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THEORETICAL INVESTIGATIONS ON PLASMA PROCESSES IN THE KAUFMAN THRUSTER

by

H. E. Wilhelm

Prepared for

LEWIS RESEARCH CENTER

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GRANT NGR-06-002-147

Annual Report

June 1, 1973

Department of Physics
Colorado State University
Fort Collins, Colorado

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FOREWORD

The theoretical work on plasma processes in the Kaufman ion propulsion system contained in this report has been supported by the NATIONAL AERONAUTICS AND SPACE ADMINISTRATION under Grant NGR - 06 - 002 - 147. This research has been monitored by Dr. John Serafini, NASA LEWIS RESEARCH CENTER.

CONTENTS

I. INTRODUCTION

This report contains theoretical investigations on plasma processes in the Kaufman ion thruster. The temporal development of the neutralization of the ion beam by electron injection is analyzed in detail for different physical conditions. The purpose of the analyses is i) to understand the formation of the steady-state, neutral ion beam, and ii) to predict theoretically the most favorable conditions for neutralization and the characteristic lengths and times required for neutralization. Among other things, the investigations indicate that the neutralization is brought about by nonlinear, discontinuous neutralization waves, and that a complete neutralization (without a periodic space charge structure) requires a strong intercomponent, collective momentum transfer between the collisionless electron and ion beams, i.e. and irreversible dissipation mechanism. Since the electron gas moves, relative to the ion beam, with supersonic drift velocity, a two-stream instability results which leads to a strong collective ("turbulent") momentum exchange between the electron and ion components. Macroscopically, this type of intercomponent momentum transfer is described by Buneman's relaxation frequency in the electrohydrodynamic equations for the electron-ion beam system. This approach results in a quantitative, transient neutralization theory and a neutralization length which is compatible with experiment.

CHAPTER II. The lateral neutralization of ion beams is treated by standard mathematical methods for first order, nonlinear partial differential equations in the special case that the electrons are injected with a density n_o equal to the ion density N. It is shown that the electron gas moves in form of a discontinuous step wave into the ion beam leaving the ion gas completely neutralized behind the head of the wave. The speed of the neutralization front is considerably increased by application of an auxiliary electric field in the direction opposite to the electron motion.

CHAPTER III. By means of a von Mises transformation, a closed form analytical solution is derived for the transient, lateral beam neutralization by electron injection with a density n_o different from the ion density N. It is shown that the electron gas moves in form of a discontinuous neutralization wave ($1/2 < n_o/N < \infty$) or neutralization shock wave ($0 < n_o/N < 1/2$) into the ion beam at the indicated density ratios. The neutralization wave produces a periodic over- and under-neutralization in the ion space traversed if $n_o \neq N$ at the injection plane. After its generation by the neutralization wave, the periodic space charge structure remains stationary if a dissipation mechanism is not considered. In this idealized, dissipation-free treatment of the transient neutralization problem, a complete neutralization is achieved only if the electrons are injected with a density n_o equal to the ion density N.

CHAPTER IV. A nonlinear theory of the longitudinal ion beam neutralization is developed by means of the von Mises transformation in which inter-component momentum transfer by strong collective interaction of the Buneman type is considered as a dissipation mechanism. It is shown that the electron gas moves in form of a discontinuous, but damped neutralization wave downstream the ion beam. As a result, the extent of the periodic over- and under-neutralization produced by the neutralization wave decreases within a few relaxation lengths downstream of the neutralizing grid to zero. In a typical case, it is shown that a complete neutralization of the ion beam is obtained within a few centimeters downstream of the grid.

CHAPTER V. By means of the Lenard-Balescu equation, the inter-component momentum transfer between stable, collisionless electron and ion components is calculated. The stable velocity distribution functions give rise to a momentum exchange between the electron and ion components which is too weak to explain the anomalous short neutralization lengths observed experimentally. For this reason, this kinetic theory is applied exclusively to the electrical current transport in stable, collisionless plasmas and a discussion of plasma conductivity.

An investigation on neutralization shock waves has been initiated in the same research period but is not included in this report. This work will be communicated as soon as it has been completed.

The investigations contained in this report represent preliminary results. The final (extended) versions of these investigations will be communicated through publications in the applied physics literature at a later date.

II. TRANSIENT ION NEUTRALIZATION BY ELECTRONS

ABSTRACT

The nonlinear initial-boundary-value problems describing the lateral neutralization of ion beams for the cases that 1) an auxiliary electric field accelerates the electrons into the ion space, and 2) the electrons are injected into the ions space at a prescribed current density are treated. Analytical solutions are derived which give the position and speed of the neutralization front in dependence of time, and the temporal development of the electron density, velocity and electric fields during the neutralization process.

The problem of the spreading and neutralization of rarified ion beams is encountered in electrostatic propulsion systems.[1-4] The neutralization has been analyzed for the homogeneous ion beam model in the one-dimensional steady-state case.[3-8] These investigations indicate that the electrons are distributed nonuniformly across the beam and that the steady-state neutralization is incomplete.[3-8]

In the following, the nonlinear initial-boundary-value problem of the neutralization of a quasi-homogeneous ion beam is treated. The purpose of the investigation is to understand the most elementary physics of the temporal development of the neutralization process. Two simple models are considered for this purpose, which permit the study of the dependence of the neutralization on 1) the electron injection mechanism and 2) the auxiliary electric field which moves the electrons of the neutralizer plasma into the ion space.

Following previous work on steady-state neutralization,[3-8] a slab type ion beam of width a and infinite length is considered, which is treated as homogeneous. The homogeneous ion beam model[3-8] is not very realistic but required for mathematical reasons in order to render an analytical discussion feasable. The electrons are injected from the side into the beam, i.e. transverse to the beam velocity (Fig.1). This arrangement is known as lateral neutralization[3-4]. Only one-dimensional electron motions transverse to the beam velocity are analyzed, i.e. downstream convection of electrons by the beam is disregarded.

THEORETICAL PRINCIPLES

The electron gas moves in form of a nonlinear wave $(\frac{1}{2} < n_o/N < \infty)$[9] or shock wave $(0 \leq n_o/N < \frac{1}{2})$[10] into the ion gas at the indicated ratios of electron density n_o to ion density N. In order to achieve a complete neutralization $(n = N)$, the condition $n_o = N$ has to be satisfied at the plane of electron injection. In this case, inertial effects play no role in the macroscopic electron motion, provided that the momentum relaxation time is small compared to the time it takes the electrons to penetrate the ion beam.

With disregard of inertial effects, the field equations for the drift velocity $\vec{v}$ and density n of the electrons and the self-consistent electric field $\vec{E}$ are in presence of a homogeneous background (N) of ions:

$$\vec{v} = (e/m)\,\tau\,\vec{E} \quad , \tag{1}$$

$$\frac{\partial n}{\partial t} = -\nabla\cdot(n\vec{v}) \quad , \tag{2}$$

$$\nabla\cdot\vec{E} = 4\pi e(n-N) \quad , \tag{3}$$

where[11]

$$\tau = (M/m)^{1/3}\,2\pi/\omega \;, \quad \omega = (4\pi N e^2/m)^{1/2} \; . \tag{4}$$

τ is the collective relaxation time of the momentum density $nm\vec{v}$ and ω is the characteristic oscillation frequency of the electrons of elementary charge $e < 0$ (m = electron mass, M = ion mass).

Equation (1) is the simplified equation of electron motion which results in the absence of inertial effects for the collisionless electron-ion system [Eq. (4)]. A velocity field $\vec{v}$ is defined by Eq. (1) only in regions where $\vec{E} \neq \vec{0}$, see Eqs. (2)-(3).

A set of first order differential equations similar (but with a collisional relaxation time) has previously been used in the analysis of the collection of electrons in the pulsed ion chamber,[12] and electron conduction phenomena in solids.[13-14] The Eqs(1)-(4) are being applied herein to the transient, non-oscillatory (n_o = N) neutralization of ion beams by electrons.

NEUTRALIZATION IN AN ELECTRIC FIELD

As a model, an ion beam $(\vec{V})$ of homogeneous density N is considered which is bounded in the planes $x = 0$ and $x = a$ by grids which are impermeable except to electrons, Fig. 1. At the beginning of the neutralization, $t = 0$, the adjacent space $x < 0$ is occupied by a homogeneous plasma reservoir of electron ion density, $n_{\pm} = N$, and a potential difference $U = \phi(a) - \phi(0) \geq 0$ is quasi-instantaneously applied to the grids at $x = 0$ and $x = a$. The auxiliary electric field $E_o = -U/a$ represents a weak test field, i.e., the voltage U is assumed to be extremely small compared to the voltage equivalent of the beam energy:

$$|e|U <<< \frac{1}{2} M\vec{V}^2 \mathrel{\hat{=}} 10^3 - 10^4 \text{ e-volt} \quad . \tag{5}$$

Under the influence of the space charge field of the ions and the auxiliary field $E_o = -U/a$, the electrons move through the grid plane $x = 0$ into the ion space, $0 \leq x \leq a$. In accordance with Eqs. (1)-(4), the resulting neutralization process is described by the nonlinear intial-boundary-value problem:

$$\frac{\partial n}{\partial t} = - \frac{\partial}{\partial x} (nE) \ , \tag{6}$$

$$\frac{\partial E}{\partial x} = \alpha(n - 1) \ , \tag{7}$$

where

$$n(x, t = 0) = H(- x) \ , \tag{8}$$

$$\int_o^1 E(x,t)\, dx = H(t) \ , \tag{9}$$

and

$$\alpha = 4\pi eaN/(-U/a) > 0 \quad . \tag{10}$$

In Eqs. (6)-(9), $n(x, t)$, $E(x, t)$, x and t are nondimensional variables which are related to the associated dimensional variables [Eqs. (1)-(4)] by

$$n = \mathcal{n}/N \ , \quad E = \mathcal{E}/(-U/a) \ , \tag{11}$$

and

$$x = \mathcal{X}/a \ , \quad t = \mathcal{t}/(-a/\frac{e}{m}\frac{U}{a}\tau) \ . \tag{12}$$

The reference constants in Eqs. (11)-(12) have an evident physical meaning, e.g., $t_o = -a/(eU\tau/ma)$ is the transit time of an electron in the field $-U/a$ for the distance a .

Combining of Eqs. (6) and (7) yields for the derivative dn/dt along the stream line of an electron fluid element

$$\frac{\partial n}{\partial t} + E\,\frac{\partial n}{\partial x} = -\alpha n(n - 1) \ . \tag{13}$$

The solution of Eq. (13), which satisfies the initial condition in Eq. (8), is obviously:

$$n(x, t) = H[s(t) - x] \ , \tag{14}$$

where

$$s(t = 0) = 0 \ , \tag{15}$$

since

$$[\,\frac{ds}{dt} - E(x, t)]\ \delta(s - x) = -\alpha H(s - x)[H(s - x) - 1]$$

$$\text{for} \quad x \lesseqgtr s(t) \ , \tag{16}$$

and

$$\frac{ds}{dt} = E(s, t) \tag{17}$$

by Eqs. (13)-(14) and Eq. (1), respectively. It represents $s(t)$ the position coordinate of the neutralization front. It is seen from Eq. (14) that the initial condition, Eq. (8), determines the form $n(x, t)$ of the solution for $t \geq 0$. $n(x, t)$ is a step function, which is discontinuous at $x = s(t)$, Fig. 2.

Integration of Eq. (7) from $x = s$ to x gives, under consideration of Eqs. (14) and (17):

$$E(x, t) = \frac{ds}{dt} + \alpha(s - x)[1 - H(s - x)] \quad , \tag{18}$$

whence

$$\frac{ds}{dt} = H(t) + \frac{1}{2}\alpha(1 - s)^2 \tag{19}$$

by substitution of Eq. (18) into the boundary condition in Eq. (9). The solution of Eq. (19), which satisfies the initial condition in Eq. (15), gives the position coordinate of the moving neutralization front:

$$s(t) = 1 + (2/\alpha)^{1/2}\, tg[(\alpha/2)^{1/2}t - arctg(\alpha/2)^{1/2}] \geq 0 \;, \quad t \geq 0 \;. \tag{20}$$

It follows for the speed ds/dt of the neutralization front in the external electric field:

$$\frac{ds}{dt} = \cos^{-2}[(\alpha/2)^{1/2}t - arctg(\alpha/2)^{1/2}] \geq 1 \;, \quad t \geq 0 \;. \tag{21}$$

The Eqs. (14) and (18), in which $s(t)$ and ds/dt are given by Eqs. (20) and (21), constitute the nondimensional solutions for the fields $n(x, t)$, $E(x, t)$ and $v(x, t)$.

The time required for the neutralization front $s(t)$ to penetrate the ion beam $(0 \leq x \leq 1)$ is determined by $s(t) = 1$. Accordingly, the dimensionless neutralization time is by Eq. (20)

$$t_N = (2/\alpha)^{1/2} \text{ arc tg}(\alpha/2)^{1/2} \propto \frac{\pi}{2}(2/\alpha)^{1/2} \quad , \quad (\alpha/2)^{1/2} >> 1 \; . \tag{22}$$

Since $t_N = \mathcal{t}_N/\mathcal{t}_o$, $\mathcal{t}_o \sim E_o^{-1}$ and $\alpha \sim E_o^{-1}$ [Eqs. (10)-(14)], the dimensional neutralization time $\mathcal{t}_N$ decreases with increasing $E_o = -U/a$.

In Fig. 3, the dimensionless position coordinate $s(t)$ of the neutralization front is shown in dependence of the dimensionless time t with α as a parameter [Eq. (20)]. The line $s(t) = 1$ intersects with the $s(t)$ curves at $t = t_N(\alpha)$, t_N being larger for smaller α-values. As to the dependence on the auxiliary field $E_o = -U/a$, it is noted that $\alpha \sim E_o^{-1}$ and $t \sim \mathcal{t}E_o$, i.e., that, for any given dimensional time $\mathcal{t}$ the dimensional position coordinate $\mathcal{s}(\mathcal{t})$ is larger for larger E_o-values.

In Fig. 4, the dimensionless velocity ds/dt of the neutralization front is shown in dependence of the dimensionless time t with α as a parameter. It is seen that, for any $t < t_N$, ds/dt is larger for larger α-values. As to dependence on the auxiliary field $E_o = -U/a$, it is noted the dimensional velocity $d\mathcal{s}/d\mathcal{t} = (a/\mathcal{t}_o) \cdot ds/dt$ increases with increasing E_o for any fixed instant $\mathcal{t} = t\mathcal{t}_o$, since $\mathcal{t}_o \sim E_o^{-1}$ and $\alpha \sim E_o^{-1}$.

In Fig. 5, the dimensionless neutralization time t_N is represented versus α [Eq. (22)]. The dimensional neutralization time $\mathcal{t}_N = t_N\mathcal{t}_o$ decreases with increasing $E_o = -U/a$, since $\mathcal{t}_o \sim E_o^{-1}$ and $\alpha \sim E_o^{-1}$. As a numerical illustration, consider a mercury ion beam. In this case:

$$\alpha = 6.036 \times 10^{-9} \frac{aN}{U/a} \; , \quad \tau^{-1} = 1.249 \times 10^2 N^{1/2} \; ,$$

$$\mathcal{t}_o = \frac{m}{|e|} \frac{a^2}{U\tau} = 2.369 \times 10^{-16} \frac{a^2}{U} \; N^{1/2}$$

Hence

$\alpha = 1.810 \times 10^{18}$, $\tau^{-1} = 1.249 \times 10^{10}$ sec^{-1}, $\mathcal{t}_o = 7.107 \times 10^{2}$ sec for $a = 10$ cm, $N = 10^{16}$ cm^{-3}, and $U = 10^{-6}/300$ cgsu. Accordingly, the neutralization time is by Eq. (22):

$$t_N = 1.105 \times 10^{-9}, \; \mathcal{t}_N = 7.853 \times 10^{-7} \text{ sec} .$$

It is seen that the neutralization time is rather short, although the driving test field is extremely weak. In this connection, it should be noted that $n(x,t) = 1$ for $x \leq s(t)$ [Eq. (14)], i.e., a net space charge field does not exist behind the neutralization front $s(t)$.

Because of the disregard of inertial effects in the electron motion [Eq. (1)], the formulas derived in this section are only applicable to situations in which the neutralization time is large compared to the momentum relaxation time, $\mathcal{t}_N \gg \tau$.

NEUTRALIZATION BY ELECTRON INJECTION

As a model, an ion beam $(\vec{V})$ of homogeneous density N is considered which is bounded in the planes $\chi = 0$ and $\chi = a$ by grids which are impermeable except to electrons, Fig. 6. Through the plane $\chi = 0$, electrons are injected quasi-instantaneously from the region $\chi \leq 0$ with a density $n_o = N$, but an arbitrary current density $j = n_o e \upsilon_o$ at the beginning $t = 0$ of the neutralization $(U = 0)$.

In accordance with Eqs. (1)-)4), the associated neutralization process is described by the nonlinear initial-boundary-value problem:

$$\frac{\partial n}{\partial t} = - \frac{\partial}{\partial x}(nE) \quad , \tag{23}$$

$$\frac{\partial E}{\partial x} = \omega\tau(n-1) \quad , \tag{24}$$

where

$$n(x, t = 0) = H(- x) \quad , \tag{25}$$

$$n(x = 0, t)v(x = 0, t) = H(t) \quad , \tag{26}$$

and τ , ω is defined in Eq. (4). In Eqs. (23)-(26), $n(x, t)$, $E(x, t)$, x and t are nondimensional variables which are related to the associated dimensional variables [Eqs. (1)-(4)] by

$$n = n/N \ , \quad v = \upsilon/\upsilon_o \ , \quad E = E/(\frac{m}{e}\frac{\upsilon_o}{\tau}) \quad , \tag{27}$$

and

$$x = \chi/(\upsilon_o/\omega) \ , \quad t = \omega t \quad . \tag{28}$$

The reference constants in Eqs. (27)-(28) have an evident physical meaning, e.g., $E_o = m\upsilon_o/e\tau$ is the electric field that accelerates an electron to the speed υ_o in τ seconds.

Combining of Eqs. (23) and (24) yields for the derivative dn/dt along the streamline of an electron fluid element

$$\frac{\partial n}{\partial t} + E\,\frac{\partial n}{\partial x} = -\omega\tau n(n-1) \quad . \tag{29}$$

The solution of Eq. (29), which satisfies the initial condition in Eq. (25), is

$$n(x,\ t) = H[\sigma(t) - x] \quad , \tag{30}$$

where

$$\sigma(t = 0) = 0 \quad , \tag{31}$$

since

$$\left[\frac{d\sigma}{dt} - E(x,\ t)\right]\ \delta(\sigma-x) = -\omega\tau H(\sigma-x)[H(\sigma-x) - 1]$$

$$\text{for} \quad x \lesseqgtr \sigma(t) \quad , \tag{32}$$

and

$$\frac{d\sigma}{dt} = E(\sigma,\ t) \tag{33}$$

by Eqs. (29)-(30) and Eq. (1), respectively. Equation (30) indicates that the initial condition, Eq. (25), determines the form of $n(x,\ t)$ for $t \geq 0$. The solution $n(x,\ t)$ is a step function which is discontinuous at $x = \sigma(t)$, the position of the neutralization front, Fig. 6.

Integration of Eq. (24) from $x = \sigma$ to x gives, under consideration of Eqs. (30) and (33):

$$E(x,\ t) = \frac{d\sigma}{dt} + \omega\tau(\sigma-x)[1-H(\sigma-x)] \quad . \tag{34}$$

The position $\sigma(t)$ of the neutralization front is determined by the boundary condition in Eq. (26), which gives

$$H(\sigma)\{\frac{d\sigma}{dt} + \omega\tau\sigma[1-H(\sigma-x)\} = H(t) \tag{35}$$

by Eqs. (30) and (34). It follows by integration of Eq. (35), since $\sigma(t) \geq 0$ for $t \geq 0$:

$$\int_0^\sigma H(\sigma') \, d\sigma' = \int_0^t H(t') \, dt' \quad . \tag{36}$$

Hence, the position coordinate of the moving neutralization front is:

$$\sigma(t) = t \ , \quad t \geq 0 \quad . \tag{37}$$

The speed $d\sigma/dt$ of the neutralization front is invariant (constant electron injection velocity):

$$\frac{d\sigma}{dt} = 1 \quad , \quad t \geq 0 \quad . \tag{38}$$

The Eqs. (30) and (34), in which $\sigma(t)$ and $d\sigma/dt$ are given by Eqs. (37) and (38), constitute the nondimensional solutions for the fields $n(x, t)$, $E(x, t)$ and $v(x, t)$.

The time required for the neutralization front $\sigma(t)$ to penetrate the ion beam $(0 \leq x \leq a\omega/v_o)$ is determined by $\sigma(t) = a\omega/v_o$ [Eq. (28)]. Hence, the dimensionless neutralization time is by Eq. (37)

$$t_N = a\omega/v_o \quad . \tag{39}$$

In Fig. 7, the (dimensionless) position coordinate $\sigma(t)$ and speed $d\sigma/dt$ of the moving neutralization front are shown versus t in accordance with Eqs. (37) and (38), respectively.

The dimensional speed of the neutralization front and the dimensional neutralization time are by Eq. (38)-(39) and Eqs. (27)-(28):

$$v_o = j_o/Ne \tag{40}$$

and

$$t_N = a/v_o \tag{41}$$

where v_o and j_o are the steady-state velocity and electrical current density with which the electrons are injected [Eq. (26)]. Accordingly,

$$v_o = 2.082 \times 10^9 \frac{|j_o|}{N} = 6.246 \times 10^7 \text{ cm/sec}$$

and

$$t_N = 4.803 \times 10^{-10} \frac{aN}{|j_o|} = 1.601 \times 10^{-7} \text{ sec}$$

for $a = 10$ cm, $N = 10^9$ cm^{-3}, and $|j_o| = 10^{-2}$ amp/cm^2 = 3×10^7 cgsu.

The speed of the neutralization front $d\sigma/dt$ is identical with the speed v_o with which the electrons are injected (n_o=N) which, in turn, determines the neutralization time t_N. In this case, the neutralization time can be shorter than the momentum relaxation time if the injection velocity is sufficiently large.

REMARKS

Closed form analytical solutions have been obtained for the nonlinear initial-boundary-value problems describing the transverse neutralization of idealized ion beams in the cases that 1) a weak auxiliary electric field pulls the electrons into the ion space, and 2) a constant electron current is injected through the neutralizer plane. In order to achieve a complete neutralization, the electron density at the neutralizer plane $(x = 0)$ has been assumed to be equal to the ion density (N). In the cases 1) and 2), the neutralization occurs through an electron flow with a discontinuous front, ahead of which the electron density is zero and behind of which the electron density is equal to the ion density (discontinuous solutions).

In case 1), the speed of the neutralization front can be considerably enhanced by increasing the intensity of the auxiliary electric field. In case 2), the speed of the neutralization front is given by the velocity with which the electrons are injected (the electron gas behaves like an incompressible fluid).

The main outcome of these simple theoretical considerations is that a short, relaxation-free neutralization is possible if the electrons are injected with a density equal to the ion density, $n_o = N$. In the general case, $n_o \neq N$, the neutralization time is much longer because of the occurrence of neutralization space charge waves[9], which exhibit only a weak damping. As a mathematical result, it is pointed out that the solutions of the nonlinear partial differential equations of first order are discontinuous. More realistic, continuous solutions are to be expected if effects associated with second order partial derivatives are taken into consideration, such as diffusion.

