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ABSTRACT

The rational use of optimality criteria is investigated for a class of structural synthesis problems where materials, configuration and applied load conditions are specified, and the minimum weight design is to be determined. This study seeks to explore the potential of hybrid methods of structural optimization for dealing with relatively large design problems involving practical complexity. The reduced basis concept in design space is used to decrease the number of generalized design variables dealt with by the mathematical programming algorithm. Optimality criteria methods for obtaining design vectors associated with displacement, system buckling and natural frequency constraints are presented. A stress ratio method is used to generate a basis design vector representing the stress constraints. The finite element displacement method is used as the basic structural analysis tool.

The optimality criteria are first derived for a general case and then modified for each type of behavior constraint. From these optimality criteria, recursive redesign relations are obtained for multiple constraints of the same behavioral type. In order to achieve high efficiency, design variable linking and temporary deletion of noncritical constraints are employed. The need for actual structural analyses is reduced by using first order Taylor series expansions to explicitly approximate the dependence of stresses and displacements on reciprocal design variables. Computer programs are written implementing some of the methods developed.

Results for several examples of truss systems subject to stress, displacement and minimum size constraints are presented. An assessment of these results indicates the effectiveness of the hybrid method developed.

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LIST OF SYMBOLS

A_i	Element size
$\vec{A}$	Element size vector
a_{ij}	Linking coefficient
C_j	Redesign ratio
c	Safety factor
D_j	Design variable
$\vec{D}$	Design variable vector
$\vec{F}$	Load vector
$\vec{f}$	Relative load vector, virtual load vector
g	Behavior constraint
i,j,k,	Subscript
J	$\{j \mid D_{jmin} < D_j < D_{jmax}\}$
J_{max}	$\{j \mid D_j = D_{jmax}\}$
J_{min}	$\{j\} D_j = D_{jmin}\}$
K_{act}	$\{k \mid g_k = 0\}$
[K]	System stiffness matrix
$[K_G]$	Geometric stiffness matrix
$[k_i]$	Element stiffness matrix
L_i	Length, surface area
[M]	System mass matrix
$[m_i]$	Element mass matrix
n	Number of basis design vector
NAC	Number of active constraints
NC	Number of behavior constraints

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NDV	Number of design variable
NM	Number of element
p	Buckling load parameter
q	Square of natural frequency
$\vec{r}$	Virtual displacement vector
s	Cycle of iteration
u	Displacement
$\vec{u}$	Displacement vector
W	Weight
w_j	Weight coefficient
X,Y,Z	Cartesian coordinates
x,y	Slack variable

Greek

α,β,ε	Scalar
δ	Increment mark
θ	Generalized design variable
$\vec{\theta}$	Generalized design variable vector
λ	Lagrange multiplier
ρ_i	Mass density
σ	Normal stress
τ	Shear stress
ω	Natural frequency
$g_{k,j}$	$\left(\frac{\partial g_k}{\partial D_j}\right)_{\vec{D}}$

$(\ )_s$	( ) at the sth iteration
*	Optimal or specified value

CHAPTER I

INTRODUCTION

During the past fifteen years, considerable progress has been made in the area of automated optimum design of structural systems. Optimum structural design methods have been studied intensively and significant contributions have been made by many investigators. These research studies can be usefully classified into three main categories as follows:

(1) Application of mathematical programming algorithms;

(2) Application of recursive redesign formulas based on fully stressed design methods and/or discretized optimality criteria concepts;

(3) Mixed or hybrid optimization methods.

In the first category, structural design problems are stated as inequality constrained minimization problems and solved numerically using mathematical programming techniques such as linear, nonlinear and dynamic programming. The most attractive feature of this approach is its generality in the sense that a broad class of structural optimization problems can, in principle, be treated in a unified manner. This approach, pioneered by Schmit[1], has actually enjoyed considerable success in a wide range of practical design problems [2,3,4 and 5]. However, by about 1970 it became apparent that the application of this method, in combination with finite element structural analysis, to large scale structural optimization problems required inordinately large numbers of analyses and long run times to solve problems of only modest proportions [6,7].

This situation led some investigators to abandon the mathematical programming approach and direct renewed effort toward implementing recursive redesign procedures based on fully stressed design concepts and discretized optimality criteria. One of the first research efforts to focus attention on the discretized optimality criteria approach was reported by Venkaya, Khot and Reddy in Ref. [8]. This early effort has in recent years been followed by several notable studies [9,10,11 and 12] which pursue this same basic line of investigation. These studies have shown that the optimality criteria approach, including fully stressed design concepts, is well suited to achieving high efficiency in appropriate specialized situations where only one type of behavior constraint dominates the optimum design. Indeed, during the past few years a widely held viewpoint has been that while mathematical programming methods are well suited to component design optimization, they are not practical for dealing with large structural systems. This assessment of the state of the art is well illustrated by the mixed optimization method for automated design of fuselage structures reported in Ref. [13]. In this work a fully stressed approach is used to obtain a gross overall distribution of material while the detailed design of rings and stiffened panels is carried out using a mathematical programming technique.

Nevertheless, other investigators, still attracted to the mathematical programming approach by its generality, focused their efforts on a quest for the efficiency improvements within the structural synthesis context [14]. With the same goal in mind, another type of mixed optimization technique was presented by Pickett. The reduced basis concept in design space and its initial exploration was reported in Refs. [15] and [16]. This work provides the foundation for a new group of mixed optimization techniques that will be called hybrid methods of structural optimization. Using the reduced basis concept it is often possible to drastically reduce the number of independent generalized design variables needed to obtain a good upper bound approximation of the optimum design. This is of crucial importance since the computational effort required to solve a mathematical programming problem grows rapidly as the number of design variables increases. It now appears that the reduced basis concept in design space is one of the most promising approximation concepts in structural synthesis, although there are many open questions that will require further study from both a theoretical and a practical point of view.

The study reported here may be classified as a mixed optimization method. The objective of this investigation is to explore the potential of hybrid methods of structural optimization using basis design vectors generated by optimality criteria and stress ratio methods. Primary attention is focused on the methods to be used in generating the basis design vectors. As previously mentioned, optimality criteria methods are well suited to the problem if only one type of behavior constraint is involved. It follows that the optimality criteria approach can ideally be used to generate basis design vectors if they are generated for each type of behavior constraint separately. Therefore, the optimality criteria approach is employed in this study as a tool for generating basis design vectors. Consequently, primary effort is concentrated on the derivation of discretized optimality criteria and the development of rational redesign procedures based on them. Although the basic statements of discretized optimality criteria are quoted from published descriptions [9-12], this work attempts to establish a general viewpoint with respect to optimality criteria and their rational usage.

CHAPTER II

PROBLEM FORMULATION

AND

SOLUTION METHOD

2.1 Formulation of Problem

Structural synthesis problems are considered for a structure which is modeled as an assemblage of a number of discrete finite elements. It is assumed that the types of elements used to model the structure are restricted to those which have the following special properties:

(1) element stiffness is proportional to its mass;

(2) there is one representative measure of intensity of stress state in each element.

Elements of this type include axial force elements, shear panels, or constant strain triangles. The material of each element and the configuration of the structure are specified beforehand. All loads are applied directly or work equivalently at the nodes of the assemblage. The cross sectional area for an axial force element, and the thickness for a shear panel or a constant strain triangular membrane are taken as the design variables, and the minimum weight design of the structure is determined subject to both behavior and geometric constraints. The former includes allowable tensile, compressive or combined stress, upper and lower bounds on joint displacement, lower bounds on general buckling loads and natural frequencies, and the latter includes minimum and maximum element sizes. Arbitrary design variable linking is also taken into account.

<u>Objective function</u>

The total weight of the structure is taken as the objective function to be minimized and it is given by

$$W = \sum_{i=1}^{NM} \rho_i L_i A_i \tag{2.1}$$

where NM is the fixed total number of elements representing the structure, ρ_i is the specified mass density of the element i, L_i is the given length for an axial force element and the preassigned surface area for a shear panel or constant strain triangle element i. The A_i denotes the single design variable sizing each element i, namely the cross sectional area for axial force elements or the thickness for shear panel and constant strain triangular elements.

<u>Constraints</u>

(1) Stress constraints

For an axial force element the stress constraints are given by

$$\sigma_{ci} \leqq \sigma_{ij} \leqq \sigma_{ti} \tag{2.2}$$

where σ_{ci} and σ_{ti} denote the allowable compressive and tensile stresses for the element i, respectively, and σ_{ij} represents the stress in member i under load condition j. For a planar element, shear panel or constant strain triangle, the stress constraint is given by

$$\sigma_{oij} \leqq \frac{\sigma_{si}}{c} \tag{2.3a}$$

where σ_{oij} is the combined stress in element i under load condition j, σ_{si} denotes the yield stress of the material used in element i, and c is a safety factor ($c > 1$). The combined stress is defined by

$$\sigma_o = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3 \tau_{xy}^2)^{\frac{1}{2}} \tag{2.3b}$$

where σ_x and σ_y are normal stresses in the x and y directions, respectively, and τ_{xy} represents the inplane shear stress referred to the x-y axes.

(2) Displacement constraints

Displacement constraints are defined by

$$u_{Lij} \leqq u_{ij} \leqq u_{Uij} \tag{2.4}$$

where u_{Lij} and u_{Uij} are the lower and upper bounds on the ith joint displacement under the jth load condition, respectively, and u_{ij} is the ith joint displacement under load condition j.

(3) Buckling constraints

Any loading system applied on the structure can be expressed as

$$\vec{F}_j = \alpha_j \vec{f}_j \tag{2.5}$$

where $\vec{F}_j$ represents the load vector for the jth loading system, $\vec{f}_j$ denotes the relative load vector in which the element of largest absolute magnitude has the value 1 or -1, and α_j is a given scaling factor. Let $p_{ij}\vec{f}_j$ be a load vector corresponding to buckling of the structure in the ith buckling mode under the jth load condition. Then the buckling constraints associated with the jth load condition are given by

$$p_{ij} \geqq c\,\alpha_j \tag{2.6}$$

where c is a safety factor. At first glance, it may appear that the design procedure need only consider a single critical buckling mode. However, it has been shown that the buckling mode corresponding to the lowest load can shift from one mode to another during the design procedure. Therefore it is appropriate to include multiple buckling mode constraints for each load condition.

(4) Natural frequency constraints

Natural frequency constraints are given by

$$\omega_{iL}^2 \leqq \omega_i^2 \leqq \omega_{iU}^2 \tag{2.7}$$

where ω_i is the ith natural frequency of the structure, and ω_{iL} and ω_{iU} are specified lower and upper bounds on the frequency, respectively.

(5) Size constraints

Simple side constraints on the design variables are given by

$$A_{i\,min} \leqq A_i \leqq A_{i\,max} \tag{2.8}$$

where A_{imin} and A_{imax} are the minimum and maximum allowable element size for element i, respectively.

(6) Linking of element sizes

In actual design problems, any group of two or more elements may be required to: (1) have the same size, or (2) maintain fixed relative proportions throughout the design process. In order to implement this practical feature, new design variables D_j are introduced such that

$$A_i = a_{ij} D_j \tag{2.9}$$

Equation (2.9) means that element size A_i is controlled by the jth design variable D_j, and its magnitude is given by $a_{ij}D_j$. It is to be understood that a_{ij} is positive and that it is the linking coefficient between A_i and D_j.

From equations (2.2) through (2.7), it is apparent that all the behavior constraints under consideration can be expressed in the following general form

$$g_k \leqq 0 \tag{2.10}$$

For example, a tensile stress constraint is given by

$$g_k = \sigma_i - \sigma_{ti} \leqq 0 \tag{2.11}$$

where k is an arbitrary subscript used to identify a given constraint. Since the behavior constraints involve structural response quantities such as σ_i, u_{ij}, p_{ij}, and ω_i, it follows that the g_k are functions of the independent design variables, that is

$$g_k = g_k(\vec{D}) \tag{2.12}$$

where $\vec{D}$ represents the vector of independent design variables.

Formulation of a design problem

Finally the problem can be stated in the following general form:

Minimize

$$W = \sum_{i=1}^{NM} \rho_i L_i A_i$$

subject to

$$\begin{aligned} & g_k \leqq 0 , \qquad k = 1, 2, \cdots, NC \\ & A_{i\,min} \leqq A_i \leqq A_{i\,max} \\ & A_i = a_{ij} D_j , \qquad \begin{cases} i = 1, 2, \cdots, NM \\ j = 1, 2, \cdots, NDV \end{cases} \end{aligned} \tag{2.13}$$

where NC is the total number of behavior constraints, and NDV is the number of independent design variable D_j, which must be less than the number of elements NM.

2.2 Solution Method

As shown in the preceeding section, the structural synthesis problem considered here can be expressed in the following mathematical form:

Minimize $\quad W = W(\vec{D})$

subject to

$$g_k(\vec{D}) \leqq 0 , \qquad k = 1, 2, \cdots, NC \tag{2.14}$$

$$A_{i\,min} \leqq A_i(\vec{D}) \leqq A_{i\,max} , \quad i = 1, 2, \cdots, NM$$

where $\vec{D}$ is a NDV dimensional design variable vector.

The optimization problem represented by equation (2.14) can not be solved analytically, because for most practical structures the behavior constraints $g_k(\vec{D})$ cannot be expressed as explicit functions of the design variables. Therefore mathematical programming techniques are usually required to solve the problem. However, as mentioned in Chapter I, mathematical programming methods are not economically feasible for large scale problems in which the number of design variables exceeds one or two hundred. This is primarily due to the high dimensionality of the design space. Therefore, a simple and effective method to overcome this difficulty is to directly reduce the number of design variables. One of the promising methods for achieving this was presented by Pickett in Ref. [15], and the essential idea is now employed here.

Consider a set of basis design vectors D_ℓ, $\ell = 1,2,\ldots,n$ with $n <$ NDV or $n <<$ NDV and define a set of generalized design variables θ_ℓ such that

$$\vec{D} = \sum_{\ell=1}^{n} \theta_\ell \vec{D}_\ell \tag{2.15}$$

The optimization problem represented by equations (2.12) is now reduced to an n dimensional optimization problem in terms of the θ_ℓ, $\ell = 1,2,\ldots,n$.

Minimize $\quad W = W(\vec{\theta})$

subject to

$$\begin{aligned} &g_k(\vec{\theta}) \leq 0, \qquad k = 1, 2, \cdots, NC \\ &A_{i\,min} \leq A_i(\vec{\theta}) \leq A_{i\,max}, \quad i = 1, 2, \cdots, NM \end{aligned} \tag{2.16}$$

As shown in Appendix B, the optimum design of the problem (2.16) is not, in general, the same as that of the problem (2.14) and it must be viewed as an approximation of the actual optimum design.

To use the reduced dimensionality technique, a set of basis design vectors must be determined at the outset. They may be generated by a variety of methods. In this study effort is directed toward generating basis design vectors in the following way:

(1) Classify the behavior constraints into several groups such as the stress group, the displacement group,

and so on. Then construct a set of subproblems, each of which has the same objective function as the original problem, while including only member size constraints and some of the behavior constraints.

(2) Solve each subproblem by methods based on optimality criteria concepts as discussed in Chapter III, then take each subproblem solution as a basis design vector.

After determining a set of basis vectors, there are several efficient mathematical programming methods that can be employed to solve the problem as stated in (2.16). In this investigation a program called CONMIN developed by Vanderplaats [17], which is based on a modified feasible directions method, was used. The overall optimization procedure employed herein is outlined in Fig. 1.

The optimality criterion approach can be useful in its own right. However, these methods encounter practical difficulties when the optimum design is governed by more than one type of critical behavior constraint. Therefore, it should be recognized that optimality criteria methods may, in the future, be employed primarily as a source of basis design vectors for hybrid methods of structural optimization.

On the other hand, as will be shown in the following chapter, problems involving size constraints and only

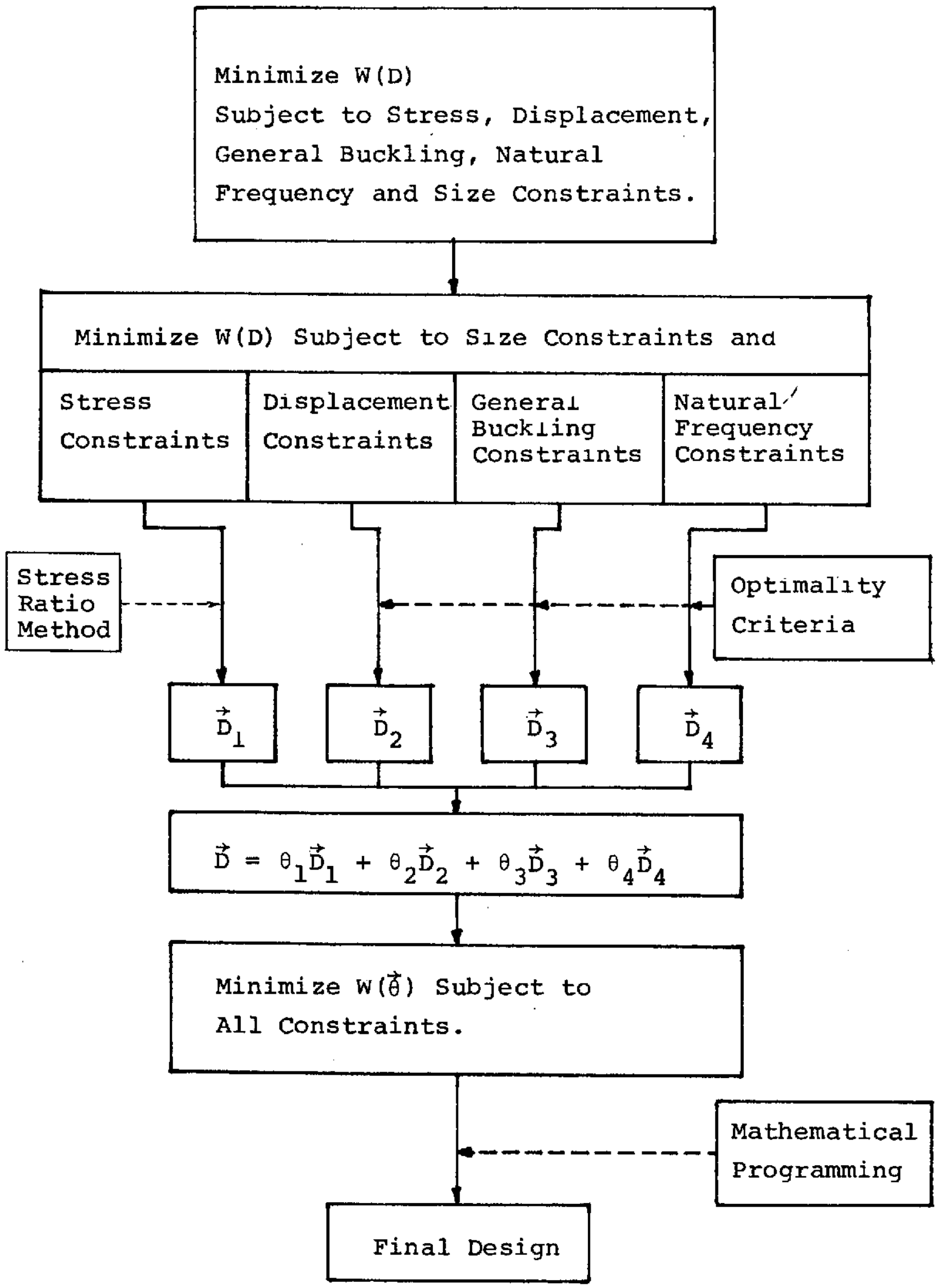

Fig. 1 Block Diagram of Solution Method

one type of behavior constraint can be treated effectively using the optimality criterion approach. In fact, including size constraints, in addition to a single type of behavior constraint, sometimes improves the computational efficiency. Furthermore, in those cases where a single type of behavior constraint dominates the optimum design of the complete problem, the appropriate basis design vector, generated including size constraints, will represent the actual optimum design exactly. Therefore, it is expected that the quality of basis design vectors may be improved if size constraints are included in each subproblem which generates a basis design vector.

