2miy

**NASA TECHNICAL NOTE** **NASA TN D-7552**

NASA TN D-7552

(NASA-TN-D-7552) ANALYTICAL EXPRESSIONS FOR POSITION ERROR IN TRIANGULATION SOLUTION OF POINT IN SPACE FOR SEVERAL STATION CONFIGURATIONS (NASA) ~~48~~ 47 p HC $3.25 CSCL 22C H1/30 N74-28286 Unclas 41837



# ANALYTICAL EXPRESSIONS FOR POSITION ERROR IN TRIANGULATION SOLUTION OF POINT IN SPACE FOR SEVERAL STATION CONFIGURATIONS

*by Sheila Ann T. Long*

*Langley Research Center*
*Hampton, Va. 23665*

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1974

| 1. Report No. | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| NASA TN D-7552 | | |

**4. Title and Subtitle**
ANALYTICAL EXPRESSIONS FOR POSITION ERROR IN TRIANGULATION SOLUTION OF POINT IN SPACE FOR SEVERAL STATION CONFIGURATIONS

**5. Report Date**
June 1974

**6. Performing Organization Code**

**7. Author(s)**
Sheila Ann T. Long

**8. Performing Organization Report No.**
L-9235

**9. Performing Organization Name and Address**
NASA Langley Research Center
Hampton, Va. 23665

**10. Work Unit No.**
879-11-36-01

**11. Contract or Grant No.**

**12. Sponsoring Agency Name and Address**
National Aeronautics and Space Administration
Washington, D.C. 20546

**13. Type of Report and Period Covered**
Technical Note

**14. Sponsoring Agency Code**

**15. Supplementary Notes**

**16. Abstract**

Analytical expressions are derived to first order for the rms position error in the triangulation solution of a point object in space for several ideal observation-station configurations. These expressions provide insights into the nature of the dependence of the rms position error on certain of the experimental parameters involved. The station geometries examined are: (1) the configuration of two arbitrarily located stations; (2) the symmetrical circular configuration of two or more stations with equal elevation angles; and (3) the circular configuration of more than two stations with equal elevation angles, when one of the stations is permitted to drift around the circle from its position of symmetry. The expressions for the rms position error are expressed as functions of the rms line-of-sight errors, the total number of stations of interest, and the elevation angles.

**17. Key Words (Suggested by Author(s))**
Position error
Triangulation
Analytical triangulation
Point in space
Observation stations

**18. Distribution Statement**
Unclassified – Unlimited

STAR Category 30

| 19. Security Classif. (of this report) | 20. Security Classif. (of this page) | 21. No. of Pages | 22. Price* |
|---|---|---|---|
| Unclassified | Unclassified | 49 | $3.25 |

*For sale by the National Technical Information Service, Springfield, Virginia 22151

# ANALYTICAL EXPRESSIONS FOR POSITION ERROR IN TRIANGULATION SOLUTION OF POINT IN SPACE FOR SEVERAL STATION CONFIGURATIONS

By Sheila Ann T. Long
Langley Research Center

## SUMMARY

Analytical expressions are derived to first order for the rms position error in the triangulation solution of a point object in space for several ideal observation-station configurations. These expressions provide insights into the nature of the dependence of the rms position error on certain of the experimental parameters involved. The station geometries examined are: (1) the configuration of two arbitrarily located stations; (2) the symmetrical circular configuration of two or more stations with equal elevation angles; and (3) the circular configuration of more than two stations with equal elevation angles, when one of the stations is permitted to drift around the circle from its position of symmetry. The expressions for the rms position error are expressed as functions of the rms line-of-sight errors, the total number of stations of interest, and the elevation angles.

## INTRODUCTION

Obtaining the errors in the triangulation solution of a point object or an elongated object in space using data from three or more arbitrarily located observation stations is a complicated problem, and numerical solutions are usually sought. Reference 1 gives formulas for the errors in the geocentric position, as calculated using the simultaneity circle, of a satellite. Reference 2 presents formulas for adjusting all measurements of a space triangulation to determine the coordinates of the stations on the basis of two or more well-known stations. In reference 3 formulas are derived for the rms errors in location, orientation, and shape in the triangulation solution of an elongated object. By programing the formulas in references 1, 2, and 3 for a high-speed computer and using data for specific object-station relationships, numerical results can be computed. References 4, 5, and 6 give numerical results for the following respective satellite triangulation nets: five specific stations in the United States; nine specific stations covering the Australian continent; and thirty-six specific stations covering the earth.

Numerical studies, however, possess the limitation that they do not provide complete insights into the nature of the dependence of the errors on the experimental parameters involved. Several analytical studies which do provide some of these insights can be found in the literature. In reference 7 analytical expressions for the position error in the solution of a point in space are presented for a system employing angle-only information for two stations and a system employing range-only information for three stations. For this latter system the three-station configuration is separated into three two-station configurations and then the separate results are combined to give the total result. Reference 8 gives analytical expressions for the position error in the range trilateration solution of a satellite for two ideal station configurations. The first configuration has three stations arranged such that the angles 3,1,2 and 1,2,3 are equal. The second configuration has four stations arranged such that the angles 3,1,2; 1,2,3; 2,4,3; and 2,3,4 are all equal. For both configurations the satellite is located at the zenith as referenced to the centroid of the triangle 1,2,3.

The purpose of the present paper is to derive analytical expressions for the rms position error in the triangulation solution of a point in space for several additional ideal observation-station configurations. These expressions are to provide additional insights into the nature of the dependence of the rms position error on certain of the parameters involved.

The first observation-station configuration examined is the configuration of two arbitrarily located stations. Since, in general, the lines of sight from two stations do not intersect, a most probable point in space must be chosen. Two choices for the most probable point – the midpoint of the shortest line between the two lines of sight and the point, on the aforementioned line, which subtends equal residual angles at the two stations – are discussed and compared. The situation in which the data from one of the stations is degraded relative to that from the other is considered. For the first choice of the most probable point, an expression for the optimum relative weighting factor for the data from the first station, which minimizes the rms position error, is derived.

The second observation-station configuration examined is the symmetrical circular configuration of two or more stations with equal elevation angles. The third configuration examined is identical to the second configuration, except that there are more than two stations and one of them is permitted to drift around the circle from its position of symmetry.

For each observation-station configuration considered, an analytical expression for the rms position error in the triangulation solution of a point in space – as a function of the rms line-of-sight errors, the total number of stations of interest, and the elevation angles – is derived to first order. Also for each configuration considered, the optimum elevation angle, which minimizes the rms position error, is found. Considerations are given as to whether the symmetrical arrangement of the circular configuration of more than two stations with equal elevation angles is the arrangement of this configuration which yields the minimum rms position error.

## SYMBOLS

$\left.\begin{matrix}A,B,C,\\ D,E,F\end{matrix}\right\}$ coefficients depending on $n$, $\gamma$, and $\theta$

$A_s$ actual point object in space

$a$ distance from each station in circular configuration to point $A_s$

$a_1,a_2$ distances from stations $S_1$ and $S_2$, respectively, to point $A_s$

$b$ distance between stations in two-station configuration

$d\theta_p,d\theta_q$ line-of-sight errors in plane of $\theta$ from pth and qth stations, respectively

$d\theta_1,d\theta_2$ line-of-sight errors in plane of $\theta_1$ and $\theta_2$ from stations $S_1$ and $S_2$, respectively

$d\phi_p,d\phi_q$ line-of-sight errors out of plane of $\theta$ from pth and qth stations, respectively

$d\phi_1,d\phi_2$ line-of-sight errors out of plane of $\theta_1$ and $\theta_2$ from stations $S_1$ and $S_2$, respectively

$e$ position error function

$\hat{e}_p$ line-of-sight unit vector, in direction of line of sight from pth station

$\left.\begin{matrix}G,H,I,\\ J,K,L\end{matrix}\right\}$ coefficients depending on $A,B,C,D,E,F$, and $\mathscr{A}$

$h$ altitude of point $A_s$

$\hat{i}_p$ actual unit vector, in direction from pth station to point $A_s$

$N$ point defined on figure 1

$n$ total number of stations in circular configuration

| | |
|---|---|
| $P_s$ | most probable point in space |
| $p,q$ | indices for sequential labeling of stations in circular configuration |
| $R$ | radius of circular configuration of stations |
| $\vec{r}_p$ | most probable vector, from pth station to point $P_s$ |
| $S_1,S_2$ | first and second stations, respectively, in configuration of two arbitrarily located stations |
| $\mathscr{S}$ | symmetrical determinant of coefficients $A,B,C,D,E,F$ |
| $X$ | $\sin^2\theta$ |
| $X_1,X_2$ | $\sin\ \theta_1$ and $\sin\theta_2$, respectively |
| $x,y,z$ | rectangular coordinate system with origin at center of circular configuration of stations, x-axis toward original position of 0th station, z-axis toward point $A_s$, and y-axis to form right-hand orthogonal triad |
| $\hat{x},\hat{y},\hat{z}$ | unit vectors in directions of increasing $x$, $y$, and $z$, respectively |
| $\alpha_p,\alpha_q$ | angles which pth and qth stations, respectively, make with x-axis |
| $\gamma$ | variable angle which drifting station makes with x-axis |
| $\vec{\Delta}_p$ | difference vector, between unit vector $\hat{i}_p$ and unit vector $\hat{e}_p$ |
| $\delta_{pq}$ | Kronecker delta |
| $\delta_x,\delta_y,\delta_z$ | x-, y-, and z-components, respectively, of vector $\vec{\delta}_r$ |
| $\vec{\delta}_p$ | line-of-sight error vector, shortest vector from line of sight from pth station to point $P_s$ |
| $\vec{\delta}_r$ | position-error vector, from point $A_s$ to point $P_s$ |
| $\epsilon$ | rms position error in triangulation solution of point in space, in general |

$\epsilon'$ rms position error in triangulation solution of point in space – for two arbitrarily located stations with point $P_s$ taken as point, on shortest line between lines of sight, which subtends equal residual angles at stations

$\zeta,\eta,\xi$ coefficients depending on a, $d\theta_p$, $d\phi_p$, $\alpha_p$, and $\theta$

$\theta$ elevation angle, in general

$\theta_1,\theta_2$ elevation angles from stations $S_1$ and $S_2$, respectively

$\sigma$ rms line-of-sight error, in general

$\sigma_1,\sigma_2$ rms line-of-sight errors from stations $S_1$ and $S_2$, respectively, when data from one station is degraded relative to that from the other

$\omega$ relative weighting factor for data from station $S_1$

A bar over an expression is used to denote the mean value.

## ANALYTICAL FORMULATION

### Position Error in Solution of Point for Two Arbitrarily Located Stations

The two arbitrarily located observation stations are denoted by $S_1$ and $S_2$, respectively, and are shown in figure 1. The two stations are separated by the distance b. The angles from the stations $S_1$ and $S_2$ to the actual point object $A_s$ in space are denoted by $\theta_1$ and $\theta_2$, respectively; these angles will be herein called elevation angles. The distances of the point $A_s$ from the stations $S_1$ and $S_2$ are denoted by $a_1$ and $a_2$, respectively. The distance from the base line to the point $A_s$ is denoted by h; this distance will be herein called the altitude. In practice, errors, herein called line-of-sight errors, will exist in the measured values of the elevation angles $\theta_1$ and $\theta_2$. The line-of-sight errors in the plane of the angles $\theta_1$ and $\theta_2$ are denoted by $d\theta_1$ and $d\theta_2$, respectively; the line-of-sight errors out of the plane of the angles $\theta_1$ and $\theta_2$ are denoted by $d\phi_1$ and $d\phi_2$, respectively.

