APPLICATION OF HAMILTON'S LAW OF VARYING ACTION

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ABSTRACT

The Law of Varying Action enunciated by Hamilton in 1834-1835 permits the direct solution of the problems of Mechanics, stationary or non-stationary. All problems of statics but only certain problems of dynamics fall under the classification of "stationary" to which direct solutions have heretofore been possible. It has been impossible to obtain direct solutions to non-stationary problems through the present state of energy theory which has been limited for 138 years to Hamilton's principle. It will be shown that Hamilton's Law permits the direct solution of non-stationary as well as stationary problems in the mechanics of solids without any knowledge or use of differential equations. Whether a system is conservative or non-conservative is of no consequence. Initial conditions, final conditions, and boundary conditions pose no problem. In this introductory paper, direct

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solutions are demonstrated for conservative and non-conservative, stationary and/or non-stationary particle motion. The generality of this theory of mechanics, which is free of the constraints imposed by the mathematics of differential equations, will be demonstrated in subsequent papers on stationary and non-stationary motion of beams and plates. All of these papers will stress three major points: 1) simplicity, 2) generality, and 3) accuracy.

NOMENCLATURE

A_i	coefficients of power series time variable
B_k	coefficients of power series time variable
c	damping force coefficient. May be function of time
c_o	damping coefficient at $t = t_o$
c_1	rate of change of damping coefficient
c_c	critical damping coefficient
F_o	force at $t = t_o$
g	Gravitational parameter. Taken as constant in this paper
$i, j; k, \ell$	subscripts
k	spring force coefficient. May be function of time
k_o	spring coefficient at $t = t_o$
k_1	rate of change of spring coefficient
m	mass. May be function of time
m_o	total mass at $t = t_o$
m_1	rate of change of mass in variable mass problem; also mass number one in two-degree of freedom problem
M, N	number of terms in truncated power series. Also degree of resulting polynomial
Q_i	general force acting in general displacement direction
q_i	dependent space variable, generalized displacement
T	kinetic energy, work of inertial forces
T	period, seconds, $T = 2\pi/\omega$
T_s	static thrust of rocket, $T_s = m_1 V_e$
t	real time
t_o	time at which observation of phenomena begins

t_1	time at which observation of phenomena ends
V_0	initial velocity at $t = t_0$
W	work of all forces other than inertial forces
y, $\dot{y}$	dependent space variable; displacement, velocity
y_0	initial displacement at $t = t_0$
δ	operates on displacement while forces are held constant
ζ	non-dimensional damping coefficient, $\zeta = c/c_c$
θ_1, θ_2	dependent space variables, angular displacement
θ_{10}, θ_{20}	angular displacement at $t = t_0$
$\dot{\theta}_{10}$, $\dot{\theta}_{20}$	angular velocity at $t = t_0$
τ	non-dimensional time, $\tau = t/t_0$
ω	circular frequency, radians/sec., harmonic motion

INTRODUCTION

Hamilton set forth the Law of Varying Action in papers concerning a general method in dynamics, published in 1834 and 1835.[1] When the varied paths were assumed to be co-terminus with the real path in both space and time, Hamilton's Law reduced to Hamilton's principle.[1] By the year 1937, when Osgood published his text, "Mechanics,"[2] Hamilton's principle had been established as "...an independent foundation of mechanics."[2] Osgood simply postulates the integral,

$$\int_{t_0}^{t_1} (T + W)\, dt \tag{1}$$

which he called Hamilton's integral and then obtains the following equation by varying Hamilton's integral and integrating the kinetic energy term by parts:

$$\delta \int_{t_0}^{t_1} (T + W)\, dt = \frac{\partial T}{\partial \dot{q}_i} \delta q_i \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i \right) \delta q_i\, dt \tag{2}$$

where generalized coordinates have been used instead of the Cartesian coordinates used by Osgood, and where δt has been set equal to zero at the outset. Following Hamilton, Osgood assumes all of the δq_i to vanish at t_0 and t_1; i.e., the end points of the varied paths are postulated to be co-terminus with the real path; thus, the first term on the right vanishes. The integrand of the second term on the right is the equations of Lagrange which are known to vanish. Eq. 2 then reduces to Hamilton's principle,

$$\delta \int_{t_0}^{t_1} (T+W)\,dt = 0 \qquad (3)$$

where it is understood that W is the work of both conservative and non-conservative forces.

When it is observed that Lagrange's equations vanish whether or not the δq_1 vanish at t_0 and t_1, eq. 2 results in the mathematical expression for the Law of Varying Action where δt has been set equal to zero for the Newtonian and/or Lagrangian mechanics to be discussed herein,

$$\delta \int_{t_0}^{t_1} (T+W)\,dt - \frac{\partial T}{\partial \dot{q}_i}\delta q_i\Big]_{t_0}^{t_1} = 0 \qquad (4)$$

Note that the zero on the right hand side of eq. 4 results from the fact that nature requires the equations of equilibrium to vanish as observed by Newton. It has nothing to do with the proof in variational calculus that the integral is an extremum. The meaning of eq. 4 is then quite clear,

> "A particle or a system or particles will follow a path and/or assume a configuration such that the equation,
>
> $$\delta \int_{t_0}^{t_1} (T+W)\,dt - \frac{\partial T}{\partial \dot{q}_i}\delta q_i\Big]_{t_0}^{t_1} = 0$$
>
> is satisfied."

Another interesting point is that the integral on the right hand side of eq. 2, when taken alone, is identical to the result obtained when Galerkin's method is applied to Lagrange's equation. Here, without any reasonings as to setting the weighted error equal to zero, it is seen that the equation,

$$\int_{t_0}^{t_1}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i\right)\delta q_i \, dt = 0 \qquad (5)$$

vanishes if and only if eq. 4 vanishes. Thus, the Ritz[3] method of solution correctly applied to eq. 4 can readily be shown to be identical to the Galerkin method applied to the differential equations of Lagrange or Euler-Lagrange, as in eq. 5. Although the Lagrange or Euler-Lagrange equations are equally available from eq. 3 and eq. 4 it is well known that the direct solution method of Ritz cannot be applied to Hamilton's principle with the same generality with which Galerkin's method can be applied to the differential equations of Lagrange or Euler-Lagrange.

It has been obvious to competent researchers that something is wrong somewhere. Bisplinghoff and Ashley on p. 36 of Ref. 4, identify the problem,

> "No difficulty is encountered when Lagrange's equations can be constructed, for these are differential equations which may, in principle, be integrated from instant to instant. But the question of how to handle the upper limit t_1 during direct application of Hamilton's principle is a more subtle one."

The wording of this statement implies two things: 1) that it may not be possible to construct Lagrange's equations for a physical system from Hamilton's principle and 2) that Bisplinghoff and Ashley did not have the slightest suspicion that the problem lies directly in the concepts with which Hamilton's principle has been passed along for 138 years.

Fung, p. 318, Ref. 5, arrives at a restricted form of Hamilton's Law for a deformable body,

> "In some applications of the direct method of calculation it is even desirable to liberalize the variations δu_i at the instant t_0 and t_1 and use Hamilton's principle in the variational form (4) which cannot be expressed elegantly as the mimimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. On introducing (5), (7), and (10), we may rewrite eq. (4) in the following form
>
> $$(13)\ \int_{t_0}^{t_1} \delta(U - K + A)dt = \int_{t_0}^{t_1}\int_V F_i\,\delta u_i\,dV dt + \int_{t_0}^{t_1}\int_S \tilde{T}_i\,\delta u_i\,dS dt + \int_V \rho \frac{\partial u_i}{\partial t}\delta u_i\,dV\Big]_{t_0}^{t_1}."$$

First, Fung's eq. (13) is not Hamilton's principle. Because of the last term and the concepts under which it was derived, it is a very restricted form of Hamilton's Law. Second, Fung gives no indication as to how the problem of the upper limit, t_1, as pointed out by Bisplingfhoff and Ashley, is to be treated. Third, this writer is not aware of any successful attempt to achieve a direct solution to any non-stationary problem of mechanics by the use of Fung's concepts and his eq. (13). For stationary problems, his eq. (13) reduces immediately, as does Hamilton's Law, to Hamilton's principle.

