LIMITING PERFORMANCE OF DYNAMIC SYSTEMS SUBJECT TO RANDOM INPUTS

Final Report on

NASA Grant No. NGR 47-005-145

Supplement No. 2

Submitted by:

Walter D. Pilkey
and
Bo Ping Wang

RESEARCH LABORATORIES FOR THE ENGINEERING SCIENCES

SCHOOL OF ENGINEERING AND APPLIED SCIENCE

UNIVERSITY OF VIRGINIA

CHARLOTTESVILLE

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FOREWORD

The research described in this report (University of Virginia Project No. 4085-2320) was performed at the University of Virginia Research Laboratories for the Engineering Sciences. W. D. Pilkey was the principal investigator. The project was performed under NASA Grant No. NGR47-005-145, Supplement Number 2 and administered at NASA Langley Research Center. This report covers some of the work performed during the third year of this grant. The valuable assistance of the technical monitors for this study, J. Sewall and R. Parrish of NASA/Langley, is appreciated.

ABSTRACT

The problem of determining the limiting performance characteristics of mechanical systems subject to random input is studied. A review is presented of the classical work in the optimal design of stochastic systems. Some recent results of stochastic optimal control theory are employed. The solution to the limiting performance problem is formulated in both the frequency and time domains. Both formulations require substantial, burdensome computations when applied to large scale systems.

1. INTRODUCTION

Random disturbances appear as complicated time-varying functions that may exhibit wide irregular variations in amplitude and frequency. Both the input disturbance and the system response must be given statistical characterization and, as expected, this complicates the optimum design problem. No all encompassing methodologies are available for optimizing realistic systems under general random environments. Since related literature from control theory on the optimization of stochastic processes is quite extensive, we will take advantage of these developments and use these techniques, both in the frequency domain and in the time domain, as a basis for this study of limiting performance of dynamic systems subject to certain classes of random inputs. The limiting performance concept for transient systems is defined in Ref. 1. Briefly, it is the theoretically optimum performance of a system, regardless of configuration.

In this study the input disturbances are characterized by spectral density matrices or correlation matrices. Although for much of this report the performance index is quadratic in both responses and control forces, other choices are possible. Most of this study is confined to linear systems.

This study begins with a review of the single-input, single-output Wiener-Hopf type approach which has been examined by others (Ref. 2,3). The governing Wiener-Hopf equation and its solution are summerized. An example is presented. Next, it is shown how the limiting performance of a multi-degree of freedom (MDF) system under random input can be determined by the extension of Wiener-Hopf techniques to the multiple-input, multiple-output case reported by Weston, et. al., (Ref. 4).

Parallel to the above frequency domain approach, the problem in the time domain will also be formulated by applying results from stochastic optimal control theory (Ref. 5). This approach has also been pointed out recently by Karnopp (Ref. 6). A single degree of freedom (SDF) problem which was previously solved by frequency domain techniques will be solved by this approach.

Also, included in this report will be dynamic programming approaches. This method is more general in the sense that it can treat performance indexes other than a mean-square type. A formulation with quadratic criteria is summarized and a non-quadratic possibility proposed.

1.1 Statement of Problem

Consider a MDF dynamical system described by

$$\underline{M}\ddot{\bar{x}} + \underline{C}\dot{\bar{x}} + \underline{K}\bar{x} + \underline{V}\bar{u} = \underline{F}\bar{f} \tag{1}$$

where

$\underline{M}$ - n x n mass matrix

$\underline{C}$ - n x n damping matrix

$\underline{K}$ - n x n stiffness matrix

$\underline{V}$ - n x nu coefficient matrix of control forces

$\underline{F}$ - n x nf coefficient matrix of disturbances

$\bar{x}$ - n x 1 state vector

$\bar{f}$ - nf x 1 disturbance vector

$\bar{u}$ - nu x 1 generic or control force vector

n - number of degree-of-freedom of the system

nu - number of controllers of the system

nf - number of disturbances (forcing functions) of the system

In Eq. (1) $\bar{f}$ is a random vector of known statistics. In order to make a limiting performance study, portions of the system have been replaced by $\bar{u}$ a vector of generic or control forces These forces can represent any sub-system. The remainder of the system must be linear as must be the system kinematics.

Since the forcing function is random in nature, the limiting performance problem must be defined in some statistical sense. The most common optimization criterion is to minimize some expected mean square response quantity while imposing constraints on other expected mean square responses. This criterion, though not as direct as the criterion used in the transient case is still meaningful. It is selected because in the analysis of stochastic processes, it is difficult to find responses other than the mean square response of the system. Another reason for using mean square criteria is that when the performance index and constraints are combined in a penalty function type objective function, the resulting optimum system will be linear. This is a well known result of optimal control theory (Ref. 5). Other criteria have been suggested by several authors. Aoki (Ref. 7) treated a simple problem based on a performance index of a maximum expected deviation over the history of a dynamic process. Trikha and Karnopp (Ref. 8) studied a single degree of freedom system with a criterion based on the values of displacement and acceleration, whose probability of being exceeded is less than a prescribed value. It does not appear as though either of these criteria can be applied to large systems.

2. REVIEW OF CLASSICAL WORK

Limiting performance characteristics of SDF systems have been reported in the literature for several inputs and are based on either expected mean square values or the probability of exceeding selected response levels. These results are summarized in Ref. 1. To make this report more self-contained, the limiting performance problem based on expected mean square responses will be treated briefly.

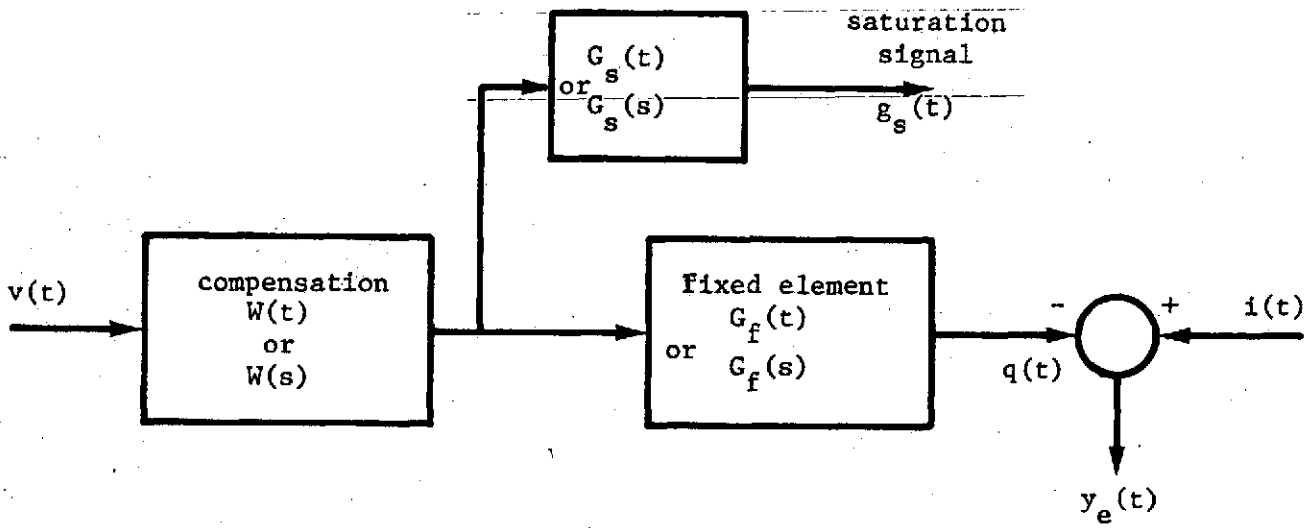
2.1 The Governing Integral Equation

Consider the configuration of Fig. 1. For a given input spectral density ϕ_{rr} the optimum compensation $W_c(t)$ (or $W_c(s)$ in the frequency domain) is to be found such that $E[y_e^2(t)]$ is minimized subject to the constraint $E[q_s^2]\leq\sigma$, where σ is the maximum allowable mean square value of $q_s(t)$ and $E[\cdot]$ stands for mathematical expectation. This constrained optimization problem will be converted to an unconstrained problem by introducing a Lagrangian multiplier ρ. The problem then becomes one of minimizing

$$F = E[y_e^2] + \rho E[q_s^2] \tag{2}$$

It is shown in Ref. 9 that the optimum compensation $W_{cm}(t)$ that minimizes (2) is governed by the integral equation

$$\int_{-\infty}^{\infty} W_{cm}(t_3)\Delta(t_1-t_3)dt_3 - \Gamma(t_1) = 0 \quad \text{for } t_1\geq 0 \tag{3}$$



v(t) = random input disturbance

q(t) = actual output

i(t) = ideal output

$y_e(t)$ = error

Fig. 1 Block diagram for deriving Wiener-Hopf integral equation.

