_{\substack{\Omega=\tilde{\Omega}\\ \rho_o=\tilde{\rho}_o}} = 0 \tag{14}$$

where $\tilde{\Omega}$, $\tilde{\rho}_o$ satisfy the dispersion relation

$$D(\tilde{\rho}_o, \tilde{\Omega}, \alpha, \beta_\parallel, \beta_\perp) = 0 \tag{15}$$

for some specific values of α, $\beta_\parallel$, $\beta_\perp$.

Note that it is possible to have an instability with zero ω_r which corresponds to purely growing modes. By setting $\tilde{\Omega} = 0$, Eqs. (13)—(15) reduce to

$$\left(\frac{\partial D}{\partial \Omega}\right)_{\Omega=0} \equiv 0 \qquad \text{(identically zero)} \quad , \tag{16}$$

$$\left(\frac{\partial D}{\partial \rho_o}\right)_{\substack{\Omega=0\\ \rho_o=\tilde{\rho}_o}} = \frac{2\omega_c^2}{\alpha}\Big[\beta_\parallel J_o(\tilde{\rho}_o) J_1(\tilde{\rho}_o) - \tilde{\rho}_o$$

$$+ \alpha\tilde{\rho}_o[J_o(\tilde{\rho}_o) J_1'(\tilde{\rho}_o) - J_1^2(\tilde{\rho}_o)]\Big] = 0 \quad , \tag{17}$$

$$D(\tilde{\rho}_o, \Omega = 0) = \frac{\omega_c^2}{\alpha} [-\beta_\perp - \tilde{\rho}_o^2 + 2\beta_\parallel\tilde{\rho}_o J_o(\tilde{\rho}_o) J_1(\tilde{\rho}_o)] = 0 \quad . \tag{18}$$

Equations (17) and (18) give a set of two simultaneous equations,

$$\frac{1}{\beta_{\parallel}} = J_0^2(\tilde{\rho}_0) - J_1^2(\tilde{\rho}_0) \quad , \tag{19}$$

$$\frac{1}{\beta_{\parallel}} + \frac{\beta_{\perp}}{\beta_{\parallel}\tilde{\rho}_0^2} = \frac{2J_0(\tilde{\rho}_0)\, J_1(\tilde{\rho}_0)}{\tilde{\rho}_0} \quad . \tag{20}$$

The right-hand side of Eqs. (19) and (20) are plotted in Figure 1. Since $\tilde{\rho}_0$ is real, Eq. (19) can be satisfied only if $\beta_{\parallel} > 1$ and for a given value of $\beta_{\parallel}$, Eq. (20) can be satisfied only when the value $\beta_{\perp}/\beta_{\parallel}$ is rendered such that the hyperbola $[1/\beta_{\parallel}][1 + (\beta_{\perp}/\tilde{\rho}_0^2)] = \text{const.}$ intersects with the curve $[2J_0(\tilde{\rho}_0)\, J_1(\tilde{\rho}_0)]/\tilde{\rho}_0 = \text{const.}$ When this occurs there exist purely imaginary roots of the dispersion relation, implying that plasma fluctuations will grow in time without propagation. The situation when these two curves just touch gives the curve of marginal instability for purely growing modes in the $\beta_{\parallel} - \beta_{\perp}$ plane and is presented in Figure 2 parameterized as $\omega_0 = 0$. Note that the curve is independent of α.

Numerical study of the dispersion relation has also been done to determine the curves of marginal instability for the first three propagating frequency bands. Solving Eqs. (13), (14), and (15) simultaneously for various values of Ω, ρ_0, $\beta_{\parallel}$ with α, $\beta_{\perp}$ as parameters, we have obtained the marginal instability boundaries in the $\beta_{\perp} - \beta_{\parallel}$ plane with α as parameter, as shown in Figures 2 and 3. It is interesting to note that these instability boundaries are essentially straight lines. Points lying on the right-hand side of these straight lines are unstable. Figure 2 is obtained for $\alpha = 0.1$. Note that these three lines, which correspond to the first three propagating frequency bands, intersect each other. The effect of the perpendicular temperature is destabilizing; increasing the value of α decreases the value of $\beta_{\parallel}$ for instability, as shown in Figure 3. Attention is also called to the fact that the instability boundaries corresponding to different values of α for the same frequency band are parallel. Instability can occur for very low $\beta_{\perp}$.

IV. DISPERSION CHARACTERISTICS

In order to make sure that these curves truly represent the instability boundaries, we have to solve the dispersion relation (11) numerically. The dispersion characteristics for the points A, B, C, D, E, F, G, and H in $\beta_\perp - \beta_\parallel$ plane for $\alpha = 0.1$, as shown in Figure 2, are shown in Figures 4(a)—4(h), respectively. Cutoff is observed at each harmonic of the cyclotron frequency and at the plasma frequency ω_p. The mode which starts from ω_p is well approximated by the cold plasma wave $\omega^2 = \omega_p^2 + c^2k^2$. Resonances are found at all harmonics of ω_c. These observations are consistent with the analysis of section II.

Now we assume $\omega \approx n\omega_c$ and $(\omega_p^2/\omega_c^2)(v_{o\parallel}^2/v_{o\perp}^2) \ll 1$, then the infinite series in Eq. (11) can be well approximated by one term and the solution of (11) is approximately given by

$$\omega \approx n\omega_c \left(1 + \frac{\omega_p^2 \dfrac{v_{o\parallel}^2}{v_{o\perp}^2} \rho_o \dfrac{d}{d\rho_o} J_n^2(\rho_o)}{n^2\omega_c^2 - c^2k^2 - \omega_p^2} \right) \tag{21}$$

which accounts the undulation of each mode near each harmonic of the cyclotron frequency. This is shown in Figure 4(a). It will be noted that Eq. (21) predicts that the modes pass through points defined by the zeros of $J_n(\rho_o)$ and $(d/d\rho)J_n(\rho_o)$, in agreement with the exact numerical solution. In order to prove that this is true in any case, it is only necessary to obtain an alternate representation of the dispersion function (11) (see Appendix B)

$$D(k,\omega) = \frac{\omega_c^2}{\alpha}\left\{\alpha\Omega^2 - \beta_\perp - \rho_o^2 - \frac{\beta_\parallel \rho_o \Omega\pi}{\sin \Omega\pi} \frac{\partial}{\partial\rho_o}\left[J_\Omega(\rho_o)\; J_{-\Omega}(\rho_o)\right]\right\} \quad . \tag{22}$$

Clearly if Ω is an integer, Eq. (22) implies

$$J_n(\rho_o)\,\frac{\partial J_n(\rho_o)}{\partial\rho_o} = 0$$

verifying that the modes pass through harmonics of the cyclotron frequency when ρ_o is a zero of the nth order Bessel function or its first derivative.

As $(\omega_p^2/\omega_c^2)(v_{o\parallel}^2/v_{o\perp}^2)$ increases, the loops above a given harmonic approach the loops below the harmonic immediately following it. The points at which the loops can couple must always lie between $\gamma_{n,m}$ and $\gamma_{n+1,m}$, where $\gamma_{n,m}$ represents the mth zero of $J_n'(\rho_o)$. At point B of Figure 2 the n = 1 mode touches the zero frequency axis, thus there is a range of ρ_o in which purely imaginary solutions exist. The imaginary solutions are indicated by a broken line, as shown in Figure 4(b).

At point C of Figure 2 there are two unstable frequency bands. In addition to the purely growing frequency band there is a complex frequency band with $\omega_c < \omega_r < 2\omega_c$. The real parts of the frequencies are indicated by fine lines and the corresponding imaginary parts are shown by broken lines in Figure 4(c). The presence of these complex solutions has the important practical significance that an individual propagating mode will grow in time to an amplitude limited only by the validity of the linear theory which has been used in obtaining the dispersion relation.

The points D, E, F, G, and H of Figure 2 give the consistent results which are shown in Figures 4(d)—4(h), respectively. It is pointed out that the real parts of the complex frequency bands are centered at about $(n + 1/2)\omega_c$ and the maximum growth rates of the unstable waves decrease with increasing ω_r and are typically of the order of the cyclotron frequency. The growth rate can be very strong for some appropriate choices of plasma parameters.

If we further increase the values of $\beta_\parallel$ and $\beta_\perp$, we reach the situation that there are two purely growing bands; they couple at $\rho_o = \rho_o'(\alpha, \beta_\parallel, \beta_\perp)$ and for $\rho_o > \rho_o'$ there always exist complex solutions with ω_r starting from zero at $\rho_o = \rho_o'$ to infinity as ρ_o approaches to infinity. This situation is typically represented by Figures 5.

