AUBURN UNIVERSITY

MULTIVARIANT FUNCTION MODEL GENERATION

Final Report

Contract NGR-01-003-008

W. A. Shaw, G. R. Harmon, J. W. Reece,
P. M. Fitzpatrick, C. H. Chen, W. F. Reiter, J. C. M. Yu,
K. D. Dannenberg, L. R. Young, J. E. Cochran,
J. J. F. Liu

School of Engineering

Department of Mechanical Engineering

Under Contract With
Electronics Research Center
National Aeronautics & Space Administration
Cambridge, Massachusetts

Administered Through

ENGINEERING EXPERIMENT STATION

AUBURN UNIVERSITY

AUBURN, ALABAMA 36830

January, 1974

FOREWORD

This compilation of papers and reports is the final report for Contract ME-(NGR-01-003-008), "Computer Techniques for Multivariant Function Model Generation Emphasizing Programs Applicable to Space Vehicle Guidance". The work was performed by those listed as authors of the papers and reports, for the National Aeronautics and Space Administration, Electronics Research Center, Cambridge, Massachusetts.

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AUBURN UNIVERSITY

A METHOD FOR DETERMINING OPTIMUM RE-ENTRY TRAJECTORIES

By

William F. Reiter, Grady R. Harmon and Joe W. Reece
Department of Mechanical Engineering

NASA Grant NGR-01-003-008

Prepared For

GUIDANCE LABORATORY
ELECTRONICS RESEARCH CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CAMBRIDGE, MASSACHUSETTS

June 1966

AUBURN, ALABAMA

N74-31316

AUBURN UNIVERSITY
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A METHOD FOR DETERMINING OPTIMUM RE-ENTRY TRAJECTORIES

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William F. Reiter
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SUMMARY

The Pontryagin Maximum Principle is used to formulate the problem of finding optimum atmospheric vehicular re-entry trajectories. The optimization problem is that of minimizing an integral which is a function of the state and control variables. The vehicle's motion is assumed to be influenced only by a gravitational force and an aerodynamic force. The problem is formulated and the necessary equations are developed simultaneously for three sets of Euler angles. Computational procedures are suggested so that numerical trajectories may be generated.

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LIST OF SYMBOLS

A	Projected cross-sectional area of vehicle
C.G.	Center of gravity
C.P.	Center of pressure
$C\alpha$	Cosine α
$C\alpha_y$	Cosine α_y
CP	Cosine $\emptyset_p$
CR	Cosine $\emptyset_r$
CY	Cosine $\emptyset_y$
C_x, C_z	Vehicle configuration factors
$\overline{F}_a$	Aerodynamic force in the aerodynamic coordinate system
$\overline{F}_{am}$	Aerodynamic force in the missile system
$\overline{F}_g$	Gravitational force in the plumbline system
$\overline{F}_{r1} = -\overline{F}_{r2}$	Roll forces in the missile system
G	Gravitational constant
m	Mass of the vehicle

M	Mass of the earth
$\overline{M}_{am}$	Aerodynamic moment (missile system)
$\overline{M}_{rm}$	Roll couple (missile system)
q	Dynamic pressure
R_o	Earth's radius
$\lvert\overline{R}\rvert$	Absolute value of the plumbline position vector
$S\alpha$	Sine α
$S\alpha_y$	Sine α_y
SP	Sine ϕ_p
SR	Sine ϕ_r
SY	Sine ϕ_y
t	Time
T	Kinetic energy of the vehicle
$\overline{V}_r$	Relative velocity vector (Aerodynamic System)
$\overline{V}_{rm}$	Relative velocity vector (Missile System)
$\overline{V}_R$	Relative velocity vector (Plumbline System)
$\overline{W}$	Velocity vector for abnormal air movement in plumbline system

$\overline{X}$	Plumbline position vector
$\overline{X}_a$	Aerodynamic system position vector
$\overline{x}_{cp}$	Position of the center of pressure in the missile system
$\overline{X}_m$	Missile system position vector
$\overline{z}_r$	Roll jet positions in the missile system
α^*	Angle of attack
α_y	Yaw angle of attack
ϕ_p	Pitch angle
ϕ_r	Roll angle
ϕ_y	Yaw angle
ρ	Density of the Atmosphere
$\overline{\omega}$	Angular velocity vector of the vehicle in the missile system
$\overline{\omega}_e$	Angular velocity vector of the attracting body in the plumbline system

LIST OF FIGURES

I. INTRODUCTION

This paper is an extension of previous work done by Grady Harmon and W. A. Shaw, presented in NASA TM X-53024, March 14, 1964.

The objectives of this paper are (1) to present a method for treating optimum re-entry problems in a simplified manner and (2) to generalize the computational scheme outlined in the aforementioned paper. The computational scheme given allows for the optimization of any functional subject to the specified constraints. Atmospheric data, vehicle configuration and aerodynamic coefficients are incorporated in the computational scheme in tabular form. Thus, different vehicles and/or atmospheres may be considered by changing the appropriate tables. The governing equations are developed for three different gimbal sets. A computational scheme is outlined for each case.

No numerical results are available at present, but development of the computer deck is underway at the Electronics Research Center.

This work is sponsored, in part, by a grant, NGR-01-003-008, from the Electronics Research Center.

II. STATEMENT OF THE PROBLEM

The problem is that of finding the optimum control process, $\alpha_y(t)$, that will transfer a vehicle from an initial state, at time t_o, in an atmosphere to a terminal state, at time t_1, in the same atmosphere so that the value of the functional

$$J = \int_{t_o}^{t_1} f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\emptyset}}, \emptyset_r, \emptyset_y, \emptyset_p, \alpha, \alpha_y, F_r)\, dt$$

is a minimum. The trajectory associated with this optimum control process is the optimum trajectory.

The rotational motion of the vehicle is treated in a simplified manner. The equations governing the vehicle's rotational motion are considered as a steady-state problem with only one component of the angular velocity vector present for any given gimbal set. A gimbal set is used to measure the Euler angles, $\emptyset_r$, $\emptyset_y$, and $\emptyset_p$. The equations of motion are developed simultaneously for three different gimbal sets.

The problem is formulated as a Pontryagin initial value problem. The relative velocity equations appear as algebraic constraints. The yaw angle of attack, $\alpha_y(t)$, is the control variable.

Additional assumptions are made as follows:

1. The motion of the vehicle is influenced by an aerodynamic force that acts through the vehicle's center of pressure.
2. The attracting body is a rotating sphere with homogeneous mass.

3. The vehicle's centroid of mass and centroid of volume are not coincident.
4. The vehicle's center of mass is invariant with respect to the vehicle.
5. The center of pressure of the vehicle is invariant with respect to the vehicle.
6. A system of roll control jets is available on the vehicle that produce a pure roll couple as required by the optimum control process.

III. COORDINATE SYSTEMS

Three rectangular coordinate systems will be used in this paper. They are:

1. The plumbline space fixed coordinate system,
2. The vehicle fixed missile coordinate system,
3. The aerodynamic coordinate system.

A. Plumbline System

The plumbline system, Figure 1, has its origin at the earth's center with the Y-axis parallel to the gravity gradient at the launch point. The X-axis is parallel to the earth fixed launch azimuth and the Z-axis is chosen to form a right-handed system.

B. Missile System

The missile system, Figure 1, is located with its origin at the center of mass of the vehicle and its y_m axis parallel to the longitudinal axis of the vehicle. The x_m and z_m axes are chosen to form a right-handed system which is parallel to the plumbline system at the launch point.

As the vehicle moves along its trajectory, the missile system undergoes a displacement with respect to the plumbline system. This displacement is given by three Euler angles as measured by a gimbal set. The Euler angles uniquely specify the orientation of the vehicle at any time. Any particular orientation of the vehicle may be described by

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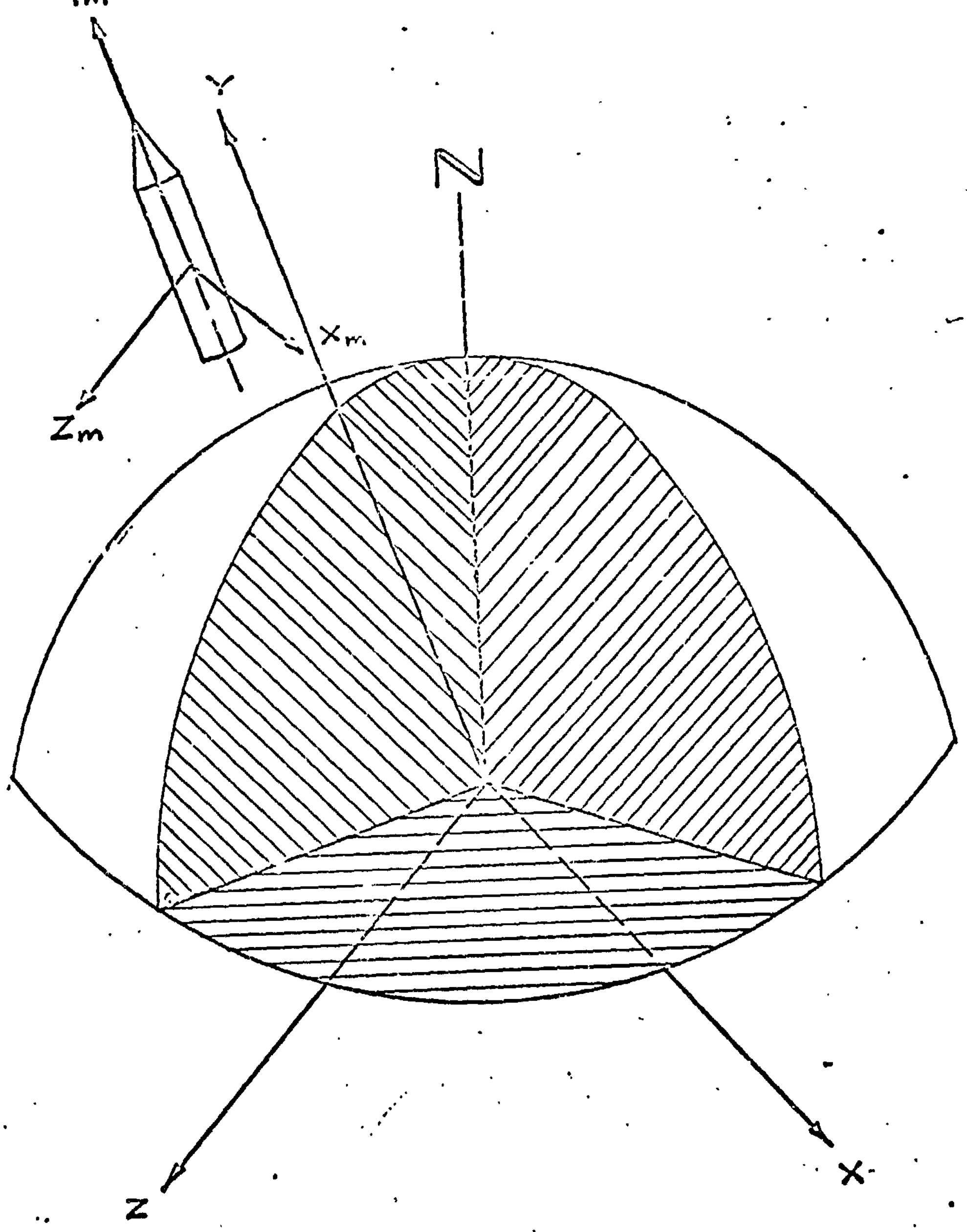