where

$$\Gamma(t_1) = \int_{-\infty}^{\infty} g_f(t_2)\phi_{vi}(t_1+t_2)dt_2$$

$$\Delta(t_1-t_3) = [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_f(t_2)g_f(t_4)dt_2dt_4$$

$$+ \rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_s(t_2)q_s(t_4)dt_2dt_4]\cdot\phi_{vv}(t_1+t_2-t_3-t_4)$$

ϕ_{vi} = cross correlation function of input to ideal output

ϕ_{vv} = autocorrelation function of input

2.2 Solution of the Governing Integral Equation

Equation (3) is of the Wiener-Hopf type for which spectral factorization can be applied to obtain an explicit solution formula in the frequency domain (Ref. 9). This is

$$W_{cm}(s) = \frac{\left[\frac{\Gamma(s)}{\Delta^-(s)}\right]_+ + \alpha}{\Delta^+(s)} \tag{4}$$

where α is an arbitrary constraint

$\Gamma(s) = 2\pi G_f(-s)\ \phi_{vi}(s)$

$\Delta(s) = 2\pi\ [G_f(s)G_f(-s) + \rho G_s(s)G_s(-s)]\ \phi_{vv}(s)$

$\Delta^+(s)$ = any factor of $\Delta(s)$ which includes all the poles and zeros in the LHP (left-half-plane)

$\Delta^-(s) = \Delta(s)/\Delta^+(s)$

$\left[\frac{\Gamma(s)}{\Delta^-(s)}\right]_+$ = component of $\frac{\Gamma(s)}{\Delta^-(s)}$ which has all its poles in the LHP such that $\frac{\Gamma(s)}{\Delta^-(s)} - \left[\frac{\Gamma(s)}{\Delta^-(s)}\right]_+$ has all its poles in the RHP.

2.3 Application to SDF System

The results of previous sections can be applied to find the limiting performance characteristics of a single degree system subject to random inputs. Consider the SDF system shown in Fig. 2. The problem is to find u such that the performance index $E[u^2] + \rho E[x^2]$ is minimized. By the present approach, we seek to find the optimum transfer function

$$W(s) = \frac{Z(s)}{Y(s)}$$

Draw the block diagram of Fig. 3. Equation (4) will provide W(s) for given input spectral density.

For an input with spectral density

$$\Phi_{yy}(s) = -\frac{A}{s^2}$$

The solution is

$$W(s) = \beta^2/(s^2 + \sqrt{2}\, s\beta + \beta^2)$$

where

$$\beta = \left(\frac{\rho}{m^2}\right)^{1/4}$$

2.4 Discussion

The classical Wiener-Hopf technique leads to tractible analytical solutions to single-input single-output systems. Since most systems do not fit in this class the usefulness of this technique is quite limited.

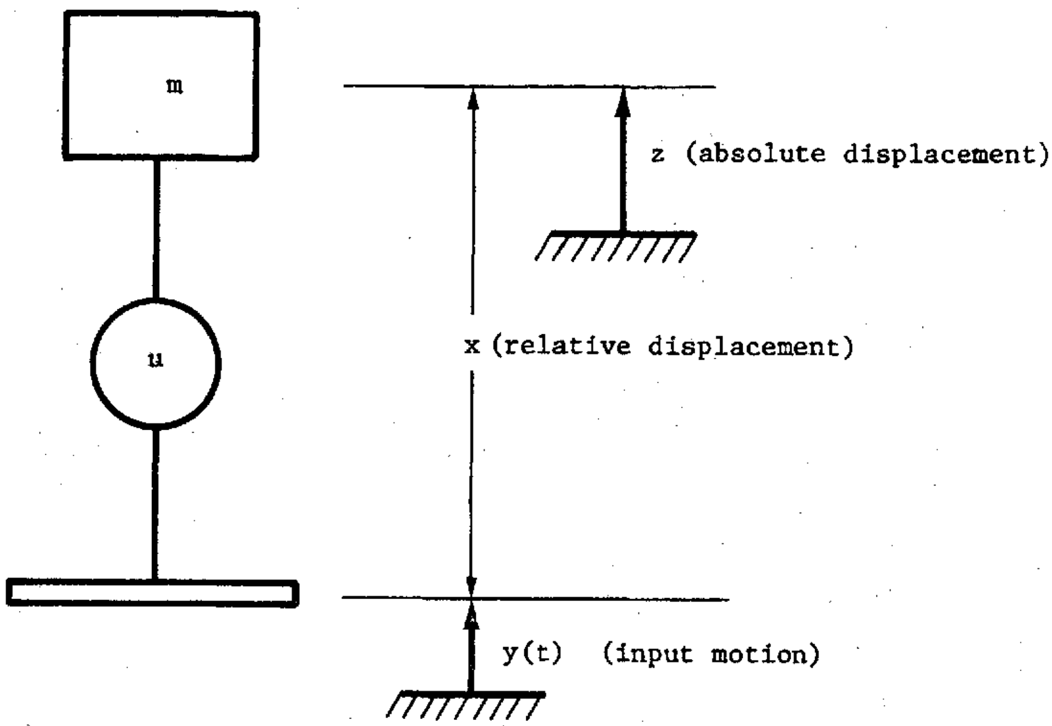
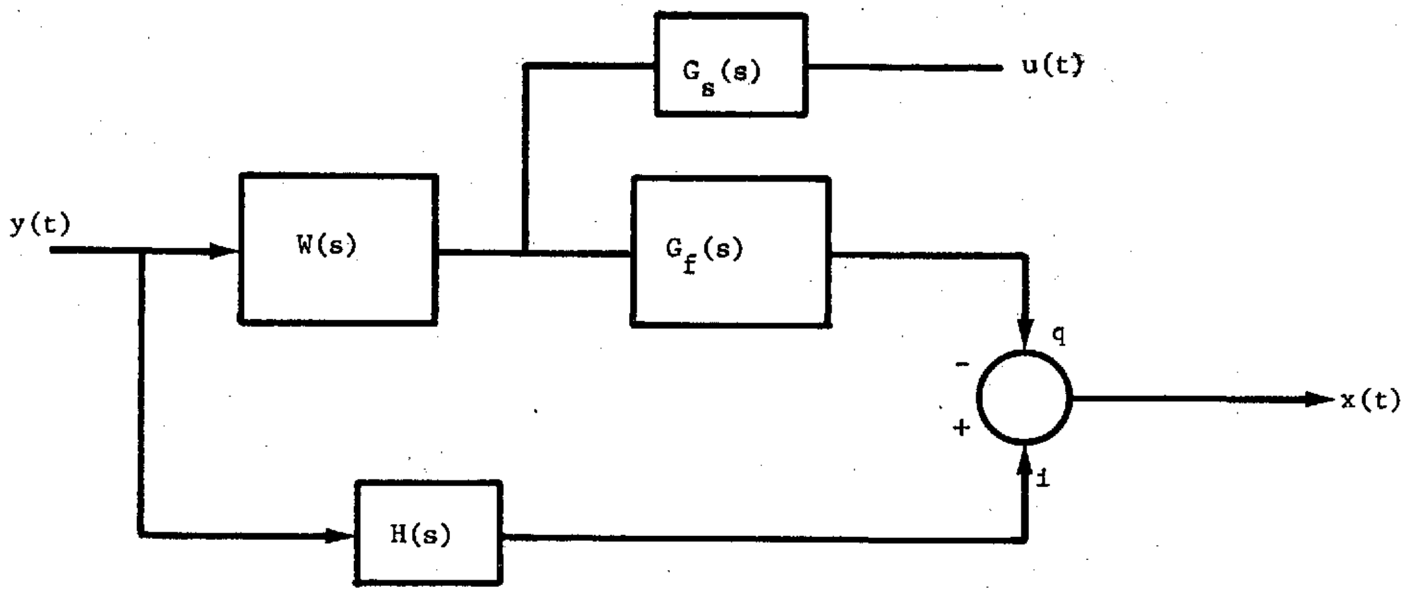


