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ACOUSTIC THEORY OF AXISYMMETRIC MULTISECTIONED DUCTS

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16. Abstract

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SUMMARY

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INTRODUCTION

Intense noise is radiated from modern turbofan aircraft engines. This noise may be broadly classified as externally generated noise and internally generated noise. The main external noise source is the engine jet. The main internal noise source is the turbomachinery stages. Steady and unsteady flow through these stages causes fluctuating loads on the blades that generate an acoustic field in the engine. This acoustic field is propagated away from the blades, transmitted through the engine ducts, and radiated to the far field.

Research in the area of internal aircraft engine noise has been traditionally divided into three fields: turbomachinery noise generation, sound transmission in ducts, and radiation from ducts. Although extensive literature is available in each of these fields, little published work which integrates these fields into a general theory exists. A first step toward organizing these fields was taken by Lansing and Zorumski (ref. 1) who treated the problem of a uniform multisectioned duct with a matrix approach. Alfredson (ref. 2) has used a similar approach to solve the problem of a hard-walled acoustic horn by matching the sound fields in a number of disk-shaped elements which approximate the horn shape.

Matrix concepts have been used advantageously in other engineering disciplines, such as structures and electrical networks, to integrate and organize complex problem areas. The electrical theory has already been carried over into acoustics in the one-dimensional approximations used in muffler theory (ref. 3); however, no accepted method has appeared previously for systematically treating multidimensional problems in duct acoustics.

This paper develops a detailed theory for interactions of the acoustic fields of a number of interconnected simple geometric regions. All acoustics problems involving sound generation in, transmission through, and radiation from duct systems are shown to be reducible to the solution of a matrix equation $[W]\{A\} = \{Q\}$. The matrix formulation is suitable for numerical solution on modern digital computers. The matrix notation also gives physical meaning to the numerical operations used in determining the acoustic field. For convenience, a number of simplifications are made in modeling the real duct systems; however, these restrictive simplifications may be removed without altering the essential structure of the resulting equations. The acoustic field is assumed to be governed by the linear convected wave equation. The geometry of the ducts is approximated by a number of interconnected circular and annular cylinders. The steady flow field is approximated by a uniform velocity along the duct axis with no temperature or density gradients. The acoustic treatment on the walls of the ducts is assumed to be acoustically linear. Point sources of sound are used in place of actual distributed sources, and the equations which couple the internal field to the external radiated field are assumed to be known. The matrix formulation is capable of representing more realistic physical situations than the one described above and will be useful as a standard for integrating various theories into a common framework.

Another important purpose of this paper is to show that the interactive effects of acoustic treatment in different regions of an aircraft engine may be used to reduce the total noise radiated from the engine. Past efforts to reduce aircraft noise by using acoustic treatment have depended on the attenuation of waves in ducts. An equally important consideration, wave reflections at impedance discontinuities, has not been utilized. It was shown in reference 1, however, that wave reflections effects are important and that these effects may be used in suppressing noise transmitted through ducts. The matrix formulation adopted in this paper allows these effects to be easily incorporated in noise reduction studies.

SYMBOLS

A wave amplitude

$a = \ell a^*$

a^*, b^*	inner and outer radii, respectively, of an annulus
$b = k b^*$	
c_a	ambient speed of sound
D/Dt	dimensionless comoving derivative based on steady flow convection (eq. (6a))
D/Dt^*	comoving derivative
$D(\Omega)$	eigenvalue determinant
$D(\phi)$, $D(\sin \phi)$	radiation directivity factors
$E(\lambda r)$	eigenfunction for annular duct (eq. (30))
$\vec{e}_z$	unit vector in z-direction
f	frequency, Hz
$\vec{f}$	scaled complex body force amplitude (eq. (4d))
$\vec{f}^*$	body force
I	eigenfunction product integral
J,Y	Bessel functions of first and second kind, respectively
$K_+^m(b,\Omega)$	infinite integral (eq. (64))
k	wave number, ω/c
M	uniform flow Mach number
N	eigenfunction normalizing factor (eqs. (33) and (34))
p	scaled complex acoustic pressure amplitude (eq. (4b))

$p_m(r,\Omega)$	axial Fourier transform of circumferential harmonic of p
p^*	pressure
Q	volumetric source strength, dimensionless, or source mode amplitude when subscripts and superscripts are used
R	reflection coefficient, used with subscripts and superscripts
$R = \lvert\vec{R}\rvert$	
$\vec{R} = k\vec{R}^*$	
$\vec{R}^*$	position vector
r,θ,z	cylindrical coordinates
T	transmission coefficient, used with subscripts and superscripts
$t = \omega t^*$	
t^*	time
$\vec{V}^*$	velocity
$\vec{v}$	scaled complex acoustic velocity amplitude (eq. (4c))
W	duct wave coefficient, used with subscripts and superscripts
β	acoustic admittance, dimensionless
γ	ratio of specific heats of gas
Δ_r	boundary condition operator (eq. (24))
$\delta(x)$	Dirac delta function
$\delta_{\mu\nu}$	Kronecker delta function

$\epsilon << 1$	small scaling parameter
λ	dimensionless radial wave number
ρ	scaled complex acoustic density amplitude (eq. (4a))
ρ_a	ambient density
ρ^*	density
$\psi(r)$	normalized eigenfunctions for annular duct (eqs. (31) and (32))
Ω	axial Fourier transform parameter (propagation constant)
ω	circular frequency, $2\pi f$, radians/sec
$\vec{\nabla} = \vec{\nabla}^*/k$	
$\vec{\nabla}^*$	divergence, or "del" operator
$\partial/\partial n$	partial derivative normal to boundary

Subscripts:

$a = ka^*$	
$b = kb^*$	
m	circumferential mode number
o	denotes source
μ	radial mode number and indicates row number in matrix
ν	radial mode number and indicates column number in matrix

Superscripts:

j,k	indicates plane number

ι, κ indicates general values of j and k

$+$ upstream waves

$-$ downstream waves

Matrix notation:

$\lfloor \; \rfloor$ row matrix

$\{ \; \}$ column matrix

$[\;]$ square matrix

$\lceil \diagdown \rfloor$ diagonal matrix

$[I]$ identity matrix

GENERAL THEORY AND CONCEPTS

This section will present the basic theory of approximating the acoustic field of a typical aircraft turbofan engine such as in figure 1. In order to bring the problem at hand within the scope of present analytical capability, the engine shown in figure 1 will be modeled by the hypothetical engine shown in figure 2. This engine model is simple enough for analytical purposes yet it retains the gross geometrical and flow characteristics of the real engine. The curved engine surfaces have been replaced by uniform cylindrical surfaces to permit a simple representation in cylindrical coordinates, the flow

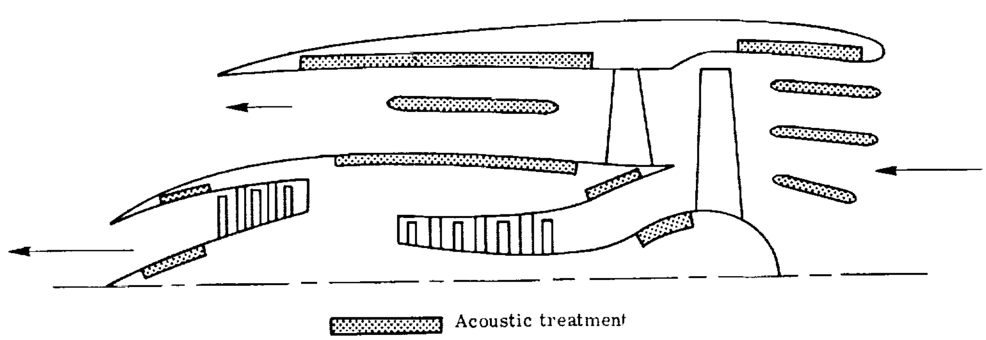

Figure 1.- Cross section of typical aircraft turbofan engine.

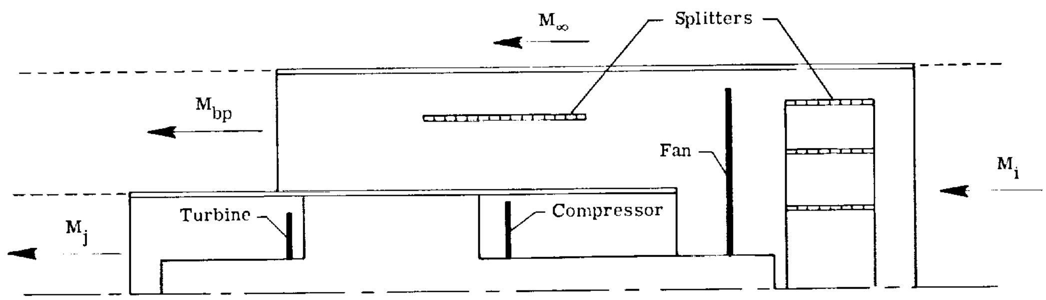

Figure 2.- Engine acoustic model.

field has been replaced by a uniform flow with Mach number M, and the finite size turbomachinery blades have been replaced by blades which have infinitesimal cross sections.