CHAPTER III

THEORY OF OPTIMALITY CRITERIA

The purpose of this chapter is to formulate the optimality criteria principle and to develop the discretized recursive procedure which will be used primarily to generate basis design vectors for each type of behavior constraint. The objective is to minimize the total weight of the structure, and all size constraints are simultaneously considered in each case.

In section 3.1 the basic optimality criteria concept and the associated recursive procedure is presented in a general manner. In the following secitons the same concepts are applied to each of several single behavior constraint types.

3.1 General Concepts

We will consider the following minimum weight design problem

Minimize
$$W = \sum_{i=1}^{NM} \rho_i L_i A_i \tag{3.1a}$$

subject to

$$g_k(\vec{A}) \leq 0 , \qquad k = 1, 2, \cdots, NC \tag{3.1b}$$

$$A_{i\,min} \leq A_i \leq A_{i\,max} \tag{3.1c}$$

$$A_i = a_{ij} D_j , \qquad j = 1, 2, \cdots, NDV \tag{3.1d}$$

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Using equation (3.1d), a design variable transformation is carried out, leading to the problem statement expressed in terms of the independent design variables after linking, namely

Minimize

$$W = \sum_{j=1}^{NDV} w_j D_j \tag{3.2a}$$

subject to

$$g_k(\vec{D}) \leq 0, \qquad k = 1, 2, \cdots, NC \tag{3.2b}$$

$$D_{j\,min} \leq D_j \leq D_{j\,max} \tag{3.2c}$$

where

$$D_{j\,min} = \underset{i \in j}{Max} \left(\frac{A_{i\,min}}{a_{ij}} \right) \tag{3.2d}$$

$$D_{j\,max} = \underset{i \in j}{Min} \left(\frac{A_{i\,max}}{a_{ij}} \right) \tag{3.2e}$$

Now the optimality criteria for the problem given by equations (3.2a) through (3.2c) will be derived. Let $\vec{D}$ be a feasible design, i.e.,

$$g_k(\vec{D}) \leq 0 \qquad , \text{ for all } k \tag{3.3a}$$

$$D_{j\,min} \leq D_j \leq D_{j\,max} \quad , \text{ for all } j \tag{3.3b}$$

but not necessarily an optimal one. It is assumed that g_k is differentiable for all k and the following notation is adopted

$$g_{k,j} = \left(\frac{\partial g_k}{\partial D_j}\right)_{\vec{D}} \tag{3.4}$$

Consider a small change of the design, given by $\delta\vec{D}$, then the corresponding change of the weight is given by

$$\delta W = \sum_{j=1}^{NDV} w_j \, \delta D_j \tag{3.5a}$$

and the change of g_k is estimated by

$$\delta g_k = \sum_{j=1}^{NDV} g_{k,j} \, \delta D_j \tag{3.5b}$$

It is possible to consider a number of small changes of design $\vec{D}$, but among them we will consider only changes which do not violate any of the constraints. Such a change is called an admissible one and is defined as a change such that

$$\begin{aligned} &\sum_{j=1}^{NDV} g_{k,j} \, \delta D_j \leqq 0 , \quad \text{for } k \in K_{act} \{ k \mid g_k = 0 \} \\ &\delta D_j \geqq 0 , \quad \text{for } j \in J_{min} \{ j \mid D_j = D_{j\,min} \} \\ &\delta D_j \leqq 0 , \quad \text{for } j \in J_{max} \{ j \mid D_j = D_{j\,max} \} \end{aligned} \tag{3.6}$$

The design $\vec{D}$ can be improved if there exists an admissible change for which δW is negative (see Fig. 2). On the contrary, if the design is optimal, δW must be nonnegative for all admissible changes. Figures 3 through 5 illustrate the typical cases of this situation schematically. Now the optimality criteria for the three cases shown in these

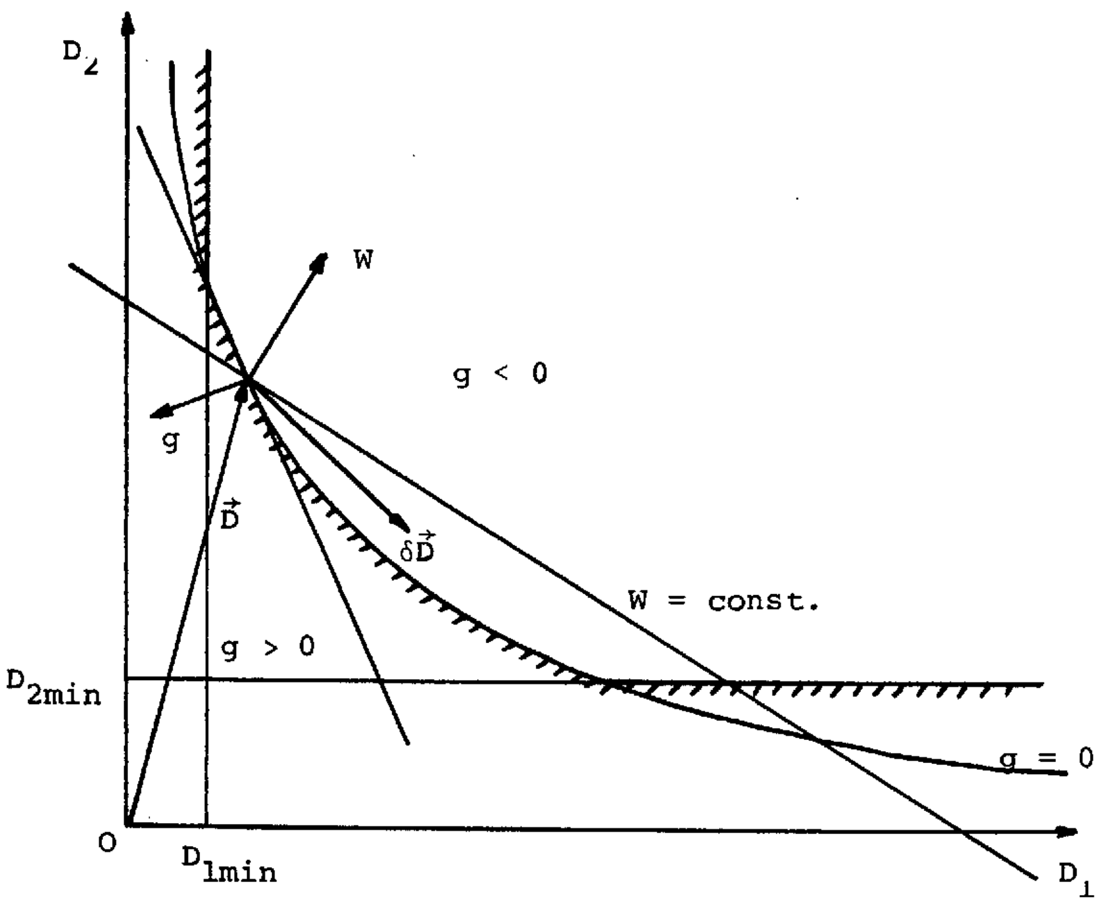

Fig. 2 Admissible Design Change reducing Objective Function

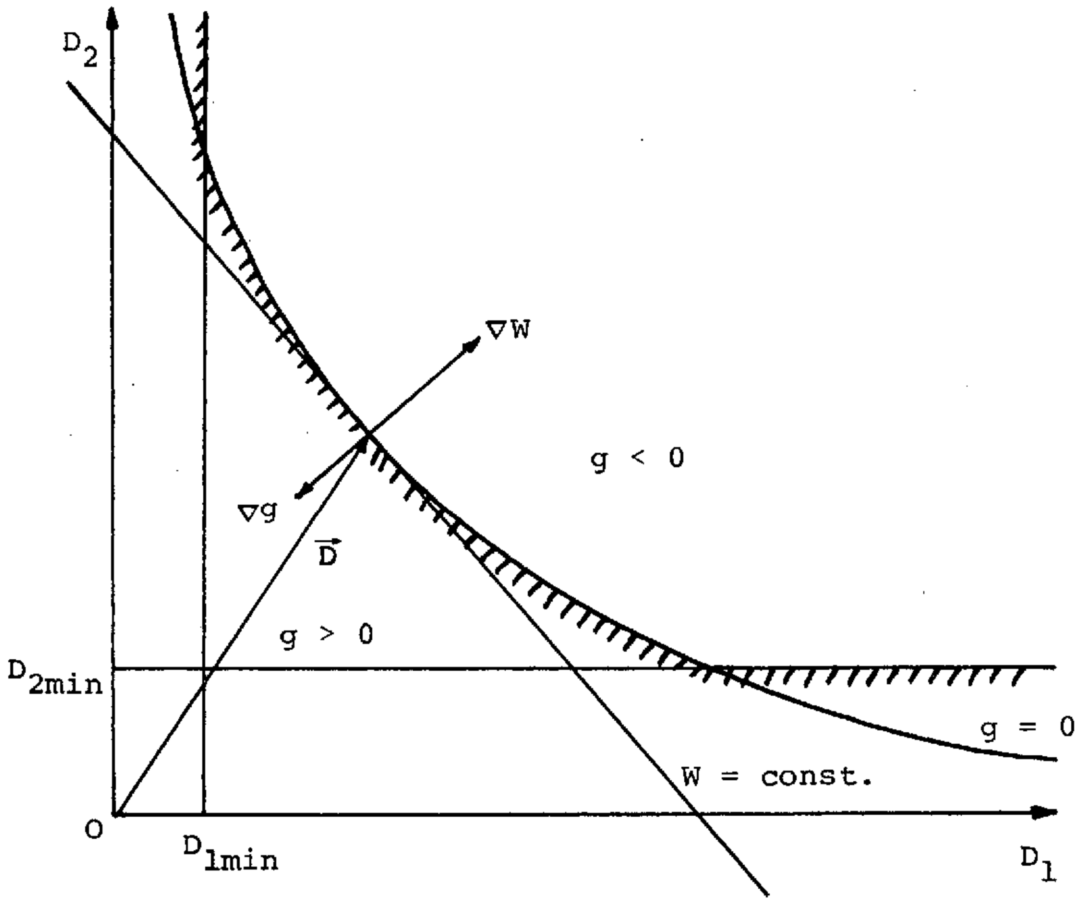

Fig. 3 Optimum Design, Case 1

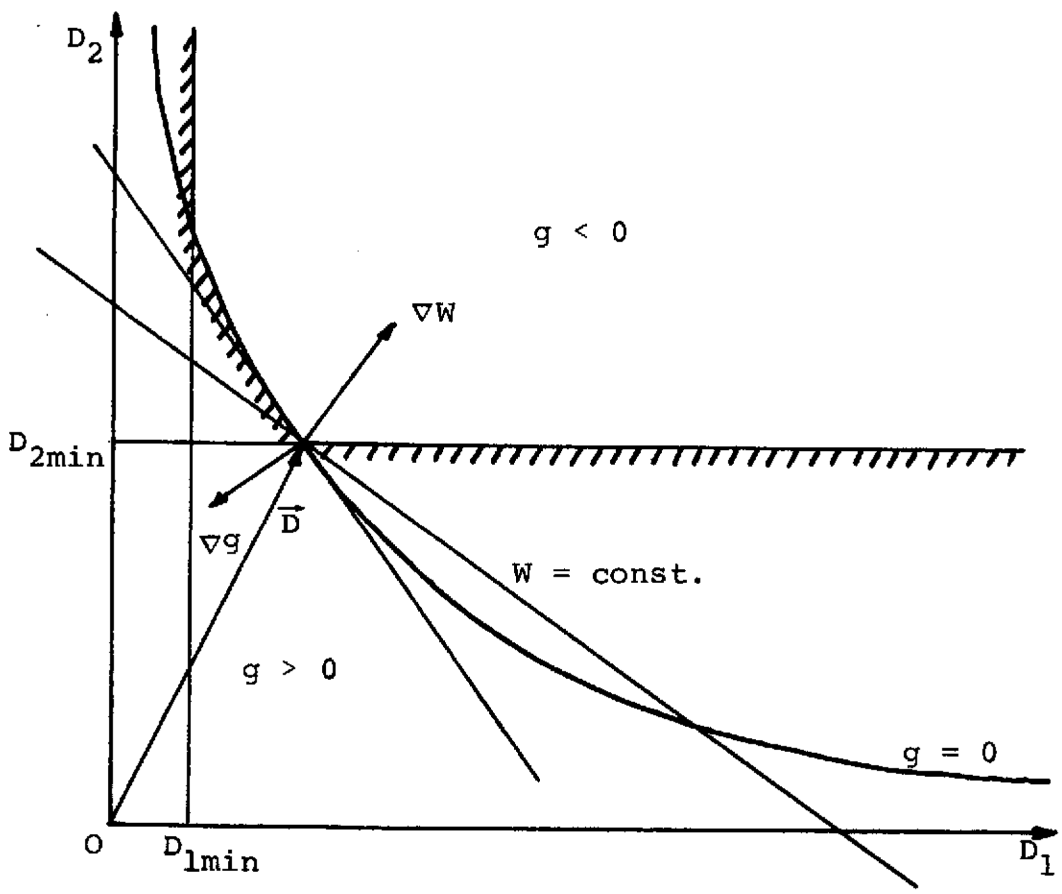

Fig. 4 Optimum Design, Case 2

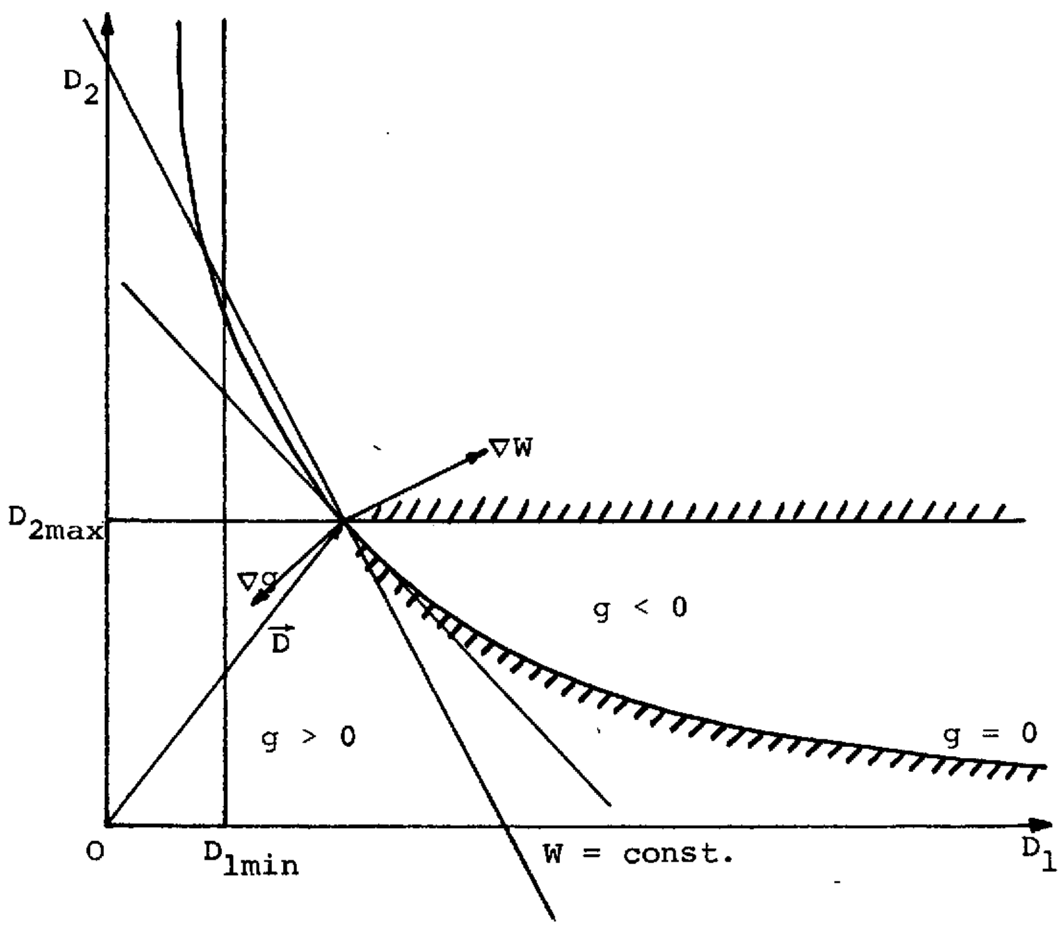

Fig. 5 Optimum Design, Case 3

figures are derived.

First in Figure 3, the admissible direction in which the increase of the weight will be minimum is perpendicular to ∇g_1, that is

$$g_{1,1}\,\delta D_1 + g_{1,2}\,\delta D_2 = 0 \tag{3.7}$$

Solving equation (3.7) for δD_1 and substituting into equation (3.5a), yields

$$\delta W = \left(-w_1 \frac{g_{1,2}}{g_{1,1}} + w_2\right)\delta D_2 \tag{3.8}$$

In this case there is no restriction on δD_2, therefore $\delta W \geq 0$ for all possible δD_2 is satisfied if and only if the factor within the parentheses is equal to zero, i.e.,

$$-w_1 \frac{g_{1,2}}{g_{1,1}} + w_2 = 0 \tag{3.9}$$

Let

$$\lambda = -\frac{w_1}{g_{1,1}} \tag{3.10}$$

then from equations (3.9) and (3.10)

$$w_1 + \lambda\, g_{1,1} = 0 \tag{3.11a}$$

$$w_2 + \lambda\, g_{1,2} = 0 \tag{3.11b}$$

where $\lambda > 0$ because from Fig. 3, $w_1 > 0$ and $g_{1,1} < 0$.