The lack of generality of the direct solutions obtainable from Hamilton's principle may have undermined confidence in the accuracy obtainable for such solutions. The writer's experience with the literature indicates that such is the case. This problem is also implied by Fung's beginning sentence on p. 336[5], "If we try to approximate...". This implies that we may not be able to approximate and would discourage one from trying. A sequence of papers on non-stationary motion of particles, beams, plates, and shells will demonstrate that Hamilton's Law will yield answers with almost unbelievable simplicity and accuracy. The breadth of the problems, both stationary and non-stationary, will serve to demonstrate the generality of Hamilton's Law when applied with the proper concepts.

It is the purpose of this paper to demonstrate the application of Hamilton's Law to achieve completely general solutions to stationary and non-stationary particle motion problems by direct application of eq. 4. To enable this to be accomplished with great simplicity, two well known observations are made:

(1) The path of any mass particle through space-time is continuous; i.e., no particle of matter can occupy two points in space at the same instant in time.

(2) The slope of the space-time path of any particle is continuous.

Thus, the space-time path of any particle of matter is continuous with continuous first derivatives. This is precisely part of the requirement for admissible functions when using the Ritz method in conjunction with stationary problems in the theory of elasticity.[3]

To introduce the simplicity of the application, only linear, one and two degree of freedom particle motion will be treated in this paper. Many degrees of freedom are treated in deformable body problems.

ONE DEGREE OF FREEDOM

Consider a particle of constant mass on which acts a linear restoring force, k(t)y, a linear viscous damping force c(t)$\dot{y}$ and a force that is a specified function of time F(t). In other words, this is the linear forced-damped-spring-mass system for which the differential equation is well-known and to which general solutions for comparison of results may be readily obtained from the differential equation. By substituting the kinetic energy and the work of the prescribed forces into eq. 4 and taking the variation, one obtains the equation,

$$\int_{t_0}^{t_1} \left\{ m\dot{y}\,\delta\dot{y} + (F - Ky - c\dot{y})\,\delta y \right\} dt - m\dot{y}\,\delta y \Big|_{t_0}^{t_1} = 0 \qquad (6)$$

The convential procedure at this point is to derive the differential equation from which the solution may be obtained. The purpose of this paper, however, is to demonstrate that the solution may be obtained without any reference to or knowledge of differential equations. By virtue of the fact that the time-space path is continuous with continuous first derivatives, it is possible to use as an admissible function a simple truncated power series,

$$y = y_0 + V_0 t + \sum_{i=2}^{N} \bar{A}_i t^i$$

Convergence is assured by Weirstrass' theorem.[6] Note that no concept of the shape of the time-space path is necessary. The power series must satisfy the specified displacement and velocity at $t = t_0$ but not at $t = t_1$. The displacement and velocity at t_1 cannot in general be known in advance because they are the result of both the initial conditions and the time history of the forces acting between t_0 and t_1. By the use of constraints, conditions can be, of course, imposed at t_1.

In eq. 6, m, k, and c, may be, any one or all, functions of time. However, for this example, assume these parameters to be constant. Put eq. 6 into more convenient computational form by non-dimensionalizing. Let, $t = t_1\tau$ and divide by m/t_1^2. Let the instant in time, t_0 at which the observation begins, be $t_0 = o$. Eq. 6 becomes,

$$\int_0^1 \left\{ \dot{y}\,\delta\dot{y} - \frac{ct_1}{m}\dot{y}\,\delta y - \frac{kt_1^2}{m} y\,\delta y + \frac{t_1^2}{m} F(t_1\tau)\,\delta y \right\} dt - \dot{y}\,\delta y \Big|_0^1 = 0 \qquad (7)$$

Now the admissible function is simply,

$$y = y_0 + V_0 t_1 \tau + \sum_{i=2}^{N} A_i \tau^i \qquad (8)$$

Substitute eq. 8 into eq. 7, note that the integral of the sum equals the sum of the integrals. Integrate (for arbitrary functions, numerical integration is used) to obtain a set of algebraic equations. These equations expressed in matrix form are,

$$\left[\frac{ij}{i+j-1} - i - \frac{ct_i}{m}\frac{i}{i+j} - \frac{Kt_i^2}{m}\frac{1}{i+j+1}\right]\left\{A_i\right\} =$$

$$-\frac{t_i^2}{m}\left\{\int_0^1 F(t_i\tau)\,\tau^j\,d\tau\right\} + \left\{\frac{(cv_0+Ky_0)}{m(j+1)}t_i^2 + \frac{Kt_i^3 v_0}{m(j+2)}\right\} \qquad (9)$$

$$i, j = 2, 3, \ldots, N$$

Eq. 9 constitutes the general solution to the system in that the time-space path yielded by the solution to this matrix equation is the sum of the particular and complementary solutions as would be obtained from the differential equation. Note the initial conditions as well as the damping coefficient appear in the non-homogeneous term. When these parameters, along with $F(t_1\tau)$ and t_1, are specified, the solution may be obtained. In particular, note that even with F = 0, the equations are not homogeneous as in the case of the differential equation.

t_1 is arbitrary. However, it is kept relatively small with the understanding that a longer period of time can be examined simply by taking the final conditions as calculated for one interval as the initial conditions of the next interval. It should be emphasized that the number of terms required in the truncated power series is not the important criteria from a practical viewpoint. The computer time required for solutions is the important criteria. Ten terms in the time variable have been found to be sufficient for all non-stationary problems of particles, beams, and plates treated to date. With this number of terms in the truncated power series, the computer time for every case of single degree of freedom particle motion was below the minimum amount ($1.68) charged for the computer and the accuracy, as will be shown, was far above expectation.

Both non-stationary problems and simple harmonic motion can be treated with equal ease. It may be of interest to point out that if $F(t_1\tau)$, c, and V_0 are set equal to zero, eq. 9 will generate a cosine curve. If $F(t_1\tau)$, c and y_0 are set equal to zero, eq. 9 will generate a sine curve. These, of course are precisely the same functions as defined by the differential equation of the simple spring-mass system. The important point to be made is this: without any knowledge of the mathematical functions involved in the answer, eq. 4 generates from the power series whatever function that is required to yield the solution which, in the case of particle motion, is the time-space path. For a deformable body, it is both the time-space path and configuration as will be demonstrated in subsequent papers.

Convergence is not only a function of the number of terms used, but is also a function of the complexity of the motion within the interval t_0 to t_1. Since t_1 is arbitrary, it may be chosen as some arbitrary number or as some characteristic time of the system; e.g., the period or a fraction of the period of one of the free vibration modes. Further, sufficient accuracy must be utilized in the calculations to insure accuracy of the results. Although many problems may be solved and the exact answers obtained by hand calculation through the use of rational numbers or by use of eight place arithmetic in the computer, fourteen to sixteen place arithmetic (double precision on the IBM 370-165 computer) is standard for this work.

Table I gives the results of a study of the accuracy obtainable from eq. 9 when $F = c = V_{\hat{0}} = 0$. The exact answer is the cosine function. The point (any point could have been used) chosen for comparison is $\mathrm{Cos}\,\omega t_{\hat{1}}\tau = 1$. The results show the extreme accuracy obtainable for three values of N, two values of $t_{\hat{1}}$ and for 16-place arithmetic and 32-place arithmetic. Few practical problems require the accuracy of calculation that is available.

Solutions to four linear, one-degree of freedom problems will be demonstrated:

1) Step forcing function with damping.
2) Polynomial forcing function with damping.
3) The rocket problem (variable mass).
4) Step force acting on a variable mass with variable damping and variable resisting force.

These examples, except for the last, were chosen because exact solutions may readily be obtained from the differential equation for comparison of results. The power of Hamilton's Law is illustrated with the last example, however, where the solution is obtained with the same ease by the use of Hamilton's Law as is the solution to a simple harmonic motion problem. The same statement cannot be made relative to the solution to the differential equation. Even in those cases where the differential equation cannot be solved, it is readily available from Hamilton's Law and the accuracy of the direct solution can readily be checked, if desired, by substitution of the direct solution into the differential equation.

