THE DIFFUSIVE IDEALIZATION OF CHARGED PARTICLE TRANSPORT IN RANDOM MAGNETIC FIELDS

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ABSTRACT

This paper examines the uniqueness and accuracy of the equation which describes the transport of charged particles diffusing in a random magnetic field parallel to a relatively large guiding field. With regard to uniqueness, it is found here that the same coefficient of diffusion is obtained by three methods that have apparently led to discrepancies in previous work. With regard to accuracy, it is found that two corrections must be added to the familiar expression in which the diffusive flux is proportional to the gradient of the density. If the temporal derivative of the flux is large, the first of these corrections is important and the diffusion equation must be replaced by the telegrapher's equation. If the gradient of the flux is large, the second correction is important and the appropriate generalization of the diffusion equation is an integral equation analogous to those invoked in classical transport theory. Explicit expressions are given for a characteristic time and a characteristic length which describe the corrections.

The well known divergence of the coefficient of diffusion, which is implied by the quasilinear analysis of pitch angle scattering, does not occur if the scattering rate is finite at 90° pitch angle. This effect is illustrated by formulas which give the coefficient of diffusion when the quasilinear expression is perturbed by a variable amount of isotropic scattering.

Subject headings: cosmic rays - hydromagnetics - magnetic fields

I. INTRODUCTION

The diffusive nature of charged particle propagation in random magnetic fields has been recognized for many years. However, there are still two unresolved difficulties that obscure the relationship between the coefficient of diffusion D and the statistical quantities that describe the random fields. In the first place, there is no concordance on the method of evaluating the Fokker-Planck coefficient $<\Delta\mu^2>/\Delta t$ which characterizes the scattering of pitch angles relative to the average "d.c." field B. The quasilinear approximation (Jokipii 1966) leads to a simple relationship

$$<\Delta\mu^2>/\Delta t = \phi\{\mu\} = A|\mu|^{q-1}(1-\mu^2) \qquad (1)$$

in which μ is the cosine of the pitch angle and q is the spectral index of the power law that gives the mean square amplitude of field fluctuations at wave number k within an interval dk, $Q_{xx}(k_o/k)^q dk$, in terms of the spectral density Q_{xx} at a reference wave number k_o. Here, the parameter A,

$$A = 2\pi \frac{V}{R^2} \; Q_{xx} \; (k_o r_L)^{-q}, \qquad (2)$$

can be expressed in terms of the particle rigidity R, velocity V and Larmor radius r_L and the spectral parameters Q_{xx}, q and k_o. The surprising implication of equation (1) that scattering vanishes at $\mu = 0$ for $q > 1$ has motivated attempts to refine the quasilinear calculation. Even the validity of the Fokker-Planck approach has been questioned (Lerche 1972; Klimas and Sandri 1971), but a consensus of recent work (Jones, Kaiser and Birmingham 1973; Jokipii and Lerche 1973; Volk 1973; Owens 1974) agrees that the formalism is applicable but that the coefficient $<\Delta\mu^2>/\Delta t$ is finite at $\mu = 0$. However, the magnitude of this coefficient is not yet firm. The second unresolved difficulty is that different methods of calculating the coefficient of diffusion corresponding to a given Fokker-Planck coefficient give different answers, in spite of the fact that the coefficient D is defined uniquely as the ratio of the flux to the gradient of the density. This paper is addressed to the second of these difficulties but not to the first.

To describe the distribution function for charged particles diffusing along a guiding field, three representations have been invoked: (1) A series expansion in terms of Legendre polynomials as in classical transport theory (Jokipii 1968; Weinberg and Wigner 1958),(2) A series expansion in terms of eigenfunctions of the scattering operator (Earl 1973), (3) A closed expression obtained by perturbation techniques (Jokipii 1966; Hasselmann and Wibberenz 1970; Kulsrud and Pierce 1969). In previous work, these approaches apparently gave different answers. In §II, this conflict is resolved by the demonstration that the correct application of either one of the first two methods leads to a result identical to that obtained by the third method. Thus, the correct expression for the coefficient of diffusion is the one derived by Jokipii (1966).

This agreement on the description of diffusive streaming motivates the consideration, in §III, of higher order effects which set limits on the accuracy of the diffusion equation. These effects lead to the intuitively plausible result that the evolution of the distribution function at a point in space and time is controlled not by the local density but instead by the average density within a spatial region comparable to the mean free path and over a period comparable to the collision time.

Jokipii (1971) has reviewed several examples in which coefficients of diffusion based upon equation (1) provide a useful description of observed phenomena. However, when $q \geq 2$, equation (1) implies that D is infinite. This divergence is disquieting for two reasons. In the first place, for $q \lesssim 2$ below the point of divergence, the coefficient D is very sensitive to variations in the spectral index. Because the index of the interplanetary magnetic power spectrum is observed to change with time within this region (Sari and Ness 1969; Siscoe et al. 1968), the predicted sensitivity implies a greater variability in the local propagation of solar and galactic cosmic rays than is observed. In the second place, the existence of a physically unreasonable divergence casts doubt upon the validity of equation (1). These doubts stimulated the development of the refined theories mentioned above. The effect of the finite scattering at $\mu = 0$ predicted by these theories is illustrated, in §III, by the explicit calculation of the coefficient of diffusion corresponding to a Fokker-Planck coefficient modified, as was suggested by Owens (1974), by the addition of an isotropic scattering term

$$\phi = A|\mu|^{q-1}(1-\mu^2) + H(1-\mu^2), \tag{3}$$

where H, the parameter that characterizes the magnitude of the perturbation, is evidently the coefficient at $\mu = 0$ for $q > 1$. Isotropic scattering could also arise from a separate effect such as binary Coulomb collisions.

An earlier paper on diffusion (Earl 1973), which introduced the concept of scattering eigenfunctions and eigenvalues, will be referred to as Paper I and equations therein will be designated here by the Roman numeral

I. A subsequent paper (Earl 1974), designated hereafter as Paper II, treated the coherent mode of particle propagation that replaces diffusion when D is infinite. The ideas in Papers I and II will be reviewed below and used extensively.