CITATIONS

1. H. R. Kaufman, An Ion Rocket with an Electron-Bombardment Ion Source, NASA TN-D-585 (1961).

2. N. R. Kerslake, D. C. Byers, and J. F. Staggs, J. Spacecraft and Rockets 1, 4 (1970).

3. E. Stuhlinger, Ion Propulsion for Space Flight, McGraw-Hill, New York 1964.

4. G. F. Au, Electric Propulsion of Space Vehicles, Verlag G. Braun, Karlstrule 1968.

5. H. R. Kaufman, One-Dimensional Analysis of Ion Rockets, NASA TN D-261 (1960).

6. R. N. Seitz, R. Shelton, and E. Stuhlinger, Present Status of the Beam Neutralization Problem, Progress in Astronautics and Rocketry 5, 383, Academic Press, New York 1961.

7. L. M. Frantz et al., Proc. IRE 14, 477 (1960).

8. L. D. Pearlstein, M. N. Rosenbluth, and G. W. Stuart, The Neutralization of Ion Beams, Progress in Astronautics and Aeronautics 9, 379, Academic Press, New York 1963.

9. H. E. Wilhelm, Nonlinear Theory of Electron-Ion Neutralization Waves; to be published, 1973.

10. H. E. Wilhelm, Nonlinear Theory of Electron-Ion Neutralization Shockwaves; to be published, 1973.

11. O. Bunemann, Phys. Rev. 115, 503 (1959).

12. S. H. Kim and W. H. Ellis, J. Appl. Phys. 43, 3027 (1972).

13. H. J. Wintle, J. Appl. Phys. 41, 4004 (1970); J. Appl. Phys. 43, 2927 (1970).

14. J. H. Calderwood and B. K. P. Scaife, Phil. Trans. Roy. Soc. A 269, 217 (1971).

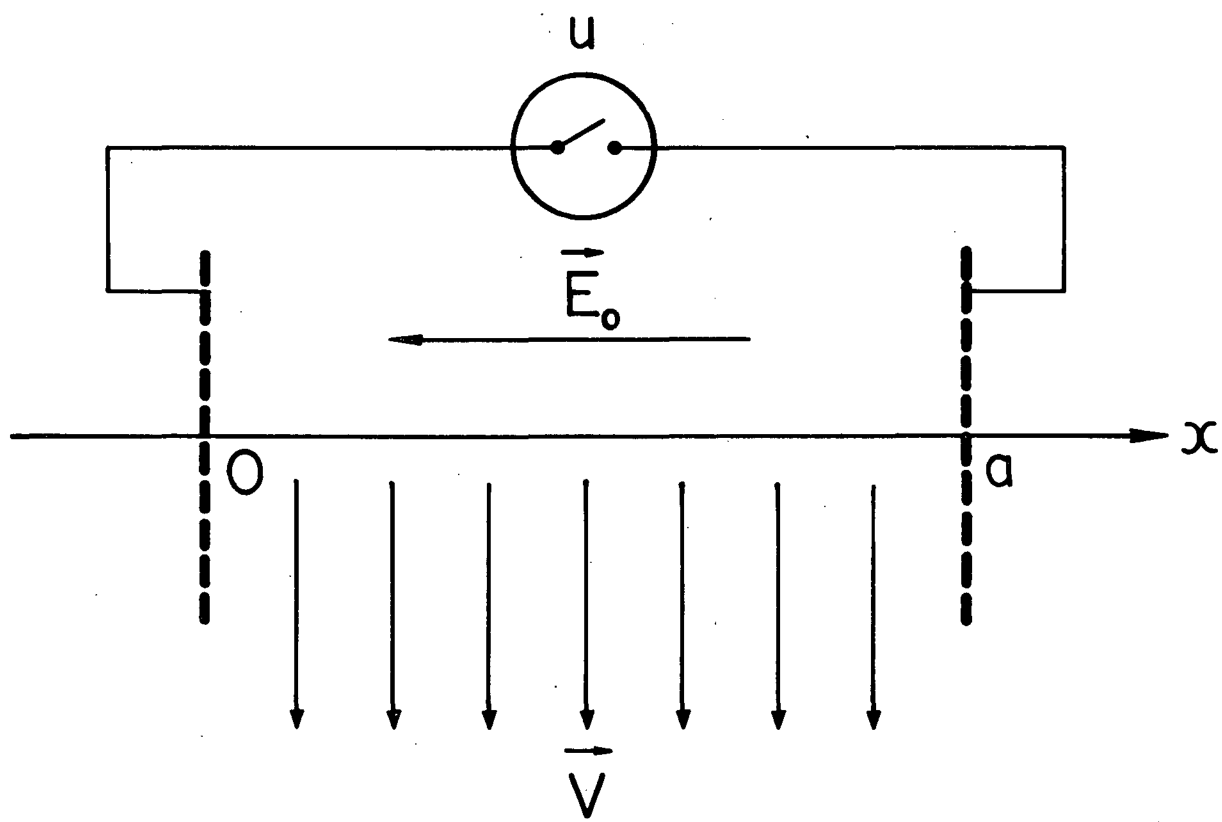

FIG. 1: Ion Beam ($\vec{V}$) Between Grids at $x = 0$ and $x = a$ in Auxiliary Electric Field $E_0 = - U/a$.

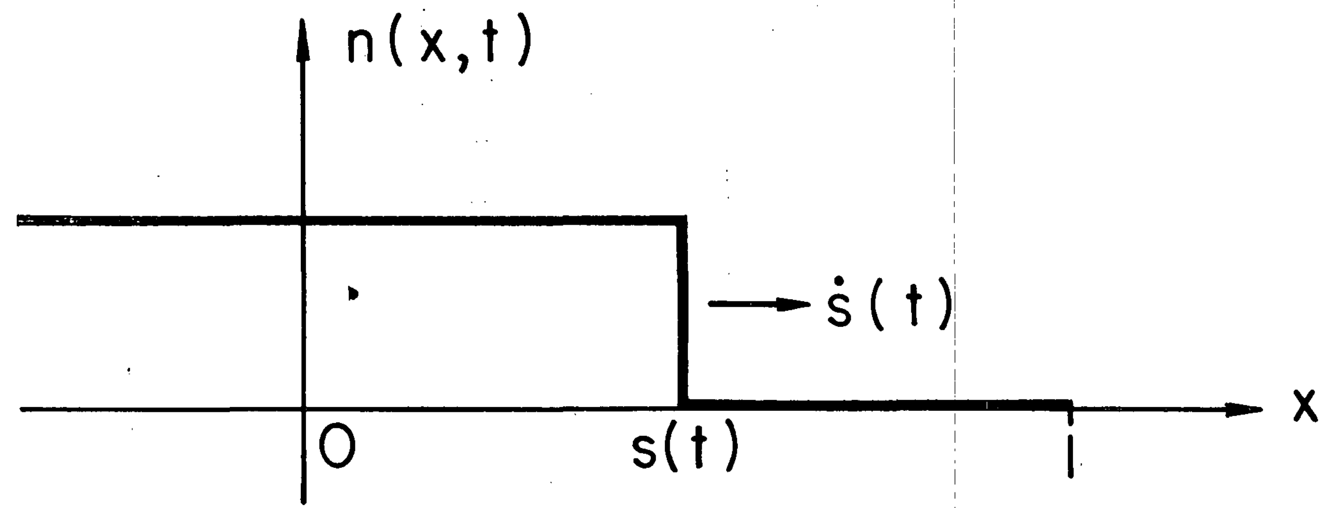

FIG. 2: Discontinuous Electron Density Field n(x, t) Versus x at Time t.

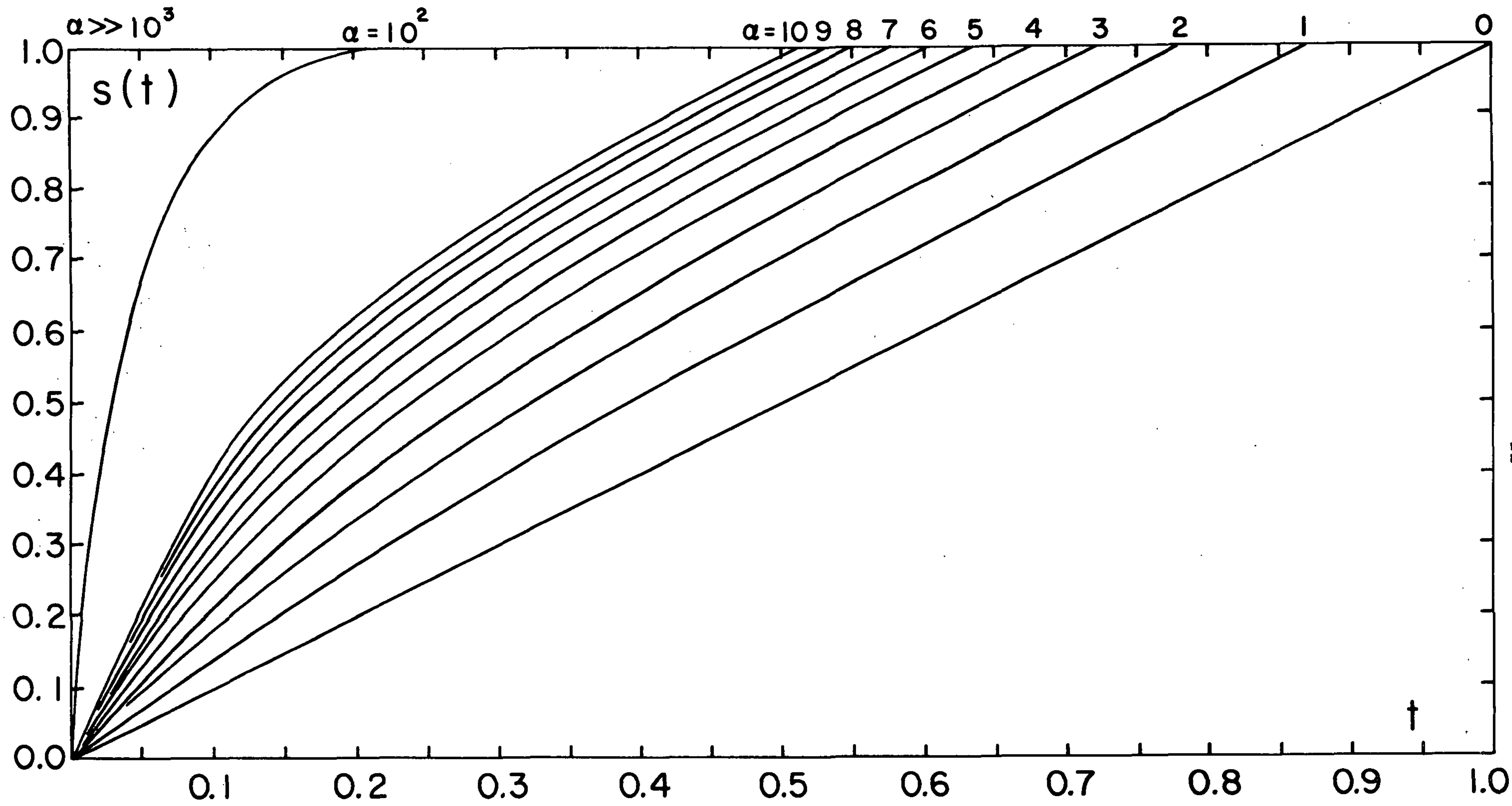

FIG. 3: Nondimensional Position Coordinate s(t) of the Neutralization Front Versus Dimensionless Time t; α= Parameter.

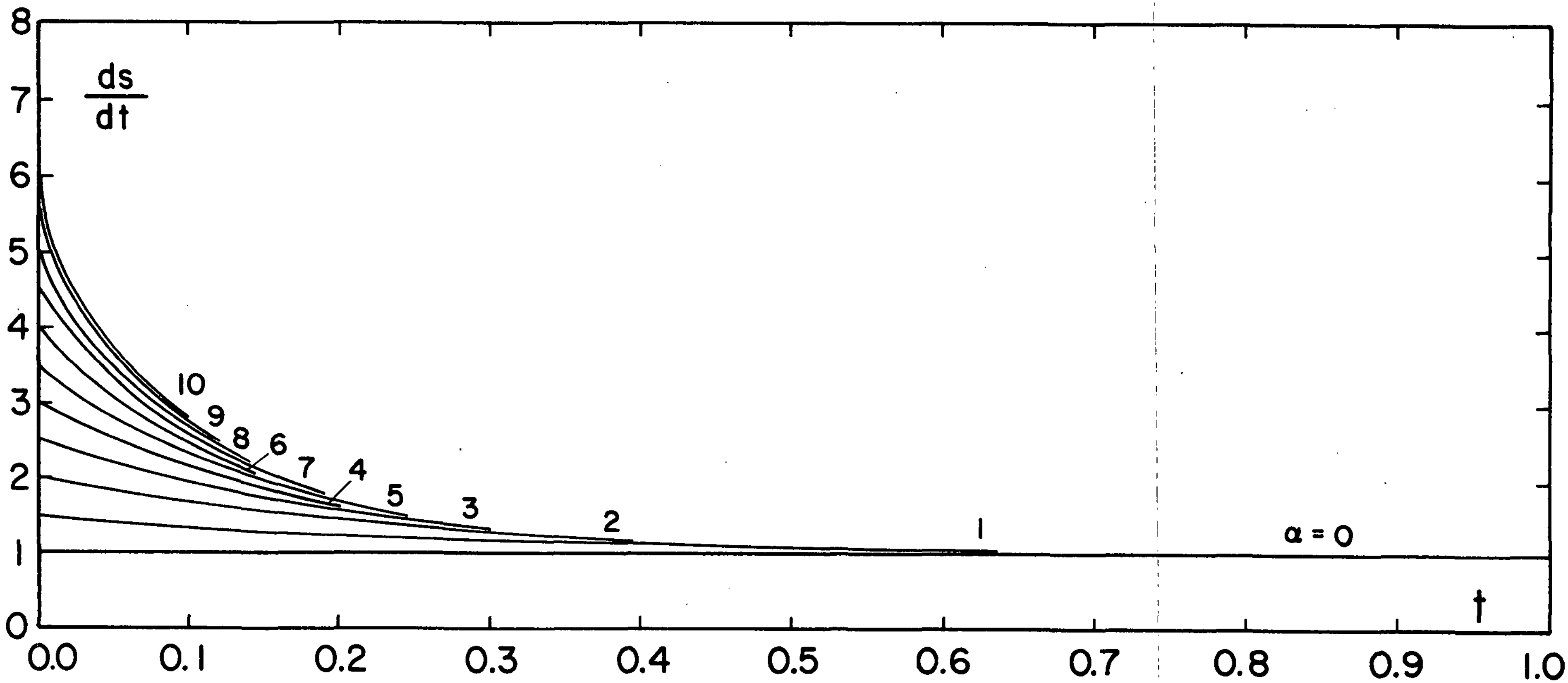

FIG. 4: Nondimensional Speed ds/dt of the Neutralization Front Versus Dimensionless Time t; α = Parameter.

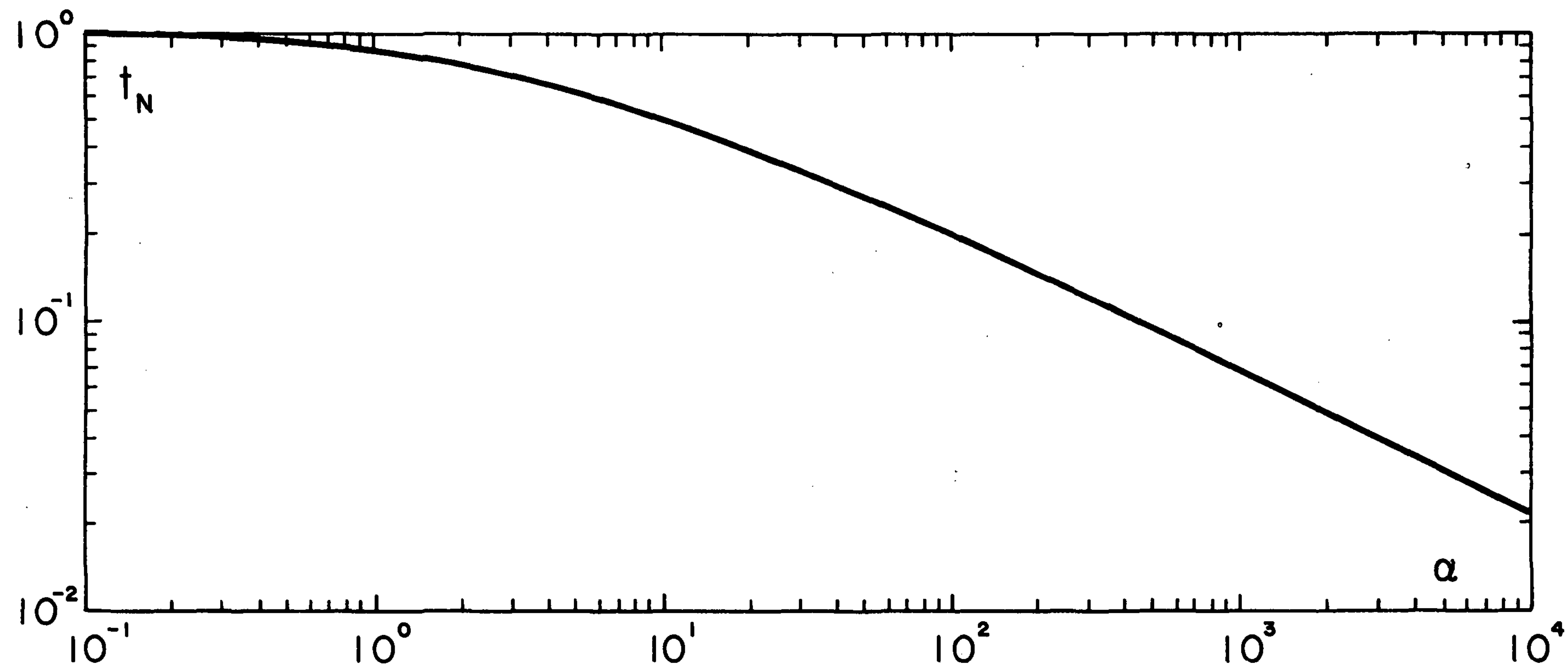

FIG. 5: Nondimensional Neutralization Time t_N Versus Parameter α.

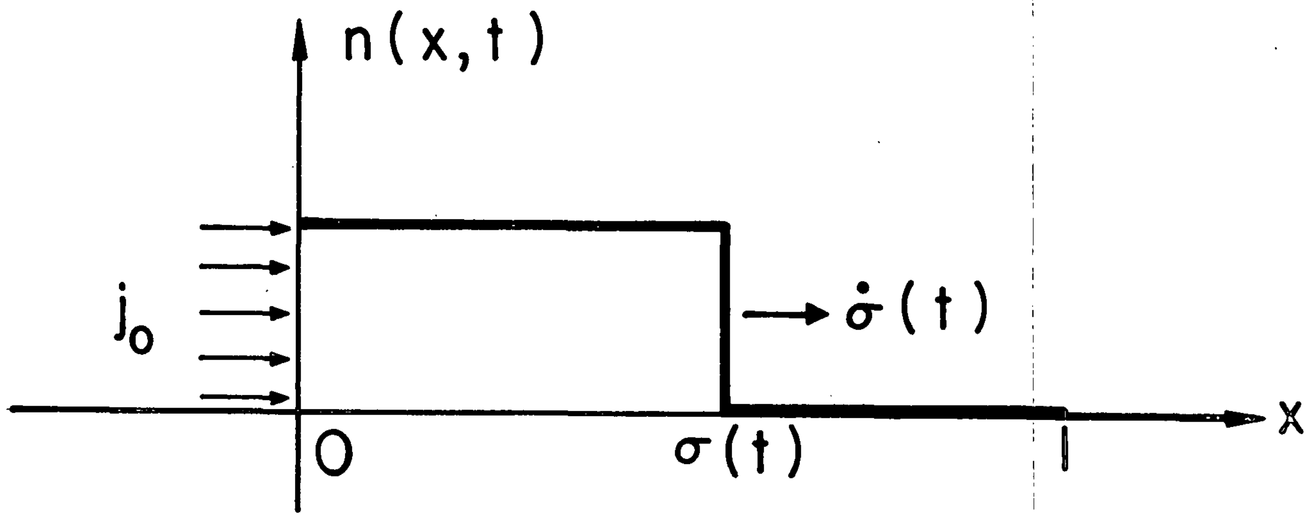

FIG. 6: Discontinuous Electron Density Field n(x, t) Versus x at Time t.

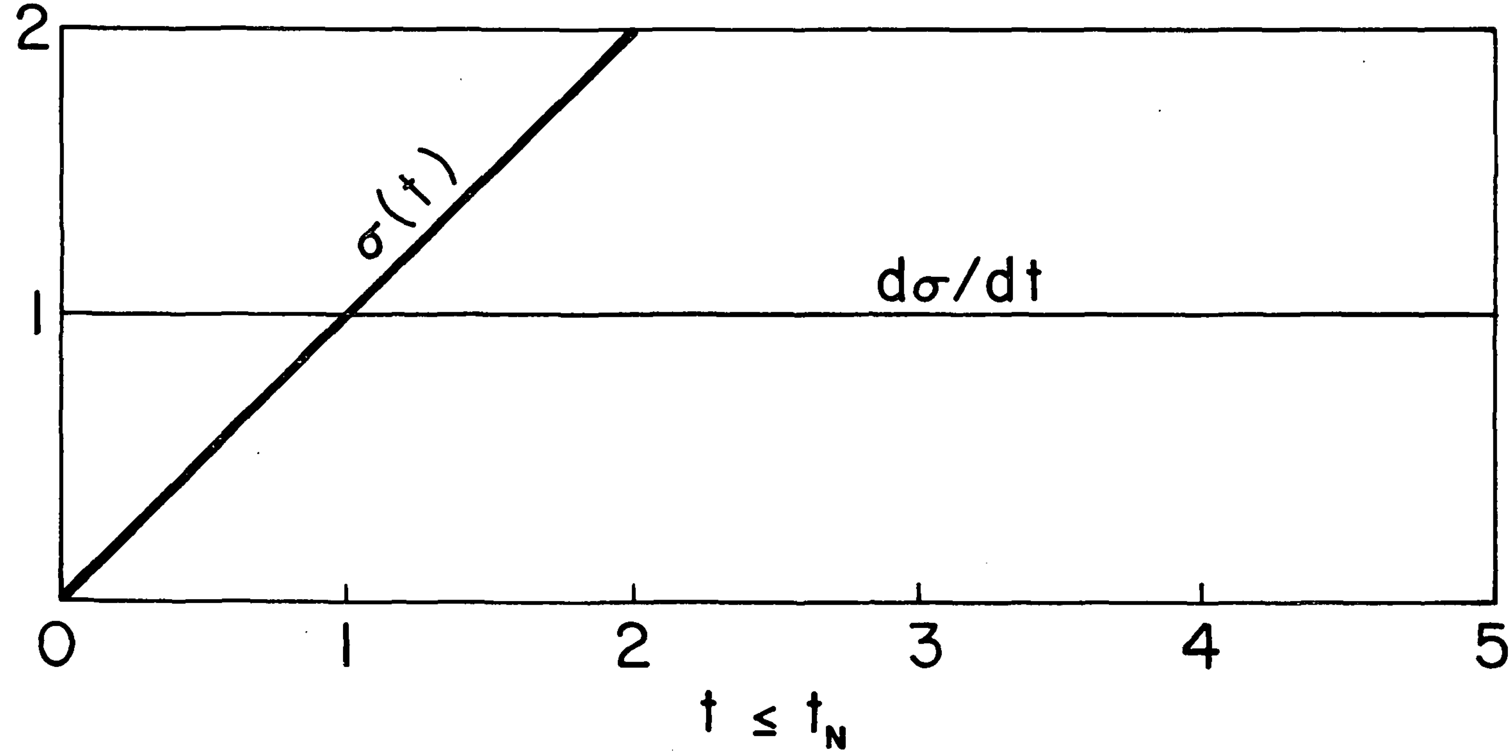

FIG. 7: Nondimensional Position Coordinate $\sigma(t)$ and Speed $d\sigma/dt$ of the Neutralization Front Versus Dimensionless Time t.