EXAMPLE I. STEP FORCE WITH DAMPING, $y_0 = V_0 = 0$

Various solutions, including the exact solution to this problem by the use of Laplace transforms is given on pages 662-672 of Ref. 7. There a finite difference solution by Houbolt is also given. The exact solution as given in Ref. 7 is,

$$yK/F_0 = 1.0 - e^{-2s}\left[\cos(19.899s) + 0.1005\sin(19.899s)\right] \qquad (10)$$

where,

$$s = \pi\zeta\tau/2(1-\zeta^2)^{1/2}\ ; \quad \tau = t/t_1$$

$$t_1 = \pi/(K(1-\zeta^2)/m)^{1/2}$$

Because of the accuracy of the direct solution, it is necessary to show the comparison of the numbers obtained in tabular form rather than plotted curves. Fig. 1 shows the results in both forms.

EXAMPLE II. POLYNOMIAL FORCING FUNCTION WITH DAMPING

A more difficult problem than the preceding, from the standpoint of the differential equation, is a forcing function varying as some arbitrary function of time. For this example, the force and the parameters are arbitrarily specified to be,

$$F = F_0 (c_1 t_1 \tau + c_2 t_1^4 \tau^4)$$

$$y_0 = V_0 = 0$$

$$\zeta = 0.25$$

$$t_1 = \pi / (K(1-\zeta^2)/m)^{1/2}$$

$$c_1 t_1 = -0.7229$$

$$c_2 t_1^4 = 1.024$$

The exact solution is found to be,

$$yK/F_0 = 0.1807 - 0.0934\tau - 0.8754\tau^2 - 0.6311\tau^3 + 1.024\tau^4 - e^{-0.8115\tau}[0.1807 \cos\pi\tau + 0.0169 \sin\pi\tau] \quad (11)$$

Fig. 2 shows a comparison of the direct solution to the exact solution.

EXAMPLE III. THE ROCKET PROBLEM (VARIABLE MASS)

The particular example given here may be found in several textbooks, in particular, Ref. 7 and Ref. 8. When the gravitational force field, g, the burning rate, m_1, and exit velocity, V_e, are taken to be constant, eq. 6 becomes:

$$\int_{t_0}^{t_1} \left\{ (m_0 - m_1 t)\dot{y}\,\delta\dot{y} - (m_0 - m_1 t) g\,\delta y + m\,(V_e - \dot{y})\,\delta y \right\} dt \qquad (12)$$

$$- (m_0 - m_1 t)\,\dot{y}\,\delta y \Big|_{t_0}^{t_1} = 0$$

The set of algebraic equations resulting from non-dimensionalization of the above equations and the substitution of eq. 8 with $y_0 = V_0 = 0$ are, in matrix form,

$$\left[m_0 \frac{i(1-i)}{i+j-1} - m_1 t_1 \frac{i(1-i)}{i+j} \right] \left\{ A_i \right\} = \left\{ \frac{(m_0 g - T_s) t_1^2}{j+1} - \frac{m_1 g t_1^3}{j+2} \right\} \qquad (13)$$

When the initial mass, m_0, the static thrust, T_s, and the rate of mass change, m_1, are specified, the time-space path (one dimensional motion in this simple case) may be found. The result for one particular choice of parameters is shown in Fig. 3 where the velocity obtained by differentiation of the direct solution is compared to the velocity from a first integration of the differential equation as given in Refs, 7 and 8.

EXAMPLE IV. FORCING FUNCTION APPLIED TO A SYSTEM WITH VARIABLE MASS, VARIABLE DAMPING, AND VARIABLE SPRING FORCE

In a general problem of this nature, numerical integration is used to evaluate the matrix elements in the event that the integrands are defined by, say, curves generated from test data. However, to illustrate the generality without getting into such details, the following functions are assumed:

$$m(t) = m_0 - m_1 t \quad ; \quad c(t) = c_0 + c_1 t$$

$$K(t) = K_0 + K_1 t$$

$$F = T_s - m_1 \dot{y} - (m_0 - m_1 t)\, g$$

The matrix equation obtained from eq. 6 is,

$$\left[m_0 \frac{i(1-i)}{i+j-1} + t_1 \frac{m_1 i(1-i) - c_0 i}{i+j} - t_1^2 \frac{c_1 i + K_0}{i+j+1} - t_1^3 \frac{K_1}{i+j+2} \right] \left\{ A_i \right\} =$$

$$\left\{ t_1^2 \frac{m_0 g - T_s + c_0 V_0 + K_0 y_0}{j+1} + t_1^3 \frac{c_1 V_0 + K_0 V_0 + K_1 y_0 - m_1 g}{j+2} + t_1^4 \frac{K_1 V_0}{j+3} \right\} \quad (14)$$

$$i, j = 2, 3, \ldots, N$$

Note that example III is a special case of this example. Fig. 4 shows the resulting displacement and velocity for an arbitrary choice of parameters. No solution to the differential equation was obtained in this case; but, substitution of the direct solution for $y, \dot{y}$ and $\ddot{y}$ into the differential equation showed equilibrium of the forces to be satisfied with the same general accuracy as in the preceding examples. The percentage error is shown for two points in time on Fig. 4.

TWO DEGREES OF FREEDOM

For the sake of brevity, only two examples of two degrees of freedom will be presented. Both are classic problems. The differential equations, but not necessarily the solutions, may be found in any text on vibration theory. The problems are,

1) The spring coupled pendulum, Fig. 5

2) The double pendulum, Fig. 6

Small angles are assumed at the outset. Non-linear motion will be treated in separate papers.

When the kinetic energy and work terms for the system with coordinates and forces acting as shown in Fig. 5, are substituted into eq. 4 with,

$$\theta_1 = \theta_{10} + \dot{\theta}_{10} t_1 \tau + \sum_{i=2}^{N} A_i \tau^i \qquad (15a)$$

$$\theta_2 = \theta_{20} + \dot{\theta}_{20} t_1 \tau + \sum_{k=2}^{M} B_k \tau^k \qquad (15b)$$

a coupled matrix equation results,

$$\begin{bmatrix} C_{ij} & D_{kj} \\ E_{il} & F_{kl} \end{bmatrix} \begin{Bmatrix} A_i \\ B_k \end{Bmatrix} = - \begin{Bmatrix} G_j \\ H_l \end{Bmatrix} \quad \begin{matrix} i,j = 2,3,\ldots,N \\ k,l = 2,3,\ldots,M \end{matrix} \qquad (16)$$

The matrix elements are given in the appendix. Although both damping and forcing functions can easily be included as in the next example, they have been omitted here to give a conservative system for which the free vibration frequencies and modes if desired could be found from the assumption of simple harmonic motion and the resulting eigenvalue problem; because, as is well known, a stationary solution exists. However, the above equation yields the solution directly without any assumption of SHM, and no eigenvalue is involved. To achieve the motion of this system in either one of its two natural modes in the laboratory, one must know the answer in advance, so that one can release the system in precisely the configuration that exists at the instant that every particle in the system would have zero velocity; or one must impart precisely the correct velocity to every particle when that particle is in the precise position at which the imparted velocity would be the correct value. If such conditions are known in advance, and are put in the above matrix equation as initial conditions, simple harmonic motion in the mode corresponding to the initial conditions will result from the calculation. In general, however, it is much more practical to prescribe an initial condition, whatever it may be, and calculate directly the resulting motion.

Fig. 5 shows the solution for an arbitrary choice of parameters when the initial conditions are taken to be $\theta_{10} = .08$ rad.; $\theta_{20} = \dot{\theta}_{10} = \dot{\theta}_{20} = 0$. Anyone who has ever observed the motion of such a pendulum system, will recognize the energy exchange in the calculated displacement curves to be precisely as observed in the physical system. If one calculates the energy, it will be found to be the same at every instant. The accuracy of the solution may also be judged from the repetitive amplitudes as the energy is exchanged between the pendulums with on-going time.

