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# THE NUMERICAL CALCULATION OF 

## LAMINAR BOUNDARY-LAYER SEPARATIONAERGNTHMTH LAMA

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## NOMENCLATURE

| A | matrix formed by difference equations |
| :---: | :---: |
| a | local speed of sound |
| $\mathrm{b}(\mathrm{x})$ | function of $x$, equation (9) |
| c | constant |
| $\mathrm{C}_{\mathrm{f}}$ | skin friction coefficient, $\frac{2 \mu\left(\partial u^{*} / \partial y^{*}\right)}{\rho_{\infty}{ }^{*} u_{\infty}{ }^{2}}$ |
| f | transformed stream function, $\int \overline{\mathrm{u}} \mathrm{d} \bar{y}$ |
| $\mathrm{f}_{1}, \mathrm{f}_{2}, \mathrm{f}_{3}, .$. | coefficients in Taylor series expansions, see equation (38) |
| g | $u v-\frac{\partial u}{\partial y}$ |
| H | conditioning matrix, also $H_{u}, H_{V}$ |
| h | parameter for relaxation scheme |
| I | identity matrix |
| i | $\sqrt{-1}$ |
| J | maximum j index |
| j | discrete index in streamwise direction |
| K | maximum $k$ index |
| k | discrete index in normal direction |
| $\ell$ | typical length scale |
| M | Mach number |
| m | normalized velocity gradient, $\frac{x}{u_{e}} \frac{d u_{e}}{d x}$ |
| $\overline{\mathrm{m}}$ | normalized Mach number gradient, $\frac{x}{M_{e}} \frac{d M e}{d x}$ |
| p | fluid pressure |
| R | Reyno1ds number, $\frac{u_{\infty}^{*} \ell}{\nu}$ |
| $\stackrel{\rightharpoonup}{\mathrm{R}}$ | residual vector, also $\vec{R}_{u}, \vec{R}_{V}$ |

$\operatorname{Re}_{x} \quad$ Reynolds number, $\frac{\rho_{e} e^{x}}{\mu_{e}}$
u
u* streamwise velocity component, physical variable
$\overline{\mathrm{u}}$
$v$ normalized component, $\frac{v^{*}}{u_{\infty}{ }^{*}} \sqrt{R}$
v* normal velocity component, physical variable
$\bar{v} \quad$ transformed component, $v \sqrt{\frac{x}{u_{e}}}+\frac{1}{2}(m-1) \bar{y} \bar{u}$
x
x* streamwise coordinate
$y$ normalized normal coordinate, $\frac{y^{*}}{\ell} \sqrt{R}$
$y^{*}$ normal coordinate
$\bar{y} \quad$ transformed normal coordinate, $y \sqrt{\frac{u_{e}}{x}}$
$z \quad u_{e}{ }^{2}-u^{2}$
$\alpha \quad$ arbitrary parameter
$\beta \quad \frac{\Delta y}{2}$
$\gamma \quad \frac{\Delta y^{2}}{2 \Delta x}$, also ratio of specific heats
$\varepsilon \quad$ truncation error term or a small parameter
$\eta \quad \sqrt{\frac{m+1}{2}} \bar{y}$
$\lambda \quad$ eigenvalue of iteration matrix $(I+h H A) ;$ also $\lambda=\int_{0}^{\bar{y}}\left(1-\bar{u}^{2}\right) d \bar{y}$ coefficient of viscosity kinematic viscosity, $\frac{\mu}{\rho}$
fluid density

| $\sigma$ | eigenvalue of HA |
| :--- | :--- |
| $\bar{\tau}$ | $\left.\frac{\partial \bar{u}}{\partial \bar{y}}\right\|_{0}$ |
| $\psi$ | stream function, $\int u d y$ |
| $\omega$ | relaxation parameter or vorticity |

## Subscripts

B backward difference operator
C central difference operator
e condition at edge of viscous layer
F forward difference operator
$\mathrm{j}, \mathrm{k} \quad$ location at a grid point or an index
$\max \quad$ with $J$ or $K$, maximum number of grid points $j$ or $k$ in the field s condition at separation
$x$. partial derivative with respect to $x$
0 constant value of $u$ or $v$; also a quantity evaluated at $\bar{y}=0$
1,2 conditions on either side of a plane in physical space
$\infty \quad$ far upstream condition
$\|\cdot\|_{2}$ Euclidean vector norm or induced matrix norm

Superscripts

| * | physical variable |
| :--- | :--- |
| - | trañsformed variátle, sé equation (5) |
| $\sim$ | perturbation term |
| (n) | iteration level |
| - | vector quantity |
| , | $\frac{\partial}{\partial y}$, also $\frac{d}{d x}$ with equations (39)-(42) |

# THE NUMERICAL CALCULATION OF LAMINAR <br> BOUNDARY-LAYER SEPARATION* <br> John M. Klineberg and Joseph L. Steger <br> Ames Research Center 

## SUMMARY

Iterative finite-difference techniques are developed for integrating the boundary-layer equations, without approximation, through a region of reversed flow. The numerical procedures are used to calculate incompressible laminar separated flows and to investigate the conditions for regular behavior at the point of separation. Regular flows are shown to be characterized by an integrable saddle-type singularity that makes it difficult to obtain numerical solutions which pass continuously into the separated region. The singularity is'removed and continuous solutions ensured by specifying the wall shear distribution and computing the pressure gradient as part of the solution. Calculated results are presented for several separated flows and the ccuracy of the method is verified. A computer program listing and complete olution case are included.

## INTRODUCTION

During the past decade, various approximate methods have been developed to calculate separated flows by using the boundary-layer equations. The most popular schemes have been integral, or moment, methods based on the early work of Abbott, Holt, and Nielsen (refs. 1-3) or Lees and Reeves (refs. 4-8). In the integral approach, the boundary-layer equations are multiplied by a power of $u$ and converted into a system of ordinary differential equations by integrating across the viscous layer. Regions of attached and separated flow are treated similarly because the average convection in the boundary layer is always in the downstream direction.

A second type of approximate method, first proposed by Reyhner and Flügge-Lotz (ref. 9) uses finite-difference techniques (refs. 10 and 11). This approach uses a forward-marching procedure, with all convective derivatives set to zero in regions of reversed flow for numerical stability. The conservation of momentum and energy is therefore violated in the portion of the separated flow bounded by the zero-velocity line, although the errors introduced by this approximation are not expected to be significant for small laminar separation bubbles. Both the finite-difference and integral methods have produced good agreement with experimental data, particularly for compression-corner flows and shock-wave/boundary-layer interactions (see the review in ref. 12).
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The first finite-difference integration of the complete boundary-layer equations through a region of reversed flow was performed by Catherall and Mangler (ref. 13). This report provides the best previous numerical evidence of flows that are regular at separation. A continuous solution was obtained by specifying the displacement thickness downstream of an appropriate point near separation and determining the pressure gradient by streamwise integration. The numerical procedure developed instabilities in the reversed-flow region, however, and the integration was continued only by decreasing the convergence criterion at each station. As the authors point out, this difficulty is to be expected because the region of separated flow should actually be integrated in the upstream direction, with boundary conditions provided from downstream.

There have also been several numerical studies of nonlinear parabolic equations of mixed type, where the direction of increasing "time" reverses in some region of the flow field. One of these investigations, by Klemp and Acrivos (ref. 14), considers the flow over a finite, stationary flat plate whose surface moves at a constant velocity opposite that of the free stream (i.e., a rotating belt). The boundary layer is divided into two regions along the unknown zero-velocity line and the equations are integrated in the appropriate flow directions, with the final solution obtained by iterating for the location of the common boundary. It is not evident that this technique would prove effective for calculating boundary-layer separation because the region of reversed flow results only from the upstream motion of the surface of the plate. Also, the pressure gradient is assumed to be zero and the shear stresses remain positive throughout the flow field. The singularities at separation and reattachment are therefore caused by discontinuities in the boundary conditions and are not associated with the vanishing of the surface skin friction.

A more useful numerical procedure for calculating the flow past an impulsively started flat plate has recently been developed by Dennis (ref. 15). For this problem, the motion at short times is given by Rayleigh's error function solution, while the final steady-state condition is given by the Blasius profile. Although the transition from the initial to the final state can be calculated directly in the three independent variables (ref. 16), Dennis formulated the problem in similarity coordinates where the governing equation is parabolic and of mixed type. The convective derivatives were approximated by backward or forward differences where appropriate, and the solution was obtained through a successive overrelaxation procedure. This numerical technique with certain modifications can also be applied to the equations that describe boundary-layer separation. The two problems are, of course, different in many important respects. In particular, there is nothing corresponding to reattachment for the impulsively started flat plate, and the downstream (large time) boundary conditions are given. One of the more interesting features of boundary-layer separation is that although there is an embedded region of reversed flow and of upstream influence, the overall problem remains parabolic in the downstream direction.

The present investigation develops a numerical procedure for integrating the laminar, incompressible boundary-layer equations, without approximation, through a region of reversed flow. Under Development of Numerical Method, a
model problem is examined to determine convergence and stability criteria, and iterative finite-difference schemes are developed to solve the nonlinear equations. Under Results and Discussion, the numerical procedures are used to investigate the conditions for regular behavior at the point of separation. The separation (and reattachment) points are shown to be saddle-type singularities in the physical plane, which make it difficult to obtain numerical solutions that pass continuously from the attached region to the separated region. The singularities are effectively removed, however, by specifying the wall shear distribution and determining the pressure as part of the solution. These inverse calculations are used to infer the type of pressure distribution required for the boundary layer to pass smoothly into a region of reversed flow. Where possible, results are compared to relevant analytical or similarity solutions to verify the accuracy of the calculations. The extension of the method to compressible flows and to the solution of complete viscousinviscid interactions is indicated in a separate section.

## DEVELOPMENT OF NUMERICAL METHOD

## The Differential Equations

The boundary-layer equations for two-dimensional, laminar, incompressible flow are

$$
\begin{gather*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0  \tag{1a}\\
u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}=u_{e} \frac{d u e}{d x}+\frac{\partial^{2} u}{\partial y^{2}} \tag{lb}
\end{gather*}
$$

where the Reynolds number has been explicitly removed by introducing the usual scaling $x=x^{*} / \ell, y=\left(y^{*} / \ell\right) \sqrt{R}, u=u^{*} / u_{\infty}, v=\left(v^{*} / u_{\infty}\right) \sqrt{R}, R \equiv u_{\infty} * \ell / \nu$. Here superscript (*) indicates the physical or untransformed variable. Boundary conditions are $u=v=0$ and $u \rightarrow u_{e}$ as $y \rightarrow \infty$. In a direct problem, $u_{e}$ is specified as a function of $x$, while, in an inverse problem, an alternate condition such as ( $\partial u / \partial y)_{0}$ or $v_{e}$ is given as a function of $x$. In this case, $u_{e}$ must be determined as part of the solution process.

The parabolic nature of the equations is evident in von Mises coordinates:

$$
\begin{align*}
\frac{\partial z}{\partial x} & =u \frac{\partial^{2} z}{\partial \psi^{2}}  \tag{2a}\\
u & =\frac{\partial \psi}{\partial y} \tag{2b}
\end{align*}
$$

with $u_{e}^{2}-u^{2}=z$ and $v=-\partial \psi / \partial x$. Equation (2a) is clearly a heat equation in which the coefficient $u$ changes sign in regions of reversed flow. Because there is no downstream boundary condition, the solution is not unique unless the separated zone is entirely confined within the domain of integration.

The equations can also be written as a system of nonlinear first-order equations in conservative form, for example,

$$
\begin{array}{r}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \\
\frac{\partial}{\partial x}\left(u^{2}-\frac{u_{e}^{2}}{2}\right)+\frac{\partial g}{\partial y}=0 \\
\frac{\partial u}{\partial y}-u v+g=0 \tag{3c}
\end{array}
$$

Because the equations are nonlinear, discontinuities may occur in the flow field even though continuous boundary conditions are specified. Equations (3a), (3b), and (3c) possess the following weak solutions:

$$
\begin{align*}
{\left[u_{2}-u_{1}\right] \sin \theta } & =\left[v_{2}-v_{1}\right] \cos \theta  \tag{4a}\\
{\left[u_{2}^{2}-u_{1}^{2}\right] \sin \theta } & =\left[g_{2}-g_{1}\right] \cos \theta  \tag{4b}\\
0 & =\left[u_{2}-u_{1}\right] \cos \theta \tag{4c}
\end{align*}
$$

where $u_{e}$ is assumed to be continuous and $\theta$ is the angle between the axis and a plane separating conditions 1 and 2. If $\theta<\pi / 2$, equation (4c) ensures that $u_{2}=u_{1}$ and, consequently, all the variables are continuous. When $\theta=\pi / 2$, the weak solutions are indeterminate. In particular, $v$ may be discontinuous with a jump of indeterminate strength even with $u$ continuous. Furthermore, if $u$ is discontinuous, then, from equation (4a), $\left[v_{2}-v_{1}\right] \rightarrow \infty$.

## Preliminary Numerical Considerations

As equation (2a) in particular shows, in the separated-flow region, information must be allowed to propagate upstream with the reversed flow velocity. A natural way to fulfill this requirement consistent with restrictions of numerical stability is to treat the $x$-derivatives with backward (upwind) finite-difference formulas in attached flow regions and with forward (downwind) finite-difference formulas in the reversed flow region. However, this means that at least a portion of the difference equations will require simultaneous solution. Furthermore, the extent of the separated region is unknown and, because the equations are nonlinear, an iterative finite-difference method appears to be the most efficient way to find a solution. Here, of course, one can rely on experience obtained with type-dependent relaxation methods employed for transonic flow fields (refs. 17 and 18).