III. NONLINEAR ELECTRON-ION NEUTRALIZATION WAVES WITHOUT DISSIPATION (LATERAL INJECTION)

ABSTRACT

The nonlinear boundary-value problem describing the dynamics of an electron gas, which is injected with a prescribed current density from a plane (emitter) into an ion gas, is solved in closed form by means of a von Mises transformation. The electrons and ions are assumed to interact predominantly through the selfconsistent space charge field, while non-conservative forces are disregarded. It is shown that the electron gas propagates in form of a regular nonlinear wave $(\frac{1}{2} < n_o/N < \infty)$ or shock wave $(0 < n_o/N < \frac{1}{2})$ into the ion gas at the indicated ratios of electron injection (n_o) to ion (N) density. Graphs of the electron density, velocity, and electric space charge field in the neutralization wave are presented for various times, as well as plots of the time-dependence of the position coordinate of the neutralization front. The neutralization waves lead to a complete neutralization i) if $n_o = N$ at the injection plane, and ii) to a spatially periodic over- and under-neutralization if $n_o \neq N$. These results are strictly valid only for times which are small compared to the relaxation time for the dissipative momentum transfer between electrons and ions.

The formation of an electrically neutral plasma by injection of electrons into a rarefield ion gas is a nonlinear process in which electron-ion interactions through the self-consistent space charge field play a predominant role. In the final satages of the neutralization, diffusion and dissipative, intercomponent momentum transfer become significant as dispersion mechanism for periodic, local neutralization unbalances.

As will be shown, the electron gas moves into a collisionless ion gas in form of a nonlinear wave $(\frac{1}{2} < n_o/N < \infty)$ or shock wave $(0 < n_o/N < \frac{1}{2})$ at the indicated ratios of electron (n_o) and ion (N) densities. For this reason, it is distinguished between i) "nonlinear neutralization waves" (no overtaking of particles) and ii) "neutralization shock waves." Both phenomena exhibit a discontinuous neutralization front (ahead of which the electron density is zero), which moves with the "neutralization speed" into the ion space.

The nonlinear theory to be presented is concerned exclusively with ordinary neutralization waves ($\frac{1}{2} < n_o/N < \infty$ at the injection plane). The ions are assumed to be distributed homogeneously at the beginning of the neutralization (initial condition). The redistribution of the ions (Coulomb repulsion) during the period which the electrons need to penetrate a distance of the order of several neutralization wave lengths (x_o) in the ion space is commonly negligible because of the large inertia of the ions $(m_i >> m_e)$.

Further, as the electrons proceed in neutralizing the ion gas, the Coulomb repulsion is more and more weakened. It is recognized that the ion gas can be treated as a homogeneous space charge background for times which are large compared to the characteristic period (t_o) of the neutralization process.

The concept of a homogeneous ion gas with uniform drift velocity bounded by a plane plate has been proposed as a model for an ion beam.[1-2] In this respect, the neutralization wave theory is applicable to the transient neutralization of ion beams of electrostatic propulsion systems.[3-4] The idealized theory should provide a qualitative understanding of the neutralization of ion beams in free space. Consideration of the inhomogeneity and spreading of free ion beams does not seem to be mathematically feasible except by means of a considerable numerical expenditure. The previous work has been concerned with the steady-state neutralization of quasi-homogeneous ion beams.[1-2, 5-6]

THEORETICAL PRINCIPLES

Subject of the considerations is a collisionless electron gas which is penetrating a quasi-homogeneous ion gas of density N. The ion gas is limited to the slab space $0 \leq x < \infty (|y| < \infty, \ |z| < \infty)$. The plane at $x=0$ serves as an emitter which injects electrons at a prescribed current density $j(t)$ as indicated in Fig. 1. The field equations describing the density $n(x,t)$, velocity $v(x,t)$ and longitudinal space charge field $E(x,t)$ are[7] (electron charge = $e < 0$, electron mass = m):

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \frac{e}{m} E \quad , \tag{1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0 \quad , \tag{2}$$

$$\frac{\partial E}{\partial x} = -4\pi e(N - n) \quad . \tag{3}$$

The injection of the electrons through the plate at $x=0$ with the current density $j(t) = en(0,t)v(0,t)$ is taken into consideration by means of the boundary conditions:

$$[n(x,t)v(x\ ,t)]_{x=0} = j(t)/e > 0 \tag{4}$$

$$n(x,t)_{x=0} = n_o \quad . \tag{5}$$

The continuity of the displacement and convection currents across the interface at $x=0$ requires

$$\left[\frac{1}{4\pi} \frac{\partial E(x,t)}{\partial t} + en(x,t)v(x,t)\right]_{x=0} = j(t) \quad . \tag{6}$$

The Eqs. (4)-(6) imply for the displacement current the boundary condition, $\partial E(x,t)/\partial t = 0$ for $x = +0$.

In Eq. (1), the pressure gradient has been neglected in comparison to the electric field force (analogous to the zero-temperature plasma approximation). The Eqs. (1)-(2) can be linearized by either transforming to Lagrangian coordinates[8] or by introducing a stream function in accordance with the von Mises transformation.[9] Both methods have been applied in the nonlinear theory of Langmuir oscillations of plasmas of infinite extension[10-11] and can be extended to the analysis of nonlinear phenomena in non-neutral particle gases of finite extension with proper boundary conditions.

The nonlinear boundary-value problem defined in Eqs. (1)-(6) is subject to the von Mises transformation. Let a stream function $\psi = \psi(x,t)$ be introduced, which automatically satisfies Eq. (2), by the relations

$$\frac{\partial\psi}{\partial t} = \frac{nv}{N} \ , \quad \frac{\partial\psi}{\partial x} = -\frac{n}{N} \ . \tag{7}$$

The stream function is constant along the trajectories of the electron fluid since

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial x}\bigg|_t \frac{dx}{dt} + \frac{\partial\psi}{\partial t}\bigg|_x = -\frac{n}{N}v + \frac{nv}{N} = 0 \ . \tag{8}$$

If the Jacobian is $J\{(x,t)/(\psi,t)\} = [\partial x/\partial\psi]_t \neq 0$, it is possible to change from the old (x,t) to the new (ψ,t) independent variables:

$$x = x(\psi,t) \ , \quad t = t \ ;$$

$$n(x,t) \to n(\psi,t) \ , \quad E(x,t) \to E(\psi,t) \ , \quad v(x,t) \to v(\psi,t) \ . \tag{9}$$

Since

$$v \equiv \frac{dx}{dt} = \frac{\partial x}{\partial\psi}\bigg|_t \frac{d\psi}{dt} + \frac{\partial x}{\partial t}\bigg|_\psi = \frac{\partial x}{\partial t}\bigg|_\psi \ , \quad \text{i.e.,}$$

$$\left.\frac{\partial v}{\partial t}\right|_\psi = \left.\frac{\partial^2 x}{\partial t^2}\right|_\psi \tag{10}$$

and

$$\frac{dv}{dt} \equiv \left.\frac{\partial v}{\partial t}\right|_x + v\left.\frac{\partial v}{\partial x}\right|_t = \left.\frac{\partial v}{\partial \psi}\right|_t \frac{d\psi}{dt} + \left.\frac{\partial v}{\partial t}\right|_\psi = \left.\frac{\partial v}{\partial t}\right|_\psi \quad , \tag{11}$$

it follows that

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = \frac{\partial^2 x}{\partial t^2} \quad . \tag{12}$$

Thus, by changing to the independent variables (ψ,t) and considering the interrelations in Eqs. (8), (9) and (12), the nonlinear Eqs. (1)-(3) are reduced to the linear system:

$$\frac{\partial^2 x}{\partial t^2} = \frac{e}{m}\,E \quad , \tag{13}$$

$$\frac{e}{m}\,\frac{\partial E}{\partial x} = -\,\omega^2\left(1 + \frac{\partial \psi}{\partial x}\right) \quad , \tag{14}$$

where

$$\omega = (4\pi N e^2/m)^{\frac{1}{2}} \tag{15}$$

is the electron plasma frequency. Integration of Eq. (14) from $x=0$ to x gives

$$\frac{e}{m}\,E(\psi,t) = -\omega^2\{\psi + x - [\psi(0,t) + \frac{e}{m\omega^2}\,E(0,t)]\} \quad . \tag{16}$$

Integration of Eqs. (2) and (3) from $x=0$ to x gives, under consideration of Eq. (7), an equation physically equivalent to Eq. (14),

$$\frac{1}{4\pi}\,\frac{\partial E}{\partial t} + eN\,\frac{\partial \psi}{\partial t} = j(t) \quad , \tag{17}$$

where the integration "constant" is $j(t)$ by Eq. (6). Integration of Eq. (17) from $t=0$ to t yields an alternate form of Eq. (16),

$$\frac{e}{m} E(\psi,t) = -\omega^2\{\psi - [\psi(x,0) + \frac{e}{m\omega^2} E(x,0)] - \frac{1}{Ne} \int_0^t j(t')dt'\} . \tag{18}$$

A comparison of Eqs. (16) and (18) reveals that the various integration "constants," $\psi(x,0),\ldots,E(0,t)$, are related in a simple way to x and $j(t)$:

$$\psi(x,0) + \frac{e}{m\omega^2} E(x,0) = -x \quad , \tag{19}$$

$$\psi(0,t) + \frac{e}{m\omega^2} E(0,t) = \frac{1}{Ne} \int_0^t j(t')dt' \quad . \tag{20}$$

Elimination of $E(\psi,t)$ from Eq. (13) by means of Eq. (16) or Eq. (18) reduces the nonlinear system in Eqs. (1)-(3) to an inhomogeneous, linear oscillator equation for $x = x(\psi,t)$:

$$\frac{\partial^2 x}{\partial t^2} + \omega^2 x = -\omega^2\psi + \omega^2 f(t) \quad , \tag{21}$$

where

$$f(t) \equiv \frac{1}{Ne} \int_0^t j(t')dt' \quad . \tag{22}$$

The general solution of Eq. (21) is by Lagrange's method of the form

$$x(\psi,t) = A(\psi)\sin \omega t + B(\psi)\cos \omega t - \psi + F(t) \tag{23}$$

where

$$F(t) = \omega \int_0^t f(t')\sin \omega(t-t')dt' \tag{24}$$

with $F(0) = 0$ and $F'(0) = 0$. The functions $A(\psi)$ and $B(\psi)$ are integration "constants" which are determined by the respective initial and/or boundary conditions for the fields $n(x,t)$ and $v(x,t)$, which are related to $\psi(0,t)$ and $\psi(x,0)$ by Eq. (7):

$$\psi(0,t) - \psi(0,0) = \frac{1}{N}\int_0^t [n(x,t')v(x,t')]_{x=0}\, dt' \ , \tag{25}$$

and

$$\psi(x,0) - \psi(0,0) = \frac{1}{N}\int_0^x [n(x',t)]_{t=0}\, dx' \ . \tag{26}$$

By means of the general solution for $x(\psi,t)$ in Eq. (23), one obtains $E(\psi,t)$ by substitution of $x(\psi,t)$ into Eq. (16), $v(\psi,t)$ by partial differentiation $\partial/\partial t)_\psi$ of $x(\psi,t)$, and $n(\psi,t)$ by partial differentiation $\partial/\partial\psi)_t$ of $x(\psi,t)$:

$$E(\psi,t) = -\frac{m}{e}\,\omega^2[A(\psi)\sin\omega t + B(\psi)\cos\omega t - f(t) + F(t)] \ , \tag{27}$$

$$v(\psi,t) = \omega[A(\psi)\cos\omega t - B(\psi)\sin\omega t] + F'(t) \ , \tag{28}$$

$$n(\psi,t) = N/[1 - \frac{dA(\psi)}{d\psi}\sin\omega t - \frac{dB(\psi)}{d\psi}\cos\omega t] \ . \tag{29}$$

The Eqs. (27)-(29) represent a parametric solution for the fields $E(x,t)$, $v(x,t)$ and $n(x,t)$ in terms of ψ and t . The function $\psi = \psi(x,t)$ is given implicitly by Eq. (23).

NEUTRALIZATION WAVES

The boundary-value problem describing the motion of the electron gas in the ion space $x \geq 0$ is defined in Eqs. (1)-(6). In order to keep the mathematical expenditure a minimum, an electron current injection corresponding to a step-impulse is considered:

$$j(t) = i_o H(t) \quad , \tag{30}$$

where

$$H(t) = 1, \quad t \geq +0 \ , \tag{31}$$

and

$$= 0, \quad t \leq -0 \ ,$$

and

$$i_o \equiv e n_o v_o < 0 \quad , \tag{32}$$

so that

$$\sigma \equiv Ne/i_o > 0 \quad . \tag{33}$$

Eq. (30) represents a quasi-instantaneous injection of the current density i_o.

Since $n(0,t)V(0,t) = j(t)/e$ is a known boundary-value [Eq. (30)], Eq. (25) is relevant which gives

$$\psi(0,t) - \psi(0,0) = \sigma^{-1} H(t)t = \psi(0,t) + \frac{e}{m\omega^2} E(0,t) \tag{34}$$

by Eq. (20). Comparison of these relations indicates that $E(0,t)$ is constant,

$$\frac{-e}{m\omega^2} E(0,t) = \psi(0,0) \equiv \psi_o \quad , \tag{35}$$

and $\partial E/\partial t = 0$ at $x=0$. In the absence of surface charges at $x = +0$, it is $\psi_o = 0$ and $E(0,t) = 0$. Equation (34) is a relation between ψ and t at $x = 0$,

$$\tilde{\psi} - \psi_o = \sigma^{-1} H(\tilde{t})\tilde{t} \; ; \quad \tilde{t} \equiv t \; , \quad \tilde{\psi} \equiv \psi(0,t) \; . \tag{36}$$

Evaluation of Eqs. (22) and (24) for the current density in Eq. (30) yields

$$f(t) = \sigma^{-1} H(t)t \quad , \tag{37}$$

$$F(t) = \frac{1}{\omega\sigma} H(t)(\omega t - \sin \omega t) \quad . \tag{38}$$

The boundary conditions in Eqs. (4)-(5) give $v(x=0,\tilde{t}) = v(\tilde{\psi},\tilde{t})$, and upon application to Eq. (28),

$$A(\tilde{\psi})\cos \omega\tilde{t} - B(\tilde{\psi})\sin \omega\tilde{t} = \frac{1}{\omega} [v(x=0,\tilde{t}) - F'(\tilde{t})] \quad , \tag{39}$$

while

$$A(\tilde{\psi})\sin \omega\tilde{t} + B(\tilde{\psi})\cos \omega\tilde{t} = \tilde{\psi} - F(\tilde{t}) \tag{40}$$

by Eq. (23) for $x = 0$. One obtains by elimination from Eqs. (39)-(40), under consideration of $v(x=0,\tilde{t}) = (N/n_o)H(\tilde{t})/\sigma$ and Eq. (38):

$$A(\psi) = \psi_o \sin \omega\tau + \frac{1}{\omega\sigma} [1 - (1- \frac{N}{n_o})\cos \omega\tau]H(\tau) \quad , \tag{41}$$

$$B(\psi) = \psi_o \cos \omega\tau + \frac{1}{\omega\sigma} [(1- \frac{N}{n_o})\sin \omega\tau]H(\tau) \quad , \tag{42}$$

where

$$\tau H(\tau) = \sigma(\psi - \psi_o) \geq 0 \quad , \tag{43}$$

since the structure of a function does not depend on the designation of the independent variable $(\tilde{\psi} \to \psi\ ,\ \tilde{t} \to \tau)$.

Upon substitution for $A(\psi)$, $B(\psi)$, and $f(t)$, $F(t)$, in accordance with Eqs. (37)-(38) and Eqs. (41)-(42), respectively, one obtains from Eqs. (27)-(29) the parametric solutions:

$$\frac{E(\psi,t)}{-m\omega/e\sigma} = -(1-\frac{N}{n_o})H(\tau)\sin\omega(t-\tau) + [H(\tau)-H(t)]\sin\omega t + \omega\sigma\psi_o\cos\omega(t-\tau)\ , \tag{44}$$

$$\frac{v(\psi,t)}{1/\sigma} = H(t) -(1-\frac{N}{n_o})H(\tau)\cos\omega(t-\tau) + [H(\tau)-H(t)]\cos\omega t - \omega\sigma\psi_o\sin\omega(t-\tau)\ , \tag{45}$$

$$\frac{n(\psi,t)}{N} = \frac{H(\tau)}{H(\tau)-(1-\frac{N}{n_o})H(\tau)\cos\omega(t-\tau)-\frac{N}{n_o}\frac{\delta(\tau)}{\omega}\sin\omega t-\omega\sigma\psi_o\sin\omega(t-\tau)}\ , \tag{46}$$

where

$$\omega\sigma x(\psi,t) = -\omega\sigma\psi + H(t)\omega t - (1-\frac{N}{n_o})H(\tau)\sin\omega(t-\tau)$$
$$+ [H(\tau) - H(t)]\sin\omega t + \omega\sigma\psi_o\cos\omega(t-\tau)\ , \tag{47}$$

and $\tau = \tau(\psi)$ is given in Eq. (43). Equation (47) gives $\psi = \psi(x,t)$ for every point (x,t) , where $t \equiv \tau$ for $x = 0$ [Eq. (35)].

For the purpose of discussion, it is suitable to introduce non-dimensional variables and fields in accordance with the substitutions

$$\begin{aligned} &x/x_o \to x\ ,\quad t/t_o \to t\ ,\quad \psi/x_o \to \psi\ ; \\ &E/E_o \to E\ ,\quad v/V_o \to v\ ,\quad n/N_o \to n\ ; \end{aligned} \tag{48}$$

$$x_o \equiv 1/\omega\sigma\ ,\quad t_o \equiv 1/\omega\ ,\quad E_o \equiv -m\omega/e\sigma\ ,\quad V_o = 1/\sigma\ ,\quad N_o = N\ .$$

If in addition surface charges are absent $(\psi_o = 0)$, the Eqs. (44)-(47) become, for $t \geq 0$ and $\tau \geq 0$:

$$E(\psi,t) = -(1-\frac{N}{n_o})\sin(t-\psi) \quad , \tag{49}$$

$$v(\psi,t) = 1 -(1-\frac{N}{n_o})\cos(t-\psi) \quad , \tag{50}$$

$$n(\psi,t) = 1/[1 -(1-\frac{N}{n_o})\cos(t-\psi)-\frac{N}{n_o}\,\delta(\psi)\sin t] \quad , \tag{51}$$

where

$$x(\psi,t) = (t-\psi) - (1-\frac{N}{n_o})\sin(t-\psi) \quad , \quad \psi > 0 \quad , \tag{52}$$

and τ has been eliminated by means of Eq. (43) for $t > 0$, i.e., $\tau \geq 0$ and $\psi \geq 0$ $(\psi_o = 0)$. Note that in Eqs. (49)-(52), N and n_o are the original, dimensionless densities.

The Eqs. (49)-(52) describe a regular nonlinear wave if $J = \partial x(\psi,t)/\partial\psi \neq 0$ $[n(\psi,t) \neq \infty$ in Eq. (51)], i.e., if

$$0 < N/n_o < 2 \quad , \quad \text{or:} \quad \tfrac{1}{2} < n_o/N < \infty \quad . \tag{53}$$

The position $\hat{x} = \hat{x}(t)$ of the head of the wave (neutralization front) obtains from Eq. (47) as the limit $\hat{x} = x(\psi \to +0,t)$,

$$\hat{x}(t) = t - (1-\frac{N}{n_o})\sin t \geq 0 \quad , \quad t \geq 0 \quad . \tag{54}$$

Accordingly, $\hat{x} = k\pi$ for $t = k\pi$, $k = 0, 1, 2, 3, \ldots,$ while $\hat{x} \lesseqgtr t$ depending on $t \neq k\pi$ and $N/n_o \lesseqgtr 1$. The speed of the neutralization front is

$$\frac{d\hat{x}(t)}{dt} = 1 -(1-\frac{N}{n_o})\cos t = v(\psi,t)_{\psi=0} \quad . \tag{55}$$

It is seen that $\hat{x}(t) \geq 0$ and $d\hat{x}(t)/dt > 0$ for $t \geq 0$ since $0 < N/n_o < 2$ by Eq. (53).

In Figs. 2, 3, and 4 the (nondimensional) fields $n(x,t)$, $E(x,t)$ and $v(x,t)$ are shown versus x with $0 \leq t \leq 10\pi$ and $\alpha = N/n_o = 0.1$ and 1.1 as parameters based on Eqs. (49)-(51). At a given time $t = \hat{t}$, the neutralization wave has penetrated the region $0 \leq x \leq \hat{x}(\hat{t})$, where $x(\hat{t})$ is the position of the neutralization front at that time. The broken curve sections in Figs. 2-4 represent, e.g., the neutralization front at time $t = 2\pi$, where $\hat{x}(2\pi) = 2\pi$ by Eq. (54). In Fig. 5, the position $\hat{x}(t)$ of the neutralization front is shown for arbitrary times t with α as a parameter [Eq. (54)].

In the region ahead of the neutralization front, $x > \hat{x}(t)$, it is $\tau < 0$ and, hence, $\psi = 0$ $(\psi_o = 0)$ or $\psi = \psi_o$ $(\psi_o \neq 0)$ by Eq. (43). Accordingly,the fields in the unneutralized ion space are by Eqs. (44-46):

$$n(x,t) = v(x,t) = 0 \ , \quad E(x,t) = x-t \ , \quad x > \hat{x}(t) \ . \tag{56}$$

The Eqs. (49)-(52) demonstrate (see Figs. 2-4) that the neutralization wave produces an incomplete neutralization of the ions in the region $0 \leq x \leq \hat{x}(t)$, if $\alpha = N/n_o \neq 1$. In periodically adjacent regions, an over- and under-neutralization is achieved (Fig. 2). The association space charge field changes periodically its direction (Fig. 3), while the velocity fields fluctuates periodically between the values $v_{\pm} = 1 \pm [1 - (N/n_o)]$, Fig. 4. If $\alpha = 1$, the neutralization wave reduces to a step wave with a neutralization speed $d\hat{x}(t)/dt = 1$ [Eqs. (49)-(62) with $N/n_o = 1$]. In the latter case, a complete neutralization is observed $[n(x,t) = 1$ for $0 \leq x \leq \hat{x}(t))]$.