In Fig. 4, $D_2 = D_{2min}$, therefore δD_2 must be non-negative. In this case $\delta W \geq 0$ is satisfied when the value in the parenthesis is not less than zero, i.e.,

$$-w_1 \frac{g_{1,2}}{g_{1,1}} + w_2 \geqq 0 \tag{3.12}$$

Using the same definition of λ, see equation (3.10), it follows that

$$w_2 + \lambda g_{1,2} \geqq 0 \tag{3.13}$$

From Fig. 5 it can be shown in a similar manner that $\delta W \geq 0$ is satisfied if

$$w_2 + \lambda g_{1,2} \leqq 0 \tag{3.14}$$

From the above discussion, the optimality criteria for these cases can be summarized as

$$w_j + \lambda g_{1,j} \begin{cases} = 0, & \text{for } j \in J\{j \mid D_{jmin} < D_j < D_{jmax}\} \\ \geqq 0, & \text{for } j \in J_{min} \\ \leqq 0, & \text{for } j \in J_{max} \end{cases} \tag{3.15a}$$

and

$$\lambda > 0 \tag{3.15b}$$

For a general case, the optimality criteria can be derived on the same basis, that is,

$$\delta W = \sum_{j=1}^{NDV} w_j \delta D_j \geqq 0 \tag{3.16a}$$

for all δD_j such that

$$\delta g_k = \sum_{j=1}^{NDV} g_{k,j}\,\delta D_j \leqq 0\,, \quad \text{for} \quad k \in K_{act}$$

$$\delta D_j \geqq 0\,, \qquad \text{for} \quad j \in J_{min} \tag{3.16b}$$

$$\delta D_j \leqq 0\,, \qquad \text{for} \quad j \in J_{max}$$

According to Farkas' lemma, a set of equations (3.16a,b) can be shown to be mathematically equivalent to the following (see Appendix A),

$$w_j + \sum_{k=1}^{NC} \lambda_k\, g_{k,j} \begin{cases} = 0\,, & \text{for} \quad j \in J \\ \geqq 0\,, & \text{for} \quad j \in J_{min} \\ \leqq 0\,, & \text{for} \quad j \in J_{max} \end{cases} \tag{3.17}$$

where

$$\lambda_k \begin{cases} \geqq 0\,, & \text{for} \quad k \in K_{act} \\ = 0\,, & \text{for} \quad k \notin K_{act} \end{cases}$$

Any design which does not satisfy (3.17) is not an optimum design because from (3.16a,b), there exists at least one admissible change of the design for which $\delta W < 0$. In this sense the set of conditions stipulated by equation (3.17) is called the optimality criterion.

A redesign equation can be obtained from the optimality criterion in various alternative ways [9], [10] and [12], but all these methods are based on the following central idea. The optimality criterion given by (3.17) can be rewritten as

$$I_j = - \sum_{k=1}^{NC} \lambda_k \frac{g_{k,j}}{w_j} \begin{cases} = 1, & \text{for } j \in J \\ \leq 1, & \text{for } j \in J_{min} \\ \geq 1, & \text{for } j \in J_{max} \end{cases} \tag{3.18}$$

Equation (3.18) suggests that if $I_j < 1$ and $D_j > D_{jmin}$ for a current design, D_j must decrease to obtain an improved design at the next iteration, and conversely if $I_j > 1$ and $D_j < D_{jmax}$, then D_j must increase. On this basis, a redesign rule is expressed as a function of I_j. In this research effort the following plausible redesign rule is used

$$(D_j)_{s+1} = \begin{cases} (I_j)_s^{\frac{1}{2}} \; (D_j)_s, & \text{for } (I_j)_s \geq 0 \\ 0, & \text{for } (I_j)_s < 0 \end{cases} \tag{3.19}$$

where $(\;)_s$ represents the s-th cycle in the iterative design process. In order to include the size constraints, the following relation must be appended to (3.19)

$$(D_j)_{s+1} = \begin{cases} (D_j)_{s+1}, & \text{if } D_{jmin} < (D_j)_{s+1} < D_{jmax} \\ D_{jmin}, & \text{if } (D_j)_{s+1} \leq D_{jmin} \\ D_{jmax}, & \text{if } (D_j)_{s+1} \geq D_{jmax} \end{cases} \tag{3.20}$$

The reason for using the square root of I_j in equation (3.19) will be discussed in subsequent sections dealing with specific constraint types. From the foregoing discussion it is obvious that if $(D_j)_{s+1} = (D_j)_s$ for all j, design $(\vec{D})_s$ satisfies the optimality criterion, and consequently it may be an optimal design. If $(D_j)_{s+1} \neq (D_j)_s$ for some j, the design can be improved further, and repeated application of the redesign equation will converge toward a design satisfying the optimality criterion.

The value of multipliers λ_k can also be obtained from the optimality criterion. From the first equation in (3.18)

$$1 + \sum_{k=1}^{NC} \lambda_k \frac{g_{k,j}}{w_j} = 0, \quad \text{for} \quad j \in J \tag{3.21}$$

Eliminating the terms corresponding to inactive constraints from equation (3.21), since those $\lambda_k = 0$, yields

$$1 + \sum_{k_{act}} \lambda_k \frac{g_{k,j}}{w_j} = 0, \quad \text{for} \quad j \in J \tag{3.22}$$

where k_{act} represents $k \in K_{act}$. Note that the values of w_j are known constants and assume that the values of the $g_{k,j}$ are known for all combinations of design variables $j \in J$ and active constraints $k \in K_{act}$, then equation (3.22) represents a set of simultaneous linear equations in which the λ_k are the unknowns. Since the number of design variables NDV is not necessarily equal to the number of active constraints, it is in general not possible to directly solve (3.22) for the λ_k. Consequently an indirect method must be employed to solve for the λ_k values.

First consider the following optimization problem in which the λ_k are the unknowns

$$\text{Minimize} \qquad I = \sum_{j \in J} \left\{ 1 + \sum_{k\,act} \lambda_k \frac{g_{k,j}}{w_j} \right\}^2 \tag{3.23}$$

If a solution is obtained which minimizes I and I_{min} has zero value, then this solution gives a set of the optimal values for the multipliers λ_k, and the design cannot be further improved. If the minimum value of I is greater than zero, it means that the value in the parentheses is not equal to zero for at least one j, and therefore the design can be improved. The problem given by (3.23) can be transformed to the following linear programming problem

$$\text{Minimize} \qquad \sum_{\ell=1}^{NAC} (x_\ell + y_\ell)$$

subject to

$$\sum_{k\,act} \lambda_k \left\{ \sum_{j \in J} \frac{g_{k,j}\, g_{\ell,j}}{w_j^2} \right\} + x_\ell - y_\ell = - \sum_{j \in J} \frac{g_{\ell,j}}{w_j} \tag{3.24}$$

$$\lambda_k \geqq 0, \quad x_\ell \geqq 0 \quad y_\ell \geqq 0$$

Now this problem can be solved by the Simplex Method, for example. The method to be used in estimating the value of the $g_{k,j}$ will be discussed in subsequent sections dealing with each of the particular constraint types considered.

In executing the computation, however, special attention must be given to identifying the maximum or minimum size elements as well as active and inactive constraints. First consider the problem of determining those elements that are to take on their maximum or minimum values. Generally no information is available on this at the outset and therefore an iterative procedure is required. The procedure is outlined as follows:

(1) Initially assume that all design variable side constraints are inactive, i.e., $D_{jmin} < D_j < D_{jmax}$, for all j, and compute the value of λ_k using (3.24);

(2) Calculate $(D_j)_{s+1}$ from equations (3.19) and (3.20), and use the results to identify the active design variable side constraints;

(3) If the distinction remains unchanged, it is done, otherwise, use the new distinction to repeat the procedure.

This iterative process must be carried out during each redesign step until the set of active side constraints has definitely stabilized and remains unchanged.

Next consider the problem of identifying the set of active behavior constraints.† The procedure discussed in the foregoing paragraph will be applicable, however, it may waste a great deal of computational effort, because in structural optimization problems, a large number of behavior constraints usually need to be considered but a relatively small number of these constraints are active.

The procedure is as follows:

(1) Select several active constraint candidates if they are known;

(2) Analyze the current design and evaluate all constraints;

(3) Find the most critical constraint, which may or may not be violated, and compare it with the list of preselected active constraint candidates;

(4) If it is already an active candidate, continue with the current list, otherwise, add it to the list of active constraint candidates;

(5) If the number of active constraint candidates exceeds the number of design variables, eliminate the one which is least critical during the redesign step.

†This is not necessarily required because even if inactive constraints participate in the process, they will be automatically eliminated by the computational result of $\lambda_k = 0$ for the corresponding k.

In the foregoing method, there is no need to compute the λ_k corresponding to inactive constraints, and this substantially reduces the computational effort required.

Now an entire optimization procedure based on optimality criteria concept is available and it is summarized as follows (see Fig. 6):

(1) Select several active constraint candidates and pick an initial design;

(2) Analyze the current design and evaluate all constraints;

(3) Find the most critical constraint and determine the active constraint candidate group for the upcoming redesign step;

(4) Compute the gradients of the active constraint candidates;

(5) Compute the value of multipliers λ_k from (3.24) and use the results to generate a new design from equations (3.19) and (3.20);

(6) Determine which elements are to take on their maximum or minimum values and repeat (5) and (6) until the set of active design variable side constraints has stabilized;

(7) Check to see if the new design satisfies the prescribed termination conditions. If so, go to (8),

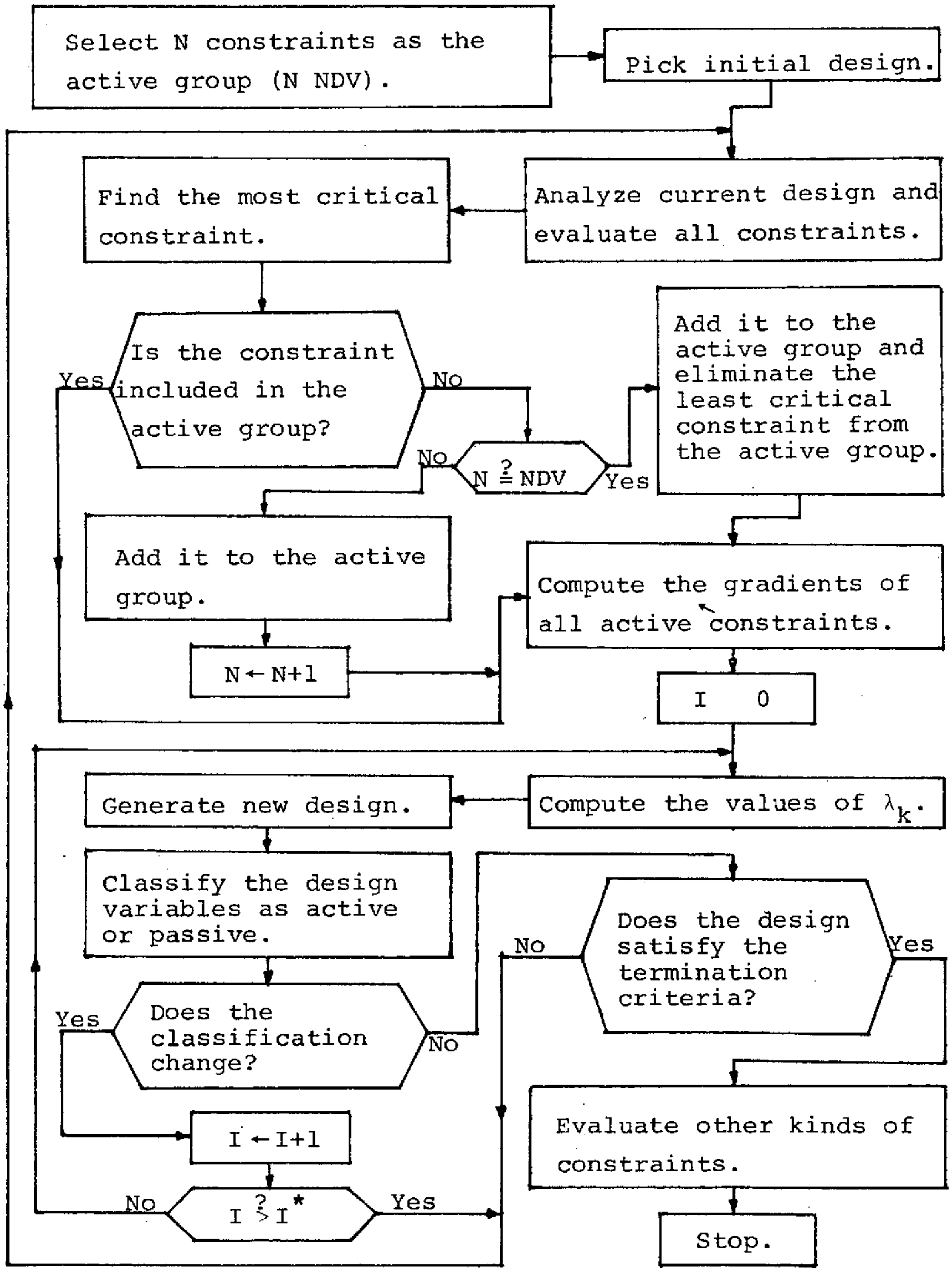

Fig. 6 Block Diagram of Optimality Criteria Approach

otherwise, go to (2) and repeat;

(8) Evaluate the remaining behavior constraints using the final design and stop.

The purpose of step (8) is to check on whether or not the final design is the optimal solution of the whole problem under consideration and to obtain information for selecting active constraint candidates in the other subproblems.

At this point some points which should be noted when applying the present method to an actual design problem are discussed. As is obvious from the derivation, the optimality criteria are nothing more than the necessary conditions for local optimality, and they do not guarantee that the design obtained by the present method is the global optimum. An example of this situation is shown in Fig. 7, where either $\vec{D}_1$, $\vec{D}_2$ or $\vec{D}^*$ will be obtained. Among them, however, only $\vec{D}^*$ is the optimum design, and the other two are obviously not optimal. But this situation can be avoided by setting up some limitations on the values of multipliers. At design $\vec{D}^*$, both constraints are active, and it follows that the values of λ_1 and λ_2 must be non-negative. At design $\vec{D}_1$, g_1 is active and g_2 is inactive but violated, and at $\vec{D}_2$, g_1 is inactive but violated. This situation suggests that the value of a multiplier λ_k corresponding to a

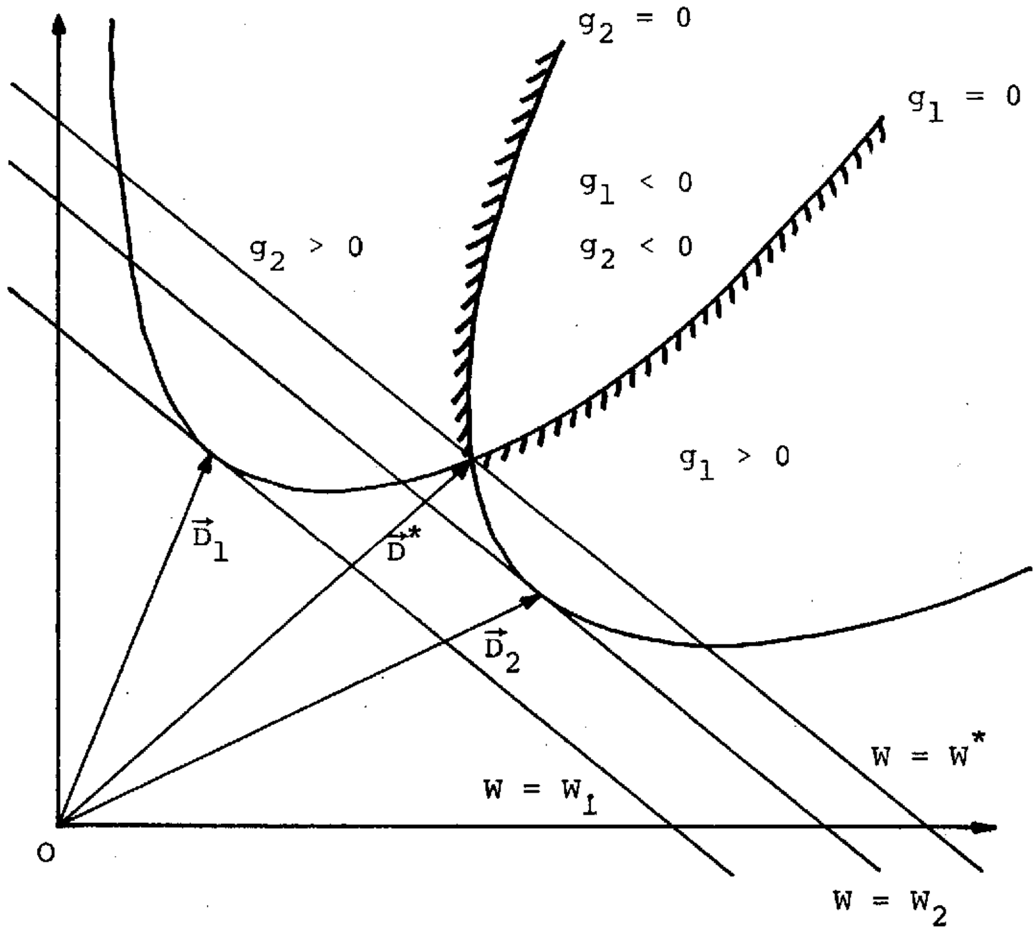

Fig. 7 Illustration of Optimum Design and Infeasible Designs

violated constraint should be positive. Based on this fact, we set up the limitation on the value of multiplier as

$$\lambda_k \geqq \epsilon_k \qquad (\epsilon_k > 0) \tag{3.25}$$

for violated constraints instead of $\lambda_k \geq 0$ in (3.24). The value of ϵ_k may be determined in various ways, but it will be better to define it as a function of $(g_k/g_k{}^*)$,† and the function should be defined for each problem. In the effort reported here

$$\epsilon_k = \alpha_k \left(\frac{g_k}{g_k^*} - 1 \right), \text{ for } k \in \{ g_k > g_k^* \} \tag{3.26}$$

was used, where α_k is an appropriate constant.

Another example is shown in Fig. 8, where the optimality criteria are satisfied by design $\vec{D}_1$, $\vec{D}_2$ and $\vec{D}_3$. Among them $\vec{D}_1$ is the global minimum, $\vec{D}_2$ is a local minimum, and $\vec{D}_3$ is a local maximum. An effective method of coping with this situation has not been found, and it would appear that this difficulty represents one of the current shortcomings of the optimality criteria approach. It should be noted, however, that the formidable difficulties posed by relative minima are not unique to the optimality criteria approach.

†For the notational convenience, we rewrite the constraint in the form $g_k - g_k{}^* \leq 0$, where $g_k{}^*$ denotes the specified upper bound on g_k.