As an alternative to a type-dependent differencing scheme, interpolative (elliptic) finite-difference formulas such as central differencing can be used over the entire flow region. In fact, in the absence of discontinuities,
parabolic and hyperbolic problems can be solved with interpolative differencing, provided the boundary conditions are properly satisfied. Of course, for a simple initial-value problem, a marching process that uses backward differencing is generally far more efficient than a simultaneous solution process.

The choice of whether to use backward-forward differencing, central differencing, or some hybrid of these will depend on the efficiency and accuracy obtainable in the iterative finite-difference method. In any case, no downstream boundary conditions can be supplied for the boundary-layer equations, so the last computed profile must be attached to allow the use of backward differencing for the $x$-derivatives.

The success of a numerical method also depends on the choice of variables into which the equations are cast. Equation (2), for example, is not suitable because the variable $\psi$ is multivalued in the separated region. For the most part, equations (1a) and (lb) appear to be the most appropriate to difference with a high probability of being readily extended to more complex (e.g., three-dimensional) flows.

Because the boundary-layer exhibits extensive growth in the x-direction, it is essential for numerical efficiency that this growth be scaled out. This can be accomplished by introducing a variable, growing grid system or by using a transformation that keeps the viscous layer of nearly uniform thickness. The following transformation is used:

$$
\left.\begin{array}{l}
\bar{y}=y \sqrt{\frac{u_{e}}{x}}  \tag{5}\\
\bar{v}=v \sqrt{\frac{x}{u_{e}}}+\frac{m-1}{2} \bar{y} \bar{u}
\end{array}\right\}
$$

so that equations (1a) and (lb) become

$$
\begin{align*}
x \frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{m+1}{2} \bar{u} & =0  \tag{6a}\\
x \bar{u} \frac{\partial \bar{u}}{\partial x}+\bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} & =m\left(1-\bar{u}^{2}\right)+\frac{\partial^{2} \bar{u}}{\partial \bar{y}^{2}} \tag{6b}
\end{align*}
$$

Boundary conditions are indicated in figure 1. These equations can also be written as a single equation for the stream function

$$
\begin{equation*}
f^{\prime \prime \prime}+\frac{m+1}{2} f f^{\prime \prime}+m\left(1-f^{\prime 2}\right)=x\left(f^{\prime} f_{x}^{\prime}-f_{x} f^{\prime \prime}\right) \tag{7}
\end{equation*}
$$



Figure 1.- Difference operators and boundary conditions for relaxation calculation.

Iterative Finte-Difference Method
In the first stages of developing a finite-difference method, it is useful to begin with the study of a model problem. A model equation is obtained here by linearizing equations (la) and (1b); the iterative convergence criteria are reviewed and an appropriate choice of difference formulas is made so that the simple model equation is iteratively stable. In the following section, the convergence of the difference equations to the differential equation is considered; and iteratively convergent differencing schemes for the nonlinear boundary-layer equations are subsequently given without analysis.

Model problem- Equations (1a) and (1b) are simplified with

$$
\left.\begin{array}{l}
u=u_{0}+\tilde{u}  \tag{8}\\
v=v_{0}+\tilde{v}
\end{array}\right\}
$$

so that the model equation becomes

$$
\begin{equation*}
\frac{\partial^{2} \tilde{u}}{\partial y^{2}}-u_{0} \frac{\partial \tilde{u}}{\partial x}-v_{0} \frac{\partial \tilde{u}}{\partial y}=b(x) \tag{9}
\end{equation*}
$$

In any local domain, $u_{0}$ and $v_{0}$ are treated as constants. Equation (9) also represents the transformed equations, equations (6a) and (6b), if an average value for $x \bar{u}_{0}$ is substituted for $u_{0}$.

If convergent difference algorithms and convergent iterative procedures can be selected for equation (9), subject to all reasonable choices of $u_{0}$ and $v_{0}$, it is assumed that such schemes can be successfully adapted to equations (1a) and (1b). While explicit and implicit marching procedures have been developed and extensively studied for parabolic equations of standard type, a comparable development does not exist for relaxation schemes. The development of such a scheme is undertaken below where the primary concern is
to ensure that the relaxation procedure is valid for both positive and negative values of $u_{0}$.

Iterative convergence criteria- Once equation (9) is differenced over a discrete network of grid points, one is left with the task of inverting the linear system of equations

$$
\begin{equation*}
A \vec{u}-\vec{c}=0 \tag{10}
\end{equation*}
$$

where the components of $\vec{u}$ consist of the dependent variables at each grid point. Then the most general first-degree iteration scheme for equation (10) is

$$
\begin{equation*}
\overrightarrow{\mathrm{u}}^{(\mathrm{n}+1)}-\overrightarrow{\mathrm{u}}^{(\mathrm{n})}=\mathrm{hH}\left[\overrightarrow{\mathrm{~A}}^{(\mathrm{n})}-\overrightarrow{\mathrm{c}}\right] \tag{11}
\end{equation*}
$$

where $H$ is a conditioning matrix usually implicitly built into the iterative solution algorithm; here we chose to extract a parameter $h$ from $H$. It should be understood that this type of iterative solution algorithm can treat nonlinear equations with the same ease as linear equations.

Equation (11) has the recursive solution:

$$
\begin{equation*}
\vec{u}^{(n)}=(I+h H A)^{n} \vec{u}^{(0)}-\sum_{m=0}^{n-1}(I+h H A)^{m} h H \stackrel{\rightharpoonup}{c} \tag{12}
\end{equation*}
$$

so if the matrix ( $I+h H A$ ) has a spectral radius (i.e., largest eigenvalue in absolute magnitude) less than 1 , then $(I+h H A)^{n} \rightarrow 0$ for $n$ sufficiently large. Furthermore, from the Neumann lemma (ref. 19, p. 26, or ref. 20, p. 82), it is evident that

$$
\begin{equation*}
-\sum_{m=0}^{n-1}(I+h H A)^{m} h H \rightarrow A^{-1} \tag{13}
\end{equation*}
$$

or $\vec{u} \rightarrow A^{-1} \stackrel{\rightharpoonup}{c}$ as required. Thus the sufficient condition for iterative convergence is that all

$$
\begin{equation*}
\left|\lambda_{j}\right| \equiv\left|1+h \sigma_{j}\right|<1 \tag{14}
\end{equation*}
$$

where $\sigma_{j}$ are the eigenvalues of HA. Hence, if all the real parts of the possibly complex. $\sigma_{j}$ are of the same sign, h can be chosen to assure convergence. This is an asymptotic convergence criterion for $n$ sufficiently large. For the scheme defined by equation (11) to be efficient, the matrix HA must not have a large condition number (refs. 19 and 21) nor should the imaginary parts of $\sigma_{j}$ be large compared to the real parts. The eigenvalueconvergence criterion does not guarantee that the norm of (I $+h H A$ ) $n \overrightarrow{\mathrm{u}}(0)$ will not grow appreciably during intermediate iterations - a situation likely to occur if the eigenvectors of $H A$ are linearly dependent or almost so.

Convergence of the model problem- The advantage of studying the model problem is that analytic expressions are obtained to describe its behavior for various choices of differencing. It is assumed that the nonlinear problem will share at least some common features. Here, let

$$
\begin{align*}
& \left.\frac{\partial^{2} u}{\partial y^{2}}\right|_{j k}=\left(\frac{1}{\Delta y}\right)^{2}\left(u_{j k-1}-2 u_{j k}+u_{j k+1}\right)+0\left(\Delta y^{2}\right)  \tag{15a}\\
& \left.\frac{\partial u}{\partial y}\right|_{j k}=\left(\frac{1}{2 \Delta y}\right)\left(u_{j k+1}-u_{j k-1}\right)+0\left(\Delta y^{2}\right) \tag{15b}
\end{align*}
$$

and

$$
\begin{equation*}
\left.\frac{\partial u}{\partial x}\right|_{j k}=\left(\frac{1}{2 \Delta x}\right)\left[-(1+\alpha) u_{j-1, k}+2 \alpha u_{j k}+(1-\alpha) u_{j+1, k}\right]+0(\Delta x) \tag{16}
\end{equation*}
$$

where $\alpha=1$ is first-order backward, $\alpha=0$ is second-order central, and $\alpha=-1$ is first-order forward. Using these approximations in equation (9) with $\beta=\Delta y / 2$ and $\gamma=(\Delta y)^{2} / 2 \Delta x$, one obtains

$$
\begin{array}{r}
\left(1+v_{0} \beta\right) u_{j k-1}-2 u_{j k}+\left(1-v_{0} \beta\right) u_{j k+1}+(1+\alpha) \gamma u_{0} u_{j-1, k}-2 \alpha \gamma u_{0} u_{j k} \\
-(1-\alpha) \gamma u_{0} u_{j+1, k}=b_{j} \\
\left(j=2,3,4 \ldots . ., J_{\max } ; \quad k=2,3,4 \ldots . K_{\max }-1\right) \tag{17}
\end{array}
$$

If $\vec{u}$ is the vector whose components are the $u_{j k}$ over the ordered grid points, equation (17) can be written as the linear system of equations, equation (10). The eigenvalues of $A$ are given by

$$
\begin{align*}
& \sigma_{j k}=-2\left[1+\sqrt{\left(1+v_{0} B\right)\left(1-v_{0} \beta\right)} \cos \left(\frac{k \pi}{K+1}\right)\right] \\
&-2 u_{0} \gamma\left[\alpha+\sqrt{-(1+\alpha)(1-\alpha)} \cos \left(\frac{j \pi}{J+1}\right)\right] \\
&\left(k=1,2 . . ., K ; \quad j=1,2 . . ., J ; \quad K=K_{\max }-2 ; \quad J=J_{\max }-1\right) \tag{18}
\end{align*}
$$

where $u$ is assumed to be given on a boundary as needed. If $\alpha$ is 0 or 1 when $u_{0}>0$ or if $\alpha$ is 0 or -1 when $u_{0}<0$, the $\sigma$ roots always have negative real parts and $A$ is a stable matrix. Thus the point-iteration scheme with $H=I$ and $h=\omega /\left(2+2 u_{0} \gamma \alpha\right)$ is proven to be convergent for an appropriate $\omega \leq 1$. As another example, the point successive overrelaxation (SOR) method has the roots

$$
\begin{gather*}
\left(1-\frac{\sigma_{j k}}{\omega}\right)\left(1+\alpha \gamma u_{0}\right)= \\
\sqrt{1-\sigma_{j k}}\left[-\sqrt{\left(1+v_{0} \beta\right)\left(1-v_{0} \beta\right)} \cos \left(\frac{k \pi}{K+1}\right)\right. \\
 \tag{19}\\
\left.-u_{0 \gamma} \sqrt{(1+\alpha)(-1+\alpha)} \cos \left(\frac{j \pi}{J+1}\right)\right] \\
(j=1,2 \ldots . . K ; \quad j=1,2 \ldots . J)
\end{gather*}
$$

and is also iteratively convergent with $h=-1$ and a proper choice of the relaxation parameter $\omega$.

Equations (18) and (19) show that the roots will be complex if $\alpha=0$ or if $\left|v_{0} \beta\right|>1$. This can be detrimental to the convergence rate of a firstdegree iteration scheme if the imaginary parts become large enough, so the central differencing should be restricted to regions where $u_{o r}$ is small. The product $v_{0} \beta$ is normally expected to be less than $l$ in absolute value and thus has the beneficial effect of reducing the term $\cos k \pi /(K+1)$.

In place of the complex roots that occur for $\alpha=0$, when $\alpha=1$ or -1 , the eigenvectors of HA appear in multiples of the number of J grid points. Under these conditions, the norm of an iteration matrix can be expected to grow before it decays; however, study of $\ell_{2}$ and $\ell_{\infty}$ induced matrix norms (ref. 19) for the point iteration scheme shows that residual growth cannot occur if the spectral radius is kept less than l. Conversely, numerical experimentation with the heat equation demonstrates that the SOR forwarddifferenced scheme ( $\alpha=-1$ ) swept from left to right can experience appreciable residual growth if $\Delta x \ll(\Delta y)^{2}$. If swept from right to left, the residuals decay rapidly.

## Convergence to the Differential Equations

Although the previous analysis shows that iteration algorithms can be used to find a solution to the system of difference equations, it does not prove that the solution of the differonce equations will converge to the solution of the differential equations as the grid is refined. However, with the exception of the central differencing scheme, all the schemes to be introduced are known to be stable and consistent for the heat equation (cf. ref. 22).

If one assumes periodic boundary conditions in $x$ and end conditions in $y$, then sufficient conditions for convergence of the centrally differenced heat equation

$$
\pm\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^{2} u}{\partial y^{2}}
$$

are $\Delta x \geq 0\left(\Delta y^{2}\right)$ and $\Delta y \geq O\left(\Delta x^{2}\right)$. (This is not an explicit leap-frog scheme.) Here convergence implies that the difference between the exact solution to the differential equation and the exact solution to the difference equation will vanish as the grid is uniformly refined over a fixed domain. That is, the summation of truncation errors given by $A^{-1} \stackrel{\rightharpoonup}{\varepsilon} \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$ where, $A$ is the matrix formed by the difference equations over both $y$ and $x$, and $\bar{\varepsilon}$ is the vector of truncation errors. While the complete convergence proof is too lengthy for this report, note that $A$ is a normal matrix and hence is unitary similar to a diagonal matrix of its eigenvalues (ref. 21). The eigenvalues are

$$
\begin{align*}
\sigma_{j k}(2 \Delta x A) & =4 \frac{\Delta x}{\Delta y^{2}}\left[1-\cos \left(\frac{j \pi}{J+1}\right)\right]+2 i( \pm 1) \sin \left(\frac{2 k \pi}{K}\right) \\
(j & =1,2 . . . J ; \quad k=1,2 . . ., K) \tag{20}
\end{align*}
$$

and $\left\|A^{-1}\right\|_{2}=\left(\min \left|\sigma_{j k}\right|\right)^{-1}$ so $\left\|A^{-1}\right\|_{2}\|\vec{\varepsilon}\|_{2}$ is simply determined.