V. CONCLUSIONS

The instability of the ordinary electromagnetic mode propagating perpendicular to an external magnetic field has been investigated for a single-species plasma with ring velocity distribution. $\partial f_o(v_\parallel, v_\perp)/\partial v_\perp > 0$ for some $v_\perp > 0$ is a necessary condition for the existence of propagating instabilities. The instability criteria give the marginal instability boundaries not only for the purely growing mode, but also for the propagating growing modes. These instabilities are all absolute because they satisfy the absolute instability criterion. The necessary condition for the purely growing mode to set in is $\beta_\parallel > 1$ and the instability boundary for purely growing modes depends on $\beta_\parallel$ and $\beta_\perp$ only, independent of α $(\alpha = v_{o\perp}^2/c^2)$. The marginal boundaries for propagating instabilities are straight lines in $\beta_\parallel - \beta_\perp$ plane with $\beta_\parallel > 1$ and α as a parameter. For different α the marginal boundaries of propagating modes with ω_r in the same harmonic branch are parallel. Increasing α decreases $\beta_\parallel$ required for instability. For appropriate choice of parameters, growth rates may be of the order of the cyclotron frequency. The growth rate is enhanced by increasing $\beta_\parallel$. The investigation of loss-cone type velocity distribution and the case of two-species plasma for the ordinary EM modes has been underway and the results will be published in the near future.

APPENDIX A

From Eq. (3), with $\Omega = \omega/\omega_c$, we have

$$D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 + \sum_{n=-\infty}^{\infty} \frac{2\pi\omega_p^2 n}{\Omega - n} \int_{-\infty}^{\infty} v_{\parallel}^2 \, dv_{\parallel}$$

$$\times \int_0^{\infty} dv_{\perp} \frac{\partial f_o}{\partial v_{\perp}} J_n^2(\rho)$$

$$= \omega^2 - c^2k^2 - \omega_p^2 - 2\pi\omega_p^2 \int dv_{\parallel} \, v_{\parallel}^2 \int d\rho f_o$$

$$\times \frac{\partial}{\partial\rho} \left\{ \sum_{n=-\infty}^{\infty} \frac{nJ_n^2(\rho)}{\Omega - n} \right\} \quad . \tag{A-1}$$

By using the identity $\sum_{n=-\infty}^{\infty} J_n^2(\rho) = 1$ and the integral representation for the Bessel function

$$J_n^2(\rho) = \frac{1}{\pi} \int_0^{\pi} J_o\left(2\rho \sin \frac{\tau}{2}\right) \cos n\tau \, d\tau \tag{A-2}$$

the sum of Bessel functions in (A-1) can be written as

$$\sum_{n=-\infty}^{\infty} \frac{nJ_n^2(\rho)}{\Omega - n} = \sum_{n=-\infty}^{\infty} \Omega^2 \frac{J_n^2(\rho)}{\Omega^2 - n^2} - 1$$

$$= \frac{1}{\pi} \int_0^{\pi} d\tau J_o\left(2\rho \sin \frac{\tau}{2}\right) \sum_{n=-\infty}^{\infty} \frac{\cos n\tau}{\Omega^2 - n^2} - 1 \quad . \tag{A-3}$$

If Ω is complex we have, by means of a Sommerfeld-Watson transformation,

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\tau}{\Omega^2 - n^2} = \frac{1}{2i} \oint_{C_1+C_2} \frac{dz}{\tan z\pi} \frac{\cos z\tau}{\Omega^2 - z^2} \tag{A-4}$$

where the contour $C_1 + C_2$ is taken around the real axis in the complex z planes as shown in Figure 6.

$$\oint_{C_1+C_2} \frac{dz}{\tan z\pi} \frac{\cos z\tau}{\Omega^2 - z^2} = \oint_{C_1+C_2} \frac{dz}{\sin z\pi} \frac{\cos (\pi - \tau)z}{\Omega^2 - z^2}$$

$$- \oint_{C_1+C_2} dz \frac{\sin \tau z}{\Omega^2 - z^2} \quad . \qquad \text{(A-5)}$$

The second integral in (A-5) vanishes because its integrand is analytic inside the contour C. Since on the large circle $C_{R_1} + C_{R_2}$ the contour integral approaches zero as the radius R approaches infinity, we have

$$\oint_{C_1+C_2} \frac{dz}{\sin z\pi} \frac{\cos (\pi - \tau)z}{\Omega^2 - z^2} = \int_{C_1+C_{R_1}} \frac{dz}{\sin z\pi} \frac{\cos (\pi - \tau)z}{\Omega^2 - z^2}$$

$$+ \int_{C_2+C_{R_2}} \frac{dz}{\sin z\pi} \frac{\cos (\pi - \tau)z}{\Omega^2 - z^2}$$

$$= \frac{2\pi i}{\Omega} \frac{\cos (\pi - \tau)\Omega}{\sin \pi\Omega} \quad . \qquad \text{(A-6)}$$

Hence (A-4) becomes

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\tau}{\Omega^2 - n^2} = \frac{\pi}{\Omega} \frac{\cos (\pi - \tau)z}{\sin \Omega\pi} \quad . \tag{A-7}$$

If Ω is real we have

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\tau}{\Omega^2 - n^2} = \frac{1}{2i} \oint_{C_1+C_2} \frac{dz}{\tan z\pi} \frac{\cos z\tau}{\Omega^2 - z^2} + \frac{\pi}{\Omega} \frac{\cos \Omega\tau}{\tan \Omega\pi} \tag{A-8}$$

and

$$\frac{1}{2i} \oint_{C_1+C_2} \frac{dz}{\tan z\pi} \frac{\cos z\tau}{\Omega^2 - z^2} = \frac{1}{2i} \oint_{C_1+C_2} \frac{dz}{\sin \pi z} \frac{\cos (\pi - \tau)z}{\Omega^2 - z^2}$$

$$- \frac{1}{2i} \oint_{C_1+C_2} dz \frac{\sin z\tau}{\Omega^2 - z^2} \quad . \tag{A-9}$$

The first integral in (A-9) vanishes because the integrand has no poles in the complex plane except on the real axis. Thus

$$\frac{1}{2i} \oint_{C_1+C_2} \frac{dz}{\tan z\pi} \frac{\cos z\tau}{\Omega^2 - z^2} = \frac{\pi}{\Omega} \sin \Omega\tau \quad .$$

Hence (A-8) becomes

$$\sum_{n=-\infty}^{\infty} \frac{\cos n\tau}{\Omega^2 - n^2} = \frac{\pi}{\Omega} \frac{\cos (\pi - \tau)z}{\sin \Omega\pi} \tag{A-10}$$

which is analytic for all Ω except possibly at $\Omega = 0, \pm 1, \pm 2, \ldots$ Note that (A-10) has the same form as (A-7). Hence (A-3) becomes

$$\sum_{n=-\infty}^{\infty} \frac{nJ_n^2(\rho)}{\Omega - n} = \int_0^{\pi} d\tau\, J_o\left(2\rho \sin \frac{\tau}{2}\right) \frac{\Omega \cos (\pi - \tau)\Omega}{\sin \pi\Omega} - 1 \ . \tag{A-11}$$

Finally Eq. (A-1) becomes

$$D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 + 4\pi\omega_p^2 \int dv_{\parallel}\, v_{\parallel}^2 \int d\rho f_o$$

$$\times \int_0^{\pi} d\tau\, J_1\left(2\rho \sin \frac{\tau}{2}\right) \frac{\Omega \sin \frac{\tau}{2} \cos (\pi - \tau)\Omega}{\sin \Omega\pi} \ . \tag{A-12}$$

Now put $\pi - \tau = \theta$:

$$\int_0^\pi d\tau\ J_1\left(2\rho\ \sin\frac{\tau}{2}\right)\ \Omega\ \sin\frac{\tau}{2}\cos\ (\pi-\tau)\Omega$$

$$= \int_0^\pi d\theta\ J_1\left(2\rho\ \cos\frac{\theta}{2}\right)\ \Omega\ \cos\frac{\theta}{2}\cos\ \Omega\theta$$

$$= \sin\ \Omega\theta\ \cos\frac{\theta}{2}\ J_1\left(2\rho\ \cos\frac{\theta}{2}\right)\Bigg|_{\theta=0}^{\pi}$$

$$- \frac{1}{2}\int_0^\pi d\theta\ \sin\ \Omega\theta\ \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[2\rho\ \cos\frac{\theta}{2}\ J_1\left(2\rho\ \cos\frac{\theta}{2}\right)\right]$$

$$= \frac{1}{2}\int_0^\pi d\theta\ \sin\ \Omega\theta\ \sin\frac{\theta}{2}\ \frac{\partial}{\partial\left(2\rho\ \cos\frac{\theta}{2}\right)}\left[2\rho\ \cos\frac{\theta}{2}\ J_1\left(2\rho\ \cos\frac{\theta}{2}\right)\right]$$

$$= \frac{\rho}{2}\int_0^\pi d\theta\ \sin\ \Omega\theta\ \sin\ \theta\ J_0\left(2\rho\ \cos\frac{\theta}{2}\right)\quad . \qquad \text{(A-13)}$$

Thus with the aid of (A-13) the dispersion function reduces to the form

$$D(k,\omega) = \omega^2 - c^2k^2 - \omega_p^2 + 2\pi\omega_p^2\int_{-\infty}^{\infty} dv_\parallel\ v_\parallel^2\int_0^\infty \frac{d\rho\ \rho f_o}{\sin\ \Omega\pi}$$

$$\int_0^\pi d\tau\ \sin\ \Omega\tau\ \sin\ \tau\ J_o\left(2\rho\ \cos\frac{\tau}{2}\right)\quad . \qquad \text{(A-14)}$$