Fig. 1. Plumbline and missile coordinate systems

different sets of Euler angles depending solely on the sequence in which the angles are measured. Therefore, it is mandatory that a specific sequence be followed in measuring the Euler angles. The three Euler angles are referred to as the yaw angle, ϕ_y, the roll angle, ϕ_r, and the pitch angle, ϕ_p. The yaw angle is measured with respect to an X axis. The roll angle is measured with respect to a Y axis, and the pitch angle is measured with respect to a Z axis. An angle is considered positive counterclockwise when viewed from the positive end of the axis about which the rotation is taken. The angles are measured by a set of gimbals on the vehicle. A gimbal set measures the Euler angles in a specific sequence such as pitch, yaw, and roll. In this paper, equations that involve the angles yaw, roll, or pitch are developed simultaneously for three different sets of Euler angles. The angles are obtained from three gimbal sets. They will be referred to as follows:

1. A gimbal set which measures in the order of pitch, yaw, roll.
2. A gimbal set which measures in the order of pitch, roll, yaw.
3. A gimbal set which measures in the order of roll, yaw, pitch.

The Euler angles are shown in Figures 2, 3, and 4.

A position vector in the missile coordinate system may be written in terms of a position vector in the plumbline coordinate system.

The equations of transformation are given by the orthogonal rotation matrices

$$[\phi_y] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & CY & SY \\ 0 & -SY & CY \end{bmatrix},$$

$$[-\phi_r] = \begin{bmatrix} CR & 0 & SR \\ 0 & 1 & 0 \\ -SR & 0 & CR \end{bmatrix},$$

$$[\phi_p] = \begin{bmatrix} CP & SP & 0 \\ -SP & CP & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The particular combination of the above rotation matrices that relate a vector in the two coordinate systems is dependent on the gimbal set used. The relationship for gimbal set 1 is

$$\overline{X}_m = [-\phi_r]\ [\phi_y]\ [\phi_p]\ \overline{X} \tag{1a}$$

or

$$\overline{X}_m = [A_d]_1\ \overline{X} \tag{1b}$$

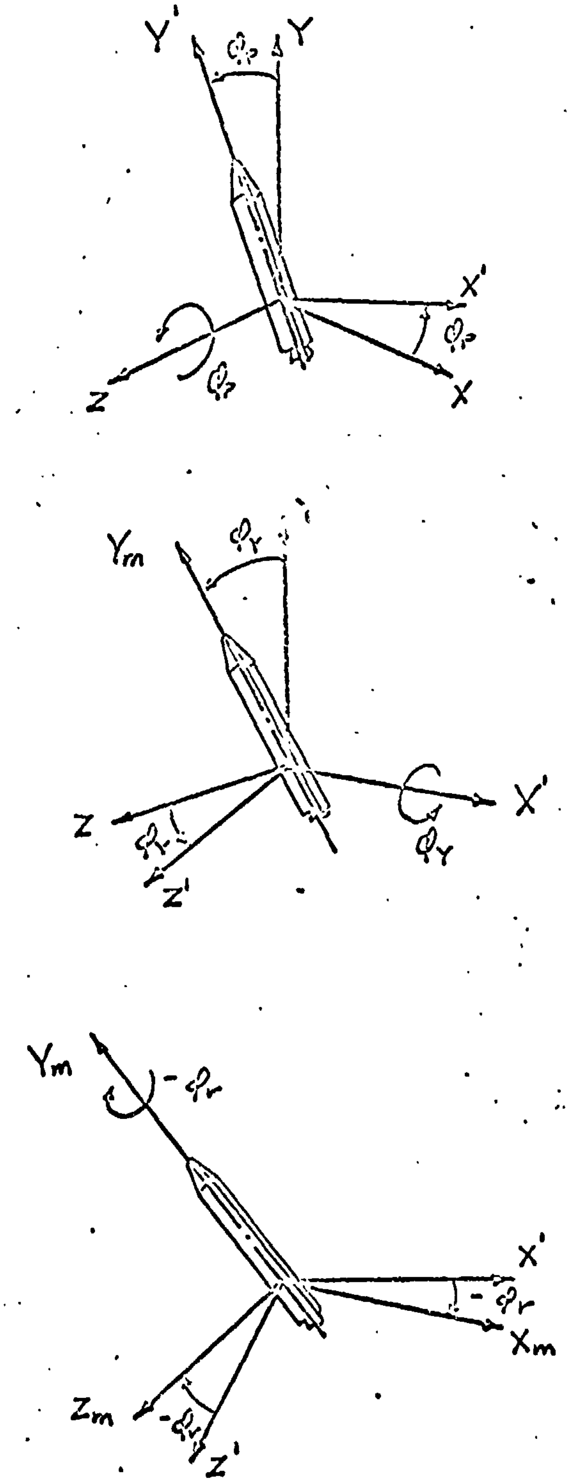

Fig. 2. Eulerian angles for gimbal set 1

where

$$[A_d]_1 = \begin{bmatrix} CRCP+SPSRSY & CRSP-SRSYCP & SRCY \\ -CYSP & CYCP & SY \\ -SRCP+CRSYSP & -SPSR-CRCPSY & CRCY \end{bmatrix} \qquad (1c)$$

is the combined product of the rotation matrices in equation (1a). When $\emptyset_y$ = 90° gimbal set 1 is oriented so that $\emptyset_r$ and $\emptyset_p$ are measured in the same direction, refer to Figure 2. This condition is referred to as gimbal lock.

The relationship for gimbal set (2) is

$$\overline{X}_m = [\emptyset_y]\ [-\emptyset_r]\ [\emptyset_p]\ \overline{X} \qquad (2a)$$

or

$$\overline{X}_m = [A_d]_2\ \overline{X} \qquad (2b)$$

where

$$[A_d]_2 = \begin{bmatrix} CRCP & CRSP & SR \\ -CYSP-SYSRCP & CYCP-SYSRSP & SYCR \\ SYSP-CYSRCP & -SYCP-CYSRSP & CYCR \end{bmatrix} \qquad (2c)$$

is the combined product of the rotation matrices in equation (2a). Gimbal set 2 is locked when $\emptyset_r$ = 90°. At this orientation, refer to Figure 3, $\emptyset_y$ and $\emptyset_p$ are measured in the same direction.

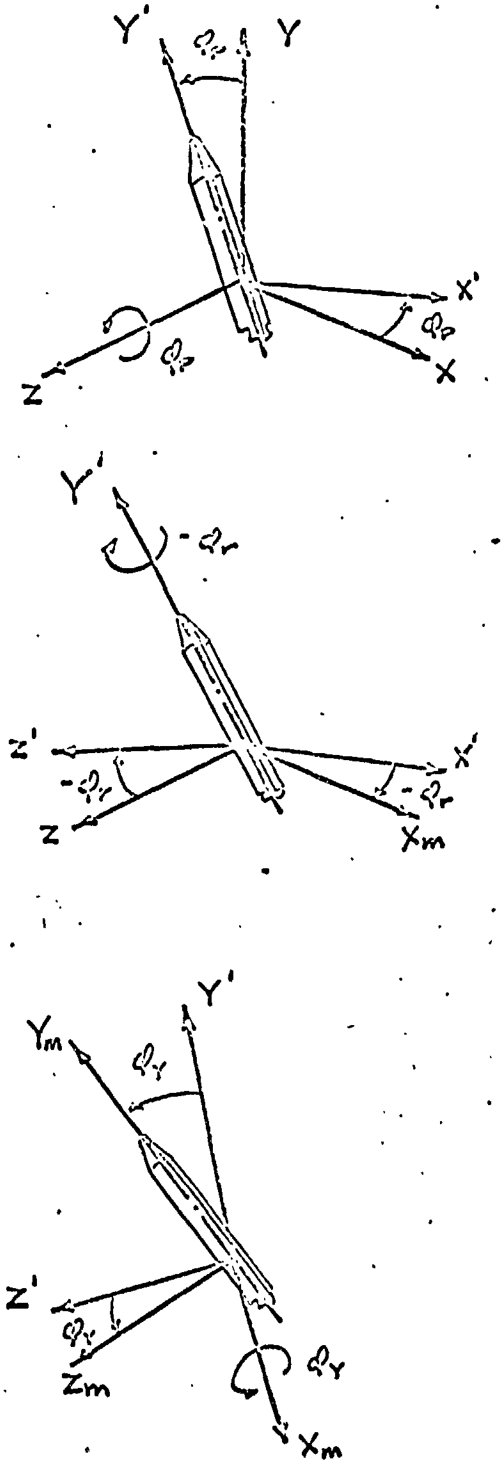

Fig. 3. Eulerian angles for gimbal set 2

The relationship for gimbal set (3) is

$$\overline{X}_m = [\emptyset_p]\ [\emptyset_y]\ [-\emptyset_r]\ \overline{X} \tag{3a}$$