2) The Double Pendulum

When the kinetic energy and work of the moments for the double pendulum shown in Fig. 6, are substituted into eq. 4 with

$$\theta_1 = \theta_{10} + \dot{\theta}_{10} t_1 \tau + \sum_{i=2}^{N} A_i \tau^i \tag{15a}$$

$$\theta_2 = \theta_{20} + \dot{\theta}_{20} t_1 \tau + \sum_{k=2}^{M} B_k \tau^k \tag{15b}$$

the matrix equation is precisely of the form as the previous example but the matrix elements are not the same,

$$\begin{bmatrix} I_{ij} & J_{kj} \\ K_{il} & L_{kl} \end{bmatrix} \begin{Bmatrix} A_i \\ B_k \end{Bmatrix} = - \begin{Bmatrix} P_j \\ Q_l \end{Bmatrix} \tag{17}$$

The matrix elements are given in the appendix for the case where the damping and spring coefficients are taken to be constant.

To demonstrate the versatility and power of Hamilton's Law, one half wave of a sinusoidal moment was applied to the second mass with no damping in the system. The half period of the sinusoidal moment was taken to be .01 seconds. Fig. 6 shows approximately two cycles of the resulting motion at which time another sinusoidal moment of the same magnitude and duration was applied to the second mass in the opposite direction, drastically decreasing the amplitude of the motion. Had this energy been put in at a different instant, the amplitude could have been increased drastically. The initial conditions and parameter values are shown in Fig. 6.

To see the effect of damping, Fig. 7 shows the same sinusoidal moment applied to the system at t=o in which damping is present. The parameters are as listed in Fig. 7. The value of damping chosen to act on the second pendulum was quite high and appears to be slightly above critical for the second pendulum. However, the first pendulum has crossed the zero reference line just before a sinusoidal moment, opposite in direction to the first, is again applied. This time, energy is added to the system by this moment instead of being dissipated as in the previous example, Fig. 6.

No detail on the initial motion during application of the force is available from Fig. 6 or Fig. 7. However, the response during this interval, $t_{\hat{0}}$ to $t_{\hat{1}}$, must be accurately calculated because the conditions at the end of this period are taken as the initial conditions for the calculation of the motion over the next interval $t_{\hat{0}}$ to $t_{\hat{1}}$. Fig. 8 is plotted to a scale that shows the very beginning of the motion during application of the moment. This feature of Hamilton's Law, i.e., being able to take $t_{\hat{1}}-t_{\hat{0}}$ arbitrarily small, permits the calculation of the initial motion and wave travel in beams and plates as will be demonstrated in subsequent papers. Fig. 8 shows the initial motion curves with and without damping. Damping, whether below critical, critical, or above critical is of no consequence in the calculation.

Hamilton's Law, properly applied, yields the solution for whatever forces that act on or within the system without second guessing and without prior knowledge of what the solution has to be. The accuracy may be checked at any time by simply substituting into the differential equations. These equations are always readily available from Hamilton's Law without regard to the theory of functionals as set forth in the variational calculus.

CONCLUSION

It has been shown that, contrary to the state of energy theory found in textbooks and in the variational calculus, direct solutions to non-stationary particle motion may be obtained through application of Hamilton's Law for both conservative and non-conservative systems. In fact, no differentation needs to be made as to whether a system is conservative, non-conservative, stationary, or non-stationary. Constraints were not treated explicitly in this first introductory paper; but, future papers will treat both holonomic and non-holonomic systems without identification and without benefit of mentioning Lagrangian multipliers.

Only linear systems will be treated in the first sequence of papers; but, this work developed from a study of what one might suppose to be a totally unrelated study of thermally stressed plates subjected to non-linear large deflections. When the solution to that problem was finally attained, the meaning of Hamilton's Law as a means of achieving direct solutions to the problems of mechanics had been discovered (or rediscovered?). This non-linear work has been temporarily set aside, but will be offered for publication when its logical place in a sequence of papers on the subject has been reached.

It should be pointed out that an integration by parts of the kinetic energy with respect to time in Hamilton's Law will yield the inertia forces and will cause the term, $\partial T/\partial \dot{q}_i \, \delta q_i \Big]_{t_0}^{t_1}$, to cancel from the equation, leaving Hamilton's Law precisely in the form obtained when Galerkin's procedure is applied to the differential equation of a particle. This result is also precisely that obtained from Hamilton's principle when, following an integration by parts, the term, $\left.\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_0}^{t_1}$, is arbitrarily set equal to zero because "... the end points must be co-terminus in both space and time".[2] The result, from which direct solutions to non-stationary problems may be obtained is, of course, no longer Hamilton's principle but eq. 5, which is completely equivalent to Hamilton's Law. When independent space variables are involved, as in deformable body problems, the application of Galerkin's method applied to the differential equations is subject to constraints not found in the application of Hamilton's Law; e.g., the treatment of the boundary conditions which will be discussed in subsequent papers.

APPENDIX

The Matrix elements for the two degree of freedom spring coupled pendulums are:

$$C_{ij} = -m_1 \ell_1^2 i(i-1)/(i+j-1) - t_1^2 (Kh^2 + m_1 g \ell_1)/(i+j+1)$$

$$D_{kj} = Kh^2 t_1^2/(j+k+1)$$

$$E_{i\ell} = Kh^2 t_1^2/(i+\ell+1)$$

$$F_{k\ell} = -m_2 \ell_2^2 k(k-1)/(k+\ell-1) - t_1^2 (Kh^2 + m_2 g \ell_2)/(k+\ell+1)$$

$$G_j = t_1^2 \left[Kh^2(\theta_{20} - \theta_{10}) - m_1 g \ell_1 \theta_{10} \right]/(j+1)$$
$$+ t_1^3 \left[Kh^2(\dot{\theta}_{20} - \dot{\theta}_{10}) - m_1 g \ell_1 \dot{\theta}_{10} \right]/(j+2)$$

$$H_\ell = t_1^2 \left[Kh^2(\theta_{10} - \theta_{20}) - m_2 g \ell_2 \theta_{20} \right]/(\ell+1)$$
$$+ t_1^3 \left[Kh^2(\dot{\theta}_{10} - \dot{\theta}_{20}) - m_2 g \ell_2 \dot{\theta}_{20} \right]/(\ell+2)$$

The Matrix elements for the forced, damped, double pendulum with springs are:

$$I_{ij} = -(m_1+m_2)\ell_1^2(i-1)i/(i+j-1) - (c_1+c_2)t_1 i/(i+j)$$
$$-\left[K_1+K_2+(m_1+m_2)g\ell_1\right]t_1^2/(i+j+1)$$

$$J_{kj} = m_2\ell_1\ell_2 jk/(i+k-1) + c_2 t_1 k/(j+k) + K_2 t_1^2/(j+k+1)$$

$$K_{i\ell} = m_2\ell_1\ell_2 i\ell/(i+\ell-1) + c_2 t_1 i/(i+\ell) + K_2 t_1^2/(i+\ell+1)$$

$$L_{k\ell} = -m_2\ell_2^2 k(k-1)/(k+\ell-1) - c_2 t_1 k/(k+\ell)$$
$$-(K_2+m_2 g\ell_2)t_1^2/(k+\ell+1)$$

$$P_j = \int_0^1 (M_1-M_2)\tau^j d\tau + t_1\dot{\theta}_{20} m_2\ell_1\ell_2 - t_1^2\left[(c_1+c_2)\dot{\theta}_{10} + c_2\dot{\theta}_{20}\right.$$
$$\left.+(K_1+K_2)\theta_{10} - K_2\theta_{20} + (m_1+m_2)g\ell_1\theta_{10}\right]/(j+1)$$
$$-t_1^3\left[(K_1+K_2)\dot{\theta}_{10} - K_2\dot{\theta}_{20} + (m_1+m_2)g\ell_1\dot{\theta}_{10}\right]/(j+2)$$

$$Q_\ell = \int_0^1 M_2\tau^\ell d\tau + t_1\dot{\theta}_{10} m_2\ell_1\ell_2 - t_1^2\left[c_2(\dot{\theta}_{20}-\dot{\theta}_{10}) + K_2(\theta_{20}-\theta_{10})\right.$$
$$\left.+m_2 g\ell_2\theta_{20}\right]/(\ell+1) - t_1^3\left[K_2(\dot{\theta}_{20}-\dot{\theta}_{10})\right.$$
$$\left.+m_2 g\ell_2\dot{\theta}_{20}\right]/(\ell+2)$$