Governing Equations and Boundary Conditions

It will be assumed that all variations in the fluid state are isentropic, except in some small regions which may be considered separately, so that the energy equation may be given as

$$\frac{Dp^*}{Dt^*} = \frac{\gamma p^*}{\rho^*} \frac{D\rho^*}{Dt^*} \tag{1}$$

In addition to equation (1), the continuity and momentum equations are required to determine the fluid motion. These equations are

$$\frac{D\rho^*}{Dt^*} + \rho^* \vec{\nabla}^* \cdot \vec{V}^* = 0 \tag{2}$$

and

$$\rho^* \frac{D\vec{V}^*}{Dt^*} + \vec{\nabla}^* p^* = \rho^* \vec{f}^* \tag{3}$$

In theoretical acoustics, it is convenient to use dimensionless variables where the length and time scales are $1/k$ and $1/\omega$, respectively. Dimensionless lengths are formed by multiplying actual lengths by the wave number, and a dimensionless time is introduced by multiplying by the circular frequency ω. The forms of the continuity and momentum equations may be preserved then if the density and pressure scales are ρ_a and $\rho_a c_a^2$ and the velocity is referred to c_a where ρ_a and c_a are the ambient density and speed of sound, respectively.

The flow field is separated into a steady part, or mean flow, and a small amplitude periodic variation of frequency ω. If a small scaling parameter $\epsilon << 1$ is introduced to account for the small amplitude of these variations, then the dependent field variables may be expressed in complex form as

$$\rho^* = \rho_a(1 + \epsilon\rho)\, e^{-it} \tag{4a}$$

$$p^* = \rho_a c_a^2\left(\frac{1}{\gamma} + \epsilon p\right) e^{-it} \tag{4b}$$

$$\vec{V}^* = c_a\left(-M\vec{e}_z + \epsilon\vec{v}\right) e^{-it} \tag{4c}$$

$$\vec{f}^* = \omega c_a \epsilon\vec{f}\, e^{-it} \tag{4d}$$

and the independent variables are

$$t^* = \frac{t}{\omega} \tag{5a}$$

and

$$\vec{R}^* = \frac{\vec{R}}{k} \tag{5b}$$

Equations (4) show that the mean flow field is assumed to have constant density, pressure, and velocity. Note that equation (4c) implies that the mean flow is conventionally in the minus z-direction.

With the approximations of equations (4) and the definitions of equations (5), the differential operator for the material derivative and "del" become

$$\frac{D}{Dt^*} = \omega\left(\frac{D}{Dt} + \epsilon\vec{v}\cdot\vec{\nabla}\right) \tag{6a}$$

and

$$\vec{\nabla}^* = k\vec{\nabla} \tag{6b}$$

In equation (6a), the part of the material derivative due to the acoustic velocity is shown explicitly so that it is understood that the dimensionless material derivative contains the convection effect of steady flow only.

Substituting equations (4) and (6) into equations (1), (2), and (3) and collecting coefficients of like powers of ϵ show that the acoustic equations are

$$\frac{Dp}{Dt} = \frac{D\rho}{Dt} \tag{7a}$$

$$\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{v} = 0 \tag{7b}$$

$$\frac{D\vec{v}}{Dt} + \vec{\nabla}p = \vec{f} \tag{7c}$$

The wave equation is derived by eliminating the velocity from the continuity and momentum equations. When the steady flow velocity is a constant, as in equation (4c), the operators $\frac{D}{Dt}$ and $\vec{\nabla}\cdot$ commute; therefore, the wave equation becomes

$$\nabla^2 p = \frac{D^2 p}{D^2 t} + \vec{\nabla} \cdot \vec{f} \tag{8}$$

where

$$\frac{D}{Dt} = -\left(i + M \frac{\partial}{\partial z}\right) \tag{9}$$

Eversman and Beckemeyer (ref. 4) have shown that an approximation to the wall boundary condition where there is a sudden change in the mean flow velocity is the so-called particle displacement boundary condition

$$i\beta \frac{D^2 p}{Dt^2} + \frac{\partial p}{\partial n} = 0 \tag{10}$$

This boundary condition will be used later in the present paper.

Concept of Reflection and Transmission

In this section, a brief heuristic discussion of transmission and reflection concepts is given. These concepts are developed from the physics of one-dimensional wave propagation. The equation forms postulated here guide the later development of a mathematical theory of transmission and reflection.

One-dimensional waves.- Figure 3 shows schematically a duct section where there is a wave represented by the symbol A, traveling forward through the plane j. Due to some properties of the duct section between planes j and k, the amplitude of this wave may be modified before it reaches plane k. This change is shown symbolically by the equation

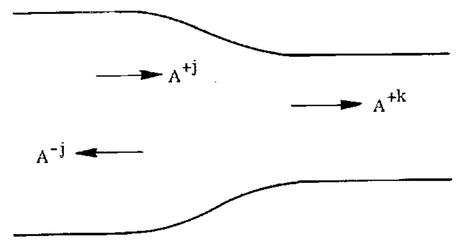

Figure 3.- Mode transmission and reflection in simple duct section.

$$A^{+k} = T^{+k+j}A^{+j} \tag{11}$$

The factor T^{+k+j} in equation (11) is called the transmission coefficient for the section. There may also be a wave reflected from the section, as shown symbolically by the backward-moving wave A^{-j} in figure 3. This reflection effect may be represented by the equation

$$A^{-j} = R^{-j+j}A^{+j} \tag{12}$$

where R^{-j+j}, which is called the reflection coefficient, depends on the properties of the duct section between planes j and k.

Similar effects may be expected if there is a backward-moving wave A^{-k} incident on the section. In this case, the transmission and reflection effects are represented by

$$A^{-j} = T^{-j-k}A^{-k} \tag{13}$$

$$A^{+k} = R^{+k-k}A^{-k} \tag{14}$$

If there are sound sources in a duct section, as shown in figure 4, there will be waves propagated away from the source in both directions. This wave generation is represented by source terms Q^{+} and Q^{-} for the forward and backward waves, respectively.

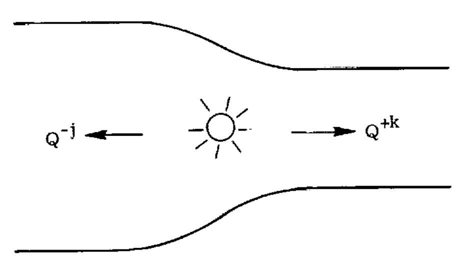

Figure 4.- Mode generation by source in duct section.

The general sound field in a duct section is a superposition of the fields due to incident waves from both directions and due to sources within the section. Consequently, two general equations may be written to describe the effects of a section on the acoustic wave at either end of the section.

$$A^{+k} = T^{+k+j}A^{+j} + R^{+k-k}A^{-k} + Q^{+k} \tag{15}$$

$$A^{-j} = T^{-j-k}A^{-k} + R^{-j+j}A^{+j} + Q^{-j} \tag{16}$$

Equations (15) and (16) are linear algebraic equations which represent the superposition of the sound fields due to various effects. Equations of this form may always be derived since the governing partial differential equations for the sound field are linear.

Two-dimensional waves.- There are many waves with characteristic wave-front shapes which may propagate in a channel. A general two-dimensional wave in a channel is formed by superimposing these characteristic waves. The amplitude of this general wave is then a vector formed from the amplitudes of the characteristic waves. Therefore, the wave generation, transmission, and reflection effects must be represented by matrix equations rather than the simple scalar forms in equations (15) and (16). These matrix equations are assumed to be

$$\left\{A^{+k}_{m\mu}\right\} = \left[T^{+k+j}_{m\mu\nu}\right]\left\{A^{+j}_{m\nu}\right\} + \left[R^{+k-k}_{m\mu\nu}\right]\left\{A^{-k}_{m\nu}\right\} + \left\{Q^{+k}_{m\mu}\right\} \tag{17}$$

$$\left\{A^{-j}_{m\mu}\right\} = \left[T^{-j-k}_{m\mu\nu}\right]\left\{A^{-k}_{m\nu}\right\} + \left[R^{-j+j}_{m\mu\nu}\right]\left\{A^{+j}_{m\nu}\right\} + \left\{Q^{-j}_{m\mu}\right\} \tag{18}$$

Some detailed examples of these equations will be presented in the next section.