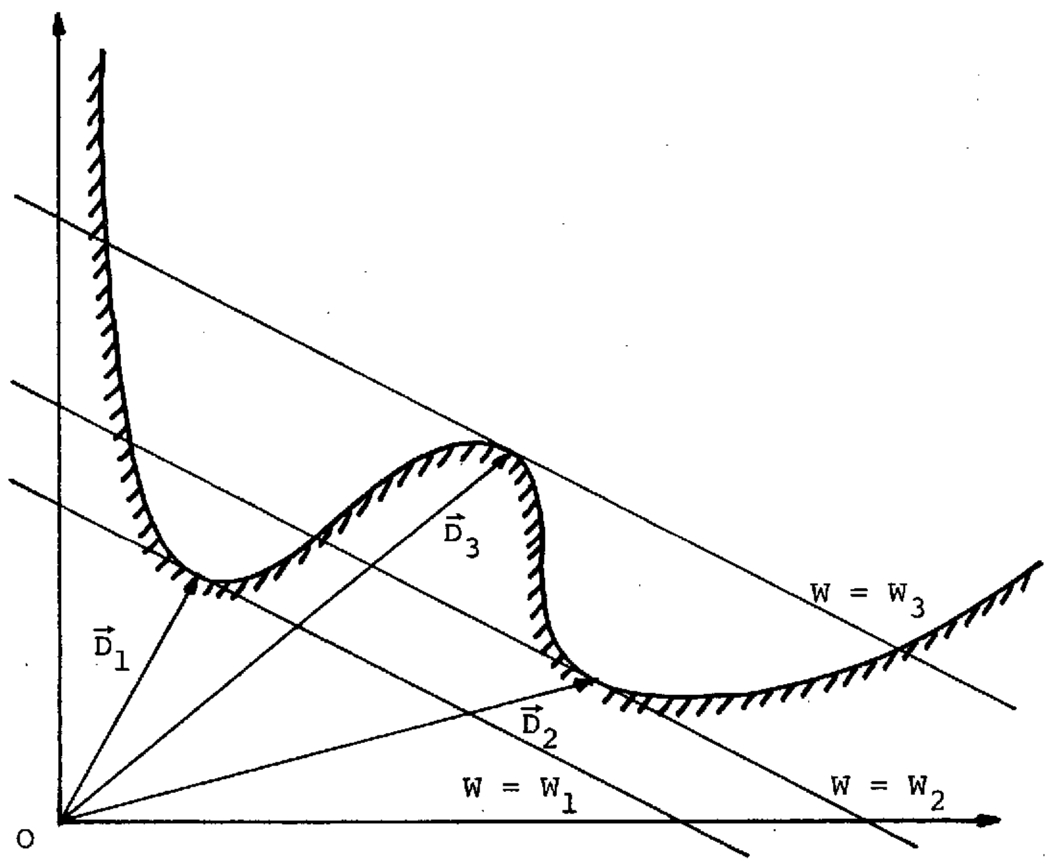

Fig. 8 Illustration of Global Minimum, Local Minimum and Local Maximum

3.2 Stress Constraint

Even for the simplest class of structures, such as trusses, the optimality criteria approach developed in section 3.1 is not practical for stress constraints. However, there is a useful method, which is commonly used, that does not require the derivatives of constraints although it deviates somewhat from the general optimization theory.

In structural design, it has often been assumed intuitively that the best design is one for which every mode of failure considered would occur simultaneously. From this idea it followed that for stress limited design problems the best design would be one in which each member is fully stressed under at least one load condition. However, it was shown by Schmit [1] that the fully stressed design is not necessarily the minimum weight design. Nevertheless, the fully stressed design scheme still has practical significance because of the following characteristics:

(1) The fully stressed design always coincides with the minimum weight design for statically determinate structures.

(2) For the case of statically indeterminate structures, the fully stressed design may be a good approximation that is often acceptable for practical purposes.

(3) The fully stressed design procedure is familiar and easy to apply in comparison with methods based on mathematical programming concepts.

For these reasons the fully stressed design concept is adopted here as the source of design basis vectors for stress constraints. That is, "the optimum design for stress constraints is assumed to be one in which each member is fully stressed under at least one of the load conditions."

The fully stressed design is usually obtained by the stress ratio method which is derived based on the assumption that the internal force distribution remains unchanged during modification of the design variables in each redesign step. This is equivalent to

$$(\sigma_i)_s (A_i)_s = \begin{cases} \sigma_{it} A_i^* , & \text{if } (\sigma_i)_s \geq 0 \\ \sigma_{ic} A_i^* , & \text{if } (\sigma_i)_s < 0 \end{cases} \tag{3.27}$$

where A_i^* denotes the optimal design for A_i. Consequently, the following redesign equation can be obtained

$$(A_i)_{s+1} = C_i (A_i)_s \tag{3.28a}$$

where

$$C_i = \begin{cases} \dfrac{(\sigma_i)_s}{\sigma_{it}} , & \text{if } (\sigma_i)_s \geq 0 \\ \dfrac{(\sigma_i)_s}{\sigma_{ic}} , & \text{if } (\sigma_i)_s < 0 \end{cases} \tag{3.28b}$$

For linked design variable D_j, the redesign equation must be modified as follows:

$$(D_j)_{s+1} = C_j\,(D_j)_s \tag{3.29a}$$

where

$$C_j = \max_{i \in j} C_i \tag{3.29b}$$

Including size constraints, equation (3.20) must be considered together with equations (3.29). The redesign procedure can now be summarized as follows:

(1) Pick an initial design;

(2) Analyze the current design and compute the stress ratio for each combination of elements and load conditions;

(3) Find the maximum stress ratio for each design variable;

(4) Generate a new design using (3.29) and (3.20);

(5) Check to see if the new design satisfies the prescribed termination conditions. If so, go to (6), otherwise, go to (2) and repeat;

(6) Evaluate the remaining behavior constraints using the final design and stop.

This procedure is shown in block diagram form in Figure 9.

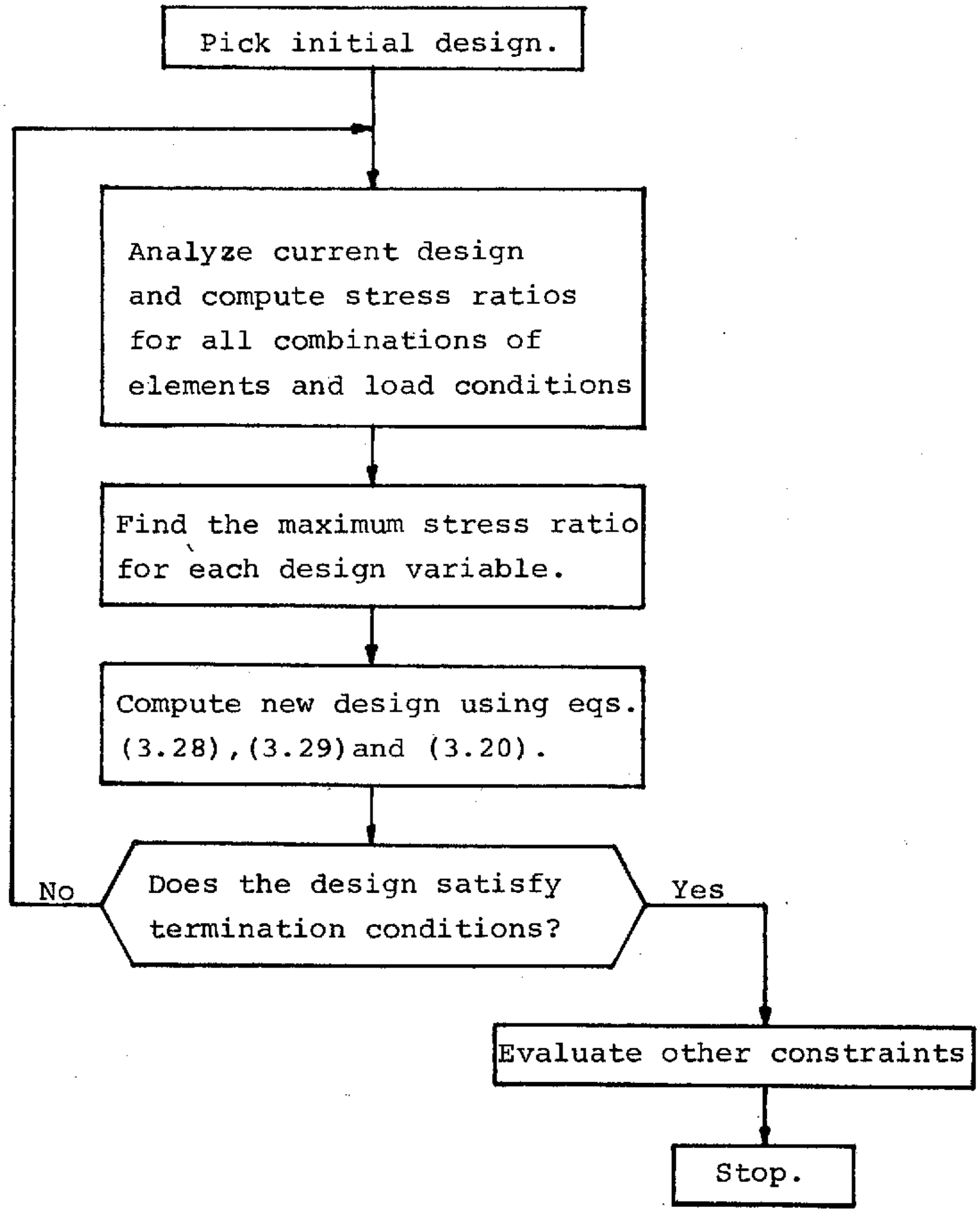

Fig. 9 Block Diagram of Stress Ratio Method

3.3 Displacement Constraint

As shown in Chapter II, a displacement constraint is given by

$$u_{Lij} \leqq u_{ij} \leqq u_{Uij} \tag{3.30}$$

In the majority of practical problems, u_{Uij} is positive and u_{Lij} is negative. Assuming this is so, the constraint can be expressed in the following form:

$$\vec{f}^{\,T} \vec{u} - u_{ij}^{*} \leqq 0 \tag{3.31}$$

where $\vec{u}$ is a displacement vector including u_{ij} in its ith row, and $\vec{f}$ is a unit force vector which has only one nonzero element in its ith row, namely

$$f_i = \begin{cases} 1 \ , & \text{if } u_{ij} \geqq 0 \\ -1 \ , & \text{if } u_{ij} < 0 \end{cases} \tag{3.32}$$

since $\vec{f}^{\,T}\vec{u}$ represents the absolute value of displacement u_{ij} and $u_{ij}{}^{*}$ denotes

$$u_{ij}^{*} = \begin{cases} u_{Uij} \ , & \text{if } u_{ij} \geqq 0 \\ u_{Lij} \ , & \text{if } u_{ij} < 0 \end{cases} \tag{3.33}$$

Hereafter equation (3.31) will be taken as the displacement constraint form because this form facilitates the derivation of partial derivatives $(\partial g/\partial D_j)$ assuming the use of a displacement type finite element method of structural analysis. For the sake of simplicity, a case with only one constraint will be considered, let

$$g = \vec{f}^T \vec{u} - u_i^* \leqq 0 \tag{3.34}$$

then

$$\frac{\partial g}{\partial D_j} = \frac{\partial}{\partial D_j} \left(\vec{f}^T \vec{u} \right) \tag{3.35}$$

From equation (3.1d)

$$\frac{\partial}{\partial D_j} = \sum_{i \in j} a_{ij} \frac{\partial}{\partial A_i} \tag{3.36}$$

Since $\vec{f}$ is independent of the design variables

$$\frac{\partial}{\partial A_i} \left(\vec{f}^T \vec{u} \right) = \vec{f}^T \frac{\partial \vec{u}}{\partial A_i} \tag{3.37}$$

The static equilibrium equations for the displacement method of structural analysis may be written in matrix form as

$$[K] \vec{u} = \vec{F} \tag{3.38}$$

where [K] is the stiffness matrix of the structure, and $\vec{F}$ is the external load vector. Assuming $\vec{F}$ is independent of A_i and differentiating both sides of equation (3.38) gives

$$\frac{\partial \vec{u}}{\partial A_i} = - [K]^{-1} \frac{\partial [K]}{\partial A_i} \vec{u} \tag{3.39}$$

Substituting equation (3.39) into equation (3.37) yields

$$\frac{\partial}{\partial A_i} \left(\vec{f}^T \vec{u} \right) = - \vec{f}^T [K]^{-1} \frac{\partial [K]}{\partial A_i} \vec{u} \tag{3.40}$$

Now define a new vector such that

$$\vec{r} = [K]^{-1} \vec{f} \tag{3.41a}$$

then $\vec{r}$ represents the response of the structure to the unit force vector $\vec{f}$. Since [K] is symmetric

$$\vec{r}^T = \vec{f}^T [K]^{-1} \tag{3.41b}$$

For the class of structures considered in this study, the stiffness matrix can be expressed in the following form

$$[K] = \sum_{i=1}^{NM} [k_i] A_i \tag{3.42}$$

where $[k_i]$ is the unit element stiffness matrix for element i, which is independent of the element size. Therefore

$$\frac{\partial [K]}{\partial A_i} = [k_i] \tag{3.43}$$

Substituting equations (3.41b) and (3.43) into equation (3.40) gives

$$\frac{\partial}{\partial A_i} (\vec{f}^T \vec{u}) = -\vec{r}^T [k_i] \vec{u} \tag{3.44}$$

From equations (3.35), (3.36) and (3.44), it follows that

$$\frac{\partial g}{\partial D_j} = -\sum_{i \in j} a_{ij} \vec{r}^T [k_i] \vec{u} \tag{3.45}$$

Using equation (3.17), the optimality criteria for a single displacement constraint is obtained, namely

$$w_j - \lambda \sum_{i \in j} a_{ij} \vec{r}^T [k_i] \vec{u} = 0, \text{ for } j \in J \tag{3.46}$$

Multiplying both sides of equation (3.46) by D_j yields the following standard form

$$W_j - \lambda U_j = 0 , \quad \text{for} \quad j \in J \tag{3.47a}$$

where

$$W_j = w_j D_j \tag{3.47b}$$

$$U_j = \sum_{i \in j} a_{ij} D_j \vec{r}^{\,T} [k_i] \vec{u} \tag{3.47c}$$

Now W_j represents the total weight of the elements in the group j, and U_j is the internal virtual work in those elements associated with the jth design variable (D_j). Equation (3.47a) can be rewritten as

$$\frac{1}{\lambda} = \frac{U_j}{W_j} \tag{3.47d}$$

Therefore the optimality criterion can be stated as "the ratio of the internal virtual work over the weight is invariant for all active element groups."

For the case of multiple constraints, the criteria can be generalized as follows

$$W_j - \sum_{k=1}^{NC} \lambda_k U_{kj} = 0 , \quad \text{for} \quad j \in J \tag{3.48a}$$

where

$$\lambda_k \begin{cases} \geqq 0 , & \text{for} \quad k \in K_{act} \\ = 0 , & \text{for} \quad k \notin K_{act} \end{cases} \tag{3.48b}$$

and U_{kj} represents U_j for the kth constraint.

Consideration is now given to the redesign equation. Initially the same assumption used in fully stressed design (see Section 3.2) is made, that is, the internal force distribution remains unchanged during modification of design variables in each redesign step. Then the following relations must be satisfied for the case of one constraint, if

$$(D_j)_{s+1} = C_j (D_j)_s \tag{3.49a}$$

then

$$(W_j)_{s+1} = C_j (W_j)_s \tag{3.49b}$$

and

$$(U_j)_{s+1} = \frac{1}{C_j} (U_j)_s \tag{3.49c}$$

At iteration s, if $W_j - \lambda U_j \neq 0$ for some j which are assumed to be active, then $(D_j)_{s+1}$ must be determined so that

$$(W_j)_{s+1} - \lambda (U_j)_{s+1} = 0 \tag{3.50}$$

Substituting equations (3.49b), (3.49c) into equation (3.50), and solving for C_j, we get

$$C_j = \left(\lambda \frac{U_j}{W_j} \right)_s^{\frac{1}{2}} \tag{3.51}$$

If the value in the parentheses in equation (3.51) is negative, it means that D_j must be inactive, i.e., $D_j = 0$. For the case of multiple constraints

$$C_j = I_j^{\frac{1}{2}} \tag{3.52a}$$

where

$$I_j = \sum_{k=1}^{NC} \lambda_k \left(\frac{U_{kj}}{W_j} \right)_s \tag{3.52b}$$

Based on the foregoing discussion, the redesign equation can be summarized as

$$(D_j)_{s+1} = \begin{cases} C_j (D_j)_s \,, & \text{if } I_j \geq 0 \\ 0 \,, & \text{if } I_j < 0 \end{cases} \tag{3.53}$$

where C_j and I_j are defined respectively by equations (3.52a) and (3.52b). Including size constraints, equation (3.20) must be employed concurrently.

Finally, consider the method used to compute the values of multipliers. Since the general idea was presented in section 3.1, it is only necessary to discuss the method used to compute U_{kj}/W_j, which corresponds to the term of $g_{k,j}/w_j$ in section 3.1. For the active constraints, the following relation must be satisfied:

$$\sum_{j=1}^{NDV} U_{kj} = u_k^* \tag{3.54a}$$

$$\sum_{j=1}^{NDV} (U_{kj}) = (u_k)_s \tag{3.54b}$$

where u_k^* is the specified value of the kth constraint and $(u_k)_s$ denotes the corresponding displacement for the current design. They can be divided into active and passive parts as follows

$$\sum_{j=1}^{NDV} U_{kj} = \sum_{j \in J} U_{kj} + \sum_{j \notin J} U_{kj} \tag{3.55a}$$

$$\sum_{j=1}^{NDV} (U_{kj})_s = \sum_{j \in J} (U_{kj})_s + \sum_{j \notin J} (U_{kj})_s \tag{3.55b}$$

Again using the same assumption employed in fully stressed design

$$\sum_{j \notin J} U_{kj} = \sum_{j \notin J} (U_{kj})_s \tag{3.56}$$

Let u_k° denote the value of $\sum_{j \notin J} (U_{kj})_s$ and let

$$U_{kj} = (U_{kj})_s \frac{u_k^* - u_k^{\circ}}{(u_k)_s - u_k^{\circ}}, \quad \text{for } j \in J \tag{3.57}$$

then U_{kj} satisfies equation (3.54a). Based on this fact the following equation can be used to estimate the value of U_{kj}/W_j,

$$\frac{U_{kj}}{W_j} = \begin{cases} \left(\dfrac{U_{kj}}{W_j}\right)_s \dfrac{u_k^* - u_k^{\circ}}{(u_k)_s - u_k^{\circ}}, & \text{if } (U_{kj})_s \geq 0 \\ \left(\dfrac{U_{kj}}{W_j}\right)_s, & \text{if } (U_{kj})_s < 0 \end{cases} \tag{3.58}$$

because $(U_{kj})_s < 0$ means that D_j is inactive at least for the constraint. After obtaining the values of the U_{kj}/W_j

the λ_k are obtained by solving the linear programming problem defined by (3.24).

3.4 Buckling Constraint

As indicated in Chapter II, a buckling constraint is given in the following form

$$g_k = -p_k + p_k^* \leqq 0 \tag{3.59}$$

and p_k is defined by

$$[K]\vec{u}_k = p_k [K_G] \vec{u}_k \tag{3.60}$$

where the subscript k denotes that the buckling load under consideration is the kth constraint. Let $p_k{}^*$ denote the prescribed lower bound on p_k, let $\vec{u}_k$ represent the corresponding buckling mode shape, and let $[K_G]$ denote the geometric stiffness matrix of the structure, which is symmetric and independent of element sizes for the class of structures considered here.