The position error function e in the triangulation solution of a point in space for two arbitrarily located observation stations is given by

$$de = \frac{\partial e}{\partial \theta_1} d\theta_1 + \frac{\partial e}{\partial \theta_2} d\theta_2 + \frac{\partial e}{\partial \phi_1} d\phi_1 + \frac{\partial e}{\partial \phi_2} d\phi_2$$

The rms position error $\epsilon$ in the solution of a point for two arbitrarily located stations is then given by

$$\epsilon^2 = \overline{\left(\frac{\partial e}{\partial \theta_1} d\theta_1 + \frac{\partial e}{\partial \theta_2} d\theta_2 + \frac{\partial e}{\partial \phi_1} d\phi_1 + \frac{\partial e}{\partial \phi_2} d\phi_2\right)^2} \tag{1}$$

where the bar denotes the mean value. It can be assumed that the line-of-sight errors $d\theta_1$, $d\theta_2$, $d\phi_1$, and $d\phi_2$ are uncorrelated; hence, equation (1) becomes

$$\epsilon^2 = \left(\frac{\partial e}{\partial \theta_1}\right)^2 \overline{d\theta_1{}^2} + \left(\frac{\partial e}{\partial \theta_2}\right)^2 \overline{d\theta_2{}^2} + \left(\frac{\partial e}{\partial \phi_1}\right)^2 \overline{d\phi_1{}^2} + \left(\frac{\partial e}{\partial \phi_2}\right)^2 \overline{d\phi_2{}^2} \tag{2}$$

Since each of the errors $d\theta_1$, $d\theta_2$, $d\phi_1$, and $d\phi_2$ is perpendicular to its line of sight, it can be assumed that

$$\overline{d\theta_1{}^2} = \overline{d\theta_2{}^2} = \overline{d\phi_1{}^2} = \overline{d\phi_2{}^2} = \sigma^2 \tag{3}$$

where $\sigma$ is the rms line-of-sight error. Therefore, equation (2) becomes

$$\epsilon^2 = \sigma^2\left[\left(\frac{\partial e}{\partial \theta_1}\right)^2 + \left(\frac{\partial e}{\partial \theta_2}\right)^2 + \left(\frac{\partial e}{\partial \phi_1}\right)^2 + \left(\frac{\partial e}{\partial \phi_2}\right)^2\right] \tag{4}$$

From figure 1, using the law of sines yields

$$\frac{a_1}{\sin\theta_2} = \frac{b}{\sin(\theta_1 + \theta_2)}$$

Hence, the distance $a_1$ of the actual point $A_s$ in space from the observation station $S_1$ is

$$a_1 = \frac{b \sin\theta_2}{\sin(\theta_1 + \theta_2)} \tag{5}$$

Similarly, the distance $a_2$ of the point $A_s$ from the station $S_2$ is

$$a_2 = \frac{b \sin\theta_1}{\sin(\theta_1 + \theta_2)} \tag{6}$$

The line-of-sight errors $d\theta_1$, $d\theta_2$, $d\phi_1$, and $d\phi_2$ are all independent and, hence, their respective effects on the rms position error in the triangulation solution of a point in space can be examined separately. The position error resulting purely from the error $d\theta_1$ is examined first. From figure 1 it is observed that the error $d\theta_1$ manifests itself in a change in the distance $a_2$, while the elevation angle $\theta_2$ is unaffected. Hence,

$$\frac{\partial e}{\partial \theta_1} = \frac{\partial a_2}{\partial \theta_1} = \frac{b \sin \theta_2}{\sin^2(\theta_1 + \theta_2)} \tag{7}$$

Similarly,

$$\frac{\partial e}{\partial \theta_2} = \frac{\partial a_1}{\partial \theta_2} = \frac{b \sin \theta_1}{\sin^2(\theta_1 + \theta_2)} \tag{8}$$

Next, the position error resulting purely from the line-of-sight error $d\phi_1$ is examined. The error $d\phi_1$ is out of the plane of the elevation angle $\theta_1$. This prevents the lines of sight from the observation stations from intersecting. For this situation a most probable point $P_s$ in space must be chosen. Two choices are treated in the present paper.

Most probable point taken as midpoint of shortest line between lines of sight.- One choice for the most probable point $P_s$ in space is the midpoint of the shortest line between the two lines of sight. This choice is used in reference 9. From figure 1 it is seen that the line segment $A_sP_s$ is

$$A_sP_s = \frac{A_sN}{2}$$

Since $d\phi_1$ is small,

$$A_sP_s = \frac{1}{2} a_1 d\phi_1 \tag{9}$$

Hence,

$$\frac{\partial e}{\partial \phi_1} = \frac{1}{2} a_1$$

Using equation (5) yields

$$\frac{\partial e}{\partial \phi_1} = \frac{b \sin \theta_2}{2 \sin(\theta_1 + \theta_2)} \tag{10}$$

Similarly,

$$\frac{\partial e}{\partial \phi_2} = \frac{b \sin \theta_1}{2 \sin(\theta_1 + \theta_2)} \tag{11}$$

Substituting equations (7), (8), (10), and (11) into equation (4) produces

$$\epsilon^2 = \sigma^2 b^2 \frac{\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left[1 + \frac{1}{4}\sin^2(\theta_1 + \theta_2)\right]}{\sin^4(\theta_1 + \theta_2)} \tag{12}$$

Figure 1 shows that the distance $b$ between the two observation stations is

$$b = a_1 \cos \theta_1 + a_2 \cos \theta_2 \tag{13}$$

where the distances $a_1$ and $a_2$ of the actual point $A_s$ in space from the stations $S_1$ and $S_2$, respectively, are

$$a_1 = \frac{h}{\sin \theta_1} \tag{14}$$

$$a_2 = \frac{h}{\sin \theta_2} \tag{15}$$

where $h$ is the altitude of the point $A_s$. Substituting equations (14) and (15) into equation (13) leads to

$$b = \frac{h \sin(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2} \tag{16}$$

Substituting equation (16) into equation (12) results in

$$\epsilon^2 = \sigma^2 h^2 \frac{\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left[1 + \frac{1}{4}\sin^2(\theta_1 + \theta_2)\right]}{\sin^2\theta_1 \sin^2\theta_2 \sin^2(\theta_1 + \theta_2)} \tag{17}$$

Therefore, the rms position error $\epsilon$ in the triangulation solution of a point in space for two arbitrarily located observation stations, with the most probable point in space taken as the midpoint of the shortest line between the two lines of sight, is

$$\epsilon = h\sigma\left\{\frac{\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left[1 + \frac{1}{4}\sin^2\left(\theta_1 + \theta_2\right)\right]}{\sin^2\theta_1\,\sin^2\theta_2\,\sin^2\left(\theta_1 + \theta_2\right)}\right\}^{1/2} \tag{18}$$

It is noted that the rms position error is directly proportional to the rms line-of-sight error. It is also noted that the rms position error is infinite for $\theta_1 + \theta_2 = 180^\circ$, which occurs when the two lines of sight are parallel.

Table I is a table of values of $\epsilon/h\sigma$ as a function of the elevation angles $\theta_1$ and $\theta_2$. For convenience, the only values of $\epsilon/h\sigma$ listed are those for the angles $\theta_1$ and $\theta_2$ at intervals of $10^\circ$. From table I it is seen that the minimum value of $\epsilon/h\sigma$ occurs for $\theta_1$ equal to $\theta_2$ in the neighborhood of $55^\circ$.

Then, the quantity $\epsilon/h\sigma$ was evaluated for values of $\theta_1$ and $\theta_2$ from $45^\circ$ to $65^\circ$ at increments of $0.001^\circ$. Table II is a table of values of $\epsilon/h\sigma$ as a function of $\theta_1$ and $\theta_2$ at increments of $0.001^\circ$. For convenience, the only values of $\epsilon/h\sigma$ listed are those for $\theta_1$ and $\theta_2$ from $56.096^\circ$ to $56.100^\circ$. From table II it is observed that the minimum value of $\epsilon/h\sigma$ occurs for $\theta_1 = \theta_2 = 56.098^\circ$. Therefore, the minimum rms position error in the triangulation solution of a point in space for two arbitrarily located observation stations, with the most probable point in space taken as the midpoint of the shortest line between the two lines of sight, occurs when the two elevation angles are such that $\theta_1 = \theta_2 = 56.098^\circ$.

Setting $\theta_1 = \theta_2 = \theta$ in equation (18) establishes the following:

$$\epsilon = \frac{\sigma h}{\sqrt{2}}\left(\frac{1 + \sin^2\theta - \sin^4\theta}{\sin^4\theta - \sin^6\theta}\right)^{1/2} \tag{19}$$

Therefore, equation (19) is the equation for the rms position error in the triangulation solution of a point in space for two arbitrarily located observation stations, with the most probable point in space taken as the midpoint of the shortest line between the two lines of sight, when the two elevation angles are set equal.

<u>Most probable point taken as point, on shortest line between lines of sight, which subtends equal residual angles at stations</u>.- A second choice for the most probable point $P_s$ in space is the point, on the shortest line between the two lines of sight, which subtends equal residual angles at the two observation stations. This choice is used in reference 8. From figure 1 it is seen that

$$\frac{A_sP_s}{a_2} = \frac{A_sN - A_sP_s}{a_1}$$

Or,

$$A_sP_s = \frac{a_2}{(a_1 + a_2)} A_sN$$

Since $d\phi_1$ is small,

$$A_sP_s = \frac{a_1 a_2 d\phi_1}{(a_1 + a_2)} \tag{20}$$

Hence,

$$\frac{\partial e}{\partial \phi_1} = \frac{a_1 a_2}{(a_1 + a_2)}$$

Using equations (5) and (6) results in

$$\frac{\partial e}{\partial \phi_1} = \frac{b \sin \theta_1 \sin \theta_2}{(\sin \theta_1 + \sin \theta_2)\sin(\theta_1 + \theta_2)} \tag{21}$$

Because of symmetry

$$\frac{\partial e}{\partial \phi_2} = \frac{b \sin \theta_1 \sin \theta_2}{(\sin \theta_1 + \sin \theta_2)\sin(\theta_1 + \theta_2)} \tag{22}$$

Changing the notation from $\epsilon$ to $\epsilon'$ in equation (4) and then substituting equations (7), (8), (21), and (22) into the equation for $(\epsilon')^2$ result in

$$(\epsilon')^2 = \sigma^2 b^2 \frac{\left[\sin^2\theta_1 + \sin^2\theta_2 + \dfrac{2 \sin^2\theta_1 \sin^2\theta_2 \sin^2(\theta_1 + \theta_2)}{(\sin \theta_1 + \sin \theta_2)^2}\right]}{\sin^4(\theta_1 + \theta_2)} \tag{23}$$

Therefore, equation (23) is the equation which gives the rms position error in the triangulation solution of a point in space for two arbitrarily located observation stations – with the most probable point in space taken as the point, on the shortest line between the two lines of sight, which subtends equal residual angles at the stations.

Comparison of position errors for two choices of most probable point.- Taking the ratio of equation (23) to equation (12) renders

$$\frac{(\epsilon')^2}{\epsilon^2} = 1 - \frac{\sin^2(\theta_1 + \theta_2)\left[\frac{1}{4}\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left(\sin\theta_1 + \sin\theta_2\right)^2 - 2\sin^2\theta_1 \sin^2\theta_2\right]}{\left(\sin\theta_1 + \sin\theta_2\right)^2\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left[1 + \frac{1}{4}\sin^2(\theta_1 + \theta_2)\right]} \tag{24}$$

Hence,

$$\frac{\epsilon'}{\epsilon} \leqq 1 \quad \text{for} \quad \frac{1}{4}\left(\sin^2\theta_1 + \sin^2\theta_2\right)\left(\sin\theta_1 + \sin\theta_2\right)^2 - 2\sin^2\theta_1 \sin^2\theta_2 \geqq 0 \tag{25}$$

For convenience, the following change of variables can be made: $X_1 = \sin\theta_1$, $X_2 = \sin\theta_2$. Since the elevation angles $\theta_1$ and $\theta_2$ are positive, then the variables $X_1$ and $X_2$ are also positive. Making this change of variables in expression (25) furnishes

$$\frac{\epsilon'}{\epsilon} \leqq 1 \quad \text{for} \quad \frac{1}{4}\left(X_1^2 + X_2^2\right)\left(X_1 + X_2\right)^2 - 2X_1^2X_2^2 \geqq 0$$

If $X_1 = X_2$, then $\left(X_1^2 + X_2^2\right) = 2X_1X_2$ and the following result can be established:

$$\frac{\epsilon'}{\epsilon} = 1, \text{ since } \frac{1}{4}\left(X_1^2 + X_2^2\right)\left(X_1 + X_2\right)^2 - 2X_1^2X_2^2 = 0$$

If $X_1 \neq X_2$, then $\left(X_1^2 + X_2^2\right) > 2X_1X_2$ and the following result can be proved:

$$\frac{\epsilon'}{\epsilon} < 1, \text{ since } \frac{1}{4}\left(X_1^2 + X_2^2\right)\left(X_1 + X_2\right)^2 - 2X_1^2X_2^2 > 0$$

Consequently,

$$\epsilon' = \epsilon \qquad \left(\theta_1 = \theta_2\right) \tag{26}$$

$$\epsilon' < \epsilon \qquad \left(\theta_1 \neq \theta_2\right) \tag{27}$$

Hence, when the two elevation angles are equal, the rms position error $\epsilon'$ is equal to the rms position error $\epsilon$. However, when the two elevation angles are not equal, then $\epsilon'$ is less than $\epsilon$. Therefore, the choice of the most probable point in space as the point, on the shortest line between the two lines of sight, which subtends equal residual angles at the two observation stations, yields a smaller value of the rms position error than does the choice of the most probable point as the midpoint of the aforementioned line, except in the special situation of equal elevation angles and then the two choices yield the same value.

Optimum relative weighting factor for data from first of two arbitrarily located stations.- Earlier in this analysis for two arbitrarily located observation stations, the mean-square line-of-sight errors $\overline{d\theta_1^2}$, $\overline{d\theta_2^2}$, $\overline{d\phi_1^2}$, and $\overline{d\phi_2^2}$ were assumed to be all equal. The situation in which the mean-square line-of-sight errors from the station $S_1$ differ from the mean-square line-of-sight errors from the station $S_2$ are now considered. Hence,

$$\overline{d\theta_1^2} = \overline{d\phi_1^2} = \sigma_1^2 \tag{28}$$

$$\overline{d\theta_2^2} = \overline{d\phi_2^2} = \sigma_2^2 \tag{29}$$

where $\sigma_1$ and $\sigma_2$ are the rms line-of-sight errors from the stations $S_1$ and $S_2$, respectively. Substituting equations (28) and (29) where appropriate into equation (2) gives

$$\epsilon^2 = \sigma_1^2\left[\left(\frac{\partial e}{\partial \theta_1}\right)^2 + \left(\frac{\partial e}{\partial \phi_1}\right)^2\right] + \sigma_2^2\left[\left(\frac{\partial e}{\partial \theta_2}\right)^2 + \left(\frac{\partial e}{\partial \phi_2}\right)^2\right] \tag{30}$$

Since the rms line-of-sight errors from the two observation stations are different, the data from the two stations must be weighted differently. The relative weighting factor for the data from the station $S_1$ can be denoted by $\omega$. The purpose of this section is to determine the optimum value of the relative weighting factor $\omega$, which minimizes the rms position error in the triangulation solution of a point in space.