SOURCE IN INFINITE UNIFORM DUCT

The sound field due to a monopole source in an infinite annular duct may be found by solving equation (8), with $\vec{\nabla} \cdot \vec{f}$ replaced by a Dirac delta function, subject to the boundary conditions in equation (10). In cylindrical coordinates, the wave equation (eq. (8)) is

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} - 2iM\frac{\partial p}{\partial z} + (1 - M^2)\frac{\partial^2 p}{\partial z^2} + p = \frac{2\pi}{r} Q\delta(r-r_o)\ \delta(\theta-\theta_o)\ \delta(z-z_o) \tag{19}$$

and the boundary conditions are

$$\left.\frac{\partial p}{\partial r} \pm i\beta_r\left(i + M\frac{\partial}{\partial z}\right)^2 p\right|_{\substack{r=b\\ \text{or}\\ r=a}} = 0 \qquad \begin{pmatrix} + \text{ for } & r = b \\ - \text{ for } & r = a \end{pmatrix} \tag{20}$$

The solution to this problem in the special case where $\beta_r = 0$ has been given previously by Drischler (ref. 5). Following Drischler's approach, it is assumed that the solution to equations (19) and (20) may be found as a circumferential series of inverse Fourier transforms

$$p(r,\theta,z) = \sum_{m=-\infty}^{\infty} e^{im\theta} \frac{1}{2\pi} \int_{-\infty}^{\infty} p_m(r,\Omega)\, e^{i\Omega z}\, d\Omega \tag{21}$$

Substituting equation (21) into equations (19) and (20) gives

$$\frac{1}{r}\frac{d}{dr}\left[\frac{r\, dp_m(r,\Omega)}{dr}\right] + \left[\lambda^2 - \frac{m^2}{r^2}\right] p_m(r,\Omega) = \frac{Q\delta(r-r_o)}{r}\, e^{-i\left(m\theta_o + \Omega z_o\right)} \tag{22}$$

where

$$\lambda = \sqrt{1 + 2M\Omega - (1 - M^2)\Omega^2} \tag{23}$$

and

$$\Delta_r\left[p_m(r,\Omega)\right] = \left.\left[\frac{dp_m(r,\Omega)}{dr} \pm i\beta_r(1 + M\Omega)^2\, p_m(r,\Omega)\right]\right|_{\substack{r=b\\ \text{or}\\ r=a}} = 0 \qquad \begin{pmatrix} - \text{ for } & r = b \\ + \text{ for } & r = a \end{pmatrix} \tag{24}$$

$$p_m(r,\Omega) = AJ_m(\lambda r) + BY_m(\lambda r) - \frac{\pi}{2} Q\, e^{-i\left(m\theta_o + \Omega z_o\right)} \begin{Bmatrix} J_m(\lambda r_o)\, Y_m(\lambda r) & (r \leqq r_o) \\ J_m(\lambda r)\, Y_m(\lambda r_o) & (r \geqq r_o) \end{Bmatrix} \tag{25}$$

The first two terms of equation (25) are homogeneous solutions of equation (22), and the last term has a slope discontinuity which is introduced by the delta function. The coefficients A and B in equation (25) must be chosen to satisfy the boundary conditions of equation (24). Therefore,

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = \frac{1}{D_m(\Omega)} \begin{bmatrix} \Delta_b\left[Y_m\right] & -\Delta_a\left[Y_m\right] \\ -\Delta_b\left[J_m\right] & \Delta_a\left[J_m\right] \end{bmatrix} \begin{Bmatrix} J_m(\lambda r_o)\ \Delta_a\left[Y_m\right] \\ Y_m(\lambda r_o)\ \Delta_b\left[J_m\right] \end{Bmatrix} \frac{\pi}{2} Q\, e^{-i\left(m\theta_o+\Omega z_o\right)} \tag{26}$$

where

$$D_m(\Omega) = \Delta_a\left[J_m\right] \Delta_b\left[Y_m\right] - \Delta_a\left[Y_m\right] \Delta_b\left[J_m\right] \tag{27}$$

Substituting equation (26) into equation (25) gives the expressions for the transforms of the pressure harmonics $p_m(r,\Omega)$ in the regions $r \leqq r_o$ and $r \geqq r_o$ as follows:

$$p_m(r,\Omega) = \frac{\pi}{2}\frac{Q\, e^{-i(m\theta_o+\Omega z_o)}}{D_m(\Omega)} \begin{Bmatrix} \left[\Delta_a\left[Y_m\right] J_m(\lambda r) - \Delta_a\left[J_m\right] Y_m(\lambda r)\right]\left[\Delta_b\left[Y_m\right] J_m(\lambda r_o) - \Delta_b\left[J_m\right] Y_m(\lambda r_o)\right] & (r \leqq r_o) \\ \left[\Delta_b\left[Y_m\right] J_m(\lambda r) - \Delta_b\left[J_m\right] Y_m(\lambda r)\right]\left[\Delta_a\left[Y_m\right] J_m(\lambda r_o) - \Delta_a\left[J_m\right] Y_m(\lambda r_o)\right] & (r \geqq r_o) \end{Bmatrix} \tag{28}$$

It may be shown that $D_m(\Omega)$ and the other expressions on the right side of equation (28) are analytic functions of Ω and that the inverse transform (eq. (21)) may be evaluated by the theory of residues. If $z > z_o$, a contour which circles the upper half of the Ω-plane is used and, if $z < z_o$, a contour circling the lower half-plane is used. Substituting equation (28) into equation (21) gives

$$p(r,\theta,z) = \pm\frac{\pi Q i}{2} \sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_o)} \sum_{\mu=1}^{\infty} e^{i\Omega_{m\mu}^{\pm}(z-z_o)} \frac{\Delta_a\left[Y_m\left(\lambda_{m\mu}^{\pm} r\right)\right] \Delta_b\left[Y_m\left(\lambda_{m\mu}^{\pm} r\right)\right]}{D_m'\left(\Omega_{m\mu}^{\pm}\right)} E_m\left(\lambda_{m\mu}^{\pm} r\right) E_m\left(\lambda_{m\mu}^{\pm} r_o\right) \tag{29}$$

where

$$E_m\left(\lambda_{m\mu}^{\pm} r\right) = J_m\left(\lambda_{m\mu}^{\pm} r\right) - \frac{\Delta_b\left[J_m\left(\lambda_{m\mu}^{\pm} r\right)\right]}{\Delta_b\left[Y_m\left(\lambda_{m\mu}^{\pm} r\right)\right]} Y_m\left(\lambda_{m\mu}^{\pm} r\right) \tag{30}$$

The constants $\Omega_{m\mu}$ are the zeros of equation (27), and equation (23) defines $\lambda_{m\mu}$.

In equations (29) and (30) the + sign is used for $z > z_o$ and the - sign is used when $z < z_o$. It will be convenient in the following analysis to use characteristic functions $\psi_{m\mu}(r)$ which are proportional to $E_m\left(\lambda_{m\mu} r\right)$ but which are normalized such that

$$\int_a^b r\psi_{m\mu}^2(r)\,dr = 1 \tag{31}$$

Consequently, $\psi_{m\mu}(r)$ is defined by

$$\psi_{m\mu}(r) = \frac{E_m\left(\lambda_{m\mu} r\right)}{N_{m\mu}} \tag{32}$$

where

$$N_{m\mu}^2 = \int_a^b rE_m^2\left(\lambda_{m\mu} r\right)\,dr \tag{33}$$

The integral (eq. (33)) may be evaluated (ref. 6, p. 135), in closed form

$$N_{m\mu}^2 = \frac{1}{2}\left\{r^2\left[1 - \frac{m^2}{\lambda_{m\mu}^2 r^2} - \beta_r^2\,\frac{\left(1 + M\Omega_{m\mu}\right)^4}{\lambda_{m\mu}^2}\right]E_m^2\left(\lambda_{m\mu} r\right)\right\}\Bigg|_a^b \tag{34}$$

The characteristic functions $\psi_{m\mu}(r)$ are not always orthogonal. If $\lambda_{m\mu} \neq \lambda_{m\nu}$, then it may be shown that if

$$I_{m\mu\nu} = \int_a^b r\psi_{m\mu}(r)\,\psi_{m\nu}(r)\,dr \tag{35}$$

then

$$I_{m\mu\nu} = \frac{2M + M^2\left(\Omega_{m\mu} + \Omega_{m\nu}\right)}{1 - \left[2M + M^2\left(\Omega_{m\mu} + \Omega_{m\nu}\right)\right]}\left[ia\beta_a\psi_{m\mu}(a)\,\psi_{m\nu}(a) + ib\beta_b\psi_{m\mu}(b)\,\psi_{m\nu}(b)\right] \tag{36}$$

If $M = 0$ or if both β_a and β_b are zero, then equation (36) shows that $\psi_m\left(\lambda_{m\mu} r\right)$ is orthogonal to $\psi_m\left(\lambda_{m\nu} r\right)$, that is, $I_{m\mu\nu} = 0$ when $\mu \neq \nu$. Equations (31) to (36) are valid for both upstream (+) modes and downstream (-) modes so that (±) superscripts may be placed in these equations when they are required.