The reference values x_o, t_o, E_o, V_o, and N_o defined in Eq. (48) determine the actual magnitude of the variables. As a numerical example, consider an ion gas (N) and current (i_o) density given by

$$N = 10^9 \text{ cm}^{-3} \ , \quad i_o \hat{=} 10^{-2} \text{ amp cm}^{-2} \ .$$

Hence:

$$x_o = 3.691 \times 10^4 \; |i_o| N^{-3/2} = 3.502 \times 10^{-2} \text{ cm}$$

$$t_o = 1.773 \times 10^{-5} \; N^{-1/2} = 5.607 \times 10^{-10} \text{ sec} \quad ,$$

and

$$E_o = 2.227 \times 10^{-4} \; |i_o| N^{-1/2} = 2.113 \times 10^{-1} \text{ cgsu} = 6.339 \times 10^{1} \text{ volt cm}^{-1} \quad ,$$

$$V_o = 2.082 \times 10^{9} \; |i_o| N^{-1} = 6.246 \times 10^{7} \text{ cm sec}^{-1},$$

$$N_o = N = 10^{9} \text{ cm}^{-3} \quad .$$

The values of x_o and t_o give the magnitude of the space and time periods of the neutralization wave $(V_o = x_o/t_o)$. The (positive) amplitudes $\hat{E}$, $\hat{v}_\pm$, and $\hat{n}_\pm$ of the dimensional fields are by Eqs. (49)-(51):

$$\hat{E} = \left|(1-\frac{N}{n_o})E_o\right| \; , \quad \hat{v}_\pm = \left|[1 \pm (1-\frac{N}{n_o})]V_o\right| \; , \; \hat{n}_\pm = \left|N/[\pm (1-\frac{N}{n_o})]\right| \tag{57}$$

In this idealized treatment of neutralization waves, dissipation mechanisms have been disregarded. Consideration of dissipation would lead to a damping of the wave amplitudes and thus to a dispersion of the local space charge unbalances, i.e. to a complete neutralization. The most important dissipation is caused by irreversible momentum exchange between the electron and ion components. By means of a dimensional argument, one finds from the intercomponent momentum relaxation time τ and the characteristic parameters i_o, N, e as order-of-magnitude of the relaxation length for neutralization

$$\lambda \cong |i_o/Ne|\tau \quad . \tag{58}$$

Citations

1. E. Stuhlinger, Ion Propulsion for Space Flight, McGraw-Hill, New York 1964.
2. G. F. Au, Electric Propulsion of Space Vehicles, Verlag G. Braun, Karlstruhe 1968.
3. H. R. Kaufman, An Ion Rocket with an Electron-Bombardment Ion Source, NASA TN-D-585-1961.
4. N. R. Kerslake, D. C. Byers, and J. F. Staggs, J. Spacecraft and Rockets, 1, 4 (1970).
5. H. R. Kaufman, One-Dimensional Analysis of Ion Rockets, NASA TN-D-261 (1960).
6. L. D. Pearlstein, M. N. Rosenbluth, and G. W. Stuart, The Neutralization of Ion Beams, Progress in Astronautics and Aeronautics 9, 379, Academic Press, New York 1963.
7. A. A. Vlasov, Many Particle Theory and its Application to Plasma, Gordon & Breach, New York 1961.
8. K. P. Stanyukovich, Unsteady Motion of Continuous Media, Pergammon Press, New York 1960.
9. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948.
10. G. Kalman, Ann. Phys. 10, 1 (1960).
11. R. W. C. Davidson and P. P. J. M. Schram, Nuclear Fusion 8, 183 (1968).

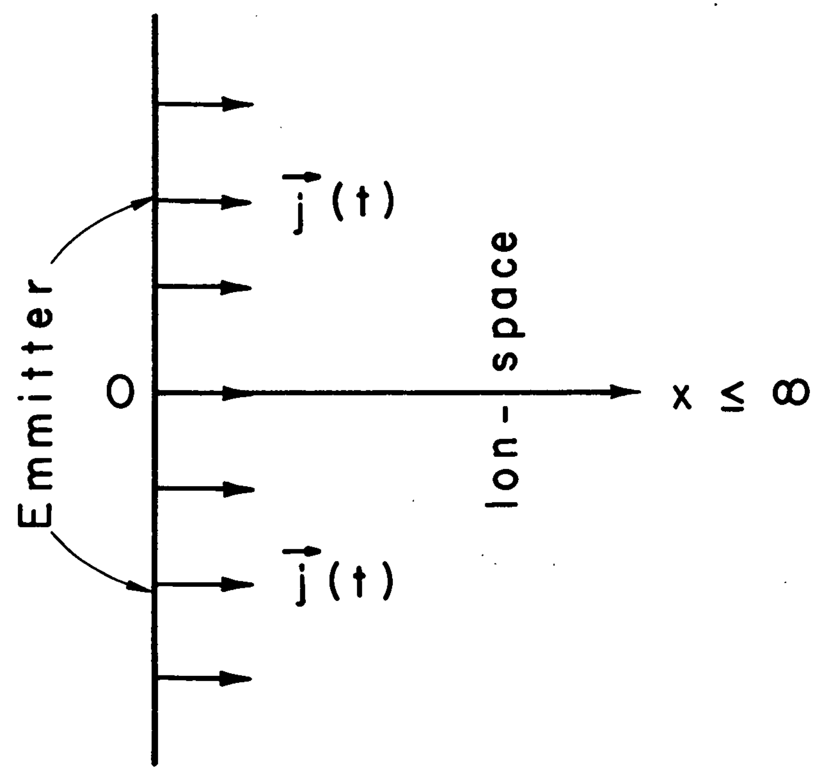

FIG. 1: Electron current injection ($\vec{j}$) from plane at $x = 0$ into ion space $x \geq 0$ (schematically).

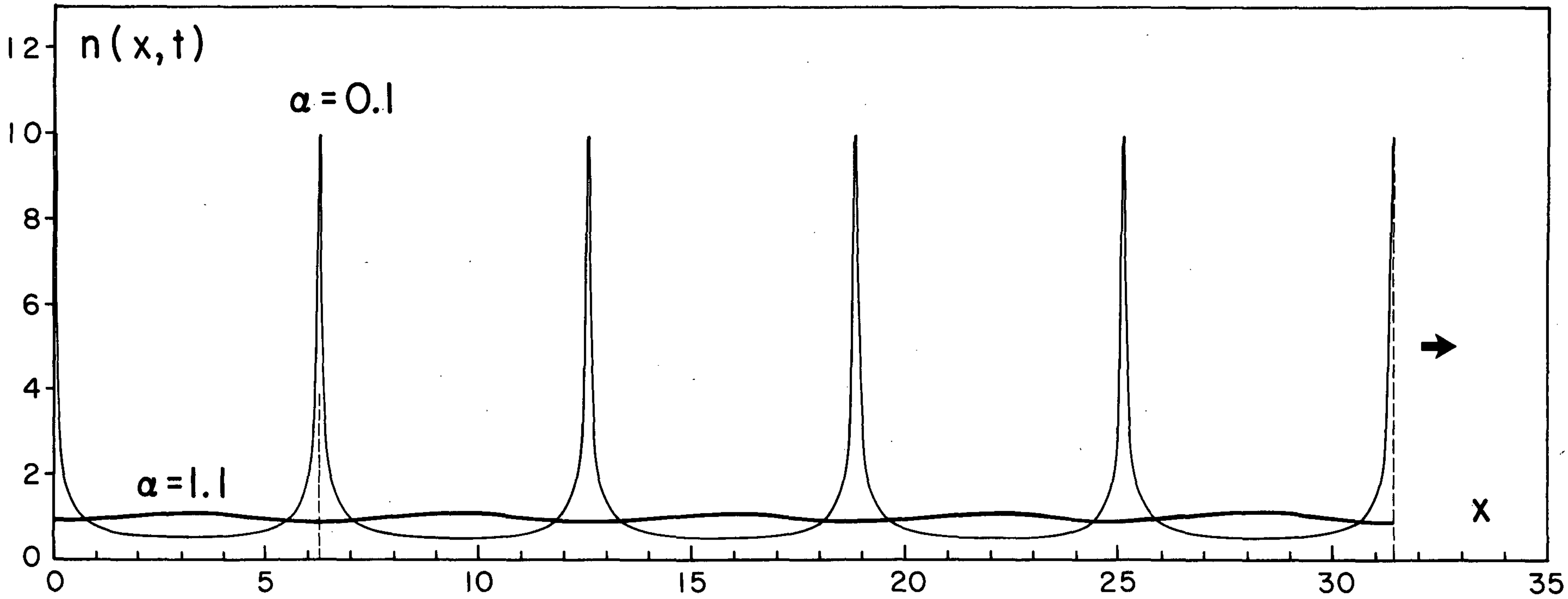

FIG. 2: Electron density field $n(x,t)$ versus x at time $t = 10\pi$ for $\alpha = N/n_o = 0.1, 1.1$.

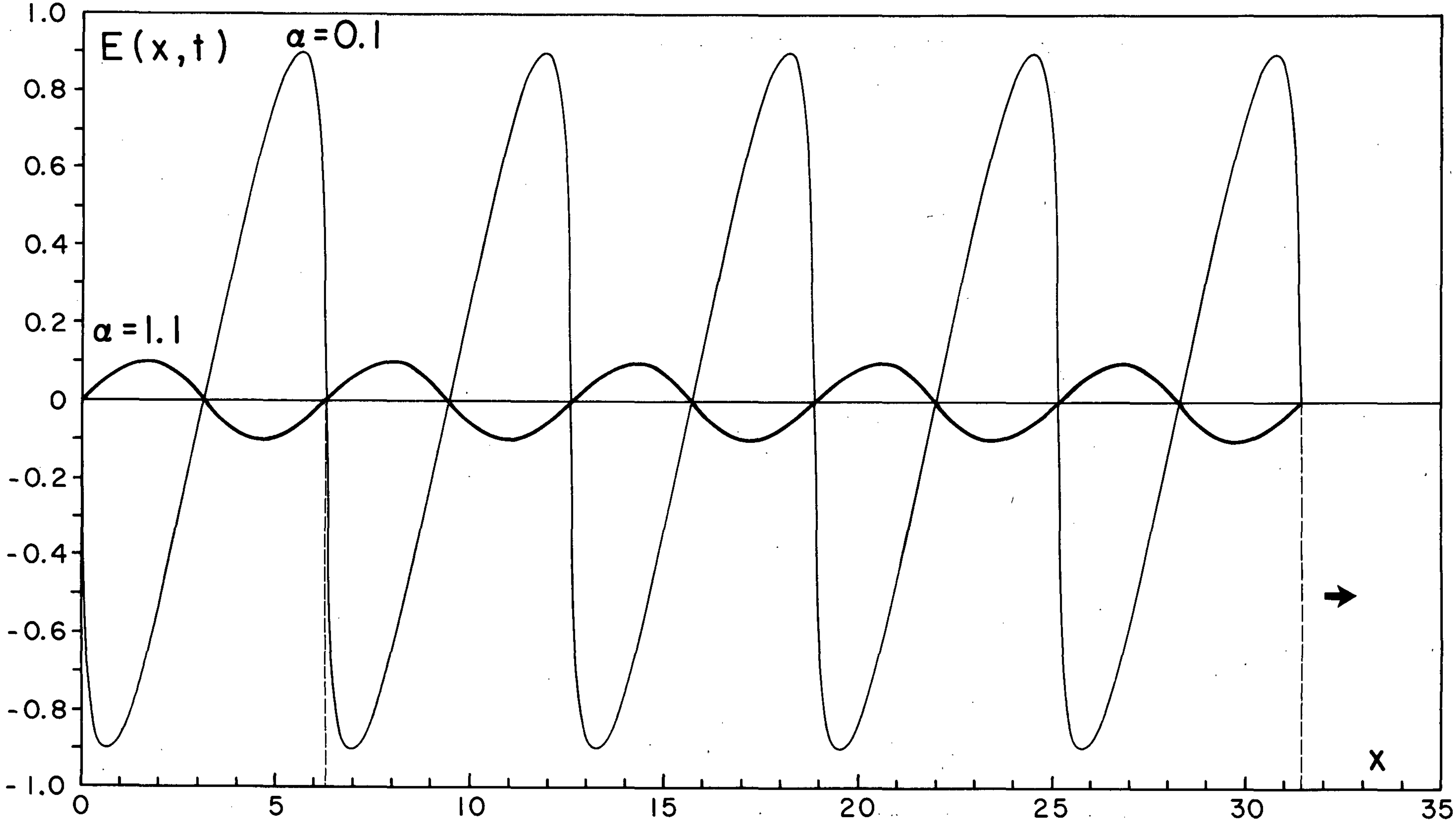

FIG. 3: Electric field $E(x,t)$ versus $0 \leq x \leq \hat{x}(t)$ at time $t = 10\pi$ for $\alpha = N/n_o = 0.1, 1.1$.

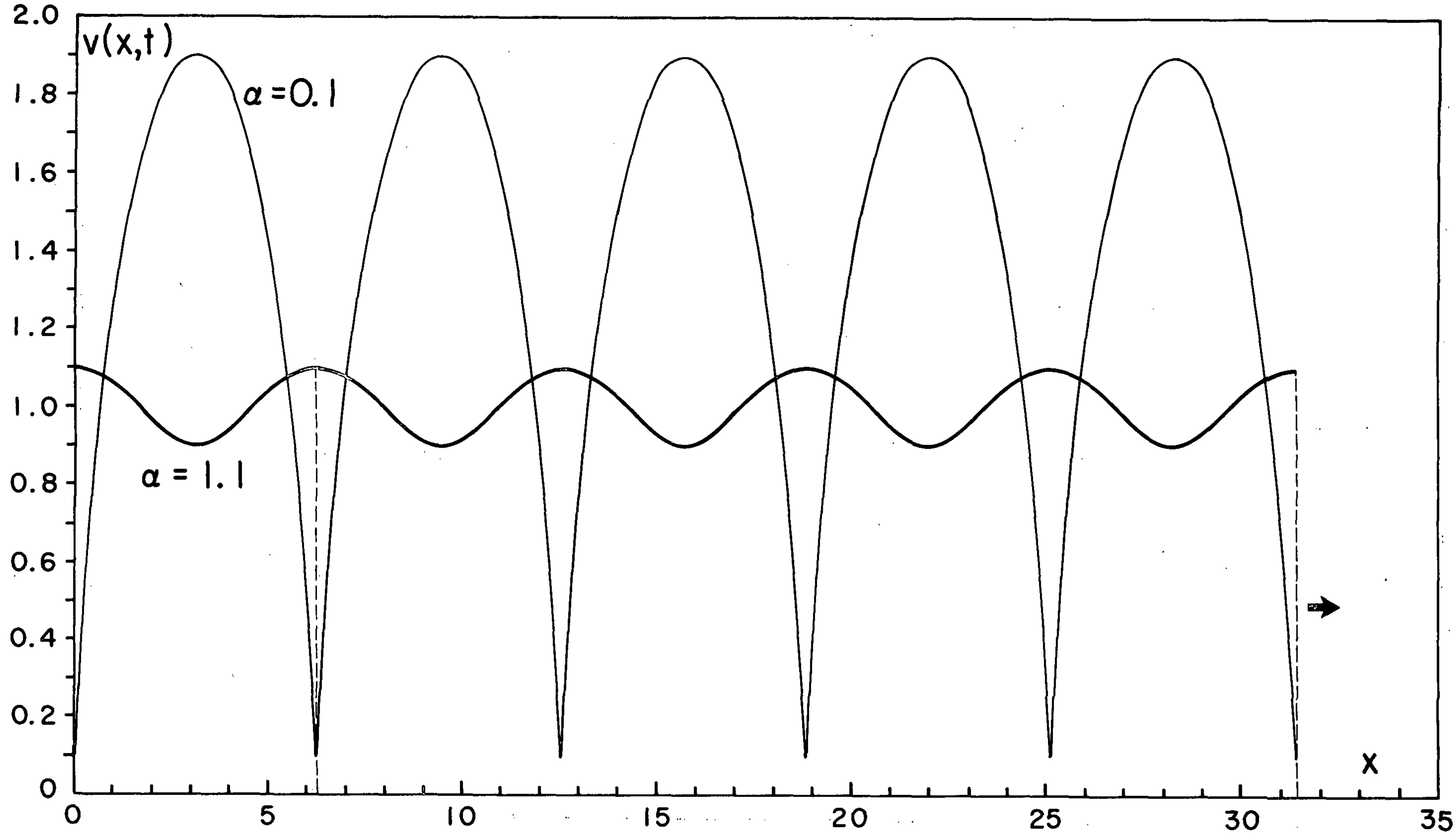

FIG. 4: Electron velocity field v(x,t) versus x at time $t = 10\pi$ for $\alpha = N/n_o = 0.1, 1.1$.

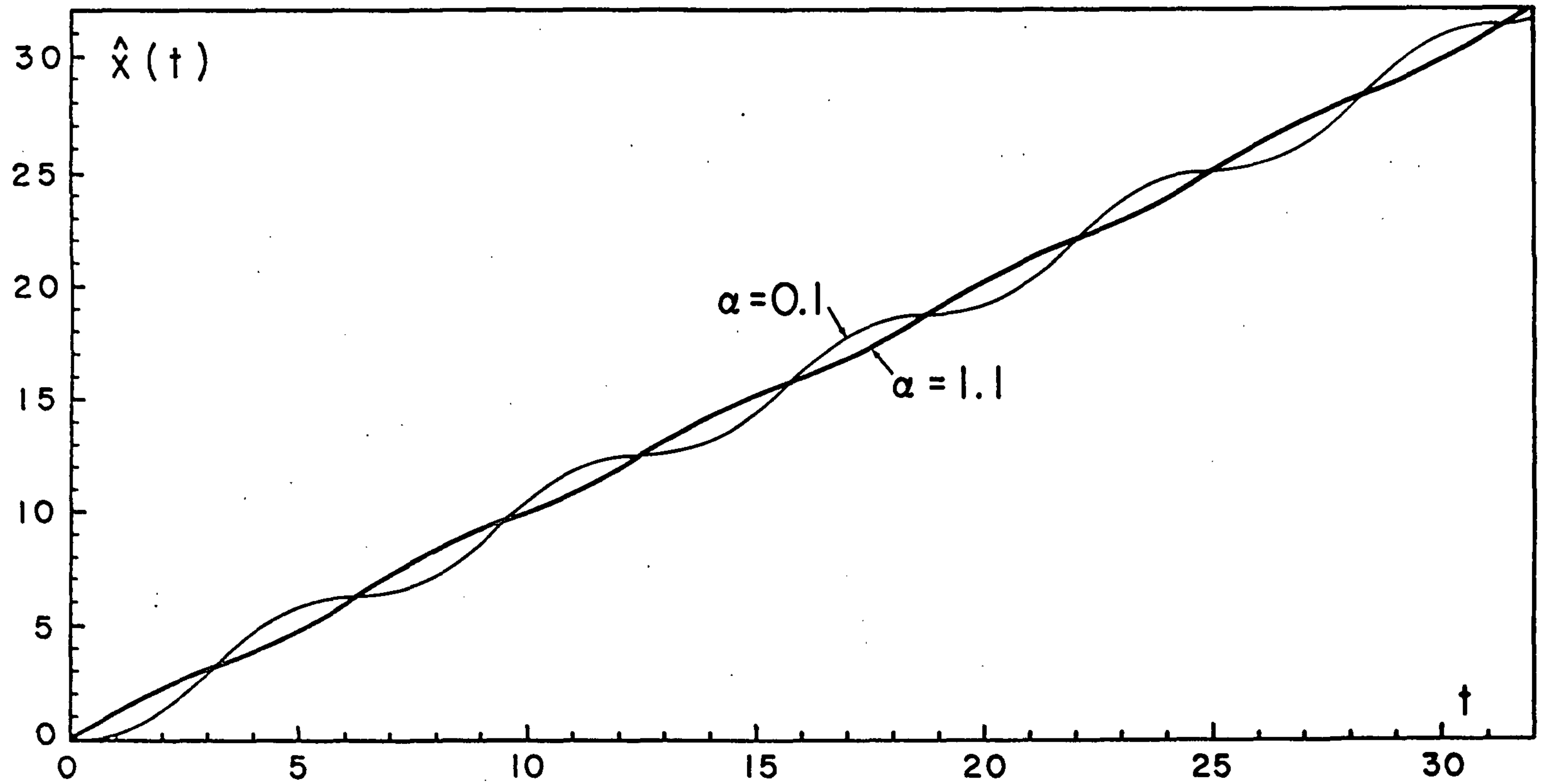

FIG. 5: Position coordinate $\hat{x}(t)$ of the neutralization front versus t for $\alpha = N/n_o = 0.1, 1.1$.

IV. NONLINEAR THEORY OF ELECTRON NEUTRALIZATION WAVES IN ION BEAMS WITH DISSIPATION (LONGITUDINAL INJECTION)

ABSTRACT

An analytical theory of nonlinear neutralization waves generated by injection of electrons from a grid in direction of a homogeneous ion beam of uniform velocity and infinite extension is presented. The electrons are assumed to interact with the ions through the self-consistent space charge field and by strong collective interactions, while diffusion in the pressure gradient is disregarded (zero-temperature approximation). The associated nonlinear boundary-value problem is solved in closed form by means of a von Mises transformation. It is shown that the electron gas moves into the ion space in form of a discontinuous neutralization wave, which exhibits a periodic field structure (incomplete neutralization). This periodic wave structure is damped out by intercomponent momentum transfer, i.e., after a few relaxation lengths a quasi-neutral plasma results. The relaxation scale in space agrees with neutralization experiments of rarefied ion beams, if the collective momentum transfer between the electron and ion streams is assumed to be of the Buneman type.

A beam of positive ions attracts electrons from the surroundings, such as system walls and rest gases of an incomplete vacuum. Thus, a partial neutralization by electrons is a quite common phenomenon in ion beams. In many technical applications,[1-2] one is interested in producing a complete neutralization of ion beams by electron injection in order to reduce beam spreading and beam current limitations by build-up of large space charge potentials.[3-4]

The neutralization of a rarified ion beam by injection of an electron-gas is a nonlinear process in which electron-ion interactions through the self-consistent space charge field play a predominant role. In the final stages of the neutralization intercomponent momentum transfer and to some extent also diffusion resulting from pressure gradients become significant as despersion mechanisms of periodic, local neutralization unbalances. As will be shown, the electron gas moves into a collisionless ion beam in form of a nonlinear wave $(1/2 \lesssim n_o/N < \infty)$ or shock wave $(0 < n_o/N \lesssim 1/2)$ at the indicated ratios of electron injection (n_o) and ion (N) densities. For this reason, it is distinguished between i) "nonlinear neutralization waves" (no overtaking of particles) and ii) "neutralization shock waves." Both phenomena exhibit a discontinuous neutralization front (ahead of which the electron density is zero), which moves with the "neutralization speed" into the ion space.