or

$$\overline{X}_m = [A_d]_3\ \overline{X} \tag{3b}$$

where

$$[A_d]_3 = \begin{bmatrix} CPCR-SPSRSY & SPCY & CPSR+SPSYCR \\ -SPCR-CPSRSY & CPCY & -SPSR+CPSYCR \\ -SRCY & -SY & CYCR \end{bmatrix} \tag{3c}$$

is the combined product of the rotation matrices in equation (3a). Gimbal set 3 is locked when $\emptyset_y = 90°$. At this orientation, refer to Figure 4, $\emptyset_p$ and $\emptyset_r$ are measured in the same direction.

The transformation matrices (1c), (2c), and (3c) will be referred to as

$[A_d]_3$ where i = 1, 2, 3.

Equations (1b), (2b), and (3b) are restated as

$$\overline{X}_m = [A_d]_i\ \overline{X}. \tag{4}$$

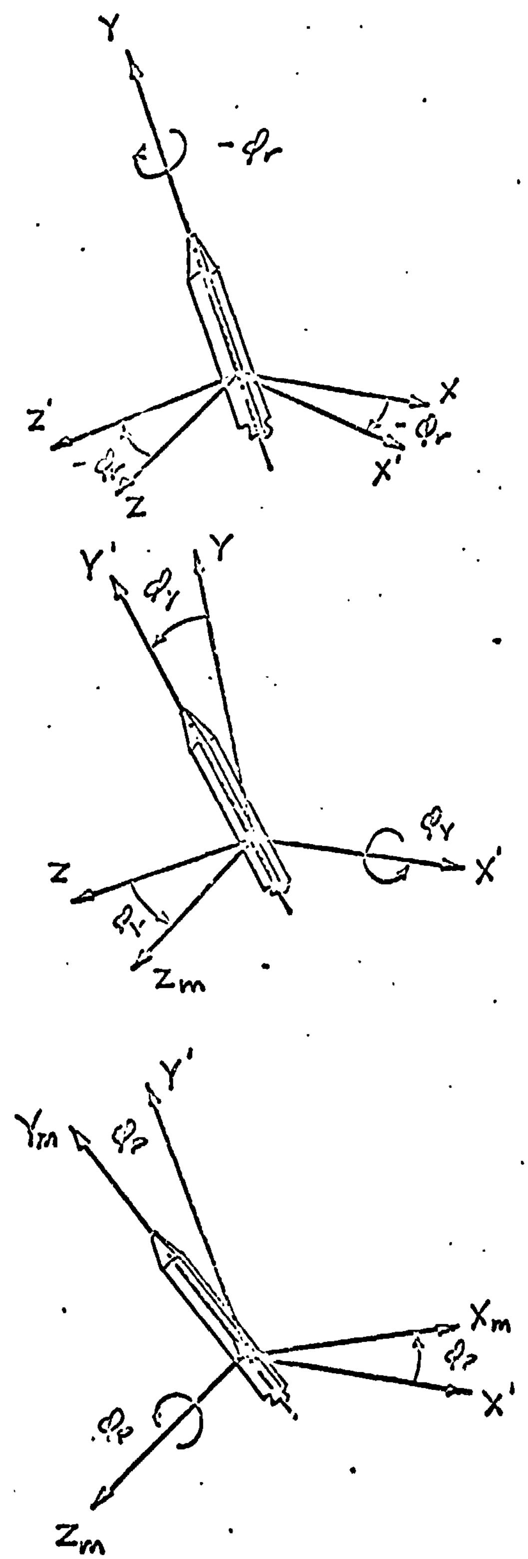

Fig. 4. Eulerian angles for gimbal set 3

C. Aerodynamic Coordinate System

The aerodynamic coordinate system is located as shown in Figure 5 with its origin at the center of pressure of the vehicle. The Y_a axis lies in the plane containing the vehicle longitudinal axis of symmetry and the relative velocity vector. The relative velocity vector, $\overline{V}_R$, is defined as the velocity of the air with respect to the vehicle as measured from the inertial reference. The X_a and Z_a axes are chosen to form a right-handed system. As the vehicle moves along its trajectory, there will be a relative displacement between the missile fixed coordinate system and the aerodynamic coordinate system. The direction of the Y_a axis is defined by the following rotations as shown in Figure 5:

1. Rotate the vehicle fixed reference frame about the Y_m axis so that the X_m axis lies in a plane parallel to the plane formed by the vehicle's longitudinal axis of symmetry and the relative velocity vector. The angle traversed is referred to as the yaw angle of attack, α_y.
2. Rotate about the new Z axis by the true angle of attack, α^*. This specifies the orientation of the aerodynamic coordinate system.

The true angle of attack, α^*, will be expressed in terms of the aerodynamic force in the next section.

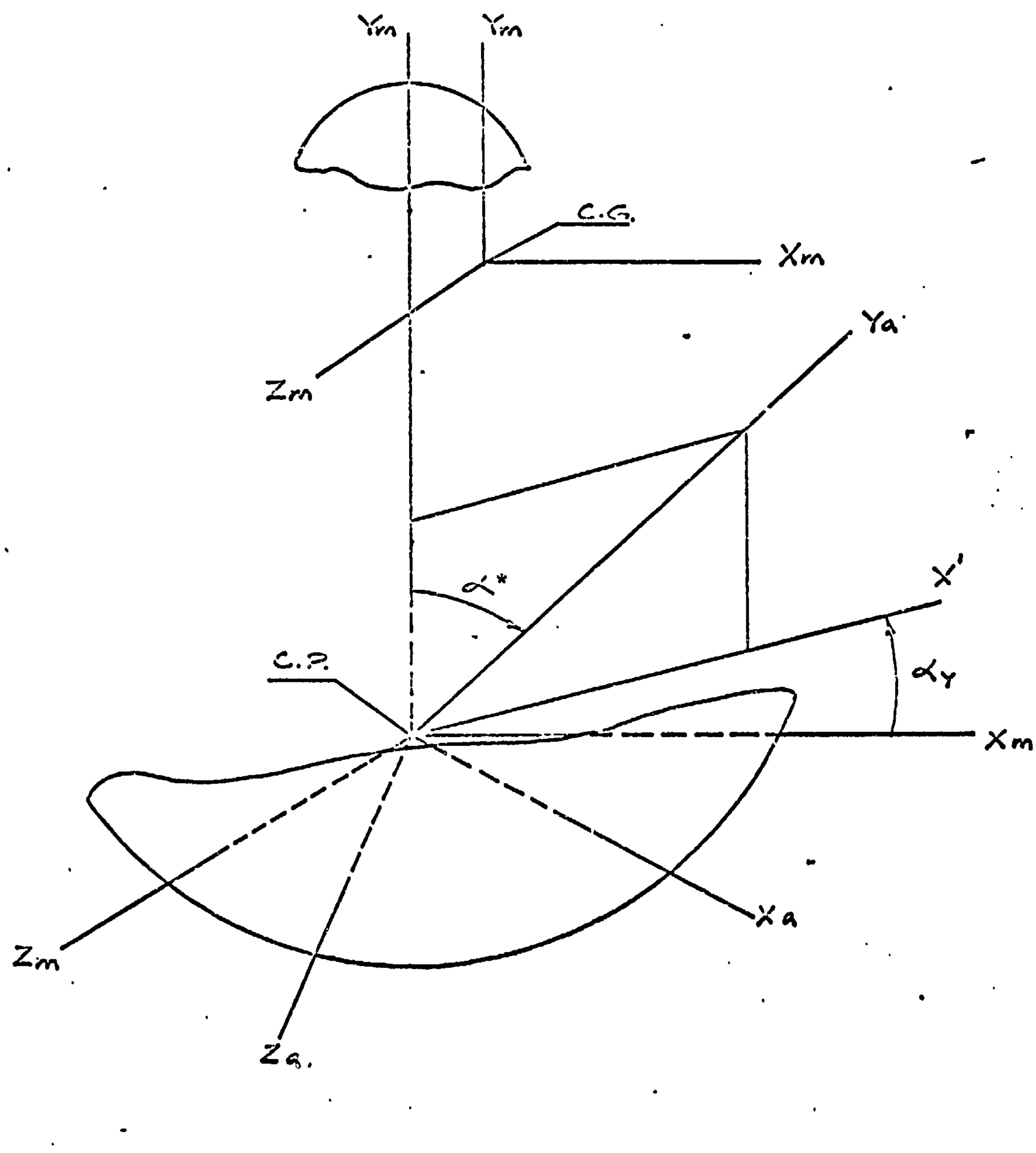

Fig. 5. Aerodynamic and missile coordinate systems

A position vector in the aerodynamic coordinate system may be written in terms of a position vector in the missile fixed coordinate system. The orthogonal transformation matrices are

$$[-\alpha^*] = \begin{bmatrix} C\alpha^* & -S\alpha^* & 0 \\ S\alpha^* & C\alpha^* & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\alpha_y] = \begin{bmatrix} C\alpha_y & 0 & -S\alpha_y \\ 0 & 1 & 0 \\ S\alpha_y & 0 & C\alpha_y \end{bmatrix} .$$

A positive vector in the aerodynamic coordinate system is expressed in terms of a position vector in the missile fixed reference as

$$\overline{X}_a = [-\alpha^*]\ [\alpha_y]\ \overline{X}_m \quad , \tag{5a}$$

or

$$\overline{X}_a = [A_a]\ \overline{X}_m \quad , \tag{5b}$$

where

$$[A_a] = \begin{bmatrix} C\alpha^* C\alpha_y & -S\alpha^* & -C\alpha^* S\alpha_y \\ S\alpha^* S\alpha_y & C\alpha^* & -S\alpha^* S\alpha_y \\ S\alpha_y & 0 & C\alpha_y \end{bmatrix} \tag{5c}$$

is the combined product of the rotation matrices in equation (5a). The aerodynamic coordinate system transformation matrix (5c) is independent of the sequence used in measuring the angles yaw, roll, and pitch.