Particle transport is described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mu V \frac{\partial f}{\partial z} = \frac{1}{2} \frac{\partial}{\partial \mu} \phi \frac{\partial f}{\partial \mu} \tag{4}$$

where f {μ, z, t} is the particle distribution function, z is distance parallel to the mean field, and t is time. Because the scattering operator on the right hand side of equation (4) has the form specified by Sturm-Liouville theory, there is a sequence of eigenfunctions $R_K\{\mu\}$ and eigenvalues $(2/\tau_K)$ which satisfy the equation

$$\frac{d}{d\mu} \phi \frac{dR_K}{d\mu} + \left(\frac{2}{\tau_K}\right) R_K = 0 \ , \tag{5}$$

where τ_K is a relaxation time that describes the temporal decay of an anisotropy proportional to R_K. These functions form an orthogonal set in terms of which the distribution function f and the streaming term $\mu V(\partial f/\partial z)$ can be expressed as series expansions

$$f\{\mu, z, t\} = \Sigma f_K\{z, t\} \ c_K R_K\{\mu\}, \tag{6}$$

$$\mu V(\partial f/\partial z) = \Sigma V_{JK}(\partial f_J/\partial z) c_K R_K\{\mu\}, \tag{7}$$

where the normalizing factors c_K defined by equation (II-7) are given in table 1 of Paper II. Here, the coefficients f_K are given by

$$f_K = c_K \int_{-1}^{+1} f R_K \, d\mu, \tag{8}$$

and the characteristic velocities V_{JK} are given by

$$V_{JK} = V_{KJ} = V c_J c_K \int_{-1}^{+1} R_J \mu R_K \, d\mu \ . \tag{9}$$

When the series representations of the distribution function (eq. [6]) and of the streaming term (eq. [7]) are substituted into equation (4), the orthogonality of the R_K implies that the coefficients f_K are described by a set of coupled differential equations the first four of which are

$$\frac{\partial f_0}{\partial t} = -V_{01}\frac{\partial f_1}{\partial z} - V_{03}\frac{\partial f_3}{\partial z} - \ldots \quad , \tag{10}$$

$$\frac{\partial f_1}{\partial t} + \frac{f_1}{\tau_1} = -V_{01}\frac{\partial f_0}{\partial z} - V_{12}\frac{\partial f_2}{\partial z} - \ldots \quad , \tag{11}$$

$$\frac{\partial f_2}{\partial t} + \frac{f_2}{\tau_2} = -V_{12}\frac{\partial f_1}{\partial z} - V_{23}\frac{\partial f_3}{\partial z} - \ldots \quad , \tag{12}$$

$$\frac{\partial f_3}{\partial t} + \frac{f_3}{\tau_3} = -V_{03}\frac{\partial f_0}{\partial z} - V_{23}\frac{\partial f_2}{\partial z} - \ldots \quad . \tag{13}$$

The matrix formulation of transport theory embodied in these equations is entirely equivalent to the Boltzmann equation.

II. DIFFUSION REVISITED

Scattering causes the distribution function to relax toward isotropy. An important stage of this evolution is the diffusive regime in which a relatively small and nearly constant anisotropy leads to slow temporal changes of the isotropic density. In this regime, the distribution function f can be approximately represented as the sum of an isotropic component $F_o = c_o f_o$ which depends upon position and time and two anisotropic components which are constant but which depend upon position

$$f\{\mu, z, t\} = F_o\{z, t\} + F_1\{\mu, z\} + F_2\{\mu, z\}, \qquad (14)$$

where F_1 is an odd function of μ and F_2 is an even function of μ. Because the form of the odd anisotropy F_1 is well known (Jokipii 1966; Kulsrud and Pearce 1969; Hasselmann and Wibberenz 1970), this section focuses upon relationships among the three methods of analysis listed in the introduction. The assumption that the even anisotropy F_2 is negligible compared to F_o and F_1, which is adopted throughout this section, will be confirmed in §III. Nevertheless, it will simplify the treatment that follows to require that

$$\int_{-1}^{+1} F_2\{z, \mu\}\, d\mu = 0 \qquad . \qquad (15)$$

Evidently, by virtue of equation (8), this condition implies that the isotropic term in the series expansion of F_2 is zero. Because F_1 is an odd function, it also satisfies equation (15).

The odd and even anisotropies are described, respectively, by the odd numbered and even numbered coefficients in the eigenfunction expansion of f (eq. [6]). Thus, if the anisotropies are constant and if the even components are negligible compared to the density f_o, then the matrix equations for the odd components take on a form

$$f_K = - \tau_K V_{OK} \frac{\partial f_o}{\partial z} \qquad (16)$$

which can be substituted into the gradients appearing on the right hand side of equation (10) to yield a diffusion equation for f_o

$$\frac{\partial f_o}{\partial t} = \frac{\partial}{\partial z} D \frac{\partial f_o}{\partial z} \qquad (17)$$

in which the coefficient of diffusion D can be expressed as

$$D = \tau_1 V_{01}^2 + \tau_3 V_{03}^2 + \tau_5 V_{05}^2 + \ldots \quad . \qquad (18)$$

In Paper I, only the first term of this series was considered. However, it is possible to estimate the contribution of the other terms with the aid of the identity

$$(1/3)V^2 = V_{01}^2 + V_{03}^2 + \ldots \qquad (19)$$

which results when the series expansion

$$\mu = (\sqrt{2}/V)(V_{01}c_1R_1 + V_{03}c_3R_3 + \ldots) \qquad (20)$$

is multiplied by itself and integrated over μ. Because $\tau_3 > \tau_5 > \ldots$, the error introduced by retaining only the first term of equation (18) is governed by the inequality

$$D - \tau_1 V_{01}^2 < \tau_3 \left(\frac{1}{3} V^2 - V_{01}^2 \right) = \left[\frac{(q-1)^2 \ (2-q)}{3(5-2q)(8-3q)} \right] \tau_1 V_{01}^2 \qquad (21)$$

in which the expression in square brackets represents, for $q < 2$, an upper limit on the fractional error. In the range $1 < q < 2$, the maximum error of 1.2% predicted by this expression at $q = 1.8$ is negligible compared to experimental uncertainties. Only in the unphysical region $q < -2.1$, where the fractional error eventually approaches $(-q/18)$, is the error larger than 10%.

According to equation (21), the error vanishes when $q = 1$ and when $q = 2$. In the former case, the eigenfunctions are Legendre polynomials for which the characteristic velocities V_{03}, V_{05}, ... are all zero. In the latter case, the fractional error is zero because the first term in equation (18) approaches infinity while all other terms remain finite. These considerations apply even when the Fokker-Planck coefficient is described by equation (3), for the eigenfunctions approach Legendre polynomials when H is large and the first term dominates when H is small. Thus, under a variety of circumstances, the flux represented by the higher order terms of equation (18) is quantitatively unimportant. However, these terms have conceptual significance, because they account for discrepancies between the analysis given in Paper I and the perturbation analysis that follows.