As a model, a quasi-homogeneous ion beam if infinite extension is considered, i.e. the density N and velocity $\vec{V}$ of the beam are assumed to be uniform. The electrons are injected with a prescribed current density $\vec{j}(t)$ in the downstream direction $(\vec{j} \parallel \vec{V})$. The homogeneous ion beam model may be justified as i) a steady state condition, and ii) an initial condition for the following reasons:

i) Since the beam is of infinite extension, a steady state exists with a homogeneous ion density N and a uniform flow velocity $\vec{V}$. Because of the relatively large inertia of the ions $(M >> m)$, the perturbation of the homogeneity of the ion gas by injection of an electron stream from a grid is negligible for a period which is large compared to the characteristic time (t_o) of the neutralization wave.

ii) The redistribution of the ions during the period which the electrons need to penetrate a distance of the order of several neutralization wave lengths (x_o) in the ion space is negligible because of the inertia of the ions $(M >> m)$. Further, as the electrons proceed in neutralizing the ion gas, the perturbing interactions through the self-consistent field are reduced.

The homogeneous ion beam model is adequate for analyzing and understanding the basic properties of transient neutralization waves. The neutralization in a finite, inhomogeneous ion beam does not seem to be treatable by analytical methods. The previous analytical work has been concerned with steady-state electron-ion neutralization without dissipation in homogeneous ion beams.[5-6]

NONLINEAR - BOUNDARY - VALUE PROBLEM

An ion gas of homogeneous density N and uniform velocity $\vec{V}$ parallel to the x - axis is considered in the space $|x| \leq \infty$, $|y| \leq \infty$, $|z| \leq \infty$. This infinite ion beam is bisected by a permeable grid in the plane $x = 0$, from which electrons are ejected with the current density $j(t)$ in the direction of V (Fig. 1). The field equations for the electron velocity $v(x,t)$, electron density $n(x,t)$, and the selfconsistent electric field $E(x,t)$ are[7]:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{e}{m}E - \nu(v-V) \ , \tag{1}$$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial x}(nv) \ , \tag{2}$$

$$\frac{\partial E}{\partial x} = 4\pi e(N-n) \ , \tag{3}$$

where $e > 0$ is the elementary charge and m the electron mass. The term $-\nu(v-V)$, describes the momentum transfer between the electrons and ions (ν = relaxation frequency). The injection of electrons with the current density $j(t)$ into the ion beam from the grid plane at $x = 0$ is taken into consideration by the boundary conditions:

$$[n(x,t)\ v(x,t)]_{x=0} = j(t)/(-e) \ , \tag{4}$$

$$n(x,t)_{x=0} = n_o \ , \tag{5}$$

The semispace $x < 0$ is assumed to be shielded from electric displacement currents by the grid at $x = 0$. Accordingly, the continuity of displacement and convection currents across the plane $x = 0$ gives

$$[\frac{1}{4\pi}\frac{\partial E(x,t)}{\partial t} - en(x,t)\ v(x,t)]_{x=0} = j(t) \ . \tag{6}$$

The Eqs. (4) - (6) imply that the displacement current satisfies the boundary condition , $\partial E(x,t)/\partial t = 0$ for $x = +0$.

The electron and ion gases are assumed to be rarefied ($n, N \lesssim 10^{12} cm^{-3}$) and of low temperature ($T < 10^{4\,o}K$) . The effects of binary collisions and linear electron-wave interactions are not discussed. Possible strong collective ("turbulence") momentum exchange between the electron and ion streams is discussed based on Buneman's relaxation frequency[8]

$$\nu = (m/M)^{1/3}\omega/2\pi \ , \tag{7}$$

where

$$\omega = (4\pi Ne^2/m)^{1/2} \ . \tag{8}$$

The Eqs. (1) - (6) describe the nonlinear-boundary-value problem of the transient neutralization process in the ion beam which takes place in the region $x \geq 0$. Thermal effects are not taken into consideration.

METHOD OF SOLUTION

The nonlinear-boundary-value problem defined in Eqs. (1) - (6) is solvable in closed form by a von Mises transformation[9-10]. Let a stream function $\psi = \psi(x,t)$ be introduced, which automatically satisfies Eq. (2), by the relations

$$\frac{\partial\psi}{\partial t} = \frac{nv}{N}, \qquad \frac{\partial\psi}{\partial x} = -\frac{n}{N}. \tag{9}$$

The stream function is constant along the trajectories of the electron fluid since

$$\frac{d\psi}{dt} = \left.\frac{\partial\psi}{\partial x}\right|_t \frac{dx}{dt} + \left.\frac{\partial\psi}{\partial t}\right|_x = -\frac{n}{N}v + \frac{nv}{N} = 0 . \tag{10}$$

The Jacobian is $J\{[(x,t)/(\psi,t)]\} = \partial x/\partial\psi|_t \neq 0$ for $n(x,t) < \infty$. For such solutions, it is permitted to introduce (ψ,t) as independent variables in place of (x,t) :

$$x = x(\psi,t) , \qquad t = t ,$$

$$n(x,t) \to n(\psi,t) , \quad E(x,t) \to E(\psi,t) , \quad v(x,t) \to v(\psi,t). \tag{11}$$

Since

$$v \equiv \frac{dx}{dt} = \left.\frac{\partial x}{\partial\psi}\right|_t \frac{d\psi}{dt} + \left.\frac{\partial x}{\partial t}\right|_\psi = \left.\frac{\partial x}{\partial t}\right|_\psi , \text{ i.e.}$$

$$\left.\frac{\partial v}{\partial t}\right|_\psi = \left.\frac{\partial^2 x}{\partial t^2}\right|_\psi \tag{12}$$

and

$$\frac{dv}{dt} = \left.\frac{\partial v}{\partial t}\right|_x + v\left.\frac{\partial v}{\partial x}\right|_t = \left.\frac{\partial v}{\partial\psi}\right|_t \frac{d\psi}{dt} + \left.\frac{\partial v}{\partial t}\right|_\psi = \frac{\partial v}{\partial t} , \tag{13}$$

it follows that

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = \frac{\partial^2 x}{\partial t^2} \ . \tag{14}$$

Thus, by changing to the independent variables (ψ,t) and considering the interrelations in Eqs. (10), (11), and (14), the nonlinear Eqs. (1)-(3) are reduced to the linear system:

$$\frac{\partial^2 x}{\partial t^2} = -\frac{e}{m}\,E - \nu\left(\frac{\partial x}{\partial t} - V\right) , \tag{15}$$

$$\frac{e}{m}\,\frac{\partial E}{\partial x} = \omega^2\left(1 + \frac{\partial \psi}{\partial x}\right) \ . \tag{16}$$

Integration of Eq. (16) from $x = 0$ to x gives

$$\frac{e}{m}\,E(\psi,t) = \omega^2\{\psi + x - [\psi(0,t) - \frac{e}{m\omega^2}\,E(0,t)]\} \ . \tag{17}$$

Integration of Eqs. (2) and (3) from $x = 0$ to x gives, under consideration of Eq. (6), an equation physically equivalent to Eq. (16),

$$\frac{1}{4\pi}\,\frac{\partial E}{\partial t} - eN\,\frac{\partial \psi}{\partial t} = j(t) , \tag{18}$$

where the integration "constant" is $j(t)$ by Eq. (4). Integration of Eq. (18) from $t = 0$ to t yields another form of Eq. (17),

$$\frac{e}{m}\,E(\psi,t) = \omega^2\{\psi - [\psi(x,0) - \frac{e}{m\omega^2}\,E(x,0)] + \frac{1}{Ne}\int_0^t j(t')dt'\} \ . \tag{19}$$

A comparison of Eqs. (17) and (19) reveals that the various integration "constants," $\psi(x,0)$, ... , $E(0,t)$, are related in a simple way to x and $j(t)$:

$$\psi(x,0) - \frac{e}{m\omega^2} E(x,0) = - x , \tag{20}$$

$$\psi(0,t) - \frac{e}{m\omega^2} E(0,t) = - \frac{1}{Ne} \int_0^t j(t')dt' . \tag{21}$$

Elimination of $E(\psi,t)$ from Eq. (15) by means of Eq. (17) or Eq. (19) reduces the nonlinear system in Eqs. (1)-(3) to an inhomogeneous, damped, linear oscillator equation for $x = x(\psi,t)$:

$$\frac{\partial^2 x}{\partial t^2} + \nu \frac{\partial x}{\partial t} + \omega^2 x = \nu V + \omega^2[f(t) - \psi] \tag{22}$$

where

$$f(t) \equiv - \frac{1}{Ne} \int_0^t j(t')dt' . \tag{23}$$

By Lagrange's method, the general solution of Eq. (23) is for $\omega > \nu/2$ [Eq. (7)]:

$$x(\psi,t) = e^{-\nu t/2}[A(\psi)\sin\Omega t + B(\psi) \cos\Omega t] - \psi + F(t) \tag{24}$$

where

$$\Omega \equiv [\omega^2 - (\nu/2)^2]^{\frac{1}{2}} > 0 , \tag{25}$$

and $F(t)$ is a particular integral of the inhomogeneous differential equation,

$$\frac{d^2F}{dt^2} + \nu \frac{dF}{dt} + \omega^2 F = \nu V + \omega^2 f(t) , \tag{26}$$

i.e.

$$F(t) = -\frac{\omega^2}{\Omega}\int_0^t f(t')e^{\nu(t'-t)/2}\sin\Omega(t'-t)dt'$$

$$+\frac{\nu V}{\omega^2}[1 - e^{-\nu t/2}(\frac{\nu/2}{\Omega}\sin\Omega t + \cos\Omega t)] \ , \qquad (27)$$

with $F(0) = 0$ and $F'(0) = 0$. $A(\psi)$ and $B(\psi)$ are arbitrary functions which are determined by the respective initial and/or boundary conditions for $n(x,t)$ and $v(x,t)$. By Eq. (9)

$$\psi(0,t) - \psi(0,0) = \frac{1}{N}\int_0^t [n(x,t')v(x,t')]_{x=0}dt' \ , \qquad (28)$$

and

$$\psi(x,0) - \psi(0,0) = \frac{1}{N}\int_0^x [n(x',t)]_{t=0}dx' \ . \qquad (29)$$

interrelates the boundary (x=0) and initial (t=0) values of $\psi(x,t)$ to those of $n(x,t)$ and $v(x,t)$.

By means of the general solution for $x(\psi,t)$ in Eq. (24), one obtains $E(\psi,t)$ by substitution of $x(\psi,t)$ into Eq. (17), $v(\psi,t)$ by partial differentiation $\partial/\partial t)_\psi$ of $x(\psi,t)$ [Eq. (12)], and $n(\psi,t)$ by partial differentiation $\partial/\partial\psi)_t$ of $x(\psi,t)$ [Eq. (9)] :

$$E(\psi,t) = \frac{m\omega^2}{e}\{e^{-\nu t/2}[A(\psi)\sin\Omega t + B(\psi)\cos\Omega t] - f(t) + F(t)\} \ , \qquad (30)$$

$$v(\psi,t) = e^{-\nu t/2}\{\Omega[A(\psi)\cos\Omega t - B(\psi)\sin\Omega t]$$

$$-\frac{\nu}{2}[A(\psi)\sin\Omega t + B(\psi)\cos\Omega t]\} + \frac{dF(t)}{dt} \ , \qquad (31)$$

$$n(\psi,t) = N\{1 - e^{-\nu t/2}[\frac{dA(\psi)}{d\psi}\sin\Omega t + \frac{dB(\psi)}{d\psi}\cos\Omega t]\}^{-1}. \qquad (32)$$

The Eqs. (30) - (32) represent a parametric solution for the fields $E(x,t)$, $v(x,t)$, and $n(x,t)$ in terms of ψ and t . The function $\psi = \psi(x,t)$ is given implicitly by Eq. (24).

LONGITUDINAL NEUTRALIZATION WAVES

The theoretical investigation of neutralization waves generated by electron injection into a homogeneous ion beam requires specification of the time dependence of $j(t)$. For mathematical convenience, an electron injection from the grid plane $x = 0$ in the form of step-impulse is considered:

$$j(t) = i_o H(t) \ , \quad t \gtrless 0 \ , \tag{33}$$

where

$$\begin{aligned} H(t) &= 1, \quad t \geq +0 \ , \\ &= 0, \quad t \leq -0 \ , \end{aligned} \tag{34}$$

and

$$i_o = - e n_o v_o < 0 \ . \tag{35}$$

Hence,

$$\sigma \equiv - Ne/i_o > 0 \ . \tag{36}$$

This model corresponds to a quasi-instantaneous electron injection with the current density i_o.

A comparison of the Eqs. (21) and (28) indicates that $E(0,t)$ is a constant independent of $j(t)$,

$$(e/m\omega^2)E(0,t) = \psi(0,0) \equiv \psi_o \ .$$

In absence of surface charges at $x = 0$, it is $E(0,t) = 0$ and $\psi_o = 0$.

The boundary-value $n(x=0,t)v(x=0,t) = -j(t)/e$ is known through Eqs. (33) and (35). Accordingly, Eq. (28) is relevant which gives

$$\tilde{\psi}(t) - \psi_o = \sigma^{-1}f(t), \quad \tilde{\psi}(t) \equiv \psi(x=0,t) \ . \tag{37}$$

The boundary conditions in Eqs. (4) - (5) give $v(x=0, t) = -j(t)/n_o e$, and upon application to Eq. (31)

$$A(\tilde{\psi})\cos\Omega t - B(\tilde{\psi})\sin\Omega t = \Omega^{-1}e^{\nu t/2}\{\tfrac{\nu}{2}[\tilde{\psi} - F(t)] - \frac{j(t)}{n_o e} - F'(t)\} \tag{38}$$

since

$$A(\tilde{\psi})\sin\Omega t + B(\tilde{\psi})\cos\Omega t = e^{\nu t/2}[\tilde{\psi} - F(t)] \tag{39}$$

by Eq. (24) for $x = 0$. The Eqs. (38) - (39) represent two independent relations from which one obtains by elimination:

$$A(\psi) = e^{\nu\tau/2}\ [\psi - F(\tau)]\sin\Omega\tau$$

$$+ \Omega^{-1}e^{\nu\tau/2}\{\tfrac{\nu}{2}[\psi - F(\tau)] - \frac{j(\tau)}{n_o e} - F'(\tau)\}\cos\Omega\tau \ , \tag{40}$$

and

$$B(\psi) = e^{\nu\tau/2}[\psi - F(\tau)]\cos\Omega\tau$$

$$-\Omega^{-1}e^{\nu\tau/2}\{\tfrac{\nu}{2}[\psi - F(\tau)] - \frac{j(\tau)}{n_o e} - F'(\tau)\}\sin\Omega\tau \tag{41}$$

where

$$\sigma(\psi - \psi_o) = f(\tau) \ , \tag{42}$$

since the structure of a function does not depend on the designation of the independent variable $(\tilde{\psi} \to \psi)$. In Eqs. (40) - (41), τ is a function of ψ defined by Eq. (42).

With the current density $j(t)$ given by Eq. (33), the associated functions $f(t)$ and $F(t)$ in Eqs. (23) and (27) become

$$f(t) = \sigma^{-1} H(t) t \ , \tag{43}$$

$$F(t) = \frac{1}{\sigma\Omega} H(t)\{\frac{\Omega}{\omega}(\omega t - \frac{\nu}{\omega}) + e^{-\nu t/2}[(1 - 2\frac{\Omega^2}{\omega^2})\sin\Omega t + \frac{\Omega\nu}{\omega^2}\cos\Omega t]\}$$

$$+ \nu V\omega^{-1}[1 - (\frac{\nu/2}{\Omega}\sin\Omega t + \cos\Omega t)e^{-\nu t/2}] \ . \tag{44}$$

Substitution of Eqs. (33) and (43) - (44) into Eqs. (40) and (41) yields:

$$A(\psi) = +\{\frac{\nu/2}{\Omega}\psi_o - \frac{H(\tau)}{\sigma\Omega}[1 - \frac{N}{n_o} - 2(\nu/2\omega)^2] - \frac{\nu^2 V}{2\omega^2\Omega}\}e^{\nu\tau/2}\cos\Omega\tau$$

$$+ [\psi_o + \frac{\nu}{\sigma\omega^2} H(\tau) - \frac{\nu V}{\omega^2}]e^{\nu\tau/2}\sin\Omega\tau$$

$$+ \frac{H(\tau)}{\sigma\Omega}[1 - 2(\nu/2\omega)^2] + \frac{\nu^2 V}{2\omega^2\Omega} \tag{45}$$

and

$$B(\psi) = -\{\frac{\nu/2}{\Omega}\psi_o - \frac{H(\tau)}{\sigma\Omega}[1 - \frac{N}{n_o} - 2(\nu/2\omega)^2] - \frac{\nu^2 V}{2\Omega^2\omega^2}\}e^{\nu\tau/2}\sin\Omega\tau$$

$$+ [\psi_o + \frac{\nu}{\sigma\omega^2} H(\tau) - \frac{\nu V}{\omega^2}]e^{\nu\tau/2}\cos\Omega\tau$$

$$- \frac{\nu}{\sigma\omega^2} H(\tau) + \frac{\nu V}{\omega^2} \tag{46}$$

where

$$\tau H(\tau) = \sigma(\psi - \psi_o) \geq 0 \tag{47}$$

by Eq. (42).

Upon substitution for $A(\psi)$, $B(\psi)$, and $f(t)$, $F(t)$, in accordance with Eqs. (45) - (46) and Eqs. (43) - (44), respectively, one obtains from Eqs. (30) - (32) the parametric solutions:

$$\frac{E(\psi,t)}{m\omega^2/\sigma\Omega e} = \sigma\Omega\psi + \sigma\Omega x(\psi,t) - \Omega t H(t) \;, \tag{48}$$

$$\begin{aligned}\frac{v(\psi,t)}{1/\sigma} &= H(t) - (1 - \frac{N}{n_o})H(\tau)e^{-\nu(t-\tau)/2}\cos\Omega(t-\tau) \\ &\quad - \{ \frac{\nu}{2\Omega}[(1 + \frac{N}{n_o})H(\tau) - 2\sigma V] + (\frac{\dot{\omega}}{\Omega})^2\sigma\Omega\psi_o\}e^{-\nu(t-\tau)/2}\sin\Omega(t-\tau) \;,\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{n(\psi,t)}{N} &= H(\tau)[H(\tau) - e^{-\nu(t-\tau)/2}\{(1 - \frac{N}{n_o})H(\tau)\cos\Omega(t-\tau) \\ &\quad + [(\nu/2\Omega)(1 + \frac{N}{n_o})H(\tau) - (\nu/\Omega)\sigma V + (\omega/\Omega)^2\sigma\Omega\psi_o]\sin\Omega(t-\tau)\} \\ &\quad - \frac{N}{n_o}\frac{\delta(\tau)}{\Omega}\, e^{-\nu t/2}\sin\Omega t]^{-1} \;,\end{aligned} \tag{50}$$

where:

$$\begin{aligned}\sigma\Omega x(\psi,t) &= \Omega t H(t) - \sigma\Omega\psi - (\nu\Omega/\omega^2)[H(t) - \sigma V] \\ &\quad - \{[1 - \frac{N}{n_o} - 2(\nu/2\omega)^2]H(\tau) + 2(\nu/2\omega)^2\sigma V - (\nu/2\Omega)\sigma\Omega\psi_o\}e^{-\nu(t-\tau)/2}\sin\Omega(t-\tau) \\ &\quad + \{(\nu\Omega/\omega)^2[H(\tau) - \sigma V] + \sigma\Omega\psi_o\}e^{-\nu(t-\tau)/2}\cos\Omega(t-\tau)\end{aligned} \tag{51}$$

by Eq. (24). In Eqs. (48) - (51), $\tau = (\psi)$ is given by Eq. (47). Eq. (51) gives $\psi = \psi(x,t)$ for every point (x,t) of the wave. The boundary condition $\psi(x=0,t)$ in Eq. (37) is satisfied by Eq. (51) since $t \equiv \tau$ for $x = 0$.

Let nondimensional variables and fields be introduced by the substitutions:

$$x/x_o \to x\ ,\quad t/t_o \to t\ ,\quad \tau/t_o \to \tau\ ,\quad \psi/x_o \to \psi\ ,$$
$$E/E_o \to E\ ,\quad v/V_o \to v\ ,\quad V/V_o \to V\ ,\quad n/N_o \to n\ , \tag{52}$$

where

$$x_o \equiv 1/\sigma\Omega\ ,\quad t_o \equiv 1/\Omega\ ,\quad E_o \equiv m\omega^2/\sigma\Omega e\ ,\quad V_o \equiv 1/\sigma\ ,\quad N_o \equiv N\ . \tag{53}$$

Further, if surface charges are absent $(\psi_o = 0)$, the Eqs. (49) - (51) become, for $t \geq 0$ and $\tau \geq 0$:

$$E(\psi,t) = \psi + x(\psi,t) - t\ , \tag{54}$$

$$v(\psi,t) = 1 - (1-\alpha)e^{-\beta(1-\beta^2)^{-\frac{1}{2}}(t-\psi)}\cos(t-\psi)$$
$$-\beta(1-\beta^2)^{-\frac{1}{2}}(1 + \alpha - 2V)e^{-\beta(1-\beta^2)^{-\frac{1}{2}}(t-\psi)}\sin(t-\psi)\ , \tag{55}$$

$$n(\psi,t) = \{1 - e^{-\beta(1-\beta^2)^{-\frac{1}{2}}(t-\psi)}[(1-\alpha)\cos(t-\psi)$$
$$+ \beta(1-\beta^2)^{-\frac{1}{2}}(1 + \alpha - 2V)\sin(t-\psi)] - \alpha\delta(\psi)e^{-\beta(1-\beta^2)^{-\frac{1}{2}}t}\sin t\}^{-1}\ , \tag{56}$$

where

$$x(\psi,t) = (t-\psi) - 2\beta(1-\beta^2)^{\frac{1}{2}}(1-V)$$
$$- [1 - \alpha - 2\beta^2(1-V)]e^{-\beta(1-\beta^2)^{-\frac{1}{2}}(t-\psi)}\sin(t-\psi)$$
$$+ 2\beta(1-\beta^2)^{\frac{1}{2}}(1-V)e^{-\beta(1-\beta^2)^{-\frac{1}{2}}(t-\psi)}\cos(t-\psi)\ , \tag{57}$$

and

$$\psi = \tau\ ,\quad \psi(x,t) \geq 0 \quad (\tau \geq 0)\ , \tag{58}$$

by Eq. (47). The dimensionless constants are defined as

$$\alpha \equiv N/n_o \ , \qquad \beta \equiv \nu/2\omega \tag{59}$$

Eq. (57) gives $\psi = \psi(x,t)$, where $\psi(x,t) \geq 0$ by Eq. (58), for every point x of that region $x \geq 0$ which is occupied by electrons at time t . The limit, $\psi(x,t) = 0$, defines a function $\hat{x} = \hat{x}(t)$, which represents the moving position coordinate of the front of the electron gas. By Eqs. (54) - (56), the fields $E(\psi,t)$, $v(\psi,t)$, and $n(\psi,t)$ depend exclusively on $(t-\psi)$, i.e. on x by Eq. (57). The spatially periodic field configuration, which grows with the speed $d\hat{x}(t)/dt$ into the space $x \geq 0$, represents a (nonlinear) "neutralization wave".

