ON THE DEFINITION OF CRATER SATURATION DIAMETER

by

Allan H. Marcus*

The Johns Hopkins University

Technical Report #168

September, 1972

*Research supported by the National Aeronautics and Space Administratio under Grant NGR 21-001-080 awarded to the Department of Mathematical Sciences, The Johns Hopkins University.

MATHEMATICAL SCIENCES
THE JOHNS HOPKINS UNIVERSITY
BALTIMORE, MARYLAND 21218

ON THE DEFINITION OF CRATER SATURATION DIAMETER

BY

Allan H. Marcus*

ABSTRACT

A theoretical model is developed for determining the diameter D_p such that the density of lunar craters exceeding D_p constitutes a specified proportion p of the maximum number that could be observed on a crater-saturated surface. It is shown that this diameter is sensitive to the value of the exponent in the fresh crater size distribution, and is much more informative than the usual log-log plot intercept definition of saturation diameter.

*Research supported by the National Aeronautics and Space Administration under Grant NGR 21-001-080 awarded to the Department of Mathematical Sciences, The Johns Hopkins University.

ON THE DEFINITION OF CRATER SATURATION DIAMETER

by

Allan H. Marcus

Theory

In order to clarify the definition of a planetary surface which is "saturated" by craters, we must consider unsaturated surfaces as well. We start therefore with the expected number of craters of diameter D to $D + dD$ formed per unit area per unit time on a surface, $n_o(D)dD$. If the crater centers are distributed at random on the surface, and if the formation of a crater of diameter y will destroy a crater of diameter D if the center of the y-crater falls within an area $A(D,y)$ centered on the center of the D-crater, then the net average destruction rate of D-craters is $\Lambda(D)$,

$$\Lambda(D) = \int_0^\infty A(D,y)\, n_o(y)dy \tag{1}$$

Hence the expected number $n(D,t)dD$ of craters of diameter D to $D + dD$, per unit area per unit time at time t, satisfies the differential equation

$$\frac{\partial}{\partial t} n(D,t) = n_o(D) - \Lambda(D)n(D,t) \tag{2}$$

and with the initial condition

$$n(D,0) = 0$$

we obtain

$$n(D,t) = n(D,\infty)\ [1 - \exp(-\ \Lambda(D)t)] \tag{3}$$

where the equilibrium number density is

$$n(D,\infty) = n_o(D)\ /\ \Lambda(D) \tag{4}$$

It can be shown (Marcus, 1970) that under a very broad range of crater obliteration models, there is an equilibrium density C such that if for some $s > 2$, $D_o > 0$, $\varphi > 0$,

$$n_o(D) = s\varphi D_o^{\ s}\ D^{-s-1} \tag{5}$$

then

$$\Lambda(D) = k\ D^{2-s} \tag{6}$$

$$n(D,\infty) = 2\ C\ D^{-3} \tag{7}$$

where

$$C = s\varphi\ D_o^{\ s}/2k \tag{8}$$

Eq. (3) can be recast into the form of a cumulative expected number density $N(D,t)$, using (4)-(8):

$$N(D,t) = \int_D^\infty n(D,t)dD$$

$$= \frac{C}{D^2}\left[\ 1 - \frac{\Gamma(\beta+1)}{\epsilon^\beta}\ P\{\gamma_\beta \leq \epsilon\}\ \right] \tag{9}$$

where

$$\Gamma(\beta+1) = \int_0^\infty u^\beta \exp(-u)du$$

$$\beta = 2/(s-2)$$

$$\epsilon = kt\ D^{-2/\beta} \quad , \quad D = (kt/\epsilon)^{\beta/2}$$

$$P\{\gamma_\beta \leq \epsilon\} = \int_0^\epsilon u^{\beta-1}\exp(-u)du/\Gamma(\beta)$$

Hence: A SURFACE is 100p % SATURATED AT DIAMETER D_p IF

$$N(D_p, T) = p\ N(D_p, \infty)$$

i.e.

$$(1-p)\,\epsilon_p^\beta / \Gamma(\beta+1) = P\{\gamma_\beta \leq \epsilon_p\} \tag{10}$$

Gault (1970) has used another definition. Some numerical examples are instructive.

EXAMPLE 1. $p \sim 0$.