If equation (14) is substituted into the Boltzmann equation, independent equations are obtained for odd and even components

$$\frac{\partial F_0}{\partial t} + \mu V \frac{\partial F_1}{\partial z} = \frac{1}{2}\frac{\partial}{\partial \mu}\,\phi\,\frac{\partial F_2}{\partial \mu} - \frac{\partial F_2}{\partial t} \qquad (22)$$

$$\mu V \frac{\partial F_0}{\partial z} + \mu V \frac{\partial F_2}{\partial z} = \frac{1}{2}\frac{\partial}{\partial \mu}\,\phi\,\frac{\partial F_1}{\partial \mu} - \frac{\partial F_1}{\partial t} \quad . \qquad (23)$$

When equation (22) is integrated over μ from -1 to +1, the scattering term on the right hand side contributes nothing because ϕ vanishes at both limits while the integral over $(\partial F_2/\partial t)$ is zero because of equation (15). Thus, the integral of the left hand side gives a fundamental relationship between the time derivative of F_o and the gradient of the flux S

$$\frac{\partial F_0}{\partial t} + \frac{\partial S}{\partial z} = 0 \qquad (24)$$

where

$$S = (V/2)\int_{-1}^{+1} \mu F_1 \, d\mu. \qquad (25)$$

Equation (24) holds even if F_1 and F_2 are large or varying rapidly. Moreover, equations (24) and (25) imply that the temporal evolution of F_o is controlled by the odd anisotropy alone. The even anisotropy influences F_o only indirectly through the effect on F_1 implied by equations (22) and (23). Because this influence is weak, the odd anisotropy will be considered in detail before the even anisotropy is discussed. Also, the time derivatives on the right hand sides of equations (22) and (23), which are zero in the diffusive regime, will be neglected in this section. However, the role of $(\partial F_1/\partial t)$ will be considered in §III.

If F_2 is negligible compared to F_o, equation (23) leads to

$$\phi \frac{\partial F_1}{\partial \mu} = -(1-\mu^2)\ V \frac{\partial F_o}{\partial z} \qquad (26)$$

in which the constant of integration was chosen to satisfy the condition $\phi\{+1\} = \phi\{-1\} = 0$. A second integration with the constant chosen to yield an odd function gives

$$F_1 = -\ V \frac{\partial F_o}{\partial z}\int_0^{\mu} \frac{1-\nu^2}{\phi\{\nu\}}\, d\nu \qquad (27)$$

a result obtained earlier by Hasselmann and Wibberenz (1970). The flux S calculated from equation (27) with the aid of equation (25) takes on the diffusive form

$$S = -D \frac{\partial F_o}{\partial z} \qquad (28)$$

in which D is given by

$$D = V^2\int_0^1 \mu \, d\mu \int_0^\mu \frac{1-\nu^2}{\phi\{\nu\}} d\nu = (V^2/2)\int_0^1 \frac{(1-\nu^2)^2}{\phi\{\nu\}} d\nu = [\frac{(V^2/A)}{(2-q)(4-q)}] \quad . \quad (29)$$

Here, the second equality follows when the order of integration over μ and ν is interchanged. Equation (29) has also been derived by Jokipii (1966), Kulsrud and Pearce (1969) and Hasselmann and Wibberenz (1970).

To link this approach to the method of eigenfunctions discussed above and in Papers I and II, expand the odd anisotropy as an eigenfunction series

$$F_1 = f_1 \, c_1 \, R_1 + f_3 \, c_3 \, R_3 + \ldots \; .$$

Then, the result of substituting this series into the right hand side of equation (23) can be evaluated with the aid of equation (5) while the variable μ on the left hand side can be replaced by equation (20) to yield

$$\frac{\partial f_o}{\partial z} (V_{01} \, c_1 \, R_1 + V_{03} \, c_3 \, R_3 + \ldots) =$$

$$(f_1/\tau_1) \, c_1 \, R_1 + (f_3/\tau_3) \, c_3 \, R_3 + \ldots \qquad (30)$$

in which the relationship $F_o = (f_o/\sqrt{2})$ has been invoked. The orthogonality of eigenfunctions implies that corresponding coefficients on opposite sides of equation (30) are equal. Because this equality leads to exactly the coefficients specified by equation (16), it follows that the two methods give identical predictions for the odd anisotropy. More specifically, the functional dependence $g\{\mu\}$ of F_1 upon μ

$$F_1 = - g\{\mu\} \frac{\partial F_o}{\partial z} = g\{\mu\}(S/D) \qquad (31)$$

can be described either by the integral appearing in equation (27) or by an eigenfunction series

$$g\{\mu\} = V\int_0^\mu \frac{1-\nu^2}{\phi\{\nu\}} d\nu = \sqrt{2}(\tau_1 V_{01} c_1 R_1 + \tau_3 V_{03} c_3 R_3 + \ldots) = [\frac{V\mu^{2-q}}{(2-q)A}] \, . \quad (32)$$

Similarly, the coefficient of diffusion expressed in terms of relaxation times and characteristic velocities by equation (18) is identical to the coefficient given in terms of an integral by equation (29).

From equations (31) and (32), it is a simple matter to calculate the anisotropy δ_1 associated with the F_1 component,

$$\delta_1 = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = g\{1\} \frac{1}{F_o} \frac{\partial F_o}{\partial z} = \frac{g\{1\}}{L_U} = \frac{Vg\{1\}}{D} \frac{S}{VF_o} = [4-q] \frac{S}{VF_o} \quad . \qquad (33)$$

In these expressions, the parameter $g\{1\} = [V/(2-q)A]$ is a characteristic length for scattering. The anisotropy is given by the ratio of this length to the scale length L_U for spatial variations of the density. The last expression, which replaces equation (I-42), reduces, for $q = 1$, to the form

$$\delta = 3\ (S/VF_o)$$

that applies to isotropic scattering.

From equation (23), it is evident that $g\{\mu\}$ is the unique odd function that gives a result proportional to μ when acted upon by the scattering operator. Only in the special case that μ is an eigenfunction of the scattering operator is the odd anisotropy itself proportional to μ. To make this point explicit, it is useful to expand the distribution function in a Legendre series

$$f = \Sigma\ a_K\{z,\ t\} \quad . \qquad (34)$$

In the diffusive regime, the odd coefficients a_K are given, as in equation (I-24), by

$$a_K = - \frac{\partial F_o}{\partial z} \frac{V(2K+1)}{K(K+1)} \int_0^1 d\nu\ \frac{(1-\nu^2)^2}{\phi\{\nu\}} \frac{d}{d\nu} P_K\{\nu\} \quad . \qquad (35)$$

In the integration specified by equation (25), the factor μ is, of course, identical to P_1 which is orthogonal to all other Legendre polynomials.

Consequently, the flux S calculated from equation (34)

$$S = \frac{V}{3} a_1$$

depends only upon the coefficient a_1. When a_1 is evaluated with the aid of equation (35), the coefficient $(Va_1/3)(\partial F_0/\partial z)^{-1}$ is exactly the same as the diffusion coefficient that appears after the second equality of equation (29). Jokipii (1968) assumed that $a_1 = (3S/V)$ and that higher order coefficients are negligible. This assumption erroneously implies that $F_1 = (3S/V)\mu$. Notwithstanding, this form can be substituted into equation (26) and the result integrated over μ to yield an expression for the diffusion coefficient

$$D = \frac{2V^2}{9\int_0^1 \phi\{\mu\}\, d\mu} \tag{36}$$

that is correct only for isotropic scattering (Jokipii 1968; also see equation [I-37].) In actuality, equation (34) is a valid representation of the distribution function whose first term is proportional to μ, but the higher order terms can not always be neglected. For example, the ratios (a_3/a_1) and (a_5/a_1) calculated with the aid of equation (35) with ϕ given by equation (1) are

$$\frac{a_3}{a_1} = \frac{7}{3}\frac{(1-q)}{(6-q)}\ ; \qquad \frac{a_5}{a_1} = \frac{11}{3}\frac{(q^2-1)}{(8-q)(6-q)(4-q)}\ . \tag{37}$$