The Eqs. (54) - (58) are based on a nonlinear transformation which exists if $J = \partial x(\psi,t)/\partial\psi \neq 0$ or $n(\psi,t) < \infty$ in Eq. (56), i.e. if

$$\begin{aligned} &0 < \alpha < 2 + \varepsilon \ , \quad \text{or:} \ \frac{1}{2+\varepsilon} < n_o/N < \infty \ , \\ &\varepsilon \equiv \exp[\beta(1-\beta^2)^{-\frac{1}{2}}\pi] - 1 \ll 1 \ , \end{aligned} \tag{60}$$

since the extrema of $n(\psi,t)^{-1}$ occur when $\sin(t-\tau) = 0$. The position $\hat{x}(t)$ of the neutralization front is obtained from Eq. (57) as the limit $\hat{x}(t) = x(\Psi \to 0, t)$,

$$\begin{aligned} \hat{x}(t) = {} & t - 2\beta(1-\beta^2)^{\frac{1}{2}}(1-V) \\ & - e^{-\beta(1-\beta^2)^{-\frac{1}{2}}t}\{[1 - \alpha - 2\beta^2(1-V)]\sin t \\ & \qquad - 2\beta(1-\beta^2)^{\frac{1}{2}}(1-V)\cos t\} \geq 0, \ t \geq 0 \ , \end{aligned} \tag{61}$$

by Eq. (60). It follows for the speed of the neutralization front,

$$\frac{d\hat{x}(t)}{dt} = 1 - e^{-\beta(1-\beta^2)^{-\frac{1}{2}}t}[(1-\alpha)\cos t + \beta(1-\beta^2)^{-\frac{1}{2}}(1 + \alpha - 2V)\sin t]$$

$$= v(\psi,t)_{\psi=0} \quad . \tag{62}$$

In the region ahead of the neutralization front, $x > \hat{x}(t)$, it is $\tau < 0$ and $\psi = 0$ $(\psi_o = 0)$ or $\psi = \text{constant}$ $(\psi_o \neq 0)$. Accordingly,

$$n(x,t) = 0 \ , \quad v(x,t) = 0 \ , \quad x > \hat{x}(t) \ ; \tag{63}$$

and

$$E(x,t) = E[\hat{x}(t),t] + x - \hat{x}(t) = x - t \ , \quad x \geq \hat{x}(t) \ , \tag{64}$$

by Eq. (54), is the space charge field in the unneutralized ion space.

In Figs. 2, 3, and 4, the nondimensional fields $n(x,t)$, $E(x,t)$, and $v(x,t)$ [Eqs. (54) - (56)] of the neutralization wave are shown versus $0 \leq x \leq \hat{x}(t)$ at time $t = 10\pi$, with $\alpha = 0.1, 1.1$ and $V = 10^2\alpha$ as parameters, and $\beta = (m/M)^{1/3}/4\pi = 1.107 \times 10^{-3}$ for mercury ions. Accordingly, it is $\hat{x}(t = 10\pi) = 10\pi - 2\beta(1-\beta^2)^{\frac{1}{2}} \times (1-V)\{1-\exp[-\beta(1-\beta^2)^{\frac{1}{2}}10\pi]\} \cong 10\pi$. For other times $t = \hat{t}$, the fields of the neutralization wave exist only up to the point $\hat{x} = \hat{x}(\hat{t})$, e.g., the broken lines in Figs. 2-4 represent the neutralization front at time $t = 2\pi$. The position coordinate $\hat{x}(t)$ of the neutralization front is given in dependence of t with $\alpha = 0.1, 1.1$, and $V/\alpha = 10^2$ as parameters ($\beta = 1.107 \times 10^{-3}$) in Fig. 5.

The momentum transfer between the electron gas and the ion beam [β-terms in Eqs. (54) - (56)] results in i) an amplitude asymmetry which is clearly visible in the case α = 1.1 of Figs. 2-4, and ii) damping of the standing wave amplitudes as shown in Figs. 2-4. According to Eqs. (54) - (57), the wave fields exhibit for large x the symptotic behavior:

$$n(x,t) \to 1, \qquad\qquad v(x,t) \to 1 \quad ,$$
$$E(x,t) \to -2\beta(1-\beta^2)^{1/2}(1-V) \quad , \quad \text{for} \qquad x \gg x_c \quad , \qquad (65)$$

where

$$x_c = (1-\beta^2)^{1/2}/\beta \quad . \qquad (66)$$

is the nondimensional length of relaxation. Accordingly, the inter-component momentum exchange leads to a dissipation of the standing wave structure a few relaxation lengths x_c downstream of the injection plane (x = 0). This result means that a complete neutralization is achieved after a length $\Delta x \simeq 3x_c$ for injection ratios $\alpha^{-1} = n_o/N \neq 1$. The dimensional length of relaxation is by Eq. (53)

$$x_c = (1-\beta^2)^{1/2}x_o/\beta = (2/\nu)|i_o|/Ne \quad . \qquad (67)$$

For neutralization experiments, it is noted that the relaxation length x_c decreases with increasing ion density N and decreasing electron current i_o. Note that the neutralization remains incomplete in the region $0 \leq x \lesssim x_c$ at all times if $\alpha = N/n_o \neq 1$.

In Figs. 6 and 7, the spatial attenuation of the nondimensional amplitude of the electron density field n(x,t) is shown at times $t \gtrsim 10^3\pi$ for α = 0.1, 1.1, $V = 10^2\alpha$, and $\beta = 1.107 \times 10^{-3}$. [In the special case, $\alpha = N/n_o = 1$, practically a relaxation free neutralization results within the wave, since $n(x,t) \cong 1$ by Eq. (56) for

$0 \leq x < x(t)$ if $\beta V << 1$.] It is seen that the ion gas is essentially neutralized at the point $x = 10^3\pi$ where $n \cong 1$.

The reference values x_o, t_o, E_o, V_o, and N_o defined in Eq. (53) determine the characteristic scales and actual magnitude of the fields of the neutralization wave. As a numerical example, consider a mercury ion beam and an electron injection current for which:

$$N = 10^9 \text{ cm}^{-3}, \; i_o \triangleq 10^{-3} \text{ amps cm}^{-2}, \; \beta = 1.107 \times 10^{-3} \quad .$$

Hence:

$$x_o = 3.691 \times 10^4 (1-\beta^2)^{-1/2} |i_o| N^{-3/2} = 3.502 \times 10^{-3} \text{ cm} \quad ,$$

$$t_o = 1.773 \times 10^{-5} (1-\beta^2)^{-1/2} N^{-1/2} = 5.607 \times 10^{-10} \text{ sec} \quad ,$$

$$E_o = 2.227 \times 10^{-4} (1-\beta^2)^{1/2} |i_o| N^{-1/2} = 2.113 \times 10^{-2} \text{cgsu} \triangleq 6.339 \times 10^0 \text{ volt cm}^{-1} ,$$

$$V_o = 2.082 \times 10^9 |i_o| N^{-1} = 6.246 \times 10^6 \text{ cm sec}^{-1} \quad ,$$

$$N_o = N = 10^9 \text{ cm}^{-3} \quad ,$$

and

$$x_c = 3.691 \times 10^4 \beta^{-1} |i_o| N^{-3/2} = 3.163 \times 10^0 \text{ cm} \quad .$$

Citations

1. E. Stuhlinger, Ion Propulsion for Space Flight, McGraw-Hill, New York 1964.

2. G. F. Au, Electric Propulsion of Space Vehicles, Verlag G. Braun, Karlsruhe 1968.

3. H. R. Kaufman, An Ion Rocket with an Electron-Bombardment Ion Source, NASA TN D-585-1961.

4. N. R. Kerslake, D. C. Byers, and J. F. Staggs, J. Spacecraft and Rockets 1, 4 (1970).

5. H. R. Kaufman, One-Dimensional Analysis of Ion Rockets, NASA TN D-261 (1960).

6. L. D. Pearlstein, M. N. Rosenbluth, and G. W. Stuart, The Neutralization of Ion Beams, Progress in Astronautics and Aeronautics 9, 319, Academic Press, New York 1963.

7. A. A. Vlasov, Many Particle Theory and its Applications to Plasma, Gordon & Breach, New York 1961.

8. O. Buneman, Phys. Rev. 115, 503 (1959).

9. R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York 1948.

10. G. Kalman, Ann. Phys. 10, 1 (1960).

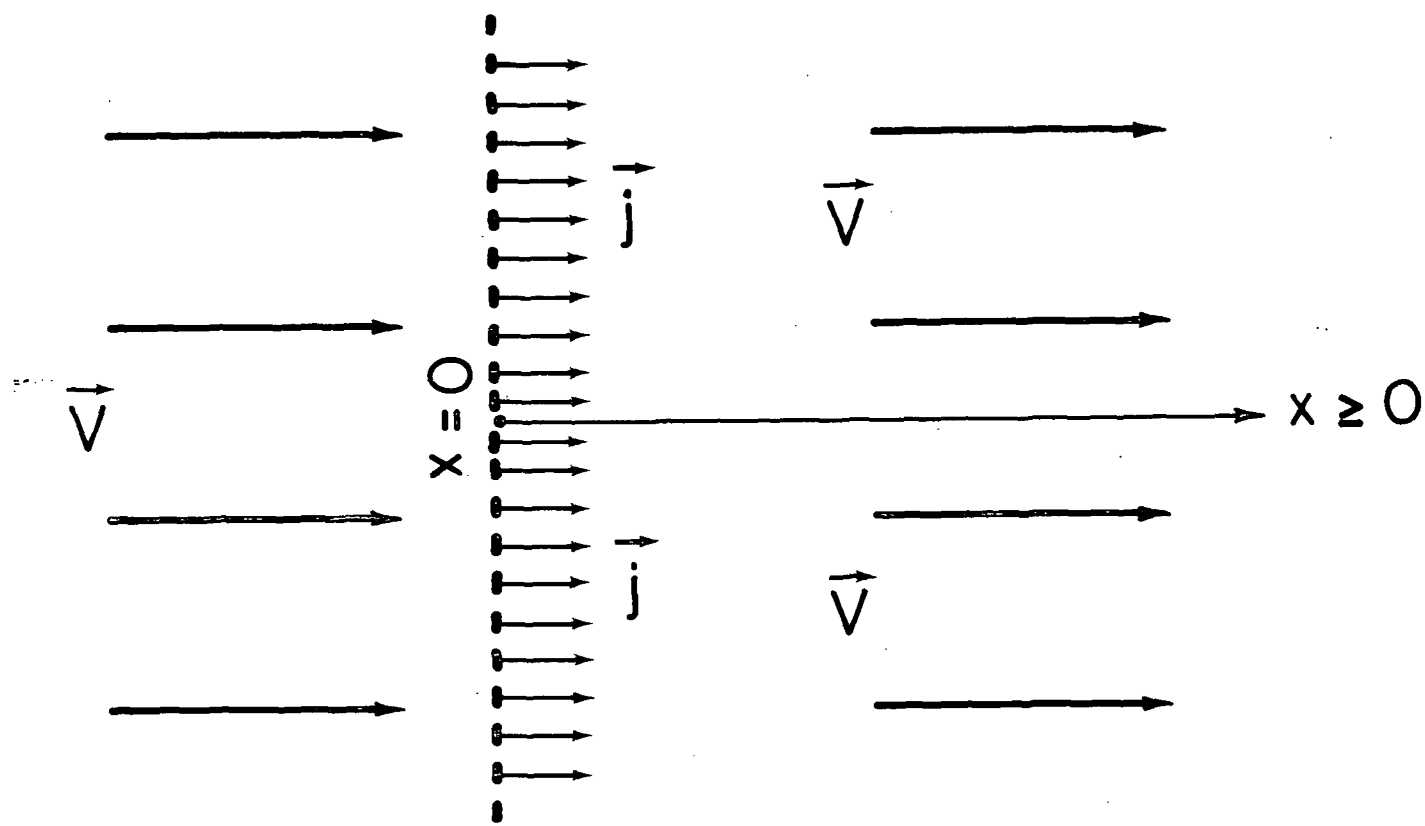

FIG. 1: Schematic representation of infinite ion beam ($\vec{V}$) with electron current injection ($\vec{j}$) from grid plane at $x = 0$.

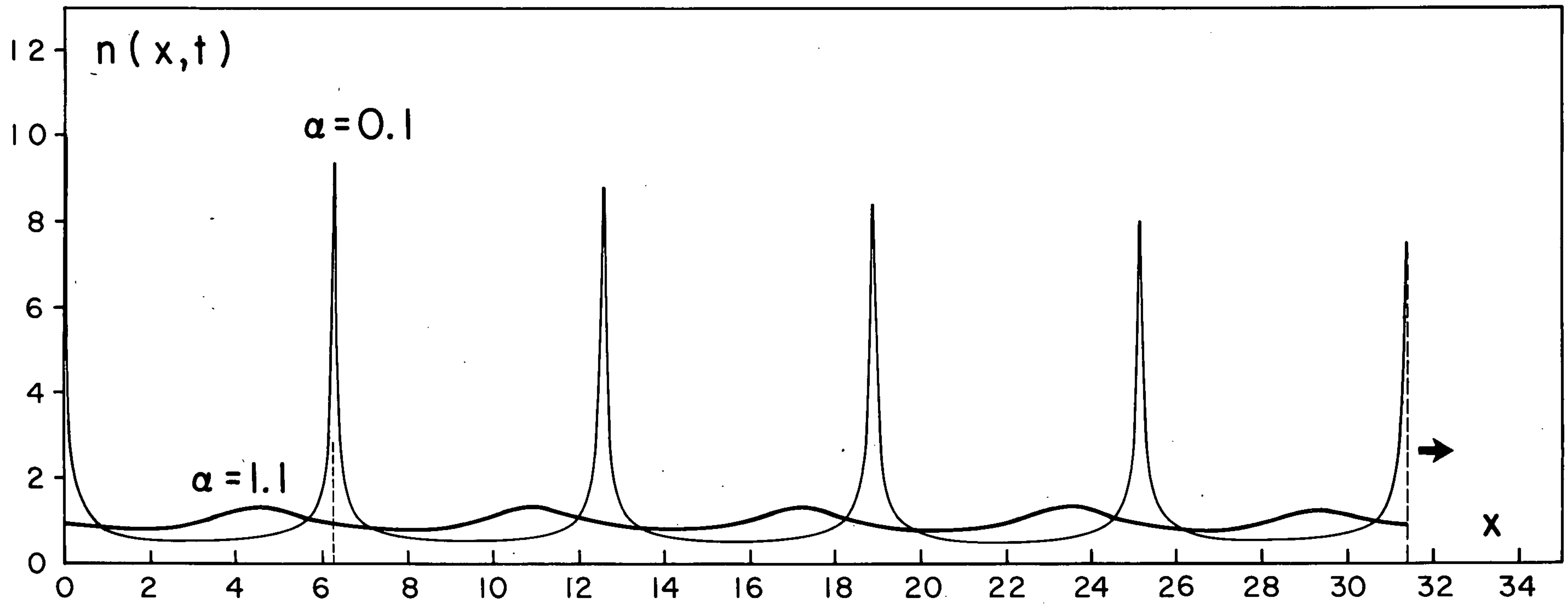

FIG. 2: Electron density wave field $n(x,t)$ versus x at time $t = 10\pi$ for $\alpha = N/n_o = 0.1,\ 1.1$ and $V = 10^2\alpha$ showing spatial attenuation.

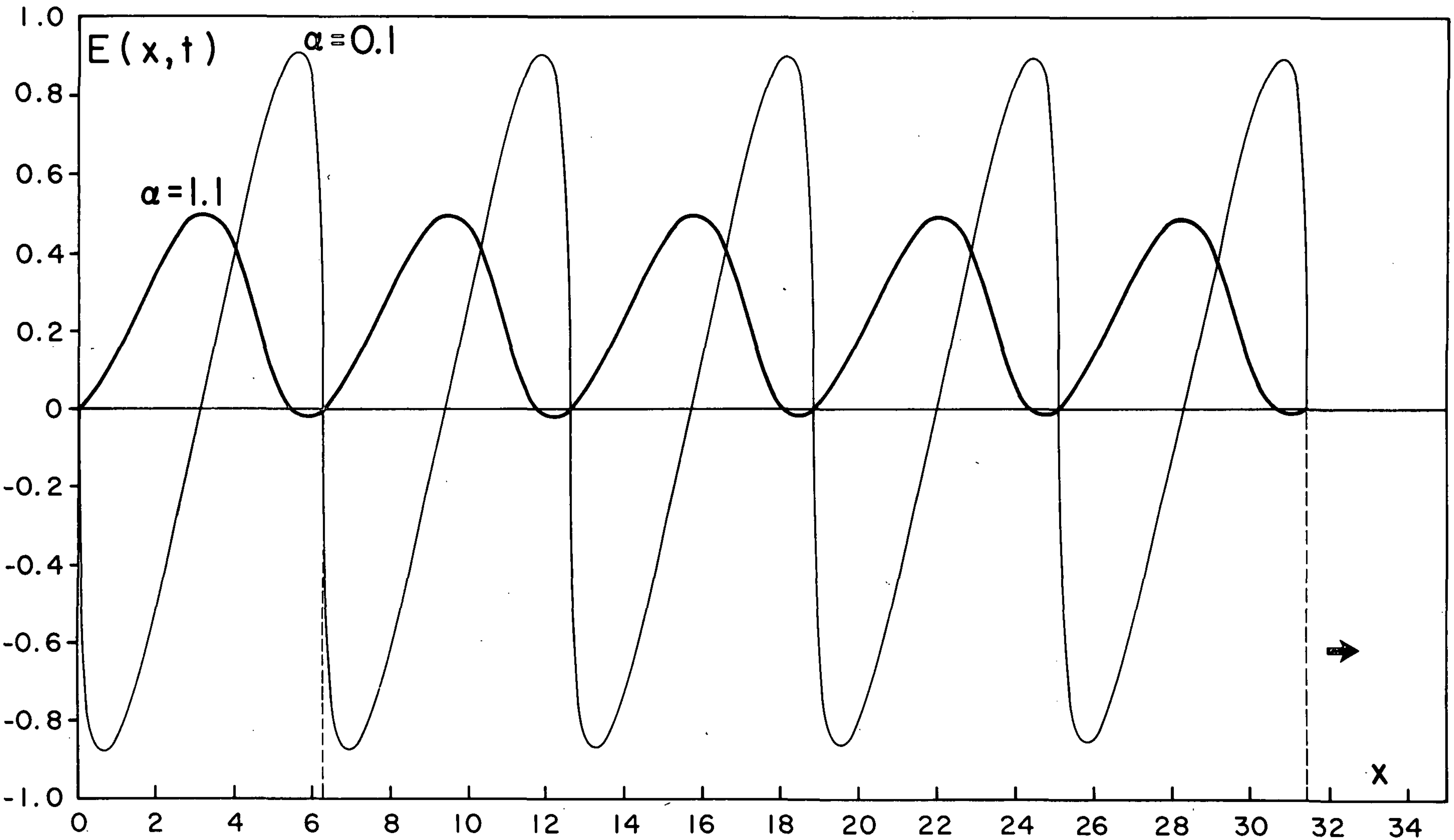

FIG. 3: Electric wave field $E(x,t)$ versus $0 \leq x \leq \hat{x}(t)$ at time $t = 10\pi$ for $\alpha = N/n_o = 0.1$, 1.1 and $V = 10^2\alpha$ showing spatial attenuation.

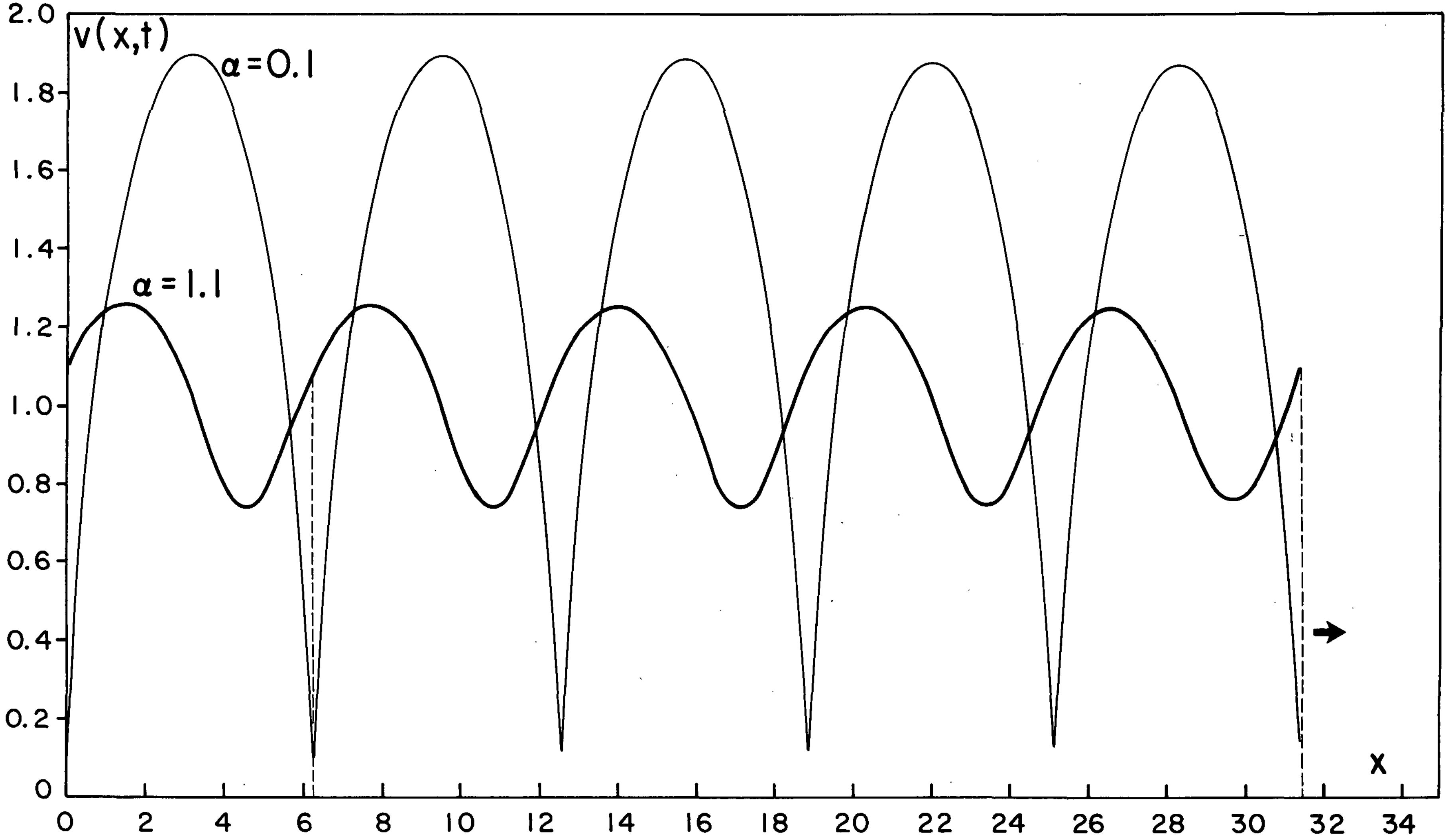

FIG. 4: Electron velocity wave field $v(x,t)$ versus x at time $t = 10\pi$ for $\alpha = N/n_o = 0.1$, 1.1 and $V = 10^2\alpha$ showing spatial attenuation.