## Finite-Difference Equations and Solution

Two second-order-accurate differencing schemes were developed to solve the boundary-layer equations (6a) and (6b). The first of these proved superior for the separated flows computed in this investigation. The second more conventional method is described because it may prove efficient for certain extensions of the present approach.

The first method employs the central-differencing schemes for $\bar{u}_{y y}$ and $\bar{u}_{y}$ given by equations (15a) and (15b). The term $x \bar{u}_{x}$ in equation ( 6 b ) is regrouped as $0.5 x\left(\bar{u}^{2}\right)_{x}$ and backward-differenced:

$$
\begin{equation*}
\left.\frac{x}{2} \frac{\partial u^{2}}{\partial x}\right|_{j k}=\frac{x_{j}}{2}\left(\frac{3 u_{j k}^{2}-4 u_{j-1, k}^{2}+u_{j-2, k}^{2}}{2 \Delta x}\right)+0\left(\Delta x^{2}\right) \tag{21}
\end{equation*}
$$

(B)
for $\overline{\mathrm{u}}>0.015$ or $\mathrm{j}=J_{\max }$. When $\bar{u}<0.005$ or if $j=2$, central differencing is used:

$$
\begin{equation*}
\left.\frac{x}{2} \frac{\partial u^{2}}{\partial x}\right|_{j k}=\frac{x_{j}}{2}\left(\frac{u_{j+1, k}^{2}-u_{j-1, k}^{2}}{2 \Delta x}\right)+0\left(\Delta x^{2}\right) \tag{22}
\end{equation*}
$$

(C)

In the intermediate zone, $0.005 \leqslant \overline{\mathrm{u}} \leqslant 0.015$, the backward and central formulas are combined according to the relation

$$
\begin{equation*}
\left.\frac{\partial u^{2}}{\partial x}\right|_{j k}=\frac{1}{2}\left[\left.(1+\alpha) \frac{\partial u^{2}}{\partial x}\right|_{\substack{j k \\(B)}}+\left.(1-\alpha) \frac{\partial u^{2}}{\partial x}\right|_{\substack{j k \\(C)}}\right] \tag{23}
\end{equation*}
$$

with $\alpha \equiv 1+200(\bar{u}-0.015)$. The difference stencils are indicated in figure 1.

We emphasize that the blending defined by equation (23) is used solely to enhance the iteration process and is not otherwise fundamental. It is obvious that when the difference equations are switched at a given value of $\overline{\bar{u}}$, a different set of data points is sampled and slightly different truncation errors result. The change in the residual error vectors at this point can be large enough to drive $\bar{u}(n+1)$ back across the value at which switching occurs. This can then start an oscillatory mode with little decay. The blending simply modifies the differencing relations in a continuous fashion so that the residuals vary smoothly. In the present scheme, the blending is completed at $\bar{u}=0.005$ to avoid a special operation at separation and reattachment. The blending can also be used between $0 \leq \overline{\mathrm{u}} \leq 0.01$ without changing the results.

The continuity equation is differenced with the modified Euler scheme (i.e., trapezoidal rule or Crank-Nicholson differencing):

$$
\begin{equation*}
v_{j k}-v_{j k-1}=\frac{\Delta y}{2}\left(\left.\frac{\partial v}{\partial y}\right|_{j k}+\left.\frac{\partial v}{\partial y}\right|_{j k-1}\right)+0\left(\Delta y^{2}\right) \tag{24}
\end{equation*}
$$

with

$$
\begin{equation*}
\left.\frac{\partial v}{\partial y}\right|_{j k}=-\left[x_{j}\left(\frac{u_{j+1, k^{-u} j-1, k}}{2 \Delta x}\right)+\frac{m_{j}+1}{2} u_{j k}\right]+O\left(\Delta x^{2}\right) \tag{25a}
\end{equation*}
$$

for $j=2, J_{\max }-1$, and

$$
\begin{equation*}
\left.\frac{\partial v}{\partial y}\right|_{j k}=-\left[x_{j}\left(\frac{3 u_{j k}-4 u_{j-1, k}+u_{j-2, k}}{2 \Delta x}\right)+\frac{m_{j}+1}{2} u_{j k}\right]+0\left(\Delta x^{2}\right) \tag{25b}
\end{equation*}
$$

at $j=J_{\text {max }}$. Note that $x \bar{u}_{x}$ is central-differenced at all times (except at $J_{\max }$ ) in both the attached and reversed flow regions. While equation (24) is generally recommended, two schemes implicit in the $y$ direction are presented as alternatives. Either the second-order-accurate "shifted" scheme

$$
\begin{equation*}
-3 v_{j k-1}+4 v_{j k}-v_{j k+1}=2 \Delta y\left(\left.\frac{\partial v}{\partial y}\right|_{j k-1}\right)+0\left(\Delta y^{2}\right) \tag{26}
\end{equation*}
$$

(where point $j k$ is updated in the relaxation) or the third-order accurate-in-y "abated Hermite" scheme:

$$
\begin{equation*}
-5 v_{j k-1}+8 v_{j k}-3 v_{j k+1}=2 \Delta y\left(\left.\frac{7}{6} \frac{\partial v}{\partial y}\right|_{j k-1}+\left.\frac{4}{6} \frac{\partial v}{\partial y}\right|_{j k}-\left.\frac{5}{6} \frac{\partial v}{\partial y}\right|_{j k+1}\right)+0\left(\Delta y^{3}\right) \tag{27}
\end{equation*}
$$

can be used with $\partial v /\left.\partial y\right|_{j k}$ again defined by equation (25). Both alternative schemes generate diagonally dominant tridiagonal blocks if a backward twopoint differencing is used at the edge where $\bar{v}$ varies linearly. Effectively, equations (24), (26), and (27) give the same results.

An additional difference algorithm must be introduced if an inverse problem is solved. To impose a specified shear distribution, the momentum equation is evaluated at the surface:

$$
\begin{equation*}
m=-\left.\frac{\partial^{2} \bar{u}}{\partial \bar{y}^{2}}\right|_{\bar{y}=0} \tag{28}
\end{equation*}
$$

The second derivative is differenced as a function of $\bar{\tau}(x)$ to generate the second-order-accurate relation:

$$
\begin{equation*}
\left.\frac{\partial^{2} u}{\partial y^{2}}\right|_{j 1}=\frac{-7 u_{j 1}+8 u_{j 2}-u_{j 3}}{2(\Delta y)^{2}}-\frac{\left.3 \tau\right|_{j}}{\Delta y} \tag{29}
\end{equation*}
$$

Wake flow is treated in the same fashion with $\bar{\tau}=0$ and the centerline velocity $\bar{u}_{0}$ specified:

$$
\begin{equation*}
\left(1-\bar{u}_{0}^{2}\right) m=-\left(\frac{\partial^{2} \overline{\mathrm{u}}}{\partial \bar{y}^{2}}\right)_{0}+x \bar{u}_{0} \frac{d \bar{u}_{0}}{d x} \tag{30}
\end{equation*}
$$

With the choice of differencing established, the solution procedure is straightforward. An approximate solution is input, usually by assuming a Blasius profile with $m=0$ everywhere. For an inverse problem, a new distribution of $m$ is then predicted for the specified boundary condition using either equation (28) or (30), with $m$ updated by the relaxation (here written for $\bar{\tau}(x)$ specified):

$$
\begin{equation*}
m_{j}^{(n+1)}=m_{j}^{(n)}-\omega\left(m_{j}^{(n)}+\frac{\left.3 \tau\right|_{j}}{\Delta y}+\frac{-8 u_{j 2}+u_{j 3}}{2(\Delta y)^{2}}\right) \tag{31}
\end{equation*}
$$

For a poor guess of the initial solution, $\omega$ is initially kept small, $\omega=0(0.05)$. New values of $\bar{u}$ are then found from relaxation of the momentum equation, while new estimates of $\bar{v}$ follow from continuity. This iteration sequence continues (with $\omega$ increased as the initial guess is improved) until an equilibrium or converged state is reached.

Solutions are found by both point and line successive underrelaxation (SUR) by using the iterative correspondence:

$$
\begin{align*}
& \vec{u}^{(n+1)}=\overrightarrow{\mathrm{u}}^{(n)}+\omega \mathrm{H}_{\mathrm{u}} \stackrel{\rightharpoonup}{\mathrm{R}}_{u}  \tag{32a}\\
& \overrightarrow{\mathrm{v}}^{(\mathrm{n}+1)}=\vec{v}^{(n)}+\omega \mathrm{H}_{v} \stackrel{\rightharpoonup}{R}_{v} \tag{32b}
\end{align*}
$$

The residual vectors $\vec{R}_{\mathbf{u}}$ and $\vec{R}_{V}$ represent the differenced momentum and differenced continuity equations, $H_{u}$ and $H_{v}$ are the conditioning matrices of the SUR algorithm, and $\omega$ is the relaxation factor. The line method (not used in eq. (32b)), in general, converges faster than the point.scheme, but it is more sensitive to changes in $m$, making it more difficult to control in a computer batch mode. For moderate reversed flows and grid spacings with $\Delta x$ approximately equal to $\Delta y$, the optimum relaxation parameter is slightly less than 1 for point SUR with $\omega=O(0.5)$ for equation (31). For line SUR, the optimum relaxation parameter is $0(0.4)$ and $\omega$ is the $O(0.15)$. The point SUR method fully converges in 400 to 800 iterations for a grid of 80 j-points and 50 k-points. Highly separated cases with rapid variations in the flow quantities require the higher iteration counts.

Note that, when $\overline{\mathrm{u}}$ is negative, it is possible to blend from the central into a three-point forward difference and that this variant of the relaxation procedure is iteratively convergent. For very large reversed-flow regions, it may be advantageous to program this additional logic. Experience also shows that switching at $\bar{u}=0$ from a three-point backward differencing into a three-point forward differencing without blending first into the central differencing is not convergent.

The second method developed is patterned after the Crank-Nicholson scheme. Equations (6a) and (6b) are first put into conservative form

$$
\begin{array}{r}
\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\partial x \bar{u}}{\partial x}+\frac{(m-1) \bar{u}}{2}=0 \\
\frac{\partial(\bar{u} \bar{v})}{\partial \bar{y}}-\frac{\partial^{2} \bar{u}}{\partial \bar{y}^{2}}+\frac{\partial\left(x \bar{u}^{2}\right)}{\partial x}+\frac{3 m-1}{2} \bar{u}^{2}-m=0 \tag{33b}
\end{array}
$$

The continuity equation is treated as before, and the $y$ derivatives in the momentum equation are again centrally differenced by use of relations (15a) and (15b). The $x$-differencing is Crank-Nicholson

$$
\begin{gather*}
\left(x u^{2}\right)_{j k}-\left(x u^{2}\right)_{j-1 k}=\frac{\Delta x}{2}\left[\left.\frac{\partial\left(x u^{2}\right)}{\partial x}\right|_{j k}+\left.\frac{\partial x u^{2}}{\partial x}\right|_{j-1, \bar{k}}\right] \\
\left(\bar{u}>0.01 \text { or } j=J_{\max }\right) \tag{34}
\end{gather*}
$$

with

$$
\begin{equation*}
\left.\frac{\partial\left(x u^{2}\right)}{\partial x}\right|_{j k}=\left.\frac{\partial^{2} u}{\partial y^{2}}\right|_{j k}+m_{j}-\left(\frac{3 m_{j}-1}{2}\right) u_{j k}-\left.\frac{\partial u v}{\partial y}\right|_{j k} \tag{35}
\end{equation*}
$$

where the appropriate central-difference formulas are substituted for the $y$ derivatives. For reversed flow, forward differencing is used

$$
\begin{align*}
\left(x u^{2}\right)_{j+l k}-\left(x u^{2}\right)_{j k} & =\frac{\Delta x}{2}\left[\left.\frac{\partial\left(x u^{2}\right)}{\partial x}\right|_{j+l k}+\left.\frac{\partial\left(x u^{2}\right)}{\partial x}\right|_{j k}\right] \\
& (\bar{u}<-0.01) \tag{36}
\end{align*}
$$

and the two schemes are linearly blended in the interval $-0.01 \leq \overline{\mathrm{u}} \leq 0.01$ (see fig. 1). As before, the blending is used solely to enhance the iteration process.

The Crank-Nicholson scheme has been solved by both point and line SUR, and for either process the relaxation parameters are approximately those described for the previous point method. This second method requires slightly more algebra per step and, in general, has a slower rate of convergence than the first method.

The conservation-law form of the Crank-Nicholson method is not considered to be an advantage, and the procedure generally predicts $m$ distributions that are slightly oscillatory. The oscillations decay as $\Delta x / \Delta y$ decreases, and they are confined to the relatively uninteresting attached flow regions. Of course, $m$ is a sensitive function of the solution and the $u$ and $\overline{\mathrm{v}}$ distributions are much smoother. A nonconservative version of the CrankNicholson scheme was also programmed. In this case, the oscillations in $m$ were negligible in attached-flow regions but observable in the separated zone.