Fig. 2 SDF system subject to random input.



$G_s(s) = ms^2$ $G_f(s) = 1$ $H(s) = 1$

$W(s) = ?$

Fig. 3 Block diagram for the configuration of Fig. 2.

3. EXTENSION OF WIENER-HOPF TYPE APPROACH TO THE LIMITING PERFORMANCE OF MDF SYSTEMS

The analytical design techniques of Newton et al. (Ref. 9), have been applied to find the limiting performance characteristics of simple systems. These techniques are extended by Weston and Bongiorno (Ref. 4). It is the purpose of this section to summarize the principal results of this extended theory. Also, an example will be given to illustrate the application of this technique.

3.1 Formulation

Consider the system configuration shown in Fig. 4. All elements of the system are assumed to be linear and time invariant. The input vectors (signal, noise, and disturbance) are generated by a stationary stochastic process whose power spectral density is known. The purpose is to find optimum compensation to minimize a performance index consisting of a weighted sum of the output mean square error plus a weighted sum of the mean square value of a set of saturation signals. The results are summarized in Theorem 1.

The following nomenclature is used. $\bar{r}_1(t)$, $\bar{n}(t)$, $\bar{d}(t)$ are n x 1 vectors of input, noise and disturbance (to the measurements) respectively, whose elements are the realization of an independent, stationary random process. The statistics of the random processes $\bar{r}_1$, $\bar{n}$, $\bar{d}$ are assumed to be known and adequately described by rational power spectral-density matrices $\underline{\Phi}_{r_1r_1}(s)$, $\underline{\Phi}_{nn}(s)$, $\underline{\Phi}_{dd}(s)$, respectively. Transfer function matrices $\underline{G}_p(s)$, $\underline{G}_s(s)$ and $\underline{H}(s)$ represent asymptotically stable systems. The dimensions of these matrices are:

$\underline{G}_p$-n x m $(m \geq n)$, $\underline{G}_c$ -m x n, $\underline{H}$-n x n, $\underline{G}_s$ -q x m

q is the dimension of $\bar{A}(s)$, the satuation signals.

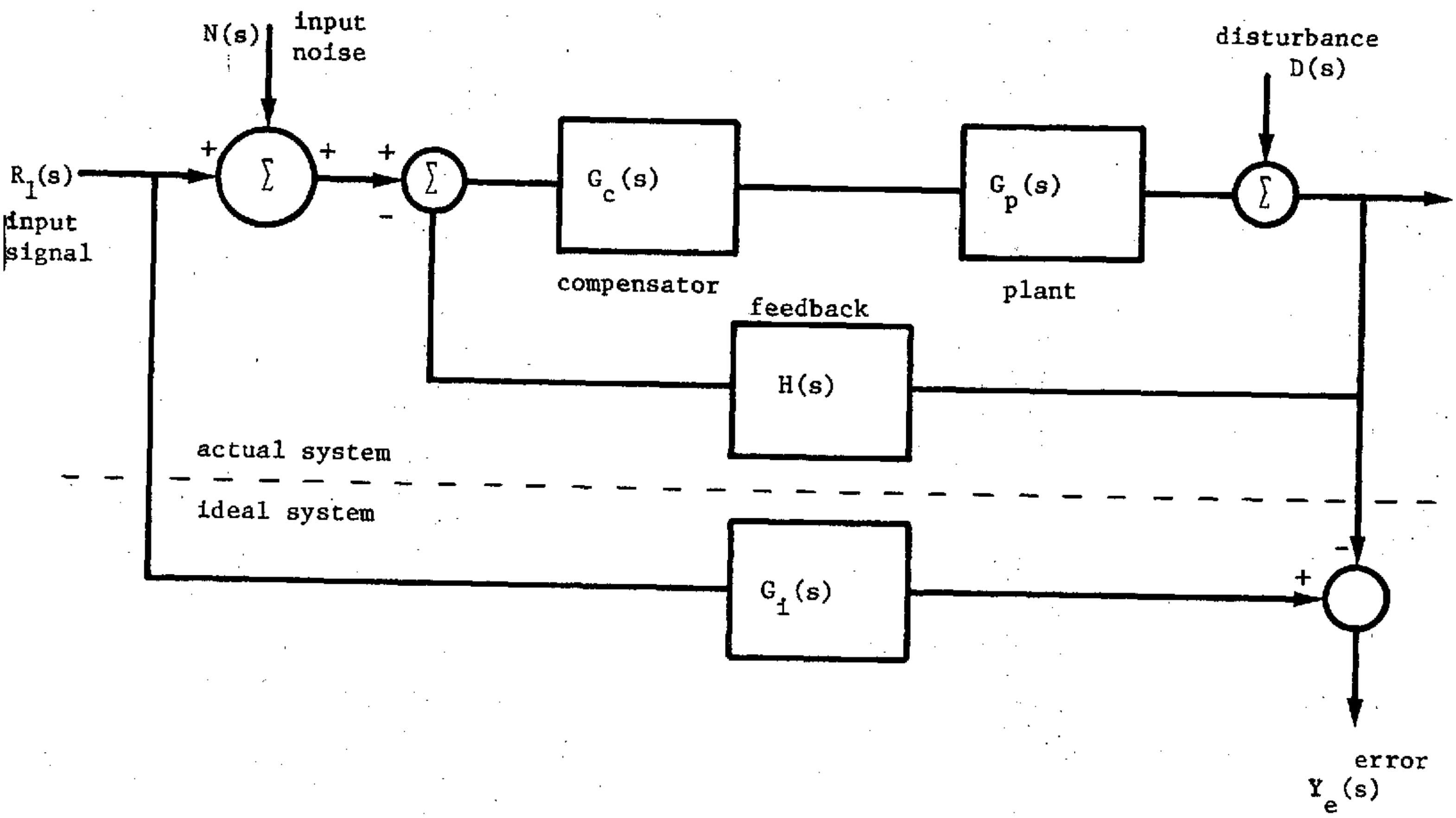


Fig. 4 Block diagram of multivariable system.

The performance index to be minimized is

$$J = E[\bar{y}_e^T Q \bar{y}_e + \bar{a}^T R \bar{a}] \tag{5}$$

where Q and R are real, constant, symmetric, positive, definite matrices. Q is n x n, R is q x q. The superscript T designates the transpose of a matrix. The problem was solved by a calculus of variation technique with the results summarized in Theorem 1.