IV. MECHANICS

A. Forces

Two forces are assumed to act on the vehicle as it moves along its trajectory. It was assumed that the attracting body is a homogeneous sphere. Thus, an inverse square gravitational force is written in terms of the plumbline coordinates as

$$\overline{F}_g = \frac{-G\, M\, m\, \overline{X}}{|\overline{R}|^3} \quad . \qquad (6)$$

The vehicle's motion is also influenced by an aerodynamic force. The force lies in the plane formed by the vehicle longitudinal axis of symmetry and the relative velocity vector and passes through the center of pressure of the vehicle, as shown in Figure 6.

The components of the aerodynamic force are defined by the equations

$$F_X = A\, q\, C_X\, (\alpha^*) \quad , \qquad (7a)$$

and

$$F_Z = A\, q\, C_Z\, (\alpha^*) \quad . \qquad (7b)$$

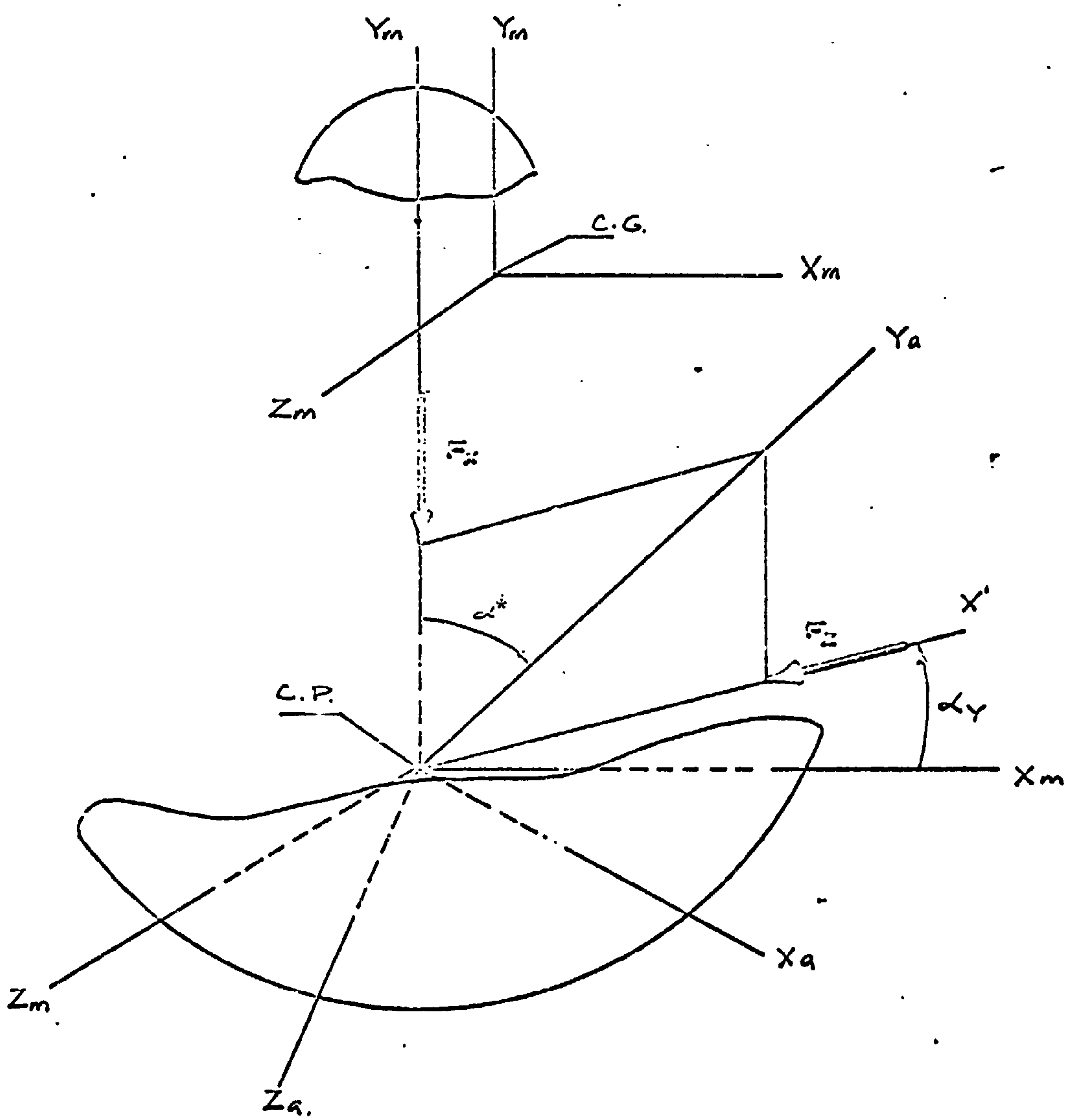

Fig. 6. Aerodynamic force components F_x and F_z

A is the projected cross-section area of the vehicle and q is the dynamic pressure. C_x and C_z are experimentally determined factors that are dependent on the vehicle's shape and the angle of attack. It is assumed that C_z and C_x are known. The aerodynamic force is expressed in the aerodynamic system as

$$\overline{F}_a = \begin{bmatrix} -F_z C\alpha^* + F_x S\alpha^* \\ -F_x C\alpha^* - F_z S\alpha^* \\ 0 \end{bmatrix} . \quad (8)$$

The aerodynamic force is expressed in terms of the missile fixed reference as

$$\overline{F}_{am} = [A_a]^T \, \overline{F}_a \quad . (9a)$$

(Note: The symbol $[A]^T$ is used to denote the transpose of matrix A.)

Equation (9a) can be written in component form as

$$\begin{bmatrix} F_{amx} \\ F_{amy} \\ F_{amz} \end{bmatrix} = \begin{bmatrix} C\alpha^* C\alpha_y & S\alpha^* C\alpha_y & S\alpha_y \\ -S\alpha^* & C\alpha^* & 0 \\ -C\alpha^* S\alpha_y & -S\alpha^* S\alpha_y & C\alpha_y \end{bmatrix} \begin{bmatrix} -F_a S\alpha C\alpha^* + F_a C\alpha S\alpha^* \\ -F_a C\alpha C\alpha^* - F_a S\alpha S\alpha^* \\ 0 \end{bmatrix} . \quad (9b)$$

When simplified, equation (9b) becomes

$$\begin{bmatrix} F_{amy} \\ F_{amy} \\ F_{amz} \end{bmatrix} = F_a \begin{bmatrix} -S\alpha C\alpha_y \\ -C\alpha \\ S\alpha S\alpha_y \end{bmatrix} . \quad (9c)$$

where the magnitude of the aerodynamic force is

$$F_a = \sqrt{F_x^2 + F_z^2} \quad , \qquad (10)$$

and α is expressed in terms of the components of the aerodynamic force through the equations

$$S\alpha = \frac{F_z}{|\bar{F}_a|} \quad , \qquad (11a)$$

$$C\alpha = \frac{F_x}{|\bar{F}_a|} \quad , \qquad (11b)$$

$$\tan\alpha = \frac{S\alpha}{C\alpha} = \frac{F_z}{F_x} = \frac{C_z(\alpha^*)}{C_x(\alpha^*)} \quad . \qquad (11c)$$

The magnitude of the aerodynamic force is related to the relative velocity through the dynamic pressure by the equation

$$q = 1/2\ \rho\ V_R^2 \quad . \qquad (12)$$

It is assumed that the atmosphere normally moves with the attracting body (6). Hence, at all times there is an air mass movement with respect to the plumbline coordinate system. $\overline{W}$ is a vector that represents any abnormal air movement. An equation expressing the velocity of the wind may be written as

$$\overline{V}_{wind} = \overline{\omega}_e \times \overline{X} + \overline{W} \quad . \tag{13}$$

The relative velocity equation is

$$\dot{\overline{X}} = \overline{V}_{wind} + \overline{V}_R \quad . \tag{14}$$

When equation (13) is substituted into equation (14), the result is

$$\overline{V}_R = \dot{\overline{X}} + \overline{X} \times \overline{\omega}_e - \overline{W} \quad , \tag{15a}$$

or, in component form,

$$\begin{bmatrix} V_{RX} \\ V_{RY} \\ V_{RZ} \end{bmatrix} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} + \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \times \begin{bmatrix} \omega_{ex} \\ \omega_{ey} \\ \omega_{eZ} \end{bmatrix} - \begin{bmatrix} W_x \\ W_y \\ W_Z \end{bmatrix} \quad . \tag{15b}$$

The relative velocity may be expressed in terms of the aerodynamic, missile, or plumbline coordinate system variables. The relative velocity vector is written in the missile coordinate system as

$$\overline{V}_{rm} = [A_d]_i \overline{V}_R = [A_a]^T \overline{V}_r \tag{16}$$

where

$$\overline{V}_{rm} = \begin{bmatrix} V_{rmx} \\ V_{rmy} \\ V_{rmz} \end{bmatrix}$$

is the relative velocity vector in the missile system and

$$\overline{V}_r = \begin{bmatrix} 0 \\ V_r \\ 0 \end{bmatrix}$$

is the relative velocity vector in the aerodynamic system. (Note that equation (16) represents three possible equations depending on i.)