Finally, we remark that a very stable first-order-accurate method can be developed by replacing the x differencing by

$$
\begin{array}{ll}
\left.\frac{x}{2} \frac{\partial u^{2}}{\partial x}\right|_{\substack{j k \\
(B)}}=\frac{x_{j}}{2}\left(\frac{u_{j k}^{2}-u_{j-1, k}^{2}}{\Delta x}\right) & (\bar{u}>0.01) \\
\left.\frac{x}{2} \frac{\partial u^{2}}{\partial x}\right|_{\substack{j k \\
(F)}}=\frac{x_{j}}{2}\left(\frac{u_{j+1, k}^{2}-u_{j k}^{2}}{\Delta x}\right) \quad(\bar{u}<-0.01) \tag{37b}
\end{array}
$$

and

$$
\begin{aligned}
& \left.\frac{x}{2} \frac{\partial u^{2}}{\partial x}\right|_{j k}=\frac{x_{j}}{4}\left[\left.(1+\alpha) \frac{\partial u^{2}}{\partial x}\right|_{\underset{j k}{(B)}}+\left.(1-\alpha) \frac{\partial u^{2}}{\partial x}\right|_{\underset{(F)}{j k}}\right] \\
& (-0.01 \leq \overline{\mathrm{u}} \leq 0.01 ; \quad \alpha=100 \overline{\mathrm{u}})
\end{aligned}
$$

This scheme, with $x \bar{u}_{x}$ of continuity also first-order-accurate and switched in an identical fashion, will generally give "computational results" for the first problem, m specified. This first-order method is not recommended because a much finer $x$-grid spacing is required to maintain accuracy. This method proved useful for the numerical experiments described in the next section.

## RESULTS AND DISCUSSION

In this section, the iterative finite-difference procedure is used to integrate the boundary-layer equations through a region of reversed flow. The separation-point singularity is investigated and conditions for regular behavior are determined. Calculated results are presented for a number of separated flows and the accuracy of the method is verified. Possible indications of the breakdown of the boundary-layer assumptions are also examined.

## Direct Solutions

One of the most extensively studied problems in separating boundary-layer flows is the response of a flat-plate boundary layer to a linearly retarded external stream. This problem has been investigated by Howarth (ref. 23), Hartree (ref. 24), and many others; recent solutions have also been obtained by Briley (ref. 25) and Leal (ref. 26) using the full Navier-Stokes equations.

A sequence of calculations for different mesh spacings is shown in figure 2. The external velocity was specified to decrease linearly from the origin and the first-order-accurate


Figure 2.- Calculation for linearly
retarded flow.。 difference scheme was used because of stability considerations. As the mesh is refined, the separation point moves upstream, with the last calculation in exact agreement with the accurate results of Smith and Clutter (ref. 27) for the flow upstream of separation. In the limit of zero-mesh spacing, it is evident that the solution is singular, with the wall shear approaching zero as the square root of the distance upstream of the separation point and the nomal velocity unlounded. This type of behavior has been discussed in detail by Goldstein (ref. 28) and by Landau and Lifshitz (ref. 29) among others.

The interesting result here is that the use of an iterative finitedifference scheme which contains type-dependent operators allows the solution
to be "continued" in the downstream direction. As the mesh is refined, it becomes evident that the flow fields upstream and downstream of separation are essentially independent, and the solution is therefore not meaningful. The wall shear jumps discontinuously to a negative value at separation and the normal velocity $\bar{v}$ becomes unbounded; all flow quantities subsequently remain continuous downstream of the jump and through reattachment. The magnitude of the discontinuity is determined by the specified pressure distribution in the separated zone. In a set of simple numerical experiments, a constant external velocity distribution was smoothly joined to the linearly retarded flow at different values of $x$. As the joining point was moved downstream, the magnitude of the jump and the extent of the reversed-flow region increased monotonically, with separation remaining at $x=0.96$.

Singular behavior at the point of separation is thus related to the fact that the wall shear $\bar{\tau} \equiv(\partial \bar{u} / \partial \bar{y})_{0}$ is nonanalytic; in particular, $\bar{\tau} \sim\left(x_{s}-x\right)^{l / 2}$ and $\partial \bar{\tau} / \partial x \rightarrow \infty$ as $x \rightarrow x_{S}$, the separation point. Therefore, the most obvious means of ensuring regular solutions at separation is to specify a continuous wall-shear distribution. The pressure distribution can then be determined as part of the solution by satisfying the momentum equation at the surface. Note that because the equations are nonlinear, it is not possible to guarantee that discontinuities will not occur in the flow field even with analytic boundary conditions prescribed (see ref. 30 for hyperbolic equations, or the weak solutions, eqs. (4a), (4b), and (4c)).

## Inverse Solutions

With the wall-shear distribution specified, $m$ can be determined from equation (31) and the second-order-accurate differencing scheme generates continuous solutions that give no indication of singular behavior at either separation or reattachment. These solutions are demonstrated to be regular under Accuracy Check. An inverse calculation cannot be duplicated by the direct method, however. Starting with a fully converged inverse solution, the calculation diverges if the iteration is continued with $m$ fixed, that is, the relaxation parameter $\omega$ is set to zero in equation (31). Two examples of this type of inverse ( $\bar{\tau}$ specified) and direct ( $m$ given) calculation sequences are shown in figure. 3. After as many as 500 iterations (less if the solution is initially perturbed), the residuals begin to grow and the relaxation procedure either becomes unstable or converges to a different "solution" of the difference equations. As the mesh is refined, the second-order scheme fails to converge while the first-order method, for moderate grid spacing, generates computational results containing a discontinuity.

The fact that the direct calculation fails to duplicate a converged inverse solution cannot be ascribed to instabilities in the numerical scheme. The only difference between the two calculations is the value of the relaxation parameter $\omega$ in equation (31), and the solution processes are essentially identical. The numerical evidence therefore strongly suggests the existence of a saddle-type singularity at the separation point. Because of this critical point, roundoff and residual errors are sufficient to cause a completely converged solution to diverge when the pressure-gradient parameter is held



Figure 3.- Inverse/direct calculations that indicate existence of saddle point.
fixed. There are no other possible sources of error in the calculations: the variation of $m$ is determined to arbitrarily high accuracy by the inverse solution, and no interpolation or differentiation is required as for computations with experimentally determined pressure distributions. With the pressure gradient corresponding to a completely regular flow field prescribed, the equations contain a saddle-type singularity at separation that makes a continuous numerical solution difficult to obtain. The saddle point is removed from the domain of integration, however, by specifying the wall shear rather than the pressure gradient as a boundary condition. A discussion of the essential differences between the two types of calculations is presented below. In the following section, the conditions for regular behavior at the point of separation are examined.

## Saddle Point

The difference between the direct- and inverse-calculation procedures can best be illustrated by examining the boundary-layer equations near the surface. Expanding the velocity profile in a Taylor series in $y$ yields

$$
\begin{equation*}
u(x, y)=f_{1} y+f_{2} \frac{y^{2}}{2!}+f_{3} \frac{y^{3}}{3!}+\ldots \tag{38}
\end{equation*}
$$

where $f_{3}=0$ and the notation is used

$$
f_{1}=\tau, \quad f_{2}=p_{x}
$$

Either $f_{1}$ or $f_{2}$ (but not both) is prescribed and all other $f_{i}$ are determined as functions of $x$ by the differential equations. The coefficients must satisfy the following set of relations:

$$
\left.\begin{array}{rl}
f_{4}-f_{1} f_{1}^{\prime} & =0  \tag{39}\\
f_{5}-2 f_{1} f_{2}^{\prime} & =0 \\
f_{6}-2 f_{2} f_{2}^{\prime} & =0 \\
f_{7}-4 f_{1} f_{4}^{\prime}+5 f_{4} f_{1}^{\prime} & =0 \\
f_{8}-5 f_{1} f_{5}^{\prime}-9 f_{2} f_{4}^{\prime}+5 f_{4} f_{2}^{\prime}+9 f_{5} f_{1}^{\prime}=0 \\
\cdot \\
\cdot
\end{array}\right\}
$$



Figure 4.- Series expansion for similar separation profile.
where the prime denotes differentiation with respect to $x$. One of the $f_{i}$ is given by the outer boundary condition that $u \rightarrow u_{e}$ as $y \rightarrow \infty$ (see ref. 28). The validity of the expansion procedure near the separation point is demonstrated in figure 4 for the particular case of similar flow with $\mathrm{m}=-0.09044$, corresponding to zero shear (see eq. (7)). For this case, the only nonzero coefficients multiply terms of the order $n^{4 n-2}(n=1,2, . .$.$) , and the$ expansion has been continued through the twenty-second power of the normal coordinate.

Direct calculations- For the pressure gradient specified, the coefficients in equations (39) must be determined by integrating the following system of first-order differential equations:

$$
\begin{align*}
f_{1} f_{1}^{\prime}= & f_{4}  \tag{40a}\\
4 f_{1}^{2} f_{4}^{\prime}= & f_{1} f_{7}+5 f_{4}^{2} \\
14 f_{1}^{3} f_{7}^{\prime}= & 2 f_{1}{ }^{2}\left[f_{10}+8 p_{x}\left(2 p_{x x}^{2}-5 p_{x} p_{x x}\right)\right] \\
& +33 f_{1} f_{4} f_{7}-35 f_{4}^{2} \tag{40b}
\end{align*}
$$

The remaining coefficients are given by the algebraic relations:

$$
\left.\begin{array}{rl}
f_{5} & =2 p_{x x} f_{1} \\
f_{6} & =2 p_{x} p_{x x}  \tag{41}\\
4 f_{1}^{2} f_{8} & =9 p_{x}\left(f_{1} f_{7}+5 f_{4}^{2}\right)+4 f_{1}^{2}\left(10 p_{x x x^{\prime}} f_{1}^{2}-13 p_{x x} f_{4}\right) \\
f_{9} & =8 f_{1}\left(5 p_{x} p_{x x x}-2 p_{x x}^{2}\right)
\end{array}\right\}
$$

If one arbitarily terminates the expansions at this point and assumes that $\mathrm{f}_{10}$ can be correctly specified, then equations (40) and (41) provide relations for all coefficients of the lower-order terms. Given the velocity profile at a particular station, standard numerical techniques can be used to integrate equations (40a) and (40b) to determine the adjacent profile provided $f_{1}$ is nonzero. As $f_{1} \rightarrow 0$, however, the solution becomes increasingly sensitive to the calculated value of $f_{1}$, and numerical errors are propagated in the direction of integration. The equations are highly nonlinear, with the coefficient $f_{1}$ of the derivatives determined by $f_{4}$, which in turn depends on $f_{7}, f_{10}$, etc., and on the outer boundary condition. Even for a pressure distribution corresponding to a regular solution at separation, the numerical integration of equations (40a) and (40b) is unlikely to result in values of $f_{1}$ and $f_{4}$ that vanish simultaneously. In that event, either $f_{l}^{\prime}$ will be infinite, leading to a square-root singularity, or $f_{1}$ will remain positive and the calculation will fail to show boundary-layer separation.

We emphasize that with the pressure gradient specified, the nonlinear equation for the wall shear (eq. (40a)) is inherent to the system of differtial equations. Even with special procedures that would guarantee that $f_{4}$ vanishes at $\tau=0$, the saddle point would remain to confound the numerical solution process. The behavior shown in figure 3 is to be expected because a converged solution perturbed by small roundoff and residual errors cannot remain converged in the presence of the saddle-point singularity.

Inverse calculations- For the wall shear specified and the pressure distribution determined as part of the solution, a different system of ordinary differential equations results:

$$
\begin{align*}
2 \tau f_{2}^{\prime}= & f_{5}  \tag{42a}\\
10 \tau f_{5}^{\prime}= & 2 f_{8}+23 \tau_{x} f_{5}-18 f_{2}\left(\tau \tau_{x x}+\tau_{x}{ }^{2}\right) \\
40 \tau f_{8}^{\prime}= & 5 f_{11}+93 \tau_{x} f_{8}+3 f_{5}\left(160 \tau \tau_{x x}-201 \tau_{x}{ }^{2}\right)  \tag{42b}\\
& +27 f_{2}\left(19 \tau_{x}{ }^{3}-16 \tau \tau_{x} \tau_{x x}-20 \tau^{2} \tau_{x x x}\right)
\end{align*}
$$

including the algebraic relations

$$
\begin{align*}
f_{4} & =\tau \tau x \\
\tau f_{6} & =f_{2} f_{5} \\
f_{7} & =\tau\left(4 \tau \tau_{x x}-\tau_{x}^{2}\right) \\
\tau f_{9} & =4\left(f_{2} f_{8}-f_{5}^{2}\right)+26 \tau_{x} f_{2} f_{5}-36 f_{2}{ }^{2}\left(\tau \tau_{x x}+\tau_{x}{ }^{2}\right)  \tag{43}\\
\tau^{2} f_{10} & =4 f_{2}\left(f_{2} f_{8}-f_{5}^{2}\right)+26 \tau_{x} f_{2}{ }^{2} f_{5}-36 f_{2}{ }^{3}\left(\tau \tau \tau_{x x}+\tau_{x}{ }^{2}\right) \\
& \ddots+\tau^{3}\left(27 \tau_{x}^{3}-24 \tau \tau_{x} \tau_{x x}+28 \tau^{2} \tau_{x x x}\right)
\end{align*}
$$

These equations are linear, with the coefficient of the derivatives $\tau$ specified as a function of $x$. The system is therefore less susceptible to numerical error, and although the matrix of coefficients still vanishes at $\tau=0$, the saddle-point singularity has been effectively removed. If the numerical integration is accurate enough to ensure that $f_{5}=0$ when $\tau$ vanishes, the solution will pass smoothly through the separation point.