For $q = 1$, the special case mentioned above applies, and these ratios are zero. In general, the higher order Legendre components are very significant. Thus, at $q = 2$, $a_3/a_1 = -0.58$ and $a_5/a_1 = 0.23$. In contrast, equation (32) implies that the higher order scattering eigenfunctions are greatly attenuated relative to the first one. Thus, the ratio of scattering eigenfunctions that corresponds to the first ratio appearing in equation (37) is

$$\frac{\tau_3 V_{03} c_3}{\tau_1 V_{01} c_1} = \frac{(q-1)(2-q)(11-4q)}{2(7-2q)(5-2q)^2} \quad . \tag{38}$$

This quantity is zero at q = 1 and q = 2. Between these points, its maximum value is 4.8% at q = 1.7. The minor role played in diffusion by higher order eigenfunctions is also illustrated by equation (21).

In principle, the coefficients a_K could also be obtained by solving the following set of matrix equations which follow from the orthogonality of Legendre polynomials when equation (34) is substituted in the Boltzmann equation

$$\frac{\partial a_K}{\partial t} + \frac{K\,V}{2K-1}\frac{\partial a_{K-1}}{\partial z} + \frac{(K+1)V}{2K+3}\frac{\partial a_{K+1}}{\partial z} = \frac{2K+1}{4}\sum_J \left(\frac{1}{\tau_{JK}}\right) a_J \tag{39}$$

where

$$\frac{1}{\tau_{JK}} = \int_{-1}^{+1} P_K \frac{d}{d\mu}\phi\frac{dP_J}{d\mu}\,d\mu = -\int_{-1}^{+1} \phi\frac{dP_J}{d\mu}\frac{dP_K}{d\mu}\,d\mu . \tag{40}$$

(Compare Weinberg and Wigner [1958, p. 246].) Streaming is described in each of equations (10) - (13) by an infinite number of gradient terms, but, in equation (39), the effect of streaming upon the evolution of each coefficient depends only upon the gradients of the next higher and the next lower coefficients. Offsetting this simplification is the complexity of the Legendre representation of the scattering operator displayed in matrix form on the right hand side of equation (39). Here, scattering has the effect that the temporal derivative of each coefficient is coupled not only to itself, as in the eigenfunction representation, but also to all other coefficients of the same parity. This comparison brings out the important point that, in the matrix representation based upon scattering eigenfunctions, the scattering matrix is diagonal. Under most circumstances this property leads to the simplest possible description of the important effect of scattering at the cost of slightly increased complexity in the description of streaming. In special circumstances, when scattering is unimportant compared to streaming, other matrix representations based upon linear combinations of eigenfunctions may be appropriate. Because the Legendre coefficients in the diffusive regime are given by equation (35), there is no reason to pursue the alternative approach of inverting the matrices in equation (39). However, it is important to realize that the set of coefficients from which equation (36) was derived does not satisfy equation (39).

In summary, the perturbation method, the Legendre method, and the eigenfunction method give mutually consistent descriptions of the diffusive odd anisotropy. The same coefficient of diffusion is obtained by all three methods. An inappropriate choice of Legendre coefficients that has been invoked in previous discussions leads to results that are severely limited in applicability. Nevertheless, the Legendre coefficients can be chosen to give a representation that is correct but slowly converging. The eigenfunction representation converges rapidly, but it offers no advantage in the diffusive regime over the simpler perturbation method.

III. LIMITATIONS OF THE DIFFUSIVE IDEALIZATION

The diffusion equation is obtained when an approximate expression for the flux, equation (28), is substituted into the fundamental equation that describes the evolution of the density, equation (24). The objective of this section is to derive a more comprehensive transport equation based upon an improved approximation to the flux. This derivation consists of an evaluation of the even anisotropy F_2 as an integral of equation (22) followed by a calculation of S which takes into account the streaming term $\mu V(\partial F_2/\partial z)$ and the time derivative $(\partial F_1/\partial t)$ which appear in equation (23) but which were neglected in the derivation of equation (28). The streaming effect is fairly straightforward to evaluate, but the temporal effect is difficult to treat exactly because the dependence on μ of $(\partial F_1/\partial t)$ is not necessarily the same as that of F_1 itself. Because this difficulty is severe only when the anisotropy changes rapidly, $|\tau_1(\partial F_1/\partial t)| \gtrsim |F_1|$, it will be assumed that the same function $g\{\mu\}$ describes both of these dependences. When this assumption is inapplicable, the perturbation method becomes unwieldly, but the method of eigenfunctions is efficacious.

By virtue of equation (24), equation (22) can be recast in the form

$$\frac{1}{2}\ \frac{\partial}{\partial\mu}\ \phi\ \frac{\partial F_2}{\partial\mu} = \frac{\partial}{\partial z}\ (\mu V F_1 - S) = -\frac{\partial}{\partial z}\ S\left(1 - \frac{V^2}{D}\ \mu\int_0^{\mu} \frac{1-\nu^2}{\phi\{\mu\}}\, d\nu\right) \qquad (41)$$

where the derivative $(\partial F_2/\partial t)$ has been neglected and where the second equality follows from equations (31) and (32). Evidently, F_2 is the even function of μ that gives a result proportional to μF_1 when acted upon by the scattering operator. The result of a first integration between limits chosen to give the odd function that is required if F_2 is to be even,

$$\frac{1}{2}\,\phi\,\frac{\partial F_2}{\partial \mu} = -\frac{\partial}{\partial z}\,S\left(\mu - \frac{V^2}{D}\int_0^\mu \rho\;d\rho\int_0^\rho \frac{1-\nu^2}{\phi\{\nu\}}\,d\nu\right), \qquad (42)$$

vanishes at $\mu = \pm 1$. When the derivative $(\partial F_2/\partial\mu)$ given by equation (42) is integrated over μ, the resulting expression for the even anisotropy can be written in a form analogous to equation (31)

$$F_2 = -\,h\{\mu\}\,\frac{\partial S}{\partial z} = h\{\mu\}\,\frac{\partial F_o}{\partial t} \qquad (43)$$

where the function $h\{\mu\}$ is given by

$$h\{\mu\} = 2\int_0^\mu \frac{\nu\;d\nu}{\phi\{\nu\}} - \frac{V^2}{D}\int_0^\mu \frac{d\nu}{\phi\{\nu\}}\int_0^\nu \frac{(\nu^2-\sigma^2)(1-\sigma^2)}{\phi\{\sigma\}}\,d\sigma - C \qquad (44)$$

in which the constant of integration C is chosen in such a way that $h\{\mu\}$ satisfies equation (15). If ϕ is described by equation (1),

$$h\{\mu\} = \frac{2}{A}\int_0^\mu \frac{\nu^{2-q} - \nu^{5-2q}}{1-\nu^2}\,d\nu - C \qquad (45)$$

where C can be expressed in terms of the digamma function ψ (Abromowitz and Stegun 1964, p. 267),

$$C = \frac{2}{A}\int_0^1 \frac{\nu^{2-q} - \nu^{5-2q}}{1+\nu}\,d\nu = \frac{1}{A}\left[\psi\{\tfrac{4-q}{2}\} + \psi\{3-q\} - \psi\{\tfrac{3-q}{2}\} - \psi\{\tfrac{7-2q}{2}\}\right]. \qquad (46)$$

Equation (43) expresses the important result that the even anisotropy is proportional to the gradient of the flux. With the aid of equation (24), this relationship can also be put in terms of the temporal derivative of the density.