Tester (ref. 7) has shown that $N_{m\mu}$ may be zero in certain cases which correspond to values of the wall admittance where $D'_m\left(\Omega_m\right) = 0$. These cases are excluded here.

The expression for the sound field due to a single source may now be written in terms of the characteristic functions as follows:

$$p(r,\theta,z) = \sum_{m=-\infty}^{\infty} e^{im\theta} \sum_{\mu=1}^{\infty} Q^{\pm}_{m\mu}(z)\, \psi^{\pm}_{m\mu}(r) \tag{37}$$

where

$$Q^{\pm}_{m\mu}(z) = \pm \frac{\pi}{2} Q\, e^{i\left[\Omega^{\pm}_{m\mu}(z-z_o) - m\theta_o\right]} \left\{ \frac{\Delta_a\left[Y_m\left(\lambda^{\pm}_{m\mu} r\right)\right] \Delta_b\left[Y_m\left(\lambda^{\pm}_{m\mu} r\right)\right]}{\left(N^{\pm}_{m\mu}\right)^{-2} D'_m\left(\Omega^{\pm}_{m\mu}\right)} \right\} \psi^{\pm}_{m\mu}(r_o) \tag{38}$$

The expression for the modal source strength (eqs. (38)) may be extended easily to represent a number of superimposed monopole sources Q_i or a source distribution $Q\left(\vec{r}_o\right)$ over some finite region. The method of extending this solution to obtain a dipole source representing the force on a rotating blade has been given by Drischler (ref. 5).

UNIFORM MULTISECTIONED DUCT (FIG. 5)

In practical problems such as the uniform multisectioned duct depicted in figure 5 there are a number of finite duct sections which are interconnected. Therefore, the infinite duct solution (eq. (37)) must be generalized to account for the effects of these finite sections and relationships must be developed to account for the acoustic coupling between sections by (1) defining the acoustic field in terms of wave

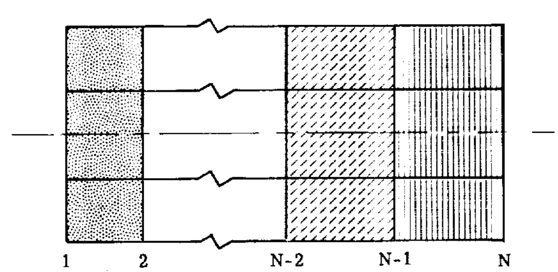

Figure 5.- Uniform multisectioned ducts.

amplitudes at a number of planes in the duct, (2) developing transmission and reflection relationships to relate these wave amplitudes, and (3) solving a matrix equation for the wave amplitudes.

Field at a Plane

In a uniform duct section containing the plane z^j = Constant, the acoustic field may be expressed as a sum of the acoustic modes $\psi_{m\mu}(r)$. Since the duct sections are axisymmetric, the discussion may be limited to the circumferential harmonics of the sound field $p_m(r,z)$ which is given by the following equation:

$$p_m(r,z) = \sum_{\mu=1}^{\infty} \left[A_{m\mu}^{+j}\, \psi_{m\mu}^{+j}(r)\, e^{i\Omega_{m\mu}^{+j}\left(z-z^j\right)} - A_{m\mu}^{-j}\, \psi_{m\mu}^{-j}(r)\, e^{i\Omega_{m\mu}^{-j}\left(z-z^j\right)} \right] \tag{39a}$$

The coefficients $A_{m\mu}^{+j}$ and $A_{m\mu}^{-j}$ in equation (39a) are the amplitudes of the acoustic modes at z^j. The negative sign in front of $A_{m\mu}^{-j}$ has been introduced so that the mode reflection coefficients will have a positive real part. In order to represent the sound field everywhere in a complicated system of ducts, a large number of planes z^j are introduced. The fundamental problem of determining the acoustic field is then reduced to finding the wave amplitudes $A_{m\mu}^{+j}$ and $A_{m\mu}^{-j}$ at these planes. Thus, if there are N planes then there must be 2N sets of equations relating these coefficients. Each of these sets contains an infinite number of equations; however, it is necessary in practice to truncate these sets to a finite number which determines a finite number of the coefficients $A_{m\mu}^{\pm j}$ in equation (39a). Matrix notation is used to write these finite sets of equations. With matrix notation, equation (39a) becomes

$$p_m(r,z) = \left\lfloor \psi_{m\mu}^{+j}(r) \right\rfloor \left[\delta_{\mu\nu}\, e^{i\Omega_{m\mu}^{+j}\left(z-z^j\right)} \right] \left\{ A_{m\nu}^{+j} \right\} - \left\lfloor \psi_{m\mu}^{-j}(r) \right\rfloor \left[\delta_{\mu\nu}\, e^{i\Omega_{m\nu}^{-j}\left(z-z^j\right)} \right] \left\{ A_{m\nu}^{-j} \right\} \tag{39b}$$

Transmission and Reflection Matrices

The second step in solving the multisectioned duct problem is to find the transmission and reflection equations which relate wave amplitudes at the specified duct planes. Two methods of finding these equations will be illustrated in this section: the method of solving the wave equation in the section between two planes and the method of matching the solutions at two planes which are close together by use of the continuity and momentum conservation conditions.

Uniform duct.- The transmission and reflection matrices for a uniform duct section may be obtained by inspection of equation (39b). Since there are no reflections in a uniform duct,

$$\left[R^{\pm j \pm k}_{m\mu\nu}\right] = \left[0\right] \tag{40}$$

and since the change of the mode amplitudes is given by the exponential factors in equation (39b),

$$\left[T^{\pm k \pm j}_{m\mu\nu}\right] = \left[\delta_{\mu\nu}\, e^{i\Omega^{\pm}_{m\mu}\left(z^k - z^j\right)}\right] \tag{41}$$

In equation (41) the positive sign is used for $z^k > z^j$ and the negative sign is used for $z^k < z^j$.

This example shows how a solution to the wave equation may be used to find transmission and reflection equations. It may be shown that, if a Green's function for a region exists, then transmission and reflection equations of the form of equations (17) and (18) may be derived from this Green's function.

Admittance discontinuity.- When there is a discontinuity in the wall admittance of an annular duct section with uniform inner and outer radii as shown in figure 6, then the conditions of conservation of mass and axial momentum at the interface may be used to derive equations for the transmission and reflection matrices. These conditions are satisfied if the axial velocity and the pressure are continuous at the interface. The axial velocities due to the waves are related to the pressures through the axial modal admittance matrix $\left[\beta_{m\nu}\right]$, which the momentum equation shows is $\left[\Omega_{m\nu}/(1 + M\Omega_{m\nu})\right]$. Consequently, for planes z^j and z^k, which are an infinitesimal distance ϵ apart as shown in figure 6,

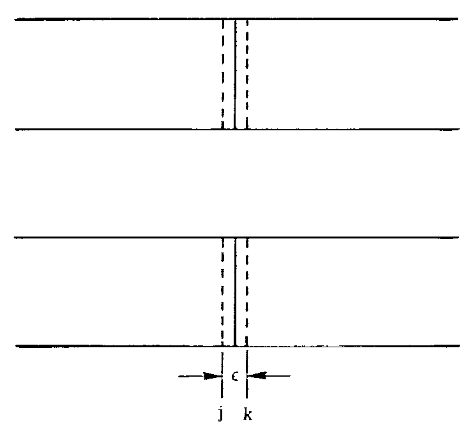

Figure 6.- Duct with admittance discontinuity.

$$\left\lfloor \psi^{-j}_{m\nu}(r) \right\rfloor \left\lceil \beta^{-j}_{m\nu} \right\rfloor \left\{ A^{-j}_{m\nu} \right\} + \left\lfloor \psi^{+k}_{m\nu}(r) \right\rfloor \left\lceil \beta^{+k}_{m\nu} \right\rfloor \left\{ A^{+k}_{m\nu} \right\} = \left\lfloor \psi^{+j}_{m\nu}(r) \right\rfloor \left\lceil \beta^{+j}_{m\nu} \right\rfloor \left\{ A^{+j}_{m\nu} \right\} + \left\lfloor \psi^{-k}_{m\nu}(r) \right\rfloor \left\lceil \beta^{-k}_{m\nu} \right\rfloor \left\{ A^{-k}_{m\nu} \right\} \tag{42}$$

$$\left\lfloor \psi^{-j}_{m\nu}(r) \right\rfloor \left\{ A^{-j}_{m\nu} \right\} + \left\lfloor \psi^{+k}_{m\nu}(r) \right\rfloor \left\{ A^{+k}_{m\nu} \right\} = \left\lfloor \psi^{+j}_{m\nu}(r) \right\rfloor \left\{ A^{+j}_{m\nu} \right\} + \left\lfloor \psi^{-k}_{m\nu}(r) \right\rfloor \left\{ A^{-k}_{m\nu} \right\} \tag{43}$$