The basic difference between the inverse and direct problems is that, for the pressure gradient prescribed, the unknown shear distribution is determined by a nonlinear equation that contains a saddle-type singularity at separation. For the wall shear specified, on the other hand, the pressure gradient is given by a linear equation that is much less sensitive to numerical error. This is probably also the case when the displacement thickness is prescribed (see ref. 13). The fact that most numerical evidence indicates a singularity at separation is therefore misleading because of the difficulty in numerically integrating through the saddle point. Of course, not all pressure distributions admit a regular solution (as discussed in the following section).

An interesting point is that, provided the correct numerical procedures are used, no difficulties are encountered at reattachment (see fig. 2 or 3). The reason for this is that any numerical errors made at the reattachment point are either integrated out of the downstream boundary or upstream toward separation. The direction of the flow, and therefore the differencing scheme, results in a solution process that allows integration away from the saddle point at reattachment but that requires integration into the singularity at separation.

Several numerical experiments were performed to verify these conclusions. In one set of computations, the velocity profiles at separation and in the immediate vicinity of that point were held fixed after converging the inverse calculation. For these cases, the inverse and direct procedures gave identical solutions. Similar results were obtained when an artificialviscosity term equal to $\varepsilon u_{x x}$ was introduced into the difference equations. As the coefficient $\varepsilon$ was decreased, however, the direct calculation would again diverge from the inverse solution.

## Pressure Gradient at Separation

As shown in the previous section, the existence of a regular solution requires that $f_{4}=0$ at the point of separation (see also refs. 31 and 32 ). The coefficients $f_{5}, f_{7}$, and $f_{9}$ must also vanish at the point of zero shear, and the pressure gradient must therefore satisfy certain specific conditions to permit the flow to pass smoothly through separation. The constraints on the pressure distribution cannot be determined directly because of the saddle point, but must be obtained from the inverse, or shear-specified, calculations.

It is reasonable to expect that only certain pressure distributions will admit regular solutions. The separation profile, for example, is determined by both the upstream and downstream flows so that some compatibility relation must be satisfied at this station. Also, from.kinematic considerations, the boundary-layer approximation to the vorticity transport equation is

$$
\begin{equation*}
u \frac{\partial \omega}{\partial x}+v \frac{\partial \omega}{\partial y}=\frac{\partial^{2} \omega}{\partial y^{2}} \tag{44}
\end{equation*}
$$

where $\omega=\partial u / \partial y$ and $p_{x}=\partial \omega / \partial y$ at $y=0$. The restriction on the pressure gradient at separation can thus be interpreted as a constraint on allowable boundary conditions: - the normal gradient of vorticity at the surface is required to satisfy some local condition for the vorticity to remain continuous at the singular point.

From physical considerations, a constraint on the allowable pressure gradient implies that the interaction between the inner viscous layer and the outer fluid essentially determines the conditions at separation. Prandtl (ref. 33) recognized this in 1938 when he stated that the pressure field could not be chosen arbitrarily for the flow downstream of separation "to agree with observation." Most numerical solutions of the Navier-Stokes equations, including the recent investigation by Leal (ref. 26) in particular, also indicate that, when the interaction with the outer flow is included, there is no evidence of singular behavior at separation.

Because of the nonlinearity of the boundary-layer equations, it is not possible to determine the precise pressure-gradient condition that permits a regular solution. Certain restrictions on the pressure distribution can be inferred from the Taylor series expansion and from the numerical solutions, however. The acceleration of a fluid particle near the surface, for example, can be approximated as follows:
$u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}=\frac{y^{2}}{2}\left[\tau \tau_{x}+2 \tau p_{x x} \frac{y}{3}+p_{x} p_{x x} \frac{y^{2}}{6}+\tau\left(4 \tau \tau{ }_{x x}-\tau \tau_{x}{ }^{2}\right) \frac{y^{3}}{60}+\ldots.\right]$

Immediately upstream of separation, $\tau$ and $p_{x}$ are positive and $\tau_{x}$ is negative. As $\tau \rightarrow 0$, the fluid in a stream tube near the surface continues to decelerate, and the streamlines continue to move away from the wall provided

$$
\begin{equation*}
p_{x x}\left(p_{x} \frac{y^{2}}{6}+2 \tau \frac{y}{3}\right)<\left|\tau \tau_{x}\right|+\left|\tau \tau_{x}^{2}\right| \frac{y^{3}}{60}+\ldots \tag{46}
\end{equation*}
$$

For the flow to separate smoothly, then, a restriction on the pressure field is that

$$
\begin{equation*}
\mathrm{p}_{\mathrm{xx}}<0 \quad \text { as } \quad \tau \rightarrow 0 \tag{47}
\end{equation*}
$$

There will therefore be an inflection on the pressure distribution upstream of the separation point. This requirement is consistent with experimental evidence, and the existence of a "knee" in the pressure curve is often taken to indicate boundary-layer separation.

The numerical evidence suggests that this condition is not sufficient, however. All regular solutions, in fact, satisfy the requirement:

$$
\begin{equation*}
\frac{d m}{d x} \geq 0 \quad \text { at } \quad \tau=0 \tag{48}
\end{equation*}
$$

This is a more restrictive condition than that given by equation (47) because $m_{x}$ can be negative for $p_{x x}$ negative. The linearly retarded flow considered under Direct Solutions, for example, satisfies equation (47) but not equation (48). In a series of papers, Meksyn (refs. 34 and 35) has contended that the existence of a minimum in $m_{x}$ was a necessary condition for regular separation. He cited Schubauer's (ref. 36) measurements of the flow over an elliptic cylinder as experimental verification of this requirement. Similar arguments have also been advanced as a result of the use of approximate methods to calculate supersonic viscous-inviscid interactions (see, e.g., ref. 37).

The most useful means of examining the numerical results is in the $\bar{\tau}-m$ phase space (fig. 5). Several typical computations are presented, including the locus of solutions for similar flow. In these coordinates, $x$ is a parameter that varies along the curves, with $\Delta x \rightarrow \infty$ for the similarity solutions. For this limiting curve, $d m / d \bar{\tau}$, and therefore $d m / d x$, is zero at the point of zero shear. All nonsimilar trajectories, on the other hand, have positive $m_{x}$ at both the separation and reattachment. points. This condition was never violated in approximately 30 different calculations using various specified shear distributions. Note that the locus of similar flows is sometimes taken to indicate singular behavior at separation because $\bar{\tau} \sim\left(m_{0}-m\right)^{1 / 2}$ and $d \bar{\tau} / d m \rightarrow \infty$ at $\bar{\tau}=0$. The similarity solutions are obtained for $m_{x}=0$, however, and the limiting value of $d \bar{\tau} / d x\left(=m_{x} d \bar{\tau} / d m\right)$ must be carefully determined if an actual flow is replaced by a sequence of similar flows. In any event, the condition for regular separation, that $m_{x} \geq 0$ at the point of zero shear, is satisfied by both the similar and the nonsimilar flows.


Figure 5.- Phase space representation.


Figure ó.- Comparison with similar solutions.

The phase-space representation of solutions presents an opportunity to verify the accuracy of the numerical procedure. The points labeled A and $B$ in figure 5, for example, have the same value of $\bar{\tau}$ and $m$ as a corresponding similarity solution. The left-hand side of equation (7), which is completely determined by $\bar{\tau}=f^{\prime \prime}(0)$ and $m$, is therefore zero. The local $x$ variation vanishes and the similar and nonsimilar profiles must be identical at those points. The velocity profiles calculated by the present scheme are compared to adjacent solutions of the similarity equation (obtained by fourth-order Runge-Kutta integration in ref. 38) in figure 6. There are essentially no differences in the results obtained by the two methods.

With a continuous shear distribution specified, the solution is constrained to be regular at both separation and reattachment. This result can be verified by comparing the calculated streamline pattern with the local solution of the Navier-Stokes equations obtained by Oswatitsch (ref. 39) (see also Dean (ref. 40) and Legendre (ref. 41)). At the point of zero shear, a regular solution of the Navier-Stokes equations requires that the angle of the dividing streamline be proportional to the ratio of the $x$ derivative of the shear and the pressure gradient. In the transformed variables, the precise condition is

$$
\begin{equation*}
\sqrt{R} \tan \theta=-3\left(\sqrt{\frac{x}{u_{e}}} \frac{\bar{\tau}_{x}}{m}\right)_{\bar{\tau}=0} \tag{49}
\end{equation*}
$$

where $\theta$ is the angle of the dividing streamline. For a prescribed shear distribution, the calculated values of $m$ can be integrated in $x$ to obtain $u_{e}$. The flow in the vicinity of separation and reattachment for a refinedmesh calculation ( $\Delta x=\Delta y=0.1$ ) is compared with equation (49) in figure 7. The calculated results agree exactly with the local Navier-Stokes solution at
the point of zero shear, again demonstrating that the boundary-layer solution is regular.



Figure 7.- Detailed flow field in the vicinity of $\tau=0$.

## Flow-Field Solutions

As previously mentioned, a number of different shear distributions were specified in an effort to determine the behavior of the boundary-layer equations in separated flow. Some of those results are presented in this section and the following one. Figure 8, for example, shows the streamlines and skinfriction variation, in physical coordinates, for a typical parabolic shear distribution. The relation between



Figure 8.- Streamlines for specified shear distribution. the physical and transformed variables is

$$
C_{f} \sqrt{R}=2 u_{e} \sqrt{\frac{u_{e}}{x}} \bar{\tau}
$$

and

$$
\begin{equation*}
\left.\psi \sqrt{\mathrm{R}}=\sqrt{x u}{ }_{e} \int_{0}^{\bar{y}} \bar{u} d \bar{y}\right\} \tag{50}
\end{equation*}
$$

In figure 9, the skin friction and streamline patterns for a different shear distribution are shown. For this case, the maximum reversed flow


Figure 9.- Streamlines for specified shear distribution.


Figure 10.- Velocity profiles for trailing-edge flow.
occurs toward the reattachment side of the separation bubble. The dividing streamline has several rapid changes in slope, and this solution would be difficult to obtain if it were necessary to explicitly iterate for the location of the $u=0$ line. Note that in all cases the normal coordinate is multiplied by the square root of the Reynolds number and that these solutions represent shallow separated regions confined to the interior of the viscous layer.

The present method can also be used to calculate flows where reattachment occurs in a wake rather than on a solid surface. The details of this type of flow field in the immediate vicinity of the trailing edge are shown in figure 10 . Here, the transition from boundary-layer flow to wake flow is assumed to occur on a scale that is small compared to the thickness of the viscous layer (see ref. 42). The prescribed boundary conditions of zero velocity and negative wall shear were thus discontinuously changed to zero shear and specified reversed-flow velocity at the trailing edge. Based on order-of-magnitude considerations, the initial reversed-flow velocity was taken to be equal to the value of the wall shear at the joining point. No attempt was made to ensure continuity of the dividing streamline or displacement thickness, although mass and momentum are conserved in the solution to the differential equations.

## Indications of Breakdown

In the previous sections, it was demonstrated that the boundary-layer equations have regular solutions at separation and reattachment. The flow structure at the separation point agrees with the limiting form of the Navier-Stokes equations, and the Goldstein solution does not appear to be relevant for real flows. The square-root singularity in the boundary-layer equations is a consequence of specifying an external pressure distribution based on an inviscid solution determined as though there were no separation. In practice, the pressure gradient is locally modified near the separation point such that the boundary-layer solution remains regular. The question that arises then concerns the manner in which the boundary-layer equations
eventually break down. Real flows tend to separate toward the rear of a closed body and vorticity is transported into the outer fluid. In some cases, the vorticity is confined to a relatively narrow region, or wake, downstream of the body. In other situations, behind a circular cylinder, for example, a large region of the fluid becomes rotational. The vorticity is no longer restricted to a thin viscous layer and the normal component of velocity ceases to be small compared with the tangential component. In the present investigation, the region of separated flow is, of course, constrained to remain close to the surface, inside a layer of order $1 / \sqrt{\mathrm{R}}$. The numerical solutions may, however, suggest when this approximation is no longer realistic.


Figure 11.- Evidence of weak solutions for highly separated flow.


Figure 12.- Normal velocity distribution for nighly separated flow.

An indication of the possible breakdown in the boundary-layer equations is shown in figure 11 for a highly separated flow. As the mesh is refined, the computed values of $m$ appear to become discontinuous at a point downstream of separation. Apparently, there are two solutions, one associated with separation and the other with reattachment, that are joined in the reversed-flow region.

The distribution of $\bar{v}_{e}$, the transformed normal velocity, is shown on an expanded scale in figure 12. The normal velocities increase rapidly downstream of the separation point, and the viscous layer begins to break away from the surface. Because of constraints imposed by the boundary conditions, however, a discontinuity in $\bar{v}$ (and in $\partial \bar{u} / \partial x$ ) occurs at the maximum value of $\bar{v}_{\mathrm{e}}$, and the remaining solution is continuous. Although there is a certain degree of smoothing in the numerical results, the discontinuity in $\overline{\mathrm{v}}_{\mathrm{e}}$ is evident in figure 12. A jump in $\bar{v}$ is an allowable weak solution of the differential equations and is apparently required for certain boundary conditions (e.g., large negative shears). If strong discontinuities occur when the shear distribution corresponding to a real flow is prescribed, however, this can be taken to indicate the breakdown of the boundary-layer assumptions.


Figure 13.- Effect of shear variation in separated region.


Figure 14.- Effect of shear variation in vicinity of reattachment.

The rapid variation of $m, \bar{v}_{e}$, and of the other flow quantities depends on the amount of reversed flow. This is illustrated in figure 13 for a sequence of solutions where the specified shear distribution was modified in the separated region. As the values of the shear become less negative, the solutions become increasingly smooth and continuous. The streamlines corresponding to $\alpha=0.1$ were previously shown in figure 8. Even for this relatively mild case, the separating flow appears to undergo a rapid transition to the reattaching portion of the flow field at $x=2.7$ approximately.