An appropriate measure of the magnitude of F_2 relative to F_1 is the ratio of anisotropies δ_2/δ_1 calculated as in equation (33) for each component separately

$$\frac{\delta_2}{\delta_1} = \frac{(h\{1\} - h\{0\})}{g\{1\}} \frac{\frac{\partial S}{\partial z}}{\frac{\partial F_o}{\partial z}} = \frac{(h\{1\} - h\{0\})}{g\{1\}} \frac{\frac{\partial F_o}{\partial t}}{\frac{\partial F_o}{\partial z}}$$

$$= \frac{D(h\{1\} - h\{0\})}{g\{1\}} \frac{1}{L_S} = \left[\frac{\psi\{3-q\} - \psi\{(3-q)/2\}}{(4-q)} \frac{V}{A}\right] \frac{1}{L_S} \qquad (47)$$

where $L_S = S/(\partial S/\partial z)$. Here, the last expression states that the ratio of anisotropies is the ratio of a characteristic length for scattering to the scale length for spatial variations of the flux L_S. Because F_1 is smaller than F_o by a comparable ratio (See eq. [33].), this implies, under most conditions, that F_2 is very small compared to F_o. However, in the coherent limit $D \to \infty$, F_o and F_1 are comparable. Under these circumstances, F_2 plays an important role that was discussed in Paper II.

To calculate F_1, substitute equation (43) into equation (23) and assume

$$\frac{\partial F_1}{\partial t} = g\{\mu\} \frac{1}{D} \frac{\partial S}{\partial t}$$

to obtain after two integrations

$$F_1 = -\int_0^{\mu} \frac{2\, d\nu}{\phi\{\nu\}} \left(\frac{1}{2} (1-\nu^2) V \frac{\partial F_o}{\partial z} + \frac{1}{D} \frac{\partial S}{\partial t} \int_{\nu}^{1} g\{\rho\}\, d\rho - V \frac{\partial^2 S}{\partial z^2} \int_{\nu}^{1} \rho h\{\rho\}\, d\rho \right). \qquad (48)$$

The flux S calculated from this expression satisfies

$$-\lambda^2 \frac{\partial^2 S}{\partial z^2} + \tau \frac{\partial S}{\partial t} + S = - D \frac{\partial F_o}{\partial z} \qquad (49)$$

where τ and λ are, respectively,

a characteristic time and a characteristic length that will be specified below. Equation (49) and

$$\frac{\partial F_o}{\partial t} = - \frac{\partial S}{\partial z}, \qquad (50)$$

which is equation (24) restated here for emphasis, are coupled partial differential equations in F_o and S that take the place of the diffusion equation. Evidently, the corrections for the streaming and temporal change of anisotropies are embodied in the first two terms in equation (49), for it reduces to the diffusive form (eq. [28]) when they can be neglected.

The characteristic time τ obtained by performing on the second term of equation (48) the integration over μ specified by equation (25) is

$$\tau = \frac{V^2}{D}\int_0^1 \frac{(1-\nu^2)\,d\nu}{\phi\{\nu\}}\int_\nu^1 g\{\rho\}\,d\rho = \frac{V^2}{D}\int_0^1 d\rho\left(\int_0^\rho \frac{1-\nu^2}{\phi\{\nu\}}\,d\nu\right)^2 = \frac{(4-q)}{A(5-2q)(2-q)} \quad (51)$$

where the first equality results from the same interchange in the order of integration over μ and ν that was invoked in equation (29) and where the second equality involves a further change in the order of integration over ν and ρ together with the specification of $g\{\rho\}$ by equation (32). Similarly, the characteristic length λ is given by,

$$\lambda^2 = V^2\int_0^1 \frac{(1-\nu^2)d\nu}{\phi\{\nu\}}\int_\nu^1 \rho h\{\rho\}d\rho$$

$$= \frac{V^2}{A^2}\frac{1}{(2-q)(4-q)}\left[2\,\psi\{\tfrac{1}{2}(7-2q)\} - \psi\{\tfrac{1}{2}(4-q)\} - \psi\{\tfrac{1}{2}(10-3q)\}\right]. \quad (52)$$

The two new transport parameters, τ and λ^2, exhibit the same dependence on $(2-q)^{-1}$ that leads to the divergence of D at q = 2. (See eq. [29].)

The first correction term in equation (49) can be evaluated in terms of F_o with the aid of equation (50) while the second term can be evaluated with the aid of equation (28) provided that the density changes slowly. Thus, equation (49) reduces to

$$S \approx -D\frac{\partial F_o}{\partial z} + (\tau D - \lambda^2)\frac{\partial^2 F_o}{\partial z\partial t} \quad (53)$$

in which the relative magnitudes of the corrections for the even anisotropy and temporal changes are respectively proportional to λ^2 and τD.

In Figure 1, where these parameters are plotted as a function of q, the quantity

$$\tau D = \frac{(V^2/A)}{(5-2q)(2-q)^2}, \qquad (54)$$

which is 3.75 times larger than λ^2 at q = 1, becomes infinitely larger at q = 2. Because of this divergence, the temporal effect dominates over the streaming correction throughout the range $1 < q < 2$. Thus, it is appropriate to consider the limit $\lambda^2 << \tau D$ in which equations (49) and (50) reduce to the telegrapher's equation

$$\frac{\partial^2}{\partial z^2}F_o - \frac{\tau}{D}\frac{\partial^2}{\partial t^2}F_o = \frac{1}{D}\frac{\partial F_o}{\partial t} \qquad (55)$$

where the parameter

$$(D/\tau)^{\frac{1}{2}} = V\,\frac{(5-2q)^{\frac{1}{2}}}{(4-q)} \qquad (56)$$

is a characteristic velocity for the coherent propagation of density inhomogeneities. This parameter is the same as the velocity V_{01} (See table II-1.) that appeared in the telegrapher's equation obtained in Papers I and II. Moreover, it was demonstrated in §II that the coefficient of diffusion appearing here is only slightly different from the one obtained in Paper I. Thus, if the term proportional to λ^2 is neglected, the implications of equations (49) and (50) are qualitatively identical and quantitatively similar to those brought out in Paper II - §III under the assumption that only the components f_o and f_1 need be considered in the eigenfunction expansion of f.