Differentiating both sides of equation (3.60) with respect to A_i, we get

$$[k_i]\vec{u}_k + [K]\frac{\partial \vec{u}_k}{\partial A_i} = \frac{\partial p_k}{\partial A_i}[K_G]\vec{u}_k + p_k [K_G]\frac{\partial \vec{u}_k}{\partial A_i} \tag{3.61}$$

Premultiplying equation (3.60) by $[\partial \vec{u}_k / \partial A_i]^T$ and premultiplying equation (3.61) by $\vec{u}_k{}^T$, then subtracting the former from the latter yields

$$\vec{u}_k^T [k_i] \vec{u}_k = \frac{\partial p_k}{\partial A_i} \vec{u}_k{}^T [K_G] \vec{u}_k \tag{3.62}$$

From equations (3.36), (3.59) and (3.62), we get

$$\frac{\partial g_k}{\partial D_j} = -\sum_{i \in j} a_{ij} \frac{\vec{u}_k^T [k_i] \vec{u}_k}{\vec{u}_k^T [K_G] \vec{u}_k} \tag{3.63}$$

Substituting equation (3.63) into equation (3.17) yields

$$w_j - \sum_{k=1}^{NC} \lambda_k \left\{ \sum_{i \in j} a_{ij} \frac{\vec{u}_k^T [k_i] \vec{u}_k}{\vec{u}_k^T [K_G] \vec{u}_k} \right\} = 0, \quad \text{for } j \in J \tag{3.64}$$

Multiplying equation (3.64) by D_j, leads to the following optimality criterion, namely

$$W_j - \sum_{k=1}^{NC} \frac{\lambda_k}{M_k} U_{kj} = 0 \quad , \quad \text{for } j \in J \tag{3.65a}$$

where

$$U_{kj} = \sum_{i \in j} a_{ij} D_j \; \vec{u}_k^T \, [k_i] \, \vec{u}_k \tag{3.65b}$$

$$M_k = \vec{u}_k^T \, [K_G] \, \vec{u}_k \tag{3.65c}$$

The redesign equation can be obtained in exactly the same manner as for displacement constraints, and that is

$$(D_j)_{s+1} = \begin{cases} C_j (D_j)_s \, , & \text{if } I_j \geq 0 \\ 0 \, , & \text{if } I_j < 0 \end{cases} \tag{3.66a}$$

where

$$I_j = \left(\sum_{k=1}^{NC} \frac{\lambda_k}{M_k} \cdot \frac{U_{kj}}{W_j} \right)_s \tag{3.66b}$$

and

$$C_j = I_j^{\frac{1}{2}} \tag{3.66c}$$

Including size constraints, equation (3.20) must be employed concurrently with equation (3.66a).

The value of λ_k can be estimated in the same manner too. For an active constraint

$$\sum_{j=1}^{NDV} \frac{U_{kj}}{M_k} = p_k^* \tag{3.67a}$$

$$\sum_{j=1}^{NDV} \left(\frac{U_{kj}}{M_k}\right)_s = (p_k)_s \tag{3.67b}$$

They can be divided into active and passive parts as follows:

$$\sum_{j=1}^{NDV} U_{kj} = \sum_{j \in J} U_{kj} + \sum_{j \notin J} U_{kj} \tag{3.68a}$$

$$\sum_{j=1}^{NDV} (U_{kj})_s = \sum_{j \in J} (U_{kj})_s + \sum_{j \notin J} (U_{kj})_s \tag{3.68b}$$

Again using the same assumption employed in fully stressed design

$$\sum_{j \notin J} U_{kj} = \sum_{j \notin J} (U_{kj})_s \tag{3.69}$$

Let p_k° denote the value of $\sum_{j \notin J} (U_{kj})_s$, and let

$$\frac{U_{kj}}{M_k} = \left(\frac{U_{kj}}{M_k}\right)_s \frac{p_k^* - p_k^{\circ}}{(p_k)_s - p_k^{\circ}} \, , \quad \text{for } j \in J \tag{3.70}$$

then equation (3.67a) is satisfied. Based on this, it follows that the following equation may be used to estimate the value of U_{kj}/M_kW_j

$$\frac{U_{kj}}{M_kW_j} = \begin{cases} \left(\frac{U_{kj}}{M_kW_j}\right)_s \cdot \frac{p_k^* - p_k^\circ}{(p_k)_s - p_k^\circ}, & \text{if } I_{kj} \geq 0 \\ \left(\frac{U_{kj}}{M_kW_j}\right)_s, & \text{if } I_{kj} < 0 \end{cases} \tag{3.71a}$$

where

$$I_{kj} = \left(\frac{U_{kj}}{M_kW_j}\right)_s \tag{3.71b}$$

3.5 Natural Frequency Constraint

A lower limit natural frequency constraint is given by

$$g_k = -q_k + q_k^* \leq 0 \tag{3.72}$$

and q_k is defined by

$$[K]\vec{u}_k = q_k [M]\vec{u}_k \tag{3.73}$$

where the subscript k denotes that the frequency under consideration is the kth constraint. Let q_k* denote the prescribed lower bound on q_k, and let $\vec{u}_k$ represent the corresponding natural mode shape. The mass matrix of the structure considered is represented by [M] and for the class of structures considered here

$$[M] = \sum_{i=1}^{NM} [m_i] A_i \tag{3.74}$$

where the $[m_i]$ are unit element mass matrices independent of the element size.

Through a development that runs parallel to that used in the case of a buckling constraint, the following optimality criterion can be obtained

$$W_j - \sum_{k=1}^{NC} \frac{\lambda_k}{T_k} (U_{kj} - q_k T_{kj}) = 0, \text{ for } j \in J \tag{3.75a}$$

where

$$U_{kj} = \sum_{i \in j} a_{ij} D_j \vec{u}_k^T [k_i] \vec{u}_k \tag{3.75b}$$

$$T_{kj} = \sum_{i \in j} a_{ij} D_j \vec{u}_k^T [m_i] \vec{u}_k \tag{3.75c}$$

$$T_k = \vec{u}_k^T [M] \vec{u}_k \tag{3.75d}$$

The redesign equation is also obtained in a manner analogous to that previously employed in the case of a buckling constraint and the result is

$$(D_j)_{s+1} = \begin{cases} C_j (D_j)_s & , \text{ if } I_j \geq 0 \\ 0 & , \text{ if } I_j < 0 \end{cases} \tag{3.76a}$$

where

$$I_j = \sum_{k=1}^{NC} \left(\frac{\lambda_k}{T_k} \cdot \frac{U_{kj} - \gamma_k T_{kj}}{W_j} \right)_s \tag{3.76b}$$

and

$$C_j = I_j^{\frac{1}{2}} \tag{3.76c}$$

If size constraints are imposed, then equation (3.20) must be used in conjunction with equations (3.76).

For an active constraint, the following relation must be satisfied

$$\sum_{j=1}^{NDV} (U_{kj} - \gamma_k^* T_{kj}) = 0 \tag{3.77}$$

Consequently

$$\sum_{j \in J} (U_{kj} - \gamma_k^* T_{kj}) = - \sum_{j \notin J} (U_{kj} - \gamma_k^* T_{kj}) \tag{3.78}$$

Now we assume that

$$\left.\begin{aligned} U_{kj} &= (U_{kj})_s \\ T_{kj} &= (T_{kj})_s \end{aligned}\right\}, \quad \text{for } j \notin J \tag{3.79}$$

then

$$\sum_{j \in J} (U_{kj} - q_k^* T_{kj}) = -\sum_{j \notin J} (U_{kj} - q_k^* T_{kj})_s \tag{3.80}$$

Define the following two quantities

$$\begin{aligned} Q_a &= \sum_{j \in J} (U_{kj} - q_k^* T_{kj})_s \\ Q_p &= -\sum_{j \notin J} (U_{kj} - q_k^* T_{kj})_s \end{aligned} \tag{3.81}$$

and let

$$\left.\begin{aligned} U_{kj} &= (U_{kj})_s \frac{Q_p}{Q_a} \\ T_{kj} &= (T_{kj})_s \frac{Q_p}{Q_a} \end{aligned}\right\}, \quad \text{for } j \in J \tag{3.82}$$

then equation (3.77) will be satisfied. Therefore the following equation is used to estimate the value of $(U_{kj} - q_k^* T_{kj})/T_k W_j$,

$$\frac{U_{kj} - q_k^* T_{kj}}{T_k W_j} = \begin{cases} \left(\dfrac{U_{kj} - q_k^* T_{kj}}{T_k W_j}\right)_s \cdot \dfrac{Q_p}{Q_a}, & \text{if } I_{kj} \geq 0 \\ \left(\dfrac{U_{kj} - q_k^* T_{kj}}{T_k W_j}\right)_s, & \text{if } I_{kj} < 0 \end{cases} \tag{3.83a}$$

where

$$I_{kj} = \left(\frac{U_{kj} - q_k^* T_{kj}}{T_k W_j} \right)_s \tag{3.83b}$$

An upper limit natural frequency constraint is given by

$$g_k = q_k - q_k^* \leq 0$$

In this case, the optimality criterion, the redesign equation, and the estimation of the value $[(U_{kj} - q_k^* T_{kj})/T_k W_j]$ can be immediately obtained from equations (3.75), (3.76), and (3.83), respectively, by replacing the term $(U_{kj} - q_k^* T_{kj})$ by $-(U_{kj} - q_k^* T_{kj})$.

CHAPTER IV

NUMERICAL EXAMPLES

Computer programs that generate basis design vectors were written implementing some of the optimization procedures presented in Chapter III. These programs were coded in FORTRAN H and were run on an IBM 360/91 computer. An optimization program called CONMIN was used to obtain the final results reported herein. The program CONMIN, developed by Vanderplaats [17], is based on a modified feasible directions method.

Several design examples are presented here to illustrate the effectiveness of the method developed in this study. These examples include two and three dimensional trusses, and in each example, except for the first one, stress, displacement and minimum size constraints are included. Some multiple load condition cases are also considered.

In order to make the method more effective, an approximation technique is employed for estimating stresses and displacements during the generation of basis design vectors. This technique, which is based on using first order Taylor series expansions to explicitly approximate stresses and displacements in terms of reciprocal design variables (see Appendix D), signi-

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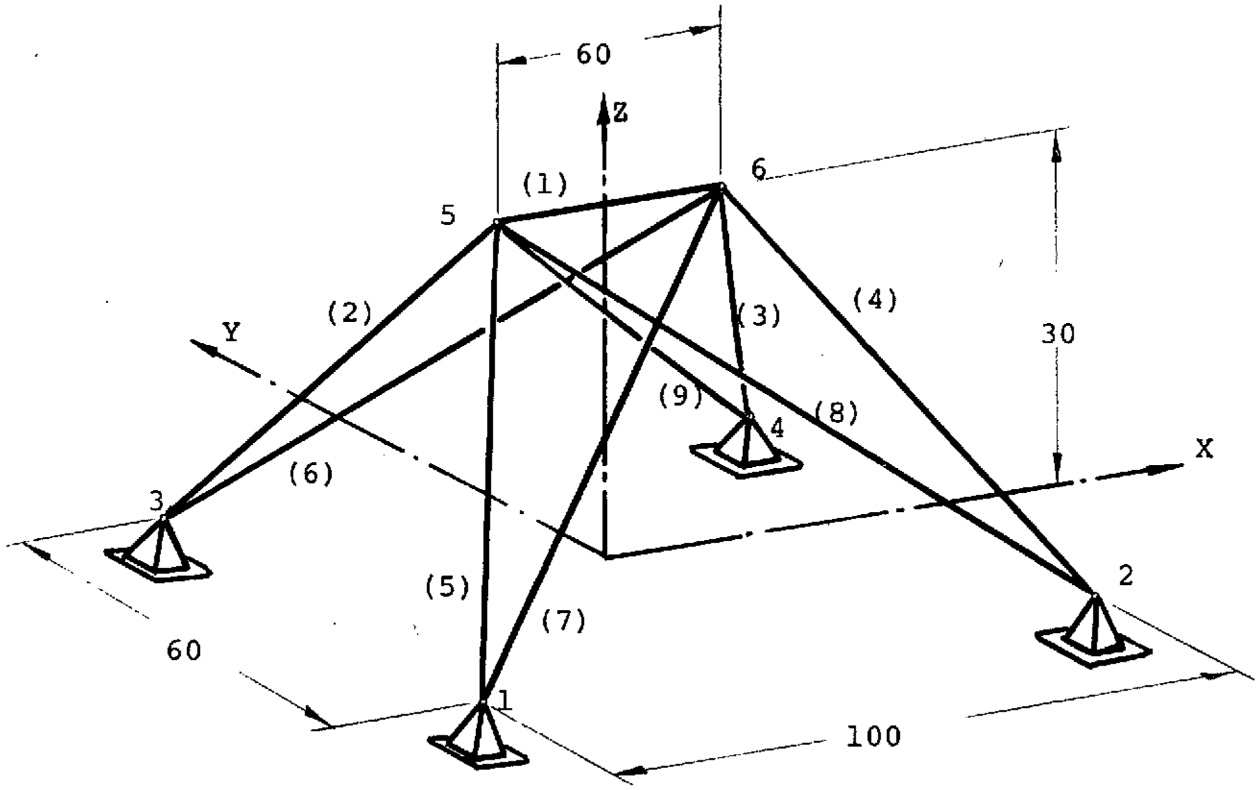
ficantly reduces the number of structural analyses needed to generate the design basis vectors. Hereafter, the optimization method combined with the Taylor series approximation technique will be referred to as revised method, and the method without the use of the Taylor series approximation technique will be designated as the ordinary method. The effectiveness of the revised method is demonstrated by comparison with the ordinary method in some examples.

4.1 9 Bar Truss

The first example problem is a nine bar space truss (see Fig. 10) which is studied to demonstrate the appropriateness of the optimization procedure developed in Chapter III in comparison with those which were given by Gellatly [9] and by Venkaya [11] (see Appendix E). For the sake of simplicity, only generalized stiffness constraints which can be called "total strain energy constraints" (see Appendix C) are considered. The material properties and the specified value of constraints (upper limit on total strain energy) are given in Fig. 10. For this example, two distinct cases are considered, and the load conditions for each case are given in Table 1(a). Design variable linking is used to impose symmetry with respect to both the x-z and y-z planes, and the number of design variables is three.

Results for these two cases are summarized in Table 1(b). In case 1, the minimum weights obtained are essentially the same, although Venkaya's design is heavier than the others by 6%. In case 2, however, the design obtained by the present method is lighter than the others by almost 20%. It is also noted that in the present design, both constraints almost reach the specified upper limit, but in the other designs the total



Material:	Aluminum, $E = 10^7$ psi, $\rho = 0.1$ pci
Minimum Size:	0.01 in^2
Maximum Size	None
Energy Limits:	100 lb-in on both load conditions

Figure 10. 9 Bar Truss

Table 1 Design Data and Results for Example 4.1

(a) Load Conditions (lb)

Case	Load Condition	Node	Direction X	Direction Y	Direction Z
1	1	5	2000.0	0.0	-3000.0
		6	0.0	0.0	-3000.0
	2	5	0.0	4000.0	0.0
		6	0.0	-4000.0	0.0
2	1	5	3000.0	0.0	0.0
		6	-3000.0	0.0	0.0
	2	5	0.0	4000.0	0.0
		6	0.0	-4000.0	0.0

(b) Summary of Results

Case	Method	No. of Analyses	Weight (lb)	Element Size (in^2) 1	2 5	6 9	Strain Energy Ld. 1	Ld. 2
1	Venkaya	10	30.5	0.678	1.180	0.158	100.0	83.2
	Gellatly	9	28.8	0.736	0.966	0.173	100.0	92.6
	Present	10	28.2	0.776	0.883	0.197	100.2	101.7
2	Venkaya	8	22.8	0.875	0.918	0.01	30.7	100.0
	Gellatly	8	22.7	0.869	0.911	0.01	30.8	100.5
	Present	8	19.1	0.275	0.911	0.01	95.5	100.5

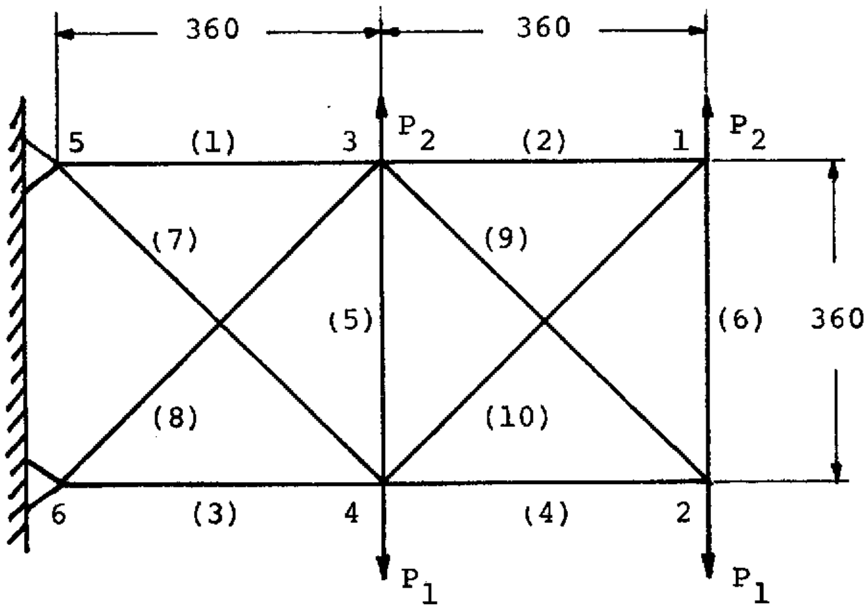
strain energy under load condition 1 is only 31% of the specified limit, which makes these designs much heavier than the present one.

As is obvious from Appendix E, both Venkaya's and Gellatly's methods are approximate, and they do not guarantee that all the active constraints achieve their limits at the same time. It follows that the design obtained by these methods cannot be expected to necessarily be optimal, and in some cases, the results may be rather far away from the optimal design . As illustrated by this example, on the other hand, the present method is able to overcome this shortcoming and it can be expected to produce better results.

4.2 10 Bar Truss

The second example problem is the familiar ten bar planar truss (see Fig. 11) for which results have been previously reported in [9],[11] and [14]. The material properties, stress limits and minimum sizes are given in Fig. 11. For this example four distinct cases are considered. In case 1-a, the truss is subject to a single load condition consisting of 100 Kip downward loads applied at joints 2 and 4 (see Fig. 11) and no displacement limitations are imposed. Case 1-b is the same as case 1-a but with vertical displacement limits of ± 2.0 in. imposed at all joints. In case 2-a the truss is subject to a single load condition consisting of 150 Kip downward loads applied at joints 2 and 4 as well as 50 Kip upward loads at joints 1 and 3, and no displacement limitations are imposed. Case 2-b is the same as case 2-a but with vertical displacement limits of ± 2.0 in. imposed at all joints. No design variable linking is employed in this example, therefore, the number of design variables is ten.