The results of an additional numerical experiment are shown in figure 14. For this case, the wall shear was varied only in the downstream portion of the separated zone and kept constant elsewhere. The nonlinearity and upstream influence of the boundary-layer equations is evident in the computed distributions of $m$ and $\bar{v}_{e}$. Note also, however, that the flow in the immediate vicinity of separation ( $x<2.5$ ) is not significantly affected by relatively large changes near reattachment.

## Upstream Influence

Part of the success of approximate methods that use forwardmarching schemes (e.g., refs. 9 and 13) may be related to the limited upstream influence discussed above, particularly for flows with small separated zones. For the cases shown in figure 14, of course, it would not be possible to obtain accurate solutions downstream of $x=2.5$ without including the boundary conditions at reattachment. To investigate this question, calculations were made with the convective term
$\bar{u} \bar{u}_{x}$ set to zero for $\bar{u} \leq 0$ with both the first- and second-order-accurate difference schemes used in a marching mode. Only backward differencing was employed for both momentum and continuity, and the equations were completely relaxed at each $x$ station before proceeding.


Figure 15.- Comparison with forwardmarching procedure.

The three point backward second-order scheme could be marched accurately a short distance into the separated zone. It always diverged rapidly, however, at approximately the location where $m_{x}$ became negative. The first-order scheme, with moderate grid spacing, could be used for small bubbles but diverged for more separated flows. A typical calculation for a mildly separated flow is compared in figure 15 with an "exact" solution obtained using the correctly differenced secondorder scheme with smaller step size. As the grid spacing was refined in $x$, the first-order marching began to diverge from the correct solution. The instability could be delayed by keeping $\Delta y \leq \Delta x$ and by accepting a less stringent iterative convergence criterion at each $x$ station, but overall, the difference equations failed to converge to a solution as the grid was refined.

This experiment indicates that backward differencing, even with $\bar{u} \bar{u}_{X}=0$ for $\bar{u} \leq 0$, is always unstable. For mild separation, the eigenvalues in the unstable range are small and dominate the numerical calculation only after a sufficient number of steps is taken. It is probable that the schemes of references 9 through 11 and 13 are also divergent, although they are useful for certain applications.

To determine the effect of neglecting the upstream convection of momentum, additional calculations were performed with the term $\bar{u} \bar{u}_{x}$ set to zero for $\overline{\mathrm{u}} \leq 0$, but with the term $\overline{\mathrm{u}}_{\mathrm{x}}$ in the continuity equation centrally differenced. In this manner, upstream influence is retained and the solution must again be obtained by relaxation methods. The results were essentially identical to the exact second-order solution, verifying that the upstream convection of momentum is not significant for laminar flows with limited separated regions.

## POSSIBLE EXTENSIONS

An important extension of the present method is to match an inner, boundary-layer solution to an outer inviscid flow to calculate complete viscous-inviscid interactions. It would also be useful to compare results of the present method to experimental measurements of laminar separating and reattaching flows. Because low-speed boundary layers rarely remain laminar through reattachment, the computations must be extended to supersonic flows.

There are, for example, a number of reliable experiments for compressioncorner interactions at supersonic speeds, as well as several different approximate solutions and Navier-Stokes calculations available for comparison (e.g., refs. 43 and 44). It is indicated below how the method can be adapted to compressible flows, and an integral relation is proposed that offers promise of allowing the treatment of complete viscous-inviscid interactions.

## Compressible Flows

To apply the method to compressible boundary layers, the following transformation can be used:

$$
\begin{align*}
& \bar{y}=\sqrt{\frac{\rho_{e} u_{e}}{\mu_{e}{ }^{x}}} \int_{0}^{y} \frac{\rho}{\rho_{e}} d y  \tag{51}\\
& \bar{v}=\sqrt{\frac{x}{\rho_{e^{\mu} e^{u}}}} \rho v+x \bar{u}\left(\frac{\partial \bar{y}}{\partial x}\right)_{y}
\end{align*}
$$

If it is assumed that the density-viscosity product is constant through the layer, the following equations result:

$$
\left.\begin{array}{c}
x \bar{u} \frac{\partial \bar{u}}{\partial x}+\bar{v} \frac{\partial \bar{u}}{\partial \bar{y}}=\bar{m}\left(1-\bar{u}^{2}\right)+\frac{\partial^{2} \bar{u}}{\partial \bar{y}^{2}} \\
x \frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial \bar{y}}+\frac{\bar{u}}{2}\left(1+\left\{\frac{1-\gamma M_{e}^{2}}{1+[(\gamma-1) / 2] M_{e}^{2}}\right) \bar{m}=0\right) \tag{52}
\end{array}\right\}
$$

where

$$
\bar{m}=\frac{x}{M_{e}} \frac{d M_{e}}{d x} \quad \text { and } \quad M_{e}=\frac{u_{e}}{a_{e}}
$$

These equations can then be solved in exactly the same fashion as equations (6a) and (6b), with $M_{e}$ calculated by integrating $\bar{m}$.

## Viscous-Inviscid Interations

The solution for a complete interaction is complicated by the fact that $\bar{\tau}$ is specified. The following integral relation can, however, be used:

$$
\begin{align*}
\bar{\tau}+\sqrt{\operatorname{Re}_{x}} \frac{v_{e}}{u_{e}}= & \left(1+\frac{\gamma-1}{2} M_{e}^{2}\right)\left(x \frac{d \lambda}{d x}+\frac{\lambda}{2}\right)+\left[\left(1+\gamma M_{e}^{2}\right) \frac{\lambda}{2}\right. \\
& \left.+\frac{M_{e}^{2}-1}{1+[(\gamma-1) / 2] M_{e}^{2}} \int_{0}^{\overline{y_{e}}} \bar{u}^{2} d \bar{y}\right] \bar{m} \tag{53}
\end{align*}
$$

For an assumed $\bar{\tau}$ distribution, the solution of equations (52) gives calculated values of $\bar{m}$ and hence of $M_{e}$ and $p_{e}$. Using an inverse inviscid procedure, the distribution $M_{e}$ can be specified to obtain a new effective body shape, that is, the streamline slopes $v_{e} / u_{e}$. Then, from equation (53), a new estimate for $\bar{\tau}$ can be determined and the procedure continued until convergence is achieved. Based on recent experience with an integral scheme (ref. 8), it will probably not be advantageous to precisely match the intermediate iterations for $v_{e} / u_{e}$.

It would, of course, be easier to specify $v_{e}$ directly for the viscous solution. For similar flows, an efficient scheme was developed by differentiating the continuity equation with respect to $\bar{y}$ and using standard secondorder central differencing for $\bar{v}$. The value of $m$ was then updated by evaluating the continuity equation at the edge of the layer. This approach failed, however, for the complete boundary-layer equations with separated regions and was much slower for attached flows than the $\bar{\tau}$ specified schemes. An alternate approach, perhaps using the vorticity equation, may be required. All analytical and numerical evidence indicates, however, that the wall shear is the optimum boundary condition for calculating separated flows.

## CONCLUDING REMARKS

The numerical procedures developed in the investigation provide an exact means for integrating the boundary-layer equations through separation and reattachment. The approach appears to be adaptable to the treatment of complete viscous-inviscid interactions for flow fields where the boundary layer remains confined to a narrow region: compression-corner flows or separation at the trailing edge of a streamlined body, for example. The method may also prove useful in evaluating different turbulence models for separated flows. As compared to complete Navier-Stokes solutions, the present approach allows an order-of-magnitude better resolution of the viscous region and requires considerably less computation time. Finally, a method based on the boundarylayer equations provides the most promising means for investigating the important problem of three-dimensional flow separation.

## Ames Research Center

National Aeronautics and Space Administration Moffett Field, Calif., 94035, March 8, 1974

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## APPENDIX

A program listing for the point-relaxation version of the first method is included. Only a description of the input and output is given; however, program variable names are the same as used in the text and should be selfexplanatory. No effort was made to optimize the code or even to use a very efficient procedure for solving the attached region of the flow. A solution case corresponding to $\alpha=0.1$ in figure 13 is included.

INPUT PARAMETERS
(Subroutine INIT)

```
JMAX = maximum number of points in x, 3\leq JMAX \leq 120
KMAX = maximum number of points in y, 4\leq KMAX \leq 100
DX = \Deltax
DY = \Deltay
XO = x-location of initial profile
UEXO = ue at XO
SMO =m at XO
ALPHU, ALPHV, ALPHM - relaxation parameters to update u, v, m
```

(Subroutine PROFL)

```
DYO = \Deltay in which initial profile is given
KMAXO = number of data points to specify initial profile
U(K),V(K) = u and v of initial profile
```

(Main)
ITERM = maximum number of iterations permitted
RMAX = calculation is terminated if the maximum residual exceeds RMAX
RMIN = residual at which iteration ceases and the converged solution is printed
ALPHM2 = after an initial number of iterations, ALPHM is reset to this value ADDAL $=$ increment to ALPHU and ALPHV after an initial number of iterations

Wall Shear is analytically input in the present program.

## OUTPUT

- The input parameters and the initial profile are printed.
- Minimum output from a marching routine that calculates the attached flow region is printed.
- Maximum residuals and their locations are printed every 10 iterations.
- The basic solution as a function of $x$ is printed; data include $j, x, m$, $u_{e}, \bar{v}_{e}$, and $\bar{\tau}$.
- The solution profiles are printed at each $x$ station. Data include $k, \bar{y}$, $\bar{u}, \bar{v}, y, u, v$, and $\psi / \sqrt{R}$ and interpolated values of $y$ at constant values of $\psi / \bar{R}$.

PROGRAM LISTING
AND
CASE RUN

```
    PROGRAM FLOSEP (INPUT,OUTPUT,TAPESEINPUT, TAPEG=OUTPUT)
C
MAIN PROGRAM
AN ITERATIVE FINITE=DIFFERENCE METHOD FOR INTEGRATING THE
LAMINAR INCOMPRESSIBLE BOUNDARYGLAYER EQUATIONS THROUGH
SEPARATION AND REATTACHMENT
    COMMON SM(120), SMC(120),U(120,100),V(120,100),XB(120)
    COMMON /PARAM/ JMAX , KMAX , DX , DY, UEXO,
    1 ALPHU , ALPHV , ALPHM , RMAX, RMIN
        COMMON/RESID/ ITER , JREST , KREST , RESU, RESV,JU, KU
C
    DIMENSION TAU(120)
C
C CONSIANTS
        ITERM = 1000
        RMAX=10.
        RMIN = 0.00005
        ALPHMZ = 0.5
        ADDAL =0.08
C
            INITIALIZATION
        CALL INIT
        SY2 = 1./(DY*OY)
C
    INPUT WALL SHEAR INTO TAU(J) ARRAY
                EXAMPLEE CASE
            ALF=0.1
            TO = 0.33238/12.
            DO 10 j=1,JMAX
            x2 = XB(J)-2.
            x6 = x2-4.
            IF(x2) 9,9,5
            5 IF(x6) 7,9,9
            7TAU(J)=TO*X2*X6*(1.*ALF*X2*X6)
            GO TO 10
            9 TAU(J) = TO*X2*X6
10 CONTINUE
            INITIALIZATION COMPLETE
MARCHING IN ATTACHED FLOW HEGIONS
    CALL MARCH (J1, J2, TAU)
```

IF (J1-J2) 12,50,50
12 CONTINUE
RELAXATION PART
WRITE $(6,500)$
C
ITER $=0$
15 CONTINUE
RMTST $=0.0$
JRM $=1$
C
UPDATE M, EQ. 31
DO 25 JEJi.J2
$R M=S M(J)+S Y 2 *(4 . * U(J, 2)=0,5 * U(J, 3)=3$. $\# D Y * T A U(J))$
SM(J) $=$ SM(J)-ALPHM*RM
$S M C(J)=0.5 *(S M(J)+1.0)$
RMP $\boldsymbol{I}$ ABS(RM)
IF (RMP-RMTST) 25,24.24
24 RESM $\Rightarrow$ RM
RMTST : RMP
$J K M=J$
25 CONTINUE
30 CONTINUE
CALL RELAXATION ROUTINE, MEYHOD ONE
ITER = ITER +1
CALL RELAX (J1. J2)
IF (ITER - (ITER/IO)*10) 32,31,32
PRINT MAXIMUM RESIDUALS AND LOCATION EVERY 10 ITERATIONS.