To make clear the significance of λ, equation (49) can be solved for S to yield an expression,

$$S = -\int_{-\infty}^{t} dt_o \int_{-\infty}^{+\infty} dz_o \left[\frac{\exp\left\{ -\frac{(t-t_o)}{\tau} - \frac{(z-z_o)^2\tau}{4\lambda^2(t-t_o)} \right\}}{[4\pi\lambda^2\tau(t-t_o)]^{\frac{1}{2}}} \right] \left(D \frac{\partial F_o}{\partial z} \right)_{z_o t_o} \qquad (57)$$

in which the Green's function within the brackets describes the effect at (z, t) of a localized, impulsive source of flux at (z_o, t_o) and in which the quantity $D(\partial F_o/\partial z)$, evaluated at (z_o, t_o), plays the role of a source function. Because an integral equation for F_o is obtained when this expression for S is substituted in equation (50), this reformulation does not simplify the problem. However, equation (57) leads to the conceptually significant implication that S is influenced not by local conditions but instead by conditions over a finite temporal interval characterized by τ and over a finite spatial interval characterized by λ. Thus, the present approach embodies a non-local quality of the transport equations similar to the attribute emphasized by Klimas and Sandri (1973). There is a temptation to identify the spatial dispersion implied by this characteristic with the dispersive spreading of coherent pulses that was discussed in Paper II - §IV. However, the latter phenomenon includes a dispersive effect which arises because the dependence of F_1 on μ differs from that of $(\partial F_1/\partial t)$. In Paper II, this difference was taken into account approximately by allowing the evolution of the third odd component f_3 to be different from that of the first odd component f_1. Because equation (57) takes into account neither this effect nor temporal variations of F_2, the method of eigenfunctions is to be preferred for the analysis of problems which involve rapid temporal variations.

On the other hand, a useful description of the steady state is obtained when the integration over t_o in equation (57) is carried out with the source function assumed to be independent of t_o,

$$S = -\int_{-\infty}^{+\infty} \frac{dz_o}{2\lambda} \exp\{-|z_o - z|/\lambda\} \left(D\frac{\partial F_o}{\partial z} \right)_{z_o} . \quad (58)$$

This expression, with S held constant by virtue of equation (50), is an integral equation for F_o that describes more accurately than does the diffusion equation the steady state spatial density profile in the vicinity of localized sources or of abrupt changes in D. The analysis of such situations is a major concern of classical transport theory, but this objective is less important in the present context than the analysis of transients.

The diffusion equation is an approximation that applies only when temporal and spatial variations of the flux are not pronounced. If significant changes occur within one characteristic time τ, then the telegrapher's equation (eq. [55]) describes a generalization of diffusion that incorporates localized disturbances propagating coherently but without dispersion. The dispersive spreading of these disturbances involves complex behavior of the anisotropy that is best described in terms of eigenfunctions. When temporal changes are gradual but significant variations occur over a distance λ, then the appropriate generalization of the diffusion equation is equation (58) treated as an integral equation for the density.

IV. THE COEFFICIENT OF DIFFUSION

Of the three transport parameters, the coefficient D is the most important because it appears by itself in the diffusive approximation to the flux. The parameters τ and λ appear only in the correction terms of a higher order approximation, but even here D plays an important role. Thus, equation (29) takes on special significance as the unique relationship between D and the Fokker-Planck coefficient ϕ. Because this relationship involves only a single integration over μ, the coefficient D is readily evaluated provided that the integral exists.

The divergence of the integral when ϕ is given by equation (1) is crucially related to the behavior of the Fokker Planck coefficient near $\mu = 0$. Specifically, when both the coefficient and its first derivative vanish at $\mu = 0$, then D is infinite. These conditions occur when $q > 2$. For $1 < q < 2$, ϕ has a cusp at $\mu = 0$ in which the coefficient vanishes, but D is finite nevertheless. For $q < 1$, D is well defined even though ϕ is infinite at $\mu = 0$.

The coefficient D is always finite if ϕ is given by equation (3). Thus, the qualitative effect of scattering at $\mu = 0$ is to eliminate the divergence of D. In this section, the detailed nature of this effect is illustrated by analytic expressions which give D as a function of the parameters $\beta = (H/A)$ and q.

Numerical integration of equation (29) is the most generally applicable method of computing D. In fact, it is the only practical method when the dependence of ϕ upon μ is very complicated as in the results of

Volk, et al (1974). Thus, the expressions presented here, although they apply to a specialized form of ϕ, may be of value as an easily implemented test of computer routines.

For $q > 1$, the parameter H is the Fokker Planck coefficient at $\mu = 0$. The theories of Jones, Kaiser and Birmingham (1973) and of Volk (1974) predict the numerical magnitude of this quantity, but the former also predicts a dependence of $<\Delta\mu^2>/\Delta t$ upon μ that can be specified in terms of Bessel functions while the latter approximates the dependence by introducing a critical value of μ below which the coefficient is constant and above which the quasilinear result applies. Both of these dependences are described with sufficient precision by equation (3) when $H << A$. However the inaccuracies that occur when this condition is not satisfied are of minor significance compared to those resulting from the neglect of scattering at $\mu = 0$. Consequently, to provide   insight, the analysis will be carried out without regard to this limitation. Moreover, the form of equation (3) is convenient because the eigenfunctions reduce to Legendre polynomials when $H >> A$. In this limit, the quasilinear term can be regarded as a small perturbation of isotropic scattering. Note that the divergence in D would also be removed by a perturbation with zero value but finite slope at the origin.

When equation (3) is substituted for ϕ, equation (29) becomes

$$D = \frac{V^2}{A}\,\frac{1}{2}\int_0^1 \frac{1-\nu^2}{\beta + \nu^{q-1}}\,d\nu = \frac{V^2}{A}\,\frac{1}{2(q-1)}\int_0^1 \frac{x^{(2-q)/(q-1)} - x^{(4-q)/(q-1)}}{x+\beta}\,dx \qquad (59)$$

where $x = \nu^{q-1}$. Evidently, D is given by a dimensionless function of β multiplied by the parameter (V^2/A) specified by equation (II-86). Although a general expression for the integral over x can be written in terms of hypergeometric functions (Gradshteyn and Ryzhik 1965, p. 285, eq. [3.194-5]), it is more appropriate in the present context to consider the expressions that apply when q takes on certain specific values for which the integral reduces to elementary functions. These expressions are given in table 1 for five values of q. In the limit $\beta >> 1$ where isotropic scattering is dominant, the following series expansion in β^{-1}, obtained from equation (59) by representing the denominator as a geometric series, is applicable:

$$\frac{DA}{V^2} = \frac{1}{3\beta} - \frac{1}{q(q+2)}\frac{1}{\beta^2} + \frac{1}{(4q^2-1)}\frac{1}{\beta^3} + \cdots \quad . \tag{60}$$

Here, the first term embodies the limit

$$D = \frac{V^2}{3H} \tag{61}$$

that applies in the case of pure isotropic scattering. Limiting forms for $\beta \to 0$ are readily obtained for specific values of q, but they are not significantly less complicated than the formulas in table 1.