The terms in equations (42) and (43) have been arranged so that the left-hand sides contain waves traveling out of the region between planes z^j and z^k while the right-hand sides contain terms traveling into the region. In order to obtain an equation like equation (17), it is necessary to eliminate $\left\{ A^{-j}_{m\nu} \right\}$ from equations (42) and (43). In the special case where $M = 0$, this elimination may be carried out by utilizing the orthogonal properties of the modes $\psi^{-j}_{m\mu}(r)$. If $M = 0$, then premultiplying equation (42) by $r \left\{ \psi^{-j}_{m\mu}(r) \right\} dr$, premultiplying equation (43) by $r \left\lceil \Omega^{-j}_{m\mu} \right\rfloor \left\{ \psi^{-j}_{m\mu}(r) \right\} dr$, integrating each equation from a to b, and subtracting the resulting matrix equations would eliminate $\left\{ A^{-j}_{m\nu} \right\}$. The elimination process will be started in the same manner in the general case where the modes are not orthogonal so that the resulting equations will reduce to the orthogonal mode case. Premultiplying equations (42) and (43) by the modal vectors and integrating give

$$\left[I^{-j-j}_{m\mu\nu} \right] \left\lceil \beta^{-j}_{m\nu} \right\rfloor \left\{ A^{-j}_{m\nu} \right\} + \left[I^{-j+k}_{m\mu\nu} \right] \left\lceil \beta^{+k}_{m\nu} \right\rfloor \left\{ A^{+k}_{m\nu} \right\} = \left[I^{-j+j}_{m\mu\nu} \right] \left\lceil \beta^{+j}_{m\nu} \right\rfloor \left\{ A^{+j}_{m\nu} \right\} + \left[I^{-j-k}_{m\mu\nu} \right] \left\lceil \beta^{-k}_{m\nu} \right\rfloor \left\{ A^{-k}_{m\nu} \right\} \tag{44}$$

$$\left[I^{-j-j}_{m\mu\nu} \right] \left\{ A^{-j}_{m\nu} \right\} + \left[I^{-j+k}_{m\mu\nu} \right] \left\{ A^{+k}_{m\nu} \right\} = \left[I^{-j+j}_{m\mu\nu} \right] \left\{ A^{+j}_{m\nu} \right\} + \left[I^{-j-k}_{m\mu\nu} \right] \left\{ A^{-k}_{m\nu} \right\} \tag{45}$$

where

$$\left[I^{\ell\kappa}_{m\mu\nu}\right] = \int_a^b r\left[\left\{\psi^{\ell}_{m\mu}(r)\right\}\left\lfloor\psi^{\kappa}_{m\nu}(r)\right\rfloor\right] dr \tag{46}$$

is called the mode integral matrix. Eliminating $\left\{A^{-j}_{m\nu}\right\}$ from equations (44) and (45) gives

$$\left[W^{-j+k}_{m\mu\nu}\right]\left\{A^{+k}_{m\nu}\right\} = \left[W^{-j+j}_{m\mu\nu}\right]\left\{A^{+j}_{m\nu}\right\} + \left[W^{-j-k}_{m\mu\nu}\right]\left\{A^{-k}_{m\nu}\right\} \tag{47}$$

where

$$\left[W^{\ell\kappa}_{m\mu\nu}\right] = \left[I^{\ell\ell}_{m\mu\nu}\right]^{-1}\left[I^{\ell\kappa}_{m\mu\nu}\right]\left[\beta^{\kappa}_{m\nu}\right] - \left[\beta^{\ell}_{m\mu}\right]\left[I^{\ell\ell}_{m\mu\nu}\right]^{-1}\left[I^{\ell\kappa}_{m\mu\nu}\right] \tag{48}$$

Now, a premultiplication of equation (47) by $\left[W^{-j+k}_{m\mu\nu}\right]^{-1}$ gives an equation of the form of equation (17) from which the transmission and reflection matrices may be identified as

$$\left[T^{+k+j}_{m\mu\nu}\right] = \left[W^{-j+k}_{m\mu\nu}\right]^{-1}\left[W^{-j+j}_{m\mu\nu}\right] \tag{49}$$

$$\left[R^{+k-k}_{m\mu\nu}\right] = \left[W^{-j+k}_{m\mu\nu}\right]^{-1}\left[W^{-j-k}_{m\mu\nu}\right] \tag{50}$$

It may be shown, by eliminating the vector $\left\{A^{+k}_{m\nu}\right\}$ from equations (42) and (43), that

$$\left[W^{+k-j}_{m\mu\nu}\right]\left\{A^{-j}_{m\nu}\right\} = \left[W^{+k-k}_{m\mu\nu}\right]\left\{A^{-k}_{m\nu}\right\} + \left[W^{+k+j}_{m\mu\nu}\right]\left\{A^{+j}_{m\nu}\right\} \tag{51}$$

and comparing equation (51) with equation (18) shows that

$$\left[T^{-j-k}_{m\mu\nu}\right] = \left[W^{+k-j}_{m\mu\nu}\right]^{-1}\left[W^{+k-k}_{m\mu\nu}\right] \tag{52}$$

$$\left[R^{-j+j}_{m\mu\nu}\right] = \left[W^{+k-j}_{m\mu\nu}\right]^{-1}\left[W^{+k+j}_{m\mu\nu}\right] \tag{53}$$

When $M = 0$ so that the modes are orthogonal, the expression for the matrix $\left[W^{\iota\kappa}_{m\mu\nu}\right]$ may be simplified since $\left[I^{\iota\iota}_{m\mu\nu}\right]$ is an identity matrix in this case. Carrying out the integration in equation (46) gives its elements

$$\begin{aligned} I^{\iota\kappa}_{m\mu\nu} = &\left\{\left(\Omega^{\iota}_{m\mu} - \Omega^{\kappa}_{m\nu}\right)\left[\left(1 - M^2\right)\left(\Omega^{\iota}_{m\mu} + \Omega^{\kappa}_{m\nu}\right) - 2M\right]\right\}^{-1}\left\{ib\left[\beta^{\iota}_{b}\left(1 + M\Omega^{\iota}_{m\mu}\right)^2\right.\right. \\ &\left. - \beta^{\kappa}_{b}\left(1 + M\Omega^{\kappa}_{m\nu}\right)^2\right]\psi^{\iota}_{m\mu}(b)\ \psi^{\kappa}_{m\nu}(b) + ia\left[\beta^{\iota}_{a}\left(1 + M\Omega^{\iota}_{m\mu}\right)^2\right. \\ &\left.\left. - \beta^{\kappa}_{a}\left(1 + M\Omega^{\kappa}_{m\nu}\right)^2\right]\psi^{\iota}_{m\mu}(a)\ \psi^{\kappa}_{m\nu}(a)\right\} \end{aligned} \tag{54}$$

and in the special case when $M = 0$, equation (54) reduces to

$$I^{\iota\kappa}_{m\mu\nu} = \frac{ib\left(\beta^{\iota}_{b} - \beta^{\kappa}_{b}\right)\psi^{\iota}_{m\mu}(b)\ \psi^{\kappa}_{m\nu}(b) + ia\left(\beta^{\iota}_{a} - \beta^{\kappa}_{a}\right)\psi^{\iota}_{m\mu}(a)\ \psi^{\kappa}_{m\nu}(a)}{\left(\Omega^{\iota}_{m\mu} - \Omega^{\kappa}_{m\nu}\right)\left(\Omega^{\iota}_{m\mu} + \Omega^{\kappa}_{m\nu}\right)} \tag{55}$$

Therefore, if $M = 0$,

$$W^{\iota\kappa}_{m\mu\nu} = \frac{ib\left(\beta^{\kappa}_{b} - \beta^{\iota}_{b}\right) \psi^{\iota}_{m\mu}(b)\, \psi^{\kappa}_{m\nu}(b) + ia\left(\beta^{\kappa}_{a} - \beta^{\iota}_{a}\right) \psi^{\iota}_{m\mu}(a)\, \psi^{\kappa}_{m\nu}(a)}{\Omega^{\iota}_{m\mu} + \Omega^{\kappa}_{m\nu}} \tag{56}$$

Equations (50), (53), and (56) show that the reflection matrices are explicitly proportioned to the admittance changes in the duct since $\left[W^{-j-k}_{m\mu\nu}\right]$ and $\left[W^{+k+j}_{m\mu\nu}\right]$ are proportioned to the admittance changes. Note that $\left[W^{-j+j}_{m\mu\nu}\right]$ and $\left[W^{+k-k}_{m\mu\nu}\right]$ are not proportional to the admittance changes since $\Omega^{-j}_{m\mu} = -\Omega^{+j}_{m\mu}$ when $M = 0$. It may be seen from equation (46) that, since $\lambda^{+j}_{m\mu} = \lambda^{-j}_{m\mu}$ if $M = 0$, $I^{-j+j}_{m\mu\nu} = I^{+j+j}_{m\mu\nu} = \delta_{\mu\nu}$; therefore, equation (48) shows that $W^{-j+j}_{m\mu\nu} = 2\delta_{\mu\nu}\Omega^{+j}_{m\nu}$ and $W^{+k-k}_{m\mu\nu} = 2\delta_{\mu\nu}\Omega^{+k}_{m\nu}$.