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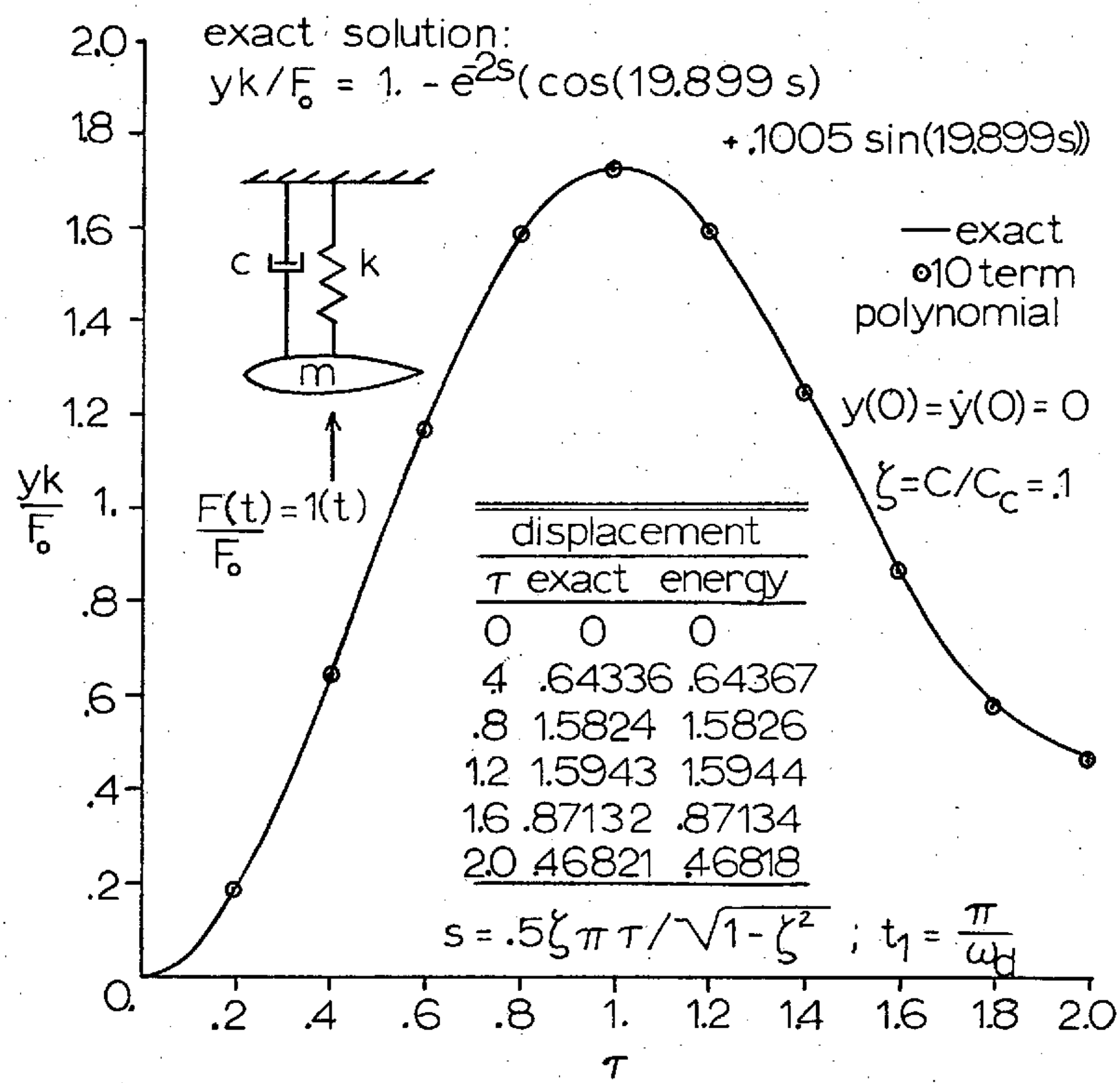
TABLE 1. CONVERGENCE

N Degree of Polynomial	Number of Cycles Calculated	$t_1 = T/4$[a] % error[b] 16-place Arithmetic	$t_1 = T/4$[a] % error[b] 32-place Arithmetic	$t_1 = 3T/4$[a] % error[b] 16-place Arithmetic	$t_1 = 3T/4$[a] % error[b] 32-place Arithmetic
			Exact, $\cos \omega t_1 \tau = 1.0$		
10	1	-2.6×10^{-2}	8.7×10^{-9}	1.36×10^{-1}	6.5×10^{-4}
10	10	-6.8×10^{-2}	8.9×10^{-8}	-1.38	4.5×10^{-3}
10	25	-1.5×10^{-1}	2.3×10^{-7}	-3.41	1.1×10^{-2}
12	1		1.0×10^{-11}		-7.4×10^{-6}
12	10		2.0×10^{-9}		-5.2×10^{-5}
12	25		1.3×10^{-8}		-1.3×10^{-4}
15	1		1.9×10^{-11}		5.5×10^{-9}
15	10		2.1×10^{-9}		4.0×10^{-8}
15	25		1.3×10^{-8}		1.1×10^{-7}

a. $T = 2\pi/\omega ; \omega = (k/m)^{1/2}$

b. % error = (1 - calculated value) x 100

top



displacement		
τ	exact	energy
0	0	0
.4	.64336	.64367
.8	1.5824	1.5826
1.2	1.5943	1.5944
1.6	.87132	.87134
2.0	.46821	.46818

