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COMMUTATIVE SEMIGROUPS OF
REAL AND COMPLEX MATRICES
BY D.R. BROWN APRIL 1974
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Abstract

The computation of divergence is greatly simplified when the covariance matrices to be analyzed admit a common diagonalization, or even triangulation; that is, when there exists an invertible matrix P such that, for each covariance matrix A under consideration, PAP^{-1} is diagonal, or triangular.

In this report, sufficient conditions are given for such phenomena to take place; the arguments cover both real and complex matrices, and are not restricted to Hermitian or other special forms. Specifically, it is shown to be sufficient that the matrices in question commute in order to admit a common triangulation. Several results hold in the case that the matrices in question form a closed and bounded set, rather than only in the finite case.

COMMUTATIVE SEMIGROUPS OF REAL AND COMPLEX MATRICES

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D.R. Brown

Commutative matrices have been examined in detail; an excellent bibliography of pertinent papers is given in [2]. In this chapter, using the Jordan form of a matrix and a theorem of Kaplansky, it is first shown that any commutative semigroup of $n \times n$ complex matrices is similar to a semigroup of lower triangular matrices (matrices which are zero above the main diagonal). Following this, commutative semigroups of real matrices are studied. For a proof of the first theorem, see [1].

1). Theorem (Kaplansky). Let S be a multiplicative semigroup of $n \times n$ matrices over a division ring, consisting of nilpotent elements. Then S is similar to a semigroup of strictly lower triangular matrices (matrices which are zero on and above the main diagonal).

2). Lemma. Let A be an $n \times n$ complex matrix in Jordan form, $A = \mathrm{diag}\{A_{11}, \dots A_{kk}\}$, $A_{ii} = \lambda_i + N_{ii}$, with each λ_i a scalar and each N_{ii} a lower triangular nilpotent having ones and zeros on the first principal subdiagonal and zeros elsewhere, and $\lambda_i \neq \lambda_j$ if $i \neq j$. Let $B = (B_{ij})$ be an $n \times n$ complex matrix decomposed in the same dimensions as A. If $AB = BA$, then $B_{ij} = 0$, $i \neq j$; that is, B is in super-diagonal form, $B = \mathrm{diag}\{B_{11}, \dots, B_{kk}\}$.

Proof: Fix $i = p$, $j = q$. Since $AB = BA$, it follows that $\lambda_k B_{pq} + N_{pp}B_{pq} = \lambda_q B_{pq} + B_{pq}N_{qq}$, whence $(\lambda_p - \lambda_q)B_{pq} = B_{pq}N_{qq} - N_{pp}B_{pq}$. Let $B_{pq} = (b_{ij})$, $i = 1, \ldots, r$; $j = 1, \ldots, s$, and let $N_{pp} = (n_{ij})$, $N_{qq} = (m_{ij})$. By the prior equality, $(\lambda_1 - \lambda_j)b_{1j} = b_{1j+1}m_{j+1j}$, $j < s$, and $(\lambda_1 - \lambda_s)b_{1s} = 0$. Since $\lambda_1 \neq \lambda_s$, $b_{1s} = 0$, and hence $b_{1j} = 0$, $j = 1, \ldots, s$. Assume the first $k - 1$ rows of B_{pq} have been proved identically zero. Then $(\lambda_k - \lambda_j)b_{ks} = 0$, from which it follows that $b_{kj} = 0, j = 1, \ldots, s$. Therefore $B_{pq} = 0$.

3). Theorem. If S is a commutative semigroup of $n \times n$ complex matrices, then S is similar to a semigroup of lower triangular matrices.

Proof: The theorem is proved by induction on the order of the matrices composing S . If $n = 1$, there is nothing to prove. Suppose the theorem has been proved for semigroups with matrices of order $k < n$, and let S be a commutative semigroup of $n \times n$ complex matrices. If there exists $A \in S$ such that A has $j > 1$ distinct eigenvalues, then by 2) S can be decomposed into super diagonal form, $S = \mathrm{diag}\{S_1, \ldots, S_j\}$. Each S_i is a commutative semigroup of order less than n , hence is similar to a semigroup of lower triangular matrices. Since the semigroups S_p, S_q, $p \neq q$, are independent of one another, it follows that S is similar to a semigroup of lower triangular matrices.

It remains to dispose of the case in which each $A \in S$ has a unique eigenvalue. By referring to the Jordan form of A, it can be seen that each $A \in S$ decomposes into the sum of a scalar (the eigenvalue of A) and a nilpotent; $A = \lambda + N$. Let $\mathcal{N} = \{N : \text{for some } A \in S, \lambda \in S(A), A = \lambda + N\}$. If $M, N \in \mathcal{N}$, then $MN = NM$; for, let $A = \lambda + N$, $B = \alpha + M$. Then $\lambda\alpha + \lambda M + \alpha N + NM = AB = BA = \alpha\lambda + \lambda M + \alpha N + MN$, hence $MN = NM$. Let $\mathcal{N}'$ be the semigroup generated by $\mathcal{N}$. Since elements of $\mathcal{N}$ commute, $\mathcal{N}'$ consists of nilpotent elements. Hence, by 1), there exists P such that $P\mathcal{N}'P^{-1}$, and therefore $P\mathcal{N}P^{-1}$, is in strictly lower triangular form. Finally, if $A = \lambda + N \in S$, then $PAP^{-1} = \lambda + PNP^{-1}$, which is lower triangular. Hence PSP^{-1} is lower triangular.