Radiation Equations

If equations like equations (17) and (18) are repeatedly written for a number of planes, the number of unknowns always exceeds the number of equations. In order to have a complete set of equations, it is necessary to specify radiation conditions at boundary planes such as the engine inlet or exhaust duct termination planes. These radiation conditions may be formulated as special cases of equations (17) and (18) where the transmission term is dropped (no incoming wave) and the source term is dropped (no sources are exterior to the regions under consideration). Thus, the general form of a radiation equation is

$$\left\{A^{-\iota}_{m\mu}\right\} = \left[R^{-\iota+\iota}_{m\mu\nu}\right]\left\{A^{+\iota}_{m\nu}\right\} \tag{57}$$

The problem of determining the radiation reflection matrices is very difficult; however, some results are available in special cases.

Flanged duct.- In reference 8, the reflection matrix is derived for the case of an annular duct without flow and with an infinite rigid flange. The sequence of analysis in this case is to use the radiation directivity factors

$$D_{m\mu}(\tau) = \left\{ r \left[\frac{\tau\psi_{m\mu}(r)\, J'_m(\tau r) - \lambda_{m\mu}\psi'_{m\mu}(r)\, J_m(\tau r)}{\lambda_{m\mu}^2 - \tau^2} \right] \right\} \Bigg|_a^b \tag{58}$$

(without flow, $\lambda_{m\mu}^- = \lambda_{m\mu}^+ = \lambda_{m\mu}$, and $-\Omega_{m\mu}^- = \Omega_{m\mu}^+ = \Omega_{m\mu}$), to calculate, first, the generalized radiation impedances by the infinite integral

$$Z_{m\mu\nu} = -i\int_0^\infty \tau(\tau^2 - 1)^{-1/2}\, D_{m\mu}(\tau)\, D_{m\nu}(\tau)\, d\tau \tag{59}$$

and, second, the radiation reflection coefficients by the matrix equation

$$\left[R_{m\mu\nu}\right] = \left[\left[I\right] + \left[Z_{m\mu\nu}\right]\left[\Omega_{m\nu}\right]\right]^{-1}\left[\left[I\right] - \left[Z_{m\mu\nu}\right]\left[\Omega_{m\nu}\right]\right] \tag{60}$$

The far-field acoustic pressure for the flanged duct may be calculated from the acoustic mode amplitudes at the duct termination and the radiation directivity factors by the summation

$$p(R,\theta,\phi) = \frac{e^{iR}}{R} \sum_{m=-\infty}^{\infty} i^{-\left(|m|+1\right)} e^{im\theta} \sum_{\mu=1}^{\infty} V_{zm\mu} D_{m\mu} \sin\phi \tag{61}$$

where

$$V_{zm\mu} = \Omega_{m\mu}\left[A_{m\mu}^+ + A_{m\mu}^-\right] \tag{62}$$

Unflanged duct.- A more realistic radiation model for aircraft applications, the semi-infinite unflanged duct, has been solved for the case of a hard-walled circular duct without flow in reference 9. In this problem, the radiation reflection coefficients are given by

$$R_{m\mu\nu} = \frac{1}{2} \frac{\psi_{m\nu}(b)}{\psi_{m\mu}(b)} \frac{b^2\lambda_{m\mu}^2}{b^2\lambda_{m\mu} - m^2} \frac{1 + \Omega_{m\mu}}{\Omega_{m\mu}} \frac{1 + \Omega_{m\nu}}{\Omega_{m\mu} + \Omega_{m\nu}} K_+^m\left(b,\Omega_{m\mu}\right) K_+^m\left(b,\Omega_{m\nu}\right) \tag{63}$$

where

$$K_+^m(b,z) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log_e\left[-2K_m'\left(\sqrt{\eta^2 - b^2}\right) I_m'\left(\sqrt{\eta^2 - b^2}\right)\right] d\eta}{\eta - bz}\right\} \tag{64}$$

The radiation directivity factors for the unflanged, hard-walled circular duct are

$$D_{m\mu}(\phi) = \frac{b\left(1 + \Omega_{m\mu}\right)\left(\Omega_{m\mu} + \cos\phi\right) K_+^m\left(b,\Omega_{m\mu}\right) \psi_{m\mu}(b)\, J_m'(b \sin\phi) \sin\phi}{2(1 + \cos\phi)\left(\lambda_{m\mu}^2 - \sin^2\phi\right) K_+^m(b,\cos\phi)} \tag{65}$$

and the far-field acoustic pressure is given in terms of the mode amplitudes which are incident on the duct termination and the directivity factors by the summation

$$p(R,\theta,\phi) = \frac{e^{iR}}{R} \sum_{m=-\infty}^{\infty} i^{-\left(|m|+1\right)} e^{im\theta} \sum_{\mu=1}^{\infty} A_{m\mu}^+ D_{m\mu}(\phi) \tag{66}$$

Application to Circular Duct

Theory.- As an example of the application of the general theory, a circular duct is used. Flow is not considered since it has been shown that the only essential effect of flow is to make the modes nonorthogonal. The duct is shown in figure 7 with the left end closed by a rigid plate and the right end terminating in an infinite flange. The flange is not essential to the solution of this problem since unflanged radiation reflection coefficients and far-field pressure formulas are available in this case. The rigid plate has a small orifice in the center through which there is an oscillating flow. The plane of the plate is designated as z^1, the plane of the right end of the first section by z^2, the left end of the second section by z^3 ($z^3 = z^2$), and so on to the plane of the flange z^6.

The velocity distribution at z^1 may be represented by the product of a Dirac delta function and a source strength Q.

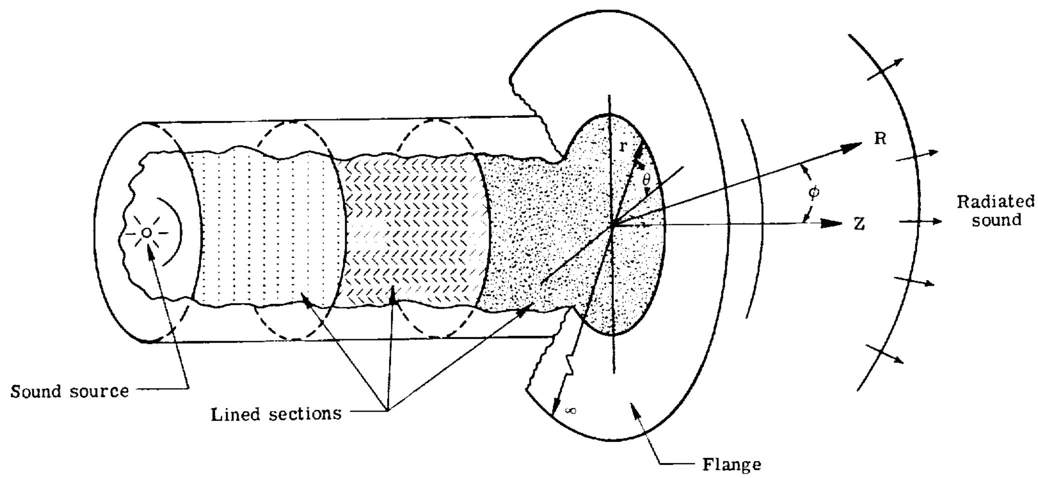

Figure 7.- Source in a multisection duct.

$$V(r) = \frac{2\pi}{r} Q\delta(r) \tag{67}$$

Since the source and the duct are axisymmetric, the acoustic field is axisymmetric and only the circumferential mode where $m = 0$ is considered. The velocity distribution (eq. (67)) must be matched by the acoustic field at z^1, consequently, the momentum equation (eq. (7c)), the acoustic field equation (eq. (39b)), and the modal orthogonal conditions (eq. (36)) are used to show that

$$\left\{A^{+1}_{o\mu}\right\} = -\left[I\right]\left\{A^{-1}_{o\nu}\right\} + Q\left\{\frac{\psi^1_{o\mu}(0)}{\Omega^1_{o\mu}}\right\} \tag{68}$$

Equation (68) is of the form of equation (17) with the transmission matrix dropped and represents a boundary condition on the acoustic field.