APPENDIX B

From the identity (Watson, 1958)

$$\frac{2}{\pi} \int_0^{\pi/2} J_{\mu+\nu} (2\rho \cos \tau) \cos (\mu - \nu)\tau \, d\tau = J_\mu(\rho) \, J_\nu(\rho) \quad . \tag{B-1}$$

If we put $\mu = -\nu = \Omega$, then

$$J_\Omega(\rho) \, J_{-\Omega}(\rho) = \frac{2}{\pi} \int_0^{\pi/2} J_o(2\rho \cos \tau) \cos \Omega\tau \, d\tau \tag{B-2}$$

and

$$\frac{\partial}{\partial p}\left[J_\Omega(\rho) \, J_{-\Omega}(\rho)\right] = \frac{-2}{\pi} \int_0^{\pi/2} J_1(2\rho \cos \tau) \cos \tau \cos 2\Omega\tau \, d\tau \quad . \tag{B-3}$$

Putting $\theta = \tau/2$, we have

$$\frac{\partial}{\partial \rho}\left[J_\Omega(\rho) \, J_{-\Omega}(\rho)\right] = \frac{-2}{\pi} \int_0^{\pi} J_1\left(2\rho \cos \frac{\theta}{2}\right) \cos \frac{\theta}{2} \cos \Omega\theta d\theta \quad . \tag{B-4}$$

By using Eq. (A-13) we have

$$\frac{\partial}{\partial\rho}\left[J_{\Omega}(\rho)\ J_{-\Omega}(\rho)\right] = \frac{-\rho}{\Omega\pi} \int_{0}^{\pi} d\theta\ \sin\Omega\theta\ \sin\theta\ J_{0}\left(2\rho\cos\frac{\theta}{2}\right) . \tag{B-5}$$

From Eq. (11) we finally have

$$D(\mathbf{k},\omega) = \frac{\omega_c^2}{\alpha}\left\{\alpha\Omega^2 - \beta_{\perp} - \rho_0 - \frac{\beta_{\parallel}\rho\Omega\pi}{\sin\Omega\pi}\frac{\partial}{\partial\rho}\left[J_{\Omega}(\rho)\ J_{-\Omega}(\rho)\right]\right\} . \tag{B-6}$$

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FIGURE CAPTIONS

Figure 1 Plot of $[2J_o(\tilde{\rho}_o)\ J_1(\tilde{\rho}_o)]/\tilde{\rho}_o$ and $J_o^2(\tilde{\rho}_o) - J_1^2(\tilde{\rho}_o)$ versus $\tilde{\rho}_o$.

Figure 2 Instability boundaries in the $\beta_{\parallel} - \beta_{\perp}$ plane. $\omega_o = 0$ curve is the instability boundary for purely growing modes, which is independent of α. $1 < \omega_o/\omega_c < 2$, $2 < \omega_o/\omega_c < 3$, $3 < \omega_o/\omega_c < 4$ curves represent the instability boundaries for the first three propagating frequency bands with $\alpha = 0.1$. Unstable regions lie on the right of the curves.

Figure 3 Instability boundaries in the $\beta_{\parallel} - \beta_{\perp}$ plane for the first propagating frequency band with α as parameter.

Figure 4 Dispersion characteristics of ordinary electromagnetic mode for $\alpha = 0.1$.

(a) $\beta_{\perp} = 0.2$, $\beta_{\parallel} = 1.6$ represents point A in Figure 2. Round points are defined by ($\omega = n\omega_c$, $\partial J_n(\rho_o)/\partial\rho_o = 0$) and triangular points by ($\omega = n\omega_c$, $J_n(\rho_o) = 0$).

(b) $\beta_{\perp} = 0.25$, $\beta_{\parallel} = 1.9$ represents point B in Figure 2. The broken line is the purely imaginary solution.

(c) $\beta_{\perp} = 0.25$, $\beta_{\parallel} = 2.25$;

(d) $\beta_{\perp} = 2.0$, $\beta_{\parallel} = 2.75$;

(e) $\beta_\perp = 7.0$, $\beta_\parallel = 4.0$;

(f) $\beta_\perp = 13.0$, $\beta_\parallel = 5.0$;

(g) $\beta_\perp = 0.75$, $\beta_\parallel = 4.0$; and

(h) $\beta_\perp = 0.75$, $\beta_\parallel = 6.0$

represent points C, D, E, F, G, and H in Figure 2, respectively. The real parts of the complex roots for frequency are indicated by fine lines and the corresponding imaginary parts by broken lines.

Figure 5 Dispersion characteristics for ordinary electromagnetic mode with $\alpha = 0.1$, $\beta_\perp = 10$, $\beta_\parallel = 80$. The real parts of the complex roots for frequency are indicated by fine lines and the corresponding imaginary parts by broken lines.

Figure 6 Contours in the complex z plane for evaluation of the integral in Eq. (A-4).

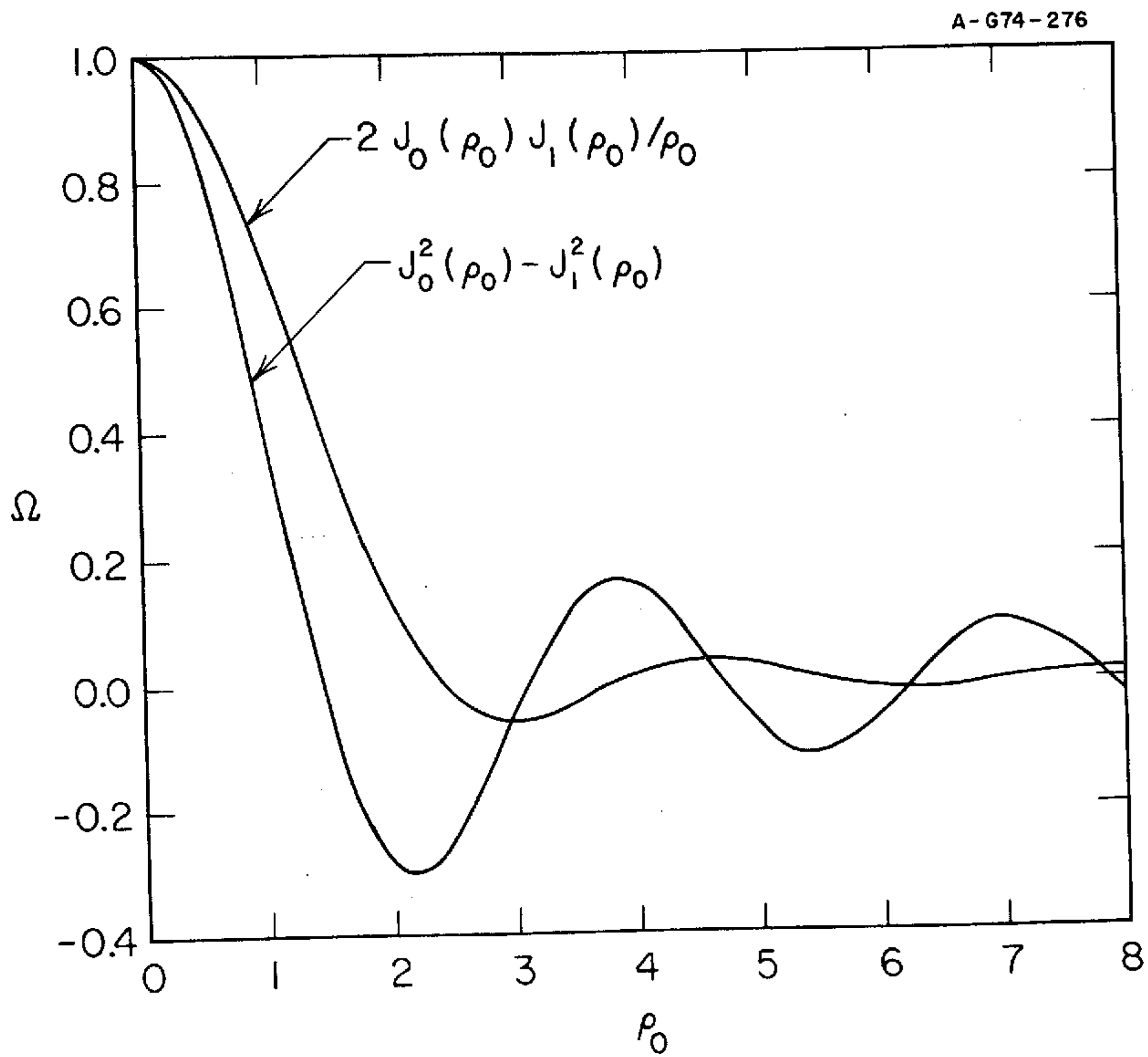

A-G74-276
2 J0 (ρ0) J1 (ρ0)/ρ0
J0² (ρ0) - J1² (ρ0)
Ω
ρ0
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0
1
2
3
4
5
6
7
8