In Figure 2, the dimensionless quantity (DA/V^2) is plotted as a function of β. Here, the curves for q = 1.5 (also considered by Owens [1974]) and q = 1.75 approach, in the limit $\beta \to 0$, the finite values of (DA/V^2) predicted by equation (29) which are 0.80 and 1.78, respectively. In contrast, the curves for q = 2 and q = 3 approach infinity as $\beta \to 0$. Thus, in this limit, the coefficient of diffusion depends, for $q \geq 2$, upon both H and A whereas it depends, for $q < 2$, only upon A. All of the curves in Figure 1 converge in the limit $\beta >> 1$ to the dashed line specified by equation (61). In particular, this line represents a limit in which, for $q >> 2$, D does not depend upon A. In this limit where the magnetic power spectrum is very steep, the quasilinear term plays a minor role and the formula for isotropic scattering applies. In any case, equation (61) gives an absolute upper limit on the value of D.

Direct observations of interplanetary magnetic power spectra (Siscoe et al. 1968; Sari and Ness 1969), suggest spectral indices in the range $1.5 < q < 2.$ and give fairly accurate values for the parameter A. Unfortunately, while existing theories agree that scattering occurs at $\mu = 0$, they disagree sharply on the predicted magnitude of β. On the one hand, Jones, Kaiser and Birmingham (1973) give

$$\beta = \left(\frac{8\alpha}{\pi}\right)^{\frac{1}{2}} \frac{<\delta B^2>}{B} = 1.6 \left(\frac{<\delta B^2>}{B^2}\right)^{\frac{1}{2}} \gtrsim 0.2 \qquad (62)$$

where $\alpha = 1$ and where δB^2 is the mean square amplitude of perpendicular fluctuations of the field. This result puts (DA/V^2) in the center of Figure 2 where the coefficient of diffusion is sensitively dependent upon both β and A. On the other hand, it follows from the work of Volk (1973, eq. [52]) that

$$\beta = \left[\frac{27}{8(3q-1)}\right]^{2/(3-q)} \left[\frac{2\pi(q-1)}{3} \frac{<\Delta B^2>}{B^2}\right]^{(q-1)/(3-q)} = .95 \frac{<\Delta B^2>}{B^2} \qquad (63)$$

where $<\Delta B^2>$ is the mean square amplitude of fluctuations above the resonant wavenumber $k_{res} = (r_L)^{-1}$ and where the second expression involving a numerical coefficient applies to the same case, $q = 2$, considered by Jones et al. These two expressions have very different implications. In the first place, their coefficients and exponents are such that for a given fractional amplitude, which would typically be less than one, the value of β predicted by the latter expression is smaller than that predicted by the former one. In the second place, the fractional amplitude appearing in equation (63) is comparatively small because it represents an integration over a limited range of wavenumbers. Because of these considerations, the formulation of Volk gives smaller values of β and larger coefficients of diffusion than the calculation of Jones et al. gives. In particular, the coherent propagation of kilovolt solar electrons, whose scattering is controlled by a relatively steep portion of the magnetic power spectrum and whose resonant wavenumbers are large, could occur, under favorable conditions, according to equation (63) but is virtually ruled out by equation (62). (See Paper II for a discussion of this coherent mode which may occur when D is very large. Lin [1974] has summarized the observations of coherent effects.) On the other hand, in the example considered by Volk (1973), the propagation of 100 MeV protons is essentially diffusive. Evidently the discrepancies in the evaluation of β must be resolved before these issues can be clarified.

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FIGURE CAPTIONS

Figure 1. The correction to the flux arising from temporal changes of the odd anisotropy is proportional to τD. The correction arising from gradients of the even anisotropy is proportional to λ^2. For $q > -0.7$, the first of these corrections is larger than the second.

Figure 2. As the spectral index q is increased, the dimensionless parameter (DA/V^2) becomes more sensitively dependent upon $\beta = (H/A)$.

TABLE 1

FORMULAS FOR THE COEFFICIENT OF DIFFUSION

q	(DA/V^2)
1	$\frac{1}{3(\beta+1)}$
(3/2)	$\frac{4}{5}+\frac{\beta}{4}-\frac{\beta^2}{3}+\frac{\beta^3}{2}-\beta^4+\beta(\beta^4-1)\ \ln\left\lvert\frac{\beta+1}{\beta}\right\rvert$
(7/4) $\beta>(1/8)$	$\frac{16}{9}+\frac{\beta}{3}-\frac{2\beta^2}{3}+\frac{2}{3}\beta^3\ln\left\lvert\frac{\beta+1}{\beta}\right\rvert-\frac{\beta^{1/3}}{3}\ln\left\{\frac{(1+\beta^{1/3})^2}{1-\beta^{1/3}+\beta^{2/3}}\right\}-\frac{2\beta^{1/3}}{3^{1/2}}\ \arctan\left\{\frac{3^{1/2}}{2\beta^{1/3}-1}\right\}$
(7/4) $\beta<(1/8)$	$\frac{16}{9}+\frac{\beta}{3}-\frac{2\beta^2}{3}+\frac{2}{3}\beta^3\ln\left\lvert\frac{\beta+1}{\beta}\right\rvert-\frac{\beta^{1/3}}{3}\ln\left\{\frac{(1+\beta^{1/3})^2}{1-\beta^{1/3}+\beta^{2/3}}\right\}-\frac{2\beta^{1/3}}{3^{1/2}}\left[\pi-\arctan\left\{\frac{3^{1/2}}{1-2\beta^{1/3}}\right\}\right]$
2	$\frac{1}{2}(\beta-\frac{1}{2})+\frac{1}{2}(1-\beta^2)\ \ln\left\lvert\frac{\beta+1}{\beta}\right\rvert$
3	$-\frac{1}{2}+\frac{\beta^{1/2}}{2}\left(\frac{\beta+1}{\beta}\right)\ \arctan\ \{\beta^{-1/2}\}$

