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Contents: Part I: Time elements

Part II: Stabilization by external energy corrections

Part III: Long-term global solutions for the synchronous satellite. A preliminary report.

Part IV: The D-S equations of motion

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Part I. Time Elements

A. Time Transformations

Time transformations of the Sundman type (ref. 1) have the form

$$dt = cr^{\alpha}ds \tag{1}$$

where t is the time, c and α are positive constants, r the magnitude of the radius-vector, and s the new independent variable.

It has been known for many years that use of the time transformations of equation (1) improves rates of convergence of analytical series solutions of gravitational systems (ref. 2 and 3). Also, when equation (1) is used in conjunction with a coordinate transformation, singularities due to collisions may be eliminated from the equations of motion (see, for example, ref 4).

Several recent published studies (ref. 5, 6, 7) and unpublished studies (ref. 8, 9) show that the use of the time transformations given by equation (1) with $1 \leq \alpha \leq 2$, substantially improves numerical integration accuracy and efficiency for satellite solutions. Moreover, it has been shown by many of these same studies that, for a large class of satellite initial conditions, α near or equal to 1.5 provides the best accuracy and efficiency (ref. 7,8,9).

B. Time Elements

It appears that time transformations lessen or weaken to some extent the dynamical in-track (Liapunov) instability associated with satellite motion. The reduction of instability in the coordinates is rendered at the expense of the introduction of a differential equation in order to obtain the time (equation (1)). The differential equation for the time exhibits some of the dynamical instability that was removed from the differential equations for the coordinates. If the perturbations are small relative to the two-body forces, then some of the instability of the time equations may be removed by use of time elements.

With zero perturbations, equation (1) may be integrated in closed form for $\alpha = 1, 1.5, 2$. These integrals define "time elements" (ref. 11). For perturbed motion, the integrals of equation (1) are then differentiated using methods of variation of parameters.

The resulting differential equations for the time elements are then integrated by means of numerical integration. A significant portion of the instability is removed from the equation that defines the time transformation (equation (1)) by taking account of the two-body solution.

It has been shown by Stiefel and Scheifele (ref. 11) that a time element may be introduced for the KS equations of motion in the four dimensional u-space. The related time transformation has $\alpha = 1$ in equation (1).

Often in practice, one does not wish to use the complex KS coordinate transformations to four dimensional space, but rather remain the three dimensional physical space and use only the time transformation of equation (1). Also, the time transformation for optimum accuracy has $\alpha \approx 1.5$ in equation (1). For these reasons we have derived a time element for $\alpha = 1.5$, as well as for $\alpha = 1$ and 2, in the three dimensional physical space. These time elements and their defining differential equations are presented here in this report.

C. The New Time Elements for $\alpha = 1, 2$

For two-body motion and for $\alpha = 1$ and 2, the integral of equation (1) is Kepler's equation for certain choices of the constant c.

For $\alpha = 1$ and $c = \sqrt{\frac{a}{\mu}}$, the integral is

$$t = \tau - \sqrt{\frac{a^3}{\mu}}\, e \sin E, \tag{2}$$

where E is the eccentric anomaly, and where a and e are the semi major axis and eccentricity, respectively, and τ is the time element. A differential equation for the time element is

$$\frac{d\tau}{ds} = \sqrt{\frac{a^3}{\mu}} + \frac{a}{r}\sqrt{\frac{a^3}{\mu}} \sin E \frac{de}{ds} + \frac{3}{2} e \sin E \sqrt{\frac{a}{\mu}} \frac{da}{ds}$$

$$+ a^3 e \cos E \{(1-e \cos E)\nabla E - \sin E \nabla e\} \cdot R \tag{3}$$

where ∇E, and ∇e denote the gradients of E and e with respect to the velocity vector, and $\vec{R}$ is the perturbing force.