For a line-of-sight error $d\phi_1$ at the observation station $S_1$, shown in figure 1, the relative weighting factor $\omega$ is

$$\omega = \frac{A_S P_S}{A_S N - A_S P_S}$$

Or,

$$A_S P_S = \left(\frac{\omega}{1 + \omega}\right) A_S N$$

Since $d\phi_1$ is small,

$$A_sP_s = \left(\frac{\omega}{1 + \omega}\right)a_1 d\phi_1 \tag{31}$$

Hence,

$$\frac{\partial e}{\partial \phi_1} = \left(\frac{\omega}{1 + \omega}\right)a_1$$

Using equation (5) renders

$$\frac{\partial e}{\partial \phi_1} = \left(\frac{\omega}{1 + \omega}\right)\frac{b \sin \theta_2}{\sin(\theta_1 + \theta_2)} \tag{32}$$

For an error $d\phi_2$ at the station $S_2$, the relative weighting factor $\omega$ is

$$\omega = \frac{A_sN - A_sP_s}{A_sP_s}$$

Or,

$$A_sP_s = \left(\frac{1}{1 + \omega}\right)A_sN$$

Since $d\phi_2$ is small,

$$A_sP_s = \left(\frac{1}{1 + \omega}\right)a_2 d\phi_2 \tag{33}$$

Hence,

$$\frac{\partial e}{\partial \phi_2} = \left(\frac{1}{1 + \omega}\right)a_2$$

Using equation (6) gives

$$\frac{\partial e}{\partial \phi_2} = \left(\frac{1}{1 + \omega}\right)\frac{b \sin \theta_1}{\sin(\theta_1 + \theta_2)} \tag{34}$$

Substituting equations (7), (8), (32), and (34) into equation (30) furnishes

$$\epsilon^2 = b^2\left[\frac{\left(\sigma_1{}^2 \sin^2\theta_2 + \sigma_2{}^2 \sin^2\theta_1\right)}{\sin^4(\theta_1 + \theta_2)} + \frac{\left(\omega^2\sigma_1{}^2 \sin^2\theta_2 + \sigma_2{}^2 \sin^2\theta_1\right)}{(1 + \omega)^2 \sin^2(\theta_1 + \theta_2)}\right] \tag{35}$$

In order to find the optimum relative weighting factor, the partial derivative of equation (35) with respect to $\omega$ is taken and then this resulting expression is set equal to zero. That is,

$$\frac{\partial \epsilon^2}{\partial \omega} = b^2 \left[ \frac{(1 + \omega)^2 \sin^2(\theta_1 + \theta_2)\left(2\omega\sigma_1{}^2 \sin^2\theta_2\right)}{(1 + \omega)^4 \sin^4(\theta_1 + \theta_2)} - \frac{\left(\omega^2\sigma_1{}^2 \sin^2\theta_2 + \sigma_2{}^2 \sin^2\theta_1\right)(2 + 2\omega)\sin^2(\theta_1 + \theta_2)}{(1 + \omega)^4 \sin^4(\theta_1 + \theta_2)} \right] = 0 \tag{36}$$

Hence,

$$\omega(1 + \omega)\sigma_1{}^2 \sin^2\theta_2 = \omega^2\sigma_1{}^2 \sin^2\theta_2 + \sigma_2{}^2 \sin^2\theta_1$$

Consequently,

$$\omega = \frac{\sigma_2{}^2 \sin^2\theta_1}{\sigma_1{}^2 \sin^2\theta_2} \tag{37}$$

Therefore, for the choice of the most probable point in space as the midpoint of the shortest line between the two lines of sight, equation (37) is the equation for the optimum relative weighting factor, which minimizes the rms position error in the triangulation solution of a point in space, for the data from the observation station $S_1$ when the data from one of the stations is degraded relative to that from the other. It is noted from equation (37) that the optimum relative weighting factor for the data from the station $S_1$ is directly proportional to the square of the ratio of the rms line-of-sight errors from the stations $S_2$ and $S_1$.

### Position Error in Solution of Point for Symmetrical Circular Configuration of $n \geqq 2$ Stations With Equal Elevation Angles

The configuration of $n$, where $n \geqq 2$, observation stations is now examined. The $n$ stations are equally spaced around the circumference of a circle, the center of which is located at the foot of the perpendicular drawn from the actual point $A_s$ in space, as shown in figure 2. For this configuration an approach different from that of the preceding section is used.

The actual unit vector $\hat{i}_p$ is in the direction from the pth observation station to the actual point $A_s$ in space. The distance between each of the n stations and the point $A_s$ is denoted by a. The most probable vector $\vec{r}_p$ is the vector from the pth station to the most probable point $P_s$ in space. The point $P_s$ is displaced from the point $A_s$ by the position error vector $\vec{\delta}_r$. The line of sight from the pth station is in the direction of the line-of-sight unit vector $\hat{e}_p$. The unit vector $\hat{e}_p$ is displaced from the unit vector $\hat{i}_p$ by the difference vector $\vec{\Delta}_p$. Lastly, the line-of-sight error vector, the shortest vector from the line of sight from the pth station to the point $P_s$, is denoted by $\vec{\delta}_p$.

From figure 2 it is observed that the line-of-sight error vector $\vec{\delta}_p$, the most probable vector $\vec{r}_p$, and the line-of-sight unit vector $\hat{e}_p$, respectively, from the pth observation station are

$$\vec{\delta}_p = \vec{r}_p - \left(\vec{r}_p \cdot \hat{e}_p\right)\hat{e}_p \tag{38}$$

$$\vec{r}_p = a\hat{i}_p + \vec{\delta}_r \tag{39}$$

$$\hat{e}_p = \hat{i}_p + \vec{\Delta}_p \tag{40}$$

Substituting equations (39) and (40) into equation (38) renders

$$\vec{\delta}_p = a\hat{i}_p + \vec{\delta}_r - \left(a\hat{i}_p + \vec{\delta}_r\right) \cdot \left(\hat{i}_p + \vec{\Delta}_p\right)\left(\hat{i}_p + \vec{\Delta}_p\right)$$

Keeping only terms to first order and noting that $\hat{i}_p \cdot \vec{\Delta}_p = 0$ to first order produce

$$\vec{\delta}_p = \vec{\delta}_r - \left(\hat{i}_p \cdot \vec{\delta}_r\right)\hat{i}_p - a\vec{\Delta}_p \tag{41}$$

Hence, again noticing that $\hat{i}_p \cdot \vec{\Delta}_p = 0$ to first order leads to

$$\left(\vec{\delta}_p\right)^2 = \left(\vec{\delta}_r\right)^2 - \left(\hat{i}_p \cdot \vec{\delta}_r\right)^2 + a^2\left(\vec{\Delta}_p\right)^2 - 2a\left(\vec{\delta}_r \cdot \vec{\Delta}_p\right) \tag{42}$$

Therefore, for the n stations equation (42) becomes

$$\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2 = \sum_{p=0}^{n-1}\left(\vec{\delta}_r\right)^2 - \sum_{p=0}^{n-1}\left(\hat{i}_p \cdot \vec{\delta}_r\right)^2 + a^2\sum_{p=0}^{n-1}\left(\vec{\Delta}_p\right)^2 - 2a\sum_{p=0}^{n-1}\left(\vec{\delta}_r \cdot \vec{\Delta}_p\right) \tag{43}$$

where $n \geqq 2$.

The elevation angle from each of the $n$ observation stations is the same, since the $n$ stations are located on the circumference of a circle with the actual point $A_s$ in space being located on the perpendicular whose foot is at the center of the circle. The elevation angle is denoted by $\theta$ and is shown in figure 2. The angles $d\theta_p$ and $d\phi_p$ are the line-of-sight errors in and out of the plane of the angle $\theta$, respectively, from the pth station. The angle $\alpha_p$ is the angle between the x-axis and the pth station, and $\alpha_p = 2\pi p/n$, where $p = 0, 1, 2, \ldots, n - 1$, as the $n$ stations are equally spaced around the circumference of the circle. The rectangular coordinate system of figure 2 has its origin at the center of the circular configuration of stations, x-axis toward the 0th station, z-axis toward the point $A_s$, and y-axis such as to form a right-hand orthogonal triad. In order to determine the rms position error in the triangulation solution of a point in space as a function of the angle $\theta$ for this configuration, equation (43) must first be written as a function of the angle $\theta$, the errors $d\theta_p$ and $d\phi_p$, the angle $\alpha_p$, and the x-, y-, and z-components of the position error vector $\vec{\delta}_p$.

The actual unit vector $\hat{i}_p$ from the pth observation station corresponds to the situation in which no line-of-sight errors exist in the line of sight from the pth station $\left(\text{i.e., when } d\theta_p = d\phi_p = 0\right)$. The unit vector $\hat{i}_p$ from the pth station is

$$\hat{i}_p = -\hat{x} \cos \theta \cos \alpha_p - \hat{y} \cos \theta \sin \alpha_p + \hat{z} \sin \theta \tag{44}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors in the directions of increasing x, y, and z, respectively. From figure 2 it is observed that the difference vector $\vec{\Delta}_p$ from the pth station is

$$\vec{\Delta}_p = \hat{e}_p - \hat{i}_p \tag{45}$$

where $\hat{e}_p$ is the line-of-sight unit vector from the pth station. Hence,

$$\vec{\Delta}_p = \hat{x}\left(d\theta_p \sin \theta \cos \alpha_p - d\phi_p \sin \alpha_p\right) + \hat{y}\left(d\theta_p \sin \theta \sin \alpha_p + d\phi_p \cos \alpha_p\right) + \hat{z}\, d\theta_p \cos \theta \tag{46}$$

The position error vector $\vec{\delta}_r$ can be expressed as

$$\vec{\delta}_r = \hat{x}\delta_x + \hat{y}\delta_y + \hat{z}\delta_z \tag{47}$$

where $\delta_x$, $\delta_y$, and $\delta_z$ are its x-, y-, and z-components, respectively.

Using equations (44), (46), and (47) establishes the desired functional dependence of each term in equation (43). Hence,

$$\sum_{p=0}^{n-1} \left(\vec{\delta}_r\right)^2 = \sum_{p=0}^{n-1} \left(\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2\right) = n\left(\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2\right) \tag{48}$$

$$\begin{aligned}
\sum_{p=0}^{n-1} \left(\hat{i}_p \cdot \vec{\delta}_r\right)^2 &= \sum_{p=0}^{n-1} \left(-\delta_x \cos\theta \cos\alpha_p - \delta_y \cos\theta \sin\alpha_p + \delta_z \sin\theta\right)^2 \\
&= \delta_x{}^2 \cos^2\theta \sum_{p=0}^{n-1} \cos^2\alpha_p + \delta_y{}^2 \cos^2\theta \sum_{p=0}^{n-1} \sin^2\alpha_p + n\delta_z{}^2 \sin^2\theta \\
&\quad - 2\delta_y\delta_z \cos\theta \sin\theta \sum_{p=0}^{n-1} \sin\alpha_p - 2\delta_x\delta_z \cos\theta \sin\theta \sum_{p=0}^{n-1} \cos\alpha_p \\
&\quad + \delta_x\delta_y \cos^2\theta \sum_{p=0}^{n-1} \sin\left(2\alpha_p\right)
\end{aligned} \tag{49}$$

$$\begin{aligned}
a^2 \sum_{p=0}^{n-1} \left(\vec{\Delta}_p\right)^2 &= a^2 \sum_{p=0}^{n-1} \Big[\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)^2 \\
&\quad + \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right)^2 + \left(d\theta_p \cos\theta\right)^2\Big]
\end{aligned} \tag{50}$$

$$\begin{aligned}
2a \sum_{p=0}^{n-1} \left(\vec{\delta}_r \cdot \vec{\Delta}_p\right) &= 2a\delta_x \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) \\
&\quad + 2a\delta_y \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right) \\
&\quad + 2a\delta_z \cos\theta \sum_{p=0}^{n-1} d\theta_p
\end{aligned} \tag{51}$$

Substituting equations (A1), (A2), (A3), (A4), and (A5), which are derived in the appendix, into equation (49) produces

$$\sum_{p=0}^{n-1}\left(\hat{i}_p \cdot \vec{\delta}_r\right)^2 = \left\{\begin{array}{ll} \frac{n}{2}\,\delta_x{}^2 \cos^2\theta + \frac{n}{2}\,\delta_y{}^2 \cos^2\theta + n\delta_z{}^2 \sin^2\theta & (n > 2) \\ 2\delta_x{}^2 \cos^2\theta + 2\delta_z{}^2 \sin^2\theta & (n = 2) \end{array}\right\} \tag{52}$$

The substitution of equation (52) into equation (43) yields two different results, one for $n > 2$ and one for $n = 2$. Hence, these two situations must be examined separately.