Theorem 1. (Ref. 4)

The physically realizable $W_c = W_{c0}$ that minimizes

$$J = E[\bar{y}_e^T Q \bar{y}_e + \bar{a}_e^T R \bar{a}]$$

associated with the system in Fig. 4, satisfies

$$\Psi W_{c0} = [\Psi_*^{-1} A^T (\Omega_*^T)^{-1}]_+ [\Omega^T]^{-1} \tag{6}$$

where

$$\Omega_* \Omega = \Phi_{rr}$$

$$\Psi_* \Psi = G_{p*} Q G_p + G_{s*} R G_s \tag{7}$$

$$A = \Phi_{ri} Q G_{p*}^T$$

The optimal compensation is then

$$G_c = [I - W_{c0} H G_p]^{-1} W_{c0} \tag{8}$$

The following notational convention is used in Theorem 1:

(1) For real matrix X, $X_*(s) = X^T(-s)$

(2) $[Z(s)]_+$ = part of function $Z(s)$ which is analytic in $R_e s \geq 0$

3.2 Application

For any multi-input, multi-output system, the application of Theorem 1 leads to the optimum transfer matrix which minimizes the performance index of (5), provided a block diagram similar to Fig. 4 can be constructed for this system. We will illustrate this using a two degree of freedom system. This same problem was solved by Bender (Ref. 3) using the classical Wiener-Hopf approach.

Consider the system shown in Fig. 5. The input to this system has the following spectral density

$$\Phi_{xx}(s) = -\frac{A}{s^2}$$

This system is governed by the differential equations

$$\begin{aligned} M\ddot{y} &= u \\ m\ddot{z} &= -u + k(y-z) \end{aligned} \qquad (9)$$

The problem is to find the optimum isolator characteristics such that the relative displacement (z-y) is minimized while the acceleration $\ddot{z}$ is bounded. Since the input is random, a meaningful performance index will be of the mean square type

$$J = E[\rho\ddot{y}^2 + (z-y)^2] \qquad (10)$$

The problem will be solved in the frequency domain.

Define

$$W(s) = \ddot{Y}(s)/\ddot{X}(s) \qquad (11)$$

$$\delta(s) = Z(s) - Y(s) \qquad (12)$$

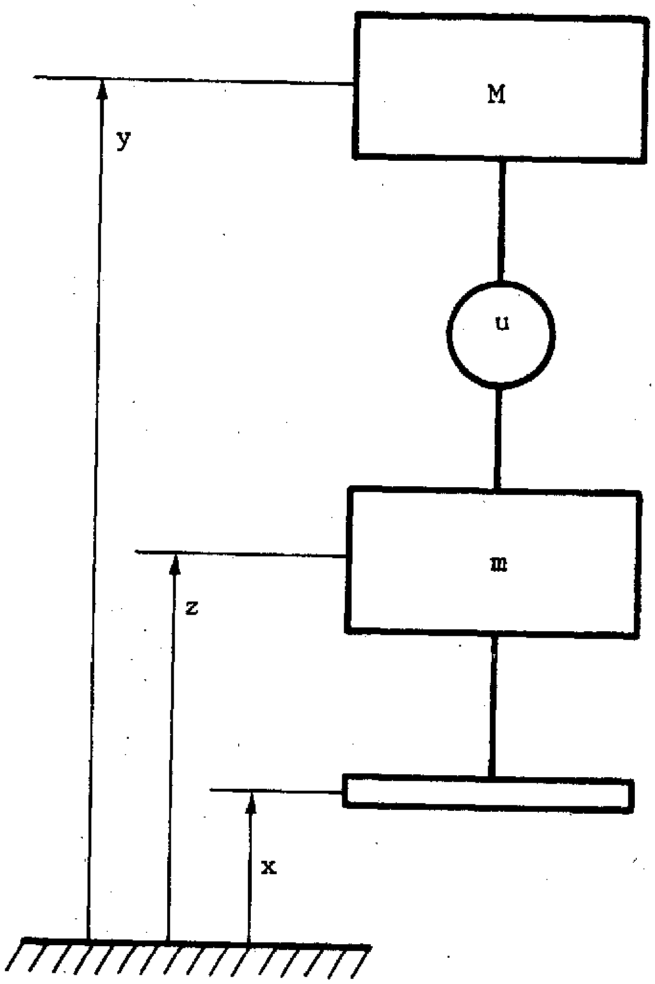


Fig. 5 Two degree of freedom vehicle model.

Transforming (9) into the frequency domain it is found that

$$Ms^2Y(s) = U(s)$$
$$ms^2Z(s) = -U(s) + K(X(s) - Z(s)) \qquad (13)$$

Solving (13) using definitions (11),

$$Z(s) = \frac{K-MW(s)}{K+s^2m} X(s) \qquad (14)$$

$$Y(s) = \frac{W(s)}{s^2} X(s) \qquad (15)$$

$$\delta(s) = Z(s) - Y(s)$$

$$= \left[\frac{1}{(\frac{s}{\omega_n})^2 + 1} - \frac{[(1+1/r)(s/\omega_n)^2 + 1]}{s^2(1+s/\omega_n)^2} W(s) \right] X(s)$$

where $\omega_n^2 = k/m$, $r = m/n$

From the block diagram of Fig. 6

$$\delta(s) = [H_2(s) - H_1(s) W(s)] \ddot{X}(s)$$

Compare the above two equations,

$$H_1(s) = \frac{1 + (1 + 1/r)(s/\omega_n)^2}{s^2(1 + (s/\omega_n)^2)} \qquad (16)$$

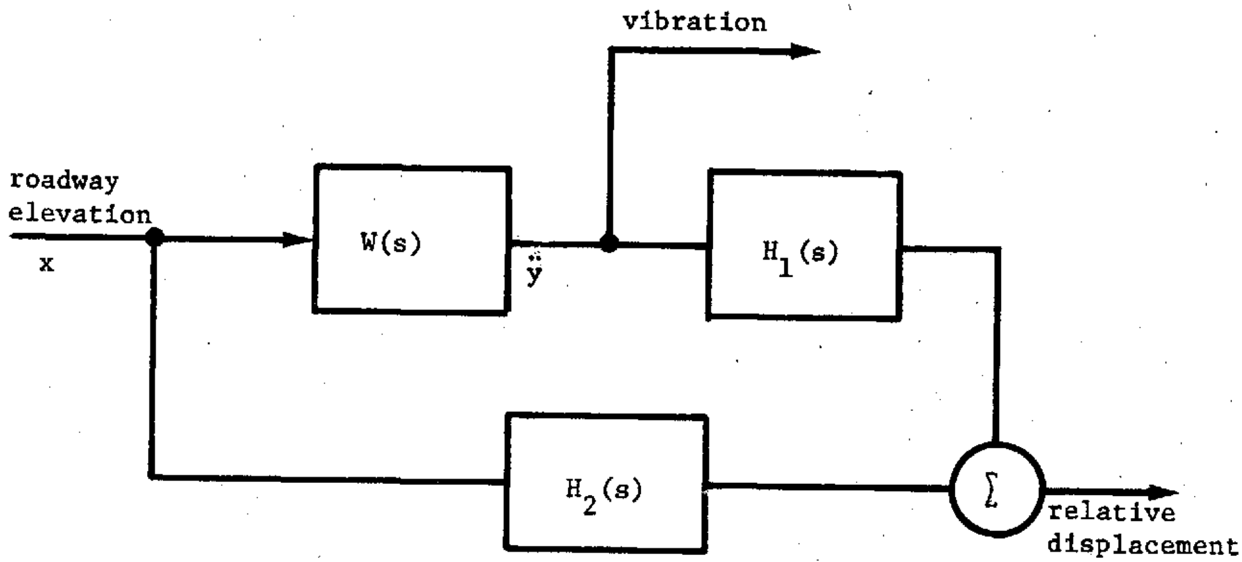


Fig. 6 Block diagram for Fig. 5.