Figure 1

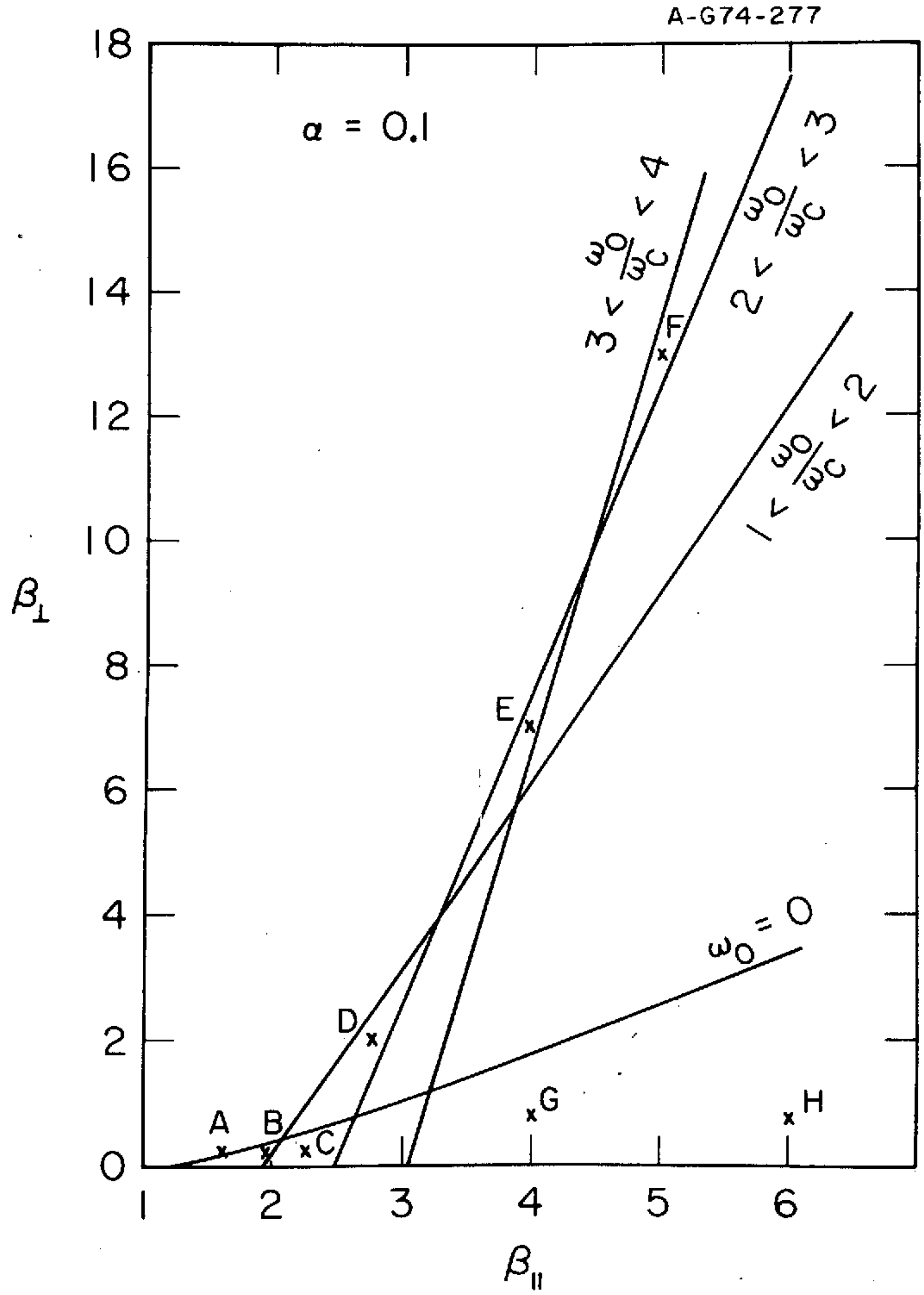

A-G74-277
α = 0.1
$3 < \omega_0/\omega_c < 4$
$2 < \omega_0/\omega_c < 3$
$1 < \omega_0/\omega_c < 2$
$\omega_0 = 0$
A
B
C
D
E
F
G
H
$\beta_\perp$
$\beta_\parallel$
0
2
4
6
8
10
12
14
16
18
1
2
3
4
5
6