Symmetrical circular configuration of $n > 2$ stations with equal elevation angles.- Substituting equation (52) for $n > 2$ along with equations (48), (50), and (51) into equation (43) leads to an expression for $\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2$ for $n > 2$ as a function of the elevation angle $\theta$, the line-of-sight errors $d\theta_p$ and $d\phi_p$, the angle $\alpha_p$, and the x-, y-, and z-components of the position error vector $\vec{\delta}_r$. That is,

$$\begin{aligned}
\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2 = {} & n\left(\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2\right) \\
& - \left(\frac{n}{2}\,\delta_x{}^2 \cos^2\theta + \frac{n}{2}\,\delta_y{}^2 \cos^2\theta + n\delta_z{}^2 \sin^2\theta\right) \\
& + a^2 \sum_{p=0}^{n-1}\left[\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)^2\right. \\
& \left. + \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right)^2 + \left(d\theta_p \cos\theta\right)^2\right] \\
& - 2a\delta_x \sum_{p=0}^{n-1}\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) \\
& - 2a\delta_y \sum_{p=0}^{n-1}\left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right) \\
& - 2a\delta_z \cos\theta \sum_{p=0}^{n-1} d\theta_p \qquad (n > 2)
\end{aligned} \tag{53}$$

In this form, $\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2$ can be minimized with respect to the x-, y-, and z-components of the vector $\vec{\delta}_r$.

To obtain the minimum with respect to $\delta_x$, the partial derivative of equation (53) with respect to $\delta_x$ is taken and then this resulting expression is set equal to zero. That is,

$$\frac{\partial}{\partial \delta_x}\left[\sum_{p=0}^{n-1} \left(\overline{\delta}_p\right)^2\right] = 2n\delta_x - n\delta_x \cos^2\theta - 2a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) = 0$$

Or,

$$\left(1 + \sin^2\theta\right) n\delta_x = 2a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) \tag{54}$$

Thus,

$$\left(1 + \sin^2\theta\right)^2 n^2 \overline{\delta_x{}^2} = 4a^2 \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \overline{\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)\left(d\theta_q \sin\theta \cos\alpha_q - d\phi_q \sin\alpha_q\right)} \tag{55}$$

where the bars denote the mean values.

It can be assumed that all of the line-of-sight errors $d\theta_p$ and $d\phi_p$ are uncorrelated. Also, since the errors $d\theta_p$ and $d\phi_p$ are all perpendicular to their respective lines of sight, it can be assumed that

$$\overline{d\theta_p d\theta_q} = \sigma^2 \delta_{pq} \tag{56}$$

$$\overline{d\phi_p d\phi_q} = \sigma^2 \delta_{pq} \tag{57}$$

$$\overline{d\theta_p d\phi_q} = \overline{d\theta_q d\phi_p} = 0 \tag{58}$$

where $\delta_{pq}$ is the Kronecker delta and $\sigma$ is the rms line-of-sight error.

Substituting equations (56) to (58) where appropriate into equation (55) results in

$$\left(1 + \sin^2\theta\right)^2 n^2 \overline{\delta_x{}^2} = 4a^2\sigma^2 \left(\sin^2\theta \sum_{p=0}^{n-1} \cos^2\alpha_p + \sum_{p=0}^{n-1} \sin^2\alpha_p\right) \tag{59}$$

Substituting equations (A4) and (A5) for $n > 2$ into equation (59) gives

$$\left(1 + \sin^2\theta\right)^2 n^2 \overline{\delta_x{}^2} = 4a^2\sigma^2 \left(\frac{n}{2} \sin^2\theta + \frac{n}{2}\right)$$

Therefore,

$$\overline{\delta_x{}^2} = \frac{2a^2\sigma^2}{n\left(1 + \sin^2\theta\right)} \tag{60}$$

Minimizing equation (53) with respect to $\delta_y$ and $\delta_z$, respectively, results in the following two equations:

$$\left(1 + \sin^2\theta\right)n\delta_y = 2a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right) \tag{61}$$

$$n\delta_z \cos\theta = a \sum_{p=0}^{n-1} d\theta_p \tag{62}$$

Using equations (61) and (62), respectively, and procedures analogous to those used for obtaining equation (60) for $\overline{\delta_x{}^2}$ yields the following equations for $\overline{\delta_y{}^2}$ and $\overline{\delta_z{}^2}$:

$$\overline{\delta_y{}^2} = \frac{2a^2\sigma^2}{n\left(1 + \sin^2\theta\right)} \tag{63}$$

$$\overline{\delta_z{}^2} = \frac{a^2\sigma^2}{n \cos^2\theta} \tag{64}$$

The rms position error $\epsilon$ in the triangulation solution of a point in space for the circular configuration of $n$ observation stations with equal elevation angles is given by

$$\epsilon^2 = \overline{\delta_x{}^2} + \overline{\delta_y{}^2} + \overline{\delta_z{}^2} \tag{65}$$

Substituting equations (60), (63), and (64) into equation (65) furnishes

$$\epsilon^2 = \frac{a^2\sigma^2}{n}\left(\frac{5 - 3\sin^2\theta}{1 - \sin^4\theta}\right) \tag{66}$$

From figure 2 it is seen that the distance $a$ from each of the $n$ stations to the actual point $A_s$ in space is

$$a = \frac{h}{\sin\theta} \tag{67}$$

Substituting equation (67) into equation (66) renders

$$\epsilon^2 = \frac{h^2\sigma^2}{n}\left(\frac{5 - 3\sin^2\theta}{\sin^2\theta - \sin^6\theta}\right) \tag{68}$$

Therefore, the rms position error $\epsilon$ in the triangulation solution of a point in space for the symmetrical circular configuration of $n > 2$ observation stations with equal elevation angles is

$$\epsilon = \frac{h\sigma}{\sqrt{n}}\left(\frac{5 - 3\sin^2\theta}{\sin^2\theta - \sin^6\theta}\right)^{1/2} \qquad (n > 2) \tag{69}$$

The rms position error from equation (69) is directly proportional to the rms line-of-sight error and inversely proportional to the square root of the total number of stations of interest. For elevation angles of $90^{\circ}$ the rms position error is infinite. For an elevation angle of $90^{\circ}$ the lines of sight from the different stations could not intersect. Figure 3 is a plot of $\epsilon/h\sigma$ for $n > 2$ as a function of the elevation angle $\theta$, for $n = 3, 4, 5, 6, 7,$ and 8.

The optimum elevation angle, which minimizes the rms position error in the triangulation solution of a point in space, for the symmetrical circular configuration of $n > 2$ observation stations with equal elevation angles can be determined by minimizing equation (68) with respect to the elevation angle $\theta$. That is,

$$\frac{\partial\epsilon^2}{\partial\theta} = \frac{2h^2\sigma^2 \sin\theta\cos\theta}{n}\left[\frac{-6\sin^6\theta + 15\sin^4\theta - 5}{\left(\sin^2\theta - \sin^6\theta\right)^2}\right] = 0 \tag{70}$$

Hence,

$$-6\sin^6\theta + 15\sin^4\theta - 5 = 0 \tag{71}$$

Making the change of variables $X = \sin^2\theta$ in equation (71) gives

$$X^3 - \frac{5}{2}X^2 + \frac{5}{6} = 0 \tag{72}$$

The three roots of this equation are

$$X = 0.675905;\ -0.524875;\ 2.348970$$

Since $X = \sin^2\theta$, the root corresponding to the physical solution must be positive and must be between zero and unity. Hence, the root corresponding to the physical solution is $X = 0.675905$. Therefore,

$$\theta = \sin^{-1}X^{1/2} = 55.300^{\circ} \qquad (n > 2) \tag{73}$$

Thus, the optimum elevation angle, which minimizes the rms position error in the solution of a point, for the symmetrical circular configuration of $n > 2$ stations with equal elevation angles is $55.300^{\circ}$. This result is also seen from figure 3. In addition, for $\theta = 55.300^{\circ} \pm 5^{\circ}$ (see fig. 3), the difference in the rms position error for $n = 3$ is 2.5 percent, while for $n = 8$ it is 0.9 percent.

The optimum elevation angle ($55.300^{\circ}$) is within a fraction of a degree of the angle ($54.667^{\circ}$) which the sloping edge of a tetrahedron makes with its base. Therefore, the optimum configuration of three observation stations for triangulating on a point in space is the configuration of the three stations plus the point which closely approximates a tetrahedron.

Given the altitude of the point in space, the value of the optimum elevation angle for $n > 2$ can be used to compute the value of the optimum radius of the symmetrical circular configuration of $n > 2$ observation stations with equal elevation angles. The radius $R$ of the circular configuration (see fig. 2) is

$$R = h \cot \theta \tag{74}$$

Hence, substituting $\theta = 55.300^{\circ}$ into equation (74) produces

$$R = (0.69243)h \qquad (n > 2) \tag{75}$$

Therefore, equation (75) furnishes the value of the optimum radius of the symmetrical circular configuration of $n > 2$ stations with equal elevation angles, for a given altitude of the point.

Symmetrical circular configuration of $n = 2$ stations with equal elevation angles.- Noticing that $\alpha_p = \pi p$ for $n = 2$ and substituting equations (48), (50), (51), and (52) for $n = 2$ into equation (43) establish an expression for $\sum_{p=0}^{1} \left(\vec{\delta}_p\right)^2$ as a function of the elevation angle $\theta$, the line-of-sight errors $d\theta_p$ and $d\phi_p$, the angle $\alpha_p = \pi p$, and the x-, y-, and z-components of the position error vector $\delta_r$. That is,

$$
\begin{aligned}
\sum_{p=0}^{1} \left(\vec{\delta}_p\right)^2 = {} & 2\left(\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2\right) - \left(2\delta_x{}^2 \cos^2\theta + 2\delta_z{}^2 \sin^2\theta\right) \\
& + a^2 \sum_{p=0}^{1} \left[\left(d\theta_p \sin\theta \cos(\pi p) - d\phi_p \sin(\pi p)\right)^2\right. \\
& \left. + \left(d\theta_p \sin\theta \sin(\pi p) + d\phi_p \cos(\pi p)\right)^2 + \left(d\theta_p \cos\theta\right)^2\right] \\
& - 2a\delta_x \sum_{p=0}^{1} \left[d\theta_p \sin\theta \cos(\pi p) - d\phi_p \sin(\pi p)\right] \\
& - 2a\delta_y \sum_{p=0}^{1} \left[d\theta_p \sin\theta \sin(\pi p) + d\phi_p \cos(\pi p)\right] \\
& - 2a\delta_z \cos\theta \sum_{p=0}^{1} d\theta_p \qquad (n = 2) \qquad (76)
\end{aligned}
$$

Using procedures analogous to those used in arriving at equation (68) from equation (53) yields

$$
\epsilon^2 = \frac{h^2\sigma^2}{2}\left(\frac{1 + \sin^2\theta - \sin^4\theta}{\sin^4\theta - \sin^6\theta}\right) \qquad (77)
$$

Therefore, the rms position error $\epsilon$ in the triangulation solution of a point in space for the symmetrical circular configuration of $n = 2$ observation stations with equal elevation angles is

$$
\epsilon = \frac{h\sigma}{\sqrt{2}}\left(\frac{1 + \sin^2\theta - \sin^4\theta}{\sin^4\theta - \sin^6\theta}\right)^{1/2} \qquad (n = 2) \qquad (78)
$$

This equation is identical to equation (19), which gives the rms position error in the solution of a point for two arbitrarily located stations, with the most probable point taken as the midpoint of the shortest line between the two lines of sight, when the two elevation angles are set equal.

From equation (78) it is noticed that the rms position error in the triangulation solution of a point in space for the symmetrical circular configuration of $n = 2$, as for $n > 2$, observation stations with equal elevation angles is directly proportional to the rms line-of-sight error and infinite for elevation angles of $90^o$. Figure 4 is a plot of $\epsilon/h\sigma$ for $n = 2$ as a function of the elevation angle $\theta$.

The optimum elevation angle, which minimizes the rms position error in the triangulation solution of a point in space, for the symmetrical circular configuration of $n = 2$ observation stations with equal elevation angles can be determined by minimizing equation (77) with respect to the elevation angle $\theta$. That is,

$$\frac{\partial \epsilon^2}{\partial \theta} = h^2\sigma^2 \sin^3\theta \cos\theta \frac{\left[\left(\sin^2\theta - \sin^4\theta\right)\left(1 - 2\sin^2\theta\right) - \left(1 + \sin^2\theta - \sin^4\theta\right)\left(2 - 3\sin^2\theta\right)\right]}{\left(\sin^4\theta - \sin^6\theta\right)^2} = 0 \tag{79}$$

Hence,

$$\left(\sin^2\theta - \sin^4\theta\right)\left(1 - 2\sin^2\theta\right) - \left(1 + \sin^2\theta - \sin^4\theta\right)\left(2 - 3\sin^2\theta\right) = 0 \tag{80}$$

Making the change of variables $X = \sin^2\theta$ in equation (80) renders

$$X^3 - 2X^2 - 2X + 2 = 0 \tag{81}$$

The root corresponding to the physical solution is $X = 0.688892$. Therefore,

$$\theta = \sin^{-1}X^{1/2} = 56.098^o \qquad (n = 2) \tag{82}$$

Thus, the optimum elevation angle, which minimizes the rms position error in the solution of a point, for the symmetrical circular configuration of $n = 2$ stations with equal elevation angles is $56.098^o$. This result is also seen from figure 4. In addition, for $\theta = 56.098^o \pm 5^o$ (see fig. 4), the difference in the rms position error for $n = 2$ is 1.5 percent.