$$H_2(s) = \frac{1}{1 + (s/\omega_n)^2} \tag{17}$$

Compare the block diagrams of Fig. 6 and Fig. 4. For the present case, the various matrices used in Theorem 1 are

$$H = 0,\ G_i = H_2,\ G_p = H_1,\ G_c = W,\ G_s = 1$$

All quantities are scalars for this system. For the performance index, $Q = 1$, $R = \rho$. From Theorem 1, we have for this case

$$W_{c0} = \Psi^{-1}[\Psi_*^{-1} A^T (\Omega_*^T)^{-1}]_+ (\Omega)^{-1} \tag{18}$$

where

$$\Omega_* \Omega = -A/s^2 \tag{19}$$

$$\Psi_* \Psi = H_{1*} H_1 + \rho$$

$$A = H_2 H_{1*}(-A/s^2) \tag{20}$$

From (19) $\quad \Omega = \sqrt{A}/s \qquad (21)$

Substitution of (19), (20), (21), into (18) gives the same equation as found in Ref. 3.

3.3 Discussion

It would appear that the results given in Theorem 1 are quite promising, in that multi-input, multi-output systems can be treated. However, there are at least two difficulties involved. First of all, to use Theorem 1, a block diagram similar to Fig. 4 must be established for the system. Unfortunately, this would be a difficult procedure to automate. Secondly, computational problems are encountered when this is extended to large systems. As indicated in Theorem 1, the solution of the optimum transfer function matrix requires matrix spectral factorization. For practical problems, this must be done computationally. An available program using Tuel's algorithm (Ref. 10) can be used to factor polynomial matrices of dimension up to 6 x 6 with order not exceeding 20 for each polynomial element. This program requires 10,000 words of storage. Apparently, this is a very limited computational capability.

4. APPLICATION OF STOCHASTIC OPTIMAL CONTROL THEORY TO THE LIMITING PERFORMANCE CHARACTERISTICS OF MDF MECHANICAL SYSTEM

The formulations described in the previous two sections are frequency domain approaches. In this section a time domain approach will be set forth, with the equations of motion in a form compatible with those used in the limiting performance study of transient systems (Ref. 11)

4.1 General Formulation

Consider a mechanical system described by Eqs. (1) with a gaussian white noise representation for $\bar{f}$. Consider a linear realizable system for $\bar{u}$. Define

$$\underline{R}_f(\tau) = E[\bar{f}(t)\bar{f}^T(t+\tau)]$$

as the correlation matrix of the input disturbance vector $\bar{f}$. For this problem

$$\underline{R}_f(\tau) = \underline{R}_0\delta(\tau) \qquad (22)$$

where δ is the usual Kronecker delta function.

Since the input disturbances are random in nature, the limiting performance characteristics will be defined in the expected mean square sense. That is, we will find $\bar{u}$ that minimizes the expected mean square of some response (objective function) while the expected mean square values of some other response variables (constraints) are bounded. Trade-off relations between the objective function and any one of the constraints can then be obtained.

To solve this problem, advantage will be taken of stochastic optimal control theory. To do so we transform our problem to the format of optimal control theory. A similar technique has been recently reported by Karnopp (Ref. 6).

Equations (1) can be written in first order form as

$$\dot{\bar{s}} = \underline{A}\bar{s} + \underline{B}\bar{u} + \underline{G}\bar{f} \tag{23}$$

where $\bar{s} = \begin{Bmatrix} \bar{x} \\ \dot{\bar{x}} \end{Bmatrix}$

$$\underline{A} = \begin{bmatrix} \underline{0} & \underline{I} \\ -\underline{M}^{-1}\underline{K} & -\underline{M}^{-1}\underline{C} \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} \underline{0} \\ -\underline{M}^{-1}\underline{V} \end{bmatrix}$$

$$\underline{G} = \begin{bmatrix} \underline{0} \\ \underline{M}^{-1}\underline{F} \end{bmatrix}$$

The limiting performance problem is to find $\bar{u}(t)$ such that the performance index

$$J = E\left[\int_{t_0}^{\infty} (\bar{s}^T \underline{Q}\bar{s} + \bar{u}^T \underline{R}\bar{u})\, dt \right] \tag{24}$$

is minimized. In (24) E[·] stands for expectation. $\underline{Q}$ and $\underline{R}$ are positive semidefinite weighting matrices. By adjusting the components of the $\underline{Q}$ and $\underline{R}$ matrices properly, the desired limiting performance problem can be formulated.

In stochastic control theory, an observation (or measurement) model is required. Since we are considering the limiting or theoretically optimum performance, it will be appropriate for us to assume noise free perfect measurements. For such problems, optimal control theory (Ref. 5) leads to the following results. For the system described by (23) subject to white noise input, the optimal control law $\bar{u}^*$ that minimizes the performance index (11) is given by

$$\bar{u}^* = -\underline{R}^{-1}\underline{B}^T\underline{Y}\bar{s}(t) \tag{25}$$

where $\underline{Y}$ is a constant matrix which is the solution of the following algebraic matrix Riccati equation

$$\underline{A}^T\underline{Y} + \underline{YA} - \underline{YBR}^{-1}\underline{B}^T\underline{Y} + \underline{Q} = 0 \tag{26}$$

Having the optimal control law, the next step is to calculate the value of the performance index and then establish the desired trade-off relations. To evaluate the correlation matrices such as $E[\bar{x}\bar{x}^T]$, a method reported by Yang and Iwan (Ref. 12) will be applied. To do so, substitute the control law into Eq. (23) and rewrite the resulting equation in second order form. Then, by application of the result given in Ref. 11, the following set of linear equations for the computation of $E[\dot{\bar{x}}\dot{\bar{x}}^T]$, $E[\bar{x}\bar{x}^T]$, and $E[\bar{x}\dot{\bar{x}}^T]$ are obtained.

$$\underline{M}E[\ddot{\bar{x}}\bar{x}^T] - \underline{C}_0 E[\dot{\bar{x}}\bar{x}^T] - \underline{K}_0 E[\bar{x}\bar{x}^T] = 0$$

$$\underline{M}E[\ddot{\bar{x}}\bar{x}]\underline{C}_0^T + \underline{C}_0 E[\dot{\bar{x}}\bar{x}^T]\underline{M}^T + \underline{M}E[\ddot{\bar{x}}\bar{x}^T]\underline{K}_0^T + \underline{K}_0 E[\bar{x}\dot{\bar{x}}^T]\underline{M}^T = \underline{F}\underline{R}_f\underline{F}^T \quad (27)$$

where

$$\underline{C}_0 = \underline{C} + \underline{TY}_{22}$$

$$\underline{K}_0 = \underline{K} + \underline{TY}_{21}$$

$$\underline{T} = \underline{VR}^{-1}\underline{M}^{-1}\underline{B}$$

$$\underline{Y} = \begin{bmatrix} \underline{Y}_{11} & \underline{Y}_{12} \\ \underline{Y}_{21} & \underline{Y}_{22} \end{bmatrix}$$

and

$$E[\bar{u}\bar{u}^T] = \underline{Z}_1 E[\dot{\bar{x}}\dot{\bar{x}}^T]\underline{Z}_1^T + \underline{Z}_1 E[\dot{\bar{x}}\bar{x}^T]\underline{Z}_2^T$$
$$+ \underline{Z}_2 E[\bar{x}\dot{\bar{x}}^T]\underline{Z}_1^T + \underline{Z}_2 E[\bar{x}\bar{x}^T]\underline{Z}_2^T \quad (28)$$

where $[\underline{Z}_2 \vdots \underline{Z}_1] = \underline{R}^{-1}\underline{BY}$

4.2 Example: Limiting Performance of an SDF Vibration Isolater

Consider the SDF system shown in Fig. 7. Suppose the disturbance $\bar{f}$ is gaussian white noise. The problem is to find the trade-off relation between mean square values of $\ddot{x}$ (or u/m) and x. This problem has been solved by Fujiwara et. al. (Ref. 14). We will solve the problem in the time domain.