Figure 2

A-G74-278

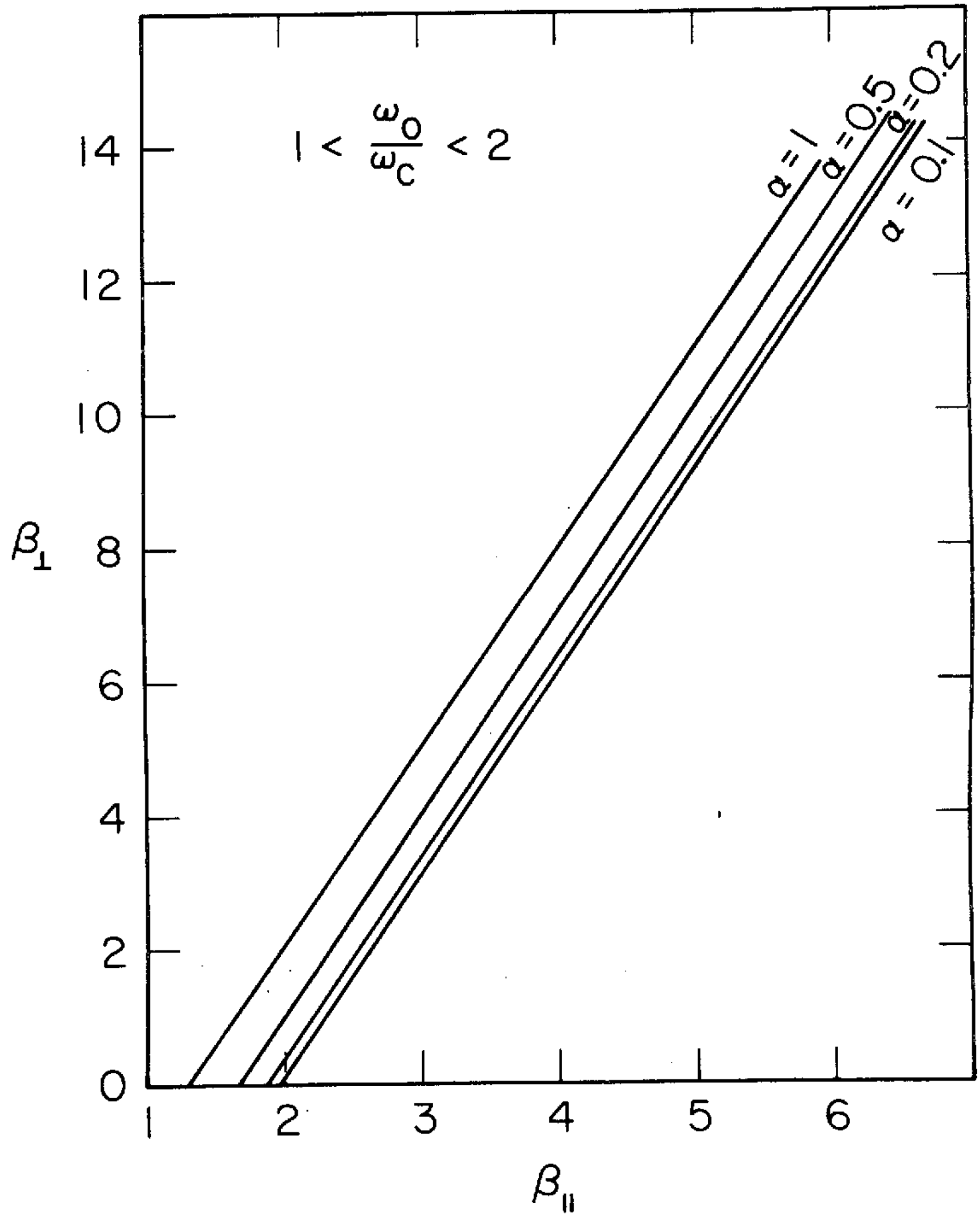


Figure 3

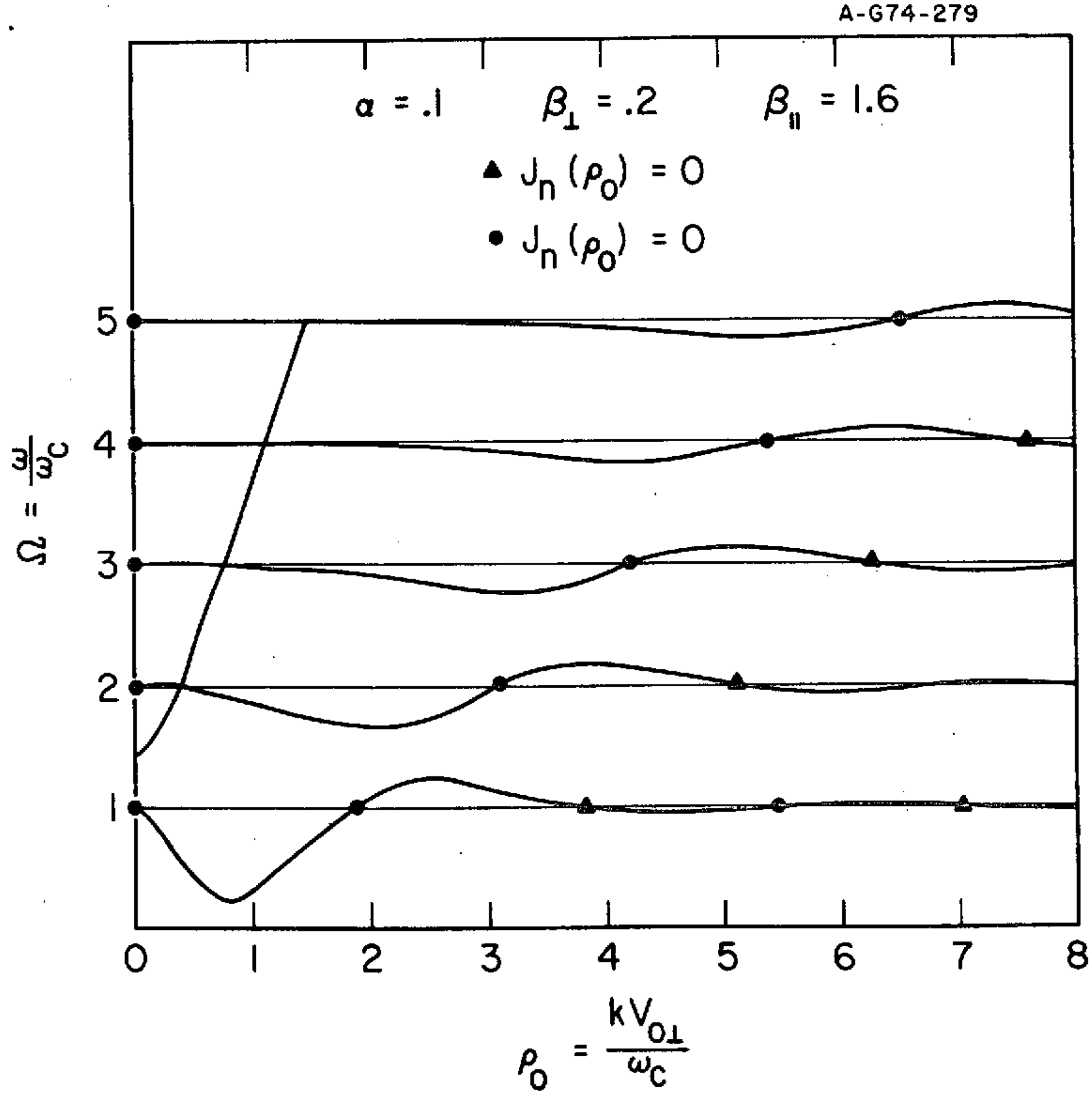


Figure 4(a)

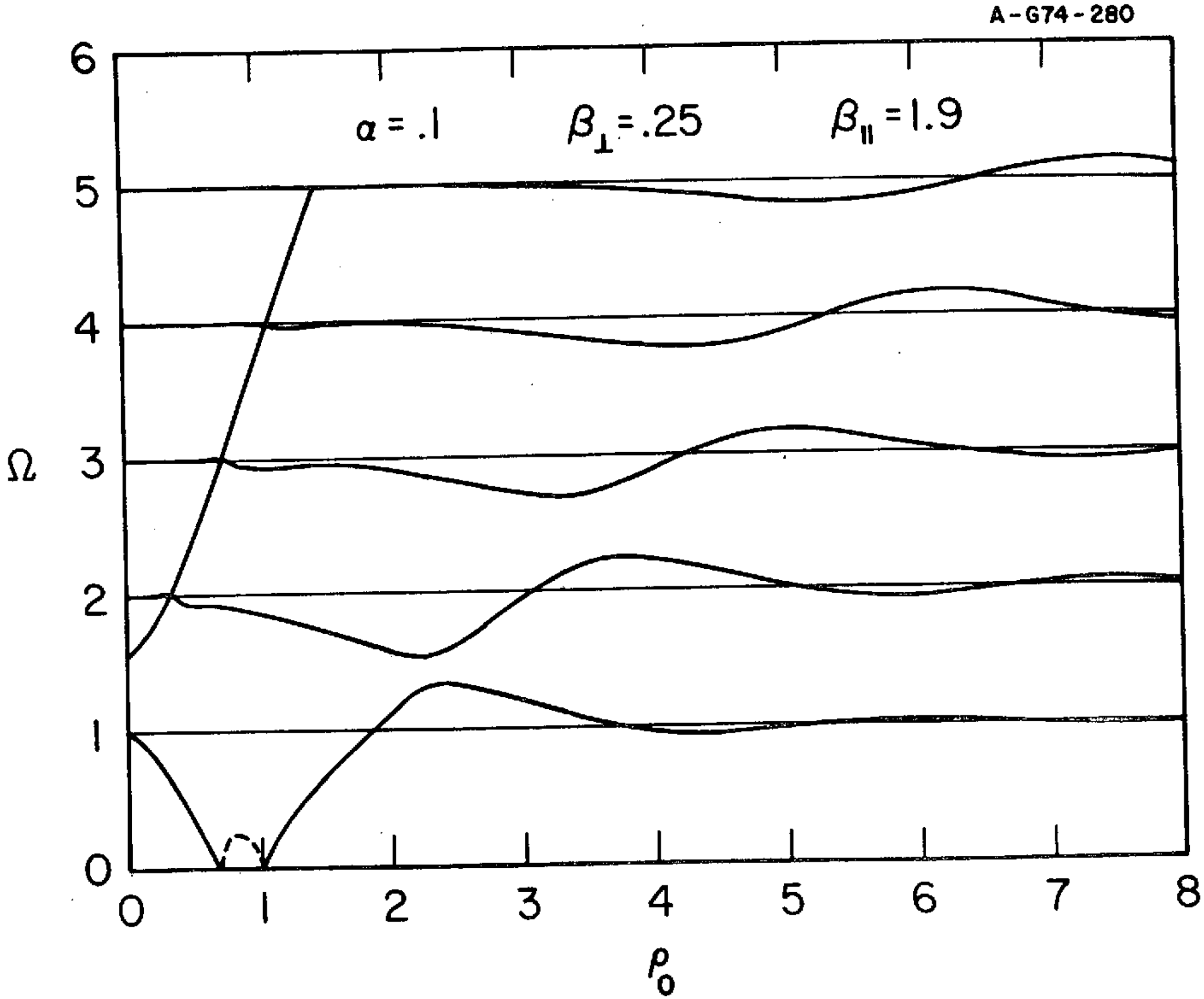


Figure 4(b)

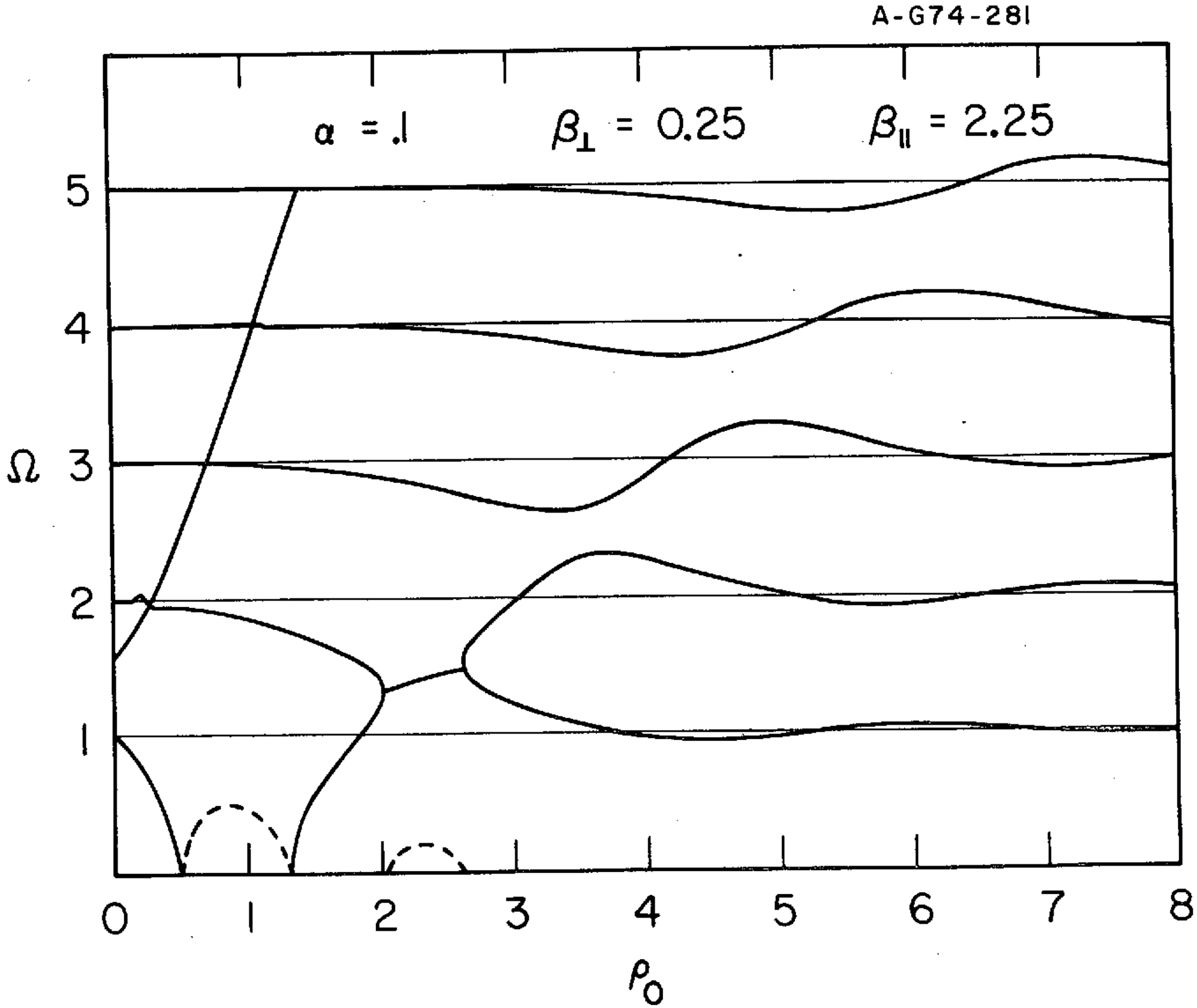


Figure 4(c)