The first term on the right hand side of equation (3) is not a constant for perturbed motion and is of the order of magnitude of the unperturbed quantities. The semi-major axis in the term must be determined from the state vector which has errors due to the numerical integration of the equations of motion. The subsequent integration of these errors in equation (3) to determine the time element compounds the error substantially. To reduce this large source of errors in the time element the following relation may be used to replace the first term on the right in equation 3:

$$\sqrt{\frac{a^3}{\mu}} = \frac{\mu}{\sqrt{-8(h-v)^3}} \tag{4}$$

where h is the total energy of the perturbed system and V is the perturbing potential defined by

$$h = h_k + V$$

where h_k is the total unperturbed or Keplerian energy.

For $\alpha = 2$ in Equation (1), and

$$c = \frac{1}{\sqrt{\mu a(1-e^2)}},$$

the integral is the same as equation (2) but here s is equal to the true anomaly, f. The differential equation for the time element is

$$\frac{d\tau}{ds} = \frac{ar}{\sqrt{\mu p}} + \sqrt{\frac{a^3}{\mu}}\,\frac{(r-ae^2)}{a(1-e^2)}\sin E\,\frac{de}{ds} + \frac{3}{2}e\sqrt{\frac{a}{\mu}}\,\sin E\,\frac{da}{ds}$$

$$+ \sqrt{\frac{a^5}{\mu}}\,\frac{e}{r}\sin E\cos E\,\frac{de}{ds} + a^2 re\cos E(1-e^2)^{-\frac{1}{2}}$$

$$\cdot\{(1-e\cos E)\,\nabla E - \sin E\nabla e\}\cdot\vec{R} \tag{5}$$

A substitution similar to that of equation (4) may be used to replace part of the first term on the right in equation (5).

D. The Intermediate Anomaly

For $\alpha = 1.5$ and $c = 1$, the integral of equation (1) for two-body motion is

$$s = \frac{2}{\sqrt{(1+e)}}\,F(\tfrac{f}{2}, k) \tag{6}$$

where s is the new independent variable, referred to here as the "intermediate anomaly," e is the eccentricity, F the incomplete elliptic integral of the first kind, k the modulus defined in terms of the eccentricity as

$$k = \sqrt{\frac{2e}{1+e}}\ ,$$

and f the true anomaly. Equation (6) may also be written as

$$\sin\tfrac{1}{2}f = \mathrm{sn}[\tfrac{1}{2}\sqrt{1+e}\ s] \tag{7}$$

where sn is the sine Jacobian elliptic function.

Equations (6) and (7) give the true anomaly as a function of s. The time is given by equation (2) where the eccentric anomaly is obtained from the true anomaly. The time element is obtained from the following differential equation.

$$\frac{d\tau}{dS} = ar^{1/2} + \frac{3}{2}\sqrt{\frac{a}{\mu}}\, e \sin u \frac{da}{dS} + \sqrt{\frac{a^3}{\mu}} \cdot \frac{de}{ds} \left\{ \sin u - e \cos u \left[\frac{\tan \frac{1}{2} f(1 + \cos u)}{(1+e)\sqrt{1-e^2}} \right.\right.$$

$$\left.\left. - \frac{1}{2e}\sqrt{\frac{r}{a}} \left[\sqrt{\mu}\, s - \frac{2E}{(1+e)^{1/2}(1-e)} \cdot \frac{sn(\frac{1}{2}s\sqrt{(1+e)\mu})\ cn\ (\frac{1}{2}s\sqrt{(1+e)\mu})}{dn(\frac{1}{2}s\sqrt{1+e)\mu})} \right]\right]\right\} \qquad (8)$$

The quantity E is the incomplete elliptic integral of the second kind; and sn, cn, and dn are the sine, cosine, and amplitude Jacobian elliptic functions, respectively.

The differential equations for any of the time elements that we have derived have no pure or mixed secular terms for unperturbed motion, therby further reducing the in-track (Liapunov) instability associated with equation (1)

E. Elliptic Function and Integral Algorithms

We have developed new computational algorithms tailored to our purpose in order to compute the necessary complete and incomplete elliptic integrals and the Jacobian elliptic functions. The algorithms are based on and modifications of those by Hofsommer and van de Riet (ref. 12) and are about five times faster than the algorithms of DiDonati and Hershey (ref 13).