The resultant force acting on the vehicle written in the plumbline coordinates is

$$\overline{F}_R = \overline{F}_g + [A_d]_i^T \overline{F}_{am} \tag{17}$$

B. Couples and Moments

The motion of the vehicle is influenced by a moment and a couple. It is assumed that the center of pressure and the center of mass are invariant with respect to the vehicle. Thus, the center of pressure is located by a constant position vector, $\overline{x}_{cp}$, in the missile fixed reference. The aerodynamic moment is given by the vector product of the position vector, $\overline{x}_{cp}$, and the aerodynamic force, $\overline{F}_{am}$. The aerodynamic moment is written in the missile fixed reference as

$$\overline{M}_{am} = \overline{x}_{cp} \times \overline{F}_{am} \quad , \qquad (18a)$$

or

$$\begin{bmatrix} M_{amx} \\ M_{amy} \\ M_{amz} \end{bmatrix} = \begin{bmatrix} F_a \; y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\ -F_a \; z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y \\ -F_a \; x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} \quad . \qquad (18b)$$

A system of roll jets is used to produce a pure roll control couple about the Y_m axis. The jets are located with respect to the missile fixed coordinate system so that

$$\overline{F}_{r1} = \begin{bmatrix} F_r \\ 0 \\ 0 \end{bmatrix} \text{ located at } \overline{Z}_r = \begin{bmatrix} 0 \\ 0 \\ Z_r \end{bmatrix}$$

and

$$\overline{F}_{r2} = \begin{bmatrix} -F_r \\ 0 \\ 0 \end{bmatrix} \quad \text{located at } -\overline{Z}_r = \begin{bmatrix} 0 \\ 0 \\ -Z_r \end{bmatrix}$$

yield a roll couple

$$\overline{M}_{rm} = 2\,(\overline{Z}_r \times \overline{F}_r) \quad , \qquad (19a)$$

which may be expressed as

$$\overline{M}_{rm} = \begin{bmatrix} 0 \\ 2z_r F_r \\ 0 \end{bmatrix} \quad . \qquad (19b)$$

The resultant moment about the center of mass of the vehicle in the missile fixed coordinate system is the sum of the roll couple and aerodynamic moment

$$\overline{M}_{Tm} = \overline{M}_{am} + \overline{M}_{rm} \quad . \qquad (20)$$

When equations (18b) and (19b) are substituted into equation (20), the result is

$$\overline{M}_{Tm} = \begin{bmatrix} F_a y_{cp} C\alpha S\alpha_y + F_a z_{cp} C\alpha \\ -F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y \\ -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} + \begin{bmatrix} 0 \\ 2z_r F_r \\ 0 \end{bmatrix} \quad , \qquad (21a)$$

which can be reduced to

$$\overline{X}_{Tm} = \begin{bmatrix} F_a y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\ 2Z_r F_r - F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y \\ -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y \end{bmatrix} . \qquad (21b)$$

C. Equations of Motion

It is possible to interpret the motion of a rigid body as the sum of two independent effects--the motion of the center of mass of the vehicle with respect to an inertial coordinate system and the rotational motion of the vehicle about its center of mass. The motion of a rigid body in general requires six independent coordinates to specify its orientation at any time. The six independent coordinates used in this problem are the three plumbline coordinates and three Eulerian angles.

The translational equations of motion are written for the center of mass in the inertial reference as

$$\overline{F}_R = m\ddot{\overline{X}} \qquad (22a)$$

or

$$\overline{F}_g + [A_d]_i^T \overline{F}_{am} = m\ddot{\overline{X}} , \qquad (22b)$$

where

$$\ddot{\overline{X}} = \begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix}$$

When expression (6) is substituted into equation (22b), the second order translational equations of motion become

$$\ddot{\overline{X}} = - \frac{G\,M\,\overline{X}}{|\overline{R}|^3} + [A_d]_i^T \frac{\overline{F}_{am}}{m} \quad . \qquad (22c)$$

The three second order differential equations, (22c), may be reduced to six first order differential equations by a change of variables. Let

$$\overline{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \dot{\overline{X}} \quad . \qquad (23)$$

When the above transformation is used, the second order differential equations of motion, (22c), reduce to

$$\dot{\overline{u}} = - \frac{G\,M\,\overline{X}}{|\overline{R}|^3} + [A_d]_i^T \frac{\overline{F}_{am}}{m} \quad . \qquad (24)$$

For convenience, the following definitions are made:

$$g = - \frac{G\,M}{|\overline{R}|^3} \quad , \qquad (25)$$

$$[A_d]_i^T \frac{\overline{F}_{am}}{m} = \frac{F_a}{m} \overline{N}_i = F_a^* \overline{N}_i \quad , \qquad (26)$$

where

$$F_a^* = \frac{F_a}{m}$$

and

$$\vec{N}_1 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_1$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_1 = \begin{bmatrix} -(S\alpha C\alpha_y)(CRCP+SRSYSP) + C\alpha CYSP + (S\alpha S\alpha_y)(-SRCP+CRSYSP) \\ -(S\alpha C\alpha_y)(CRSP-SRSYCP) - C\alpha CYCP - (S\alpha S\alpha_y)(SPSR+CRCPSY) \\ -(S\alpha C\alpha_y)(SRCY) - C\alpha SY + (CRCYS\alpha S\alpha_y) \end{bmatrix} \quad (27a)$$

and

$$\vec{N}_2 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_2$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_2 = \begin{bmatrix} -(S\alpha C\alpha_y)(CRCP) + C\alpha\ (CYSP+SYSRCP) + S\alpha S\alpha_y\ (SYSP-CYSRCP) \\ -(S\alpha C\alpha_y)(CRSP) - C\alpha\ (CYCP-SYSRSP) - (S\alpha S\alpha_y)(SYCP+CYSRSP) \\ -(S\alpha C\alpha_y)(SR) - C\alpha\ (SYCR) + (S\alpha S\alpha_y)(CRCY) \end{bmatrix} \quad (27b)$$

and

$$\vec{N}_3 = \begin{bmatrix} N \\ P \\ Q \end{bmatrix}_3$$

where

$$\begin{bmatrix} N \\ P \\ Q \end{bmatrix}_3 = \begin{bmatrix} -(S\alpha C\alpha_y)(CPCR-SPSRSY) + C\alpha\ (SPCR+CPSRSY) - S\alpha S\alpha_y\ (CYSR) \\ -(S\alpha C\alpha_y)(SPCY) - C\alpha\ (CPCY) - S\alpha S\alpha_y\ (SY) \\ -(S\alpha C\alpha_y)(CPSR+SPSYCR) - C\alpha\ (-SPSR+CPSYCR) + (S\alpha S\alpha_y)(CYCR) \end{bmatrix} \quad (27c)$$

When these definitions are used in equation (24), it may be written as

$$\dot{\overline{u}} = g\,\overline{X} + F_a^*\,\overline{N}_i \quad (28)$$

It is convenient to write the rotational equations of motion in the Lagrangian form. When the Eulerian angles (pitch, roll, yaw) are generalized coordinates, the rotational equations of motion take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_j}\right) - \frac{\partial T}{\partial \phi_j} = M_{\phi_j}. \qquad j = p,\ y,\ r \quad (29)$$

T is the rotational kinetic energy of the vehicle and M_{ϕ_j} is the moment associated with the ϕ_j rotation. Based on the assumption of an offset center of mass, all components of the inertia matrix are assumed to be non-zero. The inertia matrix is

$$[\mu] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad (30)$$

The rotational kinetic energy may be expressed with respect to the missile fixed coordinate system as

$$T = \tfrac{1}{2}\,\overline{\omega}^T\,[\mu]\,\overline{\omega} \qquad , \quad (31)$$

where $\overline{\omega}$ is the angular velocity of the vehicle in the missile fixed coordinate system.