A-G74-282

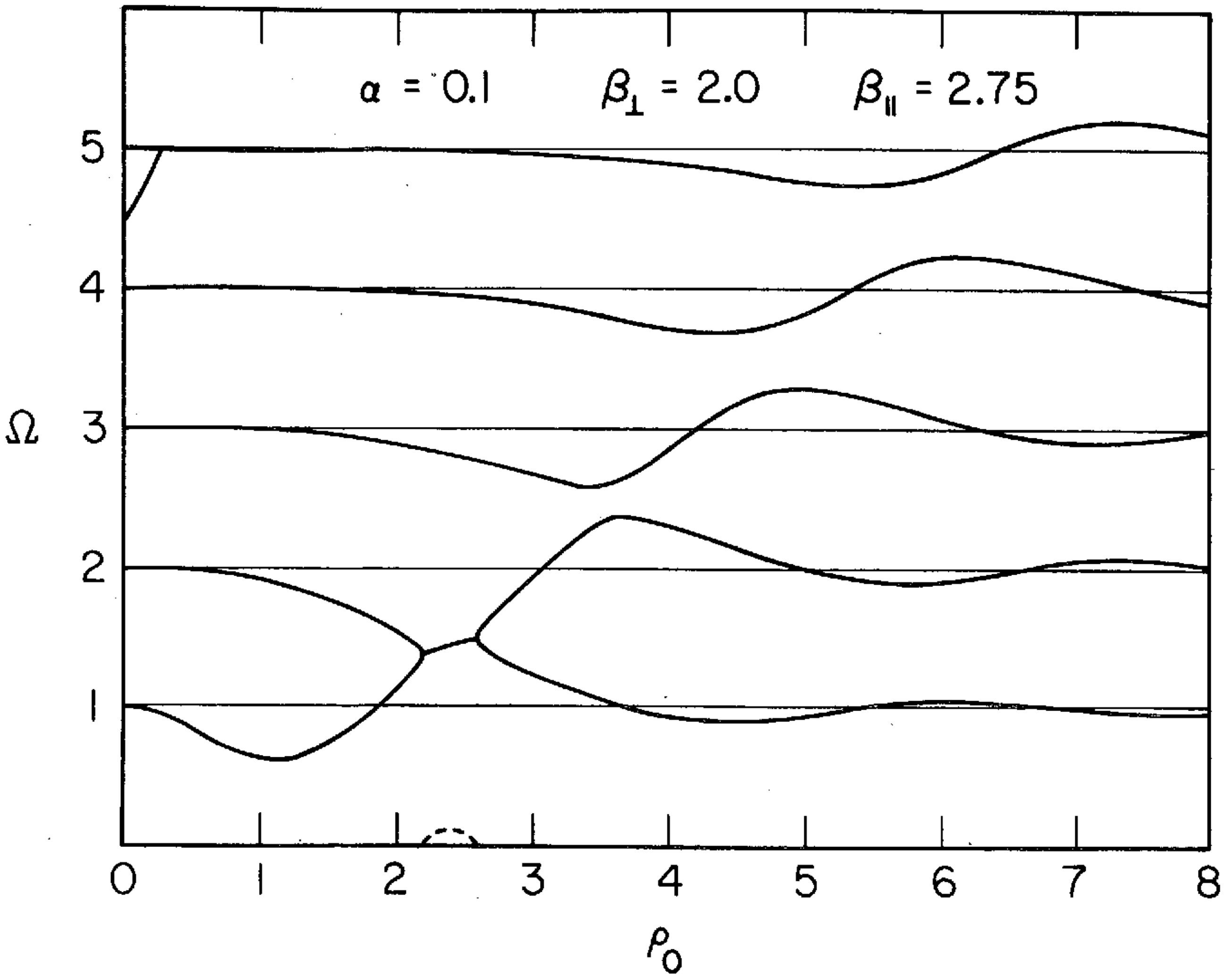
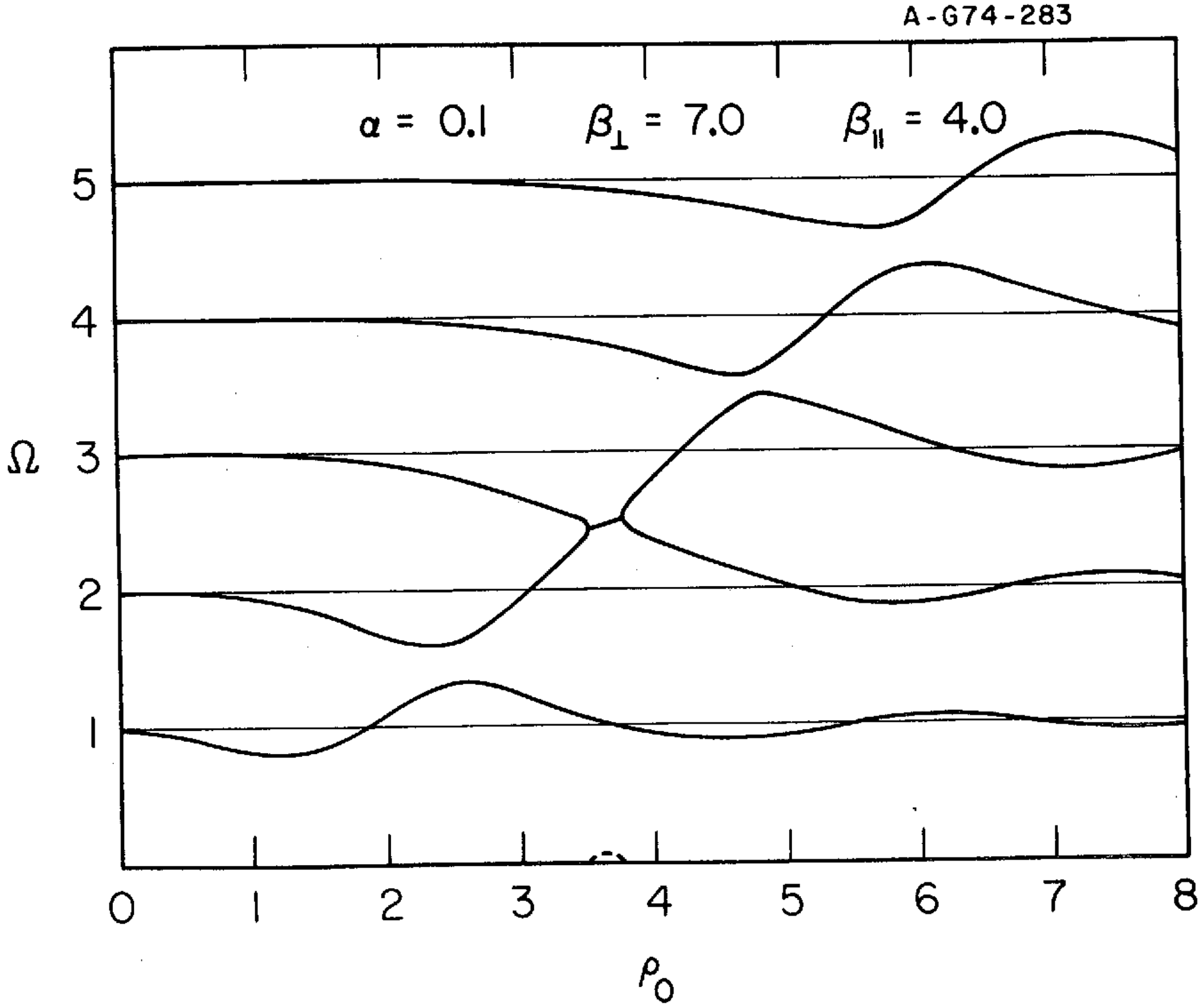


Figure 4(d)