F. Conclusion

We have compared the accuracy of integration of equation (1) directly with that of integrating equation (3) using equation (2). The result is that use of equations (2) and (3) provide about two to three times the accuracy of equation (1) after about three satellite revolutions. Further studies are presently being performed for longer integration times and for equations (5) and (8).

The full paper with a complete set of numerical comparisons will be presented at the COSPAR Symposium on satellite dynamics, June 19-21, 1974, Sao Paulo, Brazil, and at the AIAA/AAS Astrodynamics Conference, August 5-9, 1974, in Anaheim, California. The paper will be submitted for publication to the Journal of Celestial Mechanics or be published in the Proceedings of the COSPAR meeting.

Part II. Stabilization by External Energy Corrections

A. INTRODUCTION

It is shown in reference (14) that constraining solutions to satisfy exactly the integrals of motion tends to Liapunov stabilize the solutions of unstable dynamical systems. It is shown in reference(15), pp. 73-77, that complete Liapunov stabilization of Keplerian motion may be realized by regularization. Reference (15) indicates that the regularized equations of motion are stable due to the fact that the Keplerian energy is constant and that the constant appears explicitly in the differential equations. This is complete stability in the sense of Liapunov since the time equation is also stabilized along with the dependent variable equations. This point confirms the conclusions of reference (14) since reference (14) shows that use of integrals in stable dynamical systems (harmonic oscillator) does not improve accuracy. Whereas use of integrals for unstable (in the sense of Liapunov) dynamical systems (for example, Keplerian motion) improves the accuracy of the solution by several orders of magnitude. In addition, references (14) and (16) indicate that neither time nor coordinate transformations are necessary for the stabilization, only the use of the integrals of motion. Reference (15), p. 76, reference (16), reference (17), and reference (18) indicate that for unperturbed and perturbed Keplerian motion, only the energy integral is needed for stabilization. This result is in agreement with the numerical results of reference (14), which show that for an n-body problem dynamical system, by satisfying only the energy integral exactly, a solution is produced that is several orders of magnitude more accurate than by not satisfying the integral and very nearly as accurate as a solution satisfying all ten integrals of motion. Hence, it appears that there is a fundamental relation between isoenergetic solutions, regularization, and dynamical (Lyapunov) stabilization.

In reference (14), corrections are applied to the components of the state vector so that the corrected state satisfies identically the integrals of motion. The corrections are chosen so that the sum of the squares of the corrections is a minimum.

This report applies the concept of stabilization using integrals to the solution of the equations of motion of artifical satellites in an attempt to reduce the propagation of local truncation errors and improve the efficiency of a numerical integration solution process. Even though no constant energy integral exists for the full satellite equations of motion due to the presence of the tesseral harmonics, drag, and third-body perturbations; nevertheless, a slowly varying energy "integral" is present in a coördinate system rotating with the rotating Earth. The integral in the rotating coordinate system is analogous to the Jacobian integral in the restricted problem of three bodies.

The method presented in reference (14) is extended here to the satellite solution in a rotating coordinate system. An application of the method to satellite solution is presented showing accuracy improvements of at least two orders of magnitude with negligible additional computation time. In addition, a modification of the method presented in reference (14) is presented to allow the use of slowly varying integrals.

B. The Equations of Correction

The equations of correction using integrals derived in reference (14), will now be adapted to a dynamical system with three degrees of freedom using only one integral - the energy intergal.

For a dynamical system with three degrees of freedom, let $x = [x_1, x_2, x_3, x_4, x_5, x_6]$ be the state vector in the phase space, where x_1, x_2 and x_3 are the coordinates and x_4, x_5 and x_6 are the corresponding velocity components. Let

$$E(x) = 0, \tag{1}$$

be an integral of the system. Equation (1) defines a hypersurface of five dimensions imbedded in the phase space of six dimensions.