When expressions (30) and (31) are substituted into equation (29), the result is

$$\frac{d}{dt}\left(\frac{\partial\overline{\omega}}{\partial\dot{\phi}_j}^T\right)[\mu]\overline{\omega} + \frac{\partial\overline{\omega}}{\partial\dot{\phi}_j}^T[\mu]\frac{d}{dt}(\dot{\overline{\omega}})$$

$$- \frac{\partial\overline{\omega}}{\partial\phi_j}^T[\mu]\overline{\omega} = M_{\phi_j} \qquad . \quad (32)$$

The angular velocity vector, $\overline{\omega}$, is obtained from a coordinate transformation of the angular velocity components $\dot{\overline{\emptyset}}_y$, $\dot{\overline{\emptyset}}_r$, and $\dot{\overline{\emptyset}}_p$ into the missile fixed reference. The transformation is dependent on the gimbal set used. The transformation matrix is developed for gimbal set 1. (Gimbal set 1 measures the Euler angles in the order pitch, yaw, roll.) A coordinate transformation is not required for $\dot{\overline{\emptyset}}_r$ since it is measured with respect to the missile coordinate system. The angular velocity component $\dot{\bar{\phi}}_y$ is expressed in the missile fixed reference by use of the rotation matrix $[-\phi_r]^T$. The transformation is

$$\dot{\bar{\phi}}_y\Big|_{missile} = [-\phi_r]^T \begin{bmatrix} \dot{\phi}_y \\ 0 \\ 0 \end{bmatrix} .$$

The angular velocity component $\dot{\bar{\phi}}_p$ is expressed in the missile fixed reference by use of two rotation matrices as follows

$$\dot{\bar{\phi}}_p\Big|_{missile} = [-\phi_r]^T\ [\phi_p]^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix} .$$

Thus, the angular velocity vector

$$\bar{\omega} = \dot{\bar{\phi}}_r + \dot{\bar{\phi}}_y\Big|_{missile} + \dot{\bar{\phi}}_p\Big|_{missile} \tag{33a}$$

or, in component form,

$$\begin{bmatrix} \omega_{xm} \\ \omega_{ym} \\ \omega_{zm} \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\phi}_r \\ 0 \end{bmatrix} + [-\phi_r]^T \begin{bmatrix} \dot{\phi}_y \\ 0 \\ 0 \end{bmatrix} + [-\phi_r]^T\ [\phi_y]^T \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix} \tag{33b}$$

which may be expressed as

$$\bar{\omega} = [A_\omega]_1 \, \dot{\bar{\emptyset}} \qquad , \qquad (33c)$$

where

$$\dot{\bar{\emptyset}} = \begin{bmatrix} \dot{\emptyset}_y \\ \dot{\emptyset}_r \\ \dot{\emptyset}_p \end{bmatrix} \qquad ,$$

and the transformation matrix

$$[A_\omega]_1 = \begin{bmatrix} CR & 0 & SRCY \\ 0 & -1 & SY \\ -SR & 0 & CRCY \end{bmatrix} \qquad . \qquad (34a)$$

A similar argument is used to develop transformation matrices for gimbal sets 2 and 3:

$$[A_\omega]_2 = \begin{bmatrix} 1 & 0 & SR \\ 0 & -CY & CRSY \\ 0 & SY & CRCY \end{bmatrix} \qquad (34b)$$

$$[A_\omega]_3 = \begin{bmatrix} CP & -SPCY & 0 \\ -SP & -CPCY & 0 \\ 0 & SY & 1 \end{bmatrix} \qquad . \qquad (34c)$$

The angular velocity vector, $\bar{\omega}$, is restated for the three gimbal sets as

$$\bar{\omega} = [A_\omega]_i \dot{\bar{\phi}} \qquad i = 1, 2, 3 \qquad . \qquad (35)$$

It should be noted that the transformation matrices $[A_\omega]_i$ are not orthogonal.

By use of the expressions obtained above, the rotational equations of motion become

$$\ddot{\bar{\phi}}_i = [C]_i \left\{ M_{\phi_i} - \left(\left\{ \frac{d}{dt}[A_\omega]_i^T \right\} [\mu][A_\omega]_i + [A_\omega]_i^T [\mu] \frac{d}{dt}[A_\omega]_i \right) \dot{\bar{\phi}}_i + \bar{B}_i \right\} \qquad (36)$$

where

$$[C]_i = \left[[A_\omega]_i^T [\mu] [A_\omega]_i \right]^{-1} \qquad , \qquad (37a)$$

$$B_j = \dot{\bar{\phi}}_i^{\,T} \frac{\partial [A_\omega]_i^T}{\partial \phi_j} [\mu][A_\omega] \dot{\bar{\phi}}_i \qquad , \qquad (37b)$$

$$\overline{M}_{\phi_i} = [A_\omega]_i^T \, \overline{M}_{Tm} \quad , \qquad (37c)$$

$$\overline{B}_i = \begin{bmatrix} B_y \\ B_r \\ B_p \end{bmatrix}_i \quad , \text{ and } \ddot{\overline{\phi}}_i = \begin{bmatrix} \ddot{\phi}_y \\ \ddot{\phi}_r \\ \ddot{\phi}_r \end{bmatrix} \quad . \qquad (37d)$$

$$i = 1,2,3 \qquad\qquad j = p,y,r$$

Definitions for $\dot{\overline{\phi}}$ and $\ddot{\overline{\phi}}$ are introduced that conform to the simplifications referred to in the problem statement. These definitions will be used throughout the remainder of the paper. For gimbal set 1:

$$\ddot{\overline{\phi}}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\overline{\phi}}_1 = \begin{bmatrix} 0 \\ \dot{\phi}_r \\ 0 \end{bmatrix} \quad ,$$

for gimbal set 2:

$$\ddot{\overline{\phi}}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\overline{\phi}}_2 = \begin{bmatrix} \dot{\phi}_y \\ 0 \\ 0 \end{bmatrix} \quad ,$$

and for gimbal set 3:

$$\ddot{\bar{\phi}}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \dot{\bar{\phi}}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_p \end{bmatrix}$$

It is noted that each of the matrices in the matrix product of equation (37a) is non-singular. Thus, the product is non-singular, and the rotational equations of motion, (36), can be reduced to the following form for each gimbal set.

$$M_{Tm} = [A_\omega]_i^{-1} \left\{ \left(\left\{ \frac{d}{dt} [A_\omega]_i^T \right\} [\mu] \; [A_\omega]_i + [A_\omega]_i^T \; [\mu] \; \frac{d}{dt} \; [A_\omega]_i \right) \dot{\bar{\phi}}_i - \bar{B}_i \right\} \qquad (38)$$

Three rotational equations of motion are obtained for each gimbal set from equation (38). The three equations may be solved for three unknowns. Because a particular computational procedure is anticipated, the equations for each gimbal set are solved for the roll force, F_r, the angle, α, and the angular velocity component that appears.

Gimbal Set 1

The three rotational equations of motion are:

$$
\begin{aligned}
-\dot{\phi}_r^2 I_{zy} &= F_a y_{cp} S\alpha S\alpha_y + F_a z_{cp} C\alpha \\
0 &= -F_a z_{cp} S\alpha C\alpha_y - F_a x_{cp} S\alpha S\alpha_y + 2F_r z_r \quad . \qquad (39) \\
\dot{\phi}_r^2 I_{xy} &= -F_a x_{cp} C\alpha + F_a y_{cp} S\alpha C\alpha_y
\end{aligned}
$$

The first and third of equations (39) are solved for

$$\alpha = \arctan\left[\frac{I_{zy}x_{cp} - I_{xy}z_{cp}}{y_{cp}(I_{zy}C\alpha_y + I_{xy}S\alpha_y)}\right] \quad . \qquad (42a)$$

The second of equations (39) is solved for

$$F_r = \frac{F_a S\alpha\,(x_{cp}S\alpha_y + z_{cp}C\alpha_y)}{2z_r} \quad , \qquad (43a)$$

and the third is solved for

$$\dot{\phi}_r = \pm\sqrt{\frac{F_a\,(y_{cp}S\alpha C\alpha_y - x_{cp}C\alpha)}{I_{xy}}} \quad . \qquad (44a)$$

Gimbal Set 2

The rotational equations of motion are:

$$
\begin{aligned}
0 &= F_a Y_{cp} S\alpha S\alpha_y + F_a Z_{cp} C\alpha \\
\dot{\phi}_y^{\,2} I_{xz} &= -F_a Z_{cp} S\alpha C\alpha_y - F_a X_{cp} S\alpha S\alpha_y + 2F_r Z_r \\
-\dot{\phi}_y^{\,2} I_{xy} &= -F_a X_{cp} C\alpha + F_a Y_{cp} S\alpha C\alpha_y
\end{aligned}
\tag{40}
$$

The first of equations (40) is solved for

$$
\alpha = \arctan \frac{-Z_{cp}}{Y_{cp} S\alpha_y} \quad . \tag{42b}
$$

The third of equations (40) is solved for

$$
\dot{\phi}_y = \pm\sqrt{\frac{F_a (X_{cp} C\alpha - Y_{cp} S\alpha C\alpha_y)}{I_{xy}}} \quad , \tag{43b}
$$

and the second is solved for

$$
F_r = \frac{\dot{\phi}_y^{\,2} I_{xz} + F_a S\alpha (Z_{cp} C\alpha_y + X_{cp} S\alpha_y)}{2Z_r} \quad . \tag{44b}
$$

Gimbal Set 3

The three rotational equations of motion are:

$$\begin{aligned}
\dot{\phi}_p^{\,2}\, I_{yz} &= F_a Y_{cp} S\alpha S\alpha_y + F_a Z_{cp} C\alpha \\
-\dot{\phi}_p^{\,2}\, I_{xz} &= -F_a Z_{cp} S\alpha C\alpha_y - F_a X_{cp} S\alpha S\alpha_y + 2F_r Z_r \\
0 &= -F_a X_{cp} C\alpha + F_a Y_{cp} S\alpha C\alpha_y
\end{aligned} \tag{41}$$

The third of equations (41) is solved for

$$\alpha = \arctan \frac{X_{cp}}{Y_{cp} C\alpha_y} \quad . \tag{42c}$$

The first of equations (41) is solved for

$$\dot{\phi}_p = \pm\sqrt{\frac{F_a\,(Y_{cp} S\alpha S\alpha_y + Z_{cp} C\alpha)}{I_{yz}}} \quad , \tag{43c}$$

and the second is solved for

$$F_r = \frac{F_a S\,(Z_{cp} C\alpha_y + X_{cp} S\alpha_y) - \dot{\phi}_p^{\,2}\, I_{xz}}{2Z_r} \quad . \tag{44c}$$

V. THE RELATIVE VELOCITY CONSTRAINTS

The fact that the relative velocity vector may be written in terms of the three coordinate systems constitutes an algebraic constraint given by

$$\overline{V}_{rm} = [A_d]_i \, \overline{V}_R = [A_a]^T \, \overline{V}_r \tag{16}$$

where $\overline{V}_{rm}$ is the relative velocity vector expressed in the missile coordinate system. Vector equation (16) yields three equations for each gimbal set. The three equations of each set are not independent. Hence, they may not be solved for three unknowns. For each gimbal set, the three equations are solved for two angular displacements. The uniqueness of these angular displacements is discussed in Appendix B.