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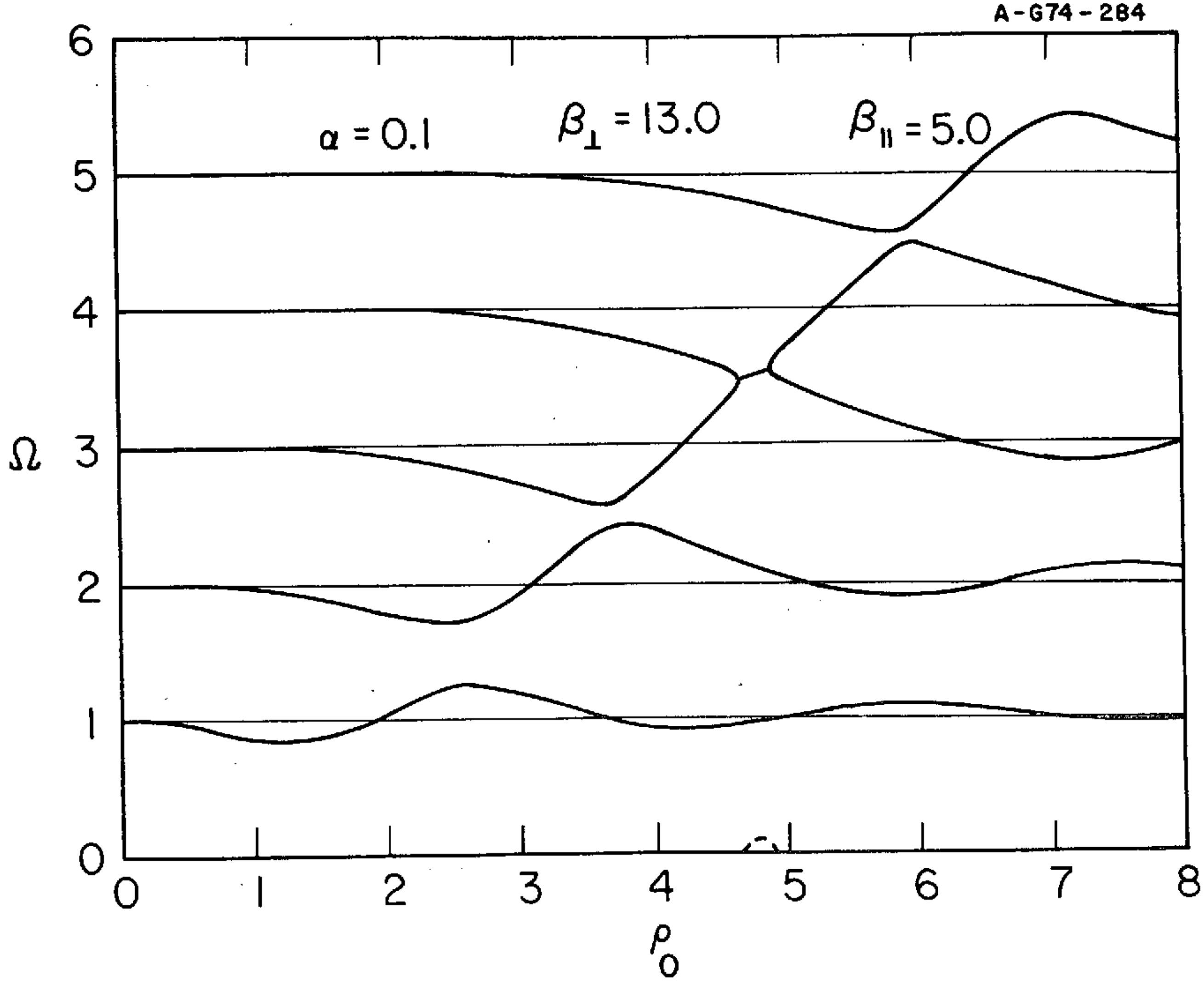

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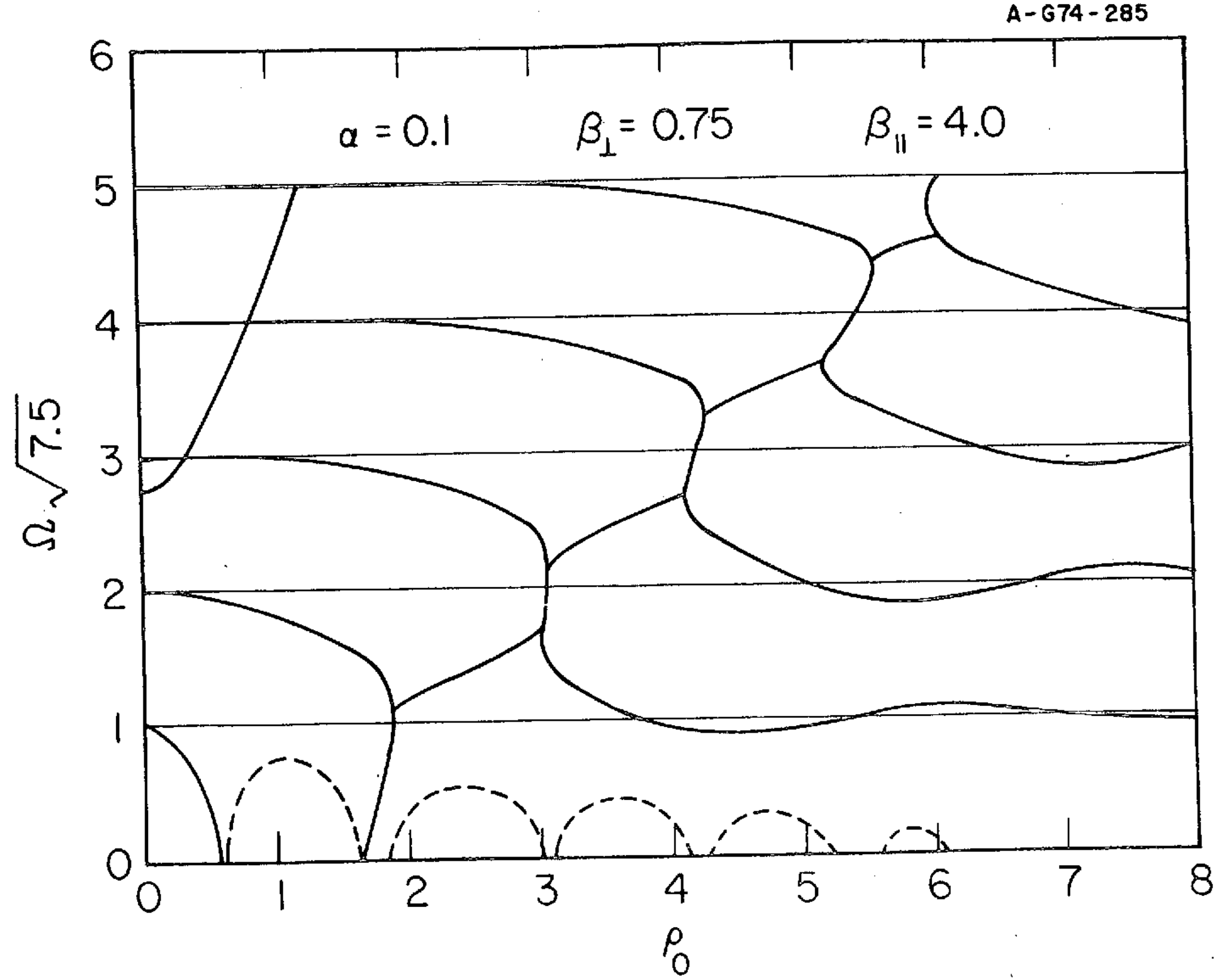

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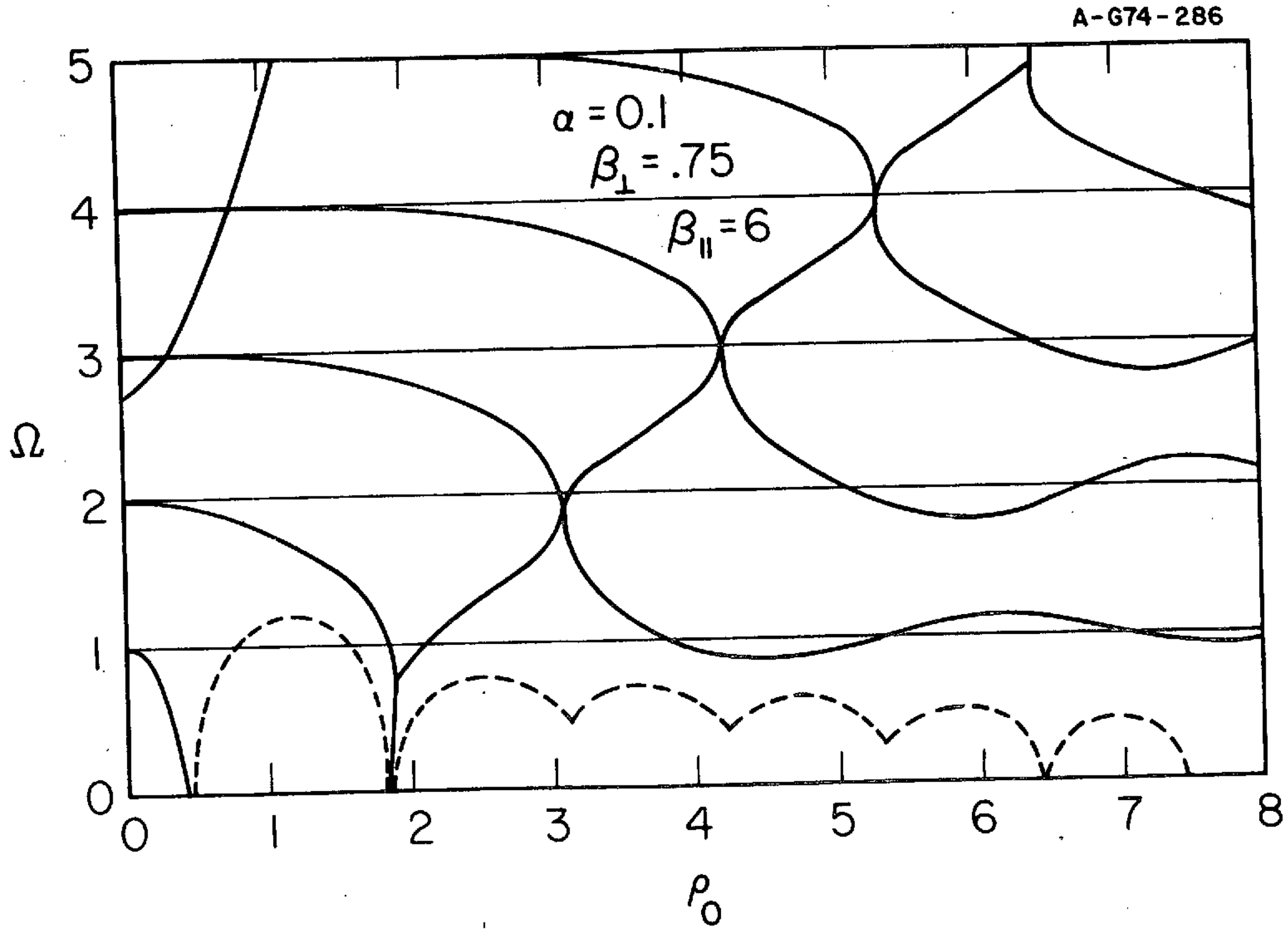