Results for cases 1-a and 2-a are summarized in Table 2, where part (a) of the table contains the results obtained by the ordinary method, part (b) of the table contains those obtained by the revised method, and part



Material	:	Aluminium, $E = 10^7$ psi, $\rho = 0.1$ pci
Minimum Size	:	0.01 in^2
Maximum Size	:	None
Load Conditions		
Case 1	:	$P_1 = 100$ kip, $P_2 = 0$
Case 2	:	$P_1 = 150$ kip, $P_2 = 50$ kip
Stress Limmits	:	25,000 psi

Fig. 11 10 Bar Truss

Table 2 Results for Example 4.2 (1)

(a) Results obtained by Ordinary Method

Case	No. of Analyses	Weight (lb)	Element Size (in^2)									
			1	2	3	4	5	6	7	8	9	10
1-a	12	1591.7	7.88	0.11	8.12	3.89	0.1	0.11	5.82	5.49	5.50	0.16
2-a	8	1664.4	5.94	0.1	10.06	3.95	0.1	2.05	8.56	2.75	5.58	0.1

(b) Results obtained by Revised Method

Case	No. of Analyses	Weight (lb)	Element Size (in^2)									
			1	2	3	4	5	6	7	8	9	10
1-a	6	1593.4	7.94	0.1	8.06	3.94	0.1	0.1	5.74	5.57	5.57	0.1
2-a	6	1664.6	5.95	0.1	10.05	3.95	0.1	2.05	8.56	2.76	5.58	0.1

(c) Stresses for Results shown in (a) and (b)

	Case	Stresses (psi)									
		1	2	3	4	5	6	7	8	9	10
(a)	1-a	25088.	19098.	-24914.	-25173.	-871.	19098.	24831.	-25180.	25173.	-19098.
	2-a	25012.	13097.	-24993.	-25006.	-112.	24998.	24989.	-25035.	25006.	-18522.
(b)	1-a	24991.	15516.	-25006.	-24997.	76.	15516.	25010.	-24980.	24997.	-21943.
	2-a	24996.	13094.	-25002	-24999.	38.	25002.	25004.	-24988.	24999.	-18454.

(c) of the table displays the stresses corresponding to the designs shown in (a) and (b). It is apparent from these results that the approximate analysis technique is rather effective in reducing the total number of analyses needed to obtain an optimum design. The results shown in part (b) of Table 2 indicate that the minimum weights obtained are just the same as those reported in Ref. [14], and it is interesting to note that the fully stressed designs obtained coincide with the actual optimal designs in this case.

Table 3 shows the results of generating basis design vectors for displacement constraints. Part (a) of the table contains the results obtained by the ordinary method, and part (b) of the table contains the results obtained by the revised method. Parts (c) and (d) of the table display the displacements corresponding to the designs shown in (a) and (b), respectively. The results obtained by both methods are essentially the same, however the number of analyses required is cut in half when the revised method is employed. Observing the optimization process, it was found that this example exhibits a particularly interesting behavior. This behavior is well illustrated by following the iteration history generated by the ordinary method. In case 1-b, initially the vertical displacement at

Table 3 Results for Example 4.2 (2)

(a) Basis Design for Displacement Constraints obtained by Ordinary Method

Case	No. of Analyses	Weight (lb)	Element Size (in^2)									
			1	2	3	4	5	6	7	8	9	10
1-b	14	5044.9	31.14	0.1	22.54	15.42	0.1	0.41	5.83	22.07	21.81	0.1
2-b	14	4502.3	24.48	0.1	23.49	14.03	0.1	0.97	9.31	14.51	19.85	0.1

(b) Basis Design for Displacement Constraints obtained by Revised Method

Case	No. of Analyses	Weight (lb)	Element Size (in^2)									
			1	2	3	4	5	6	7	8	9	10
1-b	8	5034.4	31.03	0.1	22.50	15.32	0.1	0.79	5.80	21.93	21.67	0.1
2-b	7	4504.3	24.52	0.1	23.48	14.05	0.1	0.96	9.27	14.54	19.87	0.1

(c) Displacements for Design (a)

Node	1-b		2-b	
	X	Y	X	Y
1	0.095	-2.002	-0.381	-0.205
2	-0.541	-1.942	-0.636	-2.006
3	0.233	-0.706	0.229	-0.618
4	-0.309	-1.974	-0.375	-1.937

(d) Displacements for Design (b)

Node	1-b		2-b	
	X	Y	X	Y
1	0.092	-1.996	-0.391	-0.196
2	-0.550	-1.977	-0.636	-2.002
3	0.237	-0.719	0.228	-0.617
4	-0.314	-2.000	-0.375	-1.963

joint 2 was assumed to be critical. At the 8th iteration, the vertical displacement at joint 1 became most critical, and at the 9th iteration, a feasible design of weight 5227.1 lb. was obtained. However, at the 10th iteration, the weight jumped up to 7579.0 lb. and the vertical displacement at joint 4 also became critical. After five more iterations the design weight of 5044.9 lb. was obtained, and for this design all three of the previously mentioned displacement constraints were critical. The iteration history for this case (Example 4.2 , case 1-b) is shown in Fig. 12. In case 2-b, the vertical displacement at joint 2 was initially assumed to be critical. In the 6th iteration, the vertical displacement at joint 4 became critical and seriously violated. Consequently, the weight suddenly increased in the 7th iteration and the displacement at joint 2 became most critical again. Finally a design weighing 4502.3 lb. was obtained, and for this design both the constraints were critical. The iteration history for this case (Example 4.2 case 2-b) is shown in Fig. 13. A similar phenomenon was reported in Ref. [9] for this same problem. The dramatic rise in weight may be associated with a major redistribution of the internal forces in the structure.

The final design for cases 1-b and 2-b are summarized

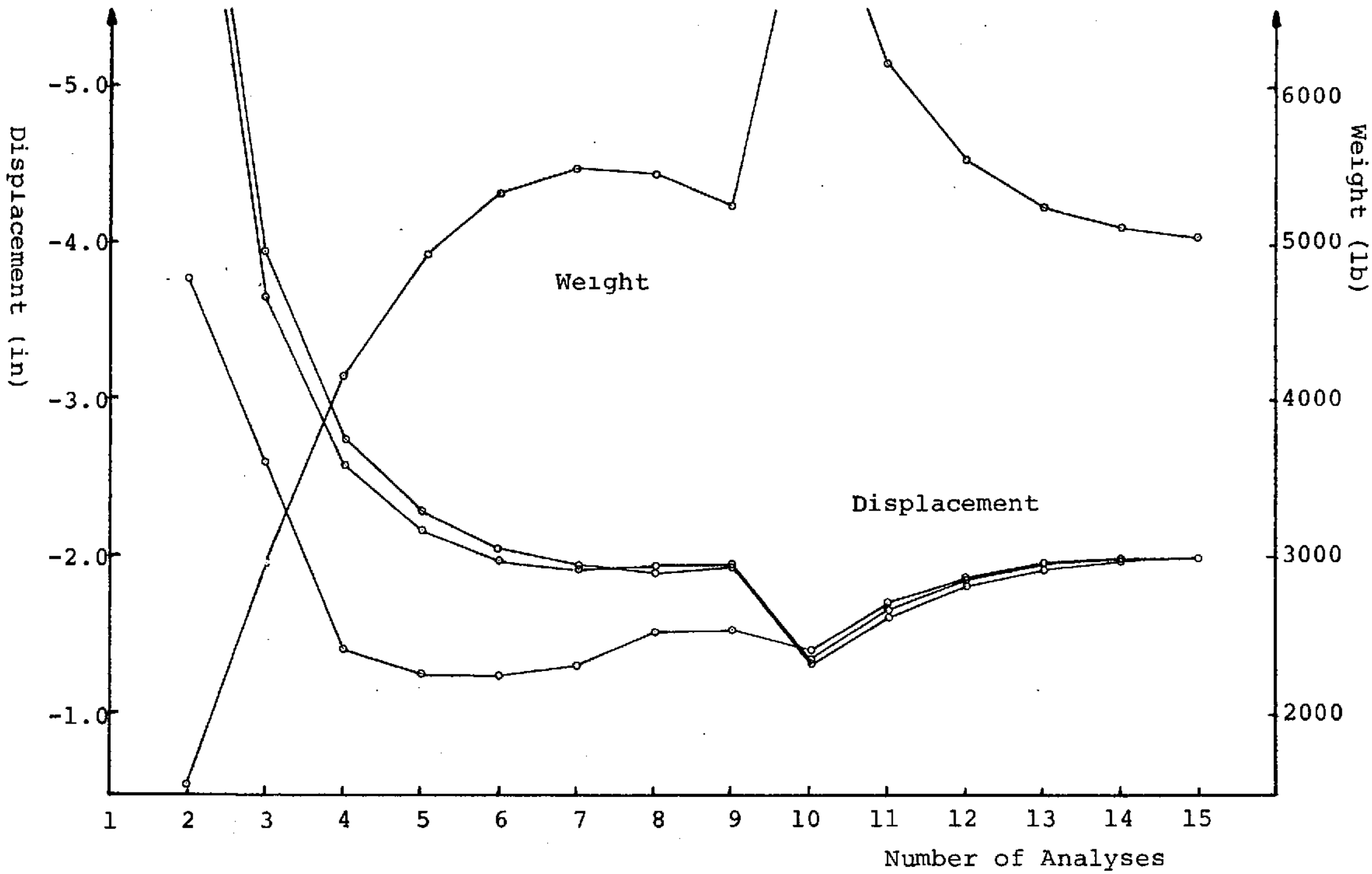


Fig. 12 Iteration History of Example 4.2, Case 1-b

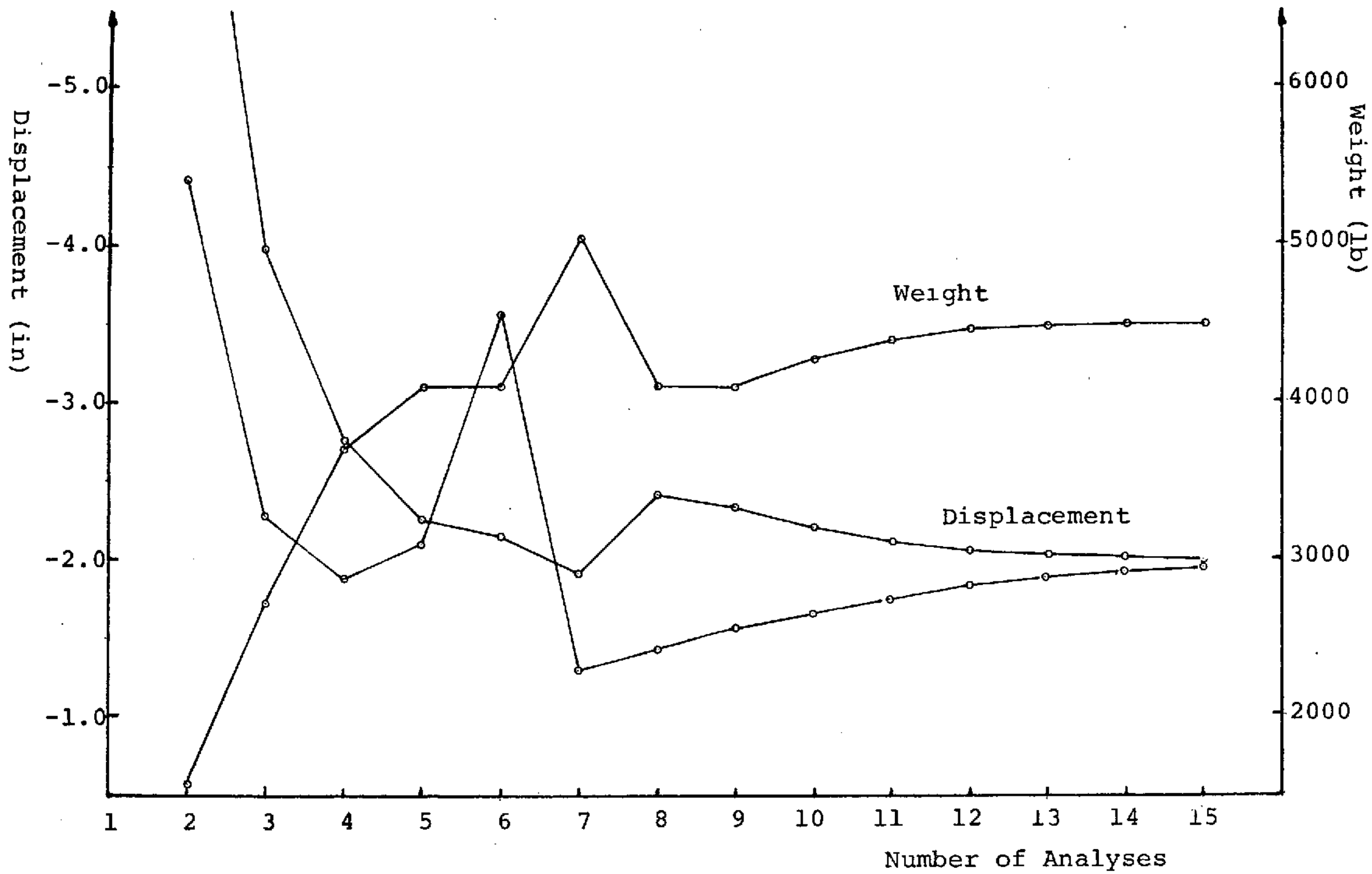


Fig. 13   Iteration History of Example 4.2, Case 2-b

together with actual stresses and displacements for the design in Table 4. In both cases the minimum weights obtained are essentially the same as those previously reported in Refs. [11] and [14]. The respective critical constraints are the tensile stress in element 5 and the vertical displacement at joints 1 and 2 in case 1-b, and the tensile stress in element 5 as well as the vertical displacement at joint 2 in case 2-b.

Table 4 Results for Example 4.2 (3)

(a) Final Design

Case	No. of Analyses	Weight (lb)	Element Size (in^2)									
			1	2	3	4	5	6	7	8	9	10
1-b	7	5077.6	30.53	0.13	23.08	15.08	0.13	0.73	7.45	21.56	21.33	0.13
2-b	7	4708.6	23.88	0.14	25.32	13.99	0.14	1.96	12.57	13.73	19.78	0.14

(b) Stresses for Design (a)

Case / Element	1-b	2-b
1	6663.	6469.
2	-1257.	-7334.
3	-8519.	-9698.
4	-6642.	-7223.
5	24962.	24747.
6	-222.	25000.
7	18334.	16371.
8	-6782.	-5611.
9	6642.	7223.
10	1778.	10372.

(psi)

(c) Displacements for Design (a)

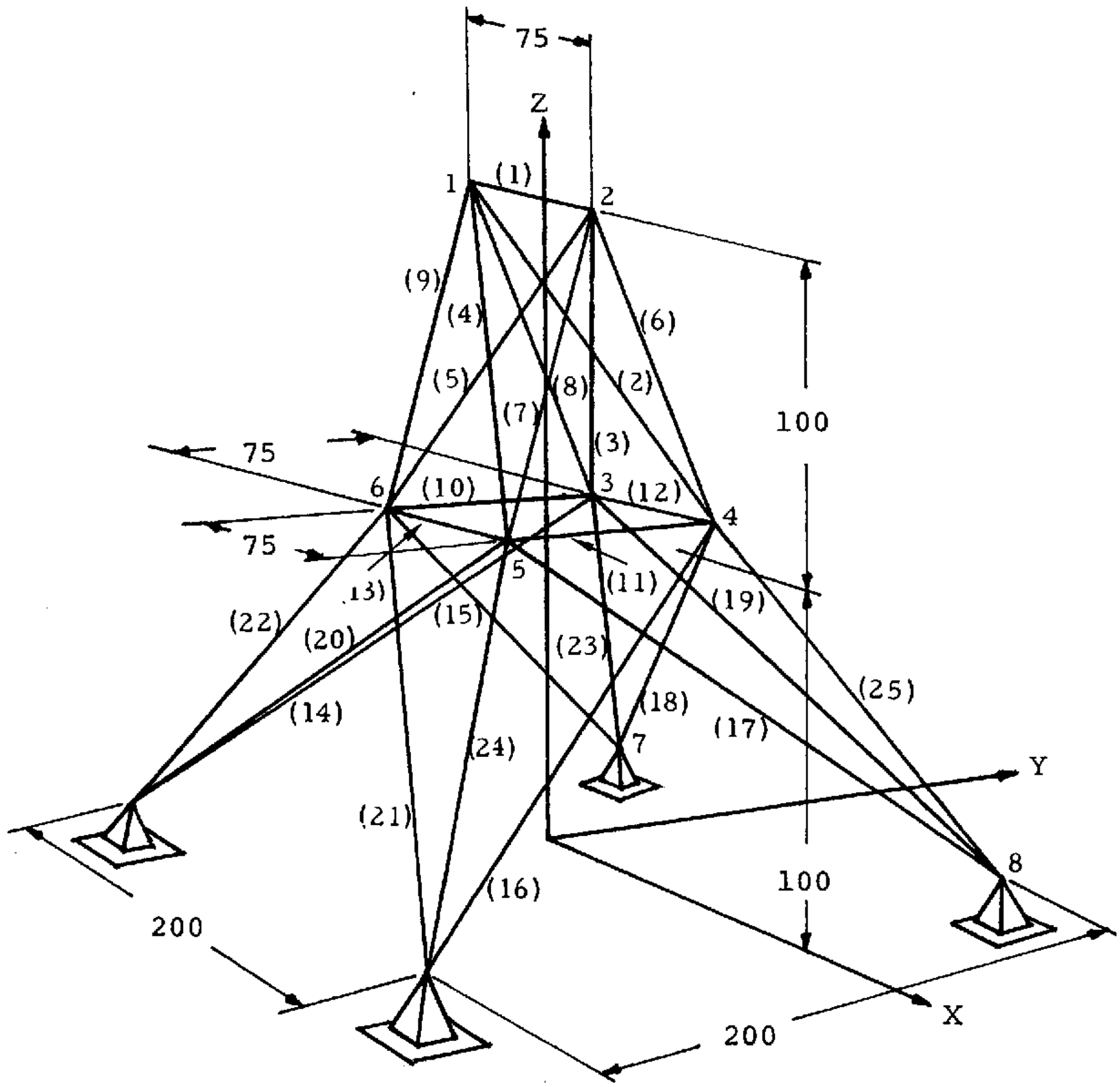
Node	1-b		2-b	
	X	Y	X	Y
1	0.195	-2.000	-0.031	-1.099
2	-0.546	-1.992	-0.609	-1.999
3	0.240	-0.728	0.233	-0.637
4	-0.307	-1.627	-0.349	-1.528

(in)

4.3 25 Bar Truss

The third example problem is a twenty five bar space truss (see Fig. 14) for which results have been previously reported in Refs. [9], and [14]. The material properties, tensile stress limits, displacement limits and minimum size constraints are given in Fig. 14. The allowable compressive stress limits are listed in Table 5(b), and they correspond to those given in Ref. [9]. The structure is subject to two distinct load conditions as given in Table 5(a). Displacement limits of ± 0.35 in. are imposed on all joints in all directions. Design variable linking is used to impose symmetry with respect to both the x-z and y-z planes, and the number of design variables is eight.