It is recalled that the minimum rms position error in the triangulation solution of a point in space for two arbitrarily located observation stations, with the most probable point taken as the midpoint of the shortest line between the two lines of sight, occurs when the two elevation angles $\theta_1$ and $\theta_2$ are such that $\theta_1 = \theta_2 = 56.098^o$. Hence, a necessary and sufficient condition for a minimum rms position error in the solution of a point for two arbitrarily located stations is that the two elevation angles be equal, and equal to $56.098^o$.

Substituting $\theta = 56.098^o$ into equation (74) renders

$$R = (0.67200)h \qquad (n = 2) \tag{83}$$

Therefore, equation (83) furnishes the value of the optimum radius of the symmetrical circular configuration of $n = 2$ observation stations with equal elevation angles, given the altitude of the point in space. Since the distance $b$ between the two stations is twice the radius $R$ of the circular configuration, then

$$b = 2R = (1.3440)h \tag{84}$$

Therefore, equation (84) produces the value of the optimum distance between two stations with equal elevation angles, given the altitude of the point.

### Variation in Position Error in Solution of Point for Circular Configuration of $n > 2$ Stations with Equal Elevation Angles, When One Station is Permitted to Drift Around the Circle

For this part of the analysis, the 0th observation station (i.e., the one, in the previous part, that was located on the x-axis and, hence, for which $\alpha_p = \alpha_0 = 0$) is permitted to drift around the circle. The variable angle which this drifting station makes with the x-axis is denoted by $\gamma$. The angle $\gamma$ can take on either positive or negative values. As the drifting station moves around the circle, for values of $\gamma$ equal to $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, it coincides with the 1st, 2d, . . ., (n-1)th stations, respectively. Each time such a coincidence occurs the number of stations is effectively reduced by one. For this reason, only the situation of $n > 2$ stations is considered in this part of the analysis, as a minimum of two stations must be present for triangulation.

Substituting equations (A6), (A7), (A8), (A9), and (A10), which are derived in the appendix, into equation (49) leads to

$$\begin{aligned}\sum_{p=0}^{n-1} \left(\hat{i}_p \cdot \vec{\delta}_r\right)^2 = {} & \delta_x^{\,2} \cos^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right) + \delta_y^{\,2} \cos^2\theta\left(\frac{n}{2} + \sin^2\gamma\right) \\ & + n\delta_z^{\,2} \sin^2\theta - 2\delta_y\delta_z \sin\theta \cos\theta \sin\gamma \\ & + 2\delta_x\delta_z \sin\theta \cos\theta(1 - \cos\gamma) \\ & + \delta_x\delta_y \cos^2\theta \sin(2\gamma) \qquad (n > 2)\end{aligned} \tag{85}$$

Substituting equations (48), (50), (51), and (85) into equation (43) results in an expression for $\sum_{p=0}^{n=1} \left(\vec{\delta}_p\right)^2$ for $n > 2$ as a function of the elevation angle $\theta$, the line-of-sight errors

$d\theta_p$ and $d\phi_p$, the angle $\alpha_p$, and the x-, y-, and z-components of the position error vector $\vec{\delta}_r$. That is,

$$\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2 = n\left(\delta_x{}^2 + \delta_y{}^2 + \delta_z{}^2\right) - \delta_x{}^2 \cos^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right)$$

$$- \delta_y{}^2 \cos^2\theta\left(\frac{n}{2} + \sin^2\gamma\right) - n\delta_z{}^2 \sin^2\theta + 2\delta_y\delta_z \sin\theta \cos\theta \sin\gamma$$

$$- 2\delta_x\delta_z \sin\theta \cos\theta(1 - \cos\gamma) - \delta_x\delta_y \cos^2\theta \sin 2\gamma$$

$$+ a^2 \sum_{p=0}^{n-1}\left[\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)^2\right.$$

$$\left. + \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right)^2 + \left(d\theta_p \cos\theta\right)^2\right]$$

$$- 2a\delta_x \sum_{p=0}^{n-1}\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)$$

$$- 2a\delta_y \sum_{p=0}^{n-1}\left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right)$$

$$- 2a\delta_z \cos\theta \sum_{p=0}^{n-1} d\theta_p \qquad (n > 2) \qquad (86)$$

In this form, $\sum_{p=0}^{n-1}\left(\vec{\delta}_p\right)^2$ can be minimized with respect to the x-, y-, and z-components of the vector $\vec{\delta}_r$.

Minimizing equation (86) with respect to $\delta_x$, $\delta_y$, and $\delta_z$, furnishes the following three equations, respectively:

$$\left[n - \cos^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right)\right]\delta_x + \left(-\cos^2\theta \sin\gamma \cos\gamma\right)\delta_y$$

$$+ \left[\sin\theta \cos\theta(\cos\gamma - 1)\right]\delta_z = a \sum_{p=0}^{n-1}\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) \qquad (87)$$

$$\left(-\cos^2\theta \sin\gamma \cos\gamma\right)\delta_x + \left[n - \cos^2\theta\left(\frac{n}{2} + \sin^2\gamma\right)\right]\delta_y$$

$$+ (\sin\theta \cos\theta \sin\gamma)\delta_z = a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right) \tag{88}$$

$$\left[\sin\theta \cos\theta(\cos\gamma - 1)\right]\delta_x + (\sin\theta \cos\theta \sin\gamma)\delta_y + \left(n \cos^2\theta\right)\delta_z = a \cos\theta \sum_{p=0}^{n-1} d\theta_p \tag{89}$$

Equations (87), (88), and (89) can be rewritten in the following forms, respectively:

$$A\delta_x + B\delta_y + C\delta_z = \xi \tag{90}$$

$$B\delta_x + D\delta_y + E\delta_z = \eta \tag{91}$$

$$C\delta_x + E\delta_y + F\delta_z = \zeta \tag{92}$$

where

$$A = n - \cos^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right) \tag{93}$$

$$B = -\cos^2\theta \sin\gamma \cos\gamma \tag{94}$$

$$C = \sin\theta \cos\theta(\cos\gamma - 1) \tag{95}$$

$$D = n - \cos^2\theta\left(\frac{n}{2} + \sin^2\gamma\right) \tag{96}$$

$$E = \sin\theta \cos\theta \sin\gamma \tag{97}$$

$$F = n \cos^2\theta \tag{98}$$

$$\xi = a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right) \tag{99}$$

$$\eta = a \sum_{p=0}^{n-1} \left(d\theta_p \sin\theta \sin\alpha_p + d\phi_p \cos\alpha_p\right) \tag{100}$$

and

$$\zeta = a \cos\theta \sum_{p=0}^{n-1} d\theta_p \tag{101}$$

The problem is now reduced to solving equations (90), (91), and (92) simultaneously for $\delta_x$, $\delta_y$, and $\delta_z$. Hence,

$$\delta_x = \frac{1}{\mathscr{A}} \begin{vmatrix} \xi & B & C \\ \eta & D & E \\ \zeta & E & F \end{vmatrix}$$

$$= \frac{1}{\mathscr{A}} \left[\xi\left(DF - E^2\right) + \eta(CE - BF) + \zeta(BE - CD)\right] \tag{102}$$

$$\delta_y = \frac{1}{\mathscr{A}} \begin{vmatrix} A & \xi & C \\ B & \eta & E \\ C & \zeta & F \end{vmatrix}$$

$$= \frac{1}{\mathscr{A}} \left[\xi(CE - BF) + \eta\left(AF - C^2\right) + \zeta(BC - AE)\right] \tag{103}$$

$$\delta_z = \frac{1}{\mathscr{A}} \begin{vmatrix} A & B & \xi \\ B & D & \eta \\ C & E & \zeta \end{vmatrix}$$

$$= \frac{1}{\mathscr{A}} \left[\xi(BE - CD) + \eta(BC - AE) + \zeta\left(AD - B^2\right)\right] \tag{104}$$

where $\mathscr{A}$ is the symmetrical determinant

$$\mathscr{A} = \begin{vmatrix} A & B & C \\ B & D & E \\ C & E & F \end{vmatrix} \tag{105}$$

Equations (102), (103), and (104) can be rewritten in the following forms, respectively:

$$\delta_x = G\xi + H\eta + I\zeta \tag{106}$$

$$\delta_y = H\xi + J\eta + K\zeta \tag{107}$$

$$\delta_z = I\xi + K\eta + L\zeta \tag{108}$$

where

$$G = \frac{1}{\mathscr{A}}\left(DF - E^2\right) \tag{109}$$

$$H = \frac{1}{\mathscr{A}}(CE - BF) \tag{110}$$

$$I = \frac{1}{\mathscr{A}}(BE - CD) \tag{111}$$

$$J = \frac{1}{\mathscr{A}}\left(AF - C^2\right) \tag{112}$$

$$K = \frac{1}{\mathscr{A}}(BC - AE) \tag{113}$$

$$L = \frac{1}{\mathscr{A}}\left(AD - B^2\right) \tag{114}$$

The coefficients $G$, $H$, $I$, $J$, $K$, and $L$ are functions only of $n$, $\gamma$, and $\theta$.

From equations (106), (107), and (108), respectively, it is recognized that the quantities $\overline{\delta_x{}^2}$, $\overline{\delta_y{}^2}$, and $\overline{\delta_z{}^2}$ are

$$\overline{\delta_x{}^2} = G^2\overline{\xi^2} + H^2\overline{\eta^2} + I^2\overline{\zeta^2} + 2GH\overline{\xi\eta} + 2GI\overline{\xi\zeta} + 2HI\overline{\eta\zeta} \tag{115}$$

$$\overline{\delta_y{}^2} = H^2\overline{\xi^2} + J^2\overline{\eta^2} + K^2\overline{\zeta^2} + 2HJ\overline{\xi\eta} + 2HK\overline{\xi\zeta} + 2JK\overline{\eta\zeta} \tag{116}$$

$$\overline{\delta_z{}^2} = I^2\overline{\xi^2} + K^2\overline{\eta^2} + L^2\overline{\zeta^2} + 2IK\overline{\xi\eta} + 2IL\overline{\xi\zeta} + 2KL\overline{\eta\zeta} \tag{117}$$

The quantities $\overline{\xi^2}$, $\overline{\eta^2}$, $\overline{\zeta^2}$, $\overline{\xi\eta}$, $\overline{\xi\zeta}$, and $\overline{\eta\zeta}$ – which are functions, not only of $\theta$ and a, but also of $\alpha_p$, $d\theta_p$, and $\overline{d\phi_p}$ – need to be evaluated. From equation (99) it can be established that the quantity $\overline{\xi^2}$ is

$$\overline{\xi^2} = a^2 \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \overline{\left(d\theta_p \sin\theta \cos\alpha_p - d\phi_p \sin\alpha_p\right)\left(d\theta_q \sin\theta \cos\alpha_q - d\phi_q \sin\alpha_q\right)}$$

Using equations (56), (57), and (58) yields

$$\overline{\xi^2} = a^2\sigma^2\left(\sin^2\theta \sum_{p=0}^{n-1} \cos^2\alpha_p + \sum_{p=0}^{n-1} \sin^2\alpha_p\right)$$

Using equations (A9) and (A10) renders

$$\overline{\xi^2} = a^2\sigma^2\left[\sin^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right) + \left(\frac{n}{2} + \sin^2\gamma\right)\right]$$

Using equation (67) results in

$$\overline{\xi^2} = h^2\sigma^2\left[\left(\frac{n}{2} - 1 + \cos^2\gamma\right) + \csc^2\theta\left(\frac{n}{2} + \sin^2\gamma\right)\right] \tag{118}$$

Similarly,

$$\overline{\eta^2} = h^2\sigma^2\left[\left(\frac{n}{2} + \sin^2\gamma\right) + \csc^2\theta\left(\frac{n}{2} - 1 + \cos^2\gamma\right)\right] \tag{119}$$

$$\overline{\zeta^2} = h^2\sigma^2\left(n \cot^2\theta\right) \tag{120}$$

$$\overline{\xi\eta} = h^2\sigma^2\left(-\frac{1}{2}\cot^2\theta \sin 2\gamma\right) \tag{121}$$

$$\overline{\xi\zeta} = h^2\sigma^2\left[\cot\theta(\cos\gamma - 1)\right] \tag{122}$$

$$\overline{\eta\zeta} = h^2\sigma^2 \cot\theta \sin\gamma \tag{123}$$

If the expressions (118), (119), (120), (121), (122), and (123) are substituted into equations (115), (116), and (117) and then these three resulting expressions are substituted into equation (65), the rms position error $\epsilon$ in the triangulation solution of a point in space for the circular configuration of $n > 2$ observation stations with equal elevation angles, when one of the stations is permitted to drift around the circle, can be computed.