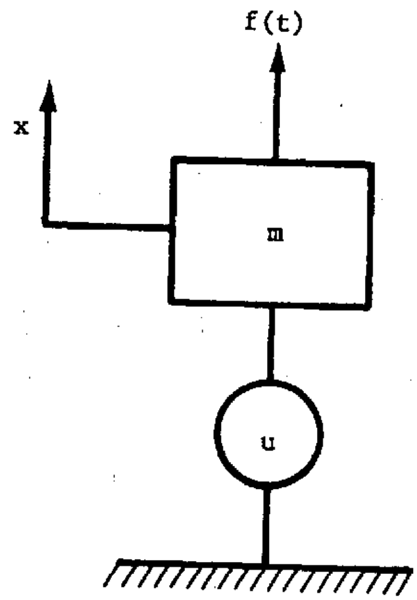


Fig. 7 SDF system subject to random force

The governing equation can be written as

$$m\ddot{x} = u + f \tag{29}$$

or in first order form

$$\dot{\bar{s}} = \underline{A}\bar{s} + \underline{B}\bar{u} + \underline{G}\bar{f} \tag{30}$$

where

$$\bar{s} = \left\{ \begin{matrix} x \\ \dot{x} \end{matrix} \right\} \qquad E[\bar{f}\bar{f}^T] = R_f \delta(\tau)$$

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$\underline{G} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

The performance index to be minimized is

$$J = E[\int_0^\infty (\bar{u}^T \underline{R} \bar{u} + \bar{s}^T \underline{Q} \bar{s})\, dt] \tag{31}$$

where

$$R = [1]$$

$$\underline{Q} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & 0 \end{bmatrix}$$

The optimum control law is

$$u^* = -\underline{R}^{-1} \underline{B}^T \underline{Y} \bar{s} \tag{32}$$

where $\underline{Y}$ is the solution of

$$\underline{A}^T \underline{Y} + \underline{Y}\underline{A} - \underline{Y}\underline{B}\underline{R}^{-1}\underline{B}^T \underline{Y} + \underline{Q} = 0 \tag{33}$$

The solution of (33) is

$$\underline{Y} = \begin{bmatrix} \lambda\sqrt{2m\lambda} & m\lambda \\ m\lambda & m\sqrt{2m\lambda} \end{bmatrix} \tag{34}$$

Substitute (34) into (32) to find

$$u^* = -\lambda s_1(t) - \sqrt{2m\lambda}\ s_2(t)$$

or

$$u^*(t) = -\lambda x(t) - \sqrt{2m\lambda}\ \dot{x}(t) \tag{35}$$

Since $E[\dot{\bar{x}}\bar{x}^T] = 0$ for an SDF system, (27) becomes

$$ME[\dot{x}^2] - K_0E[x^2] = 0$$

$$ME[\dot{x}^2]C_0 + C_0E[\dot{x}^2]M + 0 = R_f \tag{36}$$

where

$$K_0 = \lambda,\ C_0 = \sqrt{2m\lambda}$$

Thus, $E[x^2] = R_f/2\lambda\sqrt{2m\lambda}$ and

$$E[u^2] = (3R_f/2\sqrt{2m})\ \sqrt{\lambda}$$

These results are the same as obtained by Fujiwara. By eliminating λ between $E[u^2]$ and $E[x^2]$, we find the tradeoff relationship

$$\left(\frac{E[u^2]}{b}\right)^3 \left(\frac{E[x^2]}{b}\right) \approx 7$$

where

$$b = R_f/2\sqrt{2m}$$

This is plotted in Fig. 8.

4.3 Discussion

The underlying assumption for the formulation of this section is that $\bar{f}$ must be white noise. This is not so severe as it may appear, even though white noise does not actually exist. White noise does provide a good approximation for broad band random noises. The major drawback in this formulation is that its implementation encounters computational burdens for large systems.

Since the most efficient procedure of solving matrix Riccati equations (26) is by matrix spectral factorization (Ref. 4), this approach involves the same complications as described in Section 3. However, setting up this formulation for a particular problem is much easier then using the method of Section 3.

5. DYNAMIC PROGRAMMING APPROACH

In this section the application of dynamic programming techniques to the limiting performance problem of dynamic systems subject to random inputs will be discussed. This technique is more powerful in the sense it can treat problems with performance indices other than the usual expected mean square type. The essence of this technique will be discussed first and then applied to various limiting performance problems of interest.

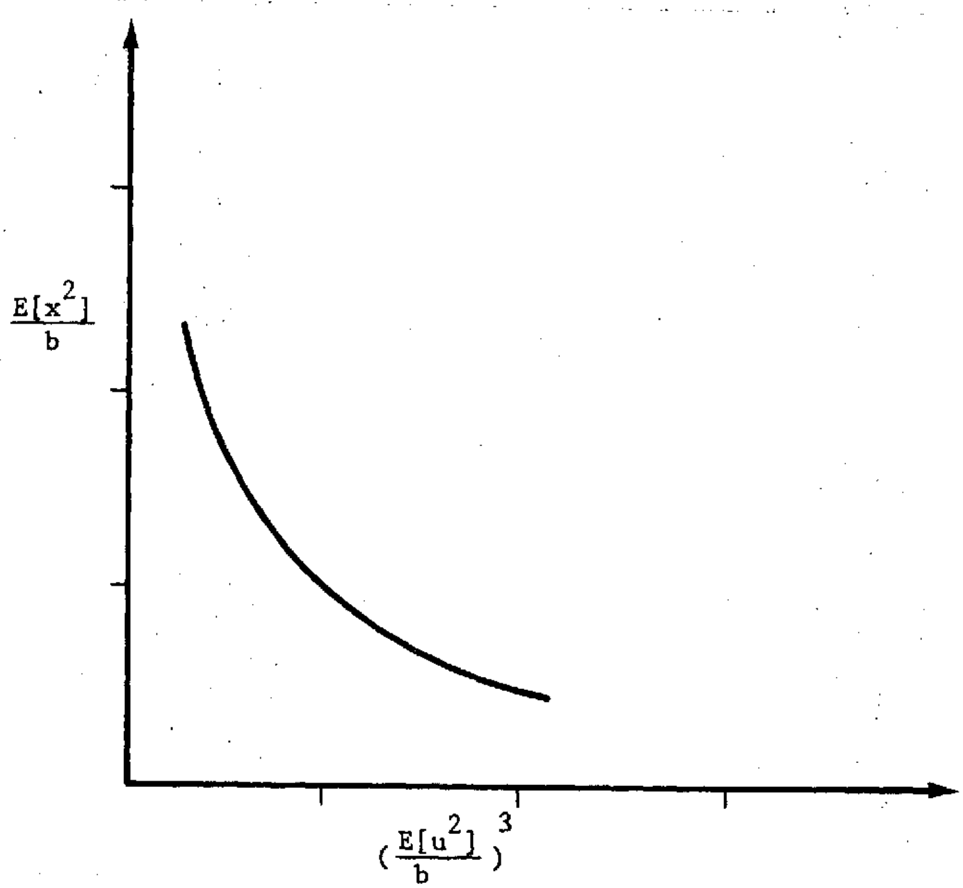


Fig. 8 A trade-off curve for SDF system subject to white noise

5.1 Definition of Dynamic Programming

Dynamic Programming is the name given by its inventor, Richard Bellman (Ref. 15), to a computationally motivated procedure for solving optimization problems through a sequence of smaller problems. It is ideal for multi-stage optimization problems since it provides an efficient algorithm for analytical and computational procedures by allowing only those continuations of multi-stage processes that constitute optimal continuations. This can be formalized as the Principle of Optimality. This principle can be applied to derive the functional relationships of dynamic programming. Many researchers have applied the dynamic programming technique to solve stochastic optimal control problems. A rather comprehensive bibliography in this regard can be found in Ref. 16.