Table 5(b) shows the basis design vector for stress constraints which was obtained by the ordinary method. The basis design vectors for displacement constraints are obtained by both the revised and the ordinary methods respectively. The results are shown in Table 6(a) and their iteration history is shown in Fig. 15. In both the designs shown in Table 6(a), critical displacements were those in the y direction at joint 1 and 2 under both the load conditions. The final design obtained is shown in Table 6(b), and the weight is very close to the lightest



Material	:	Aluminium, $E=10^7$psi, $\rho=0.1$pci
Stress Limmits	:	Tension 40,000 psi
		(See Table 5 for Compression)
Displacement Limmits	:	±0.35 in
		(All Nodes, All Directions)
Minimum Size	:	0.01 in^2

Fig. 14 25 Bar Truss

Table 5 Results for Example 4.3 (1)

(a) Load Conditions

(lb)

Load Condition	Node	Direction		
		X	Y	Z
1	1	1000.0	10000.0	-5000.0
	2	0.0	10000.0	-5000.0
	3	500.0	0.0	0.0
	6	500.0	0.0	0.0
2	1	0.0	20000.0	-5000.0
	2	0.0	-20000.0	-5000.0

(b) Basis Design for Stress Constraints obtained by Ordinary Meth0d

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1	2 ∿ 5	6 ∿ 9	10, 11	12, 13	14∿17	18∿21	22∿25
4	344.0	0.01	1.250	1.115	0.01	0.01	0.552	1.648	1.336
Allowable Stress		-35092.	-11590.	-17305.	-35092.	-35092.	-6759.	-6959.	-11082.

(psi)

Table 6 Results for Example 4.3 (2)

(a) Basis Designs for Displacement Constraints

Method	No. of Analyses	Weight (lb)	Element Size (in^2)							
			1	2 ∿ 5	6 ∿ 9	10,11	12,13	14∿17	18∿21	22∿25
Rev.	4	543.6	0.01	2.082	3.032	0.01	0.01	0.662	1.656	2.569
Ord.	8	543.6	0.01	2.082	3.032	0.01	0.01	0.662	1.656	2.569

(b) Final Design

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1	2 ∿ 5	6 ∿ 9	10,11	12,13	14∿17	18∿21	22∿25
9	551.6	0.010	2.112	3.063	0.010	0.010	0.674	1.690	2.602

(c) Final Designs obtained by using Quasi Basis Designs

Case	No. of Analyses	Weight (lb)	Element Size (in^2)							
			1	2 ∿ 5	6 ∿ 9	10,11	12,13	14∿17	18∿21	22∿25
1	6	584.9	0.013	2.442	3.169	0.013	0.316	0.663	1.947	2.396
2	5	563.0	0.013	2.078	2.667	0.013	0.013	0.748	2.070	2.547

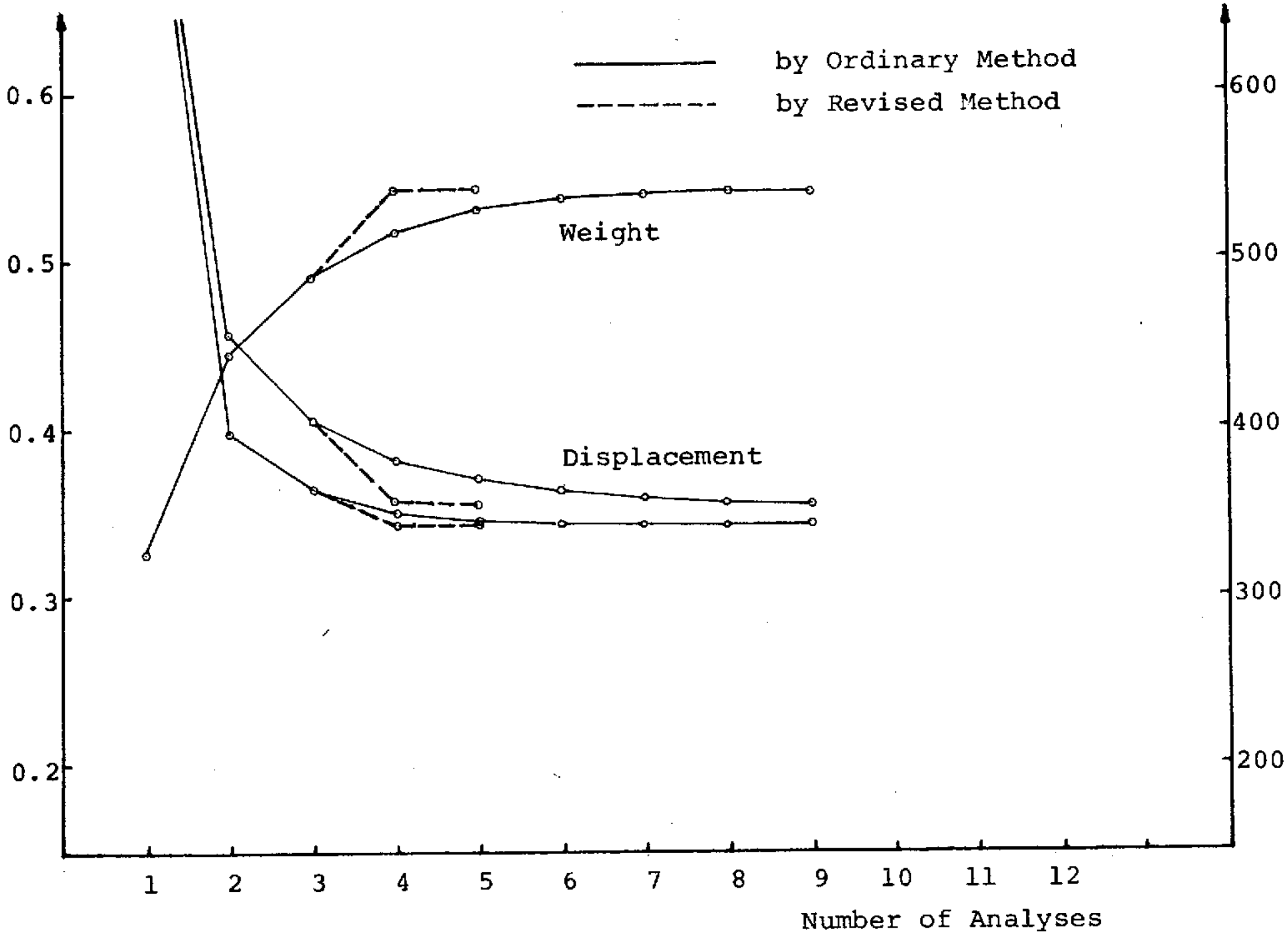


Fig. 15 Comparison of Iteration History for Ordinary and Revised Methods for Example 4.3

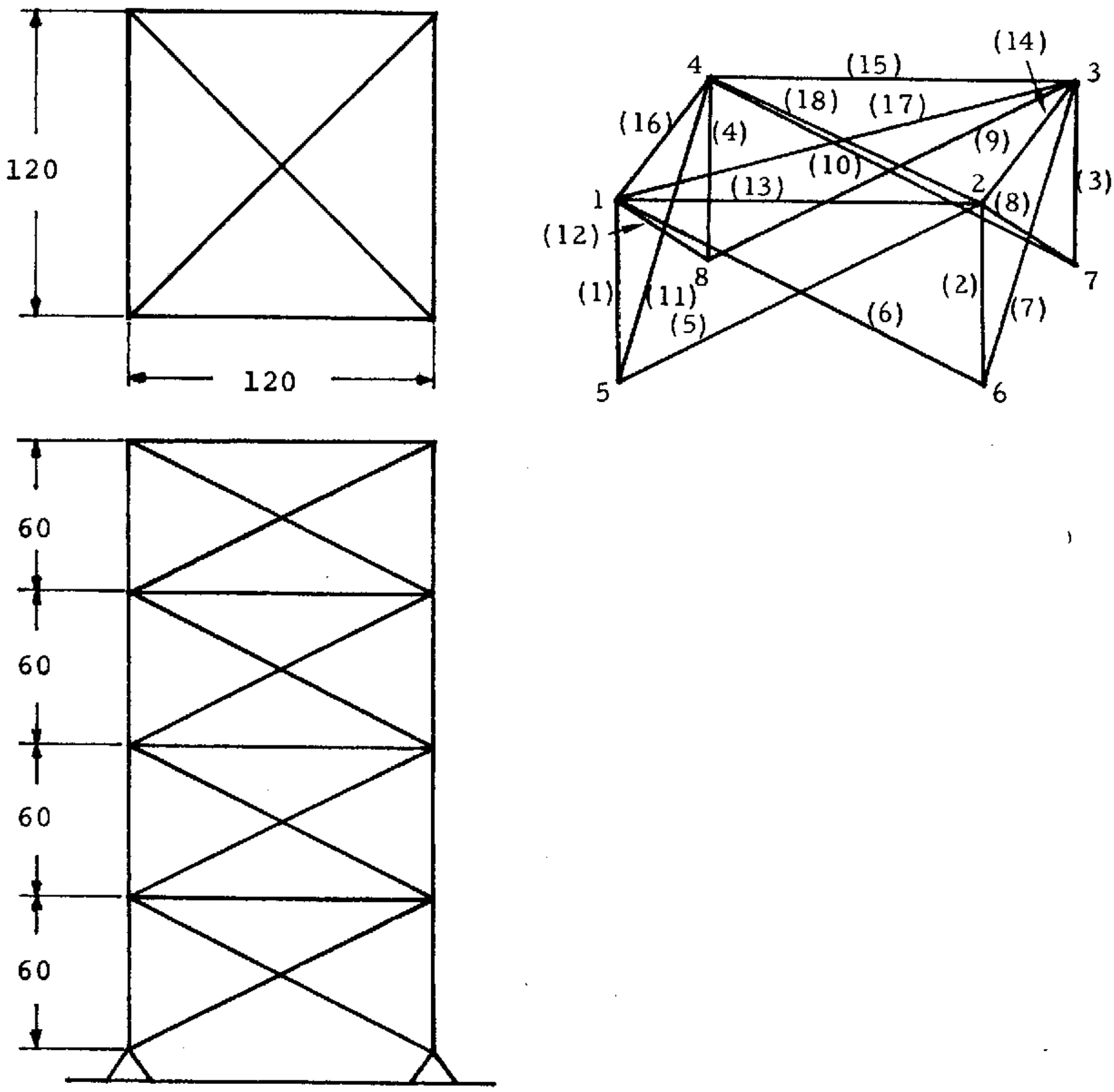
weight previously reported. The final critical constraints are the stresses in elements 19 and 20 under load condition 2, and the displacements in the y direction at joints 1 and 2 under load condition 1.

The designs obtained by using another two sets of basis design vectors are shown in Table 6(c). In case 1 the basis design vectors are obtained by executing only one iteration in each procedure for generating the basis design vectors previously used. In case 2 the basis design vectors were obtained in such a way that the redesign procedure was carried out using the Taylor series approximations after two exact structural analyses. The weights obtained are higher than that of the exact design by about 6% and 2%, respectively, however these results have practical significance since the computational effort required to obtain them is significantly less than that needed to produce the results given in Table 6(b).

4.4 72 Bar Truss

The last example problem is a seventy two bar space truss for which results have been previously reported in Ref. [9], [11] and [14]. Figure 16 shows the geometry of the structure, and the node as well as element numbering system is illustrated in detail for the upper tier. The material properties, stress limits, displacement limits and minimum sizes are given in the same figure. The structure is subject to two distinct load conditions as given in Table 7. Displacement limits of ± 0.25 in. are imposed on all joints in all directions. Design variable linking is employed, and the number of independent design variables is sixteen.

The basis design vector for stress constraints is given in Table 8(a), and that for displacement constraints is given in (b). Both the basis design vectors were obtained by the ordinary method. The final design obtained is listed in Table 9(a). The weight for the design is 384.8 lb., and is very close to the lightest weight previously reported in [11]. The final critical constraints are the compressive stresses in elements 1,2,3 and 4 under load condition 2 and the displacements in both the x and y directions at joint 1 under load condition 1. The design obtained by using another set of basis design



Material	:	Aluminium, $E=10^7$psi, $\rho=0.1$pci
Stress Limmits	:	±25,000 psi
Displacement Limmits	:	±0.25 in
		(All Nodes, All Directions)
Minimum Size	:	0.1 in^2

Fig. 16   72 Bar Truss

Table 7 Load Conditions for Example 4.4

Load Condition	Node	Direction X	Direction Y	Direction Z
1	1	5000.0	5000.0	-5000.0
2	1	0.0	0.0	-5000.0
	2	0.0	0.0	-5000.0
	3	0.0	0.0	-5000.0
	4	0.0	0.0	-5000.0

(lb)

Table 8 Results for Example 4.4 (1)

(a) Basis Design for Stress Constraints

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1∿4	5∿12	13∿16	17,18	19∿22	23∿30	31∿34	35,36
3	96.6	0.189	0.1	0.1	0.1	0.190	0.1	0.1	0.1
		37∿40	41∿48	49∿52	53,54	55∿58	59∿66	67∿70	71,72
		0.199	0.1	0.1	0.1	0.294	0.1	0.1	0.1

(b) Basis Design for Displacement Constraints

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1 ∿ 4	5 ∿12	13∿16	17,18	19∿22	23∿30	31∿34	35,36
9	376.7	0.136	0.532	0.401	0.548	0.685	0.502	0.1	0.1
		37∿40	41∿48	49∿52	53,54	55∿58	59∿66	67∿70	71,72
		1.309	0.497	0.1	0.1	1.890	0.498	0.1	0.1

Table 9 Results for Example 4.4 (2)

(a) Final Design obtained using ordinary method Basis Designs

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1 ~ 4	5 ~ 12	13~16	17,18	19~22	23~30	31~34	35,36
8	384.8	0.154	0.540	0.410	0.556	0.701	0.510	0.109	0.109
		37~40	41~48	49~52	53,54	55~58	59~66	67~70	71,72
		1.324	0.505	0.109	0.109	1.912	0.506	0.109	0.109

(b) Final Design obtained by using Quasi Basis Design

No. of Analyses	Weight (lb)	Element Size (in^2)							
		1 ∿ 4	5 ∿12	13∿16	17,18	19∿22	23∿30	31∿34	35,36
7	460.4	1.404	0.525	0.347	0.484	1.601	0.488	0.116	0.116
		37∿40	41∿48	49∿52	53,54	55∿58	59∿66	67∿70	71,72
		2.044	0.486	0.116	0.116	2.372	0.541	0.151	0.116

vectors is shown in Table 9(b). These basis design vectors are obtained by executing only one iteration in each procedure for generating the basis design vectors previously used. The weight obtianed is 460 lb. and it is heavier than the other design by almost 20%.

Finally the number of structural analyses and CPU runtime are summarized in Table 10 for each example problem.

Table 10 Summary of Number of Analyses and Run Times

Example	Case	Weight (lb)	No. of Analyses and CPU Run Time (sec)							
			Stress		Displacement		Final		Total	
4.2	1-a	1593.4	6	0.16	—	—	—	—	6	0.16
	1-b	5077.6	6	0.16	8	0.53	7	0.16	21	0.85
	2-a	1664.6	6	0.15	—	—	—	—	6	0.15
	2-b	4708.6	6	0.15	7	0.33	7	0.23	20	0.71
4.3		551.6	4	0.25	4	0.67	9	1.67	17	2.57
	*1	563.0	2	0.26	2	0.54	5	0.42	9	1.22
	*2	584.9	1	0.07	1	0.17	6	0.85	8	1.09
4.4		384.8	3	0.88	9	2.11	8	5.40	20	8.39
	*	460.4	1	0.29	1	0.24	7	4.09	9	4.62

* indicates the use of Quasi Basis Design.

CHAPTER V

CONCLUSIONS

In this study, the primary effort was focused on the derivation of optimality criteria and the development of rational recursive redesign procedures based on the optimality criteria. Optimality criteria, which are equivalent to the necessary conditions for local optimality, were derived for a general case, so that they could easily be specialized for each type of behavior constraint considered. Special attention has been given to developing design optimization procedures for basis vector generation that: (1) are rational; (2) are efficient; (3) yield feasible basis design vectors. The recursive redesign procedures presented are based on optimality criteria concepts. Furthermore the procedures presented are rational and they do not exhibit the shortcoming present in some of the previously reported methods (for example, see [9] and [11]), for cases involving multiple constraints (of the same behavior type) and multiple load conditions. As shown in example problem 4.1, this was accomplished by the use of a minimum square method to estimate the optimal values of the multipliers. It is also noted that the estimation procedure requires little additional effort because the values of the mul-

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tipliers can be obtained by an equivalent linear programming method. The procedures presented for generating basis design vectors are efficient. To attain this efficiency, a method for deleting noncritical or less critical constraints was adopted. For the example problems, the constraint deletion technique worked well because the number of active constraints at the optimum design is quite small in comparison with the total number of constraints, and most of the active constraints remain active during the entire design process. Furthermore, first order Taylor series expansions with respect to reciprocal design variables were used to provide explicit approximate representations for stresses and displacements. This high quality approximation was very effective in reducing the number of actual analyses needed to obtain an optimum design. Finally it should be noted that care has been taken to avoid the generation of infeasible basis design vectors such as the one shown in Fig. 7. This was achieved by imposing additional limitations on the values of the multipliers corresponding to violated constraints.

With regard to the results for the example problems, one notable thing is the accuracy of the approximate optimum designs. The obtained weights corresponding to the final designs obtained are very close to (less than 1% above) the best previously reported results for these example problems. It is also noted that these designs were obtained using only two basis design vectors in each case. The results reported here suggest that basis design vectors generated by optimality criteria and stress ratio methods may frequently span a subspace such that the application of mathematical programming methods to the reduced problem, cast in terms of a few generalized design variables, provides an efficient hybrid method for obtaining an excellent upper bound approximation of the optimum design. The trade off between the effort expended to refine the basis vectors (i.e., converge the subproblems) and their quality, with respect to spanning a subspace containing a good upper bound approximation of the optimum design, is an open question which will require further study. This trade off must be investigated thoroughly before hybrid methods of structural optimization can become efficient tools for the design of large practical structural systems subject to a wide range of behavioral constraint types. It should be emphasized that the significance of the results reported here is that they establish the feasibility of the hybrid method concept and they indicate the promise that these methods hold for achieving high efficiency.

Based on the study presented here, the following conclusions have been reached:

(1) The hybrid method appears to be one of the most efficient methods especially for large scale structural problems.

(2) Basis design vectors for hybrid methods can be generated efficiently by the optimality criteria methods.

(3) The stress ratio method appears to be adequate for generating the design basis vector containing information relative to the stress constraints.

(4) In many cases the optimum designs obtained for each subproblem form a reduced basis that spans a subspace containing a good upper bound approximation of the optimum design.

(5) Taylor series expansion with respect to linked reciprocal design variables can be used in the context of optimality criteria and stress ratio methods and they produce considerable improvement in efficiency by reducing the number of actual analyses needed to achieve convergence.

As a result of this study, it is suggested that the following additional work may be of interest:

(1) The method developed for buckling and natural frequency constraints should be implemented and numerical results should then be generated and examined.

(2) A method to prevent convergence to nonoptimal points such as the one shown in Fig. 8 should be sought.

(3) A trade off study employing various strategies that combine optimality criteria methods, or stress ratio methods, with the Taylor series expansion technique for approximate analysis should be carried out. The trade-off will be between the effort expended to obtain the basis vectors and their quality, with respect to spanning a subspace containing a good approximation of the optimum design.

and finally

(4) The method should be extended to other types of behavior constraints as well as a broader class of structures.