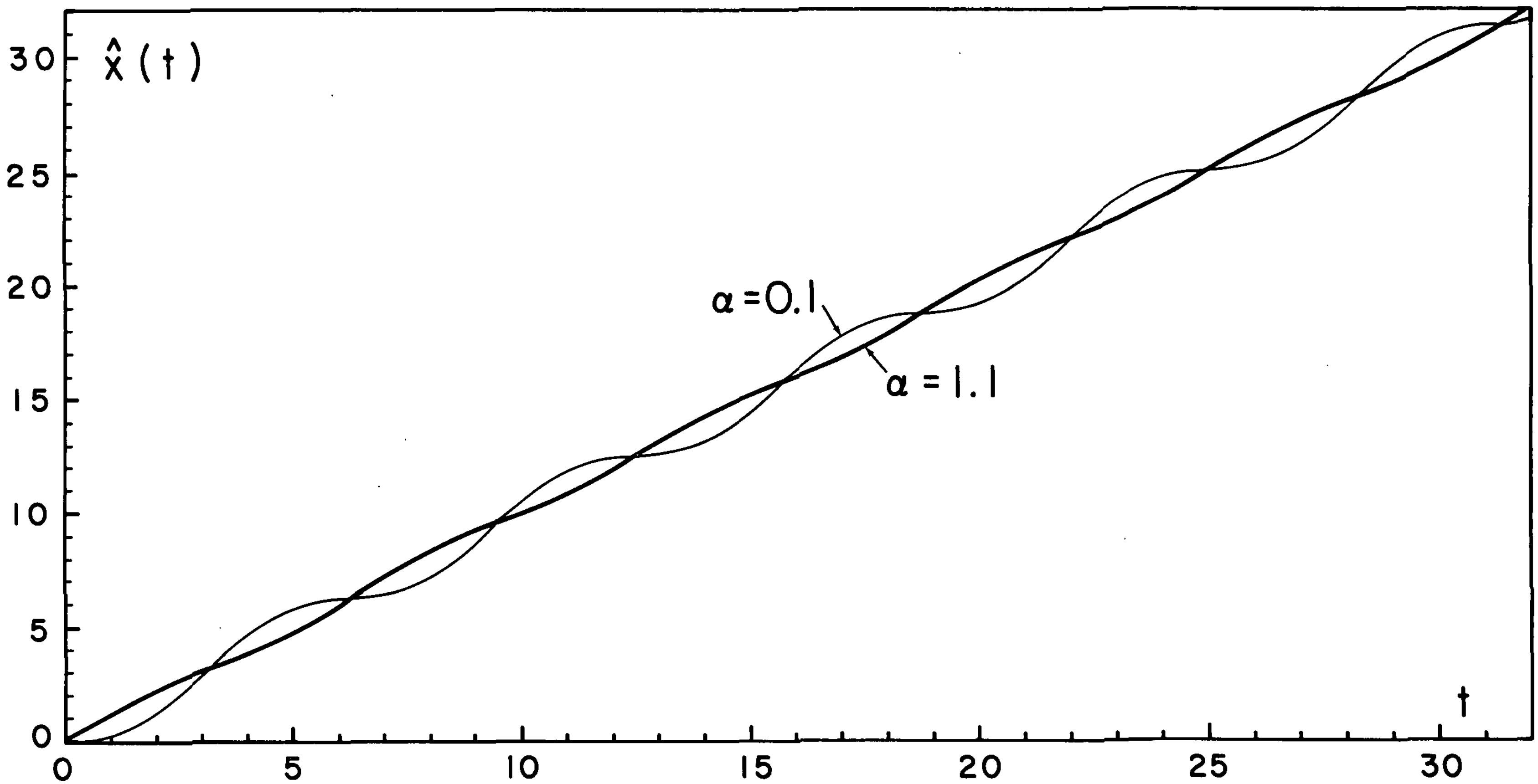

FIG. 5: Position coordinate $\hat{x}(t)$ of the neutralization front versus t for $\alpha = N/n_o = 0.1,\ 1.1$ and $V = 10^2\alpha$.

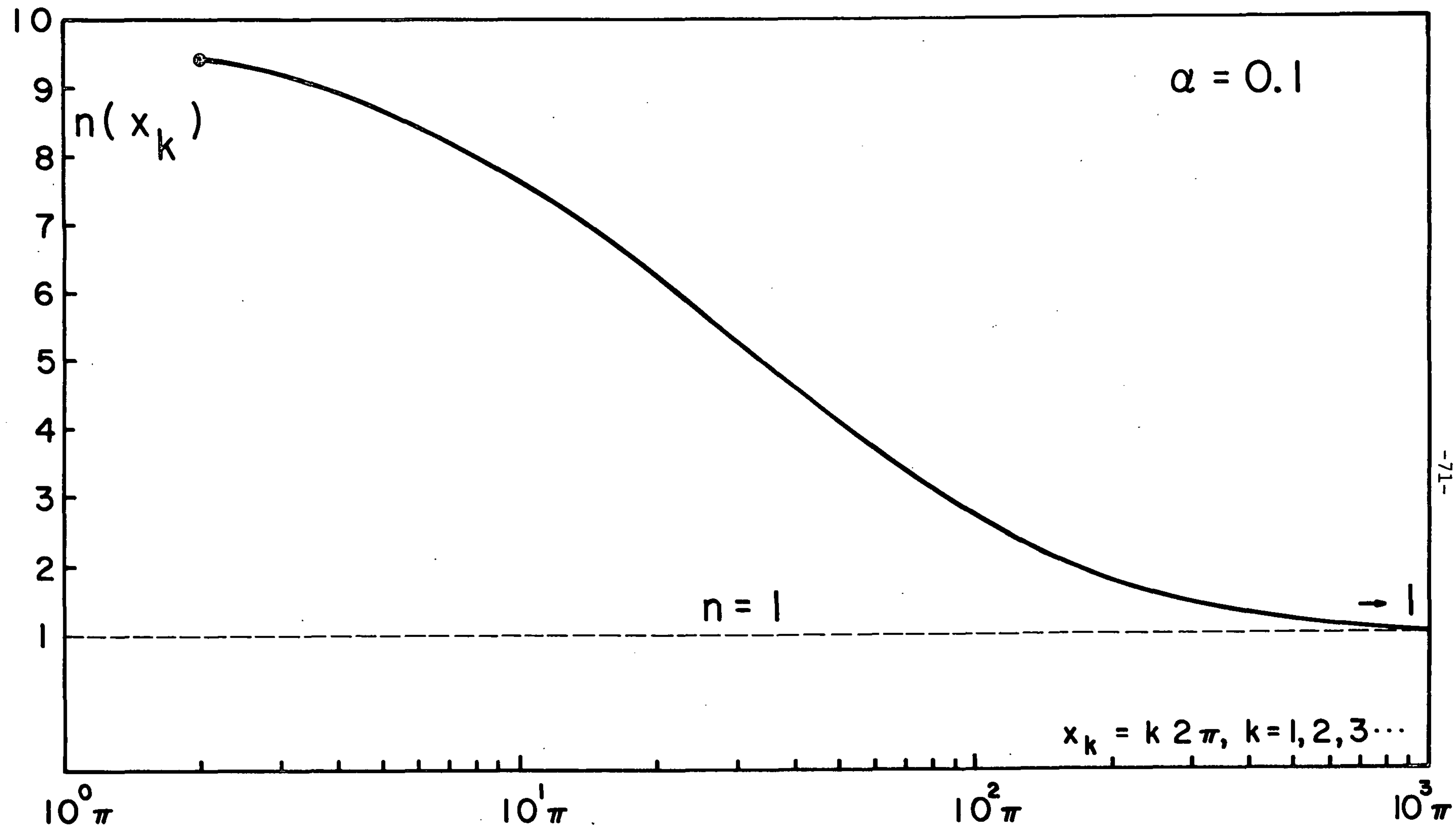

FIG. 6: Spatial attenuation curve showing amplitude $n(x_k,t)$ of electron density wave at $x_k = 2\pi k$, $k = 1, 2, 3, \ldots$, and $t \gtrsim 10^3\pi$ for $\alpha = N/n_o = 0.1$ and $V = 10^2\alpha$ (neutralization: $n \to 1$).

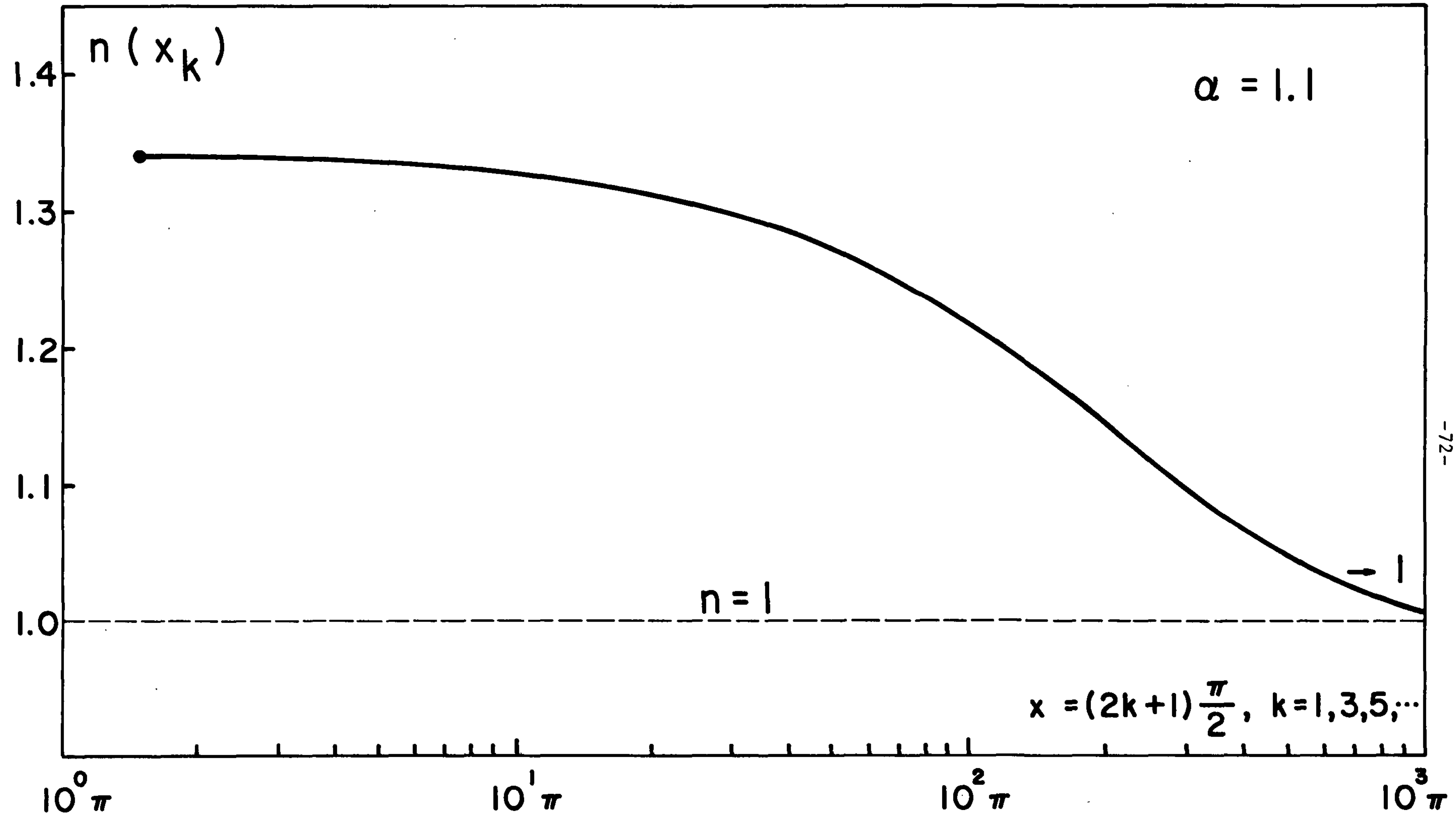

FIG. 7: Spatial attenuation curve showing amplitude $n(x_k, t)$ of electron density wave at $x_k = \frac{\pi}{2}(2k+1)$, $k = 1, 3, 5, \ldots$, and $t \gtrsim 10^3\pi$ for $\alpha = N/n_o = 1.1$ and $V = 10^2\alpha$ (neutralization: $n \rightarrow 1$).

V. INTERCOMPONENT MOMENTUM TRANSPORT AND ELECTRICAL CONDUCTIVITY OF COLLISIONLESS PLASMA

ABSTRACT

Based on the Lenard-Balescu equation, the interaction integral for the intercomponent momentum transfer in a two-component, collisionless plasma is evaluated in closed form. The distribution functions of the electrons and ions are represented in form of nonisothermal, displaced Maxwellians corresponding to the 5-moment-approximation. As an application, the transport of electrical current in an electric field is discussed for infrasonic up to sonic electron-ion drift velocities.

In a fully ionized plasma, the interaction radius of an electron or ion is of the order of magnitude of the Debye radius, D . In many plasmas of technical interest, such as glow discharges and high temperature, fully ionized gases, the Debye radius is large compared to the mean particle distance, $D \gg n^{-1/3}$, where n is the electron or ion density. In this case, a charged particle interacts simultaneously with many particles of the plasma, since the sphere of interaction contains a large number N of electrons and ions,

$$N \sim nD^3 \gg 1 \quad .$$

A kinetic equation which takes into consideration the many-particle interactions in collisionless plasmas is that of Lenard[1] and Balescu[2]. The formal mathematical deductions[1-2] of this kinetic equation apparently give no meaningful explanations of its physical foundations and its applicability to nonequilibrium situations in general. In the following, an application of the LB-equation is made which has the purpose of determining the linear momentum exchange between the electron and ion components and the associated electrical conductivity of a collisionless plasma. Since the LB-integro-differential equation can not be integrated in closed form because of its mathematical complexity, a simple plasma model is chosen by means of which intercomponent nonequilibrium effects can be studied as known from similar investigations based on the Boltzmann equation[3-4]. The electron and ion components are assumed to be in an approximate thermal equilibrium at their respective temperature ($T_r \neq T_s$) in their individual center of mass system ($\langle\vec{v}_r\rangle \neq \langle\vec{v}_s\rangle$). The drift velocity between these components can be subsonic or supersonic,

$$\frac{1}{2}\, m_{rs}(\langle\vec{v}_r\rangle - \langle\vec{v}_s\rangle)^2 \lesseqgtr KT_{rs}/m_{rs}$$

where m_{rs} and T_{rs} are the reduced mass and temperature. This idealized theory should give information on the dependence of the intercomponent momentum transfer on the thermal nonequilibrium, $T_r - T_s \neq 0$, and on the intercomponent drift velocity, $\langle\vec{g}_{rs}\rangle = \langle\vec{v}_r\rangle - \langle\vec{v}_s\rangle$. At proper supersonic drift velocities, the collective interactions will become turbulent in the plasma (two-stream instability). In this drift velocity region, the LB-equation is expected to break down or to give at best an approximate result with logarithmic accuracy.

THEORETICAL PRINCIPLES

The many particle collisions in a plasma can be regarded essentially as binary collisions complicated by the effects of the surrounding many-particle medium which becomes polarized by the respective interacting particle. If the time-dependent polarization of the plasma produced by this "test" particle is Fourier-analyzed, it is seen that the many particle collisions are equivalent to an interaction of the test particle with electrostatic waves (non-relativistic). Evidently, these interactions occur at wave numbers k or impact parameters $\rho (\rho \sim k^{-1})$ for which[5]

$$kD \gg 1 \quad , \quad \rho \ll D \quad ,$$

where D is the Debye radius. The change of the distribution function $f_r = f_r(v_r)$ of particles of type "r" as a result of many-body interactions with particles of type "s" is of the form

$$\frac{\partial f_{r/s}}{\partial t} = - \frac{\partial}{\partial \vec{v}_r} \cdot \vec{j}_{r/s} \quad . \tag{1}$$

$\vec{j}_{r/s}$ is the flux of r-particles in velocity space which can be derived intuitively from the concept of a diffusion in velocity space[6],

$$\vec{j}_{r/s} = (2e_r^2 e_s^2/m_r) \iiint_{-\infty}^{+\infty} U \cdot \left(\frac{f_r}{m_s} \frac{\partial f_s}{\partial \vec{v}_s} - \frac{f_s}{m_r} \frac{\partial f_r}{\partial \vec{v}_r}\right) d\vec{v}_s \quad , \quad \vec{j}_{s/s} \equiv \vec{0} \tag{2}$$

where

$$\overleftrightarrow{U} = \iiint_{k_{min}}^{k_{max}} \frac{\vec{k}\vec{k}\,\delta(\vec{k}\cdot\vec{v}_r - \vec{k}\cdot\vec{v}_s)}{k^4 \left|\varepsilon(\vec{k},\omega)\right|^2_{\omega=\vec{k}\cdot\vec{v}_r}} d\vec{k} \quad . \tag{3}$$

$\varepsilon(\vec{k},\omega)$ is the longitudinal dielectric constant $(|\vec{v}_r| << c)$ of the plasma[5],

$$\varepsilon(\vec{k},\omega) = 1 + k^{-2} \sum_{r,s} \frac{4\pi e_\beta^2}{m_\beta \omega} \underset{\smile}{\iiint} \frac{\vec{k}\cdot\vec{v}_\beta}{\omega - \vec{k}\cdot\vec{v}_\beta} (\vec{k}\cdot\frac{\partial f_\beta}{\partial \vec{v}_\beta}) d\vec{v}_\beta \quad . \tag{4}$$

Equation (1), together with Eqs. (2)-(4), represents the LB-equation[1-2] for a fully ionized plasma consisting of an electron (r) and ion (s) component. [The sign $\smile$ in Eq. (4) indicates that in the integration the pole $\omega - \vec{k}\cdot\vec{v}_\beta = 0$ in the complex $\vec{v}_\beta$-plane is to be traversed from below.]

The order of magnitude of the dielectric constant is estimated by evaluating the integral in Eq. (4) formally for a Maxwellian distribution of β-particles which gives

$$\varepsilon(\vec{k},\omega = \vec{k}\cdot\vec{v}_r) = 1 + \sum_{r,s} \frac{1}{k^2 D_\beta^2} \left\langle \frac{\vec{k}\cdot\vec{v}_\beta}{\vec{k}\cdot\vec{v}_r - \vec{k}\cdot\vec{v}_\beta} \right\rangle \tag{5}$$

where $D_\beta^{-2} = 4\pi n_\beta^2 e_\beta^2 / KT_\beta$. Since the average expression $\langle\beta\rangle$ is of the order one for the equilibrium distribution and of the same magnitude for nonequilibrium distributions of β-particles, the dielectric constant can be replaced by unity except in the case where f_β is unstable[5],

$$\varepsilon(\vec{k},\omega = \vec{k}\cdot\vec{v}_r) \cong 1 \ , \quad k^2 D_\beta^2 \sim k^2 D^2 >> 1 \quad . \tag{6}$$

The integration of Eq. (13) produces the (many-particle) interaction logarithm

$$\Lambda_{rs} = \int_{k_{min}}^{k_{max}} \frac{dk}{k} = \ln(\frac{k_{max}}{k_{min}}) = \ln(DKT_{rs}/|e_r e_s|) \quad , \tag{7}$$

since

$$k_{min} \cong D^{-1} \ , \quad k_{max} = KT_{rs}/|e_r e_s| \quad , \tag{8}$$

where

$$D \equiv (4\pi \sum_{r,s} n_\beta^2 e_\beta^2/KT_\beta)^{-\frac{1}{2}} \quad . \tag{9}$$

In Eq. (8), k_{min} corresponds to the maximum impact parameter $\rho_\infty = D$, i.e., k_{min} represents an extrapolation of the inequality in Eq. (6) which is permitted in view of the weak, logarithmic k-dependence of Λ_{rs} , Eq. (7).

By combining Eqs. (3) and (6), an approximation to the tensor $\overleftrightarrow{U}$ is obtained which will be used exclusively further on,

$$\overleftrightarrow{U} \simeq \iiint_{k_{min}}^{k_{max}} \frac{\vec{k}\cdot\vec{k}}{k^4} \, \delta(\vec{k}\cdot\vec{v}_r - \vec{k}\cdot\vec{v}_s) d\vec{k} \quad . \tag{10}$$

The distribution functions of the charged particle species r and s are assumed to be displaced, nonisothermal Maxwellians (5 - moment approximation)[3-4],

$$f_r(\vec{v}_r) = n_r \left(\frac{a_r}{\pi}\right)^{3/2} e^{-a_r(\vec{v}_r - \langle\vec{v}_r\rangle)^2} \quad , \quad f_s = f_{r=s} \quad . \tag{11}$$

In addition, let some frequently used abbreviations be defined in advance[3-4],

$$m_{rs} = m_r m_s/(m_r + m_s) \quad , \quad T_{rs} = m_{rs}\left(\frac{T_r}{m_r} + \frac{T_s}{m_s}\right) \quad ,$$

$$a_r = m_r/2KT_r \quad , \quad a_s = a_{r=s} \quad , \quad a_{rs} = a_r a_s/(a_r + a_s) \quad . \tag{12}$$

In connection with the analytical developments, the so-called error function is encountered which is defined by

$$\Phi(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-t^2} dt \tag{13}$$

where

$$\Phi(x) = 1 - 2\pi^{-\frac{1}{2}} x e^{-x^2} \sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)!!}{2^{m+1} x^{2(m+1)}} , \quad x > 1 , \tag{14}$$

$$= 2\pi^{-\frac{1}{2}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)m!} , \quad x \lesssim 1 . \tag{15}$$

INTERCOMPONENT MOMENTUM TRANSPORT

The force density exerted by the s-component on the r-component through particle-wave interactions is according to Eqs. (1) and (2)

$$\vec{F}_{r/s} = -2e_r^2 e_s^2 \iiint_{-\infty}^{+\infty} \vec{v}_r \left[\frac{\partial}{\partial \vec{v}_r} \cdot \iiint_{-\infty}^{+\infty} \overleftrightarrow{U} \cdot \left(\frac{f_r}{m_s}\frac{\partial f_s}{\partial \vec{v}_s} - \frac{f_s}{m_r}\frac{\partial f_r}{\partial \vec{v}_r}\right) d\vec{v}_s\right] d\vec{v}_r \qquad (16)$$

The 5-moment-approximation to the distribution functions in Eq. (11) give

$$\frac{f_r}{m_s}\frac{\partial f_s}{\partial \vec{v}_s} - \frac{f_s}{m_r}\frac{\partial f_r}{\partial \vec{v}_r} = 2n_r n_s \left(\frac{a_r a_s}{\pi^2}\right)^{3/2} \left[\frac{a_r}{m_r}(\vec{v}_r - \langle\vec{v}_r\rangle) - \frac{a_s}{m_s}(\vec{v}_s - \langle\vec{v}_s\rangle)\right]$$

$$\cdot \exp[-a_r(\vec{v}_r - \langle\vec{v}_r\rangle)^2 - a_s(\vec{v}_s - \langle\vec{v}_s\rangle)^2] \, . \qquad (17)$$

Let thermal velocities $\vec{c}_r$, $\vec{c}_s$, the relative velocity $\vec{g}_{rs}$, and the reduced thermal velocity $\vec{c}_{rs}$ be introduced by:

$$\vec{c}_r = \vec{v}_r - \langle\vec{v}_r\rangle \, , \quad \vec{c}_s = \vec{v}_s - \langle\vec{v}_s\rangle \; ; \qquad (18)$$

$$\vec{g}_{rs} = \vec{v}_r - \vec{v}_s \quad , \; \vec{g}_{sr} = -\vec{g}_{rs} \; ; \qquad (19)$$

$$\vec{c}_{rs} = \frac{a_{rs}}{a_s}\vec{c}_r + \frac{a_{rs}}{a_r}\vec{c}_s \, , \quad \vec{c}_{sr} = \vec{c}_{rs} \; . \qquad (20)$$