31 CONTINUE
WRITE (6,501) ITER,RESV,JREST,KREST,RESU,JU,KU, RESM,JRM
32 REST $=A B S(R E S V)$
IF (ITER - 200) 38,34,38
C
CHANGE RELAXATION PaRAMETER AFTER 200 ITER
34 ALPHM $=$ ALPHM2
$A L P H U=A L P H U+A D D A L$
ALPHV = ALPHV + $A D D A L$
IF (ALPHV - 1.0) 36,35,35
35 CONTINLE
ALPHU $=0.98$
ALPHV $=0.98$
36 CONTINUE
WRITE $(0,508)$ ALPHV, ALPHU, ALPHM
C TEST WHETHER TO TERMINATE CALCULATION
38 IF (REST-RMAX) $40,40,80$

```
    40 IF (ITER - ITERM) 45,45,42
    4 2 ~ C O N T I N U E ~
        WHITE (0,502)
        GO TO 50
    45 IF (REST - RMIN) 46,46,15
    4 6 ~ C O N T I N U E ~
    WRITE (6,507)
C
C
c
80 CONTINUE STOP
500 FORMATEIHO, \(35 \times 22\) HRELAXATION CALCULATION //
1 7X,4HITER,5X,5HRES V,7X,7HJV KV,6X,5HRES U, \(7 X\).
27 7HJU KU.6X,5HRES M,7X,2HJM)
501 FORMAT (5X, I5,3(E15,5,2I5))
502 FORMAT (34HO MAXIMUM ITERATIONS COMPUTED....)
507 FORMAT (25HO CONVERGED SOLUTION.....)
508 FORMAT (1HO, 17 HALPHV,ALPHU,ALPHM, 3F13.5)
END
\(c\)
C
```



```
OIMENSION UI (100), VI(100)
READ (5,500) JMAX,KMAX, DX, DY, XO, UEXO, SMO
READ (5,501) ALPHU, ALPHV, ALPHM
WRITE (6,505) JMAX, DX, XO, ALPHV, KMAX, OY, UEXO, ALPHU,SMO,ALPHM
INTERPOLATION OF INITIAL PROFILE IF NEEDED
CALL PROFL (KMAX, OY, Ui, VI)
DO 30 JE1,JMAX
\(X B(J)=X O+(J-1) \oplus D X\)
SM(J) \(=\) SMO
SMC(J) \(=0.5 *(5 M O+1\).
DO \(20 \mathrm{~K}=1, \mathrm{KMAX}\)
\(U(J, K) \equiv U I(K)\)
\(V(J, K)=V i(K)\)
20 CONTINUE
30 CONTINUE
```

C

RETURN
500 FORMAT(215,5F10.0)
501 FORMAT ( 8 F10.0)
505 FORMAT(1HI,35X,12HINPUT VALUES//
$14 X, 6$ HJMAX $=, 15,4 X, 4$ HOX $=, F 10,5,6 X, 4 H \times O=, F 10,5,4 X, 7 H A L P A Y=, F 10,5 /$
$24 X$, HHKMAX $=, 15,4 X, 4$ HOY $=, F 10,5,4 X, 6$ HUEXO $=, F 10,5,4 X, 7$ HALPHU $=$,
3F10.5/ 39X,4HMO =,F10.5,4X,7HALPHM =,F10.5)
END
c
SUBROUTINE PROFL (KMAXI, OYI, U1, V1)
C
INTERPOLATION OF INITIAL PROFILE.
DIMENSION $Y(00), U(60), V(60), C(5), S(5), T(5)$,
1 Yi(100), U1(100), Vi(100)
INT $=2$
C INPUT INITIAL PROFILE
READ (5,501) DYO,KMAXO
WRITE $(6,510)$
DO $2 \mathrm{~K}=1, \mathrm{KMAXO}$
$\operatorname{READ}(5,500) \cup(K), V(K)$
$Y(K)=(K-1) * D Y O$
WRITE ( 6,511 ) $K, Y(K), U(K), V(K)$
2 cONTINUE
IF (KMAX1 = KMAXO) 30,30,3
3 CONTINUE
ksave $=1$
DO $20 \mathrm{~K} 1=1$, KMAX1
$Y_{1}(K 1)=(K 1-1) * O Y 1$
4 DO 5 KaKSAVE, KMAXO
$k K=k$
IF $(Y(x)-Y i(K 1)) 5,5,6$
5 CONTINUE
6 1F (KSAVE-1) $9,9,7$
7 IF $(Y(K K=1)=Y 1(K 1))$ 9,9,8
8 KSAYE $=1$
GO TO 4
$9 K K=K K-(1 N T+1) / 2$
IF(KK). $10,10,11$
$10 \mathrm{KK}=1$
GO TO 13
11 $M=K K+1 N T$
IF (M-KMAXO) 13,13,12
$12 \mathrm{Kk}=\mathrm{Kk}-1$
GO TO 11
13 INTI = INT+1
kSAVE = KK
DO $14 \mathrm{~L}=1$,INT1
$C(L)=Y 1(K 1)-Y(K K)$
$S(L)=U(K K)$
$T(L)=V(K K)$
$14 k K=k k+1$

```
        DO 10 KK=1,INT
        I=Kk+1
    15D=C(KK)-C(I)
        S(I) E (C(KK)*S(I)-C(I)*S(KK))/0
        T(I) = (C(KK)*T(I)-C(I)*T(KK))/D
        I= I+1
        IF(ImINT1) 15,15,16
    16 CONTINUE
        UI(K1) =S(INT1)
        VI(K1) = T(INTI)
    20 CONTINUE
        RETUKN
C NO INTERHOLATION
    30 DO 31 K=1,KMAXI
        UI(K) = U(K)
        VI(K)=V(K)
    31 CONTINUE
        RETURN
    500 FORMAT(8F10.0)
    501 FORMAT (F10.0,I5)
    510 FORMAT(IHO,15X,14HINITIAL VALUES//4X,IHK,7X,1HY,11X,1HU,11IX,IHV)
    511 FORMAT(2X,13,3F12,6)
    END
C
C
        COMMON SM(120),SMC(120),U(120,100),V(120,100),XB(120)
        COMMON /PARAM/ JMAX , KMAX , DX , DY , UEXO .
        1 ALPHU , ALPHV , ALPHM ,RMAX, RMIN
        DIMENSION TAU(120), UX(100)
        TAUWT = 0.02
        ALPMMZ = . 5
        SY = .5*DY
        SYY = DY*DY
        SX2 =.5/DX
        SYZ = 1./(DY*DY)
    C
        N=2
        JINT = -1
        KM = KMAX=1
        IF(TAU(3) - TAUWT) 50,50,5
    C
        5N=N+1
            J = N
            J! = J + JINT
            J2 = J
            JINT = 0
```

IF $(J \cdot J M A X) 6,6,50$
TEST TO SEE IF PROFILE IS ATTACHED
6 IF(TAU(J) - TAUWT) $50,50,7$
7 CONTINUE
C OBTAIN GOOD GUESS BY USING EXTRAPOLATION OF LAST COMPUTED PROFILES