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APPENDIX A

FARKAS' LEMMA AND DERIVATION OF EQUATION (3.17)

Farkas' Lemma Ref. 18 is given as follows:

Let $\{\vec{P}_o, \vec{P}_1, \vec{P}_2, \ldots, \vec{P}_r\}$ be an arbitrary set of vectors.
There exists $\beta_i \geq 0$ such that

$$\left.\begin{array}{l} \vec{P}_o = \sum_{i=1}^{r} \beta_i \vec{P}_i \\ \text{if and only if} \\ \vec{P}_o^{\,T} \vec{y} \geqq 0 \\ \text{for all } \vec{y} \text{ satisfying} \\ \vec{P}_i^{\,T} \vec{y} \geqq 0 , \quad i = 1, 2, \cdots, r \end{array}\right\} \quad (A.1)$$

From the lemma, the following relation can easily be derived by replacing $\vec{P}_i$ by $(-\vec{P}_i)$, $i = 1,2,\ldots,r$, if and only if

$$\left.\begin{array}{l} \vec{P}_o^{\,T} \vec{y} \geqq 0 \\ \text{for all } \vec{y} \text{ satisfying} \\ \vec{P}_i^{\,T} \vec{y} \leqq 0 , \quad i = 1, 2, \cdots, r \\ \text{there exists } \beta_i \geq 0 \text{ such that} \\ \vec{P}_o + \sum_{i=1}^{r} \beta_i \vec{P}_i = 0 \end{array}\right\} \quad (A.2)$$

A set of equations (3.16) can be rewritten in the following matrix form

$$\delta W = \vec{w}^{T} \cdot \delta \vec{D}$$

for all $\delta\vec{D}$ satisfying

$$\left.\begin{aligned}
\delta \vec{g}_k &= \nabla g_k^{T} \cdot \delta \vec{D} \leqq 0 , \quad \text{for } k \in K_{act} \\
-\delta D_j &= \vec{I}_j^{T} \cdot \delta \vec{D} \leqq 0 , \quad \text{for } j \in J_{min} \\
\delta D_j &= \vec{J}_j^{T} \cdot \delta \vec{D} \leqq 0 , \quad \text{for } j \in J_{max} \\
&\text{where} \\
w^{T} &= (w_1, w_2, \cdots , w_{NDV}) \\
\delta \vec{D} &= (\delta D_1, \delta D_2, \cdots , \delta D_{NDV}) \\
\nabla \vec{g}_k^{T} &= (g_{k,1} , g_{k,2}, \cdots , g_{k,NDV})
\end{aligned}\right\} \quad (A.3)$$

$\vec{I}_j$ and $\vec{J}_j$ are unit vectors in which only the jth element has the value of -1 and 1, respectively, and all the other elements are zero.

Using the relation (A.2) and replacing $\vec{P}_o$ by $\vec{w}$, $\vec{y}$ by $\delta\vec{D}$, P_i by $\nabla\vec{g}_k$, $\vec{I}_j$ and $\vec{J}_j$, β_i by λ_k, μ_j and η_j, it is proved that the following relation is mathematically equivalent to that given by (A.3),

$$\vec{w} + \sum_{K_{act}} \lambda_k \vec{\nabla g}_k + \sum_{j \in J_{min}} \mu_j \vec{I}_j + \sum_{j \in J_{max}} \eta_j \vec{J}_j = 0 \qquad \text{(A.4)}$$

where λ_k, μ_j and η_j are nonnegative.

In order to include inactive constraints into the relation (A.4), we introduce the null multipliers such that

$$\lambda_k = 0 \,, \qquad \text{for } k \notin K_{act}$$

Using the null multipliers, the relation (A.4) can be rewritten as

$$\vec{w} + \sum_{k=1}^{NC} \lambda_k \vec{\nabla g}_k + \sum_{j \in J_{min}} \mu_j \vec{I}_j + \sum_{j \in J_{max}} \eta_j \vec{J}_j = 0 \qquad \text{(A.5)}$$

From equation (A.5), we get

$$w_j + \sum_{k=1}^{NC} \lambda_k g_{k,j} = 0 \,, \quad \text{for } j \in J$$

$$w_j + \sum_{k=1}^{NC} \lambda_k g_{k,j} = \mu_j \,, \quad \text{for } j \in J_{min}$$

$$w_j + \sum_{k=1}^{NC} \lambda_k g_{k,j} = -\eta_j \,, \quad \text{for } j \in J_{max}$$

Noting that μ_j and η_j are nonnegative, we get the following relations

$$w_j + \sum_{k=1}^{NC} \lambda_k g_{kj} \begin{cases} = 0 , & \text{for } j \in J \\ \geqq 0 , & \text{for } j \in J_{min} \\ \leqq 0 , & \text{for } j \in J_{max} \end{cases}$$

where

$$\lambda_k \begin{cases} \geqq 0 , & \text{for } k \in K_{act} \\ = 0 , & \text{for } k \notin K_{act} \end{cases}$$

This relation is just the same as that given by (3.17).

APPENDIX B

MATHEMATICAL CONSIDERATION

ON THE HYBRID METHOD

Here some mathematical considerations for the hybrid method (basis design vector method) are presented. As shown in Chapter II, the structural optimization problem considered in this paper is given in the following form

Minimize $W = W(\vec{D})$

subject to (B.1)

$$g_k(\vec{D}) \leqq 0 , \qquad k = 1, 2, \cdots, K$$

where $\vec{D}$ represents an M dimensional design variable vector, and $g_k(\vec{D})$ includes both behavior and size constraints and K denotes the total number of constraints.

Let $\vec{D}_1, \vec{D}_2, \ldots, \vec{D}_N$, $N < M$, be an arbitrary set of M dimensional vectors, and define a set of new design variables θ_j such that

$$\vec{D} = \sum_{j=1}^{N} \theta_j \vec{D}_j \tag{B.2}$$

Assume that we get the optimal solution of problem (B.1) for $\vec{\theta}$, let it be $\vec{\theta}^*$, then the following optimality criteria must be satisfied

$$\frac{\partial W(\vec{D}^*)}{\partial \theta_j} + \sum_{k=1}^{K} \lambda_k \frac{\partial g_k(\vec{D}^*)}{\partial \theta_j} = 0, \quad j = 1, 2, \cdots, N \tag{B.3}$$

where

$$\vec{D}^* = \sum_{j=1}^{N} \theta_j^* \vec{D}_j \tag{B.4}$$

From equation (B.2)

$$\frac{\partial}{\partial \theta_j} = \sum_{i=1}^{M} D_{ji} \frac{\partial}{\partial D_i} \tag{B.5}$$

where D_{ji} denotes the ith element of $\vec{D}_j$. From equations (B.3) and (B.5)

$$\sum_{i=1}^{M} D_{ji} \left\{ \frac{\partial W(\vec{D}^*)}{\partial D_i} + \sum_{k=1}^{K} \lambda_k \frac{\partial g_k(\vec{D}^*)}{\partial D_i} \right\} = 0 \tag{B.6}$$

If $\vec{D}^*$ is the actual optimal solution of problem (B.1), the term in the parenthesis should be zero for all i. However, equation (B.5) does not guarantee it because the number of equations N is less than the number of the terms in the parentheses which is M. It follows that the actual optimal solution does not necessarily exist in the subspace defined by equation (B.2) for any choice of D_j, j = 1,2,...,N. Therefore, the solution given by equation (B.4) must be an approximation of the actual optimal solution.

Next we will consider the case that $\vec{D}_j$ is given by the optimal solution of the jth subproblem which is defined as follows

Minimize $W = W(\vec{D})$

subject to

$$g_{jk}(\vec{D}) \leq 0 \ , \qquad k = 1, 2, \cdots, K_j$$

$$g_{ok}(\vec{D}) \leq 0 \ , \qquad k = 1, 2, \cdots, K_o$$

where $g_{jk}(\vec{D})$ represents the constraints which are to be imposed only in the jth subproblem, and $g_{ok}(\vec{D})$ denotes the constraints to be imposed in every subproblem. Let K_j and K_o be the number of corresponding constraints, respectively, and investigate the conditions under which the method will give the actual optimal solution. From the assumption, the following optimality criteria must be satisfied for each j,

$$\frac{\partial W(\vec{D}_j)}{\partial D_i} + \sum_{k=1}^{K_j} \lambda_{jk} \frac{\partial g_{jk}(\vec{D}_j)}{\partial D_i} + \sum_{k=1}^{K_o} \mu_{jk} \frac{\partial g_{ok}(\vec{D}_j)}{\partial D_i} = 0,$$

$$i = 1,2,\ldots,M \tag{B.7}$$

If $\vec{D}^*$ defined by equation (B.4) is the actual optimal solution, then

$$\frac{\partial W(\vec{D}^*)}{\partial D_i} + \sum_{k=1}^{K-K_o} \lambda_k \frac{\partial g_k(\vec{D}^*)}{\partial D_i} + \sum_{k=1}^{K_o} \mu_k \frac{\partial g_{ok}(\vec{D}^*)}{\partial D_i} = 0, \tag{B.8}$$

$$i = 1, 2, \cdots, M, \qquad K = \sum_j K_j + K_o$$

If equation (B.8) can be expressed by the linear combination of equation (B.7), there exist λ_k and μ_k satisfying equation (B.8), and consequently $\vec{D}^*$ gives the actual optimal solution. Generally, this is possible only when g_k is a linear function of $\vec{D}$ for all k, and W is either of the following

(1) linear function of $\vec{D}$

(2) $W = \frac{1}{2}\vec{D}^T[A]\vec{D} + W_o$

where [A] is the MxM matrix and W_o is constant. Under these conditions the hybrid method may give the actual optimal solution. In other cases, it may depend on the problem itself, and general conditions have not yet been derived.

APPENDIX C

OPTIMALITY CRITERIA FOR GENERALIZED STIFFNESS CONSTRAINTS

Generally speaking, the total strain energy stored in a structure represents an inverse measure of the stiffness of the structure. Therefore, the stiffness requirement can be set up by restricting the value of total strain energy. This is called "generalized stiffness constraint" [11], and it is usually given by

$$g = \frac{1}{2}\vec{u}^T[K]\vec{u} - U^* \leqq 0 \tag{C.1}$$

where U* denotes the specified upper limit of total strain energy.

Differentiating both sides of equation (C.1) with respect to A_i, we get

$$\frac{\partial g}{\partial A_i} = \frac{1}{2}\vec{u}^T[k_i]\vec{u} + \vec{u}^T[K]\frac{\partial \vec{u}}{\partial A_i} \tag{C.2}$$

Substituting equations (3.39) and (3.43) into equation (C.2) yields

$$\frac{\partial g}{\partial A_i} = -\frac{1}{2}\vec{u}^T[k_i]\vec{u} \tag{C.3}$$

Using the relation given by (3.36), we get

$$\frac{\partial g}{\partial D_j} = -\frac{1}{2}\sum_{i\in j} a_{ij}\,\vec{u}^T[k_i]\vec{u} \tag{C.4}$$

From equation (3.17), we obtain the optimality criteria for generalized stiffness constraints such that

$$w_j - \frac{\lambda}{2} \sum_{i \in j} a_{ij} \vec{u}^T [k_i] \vec{u} = 0, \quad \text{for } j \in J \qquad \text{(C.5)}$$

Multiplying both sides of equations (C.5) by D_j, we get the following standard form,

$$W_j - \lambda U_j = 0, \qquad \text{for } j \in J \qquad \text{(C.6)}$$

where

$$U_j = \frac{1}{2} \sum_{i \in j} a_{ij} D_j \vec{u}^T [k_i] \vec{u}$$

then U_j denotes the total strain energy stored in the jth group of elements.

For multiple constraints, the criteria can be generalized as

$$W_j - \sum_{k=1}^{NC} \lambda_k U_{kj} = 0, \quad \text{for } j \in J \qquad \text{(C.7)}$$

where

$$\lambda_k \begin{cases} \geq 0, & \text{for } k \in K_{act} \\ = 0, & \text{for } k \notin K_{act} \end{cases}$$

and U_{kj} represents U_j for the kth constraint.

The redesign equation can be obtained in similar to that used for displacement constraints, and it is

$$\left.\begin{aligned} (D_j)_{s+1} &= C_j (D_j)_s \\ \text{where} \\ C_j &= \left(\sum_{k=1}^{NC} \lambda_k \frac{U_{kj}}{W_j} \right)_s^{\frac{1}{2}} \end{aligned}\right\} \qquad (C.8)$$

If size constraints are imposed, equation (3.20) must be used together with equation (C.8).

The value of U_{kj}/W_j can be estimated in the following manner. For active constraints

$$\sum_{j=1}^{NDV} U_{kj} = U_k^* \qquad (C.9a)$$

$$\sum_{j=1}^{NDV} (U_{kj})_s = (U_k)_s \qquad (C.9b)$$

where $(U_k)_s$ represents the total strain energy of a whole structure at the sth iteration.

Let

$$U_{kj} = (U_{kj})_s \frac{U_k^*}{(U_k)_s} \qquad (C.10)$$

then equation (C.9a) is satisfied. Thus we will use the following equation to estimate the value of U_{kj}/W_j

$$\frac{U_{kj}}{w_j} = \left(\frac{U_{kj}}{w_j}\right)_s \cdot \frac{U_k^*}{(U_k)_s} \tag{C.11}$$

APPENDIX D

LINEAR APPROXIMATION OF STRESS AND DISPLACEMENT

It has been recognized that structural behaviors such as stress, displacement and so on can be estimated by using a first order Taylor series expansion, which is given in the following form

$$f(\vec{x}) \doteq f(\vec{x}_o) + \sum_{j=1}^{N} \frac{\partial f(\vec{x}_o)}{\partial x_j}(x_j - x_{jo}) \tag{D.1}$$

where $f(\vec{x})$ is an arbitrary differentiable function of variable $\vec{x} = \{x_j\}$, and $\vec{x}_o = \{x_{jo}\}$ is an arbitrary given point. Applying equation (D.1) to displacement yields

$$u(\vec{x}) \doteq u(\vec{x}_o) + \sum_{j=1}^{N} \frac{\partial u(\vec{x}_o)}{\partial x_j}(x_j - x_{jo}) \tag{D.2}$$

where $\vec{x}$ represents an appropriate design variable vector. Equation (D.2) must be applicable to any displacement, thus we get

$$\vec{u}(\vec{x}) = \vec{u}(\vec{x}_o) + \sum_{j=1}^{N} \frac{\partial \vec{u}(\vec{x}_o)}{\partial x_j}(x_j - x_{jo}) \tag{D.3}$$

It has also been found that the use of reciprocals of the sizing type design variables is very effective in increasing the accuracy of this estimation. Therefore, x_j is selected as

$$X_j = \frac{1}{D_j}, \qquad j = 1, 2, \cdots, N \tag{D.4}$$

From equation (D.4)

$$\frac{\partial \vec{u}}{\partial X_j} = - D_j^2 \frac{\partial \vec{u}}{\partial D_j} \tag{D.5}$$

Substituting equations (D.4) and (D.5) into equation (D.3) yields

$$\vec{u}(\vec{D}) = \vec{u}(\vec{D}_o) - \sum_{j=1}^{N} D_{jo}^2 \frac{\partial \vec{u}(\vec{D}_o)}{\partial D_j} \left(\frac{1}{D_j} - \frac{1}{D_{jo}} \right) \tag{D.6}$$

From equations (3.36), (3.39) and (3.43)

$$\frac{\partial \vec{u}}{\partial D_j} = - [K]^{-1} \sum_{i \in j} a_{ij} [k_i] \vec{u} \tag{D.7}$$

Substituting equation (D.7) into (D.6), we get

$$\vec{u}(\vec{D}) \doteq \vec{u}(\vec{D}_o) + [K(\vec{D}_o)]^{-1} \left(\sum_{j=1}^{N} \sum_{i \in j} a_{ij} [k_i] \right) \vec{u}(\vec{D}_o) \left(\frac{1}{D_j} - \frac{1}{D_{jo}} \right) \tag{D.8}$$

If we know $\vec{u}$ and [K] for design $\vec{D}_o$, then displacement for a new design $\vec{D}$ can be estimated by using equation (D.8).

In the displacement method of analysis, stress is readily expressed as a function of displacements. Therefore stress can also be estimated using equation (D.8). For an axial force element, stress is given by

$$\vec{\sigma} = [S] \vec{u} \tag{D.9}$$

where [S] is a geometrically determined matrix. Substituting equation (D.8) into equation (D.9), we get

$$\vec{\sigma}(\vec{D}) = \vec{\sigma}(\vec{D}_o)$$

$$+ [S][K(\vec{D}_o)]^{-1}\left(\sum_{j=1}^{N}\sum_{i\in j} a_{ij}[k_i]\right)\vec{u}(\vec{D}_o)\left(\frac{1}{D_j} - \frac{1}{D_{jo}}\right) \quad (D.10)$$

APPENDIX E

REDESIGN PROCEDURES BASED ON OPTIMALITY CRITERIA PREVIOUSLY PRESENTED

Here a brief explanation of two representative redesign procedures based on discretized optimality criteria is presented. Among these two procedures, one was given by Venkaya [11], and the other was given by Gellatly [9].

Venkaya has presented a redesign equation for generalized stiffness constraints under multiple load conditions in Ref. [11], which is

$$(\alpha_i \Lambda)_{s+1} = \left(\sum_{k=1}^{p} c_k \frac{u_i^{(k)\prime}}{\tau_i'} \right)_s^{\frac{1}{2}} (\alpha_i)_s \tag{E.1}$$

where α_i is the ith relative design variable, and Λ is a scaling parameter. $u_i^{(k)\prime}$ is the total strain energy of the ith element under the kth load condition for the relative design, and τ_i' denotes the weight of the ith element for the relative design. s represents the cycle of iteration. The weighting parameter c_k is given by

$$c_k = \frac{W}{Z_k}, \quad k = 1, 2, \cdots, p \tag{E.2}$$

where W is the current total weight of the structure and z_k is the specified value for the kth constraint. p is the number of constraints.

Equation (E.1) can be rewritten as

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$$(\alpha_i \Lambda)_{s+1} = \left(\sum_{k=1}^{p} C_k \frac{u_i^{(k)}}{\tau_i} \right)_s^{\frac{1}{2}} (\alpha_i \Lambda)_s \tag{E.3}$$

because

$$u_i^{(k)} = \frac{u_i^{(k)\prime}}{\Lambda}$$

$$\tau_i = \Lambda \tau_i'$$

where $u_i^{(k)}$ is the strain energy of the ith element under the kth load condition, and τ_i is the weight of the element. Under design variable linking, equation (E.3) can be modified by using the notation defined in this paper as

$$(D_j)_{s+1} = \left(\sum_{k=1}^{p} C_k \frac{U_{kj}}{W_j} \right)_s^{\frac{1}{2}} (D_j)_s \tag{D.4}$$

where U_{kj} represents the total strain energy of the jth group elements, and W_j is the total weight of the group.

No redesign equation for generalized stiffness constraints has been given by Gellatly. However, he has presented a redesign procedure for a combination of stress and displacement constraints. The basic concept of his method can be summarized as follows: Compute a new value of each design variable for each constraint and select the largest one for each design variable [9]. This concept was applied to the first example problem in Chapter IV as Gellatly's redesign procedure.