Fig. 1 C. D. Bailey

top

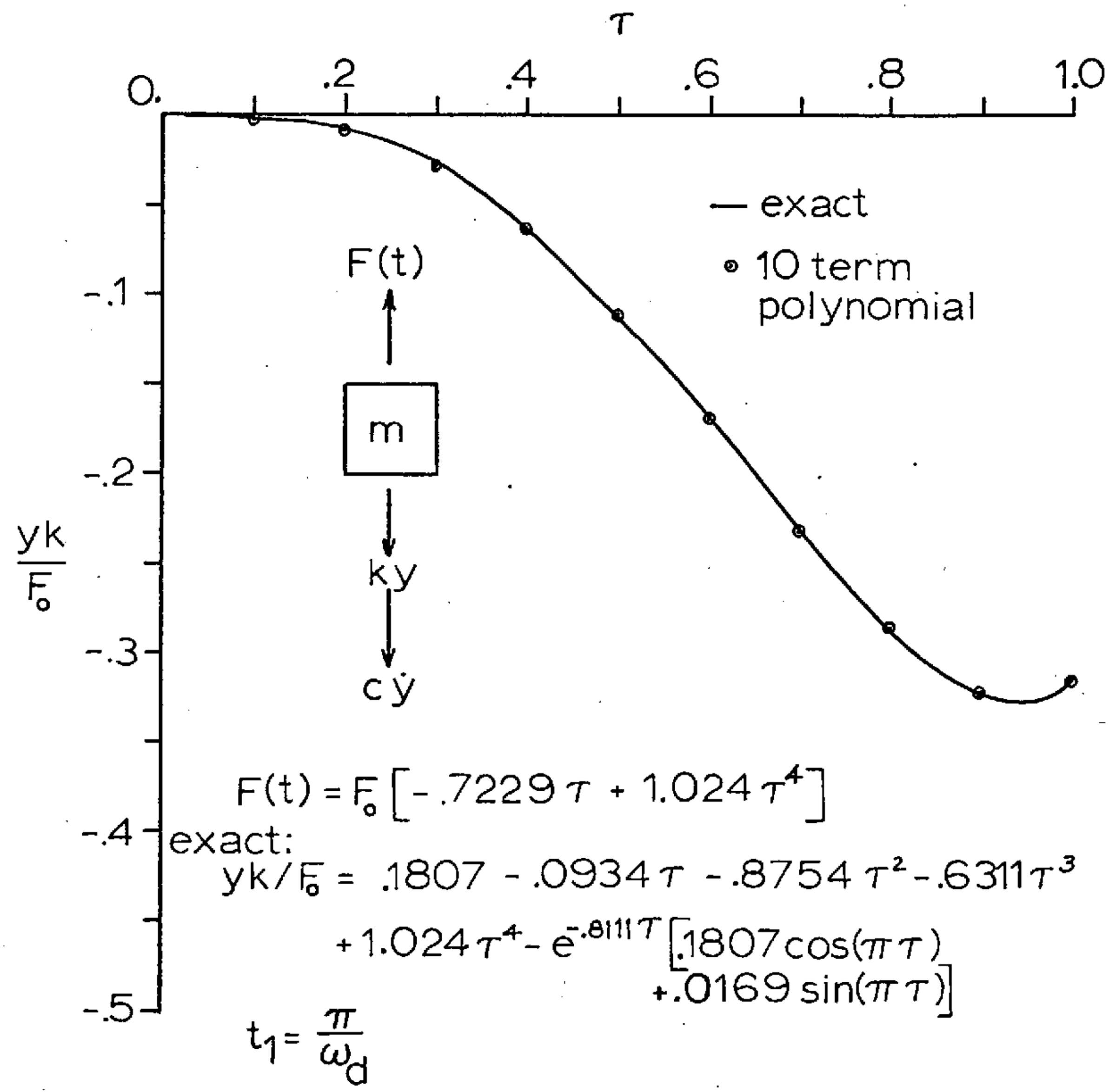
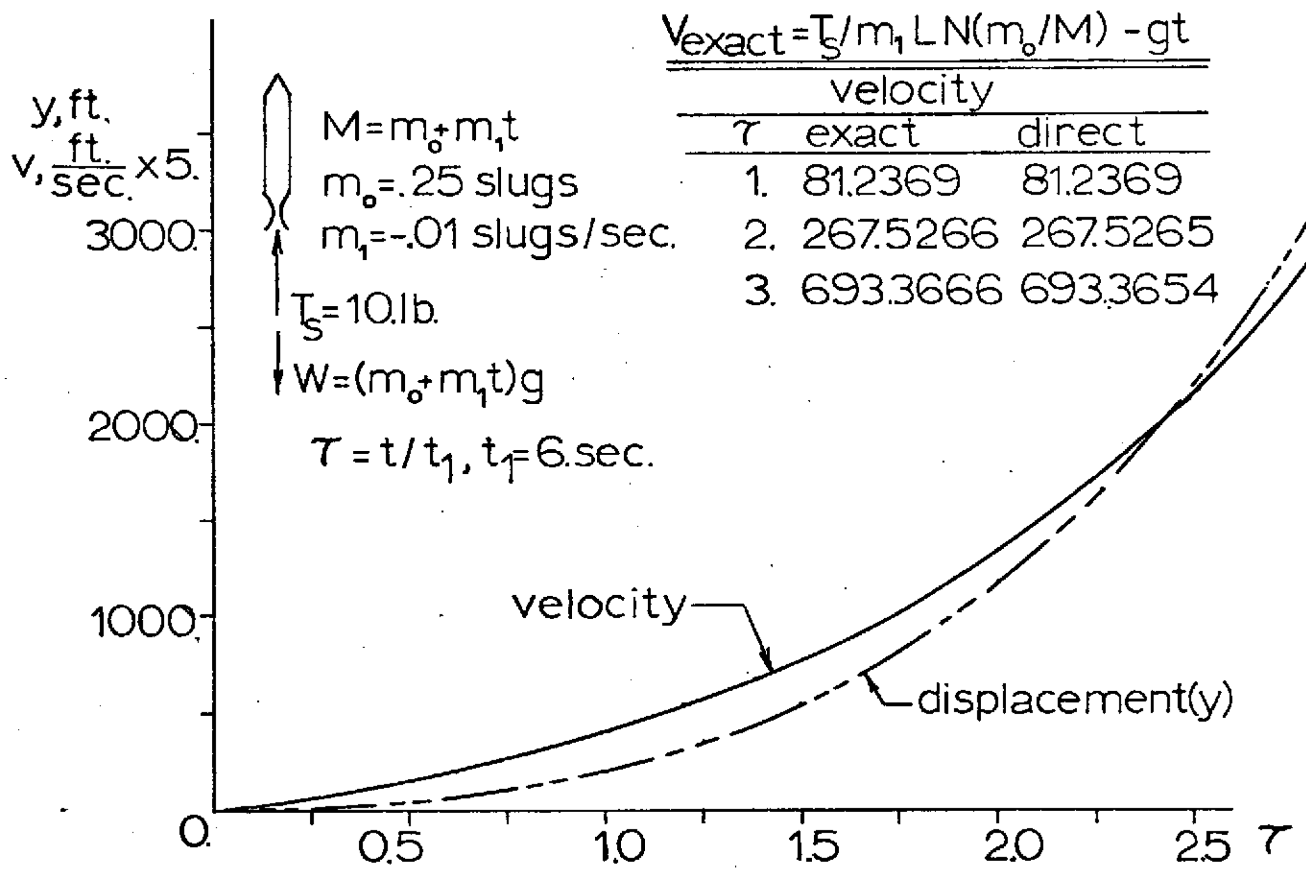


Fig. 2, C.D. BAILEY, Particle Paper


y, ft.
v, ft./sec. ×5.
3000.
2000.
1000.
0.
0.5
1.0
1.5
2.0
2.5
τ
M=m₀+m₁t
m₀=.25 slugs
m₁=-.01 slugs/sec.
Ts=10.lb.
W=(m₀+m₁t)g
τ = t/t₁, t₁=6.sec.
Vexact=Ts/m₁ LN(m₀/M) -gt
velocity
τ exact direct
1. 81.2369 81.2369
2. 267.5266 267.5265
3. 693.3666 693.3654
velocity
displacement(y)