4) Corollary. Let S be as in 3). If there is an $A \in S$ having n distinct eigenvalues, then S is similar to a semigroup of diagonal matrices. If, furthermore, the matrices in S are real and A has a real spectrum, then S is similar to a semigroup of real diagonal matrices.

Proof: The Jordan form of A is diagonal; by 2) it follows that S can be diagonalized. The latter part of the corollary follows from the fact that the Jordan form of A is real.

Note that a semigroup of $n \times n$ diagonal matrices is iseomorphic to a subsemigroup of the Cartesian product of n copies of the scalar field under coordinate multiplication. Sufficient conditions, different from those in 4) for a semigroup of complex (real) matrices to be similar to a diagonal semigroup are now investigated.

5). <u>Lemma</u>. Let $\{E_i\}$, $i = 1, \ldots, k$ be $n \times n$ complex (real) idempotents such that $E_iE_j = E_jE_i = E_i$, $i \leqq j$, rank $E_i = r(i)$. Then there exists a complex (real) matrix P such that

$$PE_iP^{-1} = \begin{pmatrix} I_{r(i)} & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, k .$$

Proof: Note if $i < j$, then $r(i) < r(j)$. The lemma is proved by induction on n. If $n = 1$, there is nothing to prove. Assume the lemma has been proved for matrices of order less than n. As remarked above, E_k is the idempotent of maximal rank $r(k)$ in the set $\{E_i\}$. Hence there exists a complex (real) matrix P such that $PE_kP^{-1} = \begin{pmatrix} I_{r(k)} & 0 \\ 0 & 0 \end{pmatrix}$. Since E_k is an identity for E_i, $i = 1, \ldots, k$, it follows that $PE_iP^{-1} = \begin{pmatrix} X_i & 0 \\ 0 & 0 \end{pmatrix}$, where X_i is some $r(k) \times r(k)$ submatrix. If $r(k) < n$, then the system PE_iP^{-1} is isomorphic to a system of dimension less than n, and the lemma is proved, otherwise, $E_k = I_n$, in which case the above argument is applied to E_{k-1}. Since $r(k-1) < n$, the inductive hypothesis can be applied to the system $\{E_i\}$, $i = 1, \ldots, k-1$, to complete the proof of the lemma.

6). <u>Corollary</u>. Let S be a commutative semigroup of complex (real) $n \times n$ matrices with zero E_1, rank $E_1 = s$; identity E_k, rank $E_k = s + k - 1$. Suppose, also, that S contains a system of k idempotents $\{E_i\}$ as in 6) between E_1 and E_k with rank $E_i = s + i - 1$. Then S is isomorphic to a diagonal semigroup of $(k - 1)\times(k - 1)$ complex (real) matrices.

Proof: Let f be the function defined by $f(X) = X - E_1$, $X \in S$. It is clear that f is an isomorphism of S into the $n\times n$ matrices such that rank $f(E_i) = i - 1$. Let P be the matrix of 5) such that $Pf(E_i)P^{-1} = \begin{pmatrix} I_{i-1} & 0 \\ 0 & 0 \end{pmatrix}$. Now $Pf(S)P^{-1}$ has identity $\begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$; Hence if $X \in S$, then $Pf(X)P^{-1} = \begin{pmatrix} X' & 0 \\ 0 & 0 \end{pmatrix}$, where X' is a $(k - 1)\times(k - 1)$ submatrix. Since X' must commute with I_{i-1}, $i = 1, \ldots, k$, it follows by direct computation that X' must be diagonal. The details are omitted. The function f and the similarity generated by P are clearly continuous.

Note that, if S satisfies the hypotheses of 6) and is, in addition, compact, then the diagonal entries of S are bounded above in modulus by 1 . Hence, in the complex case, S is a subsemigroup of the Cartesian product of unit discs; in the real case, S is a subsemigroup of the Cartesian product of real intervals [-1,1]. If S has stochastic entries, the full interval [-1, 1] may still be realized, as is shown by the 2×2 example $\begin{pmatrix} x & 1 - x \\ 1 - x & x \end{pmatrix}$, $x \in [0,1]$.

7). Corollary. If S is a semigroup of $n\times n$ real matrices, and if S has a zero E and an identity F whose ranks differ by one, then S is isomorphic to a subsemigroup of the real numbers, and hence commutative. If S is also connected, then S contains the convex arc between E and F .

Proof: In the argument establishing 6) the commutativity is not needed when there are less than 3 idempotents in the chain. Hence S is iseomorphic to a semigroup of 1×1 real matrices, which are essentially the real numbers. If S is connected, then the iseomorphic copy of S must contain the arc between (1) and (0). Since the function f and the similarity generated by P of 6) are affine mappings, S must contain the convex arc from E to F .

If the ranks of E and F differ by more than one, then S need not even be diagonal. Indeed, the mapping

$$f(x) = \begin{pmatrix} x & 0 \\ (-\ln x)x & x \end{pmatrix}, \quad x \in (0,1], \quad f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{is an}$$

iseomorphic imbedding of the unit interval demonstrating this fact.

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