The expression for $\epsilon/h\sigma$ for $n = 3$ was programed for a high-speed computer to get numerical values.

The quantity $\epsilon/h\sigma$ was evaluated for values of $\gamma = 0^\circ$ to $360^\circ$, at $5^\circ$ intervals, for each of the values of $\theta = 5^\circ$ to $85^\circ$, at $5^\circ$ intervals. For each particular value of $\theta$ examined, the quantity $\epsilon/h\sigma$ was the smallest when $\gamma = 0^\circ$ (and $360^\circ$). Table III is a table of values of $\epsilon/h\sigma$ as a function of both $\theta$ and $\gamma$. For convenience, the values listed in the table are only those for $\theta$ and $\gamma$ at $20^\circ$ intervals. It is observed from table III that for each particular value of $\theta$, the smallest value of $\epsilon/h\sigma$ occurs when $\gamma = 0^\circ$ (and $360^\circ$).

Figure 5 is a plot of $\epsilon/h\sigma$ as a function of $\gamma$ for $\theta = 5^\circ$ to $85^\circ$. Again, for convenience, $20^\circ$ intervals in $\theta$ were used. It is again seen from figure 5 that for each particular value of $\theta$ the quantity $\epsilon/h\sigma$ is a minimum when $\gamma = 0^\circ$ (and $360^\circ$), which represents the symmetrical arrangement of the circular configuration of observation stations with equal elevation angles.

It can be seen from table III and figure 5 that when $\gamma = 180^\circ$ (i.e., when the drifting observation station is again equidistant from the two fixed stations), then $\epsilon/h\sigma$ is either a relative minimum or a maximum, depending on the value of the elevation angle $\theta$. It is a relative minimum for $\theta$ equal to $5^\circ$ and $25^\circ$; and it is a maximum for $\theta$ equal to $45^\circ$, $65^\circ$, and $85^\circ$. For $\theta$ equal to $5^\circ$ and $25^\circ$, the quantity $\epsilon/h\sigma$ is a maximum when $\gamma$ is equal to $90^\circ$ and $270^\circ$.

Figure 6 is a plot of $\epsilon/h\sigma$ as a function of the elevation angle $\theta$ for $\gamma = 0^\circ$, $60^\circ$, $120^\circ$, and $180^\circ$. The quantity $\epsilon/h\sigma$ was examined for $\gamma$ only through $180^\circ$, as $\epsilon/h\sigma$ is symmetrical with respect to $\gamma = 180^\circ$, which can be seen from table III. It is observed from figure 6 that for each particular value of $\gamma$ the quantity $\epsilon/h\sigma$ is a minimum for $\theta = 55.300^\circ$. It is remembered that for the symmetrical circular configuration of $n > 2$ observation stations with equal elevation angles the optimum elevation angle, which minimizes the rms position error in the triangulation solution of a point in space, is $55.300^\circ$.

Therefore, the symmetrical arrangement for $n = 3$ observation stations with equal elevation angles in a circular configuration gives the minimum rms position error in the triangulation solution of a point in space. It may be plausibly assumed, then, that the symmetrical arrangement is also the best arrangement for $n > 3$ stations with equal elevation angles in a circular configuration.

## CONCLUSIONS

The purpose of this paper was to derive, to first order, analytical expressions for the rms position error in the triangulation solution of a point object in space for several ideal observation-station configurations to provide insights into the nature of the dependence of the rms position error on certain of the experimental parameters involved.

For two arbitrarily located observation stations, the rms position error in the triangulation solution of a point in space is directly proportional to the rms line-of-sight error. The minimum rms position error occurs when the two elevation angles are both equal to $56.098^{\circ}$. For the configuration of the two arbitrarily located stations, two choices of the most probable point in space – the midpoint of the shortest line between the two lines of sight, and the point, on the aforementioned line, which subtends equal residual angles at the two stations – were used. The latter choice of the most probable point yields the smaller value of the rms position error, except in the special situation of equal elevation angles and then the two choices yield the same value. For the first choice of the most probable point, the optimum relative weighting factor, which minimizes the rms position error, for the data from the first station when the data from one of the stations is degraded relative to that from the other, is directly proportional to the square of the ratio of the rms line-of-sight errors from the second and first stations, respectively.

For the symmetrical circular configuration where the number $n$ of observation stations with equal elevation angles is two or more, the rms position error in the triangulation solution of a point in space is directly proportional to the rms line-of-sight error. For $n > 2$ the rms position error is inversely proportional to the square root of the total number of stations of interest. For $n = 2$ the optimum elevation angle, which minimizes the rms position error, is $56.098^{\circ}$. For elevation angles of $56.098^{\circ} \pm 5^{\circ}$, the difference in the rms position error for $n = 2$ is 1.5 percent. For $n > 2$ the optimum elevation angle, which minimizes the rms position error, is $55.300^{\circ}$ regardless of how many (greater than two) stations are present. For elevation angles of $55.300^{\circ} \pm 5^{\circ}$, the percent difference in the rms position error for $n = 3$ is 2.5 percent, while for $n = 8$ it is 0.9 percent. The value of the optimum elevation angle for $n > 2$ is within a fraction of a degree of the angle ($54.667^{\circ}$) which the sloping edge of a tetrahedron makes with its base. Therefore, the optimum configuration of three stations for triangulating on a point in space is the configuration of the three stations plus the point which closely approximates a tetrahedron.

When one of the stations in the circular configuration of $n = 3$ observation stations with equal elevation angles is permitted to drift around the circle from its position of symmetry, the rms position error in the triangulation solution of a point in space is a minimum when the variable angle of the drifting station equals zero degrees, which represents the symmetrical arrangement. For each value of the variable angle of the drifting station, the rms position error is a minimum when the elevation angle equals $55.300^{\circ}$.

Based on the assumptions used in this study, a necessary and sufficient condition for a minimum rms position error in the triangulation solution of a point in space for two arbitrarily located observation stations is that the two elevation angles be equal, and equal to 56.098°. A necessary and sufficient condition for a minimum rms position error in the solution of a point for three stations with equal elevation angles in a circular configuration is that the three stations be symmetrically arranged around the circle. It may be plausibly assumed, then, that the symmetrical arrangement of the circular configuration of $n > 3$ stations with equal elevation angles is the arrangement of this configuration which yields the minimum rms position error.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., March 14, 1974.

## APPENDIX

## DERIVATION OF CERTAIN TRIGONOMETRIC SERIES USED FOR CIRCULAR CONFIGURATION OF $n \geqq 2$ STATIONS WITH EQUAL ELEVATION ANGLES

For the symmetrical circular configuration of $n \geqq 2$ observation stations with equal elevation angles, the angle between the x-axis and the pth station is $\alpha_p = 2\pi p/n$, as shown in figure 2. Now,

$$\sum_{p=0}^{n-1} \cos \alpha_p = \mathrm{Re}\left[\sum_{p=0}^{n-1} \exp\left(i \frac{2\pi p}{n}\right)\right] = \mathrm{Re}\left[\frac{\exp(i2\pi) - 1}{\exp\left(i \frac{2\pi}{n}\right) - 1}\right]$$

Therefore,

$$\sum_{p=0}^{n-1} \cos \alpha_p = 0 \qquad (n \geqq 2) \qquad \text{(A1)}$$

Similarly,

$$\sum_{p=0}^{n-1} \sin \alpha_p = 0 \qquad (n \geqq 2) \qquad \text{(A2)}$$

Also,

$$\sum_{p=0}^{n-1} \sin 2\alpha_p = \mathrm{Im}\left[\sum_{p=0}^{n-1} \exp\left(i \frac{4\pi p}{n}\right)\right] = \mathrm{Im}\left[\frac{\exp(i4\pi - 1)}{\exp\left(i \frac{4\pi}{n} - 1\right)}\right] = 0 \qquad (n > 2)$$

For $n = 2$,

$$\sum_{p=0}^{n-1} \sin 2\alpha_p = \sum_{p=0}^{1} \sin(2\pi p) = 0$$

Therefore,

$$\sum_{p=0}^{n-1} \sin 2\alpha_p = 0 \qquad (n \geqq 2) \qquad \text{(A3)}$$

In addition,

$$\sum_{p=0}^{n-1} \cos 2\alpha_p = \mathrm{Re}\left[\sum_{p=0}^{n-1} \exp\left(i\,\frac{4\pi p}{n}\right)\right] = \mathrm{Re}\left[\frac{\exp(i4\pi - 1)}{\exp\left(i\,\frac{4\pi}{n} - 1\right)}\right] = 0 \qquad (n > 2)$$

For $n = 2$,

$$\sum_{p=0}^{n-1} \cos 2\alpha_p = \sum_{p=0}^{1} \cos(2\pi p) = 2$$

Hence,

$$\sum_{p=0}^{n-1} \cos 2\alpha_p = \begin{cases} 0 & (n > 2) \\ 2 & (n = 2) \end{cases}$$

Since

$$\sum_{p=0}^{n-1} \cos^2\alpha_p = \frac{1}{2} \sum_{p=0}^{n-1} \left(1 + \cos 2\alpha_p\right)$$

then

$$\sum_{p=0}^{n-1} \cos^2\alpha_p = \begin{cases} \frac{n}{2} & (n > 2) \\ 2 & (n = 2) \end{cases} \qquad \text{(A4)}$$

Since

$$\sum_{p=0}^{n-1} \sin^2\alpha_p = \frac{1}{2}\sum_{p=0}^{n-1}\left(1 - \cos 2\alpha_p\right)$$

then

$$\sum_{p=0}^{n-1} \sin^2\alpha_p = \begin{cases} \frac{n}{2} & (n > 2) \\ 0 & (n = 2) \end{cases} \tag{A5}$$

If, for $n > 2$ observation stations, the 0th station (i.e., the one previously located on the x-axis and for which $\alpha_p = \alpha_0 = 0$) is permitted to drift around the circle, making the variable angle $\gamma$ with the x-axis, then equations (A1), (A2), (A3), (A4), and (A5) must be modified. Now,

$$\sum_{p=0}^{n-1} \cos \alpha_p = \cos \alpha_0 + \sum_{p=1}^{n-1} \cos \alpha_p$$

Therefore,

$$\sum_{p=0}^{n-1} \cos \alpha_p = \cos \gamma - 1 \qquad (n > 2) \tag{A6}$$

Also,

$$\sum_{p=0}^{n-1} \sin \alpha_p = \sin \alpha_0 + \sum_{p=1}^{n-1} \sin \alpha_p$$

Therefore,

$$\sum_{p=0}^{n-1} \sin \alpha_p = \sin \gamma \qquad (n > 2) \tag{A7}$$

In addition,

$$\sum_{p=0}^{n-1} \sin 2\alpha_p = \sin 2\alpha_0 + \sum_{p=1}^{n-1} \sin 2\alpha_p$$

Therefore,

$$\sum_{p=0}^{n-1} \sin 2\alpha_p = \sin 2\gamma \qquad (n > 2) \qquad \text{(A8)}$$

Furthermore,

$$\sum_{p=0}^{n-1} \cos^2\alpha_p = \cos^2\alpha_0 + \sum_{p=1}^{n-1} \cos^2\alpha_p$$

Therefore,

$$\sum_{p=0}^{n-1} \cos^2\alpha_p = \cos^2\gamma + \frac{n}{2} - 1 \qquad (n > 2) \qquad \text{(A9)}$$

Finally,

$$\sum_{p=0}^{n-1} \sin^2\alpha_p = \sin^2\alpha_0 + \sum_{p=1}^{n-1} \sin^2\alpha_p$$

Therefore,

$$\sum_{p=0}^{n-1} \sin^2\alpha_p = \sin^2\gamma + \frac{n}{2} \qquad (n > 2) \qquad \text{(A10)}$$

## REFERENCES


1. Dinescu, A.: Model of Space Triangulation Based on Nonsimultaneous Observations of Satellites. Observations of Artificial Earth Satellites, No. 2, 1963, Stuart W. Kellogg, transl., Smithsonian Inst. Astrophys. Observ., May 1968, pp. 43-74.
2. Ustinov, G. A.: Adjustment of a Space Triangulation. Observations of Artificial Earth Satellites, No. 2, 1963, Stuart W. Kellogg, transl., Smithsonian Inst. Astrophys. Observ., May 1968, pp. 31-41.
3. Long, Sheila Ann T.: Derivation of Formulas for Root-Mean-Square Errors in Location, Orientation, and Shape in Triangulation Solution of an Elongated Object in Space. NASA TN D-7477, 1974.
4. Wong, Kam W.: Satellite Triangulation Accuracy. Photogram. Eng., vol. XXXIII, no. 1, Jan. 1967, pp. 100-108.
5. Lambeck, Kurt: A Hypothetical Application of the Geometric Method of Satellite Geodesy. Rep. 804-152 (NSG-87-60), Smithsonian Inst. Astrophys. Observ., June 1968. (Available as NASA CR-97083.)
6. Lortie, Edna L.: Error Study of a Worldwide Satellite Triangulation Net. BRL No. 1310, U.S. Army, Jan. 1966. (Available from DDC as AD 630 781.)
7. Perret, Edouard E.: Error Analysis for Determination of Target Position and Velocity From Two or More Observers. WADD Tech. Note 60-70, U.S. Air Force, June 1960.
8. Powers, J. W., Jr.: Range Trilateration Error Analysis. 1966 Aerospace Systems Conference Record, IEEE Publ. 10 C 22, pp. 572-585.
9. Lloyd, K. H.: Concise Method for Photogrammetry of Objects in the Sky. WRE-TN-72, Aust. Def. Sci. Serv., Aug. 1971, pp. 6-8.