During a process of numerical integration of the system, a computed solution is obtained at time t:

$$\eta = \eta(t) = [\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6] \quad ,$$

where η_1, η_2, and η_3 are the computed position components and η_4, η_5, and η_6 are the computed velocity components. Due to errors in the computational procedure, the integral may not be satisfied exactly but

$$E(\eta) = \varepsilon \ , \tag{2}$$

where ε is the small quantity. The solution has left the integral surface defined by Equation (1) and is on the surface defined by Equation (2). It is desired to make corrections $\Delta\eta = [\Delta\eta_1, \Delta\eta_2, \Delta\eta_3, \Delta\eta_4, \Delta\eta_5, \Delta\eta_6]$ to the computed vector η to obtain the vector

$$x = \eta + \Delta\eta \quad ,$$

such that

$$E(x) = 0 \quad . \tag{3}$$

The square of the magnitude of the correction vector $\Delta\eta$ may be written as

$$f(\Delta\eta) = \sum_{i=1}^{6} (\Delta\eta_i)^2 \quad . \tag{4}$$

The corrections are uniquely chosen so that the function f of Equation (4) is minimized, subject to the constraint of Equation (3). The solution may be obtained by solving the following equation

$$\Delta\eta_i - \lambda\frac{\partial E}{\partial\eta_i} = 0 \ , \ i = 1,2,3,4,5,6 \tag{5}$$

along with Equation(3) for the seven unknowns λ, and $\Delta\eta_i$, i = 1,2,3,4,5, and 6. The quantity λ is the Lagrange multiplier. Unless the integral given by Equation (3) is a simple function of the variables (for instance linear), the solution of the system may be complex (or perhaps not obtainable) as is the case when the integral is the integral of energy of a gravitational system. The solution may be simplified by an expansion of the integral in powers of the corrections. The expansion becomes

$$E(x) = E(\eta) + \sum_{i=1}^{6} \frac{\partial E}{\partial\eta} \Delta\eta_i + \dots \quad . \tag{6}$$

Since the errors of the computation and hence the necessary corrections, $\Delta\eta_i$, are generally small, second and higher-order terms may be neglected.

Solving Equations (5) and (6) for the correction $\Delta\eta_i$, with $E(x) = 0$ and $E(\eta) = \varepsilon$, yields

$$\Delta\eta_i = \frac{-\varepsilon\frac{\partial E}{\partial\eta_i}}{\sum_{j=1}^{6}\left[\frac{\partial E}{\partial\eta_j}\right]^2} \quad , \quad i = 1,2,3,4,5,6. \tag{7}$$

The correction vector $\Delta\eta$ is added to the computed state vector η to obtain a new state vector x which satisfies the integral $E(x) = 0$, with an error of order $|\Delta\eta|^2$. Geometrically, minimizing Equation (4) subject to Equation (6) causes the vector $\Delta\eta$ of Equation (7) to be normal to a five-dimensional plane which is approximately tangent to the surface $E = 0$ at the point x. The equation of the plane is given by Equation (6), neglecting the second and higher-order terms. Equation (7) is generalized in reference (1) to a system having n degrees of freedom and p integrals of motion, where n and p are any positive integers. Also, a more detailed derivation of Equation (7) is given in reference (14).

The energy integral of a dynamical system (per unit mass) may be written in terms of a potential energy U and the state vector x as

$$E(x) = \frac{1}{2}(x_4^2 + x_5^2 + x_6^2) + U - C = 0$$

where C is the value of the energy for a set of initial conditions.

A numerical integration of the system yields the solution vector η at time t. The errors in the computation may cause E to be nonzero:

$$E(\eta) = \varepsilon, \tag{8}$$

where ε is the error of the integral. The correction vector $\Delta\eta$ may be calculated by using Equation (7), so that $E(\eta + \Delta\eta) = 0$. For the calculation, the quantities ε and $\frac{\partial E}{\partial \eta_i}$ are needed.

The quantities $\frac{\partial E}{\partial \eta_i}$ are the partial derivatives of E with respect to η(or x) and are easily computed. The partial derivatives of E_1 with respect to x_4, x_5, and x_6 are equal to x_4, x_5, or x_6, respectively. The partial derivatives of E_1 with respect to x_1, x_2, and x_3 are simply minus the respective components of the force F, already computed during the integration step.