ϵ_p must be small, so expanding the exponential in $P\{\gamma_\beta \leq \epsilon_p\}$ as a MacLaurin series in u, we obtain

$$\epsilon_p \sim (\beta+1)\,p/\beta \tag{11}$$

$$D_p \sim (\beta kt/(\beta+1)\,p)^{\beta/2} = C_p \tag{12}$$

EXAMPLE 2. The usual definition of the diameter D^* at which a surface is saturated (Gault et. al., 1970) is

$$N(D^*, \infty) = t \int_{D^*}^{\infty} n_o(D)dD \tag{13}$$

whence, from (5) and (7),

$$D^* = C_1$$

where C_1 is defined by (12) with $p = 1$. However, it is evident from (3) that no surface is ever 100% saturated.

EXAMPLE 3. $p \sim 1$

ϵ_p must be large, so $P\{\gamma_\beta \leq \epsilon\} \sim 1$ and

$$\epsilon_p \sim (\Gamma(\beta + 1) / (1 - p))^{1/\beta} \tag{15}$$

$$D_p \sim (kt)^{\beta/2}((1 - p) / \Gamma(\beta + 1))^{1/2} = B_p \tag{16}$$

EXAMPLE 4. As a preliminary estimate of D_p, use saturation ratio for the incremental number density

$$p = n(A_p, t) / n(A_p, \infty) \tag{17}$$

whence from (3) and (6)

$$A_p = (-kt/\log_e(1 - p))^{\beta/2} \tag{18}$$

We obtain after some numerical calculations

ϵ_p		$s = 4$, $\beta = 1$	$s = 3$, $\beta = 2$	$s = 8/3$, $\beta = 3$
p	0.5	2.25	1.15	0.97
	0.8	4.965	2.73	2.25

Thus:

	p	$s = 4$, $\beta = 1$	$s = 3$, $\beta = 2$	$s = 8/3 = 2.67$, $\beta = 3$
C_p/D_p	0.5	1.50	1.53	1.75
	0.8	1.74	2.28	3.06
D^*/D_p	0.5	1.06	0.767	0.620
	0.8	1.55	1.82	2.18
B_p/D_p	0.5	1.06	1.15	1.65
	0.8	0.996	0.862	0.616
A_p/D_p	0.5	1.80	1.66	1.65
	0.8	1.75	1.70	1.65

Discussion

The results are rather sensitive to the choice of a definition for crater saturation. D_p, the diameter such that the number of large craters exceeding this size is 100 p percent of the total number that could be observed, is a well-defined, computable, and intuitively satisfactory characterization of saturation. The same could be said of the diameter A_p defined by incremental number densities, and it can be shown that if s not too much larger than 2, then

$$A_p \sim (\log_e 2)\ D_p = 1.61\ D_p \tag{19}$$

as the numerical example shows.

Both of these quantities must be estimated from an empirical analysis of crater densities, based on the result

$$\begin{aligned} N(D,t) &\sim C / D^2 && \text{if } D << D_p \\ &\sim 2Ckt / sD^s && \text{if } D >> D_p \end{aligned} \tag{20}$$

We usually plot $\log N(D,t)$ vs. $\log D$ and define D^* as the intersection of these two (approximately straight line segments. The curved transition between the straight line segments covers a large range of diameters and is obscured by the logarithmic plot.

It is interesting to compare crater counts from the Apollo 11 and 12 crater count (Shoemaker, 1970), for D in meters:

	s	C	2Ckt/s	D^*	β	$D_{0.5}$	$D_{0.8}$
Apollo 11	2.93	$0.10/_m 2$	$8.0/_m 2$	141_m	2.15	189_m	75_m
Apollo 12	2.86	$0.10/_m 2$	$2.0/_m 2$	32.6_m	2.33	45.5_m	16.8_m

The D_p criterion is evidently more informative.

REFERENCES

1. Gault, D.E. (1970). Saturation and equilibrium conditions for impact cratering on the lunar surface: Criteria and implications. Radio Science, 5, 273-291.
2. Marcus, A. H. (1970). Comparison of equilibrium size distributions for lunar craters. J. Geophys. Res., 75, 4977-4984.
3. Shoemaker, E. M. (1970). Origin of fragmental debris on the lunar surface and the bombardment history of the Moon. Publ. Inst. Investig. Geol. (Barcelona), 24. Revised, 1971.