The transmission and reflection matrices for the uniform sections (eqs. (40) and and (41)), for the wall admittance changes (eqs. (49), (50), (51), and (52)), and for the radiation reflections (eq. (57)) are combined with the source equation (eq. (67)) to give a complete set of equations for the wave amplitudes in the duct as follows:

$$
\left[\begin{array}{cccccc|cccccc}
[I] & & & & & & [I] & & & & & \\
-[T^{+2+1}_{o\mu\nu}] & [I] & & & & & & & & & & \\
 & -[T^{+3+2}_{o\mu\nu}] & [I] & & & & & & -[R^{+3-3}_{o\mu\nu}] & & & \\
 & & -[T^{+4+3}_{o\mu\nu}] & [I] & & & & & & & & \\
 & & & -[T^{+5+4}_{o\mu\nu}] & [I] & & & & & & -[R^{+5-5}_{o\mu\nu}] & \\
 & & & & -[T^{+6+5}_{o\mu\nu}] & [I] & & & & & & \\
\hline
 & & & & & & [I] & -[T^{-1-2}_{o\mu\nu}] & & & & \\
 & -[R^{-2+2}_{o\mu\nu}] & & & & & & [I] & -[T^{-2-3}_{o\mu\nu}] & & & \\
 & & & & & & & & [I] & -[T^{-3-4}_{o\mu\nu}] & & \\
 & & & -[R^{-4+4}_{o\mu\nu}] & & & & & & [I] & -[T^{-4-5}_{o\mu\nu}] & \\
 & & & & & & & & & & [I] & -[T^{-5-6}_{o\mu\nu}] \\
 & & & & & -[R^{-6+6}_{o\mu\nu}] & & & & & & [I]
\end{array}\right]
\left\{\begin{array}{c}
\{A^{+1}_{o\nu}\} \\ \{A^{+2}_{o\nu}\} \\ \{A^{+3}_{o\nu}\} \\ \{A^{+4}_{o\nu}\} \\ \{A^{+5}_{o\nu}\} \\ \{A^{+6}_{o\nu}\} \\ \hline \{A^{-1}_{o\nu}\} \\ \{A^{-2}_{o\nu}\} \\ \{A^{-3}_{o\nu}\} \\ \{A^{-4}_{o\nu}\} \\ \{A^{-5}_{o\nu}\} \\ \{A^{-6}_{o\nu}\}
\end{array}\right\}
=
\left\{\begin{array}{c}
Q\dfrac{\psi^{+1}_{o\mu}(0)}{\Omega^{+1}_{o\nu}} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \hline \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \\ \{0\}
\end{array}\right\}
\tag{69}
$$

In equation (69), only nonnull matrices are shown.

The matrix equation (69) is equivalent to equation (23) of reference 1. In order to reduce equation (69) to the form of equation (23) of reference 1, it is necessary to eliminate the vectors $\left\{A_{o\nu}^{-5}\right\}$, $\left\{A_{o\nu}^{-1}\right\}$, $\left\{A_{o\nu}^{-3}\right\}$, $\left\{A_{o\nu}^{+2}\right\}$, $\left\{A_{o\nu}^{+4}\right\}$, and $\left\{A_{o\nu}^{+6}\right\}$ from equation (69). This operation is easy since these vectors are not reflected in equation (69); that is, they are not multiplied by reflection matrices.

Results.- This problem was considered previously in reference 1 and it was shown, by using idealized wall admittance values, that a duct with three liner sections with different wall admittances can be made to radiate less acoustic energy than a duct with an optimum uniform liner. The wall admittances were chosen to increase the effect of the reflection matrices by having large admittance changes between adjacent sections.

Based on these previous results, a set of duct liners were designed which would have large admittance changes at a specified frequency. The liners were made with single layers having different backing depths. The first and third liners had backing depths of about $\frac{3}{8}\lambda$, whereas the center liner had a depth of about $\frac{1}{8}\lambda$. The acoustic resistance of the center liner was chosen to be twice the resistance of the end liners. The length of the center section was made equal to the combined lengths of the end sections. The total duct length was chosen to be equal to its diameter.

The radiation patterns for a point source in this duct are shown in figure 8 and are compared with the radiation from a hard-walled duct and from a duct with a uniform liner. Radiation patterns for both the flanged and unflanged termination theories are shown for comparison. The unflanged duct radiation patterns tend to be 5 to 6 dB lower at 90° than the flanged patterns and give results for angles greater than 90°. The admittance for the uniform liner is the optimum admittance for a plane wave source in an infinite duct. This admittance was given by Rice in reference 10. The results in figure 8 show that this uniform liner is not effective for the point source in a short duct. The three-section liner configuration shows substantial reductions in the radiated noise. This single example demonstrates the potential usefulness of multisection liners. A complete analytical and experimental study of this concept is desirable but is beyond the scope of the present paper.

UNIFORM MULTICHANNELED DUCT (FIGS. 2 AND 9)

General Equations

When the duct is divided into many channels, such as the example duct in figure 9, it is necessary to write a more general set of transmission and reflection equations than equations (17) and (18). For example, right-moving waves at station 6 (fig. 9) may be

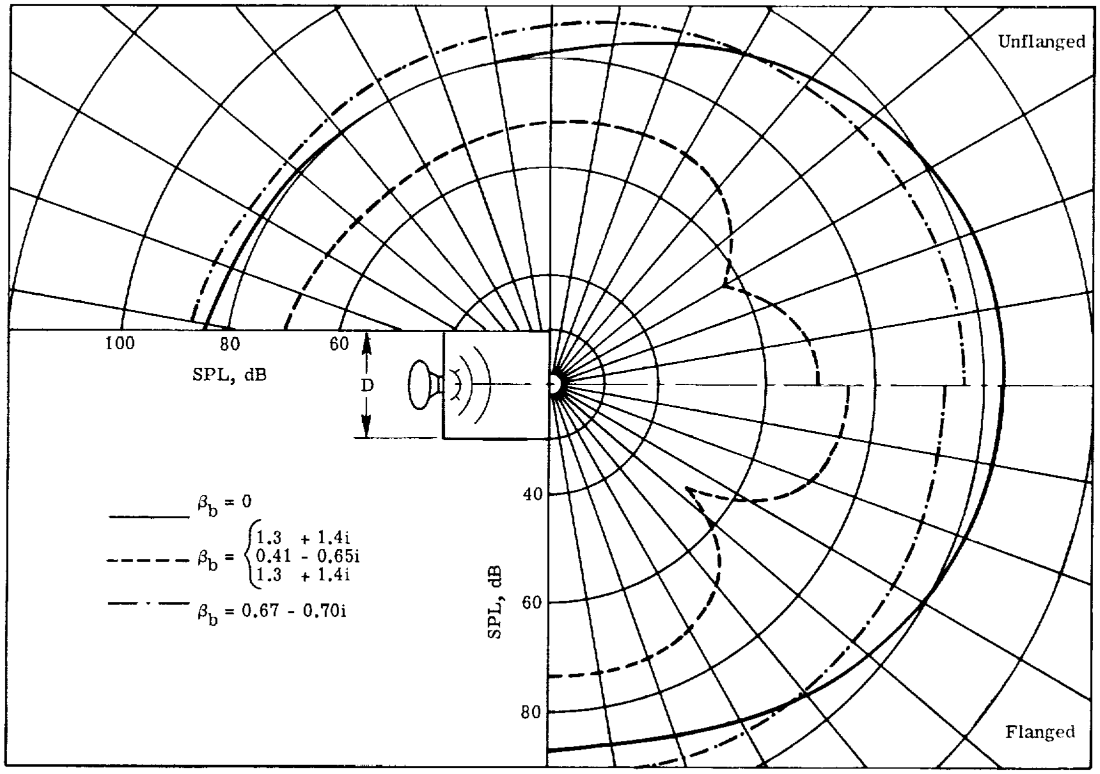

Figure 8.- Radiation from source in short duct. $f = 1000$ Hz; $c_a = 338$ m/sec; $D = 0.39$ m.