TOP

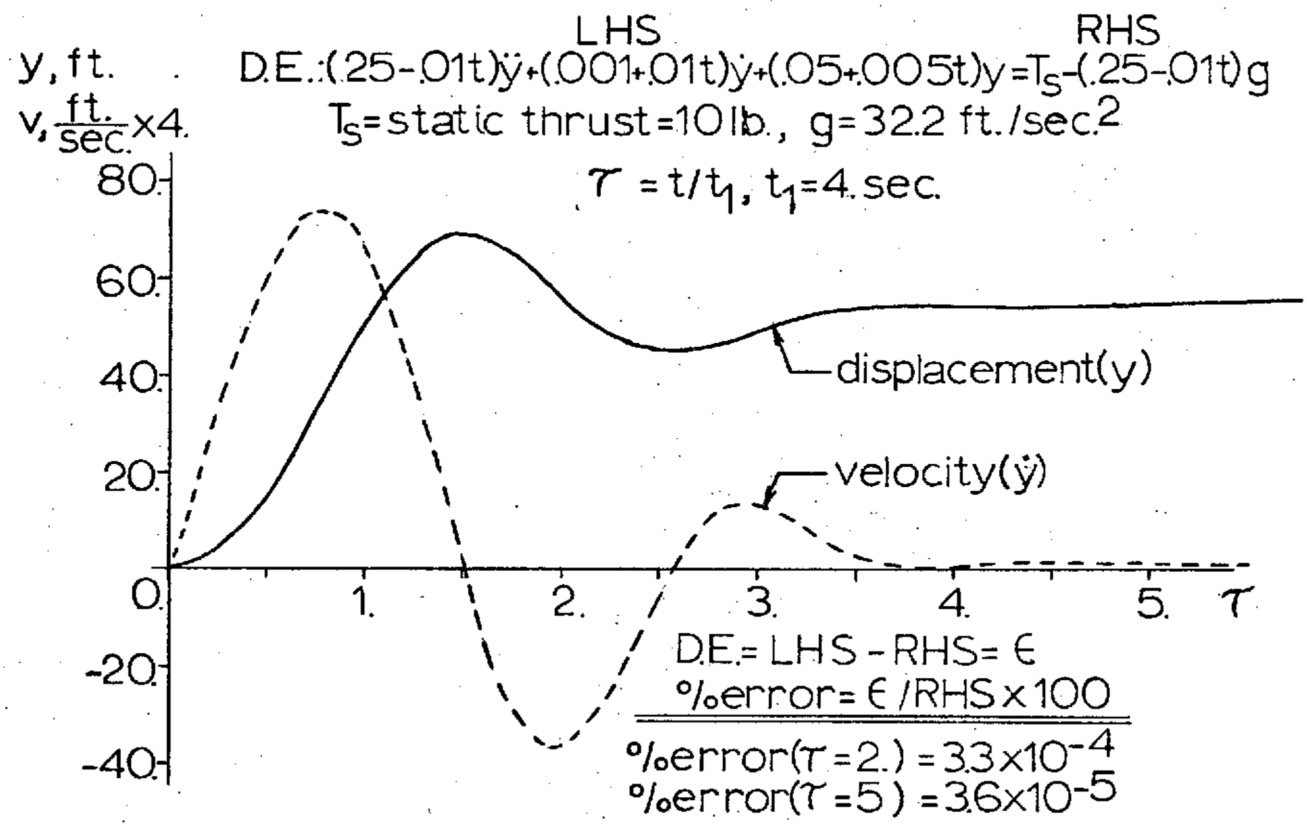


Fig. 4, C. D. Bailey

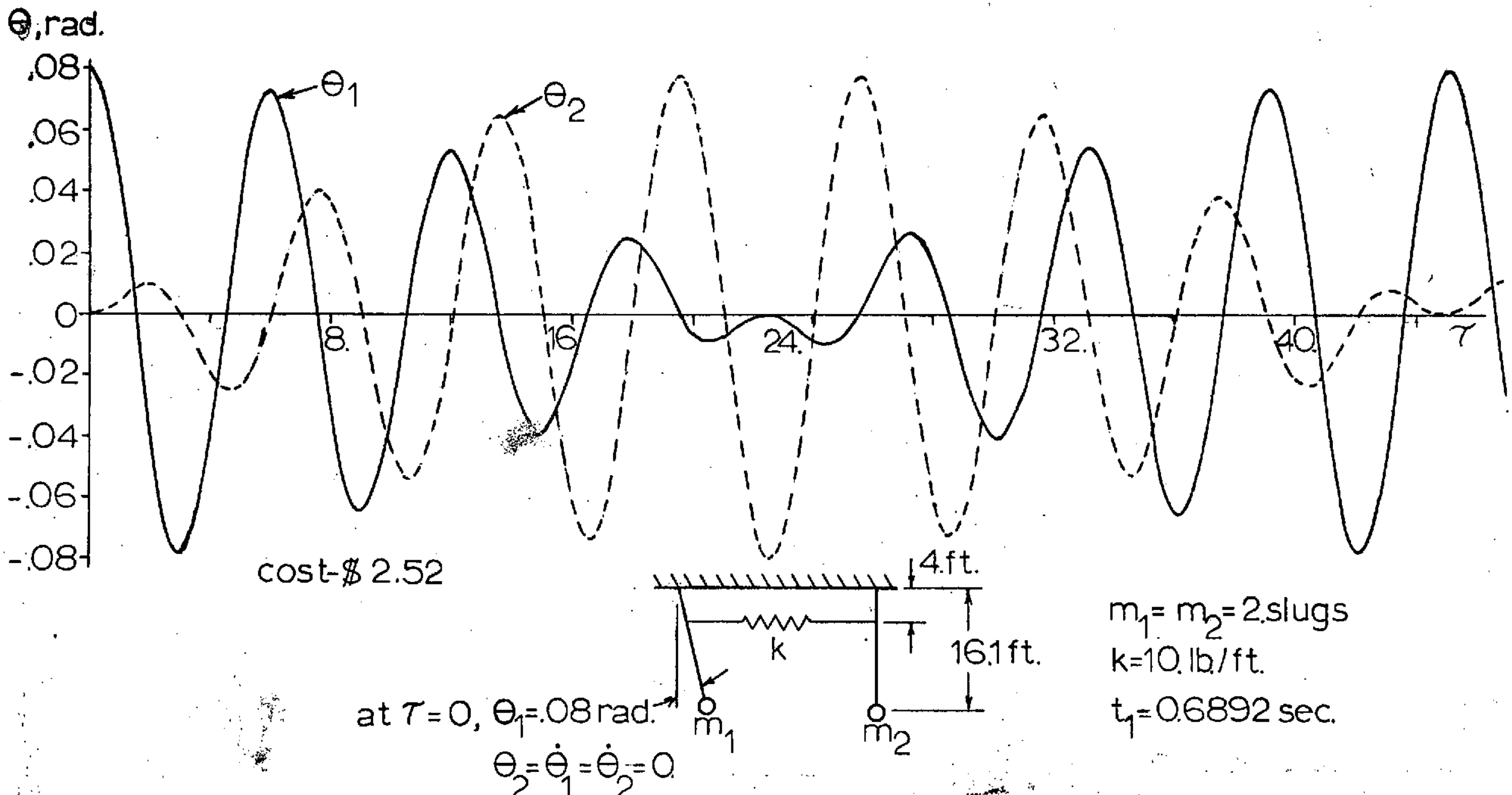

θ, rad.
.08
.06
.04
.02
0
-.02
-.04
-.06
-.08
θ1
θ2
8.
16.
24.
32.
40.
τ
cost-$ 2.52
4 ft.
k
16.1 ft.
at τ = 0, θ1 = .08 rad.
θ2 = θ̇1 = θ̇2 = 0
m1
m2
m1 = m2 = 2 slugs
k = 10. lb./ft.
t1 = 0.6892 sec.

Fig. 5, C.D. [illegible]