C. Previous Applications and Evaluations of the Method

As presented in reference (14), the method was applied to the numerical integration of several dynamical systems to determine its paractical value. The systems considered were the harmonic oscillator, the gravitational system of two-bodies, and the gravitational system of 25-bodies.

The application of the correction method to the harmonic oscillator in a phase space of two dimensions showed no noticeable differences in accuracy between the corrected and uncorrected solutions (reference 1).

However, the application of the method to the system of two-bodies in a phase space of four dimensions over a range of initial conditions showed a large difference in accuracy between corrected and uncorrected solutions. The corrected solutions were about three orders of magnitude more accurate than the uncorrected solutions (reference (14)).

The different results obtained for the harmonic oscillator and the system of two-bodies offers an explanation of when and why the method appears to be of value, as presented in reference (14). The errors in the integral of the harmonic oscillator are small compared to the error in the state variables of the solution. Since the harmonic oscillator is a stable system, a solution with a small error will not diverge from a system without the error. This would explain the result indicating no difference between the corrected and the uncorrected solutions for the harmonic oscillator. The errors in the energy integral of the system of two-bodies are also small relative to the total errors in the state variables of the solution. But the system of two-bodies is unstable in the Lyapunov sense and hence the system with the energy errors will diverge from the system without the energy errors.

The method was applied to a gravitational system of 25-bodies. The results show that the method yields a more efficient numerical integration process of the n-body problem. A greater accuracy of about two orders of magnitude is obtained with the method while using the same time of calculation then without using the method. Or the same accuracy is obtained with the method while using about 25% less time of calculation (reference 14).

D. Application of the Method to Satellite Motion

Recently, the method has been applied to the solution of an artificial satellite of the Earth. The equations of motion of the satellite included all zonal, sectoral, and tesseral spherical harmonics. The equations of motion did not include luni-solar nor drag effects. With all of the spherical harmonics included, no energy integral exists. However, an integral exists in a coordinate system rotating with the rotation of the Earth. The integral is analogous to the Jacobian integral in the rotating coordinate system of the restricted problem of three bodies. If the integral is formulated in the rotating coordinate system using position and velocity relative to the rotating coordinate system and then transformed into the fixed system, in terms of position and a velocity relative to the fixed system, the integral has the form

$$E(\vec{\gamma}\ \dot{\vec{\gamma}}) = \frac{1}{2}\dot{\vec{\gamma}}\cdot\dot{\vec{\gamma}} + U - C + \vec{\gamma} \otimes \dot{\vec{\gamma}} \cdot \vec{W} = 0 \qquad (9)$$

where U is the potential, $\vec{\gamma}$, is the position vector, and $\dot{\vec{\gamma}}$ is the velocity vector relative to the fixed frame. The quantity C is the value of the integral for a certain set of initial conditions and $\vec{W}$ is the angular velocity vector of the rotation of the Earth. The quantity $\vec{\gamma} \otimes \dot{\vec{\gamma}} \cdot \vec{W}$ is the component of the angular momentum in the direction of the rotational axis of the Earth times the rotation rate of the Earth. That equation (9) is a constant of the motion may be proved in several ways. One way is to formulate the Hamiltonian of the satellite system in extended phase space. Equation (9) is that Hamiltonian, neglecting the constant C.

In addition, a modification of the method presented in reference (1) may be be incorporated to allow the use of slowly varying integrals. A differential equation may be formulated giving the rate of change of an integral. This equation may be integrated numerically along with the state vector. Then, at any time, the value of the integral is known and the correction procedure of reference (1) is applied. It appears that such a modification induces stabilization and does so only for slowly varying integrals.

E. Numerical Results

The method was applied to the satellite solution by the following procedure. Two solutions were obtained by numerical integration. One solution did not utilize the integral while the other introduced corrections determined by equations (7), (8), and (9). The corrections were applied to the state vector after each integration step, as in the discussion following Equation (1) above. Both solutions were compared with a more accurate solution determined by smaller integration step-sizes. Both solutions were obtained using a high-order predictor-corrector integration algorithm of the STDS system of the NASA Goddard Space Flight Center (reference 19). Both uncorrected and corrected solutions were integrated with the same step-size and with the same number of integration steps.