C

C
10 ITER $=$ ITER +1
DO $20 \mathrm{~J}=\mathrm{J} 1, \mathrm{~J} 2$
$J R=J=1$
$J R R=j-2$
REST = 0 。
$R M=S M(J)+S Y 2 *(4 * * U(J, 2)=0.5 * U(J, 3)=3 . * D Y * T A U(J))$
SM(J) $=S M(J)-A L P H M \oplus R M$
$S M C(J)=0.5 *(S M(J)+1,0)$
$0018 \mathrm{~K}=2, \mathrm{KM}$
$K R=K-1$
$K P=K+1$
C
IF $(J-2) 12,12,13$
$12 U X(K)=S \times 2 *(U(J+1, K)=U(J R, K))$
$U 2 X=U X(K) *(U(J+1, K)+U(J R, K))$
DIAX $=0$.
GOTO 14
$13 U X(K)=s \times 2 *(3 . * U(J, K)-4, * U(J R, K)+U(J R R, K))$ $U 2 X=S \times 2 *(3 * * U(J, K) * * 2-4, * U(J R, K) * * 2+U(J R R, K) * * 2)$
DIAX $=3, * \times B(J) * S X 2$
14 CONTINUE
$U Y=S Y *(U(J, K P)=U(J, K R))$
$F U=U Y * V(J, K)+S Y Y *(0,5 \star \times B(J) * U 2 X=S M(J) *(1,-U(J, K) * * 2))$
$R U=U(J, K R)-2, \star U(J, K)+U(J, K P)=F U$
$R V=V(J, K)-V(J, K R)+S Y *(X B(J) *(U X(K)+U X(K R))+S M C(J) *(U(J, K)+$
1 U(J,KR) J)
OT $=2,+S Y Y$ U(S,K)*CIAX
DU $=$ RU/DT
$D V=-R V$
$U(J, K)=U(J, K)+D U$
$V(J, K)=V(J, K)+O V$
$R T=A B S(R V)$
IF(RT-REST) $18,18,15$
15 REST = RT
18 CONTINUE
20 CONTINUE
IF (ITER - 20) 20,26,25

```
    25 ALPHM = ALPHMZ
    26 CONTINUE
    IF( REST = RMAX) 27,100,100
    27 IF( REST - RMIN) 30,30,28
    28 IF( ITER = 600) 10,100,100
    30 CONTINUE
    IF(N-3) 35,35,40
    35 WRITE(6,501)
    I2 = 0
    RZ = 0,
    DO 30 J = 1,2
    V(J,KMAX) = V(J,KM) - DY*SMC(J)
    TAUW = .5*( - 3,*U(J,1) +4.*U(J,2) -U(J,3))/DY
    WRITE(6,500) J,IZ,RZ,V(J,KMAX),TAUW,SM(J)
    36 CONTINUE
    40 J = N
        TAUW=.5*( - 3.*U(J,1) +4.*U(J,2) - U(J,3))/DY
        V(J,KMAX) = V(J,KM) - DY*SMC(J)
        WRITE(6,500) J,ITER,REST,V(J,KMAX),TAUW,SM(J)
        GO TO 5
c
    50 JI = N
    J2 = JMAX
    RETURN
    100 J = N
    TAUW = 5*( - 3.*U(J,1) +4.*U(J,2) -U(J,3))/OY
    V(J,KMAX) = V(J,KM) - DY*SMC(J)
    WRITE(6,500) J,ITER,REST,V(J,KMAX),TAUW,SM(J)
    STOP
    500 FORMAT(1H ,215,4F13,5)
    501 FORMAT(1H1, 35XI&HMARCHING PROCEDURE // 5X1HJ, 2X4HITER,
    I 7X5HRES V, 8X4HVMAX, 10X3HTAU,10X1HM J
    END
    =.015
    EPS2 = 0.005
    KM = KMAX -1
C
    EPS = EPS1-EPS2
    RESU = 0.
```

```
RESV = 0.0
RTEST = 0.
REST = 0.0
SY = 0.5*DY
SYY = DY*DY
SX2 = 0.5/DX
00 50 Jaj1.J2
JR = J-I
JP = J+1
Ux(1) = 0.0
DO 30 KE 2,KM
KR = K-1
KP = K&1
10 UX(K)=SX2*(U(JP,K)-U(JR,K))
    IF(J-2) 11,11,12
    11 U2X = sx2*(U(JP,K)**2*U(JR,K)**2)
        DIAX = 0.
        GO TO 20
    12T=U(J,K)
    IF(T-EPS1) 16,16,14
13 UX(K) a Sx2*(3.*U(J,K)=4.*U(JR,K)+U(JR-1,K))
14 JG = J-2
    U2x=Sx2*(U(JO,K)**2*4.*U(JR,K)**2*3,*U(J,K)**2)
    DIAX = 3,*XB(J)*SX2
    GO TO 20
    SEPARATED FLOW
16 U2x = Sx2*(U(JP,K)**2-U(JR,K)**2)
    OIAX = 0,
    IF(T*EPS2) 20,20,18
    SEPARATION POINT
    REATtachmENT POINT
18 JG= J-2
    U2P = U2X
    U2x= sx2*(U(JQ,K)**2-4,*U(JR,K)**2*3.*U(J,K)**2)
    TA = (T-EPS2)/EPS
    U2X = TA*U2X+(1.-TA)*U2P
    DIAX = 3.*TA*XB(J)*SX2
c
20 continue
    UY = SY*(U(J,KP)=U(J,KR))
    FU=UY*V(J,K)+SYY*(O,5*XB(J)*UZX-SM(J)*(1,-U(J,K)**2))
    RU = U(S,KR)-2,*U(J,K)+U(P,KP)-FU
    RV = V(J,K)=V(J,KR)*SY*(XB(J)* (UX(K)+UX(KR))+SMC(J)*(U(J,K)+
    1 U(J,KR)))
    DT = 2. + SYY*U(J,K)*DIAX
```

```
    DU = RU/DT
    OV = - RV
    U(J,K)=U(J,K)+ALPHU*DU
    V(J,K) = V(J,K) +ALPHV*DV
    RT = ABS(RU)
    IF(RT=RTEST) 22,22,21
    21 RESU = RU
        RTEST = RT
        JU = J
        KU = K
    22 CONTINUE
        RTT = ABS(RV)
        IF(RTT=REST) 27,27,26
    26 RESV = RV
        REST = RTT
        JREST = J
        KREST = K
        IF(REST-RMAX) 27,27.100
    27 continue
    30 CuNTINUE
        K = KMAX
        RV = V(J,KMAX)=V(J,KM)+SY*(XB(J)*UX(KM)+SMC(J)*(1,+U(J,KM)))
        V(J,KMAX) = V(J,KMAX)=ALPHV*RV
        RTT = ABS(RV)
        IF(RTT-REST) 35,35,34
    34 RESV = RV
        REST = RTT
        JREST = J
        KREST = K
        35 CONTINUE
        5O CONTINUE
    100 RETURN
        END
c
                OUTPUT SUBROUTINE
        COMMON SM(120), SMC(120).U(120.100),V(120,100),XB(120)
        COMMON /PARAM/ JMAX , KMAX , DX , DY , UEXO ,
    1
        COMMON /RESID/ ITER ', JPREST ', KREST, 'RMESU', RMESV,JU, KU
        COMMON /STRM, PSI(100), YST(100), POUT(60), YI(60), Y2(60),
    1 K1M , K2M
        DIMENSION UY(120), UE(120), F(120), FX(120), UST(100),
    1
            VST(100), ETA(100)
        HY}=0.5/D
        KEND = KMAX
        Sx = 0.5*Dx
        DO 10 J=1,JMax
        Ur(J) = HY*(-3.*U(J,1)+4.*U(J,2)-U(J,3))
    10 cUNTINUE
```

```
        REST = RESV
        WRITE(6,500) ITER,JREST,KREST,REST
C
INTEGRATION FOR UE(X)
    UE(I) = UEXO
    F(1)=ALOG(UEXO)
    IF(XB(1) =.00001) 12,13,13
    12 XB(1) =.00001
    13 CONTINUE
    FX(1) = SM(1)/XB(1)
    DO 15 J=2,JMAX
    FX(J) = SM(J)/XB(J)
    F(J)=F(J-1)+SX*(FX(J)+FX(J-1))
    UE(J) = EXP(F(J))
I5 CONTINUE
WRITE (6,502)
WRITE (6,503) (J,XB(J),SM(J),UE(J),V(J,KMAX),UY(J),J=1,JMAX)
DO 20 K=1,KMAX
ETA(K) E (K-1)*DY
2O CONTINUE
    YST(1) = 0.
    VST(1) =0.
    PSI(1)=0.
    DO 50 J=1.JMAX
    Cl = SGRT(XB(J)/UE(J))
    C2 = 1./C1
    C3 = SQRT(XB(J)*UE(J))
    Si = 0.5*(SM(J)-1.)
    S2 = 0.5*DY*C3
    UST(1)=UE(J)*U(J,1)
    DO 30 K=2,KMAX
    Y E ETA(K)
    YST(K) = Cl*Y
    UST(K)=U(J,K)*UE(J)
    VST(K) = (V(J,K)-SI*Y*U(J,K))*CZ
    PSI(K)=PSI(K-1)*SZ*(U(J,K)*U(J,K*I))
    30 CONTINUE
    OSTR = YST(KMAX)-PSI(KMAX)/UE(J)
    WRITE(6,510) XB(J),DSTR
    CALG STREAM (KEND)
        WRITE (6,511)
        IF (KIM ,EQ, O) GO TO 40
        WRITE (6,512) (K,ETA(K),U(J,K),V(J,K),YST(K),UST(K), VST(K),
    1 PSI(K), POUT(K), YI(K), YZ(K), K=1,K\M)
        KIM - NUMBER OF POINTS WITH SEPARATION
40 CONTINUE
```

```
        K2 = KIM + 1
        WRITE (6,514) (K,ETA(K),U(J,K),V(J,K),YST(K),UST(K), VST(K),
    1 PSI(K), POUT(K), Y1(K), K=KZ,K2M)
C
        K3 = K2M + 1
        WRITE (6,513) (K,ETA(K),U(J,K),V(J,K),YST(K),UST(K), VST(K),
    1 PSI(K) , K=K3,KMAX)
        5O CONTINUE
        RETURN
    500 FORMAT(9HI ITER =,I5,9H JRESV =,I5,9H KRESV =,I5,8H RESV E,
    1 F13.5
    502 FORMAT(SHO J,GX,4HX(J),8X,2HSM,9X,6HU EDGE,7X,GHV EDGE,6X,
    & 1OHDU/DY WALL)
    503 FORMAT(2X,I3,F10,3,4F13.6)
    510 FORMAT(IHI,GHX(J) = F!2.5,2XBHDELSTR = F12.6 )
    51: FORMAT(102X,12HINTERPOLATED/
    13X,1HK,4X,3HETA,9X,1HU,10X,1HV, 15X,1HY,11X,3HUST,9X,3HVST,9X,
    2 3HPSI,15X,3HPSI,7X,1HY,9X,1HY)
    512 FORMAT(I4,F8,3,2F12,6,4X,4F12.6,10X,F7,4,2F10,5)
    513 FORMAT(I4,F8,3,2F12,6,4X,4F12,6)
    514 FORMAT(I4,F8,3,2F12,6,4X,4F:2,6,10X,F7,4,F10,5)
    END
C
C
    COMMON/STRM/ PSI(100), YST(100), POUT(60), YI(60), YZ(00),
    DIMENSION C(4) ,S(4)
C
C
    PSMIN = -0.10
    PSMAX = 10.0
    NO = - 200.*PSMIN
    NO = NO&1
    N1 = NO+9
    N2=N1+10
    N3=N2+10
    DO 8 N=1,60
    IF(N-N3) 2,1,1
    1POUT(N) = 6+N-N3
    NMAX =N
    IF(POUT(N)-PSMAX) 8,9,9
    2 IF(N-N2) 4,3,3
    3 POUT(N) = 0.5*(N-N2+1)
    GO TO 8
```

```
    4 IF(N-NI) 6,6,5
    5 POUT(N) = 0.05*(N-N1)
        GO TO 6
    6POUT(N) = 0.005*(N-NO)
    8 CONTINUE
    9 ~ C O N T I N U E ~
C
C*****FIND MINIMUM PSI
    INT =2
        OO 10 K=2,KMAX
        KK =K
        IF(PSI(KK)=PSI(KK-1)) 10,10,11
    10 CONTINUE
    11 KPMIN = KK=1
        IF(KPMIN-INT) 12,12,15
    12 NMIN = NO
        KPMIN =1
        GO PO 40
    15 PMIN = PSI(KPMIN)
C
C*****FIND INITIAL PRINTOUT VALUE
    DO 17 N=1,NMAX
        NN E N
        IF(POUT(N)-PMIN) 17,17,18
    17 CONTINUE
    18 NMIN = NN
    NI = NO-NMIN
C
C*****INTERPOLATION...PSI FROM WALL TOU = O
    DO 30 LEI,N1
    NN=NO-L
    DO 2O K=I,KPMIN
    KK = K
    IF(POUT(NN)*PSI(KK)) 20,20,21
    20 CONTINUE
    21 KK = KK*(INT+1)/2
    IF(KK) 22,22,23
    22 KK = 1
    GO TO 25
    23M=KK+INT
        IF(M-KPMIN) 25,25,24
    24 KK=KK=1
        GO TO 23
    25 INTI=INT+I
        DO 26 J=1.INTI
        C(J) = POUT(NN)-PSI(KK)
        S(J) = YST(KK)
    26 KK = Kk+1
        DO 28 J m 1,INT
        I= J+1
    27S(I) = (C(J)*S(I)-C(I)*S(J))/(C(J)-C(I))
        I=I+1
```

```
            IF(I-INTI) 27,27,28
        28 CONTINUE
        YZ(NN) = S(INT!)
    30 CONTINUE
        KPMIN = KPMIN+1
    4O CONTINUE
        KSAVE = KPMIN
C
C*****INTERPOLATIUN...PSI FROM U = O TO EDGE
    DO 6O NENMIN,NMAX
        NN = N
        DO 45 K=KSAVE,KMAX
        KK = K
        IF(PSI(KK)-POUT(NN)) 45,45,46
    4 5 ~ C O N T I N U E ~
    46 KK = KK=1
        IF(KK-KPMIN) 47,47,48
    47 KK = KPMIN
    GO TO 49
    48 M = KK+2
        IF(M-KMAX) 49,49,46
    49 KSAVE = KK
        DO 50 J=1.3
        C(J) = POUT(NN)-PSI(KK)
        S(J) = YST(KK)
    50 KK = Kk+1
        S(2)=(C(1)*S(2)-C(2)*S(1))/(C(1)-C(2))
        S(3)=(C(1)*S(3)*C(3)*S(1))/(C(1)-C(3))
        Y1(NN)=(C(2)*S(3)-C(3)*S(2))/(C(2)-C(3))
    OO CONTINUE
        K1 = 0
        IF(NO-NMIN) 75,75,65
    65 NOI = NO-1
        DO 70 L=NMIN,NOI
        K1 = K1 + 1
        POUT(K1) = POUT(L)
        YI(KI) = YI(L)
        YZ(K1)=YZ(L)
    7O CONTINUE
    75 CONTINUE
        KlM = Kl
        K2 =KIM
        DO 80 L = NO,NMAX
        K2 = K2 +1
        POUT(K2) E POUT(L)
        YI(K2)= YI(L)
    80 CONTINUE
        K2M = KZ
        RETURN
        END
cccccc}\begin{array}{cccc}{52}&{52.125 0.24}&{0.0.}&{0.9}
```

|  | 50 |
| :---: | :---: |
| 0,00000 | 0.00000 |
| 0.08304 | -0.00519 |
| 0.16597 | -0.02075 |
| 0.24848 | -0.04665 |
| 0.33003 | -0.08281 |
| 0.40991 | -0.12906 |
| 0.48726 | -0.18513 |
| 0.56111 | -0.25065 |
| 0.63047 | -0.32513 |
| 0.69442 | -0.40793 |
| 0.75216 | -0.49834 |
| 0.80313 | $\therefore 0.59555$ |
| 0.84704 | -0.69869 |
| 0.88390 | -0.80687 |
| 0.91400 | -0.91924 |
| 0.93790 | -1.03498 |
| 0.95632 | -1.15337 |
| 0.97010 | -1. 27377 |
| 0.98010 | -1.39566 |
| 0.98712 | -1.51861 |
| 0.99191 | -1.64230 |
| 0.99506 | -1.76649 |
| 0.99708 | -1.89100 |
| 0.99832 | -2.01571 |
| 0.99906 | -2.14055 |
| 0.99949 | -2.26546 |
| 0.99973 | -2.39041 |
| 0.99986 | -2.51538 |
| 0.99993 | -2.64037 |
| 0.99996 | -2.76537 |
| 0.99998 | -2.89036 |
| 0.99999 | -3.01536 |
| 0.99999 | -3.14036 |
| 0.99999 | -3.26536 |
| 0.99999 | -3.39036 |
| 0.99999 | -3.51536 |
| 0.99999 | -3.64036 |
| 0.99999 | -3.76536 |
| 0.99999 | -3.89036 |
| 0.99999 | -4.01536 |
| 0,99999 | -4.14036 |
| 0.99999 | -4.26536 |
| 1.00000 | -4.39036 |
| 1.00000 | -4.51535 |
| 1.00000 | - 4.64035 |
| 1.00000 | -4.76535 |
| 1.00000 | -4.89035 |
| 1.00000 | -5.01535 |
| 1.00000 | -5.14035 |
| 1.00000 | -5.26535 |
| \%ENDOS |  |

## INPUT VALUES

| JMAX $=$ | 52 | $D X=$ | .12500 | $\times 0$ | 0.00000 | ALPHV | = | .90000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| KMAX = | 52 | DY $=$ | .24000 | UEXO | 1.00000 | ALPMU | = | .90000 |
|  |  |  |  | MO | 0.00000 | ALPHM | = | .05000 |

INITIAL VALUES

| $k$ | $Y$ | U | $V$ |
| :---: | :---: | :---: | :---: |
| 1 | 0.000000 | 0.000000 | 0.000000 |
| 2 | . 250000 | .083040 | . .005190 |
| 3 | .500000 | .165970 | .. 020750 |
| 4 | .750000 | . 248480 | -. 046650 |
| 5 | 1.000000 | .330030 | .. 082810 |
| 6 | 1.250000 | .409910 | -. 129060 |
| 7 | 1.500000 | .487260 | .. 185130 |
| 8 | 1.750000 | . 501110 | . 2550650 |
| 9 | 2.000000 | .630470 | . .325130 |
| 10 | 2.250000 | .694420 | .. 407930 |
| 11 | 2.500000 | .752160 | -. 498340 |
| 12 | 2.750000 | .803130 | . .595550 |
| 13 | 3.000000 | .847040 | . .698690 |
| 14 | 3.250000 | .883900 | -. 806870 |
| 15 | 3.500000 | .914000 | . 919240 |
| 16 | 3.750000 | .937900 | -1.034980 |
| 17 | 4.000000 | .956320 | -1.153370 |
| 18 | 4.250000 | .970100 | -1.273770 |
| 19 | 4.500000 | .980100 | -1.395660 |
| 20 | 4.750000 | .987120 | -1.518610 |
| 21 | 5.000000 | .991910 | -1.642300 |
| 22 | 5.250000 | .995060 | -1.766490 |
| 23 | 5.500000 | .997080 | -1.891000 |
| 24 | 5.750000 | .998320 | -2.015710 |
| 25 | 6.000000 | .999060 | -2,140550 |
| 26 | 6.250000 | .999490 | -2.265460 |
| 27 | 6.500000 | .999730 | -2,390410 |
| 28 | 6.750000 | .999860 | -2.515380 |
| 29 | 7.000000 | .999930 | -2.640370 |
| 30 | 7.250000 | .999960 | -2.765370 |
| 31 | 7.500000 | .999980 | -2.890360 |
| 32 | 7.750000 | .999990 | -3.015360 |
| 33 | 8.000000 | .999990 | -3.140360 |
| 34 | 8.250000 | .999990 | -3.265360 |
| 35 | 8.500000 | .999990 | -3.390360 |
| 36 | 8.750000 | .999990 | - 3.515360 |
| 37 | 9.000000 | .999990 | -3.640360 |
| 38 | 9.250000 | .999990 | -3.765360 |
| 39 | 9.500000 | . 9.99990 | - 3.890360 |
| 40 | 9.750000 | .999990 | -4.015360 |
| 41 | 10.000000 | .999990 | -4.140360 |
| 42 | 10.250000 | .999990 | -4.265360 |
| 43 | 10.500000 | 1.000000 | -4.390360 |
| 44 | 10.750000 | 1.000000 | -4.515350 |
| 45 | 11.000000 | 1.000000 | -4.640350 |
| 46 | 11.250000 | 1.000000 | -4.765350 |
| 47 | 11.500000 | 1.000000 | -4.890350 |
| 48 | 11.750000 | 1.000000 | -5,015350 |
| 49 | 12.000000 | 1.000000 | -5.140350 |
| 50 | 12.250000 | 1.000000 | -5.265350 |























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[^0]:    *For sale by the National Technical Information Service, Springfield, Virginia 22151

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