To illustrate the principle of optimality, consider the following simple example: Given:

$$\bar{s}_{i+1} = T(\bar{s}_i, \bar{u}_i) \quad \text{with } \bar{s}_0 = \bar{c} \quad \text{where} \quad \bar{s} = \text{state vector}, \ \bar{u} = \text{control vector} \tag{37}$$

find $\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1}$ to minimize

$$\Psi = \sum_{i=1}^{N} \phi(s_i) \tag{38}$$

Define

$$f_N(c) = \underset{\bar{u}_0}{\text{Min}} \ \underset{\bar{u}_1}{\text{Min}} \ldots \underset{\bar{u}_{N-1}}{\text{Min}} [\phi(\bar{s}_1) + \ldots + \phi(\bar{s}_N)] \tag{39}$$

Then, by definition,

$$f_1(\bar{c}) = \underset{\bar{u}_0}{\text{Min}}\ \phi(T(\bar{c},\ \bar{u}_0)) \tag{40}$$

for $N \geq 2$

$$f_N(\bar{c}) = \underset{\bar{u}_0}{\text{Min}}\ [\phi(T(\bar{c},\bar{u}_0) + f_{N-1}(T(\bar{c},\bar{u}_0)] \tag{41}$$

The principal of optimality, when applied to the above example, specifies that when only one more decision ($\bar{u}$'s) stage remains, the decision must be made in such a way as to minimize $\phi(\bar{s}_1)$. When more than one decision stage remains, make the immediate decision $\bar{u}_0$ in such a way as to minimize the overall sum consisting of the terms of the immediate contribution $\phi(\bar{s}_1)$ and the term $f_{N-1}(T(\bar{c},\bar{u}_0))$ representing sums obtainable from the resulting state $T(\bar{c},\bar{u}_0)$ by an optimal continuation. Solution of the functional relations (41) gives the optimal control sequences.

In the following subsections, several limiting performance problems will be formulated with different optimization criteria. These criteria include: mean quadratic performance index, probability of maximum deviation.

5.2 Quadratic Performance Index

Consider a mechanical system described by (23) in which $\bar{f}$, the disturbance, is an independent random variable. The discrete solution of (23) can be written as

$$\bar{s}(k+1) = \Phi\bar{s}(k) + \underline{G}\bar{u}(k) + \bar{h}(k) \tag{42}$$

where

$$\underline{\Phi} = e^{\underline{A}T} \qquad T = \text{sampling period}$$

$$\underline{G} = \int_{kT}^{(k+1)T} \underline{\Phi}(t-\tau)\underline{B}\, d\tau$$

$$\bar{h}(k) = \int_{kT}^{(k+1)T} \underline{\Phi}(t-\tau)\underline{D}\bar{f}(\tau)d\tau$$

The performance index of interest is the mean quadratic form

$$I_N = E \sum_{k=1}^{N} [\bar{s}^T(k)\underline{Q}\bar{s}(k)+\bar{y}^T(k-1)\underline{R}\bar{u}(k-1)] \tag{43}$$

where $\underline{Q}$, $\underline{R}$ are positive definite symmetric matrices. By adjusting the values in the elements of $\underline{Q}$ and $\underline{R}$ matrices, relative weight can be imposed on state as well as control. Following a dynamic programming approach as used by Tou (Ref. 17), the following results can be derived.

The optimal control law that minimizes the performance index (43) can be obtained from the algorithm

$$\bar{u}^*(j) = \underline{K}\ (N-j)\ \bar{x}(j) \tag{44}$$

where

$$\underline{K}(N-j) = -(\underline{L}_{GG}(N-(j+1)+R)^{-1}\underline{L}_{G\phi}(N-(j+1))$$

$$\underline{P}\ (N-(j+1)) = \underline{L}_{\phi\phi}(N-(j+1)) + \underline{L}_{\phi G}(N\ (j+1))B(N-j)$$

$$\underline{L}_{\phi\phi}(N-(j+1)) = \underline{\Phi}^{T}\underline{S}(N-(j+1))\Phi(j) \qquad (45)$$

$$\underline{L}_{GG}(N-(j+1)) = \underline{G}^{T}\underline{S}(N-(j+1))\underline{G}$$

$$\underline{L}_{G\phi}(N-(j+1)) = \underline{G}^{T}\underline{S}(N-(j+1))\underline{\Phi}$$

$$\underline{L}_{\phi G}(N-(j+1)) = \underline{\Phi}^{T}\underline{S}(N-(j+1))\underline{G}$$

$$\underline{S}(N-(j+1)) = \underline{Q} + \underline{P}(N-(j+1))$$

with

$$\underline{P}(0) = \underline{0}$$

Thus, from $j = N - 1$, with $\underline{P}(0) = \underline{0}$ we can generate $\underline{K}(1)$, $\underline{P}(1)$, $\underline{K}(2)$, $\underline{P}(2)$, . . . recursively. Note that even when the system is invariant, the feedback gain matrix is still time varying. The performance index can be evaluated from

$$f_N[\bar{x}(0)] = \bar{x}^{T}(0)\underline{P}(N)\bar{x}(0)$$

Such problems as the airplane taxiing during landing can be treated by this technique. The landing velocity provides an initial condition and the runway is the random input. Also, taxiing stops after a finite time. This approach can be used to evaluate the limiting performance for such systems.

5.3 Minimizing the Probability of a Maximum Deviation

Bellman (Ref.18) pointed out that the problem of minimizing the probability that the maximum response will exceed a preassigned value z, can be solved by dynamic programming. For this case

$$f_N(c) = \text{Prob}\{\max(|\bar{x}_1|\,|\bar{x}_2|\ . . ., |\bar{x}_N|) \leq \bar{x}) \qquad (46)$$

$$\text{where} \quad |\bar{x}| \leq z$$

If we interprete $|\bar{x}| \leq z$ in the component by component sense, then this problem, when solved for a range of z, will result in an interesting trade-off diagram (Fig. 9). For example, for z = A in Fig. 9, the probability that max $|x_i|$ will not exceed A will be P_1. The probability of exceeding A is $(1-P_1)$. This could be correlated to some other properties of the system, such as reliability.

5.4 Discussion

Dynamic programming provides a very general limiting performance problem formulation. However, due to its inherent "curse of dimensionality"[4] and the concomitant computational burden, the application of this technique to large scale problems is limited.

6. CONCLUSIONS

The limiting performance characteristics for dynamic systems subject to random inputs has been solved for a class of problem by various control theory optimization techniques. In most cases, these problems are restricted to linear systems subject to inputs with known power spectral densities, either white noise or representable by rational functions.