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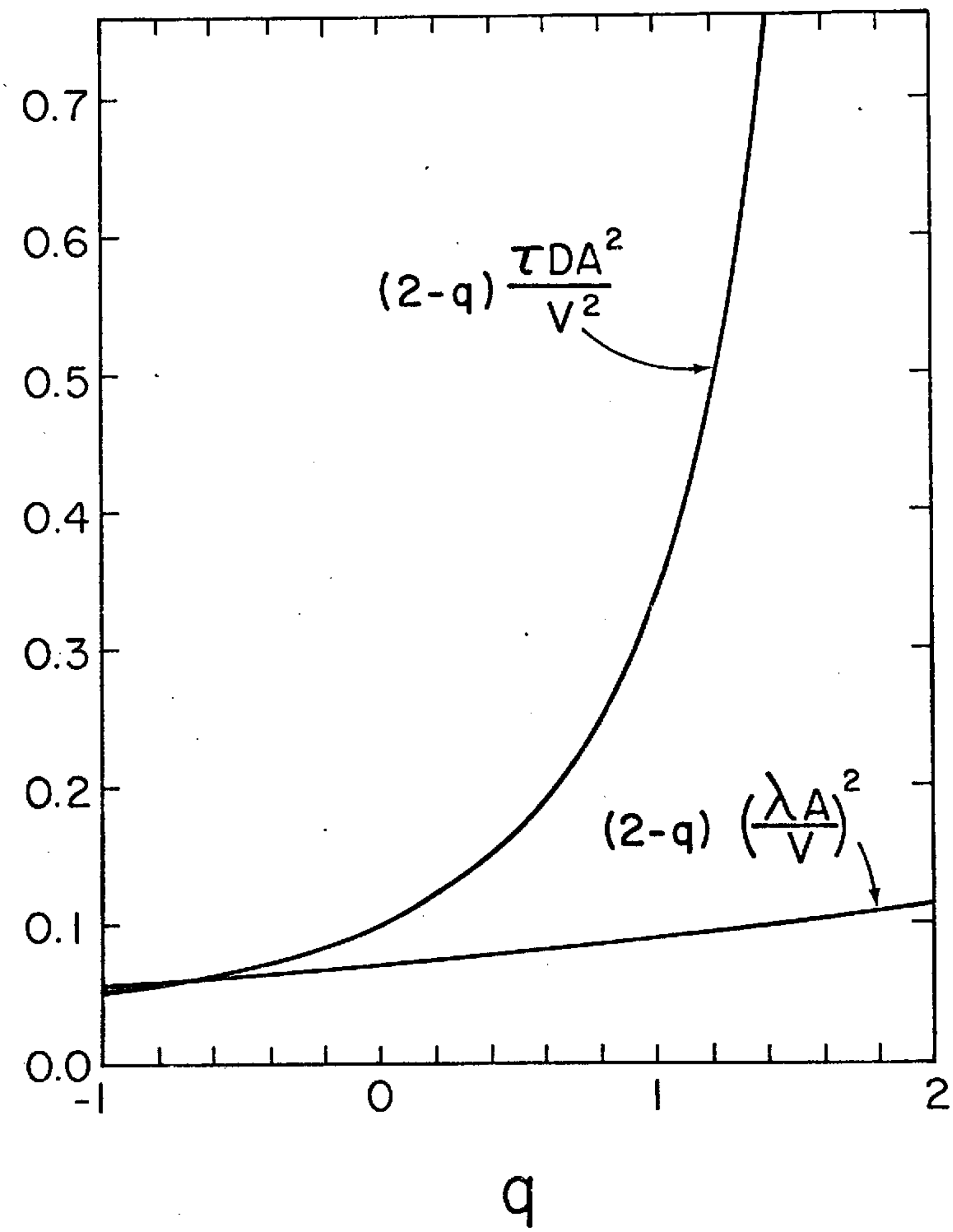

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-1
0
1
2
q
$(2-q)\,\frac{\tau D A^2}{V^2}$
$(2-q)\left(\frac{\lambda A}{V}\right)^2$

FIGURE 1

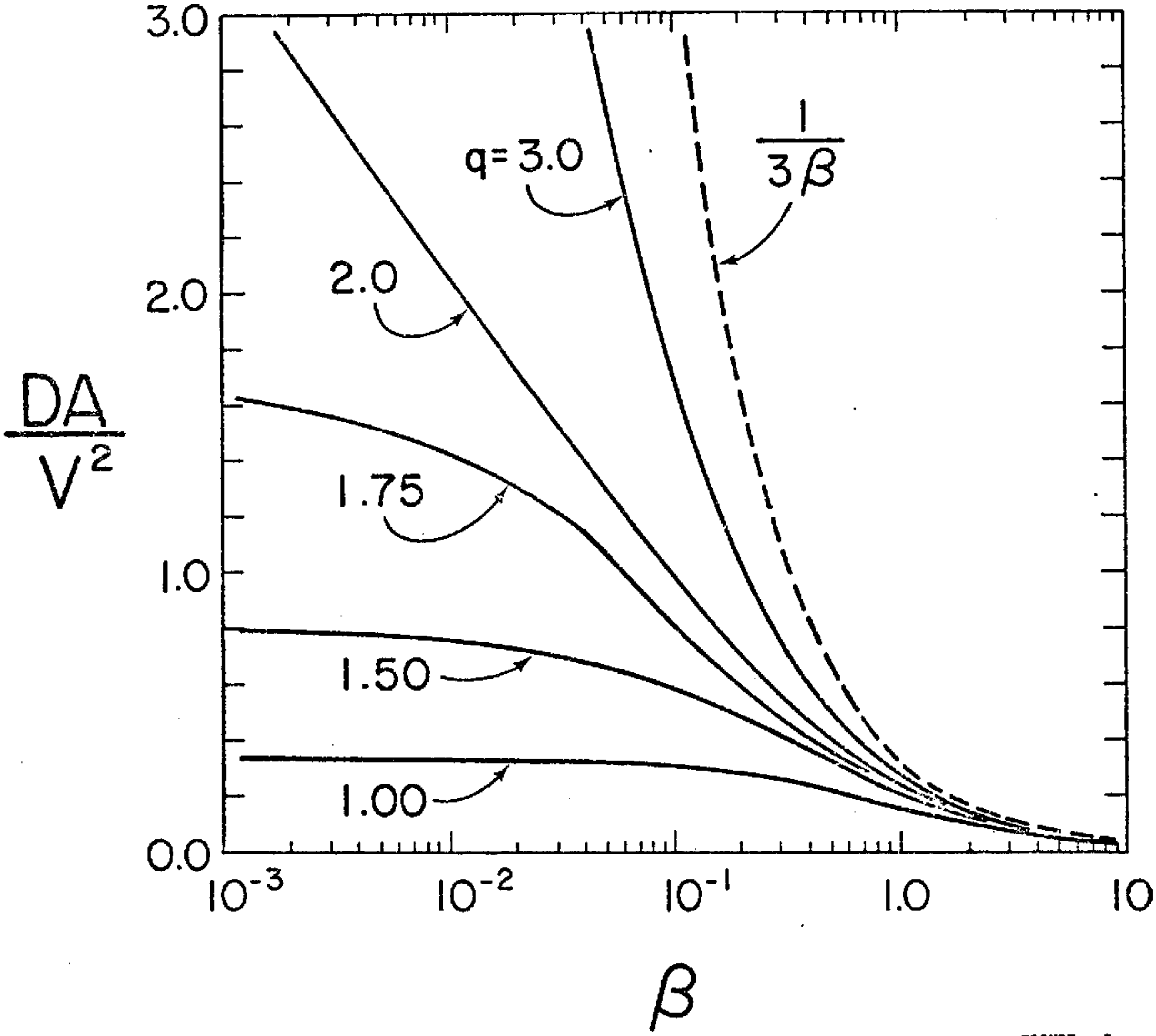

FIGURE 2