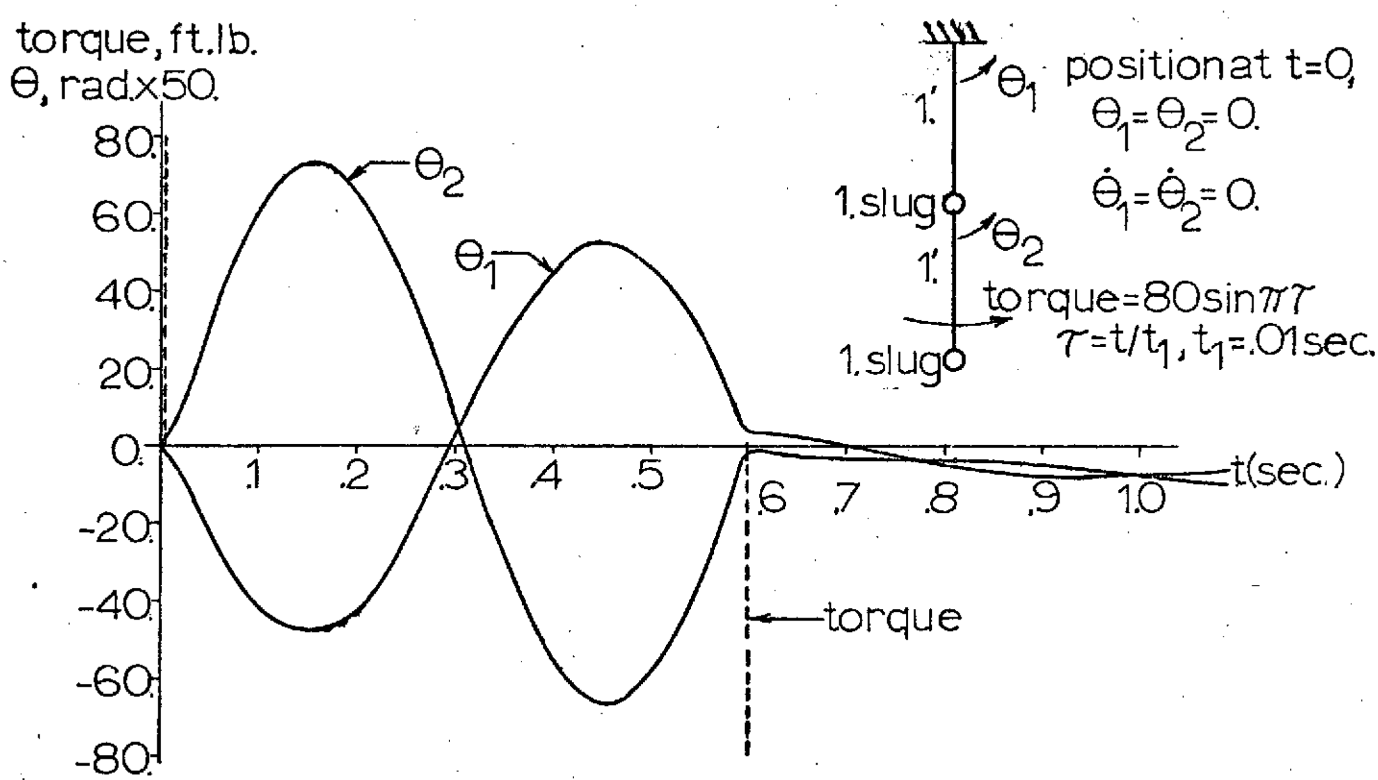


Fig. 6, C.D. Bailey, Particle

Top

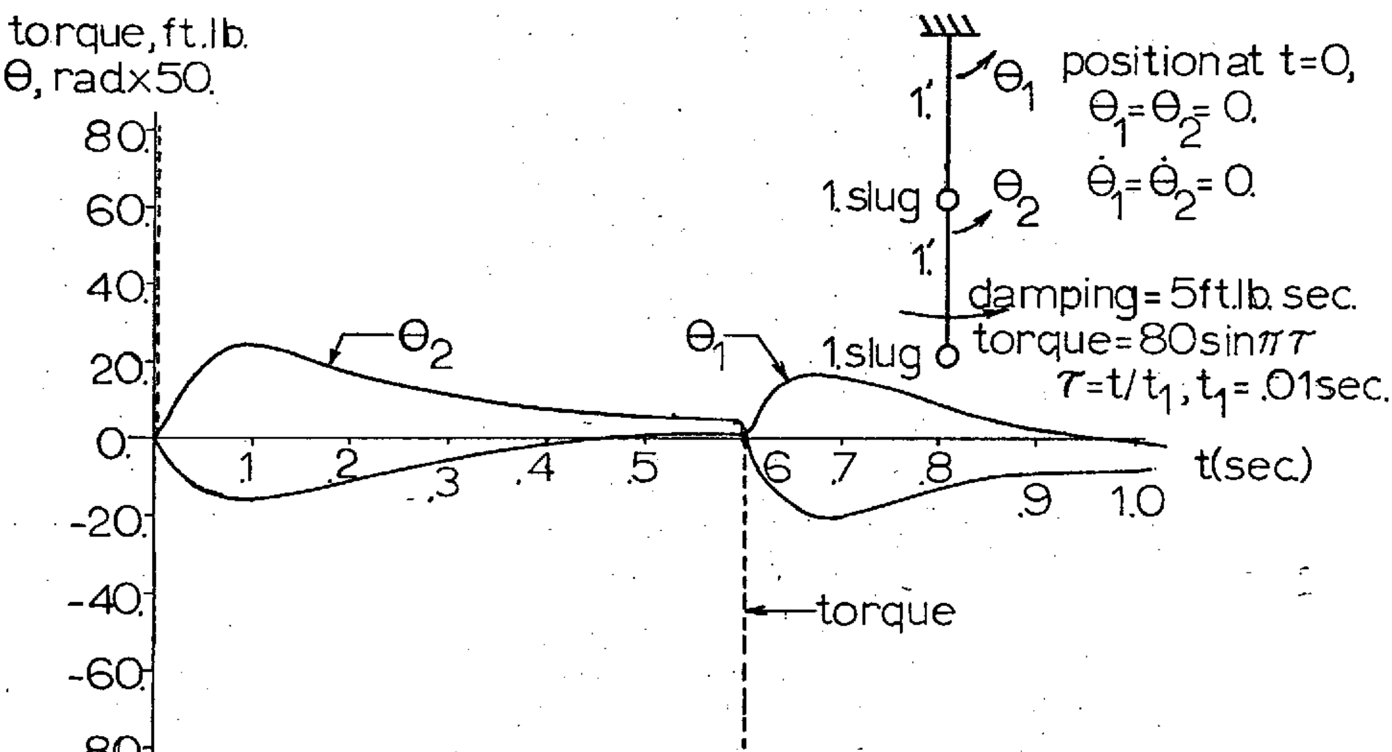


Fig. 7, C. D. Bailey, particle Paper

Top

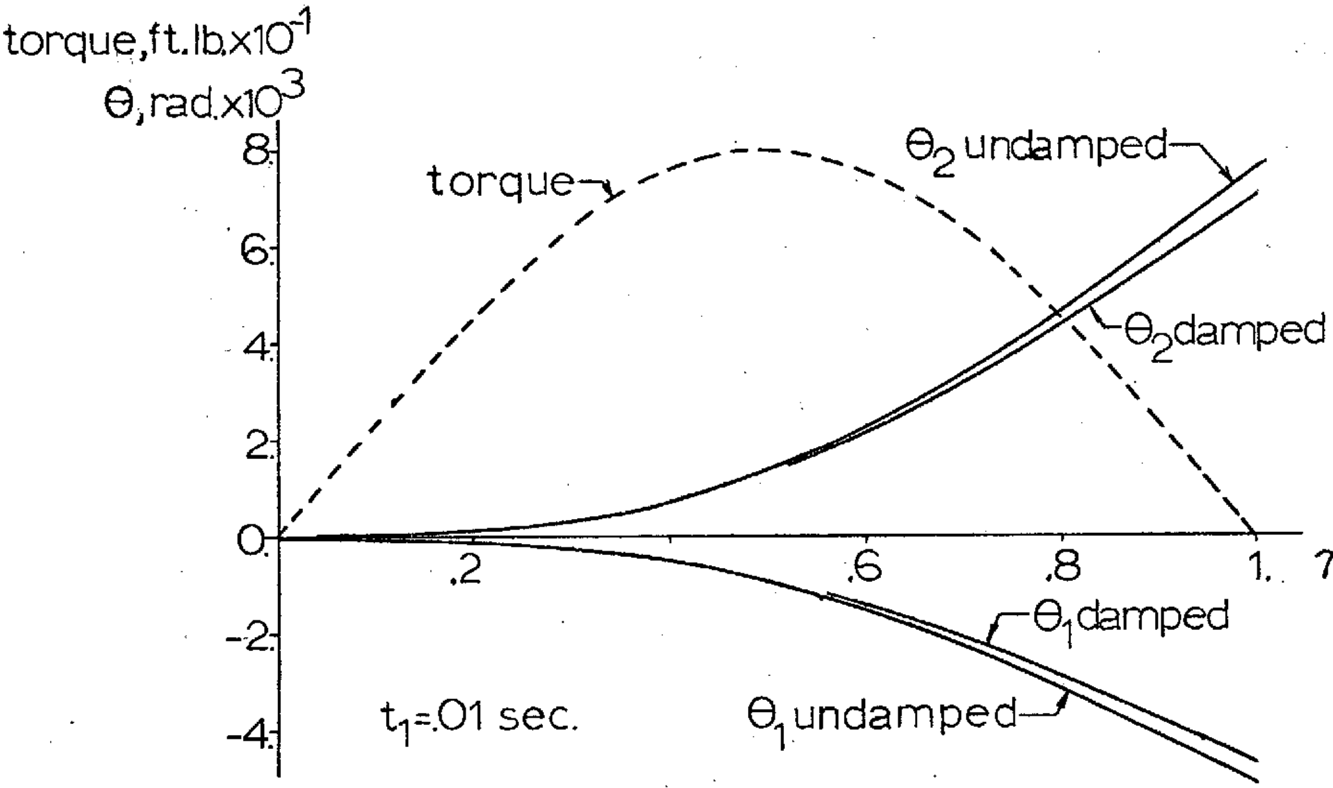


Fig. 5, C.D. Bailey, Particle Paper

LIST OF FIGURES