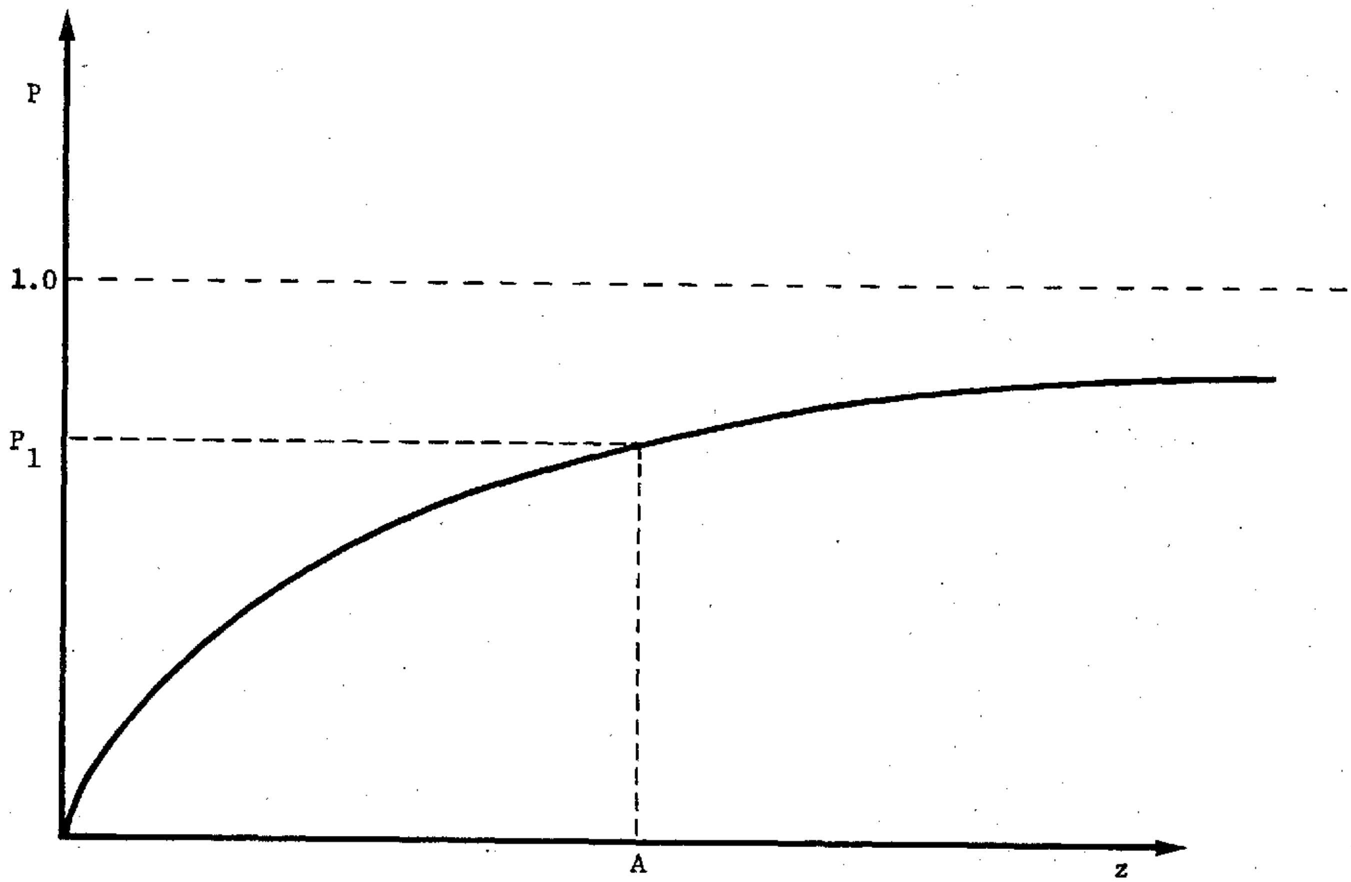


Fig. 9 A trade-off diagram.

In the case of white noise, stochastic optimal control theory in the time-domain can be applied to find the optimal law leading to limiting performance. This approach has been formalized for vibration isolation systems and only requires matrix manipulation and solution of a set of algebraic Riccati equations.

The analytic design technique extended by Weston is more general in the sense that it can solve problems subject to colored noise inputs. The drawback in this approach is that the system configuration must be put in a special block-diagram form. However, for the computation of an optimum transfer function, existing spectral factorization programs can be applied.

Dynamic programming, which appears to be the most versatile formulation, is of limited value due to its computational inefficiency.

7. REFERENCES

1. Sevin, E., and Pilkey, W., Optimum Shock and Vibration Isolation, SVIC Monograph No. 6, Shock and Vibration Information Ctr., Washington, 1971.

2. Karnopp, D. C., and Trikha, A. K., "Comparative Study of Optimization Techniques for Shock and Vibration Isolation," J. Eng. Ind., 91, 1128-1132, 1969.

3. Bender, E. K., "Optimization of the Random Vibration Characteristics of Vehicle Suspensions," Ph.D. dissertation, Mass. Institute of Technology, 1966.

4. Weston, J. E. and Bongiorno, J. J., Jr., "Extension of Analytical Design Techniques to Multivariable Feedback Control Systems" IEEE Tran. Aut. Contr. Vol. AC-17, No. 5, Oct. 1972.

5. Bryson, A. E., Jr., and Ho, Y. C., Applied Optimal Control. Blaisdell Publishing Co., Waltham, Massachusetts, 1969.

6. Karnopp, D. C., "Active and Passive Isolation of Random Vibration Systems" in Isolation of Shock and Vibration Systems, Eds. J. Snowden and E. Ungar, ASME, New York, 1973.

7. Aoki, M., "On Minimum of Maximum Expected Deviation form an Unstable Equilibrium Position of a Randomly Perturbed Control System," IRE Trans. on Auto. Contr. Vol. AC-7, March, 1962.

8. Trikha, A. K., and Karnopp, D. C., "A New Criterion for Optimizing Linear Vibration Isolator Systems Subject to Random Input," J. Engr. Ind. Vol. 91, 1969.

9. Newton, G. C., Gould, L. A., and Kaiser, J. F., Analytical Design of Linear Feedback Control, John Wiley and Sons, Inc., New York, N.Y., 1964.

10. Tuel, W. G., Jr., "Computer algorithm for Spectral Factorization of Rational Matrices," IBM J. Res. Develop. Vol. 12, pp. 115-120, Mar. 1968.

11. Pilkey, W. D., Wang, B. P., Yoo, Y., Clark, B., "PERFORM - A Performance Optimizing Program for Dynamic Systems Subject to Transient Inputs" NASA CR-2268.

12. Yang, I. M., and Iwan, W. D., "Calculatinn of Correlation Matrices for Linear Systems Subjected to Non-white Excitation," J. of Applied Mechanics, June 1972.

13. Fujiwara, N., Murotsu, Y., and Nakagawa, K., "Optimization of Vibration Absorbers for Systems with Random Excitations," Proc. 18th Japan Nati. Cong. Appl. Mech. 1968, p. 63-76.

14. Chang, S. S. L., Synthesis of Optimal Control Systems, McGraw Hill, New York, 1961.

15. Bellman, R., and Dreyfus, S., Applied Dynamic Programming, Princeton University Press, Princeton, N. J., 1962.

16. Mendel, J. M., and Gieseking, D. L., "Bibliography of the Linear-Quadratic-Gaussian Problem," IEEE Trans. Auto. Contr. Vol. AC-16, No. 6, Dec. 1971.

17. Tou, J. T., Optimum Design of Digital Control System, Academic Press, New York, N.Y., 1963.

18. Bellman, R., "On Minimizing the Probability of a Maximum Deviation," IRE Trans. Automat. Contr. Vol. AC-7, 1962.