Hence,

$$\vec{c}_r = \vec{c}_{rs} + \frac{a_{rs}}{a_r}(\vec{g}_{rs} - \langle\vec{g}_{rs}\rangle) \; , \qquad (21)$$

$$\vec{c}_s = \vec{c}_{rs} - \frac{a_{rs}}{a_s}(\vec{g}_{rs} - \langle\vec{g}_{rs}\rangle) \; , \qquad (22)$$

and

$$\vec{c}_r - \vec{c}_s = \vec{g}_{rs} - \langle\vec{g}_{rs}\rangle \; . \qquad (23)$$

Since the Jacobians are $J_1 = |[\partial(\vec{v}_r,\vec{v}_s)/\partial(\vec{c}_r,\vec{c}_s)]| = 1$ and $J_2 = |[\partial(\vec{c}_r,\vec{c}_s)/\partial(\vec{c}_{rs},\vec{g}_{rs})]| = 1$, the velocity space elements in the various variables equal,

$$d\vec{v}_r d\vec{v}_s = d\vec{c}_r d\vec{c}_s = d\vec{c}_{rs}\, d\vec{g}_{rs} \quad . \tag{24}$$

By means of Eqs. (18)-(22), the bracket expressions in Eq. (17) are transformed as

$$\frac{a_r}{m_r}\vec{c}_r - \frac{a_s}{m_s}\vec{c}_s = (\frac{a_r}{m_r} - \frac{a_s}{m_s})\vec{c}_{rs} + \frac{a_{rs}}{m_{rs}}(\vec{g}_{rs} - \langle\vec{g}_{rs}\rangle) \tag{25}$$

and

$$a_r\vec{c}_r^{\,2} + a_s\vec{c}_s^{\,2} = (a_r + a_s)\vec{c}_{rs}^{\,2} + a_{rs}(\vec{g}_{rs} - \langle\vec{g}_{rs}\rangle)^2 \tag{26}$$

The relations $\vec{c}_{rs} = \vec{c}_{rs}(\vec{v}_r,\vec{v}_s)$ and $\vec{g}_{rs} = \vec{g}_{rs}(\vec{v}_r,\vec{v}_s)$ are given in Eqs. (20) and (19), respectively. Hence, the divergence operator becomes

$$\frac{\partial\cdot}{\partial\vec{v}_r} = \frac{a_{rs}}{a_s}\frac{\partial\cdot}{\partial\vec{c}_{rs}} + \frac{\partial\cdot}{\partial\vec{g}_{rs}} \quad . \tag{27}$$

The tensor $\overleftrightarrow{U}$ in Eq. (10) is evaluated by means of spherical coordinates $(1,\theta,\phi)$ with the polar axis parallel to $\vec{g}_{rs}(\vec{k}^o = \vec{k}/k)$,

$$\overleftrightarrow{U} = \frac{1}{g_{rs}}\int_{k_{min}}^{k_{max}}\frac{dk}{k}\int_0^{\pi}\int_0^{2\pi}\vec{k}^o\,\vec{k}^o\ \delta(\cos\theta)\sin\theta d\theta d\phi$$

$$= \pi(\Lambda_{rs}/g_{rs})\int_0^{\pi}[\overleftrightarrow{\delta}\sin^2\theta + (2\cos^2\theta - \sin^2\theta)\frac{\vec{g}_{rs}\vec{g}_{rs}}{g_{rs}^2}]\ \delta(\cos\theta)\sin\theta\ d\theta \ ,$$

i.e.,

$$\overleftrightarrow{U} = \pi(\Lambda_{rs}/g_{rs})(\overleftrightarrow{\delta} - \frac{\vec{g}_{rs}\cdot\vec{g}_{rs}}{g_{rs}^2}) \tag{28}$$

where Λ_{rs} is defined in Eq. (7), and $\delta_{ij} = 1$ or 0 for $i = j$ or $i \neq j$.

The integral over $\vec{v}_r$ in Eq. (16) reduces to one over $\vec{c}_r = \vec{v}_r - \langle\vec{v}_r\rangle$, since the term proportional to $\langle\vec{v}_r\rangle$ vanishes by Gauss's law. Since $\vec{v}_r$ and $\vec{v}_s$ are independent variables, the operator $\partial\cdot/\partial\vec{v}_r$ can be put under the $d\vec{v}_s$-integral. Thus, one obtains, under consideration of Eqs. (17) and (25)-(28),

$$\vec{F}_{r/s} = -4\pi e_r^2 e_s^2 (a_r a_s/\pi^2)^{3/2} n_r n_s \Lambda_{rs} \int_{-\infty}^{+\infty} \cdots \int (\vec{c}_{rs} + \frac{a_{rs}}{a_r}\vec{g}_{rs})[(\frac{a_{rs}}{a_s}\frac{\partial}{\partial\vec{c}_{rs}} + \frac{\partial}{\partial\vec{g}_{rs}})\cdot(A\vec{B})]d\vec{c}_{rs}d\vec{g}_{rs} \tag{29}$$

where

$$A \equiv \exp[-(a_r+a_s)\vec{c}_{rs}^{\,2} - a_{rs}(\vec{g}_{rs}-\langle\vec{g}_{rs}\rangle)^2] , \tag{30}$$

$$\vec{B} \equiv (\frac{a_r}{m_r} - \frac{a_s}{m_s})[\frac{\vec{c}_{rs}}{g_{rs}} - \frac{(\vec{g}_{rs}\cdot\vec{c}_{rs})}{g_{rs}^3}\vec{g}_{rs}] + \frac{a_{rs}}{m_{rs}}[\frac{(\vec{g}_{rs}\cdot\langle\vec{g}_{rs}\rangle)}{g_{rs}^3}\vec{g}_{rs} - \frac{\langle\vec{g}_{rs}\rangle}{g_{rs}}] . \tag{31}$$

Accordingly,

$$\vec{F}_{r/s} = -4\pi e_r^2 e_s^2 (a_r a_s/\pi^2)^{3/2} n_r n_s \Lambda_{rs} (\vec{I}_1+\vec{I}_{12}+\vec{I}_{21}+\vec{I}_2) \tag{32}$$

where

$$\vec{I}_1 \equiv \frac{a_{rs}}{a_s}\int_{-\infty}^{+\infty}\cdots\int c_{rs}\frac{\partial}{\partial\vec{c}_{rs}}\cdot(A\vec{B})\, d\vec{c}_{rs}\, d\vec{g}_{rs} , \tag{33}$$

$$\vec{I}_{12} \equiv \frac{a_{rs}^2}{a_r a_s}\iiint_{-\infty}^{+\infty}\vec{g}_{rs}[\iiint_{-\infty}^{+\infty}\frac{\partial}{\partial\vec{c}_{rs}}\cdot(A\vec{B})d\vec{c}_{rs}]d\vec{g}_{rs} = \vec{0} , \tag{34}$$

$$\vec{I}_{21} = \iiint_{-\infty}^{+\infty}\vec{c}_{rs}[\iiint_{-\infty}^{+\infty}\frac{\partial}{\partial\vec{g}_{rs}}\cdot(A\vec{B})d\vec{g}_{rs}]d\vec{c}_{rs} = \vec{0} , \tag{35}$$

$$\vec{I}_2 = \frac{a_{rs}}{a_r} \int_{-\infty}^{+\infty} \ldots \int \vec{g}_{rs} \frac{\partial}{\partial \vec{g}_{rs}} \cdot (A\vec{B}) d\vec{c}_{rs}\, d\vec{g}_{rs} \quad . \tag{36}$$

Now,

$$\frac{\partial}{\partial \vec{c}_{rs}} \cdot (A\vec{B}) = 2A[-(a_r+a_s)\vec{c}_{rs} \cdot \vec{B} + (\frac{a_r}{m_r} - \frac{a_s}{m_s})\frac{1}{g_{rs}}] \tag{37}$$

and

$$\frac{\partial}{\partial \vec{g}_{rs}} \cdot (A\vec{B}) = 2A\{-a_{rs}(\vec{g}_{rs} - <\vec{g}_{rs}>) \cdot \vec{B}$$

$$+ [\frac{a_{rs}}{m_{rs}} \frac{(\vec{g}_{rs} \cdot <g_{rs}>)}{g_{rs}^3} - (\frac{a_r}{m_r} - \frac{a_s}{m_s}) \frac{(\vec{g}_{rs} \cdot \vec{c}_{rs})}{g_{rs}^3}]\} \tag{38}$$

by Eqs. (30) and (31). Since A is an even function of $\vec{c}_{rs}$, it follows

$$\vec{I}_1 = -2a_r \frac{a_{rs}}{m_{rs}} \int_{-\infty}^{+\infty} \ldots \int A\vec{c}_{rs} [\frac{(\vec{g}_{rs} \cdot <\vec{g}_{rs}>)(\vec{g}_{rs} \cdot \vec{c}_{rs})}{g_{rs}^3}$$

$$- \frac{(\vec{c}_{rs} \cdot <\vec{g}_{rs}>)}{g_{rs}}] d\vec{c}_{rs}\, d\vec{g}_{rs} \quad . \tag{39}$$

and

$$\vec{I}_2 = 2 \frac{a_{rs}}{a_r} \frac{a_{rs}}{m_{rs}} \int_{-\infty}^{+\infty} \ldots \int A\vec{g}_{rs} \{a_{rs} [\frac{(\vec{g}_{rs} \cdot <\vec{g}_{rs}>)^2}{g_{rs}^3} - \frac{<g_{rs}>^2}{g_{rs}}]$$

$$+ \frac{(\vec{g}_{rs} \cdot <\vec{g}_{rs}>)}{g_{rs}^3}\}\, d\vec{c}_{rs}\, d\vec{g}_{rs} \quad . \tag{40}$$

Upon performing the $d\vec{c}_{rs}$-integrations, the Eqs. (39) and (40) reduce to

$$\vec{I}_1 = -\frac{\pi^{3/2}}{m_{rs}} \frac{a_{rs}^2/a_s}{(a_r+a_s)^{3/2}} \iiint_{-\infty}^{+\infty} e^{-a_{rs}(\vec{g}_{rs}-\langle\vec{g}_{rs}\rangle)^2}$$

$$\times \left[\frac{\vec{g}_{rs}\cdot\langle\vec{g}_{rs}\rangle}{g_{rs}^3}\vec{g}_{rs} - \frac{\langle\vec{g}_{rs}\rangle}{g_{rs}}\right]d\vec{g}_{rs} \tag{41}$$

and

$$\vec{I}_2 = 2\frac{\pi^{3/2}}{m_{rs}} \frac{a_{rs}^2/a_r}{(a_r+a_s)^{3/2}} \iiint_{-\infty}^{+\infty} e^{-a_{rs}(\vec{g}_{rs}-\langle\vec{g}_{rs}\rangle)^2}\vec{g}_{rs}$$

$$\times \left\{a_{rs}\left[\frac{(\vec{g}_{rs}\cdot\langle\vec{g}_{rs}\rangle)^2}{g_{rs}^3} - \frac{\langle\vec{g}_{rs}\rangle^2}{g_{rs}}\right] + \frac{\vec{g}_{rs}\cdot\langle\vec{g}_{rs}\rangle}{g_{rs}^3}\right\}d\vec{g}_{rs} \tag{42}$$

Let a spherical coordinate system (g_{rs},α,β) with the polar axis parallel to $\langle\vec{g}_{rs}\rangle$ be introduced, in which

$$\vec{g}_{rs} = g_{rs}\ (\sin\alpha\ \cos\beta,\ \sin\alpha\ \sin\beta,\ \cos\alpha)\ ,$$

$$d\vec{g}_{rs} = g_{rs}^2\ dg_{rs}\ \sin\alpha\ d\alpha\ d\beta\ , \tag{43}$$

and the substitutions,

$$\tau = \cos\alpha\ ,$$

$$x = a_{rs}^{1/2}\ g_{rs}\ ,\quad \gamma_{rs} = a_{rs}^{1/2}|\langle\vec{g}_{rs}\rangle|\ . \tag{44}$$

These operations transform the Eqs. (41) and (42) to:

$$\vec{I}_1 = -2\frac{\pi^{5/2}}{m_{rs}}\frac{a_{rs}/a_s}{(a_r+a_s)^{3/2}}(G_{12}-G_{10})\langle\vec{g}_{rs}\rangle \tag{45}$$

and

$$\vec{I}_2 = 4\frac{\pi^{5/2}}{m_{rs}}\frac{a_{rs}/a_r}{(a_r+a_s)^{3/2}}[\gamma_{rs}(G_{23}-G_{21}) + G_{12}]\langle\vec{g}_{rs}\rangle \tag{46}$$

where

$$G_{mn} \equiv \int_{\tau=-1}^{+1} \int_{x=0}^{\infty} e^{-(x^2+\gamma_{rs}^2-2\gamma_{rs}x\tau)} x^m \, dx \, \tau^n \, d\tau \quad . \tag{47}$$

By means of an obvious substitution, this integral is brought into the form

$$G_{mn} = \int_{-1}^{+1} H_m(\tau) \, e^{\gamma_{rs}^2(\tau^2-1)} \tau^n \, d\tau \quad , \tag{48}$$

where

$$H_m(\tau) = \int_{-\gamma_{rs}\tau}^{\infty} e^{-\xi^2} (\xi+\gamma_{rs}\tau)^m \, d\xi \quad , \tag{49}$$

i.e.,

$$H_m(\tau) = \sum_{n=0}^{m} \binom{m}{n} (\gamma_{rs}\tau)^{m-n} \int_{-\gamma_{rs}\tau}^{\infty} e^{-\xi^2} \xi^n \, d\xi \quad .$$

Hence,

$$G_{10} = \frac{1}{\gamma_{rs}} \frac{\pi^{\frac{1}{2}}}{2} \Phi(\gamma_{rs}) \quad , \tag{50}$$

$$G_{12} = \frac{\gamma_{rs}^2 - 1}{\gamma_{rs}^3} \frac{\pi^{\frac{1}{2}}}{2} \Phi(\gamma_{rs}) + \frac{1}{\gamma_{rs}^2} e^{-\gamma_{rs}^2} \quad , \tag{51}$$

$$G_{21} = \frac{2\gamma_{rs}^2 - 1}{2\gamma_{rs}^2} \frac{\pi^{\frac{1}{2}}}{2} \Phi(\gamma_{rs}) + \frac{1}{2\gamma_{rs}} e^{-\gamma_{rs}^2} \quad , \tag{52}$$

$$G_{23} = \frac{3 - 3\gamma_{rs}^2 + 2\gamma_{rs}^4}{2\gamma_{rs}^4} \frac{\pi^{\frac{1}{2}}}{2} \Phi(\gamma_{rs}) + \frac{\gamma_{rs}^2 - 3}{2\gamma_{rs}^3} e^{-\gamma_{rs}^2} \quad , \tag{53}$$

and

$$\frac{a_s\vec{I}_1}{a_r\vec{I}_2} = 2 \frac{\pi^{5/2}}{m_{rs}} \frac{a_{rs}}{(a_r+a_s)^{3/2}} \gamma_{rs}^{-3} \frac{\pi}{2}^{\frac{1}{2}} \left[\Phi(\gamma_{rs}) - \gamma_{rs}^{1} \frac{2}{\pi^{\frac{1}{2}}} e^{-\gamma_{rs}^2}\right] < \vec{g}_{rs} > \quad . \tag{54}$$

Substitution of Eq. (54) into Eq. (32) gives the following expression for the force density exerted by the s-component on the r-component:

$$\vec{F}_{r/s} = - M(a_{rs}^{\frac{1}{2}}|<\vec{g}_{rs}>|)\tau_{rs}^{-1} n_r m_r(<\vec{v}_r> - <\vec{v}_s>) \tag{55}$$

where

$$M(\gamma_{rs}) = \frac{3\pi^{\frac{1}{2}}}{4} \gamma_{rs}^{-3}[\Phi(\gamma_{rs}) - \frac{2}{\pi^{\frac{1}{2}}} \gamma_{rs} e^{-\gamma_{rs}^2}] \quad , \tag{56}$$

$$\tau_{rs}^{-1} = \frac{8}{3}(\frac{2KT_{rs}}{\pi m_{rs}})^{\frac{1}{2}} \frac{m_{rs}}{m_r} n_s Q_{rs} \quad , \tag{57}$$

$$Q_{rs} = \frac{\pi}{2}(\frac{e_r e_s}{KT_{rs}})^2 \Lambda_{rs} \quad . \tag{58}$$

This result indicates that the intercomponent friction force $\vec{F}_{r/s} = -\vec{F}_{s/r}$ (action-reaction principle) is in general a transcendental function of the drift velocity $<\vec{g}_{rs}> = <\vec{v}_r> - <\vec{v}_s>$ [Eqs. (55)-(56)]. The relaxation time τ_{rs} is an expression analogous in structure to that for binary Coulomb collisions[3-4] [Eq. (57)]. Q_{rs} is the many-body Coulomb transport cross section [Eqs. (58) and (7)].

The transcendental function $M(\gamma_{rs})$ in Eq. (56) has the limiting properties [Eqs. (14)-(15)]:

$$M(\gamma_{rs}) \cong 1 \quad , \quad \gamma_{rs} << 1 \quad , \tag{59}$$

$$M(\gamma_{rs}) \cong \frac{3}{4} \pi^{\frac{1}{2}} \gamma_{rs}^{-3} \quad , \quad \gamma_{rs} >> 1 \quad . \tag{60}$$

Accordingly,

$$\vec{F}_{r/s} = -\tau_{rs}^{-1} n_r m_r(<v_r> - <v_s>) \quad , \quad \gamma_{rs} << 1 \quad , \tag{61}$$

and

$$\vec{F}_{r/s} = -\tau_{rs}^{-1} n_r m_r(\frac{3\pi^{\frac{1}{2}}}{4a_{rs}^{3/2}}) \frac{<\vec{v}_r> - <\vec{v}_s>}{|<\vec{v}_r> - <\vec{v}_s>|^3} \quad , \quad \gamma_{rs} >> 1 \quad . \tag{62}$$

It is recognized that the intercomponent friction force i) increases proportional to the drift velocity in the infrasonic region, $|\vec{F}_{r/s}| \sim g_{rs}$ by Eq. (61), and ii) decreases inversely with the square of the drift velocity in the suprasonic region, $|\vec{F}_{r/s}| \sim g_{rs}^{-2}$ by Eq. (62). It should be noted that Eq. (62) is not an exact extrapolation to suprasonic drift velocities, since the LB-equation is no longer rigorously applicable for supersonic drift velocities due to the appearance of the two-stream instability ($\gamma_{rs} > 1$). The Eqs. (55) and (62) can be used, however, as approximations in the supersonic and suprasonic region of drift velocities, respectively. As a justification, it is noted that Eq. (55) and Eq. (62) give not only a qualitative but also a quantitative description (within the experimental uncertainties) of the so-called run-away-effect.[9]

ELECTRICAL CONDUCTIVITY

In order to discuss the concept of electrical conductivity in a simple way, a quasi-homogeneous steady-state electron-ion plasma $(r = e\ ,\ \ s = i)$ is considered in a frame of reference in which an electric field $\vec{E}$ exists but no magnetic field $(\vec{B} = \vec{0})$. The electrical current density,

$$\vec{j} = \sum_{e,i} n_\beta e_\beta \ \langle\vec{v}_\beta\rangle = -ne\langle\vec{g}_{ei}\rangle \quad , \tag{63}$$

$$n_i = n_e = n\ ,\quad e_i = -e_e = e \quad ,$$

is related to the argument γ_{rs} defined in Eq. (44) by

$$\gamma_{ei} = j/j_T \tag{64}$$

where

$$j = |\vec{j}|\ ,\quad j_T = ne/a_{ei}^{\frac{1}{2}} \doteq ne(2KT_e/m_e)^{1/2} \tag{65}$$

In the steady-state of a quasi-homogeneous plasma the intercomponent friction and electric force densities are in balance, $\vec{F}_{e/i} - ne\vec{E} = \vec{0}$. The corresponding Ohm's law is by Eqs. (55) and (63)-(64):

$$\vec{j}\ M(j/j_T) = \sigma\ \vec{E}\ ,\quad j \lesssim j_T\ , \tag{66}$$

where

$$\sigma = (ne^2/m_e)\tau_{ei} \tag{67}$$

and

$$\tau_{ei}^{-1} \simeq \frac{8}{3}\left(\frac{2KT_e}{\pi m_e}\right)^{\frac{1}{2}} n\quad Q_{ei}\ ,\quad m_e \ll m_i \quad , \tag{68}$$

$$Q_{ei} \doteq \frac{\pi}{2}\left(\frac{e^2}{KT_e}\right)^2 \Lambda_{ei}\ ,\quad \Lambda_{ei} = \ln(DKT_e/e^2)\ ,\quad m_e \ll m_i\ , \tag{69}$$

by Eqs. (57)-(58) and (7). According to Eq. (66), the relation between current density $\vec{j}$ and electric field $\vec{E}$ is nonlinear if j is not small compared to the thermal current density [Eq. (56)].

In the infrasonic drift limit, $j \ll j_T$, σ has the meaning of an electrical conductivity [Eq. (67)]. If the electron drift velocity is small compared to the thermal speed $a_{ei}^{-1/2} \cong (2KT_e/m_e)^{1/2}$, Eq. (66) reduces to the familiar linear Ohm's law:[10]

$$\vec{j} = \sigma\vec{E} \ , \ j \ll j_T \ . \tag{70}$$

It should be noted that Eq. (66) is not applicable to suprasonic drift velocities, $j \gg j_T$. In the latter case, the run-away-effect occurs, and an Ohm's law does no longer exist in steady state.

The results obtained in this investigation by means of the LB-equation justify the use of the Boltzmann equation with a Rutherford cross section in plasma kinetics as long as the dielectric constant $\varepsilon(\vec{k},\omega)$ can be approximated by one. It can be shown that the collective effects are only important if the ratio of the electron to the ion temperature is very large, $T_e/T_i > 10^2$. A similar conclusion concerning the significance of the collective contributions has been obtained in connection with the thermal relaxation in anisotropic two-temperature plasmas.[11] It is noted that electron temperatures, $T_e > 10^2 T_i$, are extremely difficult to realize in fully ionized plasmas with stable distribution functions.

CITATIONS

1. A. LENARD, Ann. Phys. 3, 390 (1960).
2. R. BALESCU, Phys. Fluids 3, 250 (1960).
3. H. E. WILHELM, Z. NATURFORSCHUNG 25a, 322 (1970).
4. H. E. WILHELM, Il Nuovo Cimento 68B, 189 (1970).
5. D. C. MONTGOMERY, Theory of the Unmagnetized Plasma, Gordon & Breach, New York (1971).
6. S. CHANDRASEKHAR, Principles of Stellar Dynamics, University of Chicago Press, Chicago (1942).
7. M. ABRAMOWITZ and I. A. STEGUN, Handbook of Mathematical Functions, Dover Publications, New York (1965).
8. A. I. AKHIEZER, Collective Oscillations in a Plasma, The M.I.T. Press, Cambridge, Mass. (1967).
9. H. DREICER, Phys. Rev. 115, 238 (1959).
10. L. SPITZER, JR., The Physics of Fully Ionized Gases, Interscience, New York (1966).
11. G. Lehner and F. Pohl, Z.Phys. 216, 488(1968).

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