TABLE I.- THE QUANTITY $\epsilon/h\sigma$ IN SOLUTION OF POINT FOR TWO ARBITRARILY LOCATED STATIONS, WITH MOST PROBABLE POINT TAKEN AS MIDPOINT OF SHORTEST LINE BETWEEN LINES OF SIGHT, AS A FUNCTION OF THE TWO ELEVATION ANGLES IN INTERVALS OF $10^{\circ}$

| $\theta_1$, deg | $\theta_2$, deg | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 |
| 5 | 93.79 | 35.91 | 24.15 | 18.96 | 16.16 | 14.52 | 13.55 | 13.04 | 12.88 |
| 15 | 35.91 | 11.26 | 7.40 | 5.93 | 5.18 | 4.76 | 4.55 | 4.47 | 4.52 |
| 25 | 24.15 | 7.40 | 4.68 | 3.70 | 3.24 | 3.01 | 2.92 | 2.92 | 3.02 |
| 35 | 18.96 | 5.93 | 3.70 | 2.90 | 2.54 | 2.38 | 2.34 | 2.38 | 2.53 |
| 45 | 16.16 | 5.18 | 3.24 | 2.54 | 2.24 | 2.11 | 2.11 | 2.21 | 2.42 |
| 55 | 14.52 | 4.76 | 3.01 | 2.38 | 2.11 | 2.03 | 2.07 | 2.24 | 2.58 |
| 65 | 13.55 | 4.55 | 2.92 | 2.34 | 2.11 | 2.07 | 2.18 | 2.47 | 3.08 |
| 75 | 13.04 | 4.47 | 2.92 | 2.38 | 2.21 | 2.24 | 2.47 | 3.02 | 4.28 |
| 85 | 12.88 | 4.52 | 3.02 | 2.53 | 2.42 | 2.58 | 3.08 | 4.28 | 8.21 |

TABLE II.- THE QUANTITY $\epsilon/h\sigma$ IN SOLUTION OF POINT FOR TWO ARBITRARILY LOCATED STATIONS, WITH MOST PROBABLE POINT TAKEN AS MIDPOINT OF SHORTEST LINE BETWEEN LINES OF SIGHT, AS A FUNCTION OF THE TWO ELEVATION ANGLES IN INCREMENTS OF 0.001°

| $\theta_1$, deg | $\theta_2$, deg | | | | |
|---|---|---|---|---|---|
| | 56.096 | 56.097 | 56.098 | 56.099 | 56.100 |
| 56.096 | 2.0278919610 | 2.0278919578 | 2.0278919558 | 2.0278919550 | 2.0278919554 |
| 56.097 | 2.0278919578 | 2.0278919550 | 2.0278919535 | 2.0278919532 | 2.0278919541 |
| 56.098 | 2.0278919558 | 2.0278919535 | 2.0278919525 | 2.0278919527 | 2.0278919540 |
| 56.099 | 2.0278919550 | 2.0278919532 | 2.0278919527 | 2.0278919533 | 2.0278919552 |
| 56.100 | 2.0278919554 | 2.0278919541 | 2.0278919540 | 2.0278919552 | 2.0278919575 |

TABLE III.- THE QUANTITY $\epsilon/h\sigma$ IN SOLUTION OF POINT FOR CIRCULAR CONFIGURATION OF $n = 3$ STATIONS WITH EQUAL ELEVATION ANGLES, WHEN ONE STATION IS PERMITTED TO DRIFT AROUND THE CIRCLE, AS A FUNCTION OF THE ELEVATION ANGLE AND THE VARIABLE ANGLE

| $\gamma$, deg | $\theta$, deg | | | | |
|---|---|---|---|---|---|
| | 5 | 25 | 45 | 65 | 85 |
| 0 | 14.77913 | 2.93361 | 1.76383 | 1.77858 | 6.70017 |
| 20 | 15.08990 | 2.96540 | 1.77248 | 1.78803 | 6.74501 |
| 40 | 16.01071 | 3.05499 | 1.79750 | 1.81630 | 6.87938 |
| 60 | 17.38405 | 3.17939 | 1.83586 | 1.86287 | 7.10196 |
| 80 | 18.56983 | 3.28970 | 1.88192 | 1.92612 | 7.40728 |
| 100 | 18.58380 | 3.32772 | 1.92795 | 2.00217 | 7.78023 |
| 120 | 17.41489 | 3.27762 | 1.96638 | 2.08354 | 8.18797 |
| 140 | 16.04393 | 3.18378 | 1.99285 | 2.15846 | 8.57282 |
| 160 | 15.12098 | 3.10440 | 2.00725 | 2.21226 | 8.85561 |
| 180 | 14.80908 | 3.07453 | 2.01166 | 2.23197 | 8.96061 |
| 200 | 15.12098 | 3.10440 | 2.00725 | 2.21226 | 8.85561 |
| 220 | 16.04393 | 3.18378 | 1.99285 | 2.15846 | 8.57282 |
| 240 | 17.41489 | 3.27762 | 1.96638 | 2.08354 | 8.18797 |
| 260 | 18.58380 | 3.32772 | 1.92795 | 2.00217 | 7.78023 |
| 280 | 18.56983 | 3.28970 | 1.88192 | 1.92612 | 7.40728 |
| 300 | 17.38405 | 3.17939 | 1.83586 | 1.86287 | 7.10196 |
| 320 | 16.01071 | 3.05499 | 1.79750 | 1.81630 | 6.87938 |
| 340 | 15.08990 | 2.96540 | 1.77248 | 1.78803 | 6.74501 |
| 360 | 14.77913 | 2.93361 | 1.76383 | 1.77858 | 6.70017 |

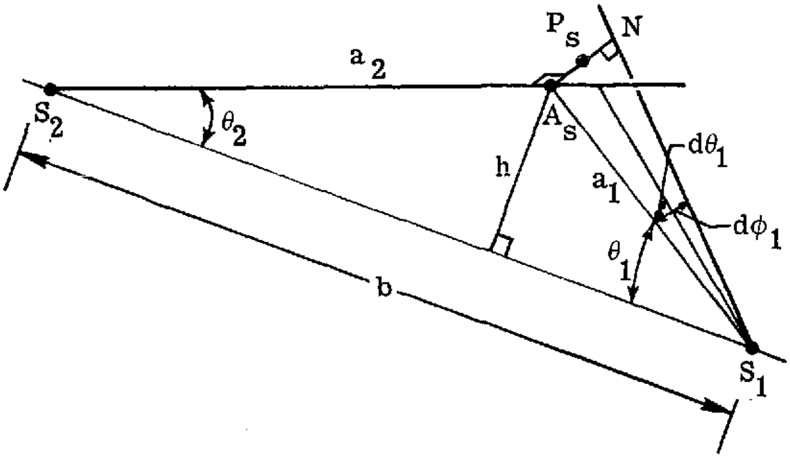


Figure 1.- Two arbitrarily located stations.

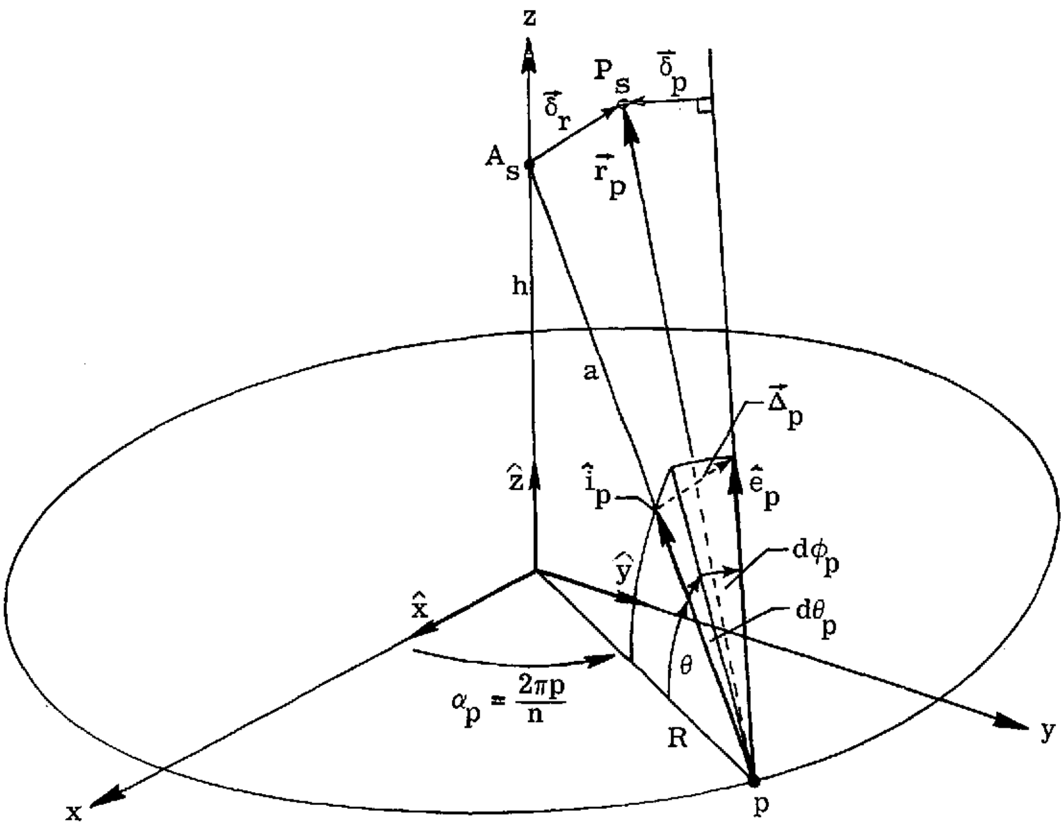


Figure 2.- Symmetrical circular configuration of $n \geqq 2$ stations with equal elevation angles.

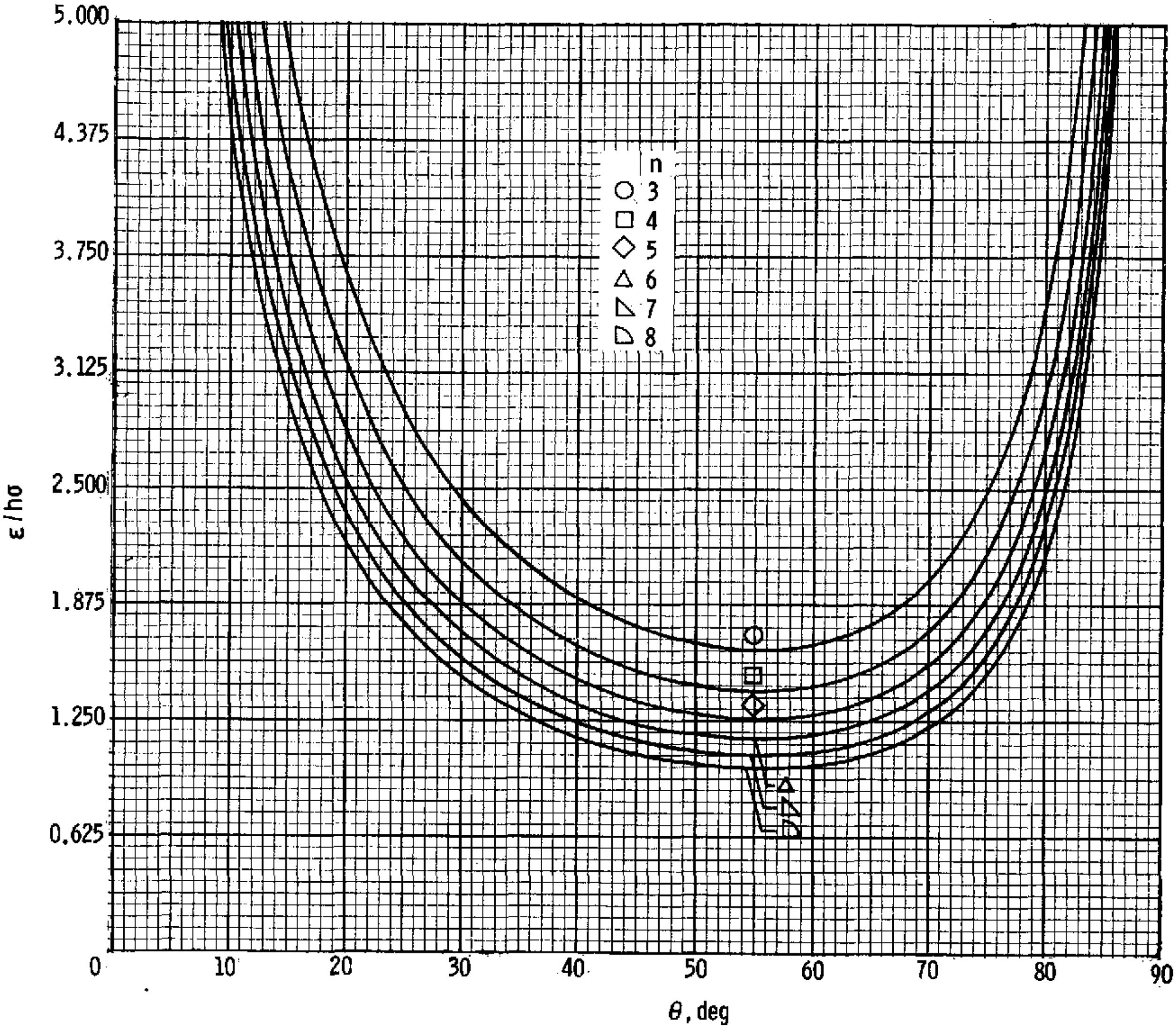


Figure 3.- The quantity $\epsilon/h\sigma$ in solution of point for symmetrical circular configuration of $n > 2$ stations with equal elevation angles as a function of the elevation angle, for $n = 3, 4, 5, 6, 7,$ and 8.