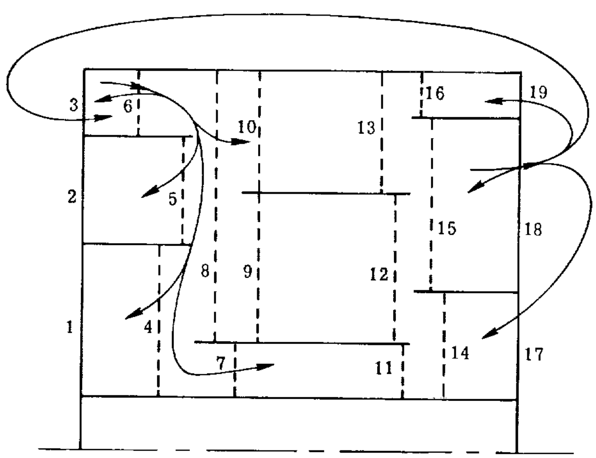

Figure 9.- Uniform multichanneled duct.

transmitted to stations 8 and 7 and reflected into planes 6, 5, and 4 as backward-moving waves. Similar possibilities exist for other regions which are bounded by planes within the duct. Communication between the various terminal planes is also possible as shown in figure 9 where right-moving waves at plane 18 are reflected into planes 17, 18, and 19 and transmitted into planes 1, 2, and 3. In general, a matrix may be written for the duct mode amplitudes in the form

$$
\left[\begin{array}{c|c}
\vdots & \vdots \\
\cdots\left[W^{+\iota+\kappa}_{m\mu\nu}\right]\cdots & \cdots\left[W^{+\iota-\kappa}_{m\mu\nu}\right]\cdots \\
\vdots & \vdots \\
\hline
\vdots & \vdots \\
\cdots\left[W^{-\iota+\kappa}_{m\mu\nu}\right]\cdots & \cdots\left[W^{-\iota-\kappa}_{m\mu\nu}\right]\cdots \\
\vdots & \vdots
\end{array}\right]
\left\{\begin{array}{c}
\vdots \\
\left\{A^{+\kappa}_{m\nu}\right\} \\
\vdots \\
\hline
\vdots \\
\left\{A^{-\kappa}_{m\nu}\right\} \\
\vdots
\end{array}\right\}
=
\left\{\begin{array}{c}
\vdots \\
\left\{Q^{+\iota}_{m\mu}\right\} \\
\vdots \\
\hline
\vdots \\
\left\{Q^{-\iota}_{m\mu}\right\} \\
\vdots
\end{array}\right\}
\tag{70}
$$

Equation (70) contains all possible interactions of the waves in different planes. Equations (17) and (18) for the single channel duct are special cases of the rows of equation (70).

Axisymmetric Duct

Splitters and bifurcations.- When an axisymmetric duct is divided into multiple channels with circumferential splitters, reflection and transmission matrices must be derived which are more general than the ones for the admittance discontinuity; however, the general method of derivation, with conservation conditions, is still applicable.

Consider a duct which is divided by splitters as shown in figure 10. At some plane in the duct, one set of splitters ends and another begins. Again, planes spaced a small distance ϵ apart about the discontinuity are used to derive the governing equations. Each plane which cuts through the splitters defines a number of annular regions in which there are modes $\psi^{\pm j}_{m\nu}(r)$, but these modes have no meaning except in the range $\left(a^j \leqq r \leqq b^j\right)$. Therefore the modes are defined to be zero outside of their range, that is

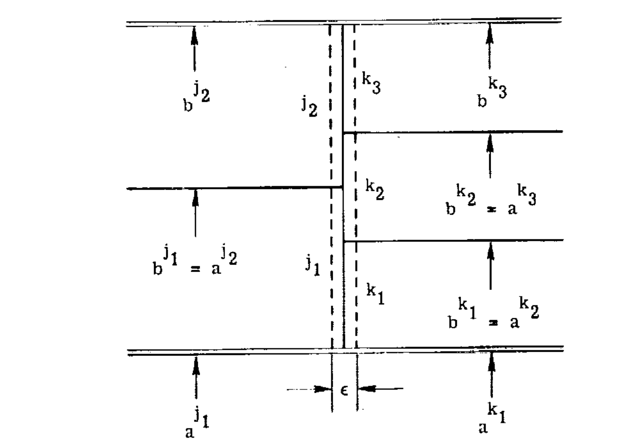

Figure 10.- Duct with splitters.

$$\psi^{\pm j}_{m\nu}(r) = 0 \qquad \left(\text{if} \quad r < a^j \quad \text{or if} \quad r > b^j\right) \tag{71}$$

The conditions of continuity of axial velocity and pressure (eqs. (42) and (43)) are still valid if

$$\left\lfloor \psi^{\iota}_{m\nu}(r) \right\rfloor = \left\lfloor \left\lfloor \psi^{\iota_1}_{m\nu}(r) \right\rfloor \left\lfloor \psi^{\iota_2}_{m\nu}(r) \right\rfloor \ldots \left\lfloor \psi^{\iota_n}_{m\nu}(r) \right\rfloor \right\rfloor \qquad (a \leqq r \leqq b) \tag{72}$$

$$\left[\beta^{\iota}_{m\nu}\right] = \begin{bmatrix} \left[\beta^{\iota_1}_{m\nu}\right] & & & & \\ & \left[\beta^{\iota_2}_{m\nu}\right] & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & \left[\beta^{\iota_n}_{m\nu}\right] \end{bmatrix} \tag{73}$$

$$\left\{A^{\iota}_{m\nu}\right\} = \begin{Bmatrix} \left\{A^{\iota_1}_{m\nu}\right\} \\ \left\{A^{\iota_2}_{m\nu}\right\} \\ \cdot \\ \cdot \\ \cdot \\ \left\{A^{\iota_n}_{m\nu}\right\} \end{Bmatrix} \tag{74}$$

where $\iota = \pm j$ or $\pm k$. Also, since the matching equations (42) and (43) are valid with this generalized notation, equations (46) to (53) are also valid.

Matrix partitioning.- When there are duct splitters, the mode integral matrix given by equation (46) must be partitioned for computational purposes. The partitioning is done so that the mode integral submatrices are compatible with the modal subvectors $\psi_m\left(\lambda_{m\nu}^{\iota_n} r\right)$ in equation (72). The partitioning serves to define limits on the integrals which generate the submatrices. Since the range of the integral in equation (46) is $(a \leqq r \leqq b)$ and the modal functions associated with any two planes ι_n and κ_n are nonzero only in the ranges $\left(a^{\iota_n} \leqq r \leqq b^{\iota_n}\right)$ and $\left(a^{\kappa_n} \leqq r \leqq b^{\kappa_n}\right)$, respectively, the range of the integral for the submatrix is $\left(\left(a^{\iota_n} \leqq r \leqq b^{\iota_n}\right) \cap \left(a^{\kappa_n} \leqq r \leqq b^{\kappa_n}\right)\right)$. When this intersection is the null set, the submatrix is null. As examples, the modal submatrix integrals for planes j_1 and k_2 in figure 10 are taken from a^{k_2} to b^{j_1} and the modal submatrices for planes j_1 and k_3 are null since planes j_1 and k_3 do not have a common range in r.

Variable degrees of freedom.- It has been tacitly assumed that the number of degrees of freedom is the same at adjacent planes such as j and k in figure 6. This is not a necessary requirement for a matching of the acoustic fields and it may be seen that equations (47) and (51) are valid even if the orders of the modal vectors at z^j are different from those at z^k. In this case, however, it is not possible to obtain equations like equations (49), (50), (52), and (53). Consequently, equations (17) and (18) should be replaced by the more general equations (47) and (51) which will permit variable degrees of freedom to be used in computational work.

CONCLUDING REMARKS

A set of acoustic equations has been derived for the sound field in and radiated from multisection ducts. The equations are in matrix form and are based on the use of acoustic modes as generalized coordinates. The results of this paper are directly applicable to duct systems which are made from a number of joined circular and annular sections. Duct systems of this type may be used as a mathematical model of an aircraft engine to study the interactive effects of internal noise sources, transmission through complex duct liner and splitter configurations, and radiation to the far field.

There are several advantages to this formulation of the duct acoustics problem. The matrix notation used is convenient for programing and digital computation. The notation may also be used in variable-area ducts with flow gradients so that the extension of the present work to this more general case be done in a routine manner. In addition, the matrix formulation may be extended to include ducts with radial splitters. This extension could utilize the acoustic modes in annular sectors and the matrix equations would have terms coupling the circumferential modes.

The matrix formulation of the acoustic equations gives a format for integrating the results of several areas of research. Turbomachinery noise sources can be described as sources of acoustic modes in an infinite duct. The matrix theory then shows how to incorporate these source effects in finite ducts. Transmission through a duct section of any type, such as a variable duct or a constant area duct, may be studied without concern for the ducts to which it is to be joined since the matrix notation facilitates the joining of different duct sections. A variety of mathematical methods may be used: numerical methods for very irregular duct sections, perturbation methods for ducts with slowly-varying properties, and analytical methods for uniform ducts. The theory of radiation from ducts may also be developed using semi-infinite duct models. The matrix equations then combine the radiation effects into the finite ducts. Finally, the equations for joining different duct sections are identical in form to the equations for transmission through a duct section; therefore, all source, duct transmission, and radiation effects are combined in a unified format that makes the present theory suitable for programing and computation.

It has been shown by Lansing and Zorumski (Journal of Sound and Vibration, March 8, 1973) that multisection liners can be more effective in reducing radiated noise than uniform liners. The present theory shows the two important physical effects by which multisection liners reduce radiated noise. These are the transmission effects and the reflection effects. In the past, all practical efforts to reduce aircraft noise using acoustic treatment have utilized transmission effects. No effort has been made to take advantage of reflection effects; however, it has been shown here that they can be important. In a uniform circular duct without flow, the mode reflection matrices at the interface between two liner sections are shown to be proportional to the differences in the acoustic admittances of the liners.

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National Aeronautics and Space Administration,
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