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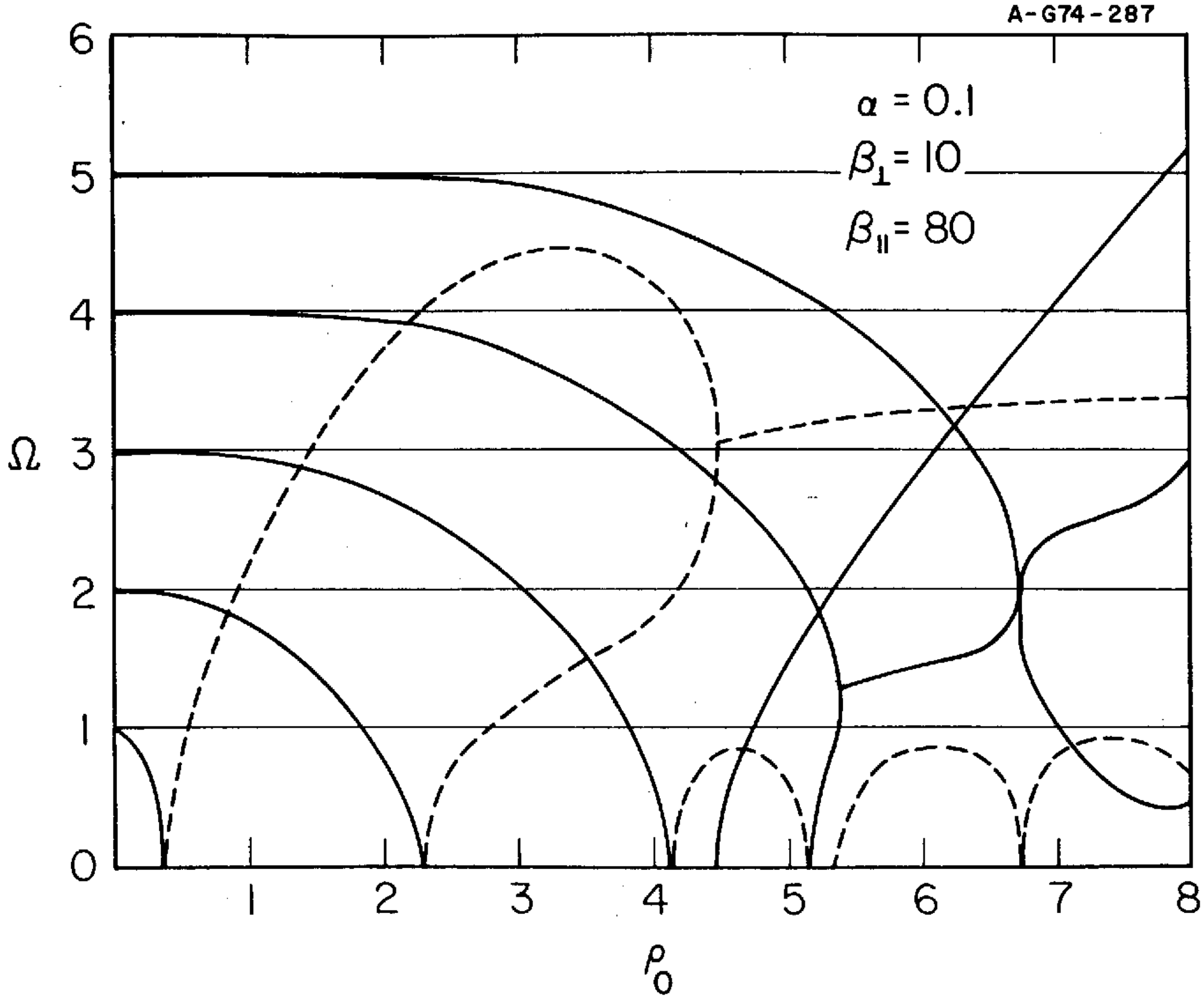

Figure 5

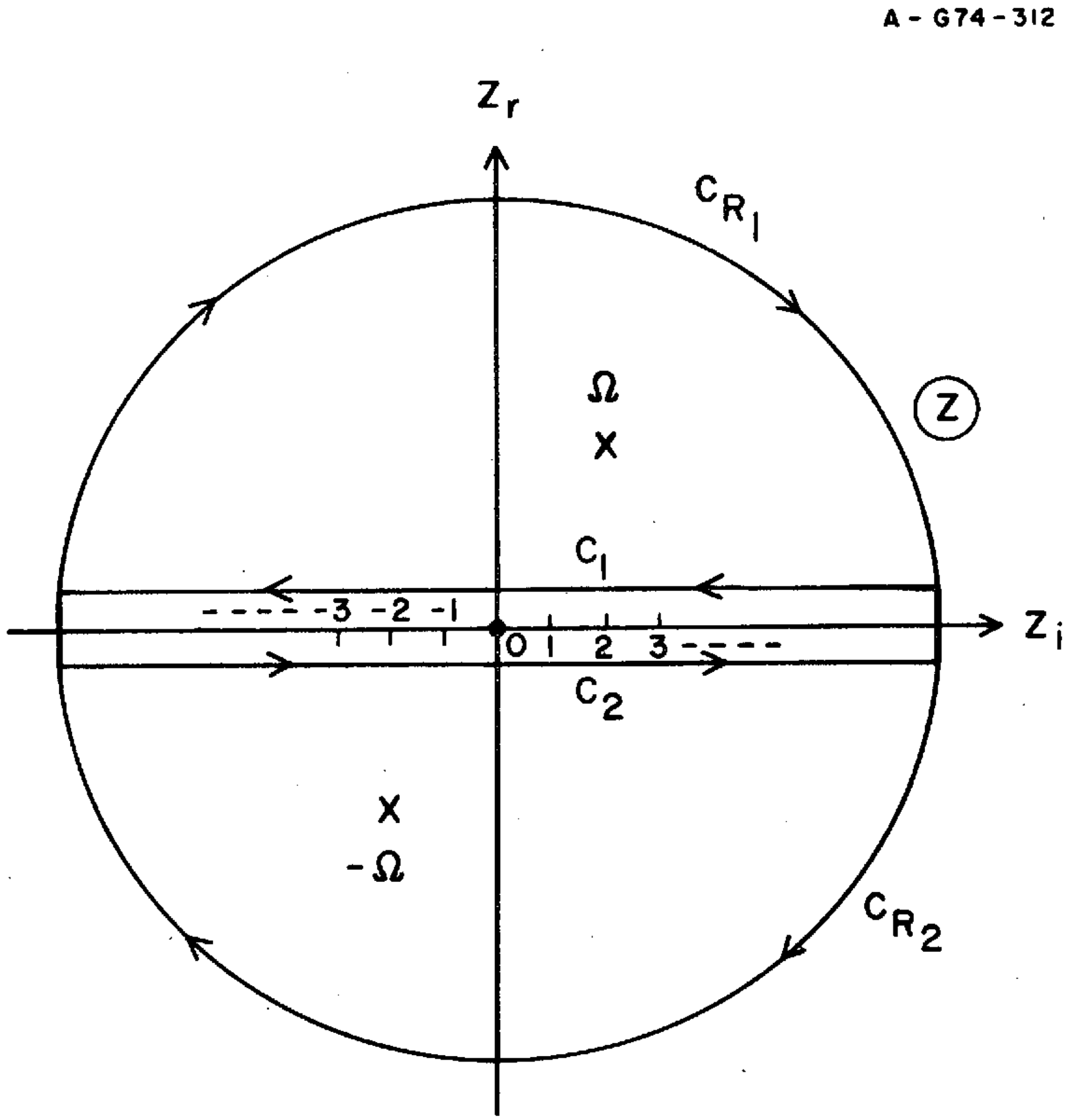

Figure 6