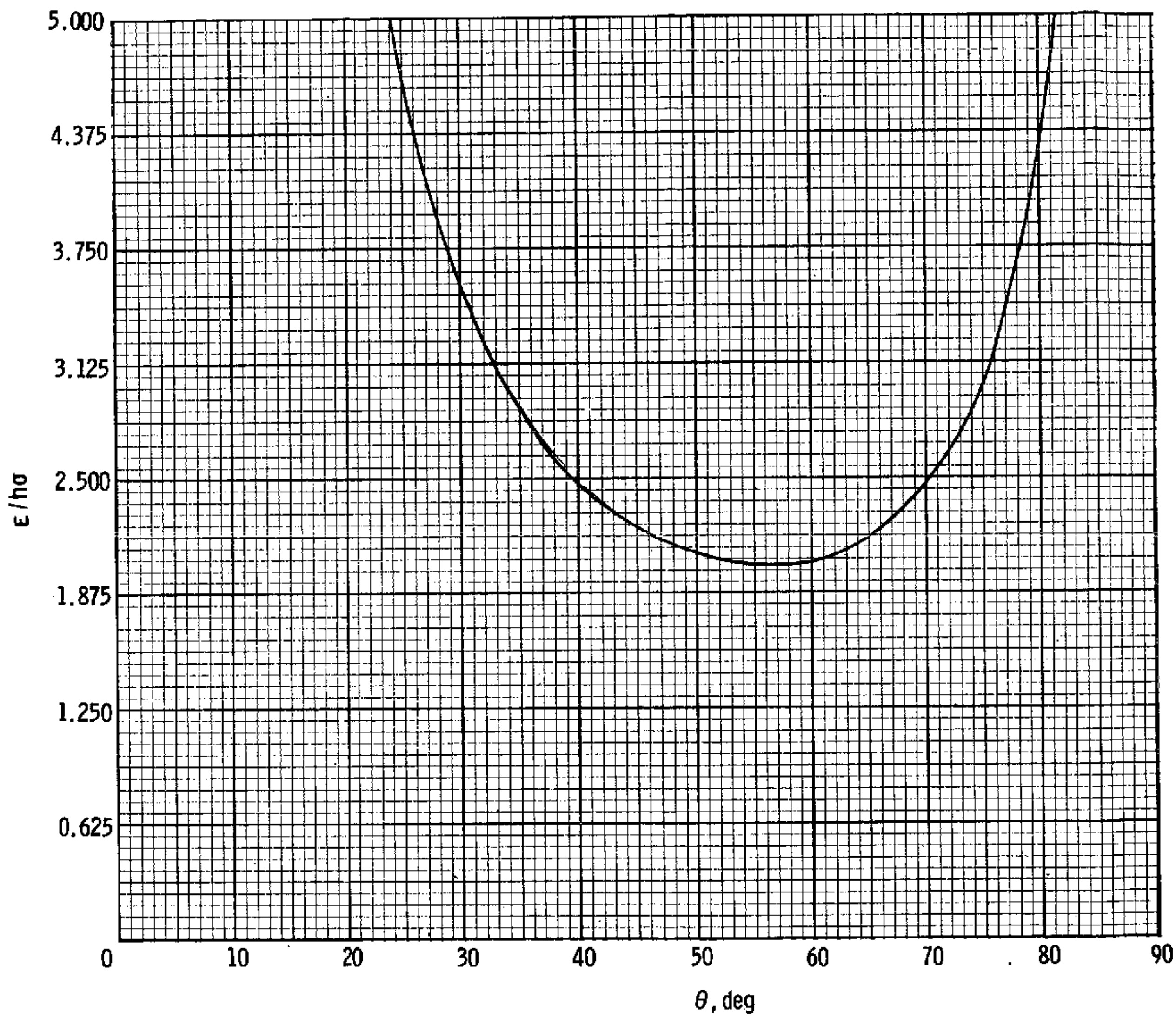


Figure 4.- The quantity $\epsilon/h\sigma$ in solution of point for symmetrical circular configuration of $n = 2$ stations with equal elevation angles as a function of the elevation angle.

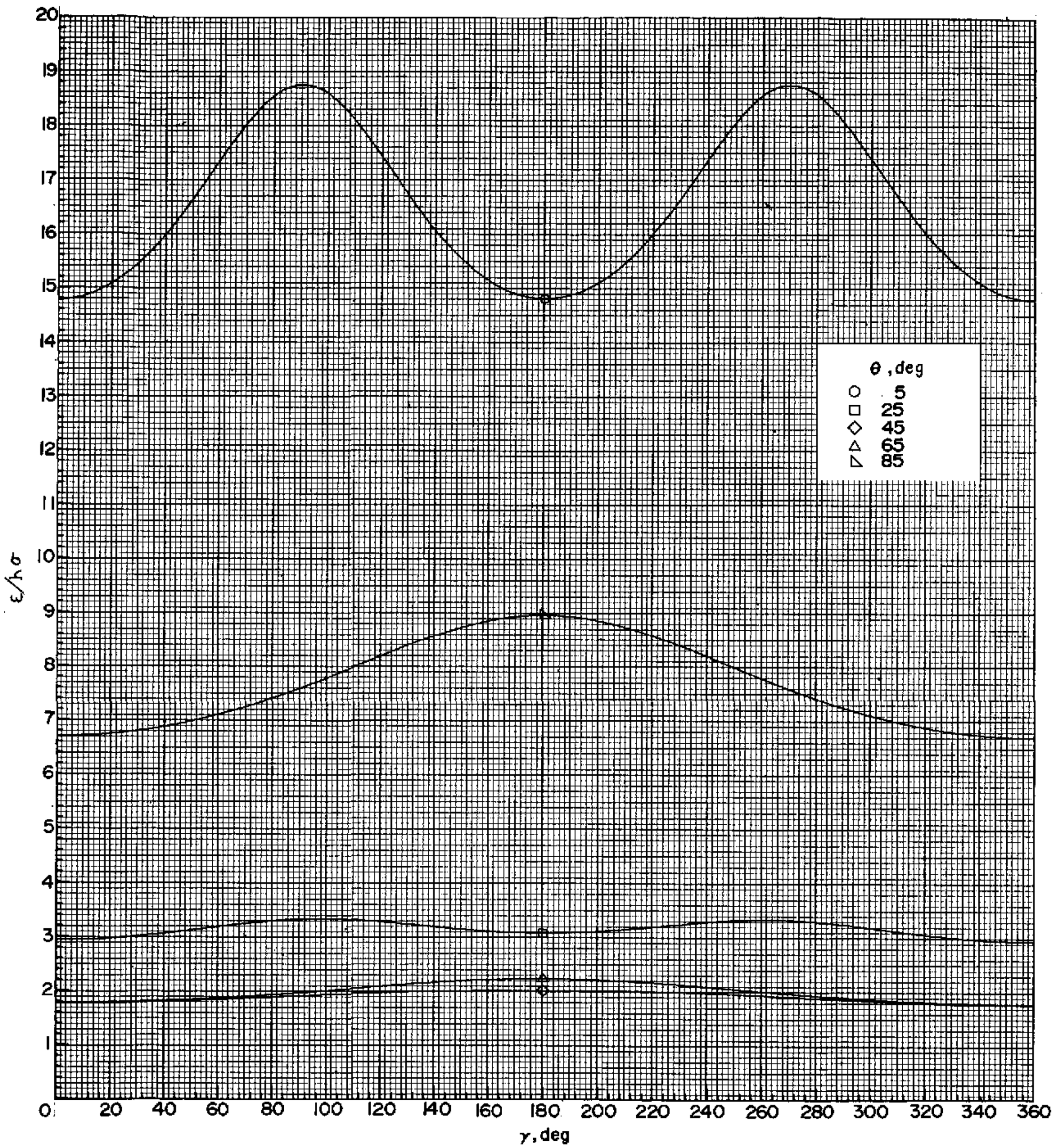


Figure 5.- The quantity $\epsilon/h\sigma$ in solution of point for circular configuration of $n = 3$ stations with equal elevation angles, when one station is permitted to drift around the circle, as a function of the variable angle for different values of the elevation angle.

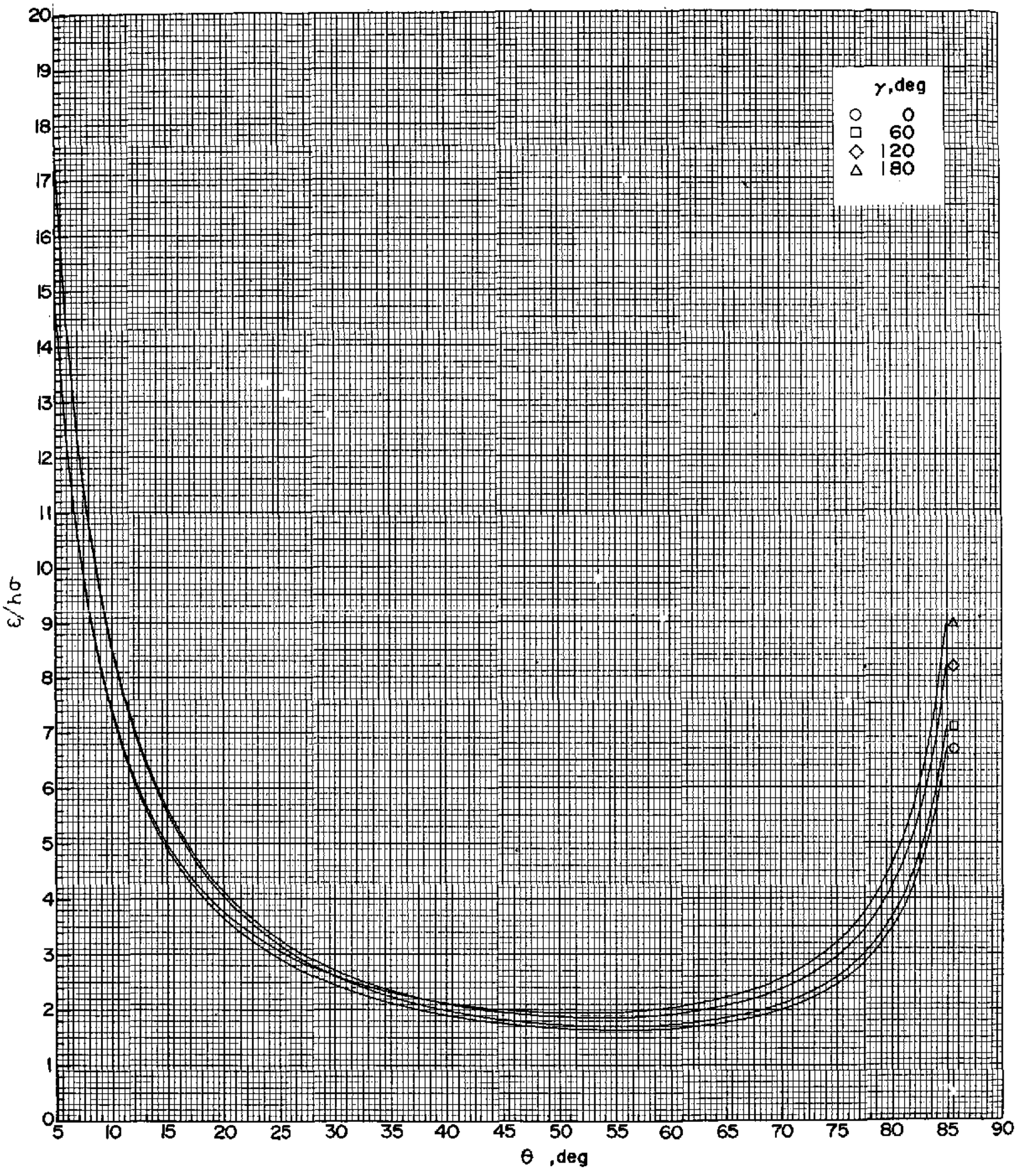


Figure 6.- The quantity $\epsilon/h\sigma$ in solution of point for circular configuration of $n = 3$ stations with equal elevation angles, when one station is permitted to drift around the circle, as a function of the elevation angle for different values of the variable angle.



NASA-Langley, 1974 L-9235