The satellite had initial orbital parameters as follows: A semimajor axis of 136000 km; an orbital period of 120 hours; an eccentricity of .8; and an inclination of 40°. The solutions were found for about 5 revolutions or over a time span of 3 weeks. The corrected solutions were found to be more accurate than the uncorrected by at least two orders of magnitude. That is, the position and velocity components of the corrected solution had at least two more digits of agreement with the true solution than the uncorrected solution.

In addition, solutions obtained by a 4th-order Runge-Kutta were performed for a satellite with an eccentricity equal to 0.6 and a period equal to 3 hours. Two-body forces and forces due to the second zonal harmonic were included.

Various step sizes were used: from 500 steps per revolution to 4000 steps per revolution. After 10 revolutions, the corrected solution was at least one, and often two, orders of magnitude more accurate than the uncorrected solution, for all step sixes used.

Also, the following study was performed. Corrections were made according to the following two techniques: (1) corrections were made to both the position and velocity components with various step sizes; and (2) corrections were made to only the position components. This study was initiated so as: (1) to study whether there is a difference if corrections are used with a Class I integrator or with a Class II integrator; and (2) to determine whether reapportioning the corrections from the velocity vector to the position vector could cause differences.

Result: No significant difference was found between the corrected final state for technique (1) and for technique (2).

Part III. Long-term Global Solutions for the Synchronous Satellite. A preliminary report.

A. Introduction

A new method has been developed by one of the principal investigators (Nacozy) to yield semi-analytical solutions of the long-term solution of resonant satellites, in deep resonance with the tesseral harmonics.

The method is able to yield global solutions for all eccentricities, inclinations, and commensurability ratios for resonant satellites, including the synchronous satellite. The method is not restricted to moderate or small eccentricities and inclinations nor to just synchronous satellites as is the analytical solution of Musen and Baile (reference (21)). In addition, the method does not yield particular solutions as are given by Gedeon (reference (22)) and others.

Our method is based on a semi-analytical approach to resonant perturbations introduced by Hori (reference (23)), developed by Giacaglia (reference (24)), and extended and applied by Giacaglia and Nacozy (reference (25)), and Nacozy and Diehl (reference (26) (and partially used by Musen and Bailie reference (21)). The method numerically averages the Hamiltonian and uses the fact that the averaged Hamiltonian has a minimum with respect to the resonant (critical) argument at the stable, stationary value of the resonant argument.

Our semi-analytical method also appears to have the advantage that luni-solar and radiation pressure effects may be easily added numerically without the need for analytical developments.

We are presently applying the method to synchronous satellites, to determine the practicality and potential of the method. For the J_2-J_{22} problem, we have now successfully obtained long-term solutions for e, i, ω, and Ω, for <u>all</u> e and i, for a synchronous satellite in the vicinity of the stable stationary solution. We have discovered a family of periodic solutions for the J_2-J_{22} problem. These orbits are inclined and eccentric orbits and have a repeating ground track.

We next plan to include more zonals and Tesserals and luni-solar perturbations and radiation pressure. We also have the capability to consider the 4, 6, 12, and 18 hour resonant satellites.

Part IV. The D-S Equations of Motion

Abstract

A new set of canonical two-body elements referred to as Delaunay-similar (D-S) elements is presented in references (27) and (28). In contrast to the

classical Delaunay theory which has time as the independent variable, the D-S theory uses an independent variable which is a generalized true anomaly. This study is concerned with numerical integration of the canonical perturbation equations of these elements. A description of the derivation of the D-S elements is given. Two modifications are introduced which increase the numerical stability of the system. The differential equations are established in Gaussian form fit for numerical integration. All associated transformation formulae and partial derivatives are described in detail.

(This is an abstract of a study performed by G. Scheifele and R. Samway. The complete study is given in the Masters Thesis of Robert C. Samway, The University of Texas at Austin, August, 1973. The Thesis may be obtained through the Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin.)

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