<u>Gimbal Set 1</u>

The constraint equations are:

$$\begin{aligned}
(CRCP+SRSYSP)\,V_{Rx} + (CRSP-SRSYCP)\,V_{Ry} + SRSYV_{RZ} &= V_{rmx} \\
(-CYSP)\,V_{Rx} + CYCP\,V_{Ry} + SYV_{RZ} &= V_{rmy} \\
(-SRCP+CRSYSP)\,V_{Rx} - (SRSP+CRCPSY)\,V_{Ry} + CYV_{RZ} &= V_{rmz}
\end{aligned} \tag{45}$$

The first and third of equations (45) are solved for

$$SP = \frac{J\,V_{RY} - V_{RX}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} , \tag{46a}$$

and

$$CP = \frac{J\,V_{RX} + V_{RY}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(R_{RX}^2 + V_{RY}^2)} \tag{46b}$$

where

$$J = CR\;V_{rmx} - SR\;V_{rmz} ,$$

$$\phi_p = \arctan\frac{SP}{CP} . \tag{46c}$$

As shown in Appendix B, equations (46) may be solved for a unique value of ϕ_p only if

$$-\Pi \leq \phi_p \leq \Pi$$

The second set of equations (45) is solved for

$$SY = \frac{V_{rmy}\,V_{RZ} - K\sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} , \tag{47a}$$

and

$$CY = \frac{V_{rmy}\ K + V_{RZ}\sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} \quad , \qquad (47b)$$

where

$$K = CP\ V_{RY} - SP\ V_{RX} \quad ,$$

$$\phi_y = \arctan\frac{SY}{CY} \quad . \qquad (47c)$$

As shown in Appendix B, equations (47) may be solved for a unique value of ϕ_y only if

$$-\Pi \leq \phi_y \leq \Pi \quad .$$

<u>Gimbal Set 2</u>

The constraint equations are:

$$V_{RX}CRCP + V_{RY}CRSP + V_{RZ}SR = V_{rmx}$$

$$-V_{RX}CPSRSY - V_{RX}CYSP + V_{RY}(CYCP\text{-}SYSRSP) + V_{RZ}SYCR = V_{rmy} \qquad (48)$$

$$V_{RX}\ (SYSP\text{-}CYSRCP) - V_{RY}\ (SYCP\text{+}CYSRSP) + V_{RZ}CRCY = V_{rmz}$$

The second and third of equations (48) are combined to give

$$V_{RX}\,SP - V_{RX}\,CP = -V_{rmy}\,CY + V_{rmz}\,SY$$

which is solved for

$$SP = \frac{F\,V_{RX} + V_{RY}\sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} , \qquad (49a)$$

and

$$CP = \frac{F\,V_{RY} + V_{RX}\sqrt{V_{RX}^2 - F^2 - V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \qquad (49b)$$

where

$$F = -V_{rmy}\,CY + V_{rmz}\,SY ,$$

$$\phi_p = \arctan \frac{SP}{CP} . \qquad (49c)$$

As shown in Appendix B, equations (49) may be solved for a unique value of ϕ_p only if

$$-\pi \leq \phi_p \leq \pi .$$

The first of equations (48) is solved for

$$SR = \frac{V_{RZ} V_{rmx} - G\sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 - G^2)} , \qquad (50a)$$

and

$$SR = \frac{G V_{rmx} + V_{RZ}\sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \qquad (50b)$$

where

$$G = V_{RX} CP + V_{RY} SP ,$$

$$\emptyset_r = \arctan \frac{SR}{CR} . \qquad (50c)$$

As shown in Appendix B, equations (50) may be solved for a unique value of $\emptyset_r$ only if

$$-\Pi \leq \emptyset_r \leq \Pi .$$

Gimbal Set 3

The constraint equations are:

$$V_{RX}(CPCR-SPSRSY) + V_{RY}SPCY + V_{RZ}(CPSR+SPSYCR) = V_{rmx}$$

$$-V_{RX}(SPCR+CPSRSY) + V_{RY}CPCY + V_{RZ}(-SPSR+CPSYCR) = V_{rmy} \qquad (51)$$

$$-V_{RX}CYSR - V_{RY}SY + V_{RZ}CYCR = V_{rmz}$$

The first and second of equations (51) are solved for

$$SR = \frac{V_{RZ} A - V_{RX}\sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} , \qquad (52a)$$

and

$$CR = \frac{V_{RX} A + V_{RZ}\sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} \qquad (52b)$$

where

$$A = CP\, V_{rmx} - SP\, V_{rmy} ,$$

$$\phi_r = \arctan \frac{SR}{CR} . \qquad (52c)$$

As shown in Appendix B, equations (52) may be solved for a unique value of ϕ_r only if

$$-\pi \leq \phi_r \leq \pi .$$

The third of equations (51) is solved for

$$SY = \frac{-V_{RY} V_{rmz} + B\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} , \qquad (53a)$$

and

$$CY = \frac{B\ V_{rmz} + V_{RY}\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \qquad (53b)$$

where

$$B = V_{RZ}CR - V_{RX}SR \quad ,$$

$$\phi_y = \arctan\frac{SY}{CY} \quad . \qquad (53c)$$

As shown in Appendix B, equations (53) may be solved for a unique value of ϕ_y only if

$$-\Pi \leq \phi_y \leq \Pi \quad .$$

VI. FORMULATION OF THE OPTIMIZATION PROBLEM

The optimization problem is that of finding the optimum control process, $\alpha_y(t)$, that will transfer a vehicle from an initial state to a terminal state in an atmosphere in a manner so that the functional

$$J = \int_{t_o}^{t_1} f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\phi}}, \phi_r, \phi_y, \phi_p, \alpha, \alpha_y, F_r)\, dt$$

is a minimum. Since the Pontryagin formulation is to be used, it is necessary to write the Pontryagin H function for each gimbal set (5).

Gimbal Set 1

The Pontryagin H function is

$$H_1 = \overline{\lambda}_1 \cdot \dot{\overline{X}} + \overline{\lambda}_{II} \cdot \dot{\overline{u}} \pm \lambda_7 \dot{\phi}_r + \lambda_8 \dot{J} \quad , \qquad (55a)$$

which may be expressed as

$$H_1 = \overline{\lambda}_I \cdot \dot{\overline{X}} + \overline{\lambda}_{II} \cdot (g\, \overline{X} + F_a^* \overline{N}_1)$$

$$\pm \lambda_7 \sqrt{\frac{F_a (y_{cp}\, S\alpha\, C\alpha_y - x_{cp}\, C\alpha)}{I_{xy}}} + \lambda_8\, f(\overline{X}, \dot{\overline{X}}, \dot{\overline{\phi}}_1, \overline{\phi}, \alpha, \alpha_y, F_r) \quad (55b)$$

where

$$\bar{\lambda}_I = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} , \text{ and } \bar{\lambda}_{II} = \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} .$$

The $\lambda(t)$ are auxiliary variables used in a manner analogous to Lagrangian multipliers in the classical calculus of variations.

Gimbal Set 2

The Pontryagin H function is

$$H_2 = \bar{\lambda}_{III} \cdot \dot{\bar{X}} + \bar{\lambda}_{IV} \cdot \dot{\bar{u}} \pm \lambda_{15} \dot{\phi}_y + \lambda_{16} \dot{J} , \qquad (56a)$$

which may be expressed as

$$H_2 = \bar{\lambda}_{III} \cdot \dot{\bar{X}} + \bar{\lambda}_{IV} \cdot (g\bar{X} + F_a^* \bar{N}_2)$$

$$\pm \lambda_{15} \sqrt{\frac{F_a(x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}} + \lambda_{16} f(\bar{X}, \dot{\bar{X}}, \dot{\bar{\phi}}_2, \bar{\phi}, \alpha, \alpha_y, F_r) . \qquad (56b)$$

where

$$\bar{\lambda}_{III} = \begin{bmatrix} \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \end{bmatrix} , \text{ and } \bar{\lambda}_{IV} = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix} .$$

Gimbal Set 3

The Pontryagin H function is

$$H_3 = \bar{\lambda}_V \cdot \dot{\bar{X}} + \bar{\lambda}_{VI} \cdot \dot{\bar{u}} \pm \lambda_{23}\dot{\phi}_p + \lambda_{24}\dot{J} \quad , \text{(57a)}$$

which may be expressed as

$$H_3 = \bar{\lambda}_V \cdot \dot{\bar{X}} + \bar{\lambda}_{VI} \cdot (g\,\bar{X} + F_a^*\bar{N}_3)$$

$$\pm\lambda_{23}\sqrt{\frac{F_a(y_{cp}S\alpha S\alpha_y + z_{cp}C\alpha)}{I_{yz}}} + \lambda_{24}\, f(\bar{X}, \dot{\bar{X}}, \dot{\bar{\phi}}_3, \bar{\phi}, \alpha, \alpha_y, F_r) \quad \text{(57b)}$$

where

$$\bar{\lambda}_V = \begin{bmatrix} \lambda_{17} \\ \lambda_{18} \\ \lambda_{19} \end{bmatrix}$$

and

$$\bar{\lambda}_{VI} = \begin{bmatrix} \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix}$$

The expressions for the auxiliary variables are obtained from the H functions as follows:

Gimbal Set 1

$$-\dot{\bar{\lambda}}_I = \frac{\partial H_1}{\partial \bar{x}}$$

$$= F_o^* \frac{\partial(\bar{\lambda}_{II} \cdot \bar{N}_1)}{\partial \bar{x}} + (\bar{\lambda}_{II} \cdot \bar{N}_1)\frac{\partial F_o^*}{\partial \bar{x}} - \bar{\lambda}_{II} g$$

$$+ (\bar{\lambda}_{II} \cdot \bar{x})\frac{\partial g}{\partial \bar{x}} \pm \lambda_7 \sqrt{\frac{y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha}{I_{xy}}} \left\{\frac{\partial (F_o^*)^{1/2}}{\partial \bar{x}}\right\}$$

$$+ \lambda_8 \frac{\partial F_o^*}{\partial \bar{x}}. \tag{58a}$$

$$-\dot{\bar{\lambda}}_{II} = \frac{\partial H_1}{\partial \bar{u}}$$

$$= \bar{\lambda}_I + F_o^* \frac{\partial(\bar{\lambda}_{II} \cdot \bar{N}_1)}{\partial \bar{u}} + (\bar{\lambda}_{II} \cdot \bar{N}_1)\frac{\partial F_o^*}{\partial \bar{u}}$$

$$+ \lambda_7 \sqrt{\frac{y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha}{I_{xy}}} \left\{\frac{\partial (F_o^*)^{1/2}}{\partial \bar{u}}\right\}$$

$$+ \lambda_8 \frac{\partial F_o^*}{\partial \bar{u}}. \tag{59a}$$

$$-\dot{\lambda}_7 = \frac{\partial H_1}{\partial \phi_r} = F_a^* \left\{ \bar{\lambda}_{II} \cdot \frac{\partial \bar{N}_1}{\partial \phi_r} \right\}. \tag{60a}$$

$$-\dot{\lambda}_8 = \frac{\partial H_1}{\partial J} = 0. \tag{61a}$$

Gimbal Set 2

The expressions for the auxiliary variables are:

$$\begin{aligned} -\dot{\bar{\lambda}}_{III} &= \frac{\partial H_2}{\partial \bar{x}} \\ &= F_a^* \frac{\partial(\bar{\lambda}_{IV} \cdot \bar{N}_2)}{\partial \bar{x}} + (\bar{\lambda}_{IV} \cdot \bar{N}_2) \frac{\partial F_a^*}{\partial \bar{x}} \\ &\quad + \bar{\lambda}_{IV} g + (\bar{\lambda}_{IV} \cdot \bar{x}) \frac{\partial g}{\partial \bar{x}} \\ &\quad \pm \lambda_{15} \sqrt{\frac{x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y}{I_{xy}}} \left\{ \frac{\partial (F_a^*)^{1/2}}{\partial \bar{x}} \right\} \\ &\quad + \lambda_{16} \frac{\partial F_a^*}{\partial \bar{x}}. \end{aligned} \tag{58b}$$

$$-\dot{\bar{\lambda}}_{IV} = \frac{\partial H_2}{\partial \bar{u}}$$

$$= \bar{\lambda}_{III} + F_o^* \frac{\partial(\bar{\lambda}_{IV} \cdot \bar{N}_2)}{\partial \bar{u}} + (\bar{\lambda}_{IV} \cdot \bar{N}_2) \frac{\partial F_o^*}{\partial \bar{u}}$$

$$\pm \lambda_{15} \sqrt{\frac{x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y}{I_{xy}}} \left\{ \frac{\partial (F_o^*)^{1/2}}{\partial \bar{u}} \right\}$$

$$+ \lambda_{16} \frac{\partial F_o^*}{\partial \bar{u}}. \qquad (59b)$$

$$-\dot{\lambda}_{15} = \frac{\partial H_2}{\partial \phi_y} = F_o^* \left(\bar{\lambda}_{IV} \cdot \frac{\partial \bar{N}_2}{\partial \phi_y} \right). \qquad (60b)$$

$$-\dot{\lambda}_{16} = \frac{\partial H_3}{\partial J} = 0. \qquad (61b)$$

Gimbal Set 3

The expressions for the auxiliary variables are:

$$-\dot{\bar{\lambda}}_{V} = \frac{\partial H_3}{\partial \bar{x}}$$

$$= F_c^* \frac{\partial(\bar{\lambda}_{VI} \cdot \bar{N}_3)}{\partial \bar{x}} + (\bar{\lambda}_{VI} \cdot \bar{N}_3) \frac{\partial F_o^*}{\partial \bar{x}} + \bar{\lambda}_{VI}\, g$$

$$+ (\bar{\lambda}_{VI} \cdot \bar{x}) \frac{\partial g}{\partial \bar{x}} \pm \lambda_{23} \sqrt{\frac{y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha}{I_{yz}}} \left\{ \frac{\partial (F_o^*)^{1/2}}{\partial \bar{x}} \right\}$$

$$+ \lambda_{24} \frac{\partial F_o^*}{\partial \bar{x}}. \qquad (58c)$$

$$-\dot{\lambda}_{VI} = \frac{\partial H_3}{\partial \bar{u}}$$

$$= \bar{\lambda}_{V} + F_a^{*}\frac{\partial(\bar{\lambda}_{VI}\cdot\bar{N}_3)}{\partial \bar{u}} + (\bar{\lambda}_{VI}\cdot\bar{N}_3)\frac{\partial F_a^{*}}{\partial \bar{u}}$$

$$\pm \lambda_{23}\sqrt{\frac{y_{cp}S\alpha S\alpha_y + z_{cp}C\alpha}{I_{yz}}}\left\{\frac{\partial (F_a^{*})^{\frac{1}{2}}}{\partial \bar{u}}\right\}$$

$$+ \lambda_{24}\frac{\partial F_a^{*}}{\partial \bar{u}}. \qquad (59c)$$

$$-\dot{\lambda}_{23} = \frac{\partial H_3}{\partial \phi_p} = F_a^{*}\left\{\bar{\lambda}_{VI}\cdot\frac{\partial \bar{N}_3}{\partial \phi_p}\right\}. \qquad (60c)$$

$$-\dot{\lambda}_{24} = \frac{\partial H_3}{\partial J} = 0. \qquad (61c)$$

Equations (61a), (61b) and (61c) imply that λ_8, λ_{16}, and λ_{24} are constant. The constant in each case is taken equal to plus one. This insures that a minimization of the H function is also a minimization of the payoff function.

The necessary condition for a critical value of J is

$$\frac{\partial H_i}{\partial \alpha_y} = 0 \tag{62}$$

where

$$i = 1, 2, 3$$

The inequality

$$\frac{\partial^2 H_i}{\partial \alpha_y^2} \geq 0 \tag{63}$$

must also be satisfied to insure a minimum of the payoff function. (Note: The criteria expressed in (62) and (63) are valid only if the H function is differentiable at each point on the trajectory.) Partial differentiation of the H functions as indicated in (62) and (63) produces the equations given on the following page.

$$\frac{\partial H_1}{\partial \alpha_y} = \bar{\lambda}_{\dot{II}} F_a^* \frac{\partial \bar{N}_1}{\partial \alpha_y} \pm \lambda_7 \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha)}{I_{xy}}}$$

$$+ \lambda_8 \frac{\partial}{\partial \alpha_y} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_1, \alpha_y, \alpha, F_r) = 0. \quad \text{(64a)}$$

$$\frac{\partial H_2}{\partial \alpha_y} = \bar{\lambda}_{\dot{IV}} F_a^* \frac{\partial \bar{N}_2}{\partial \alpha_y} \pm \lambda_{15} \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}}$$

$$+ \lambda_{16} \frac{\partial}{\partial \alpha_y} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_2, \alpha_y, \alpha, F_r) = 0. \quad \text{(64b)}$$

$$\frac{\partial H_3}{\partial \alpha_y} = \bar{\lambda}_{\dot{IV}} F_a^* \frac{\partial \bar{N}_3}{\partial \alpha_y} \pm \lambda_{23} \frac{\partial}{\partial \alpha_y} \sqrt{\frac{F_a (y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha)}{I_{yz}}}$$

$$+ \lambda_{24} \frac{\partial}{\partial \alpha_y} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_3, \alpha_y, \alpha, F_r) = 0. \quad \text{(64c)}$$

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{II}} \bar{F}_0^{*} \frac{\partial^2 \bar{\Pi}_1}{\partial \alpha_y^2} \pm \lambda_7 \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{\bar{F}_0 (y_{cp} S\alpha C\alpha_y - x_{cp} C\alpha)}{I_{xy}}}$$

$$+ \lambda_8 \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_1, \alpha, \alpha_y, F_r) > 0. \quad \text{(65a)}$$

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{IV}} \bar{F}_0^{*} \frac{\partial^2 \bar{\Pi}_2}{\partial \alpha_y^2} \pm \lambda_{15} \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{\bar{F}_0 (x_{cp} C\alpha - y_{cp} S\alpha C\alpha_y)}{I_{xy}}}$$

$$+ \lambda_{16} \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_2, \alpha, \alpha_y, F_r) > 0. \quad \text{(65b)}$$

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} = \bar{\lambda}_{\dot{VI}} \bar{F}_0^{*} \frac{\partial^2 \bar{\Pi}_3}{\partial \alpha_y^2} \pm \lambda_{23} \frac{\partial^2}{\partial \alpha_y^2} \sqrt{\frac{\bar{F}_0 (y_{cp} S\alpha S\alpha_y + z_{cp} C\alpha)}{I_{yz}}}$$

$$+ \lambda_{24} \frac{\partial^2}{\partial \alpha_y^2} f(\bar{x}, \dot{\bar{x}}, \bar{\phi}, \dot{\bar{\phi}}_3, \alpha, \alpha_y, F_r) > 0. \quad \text{(65c)}$$

The algebraic and differential constraint equations (28), (42), (43), (44), (46c), (47c), (49c), (50c), (52c), and (53c), and the characteristic equations (58), (59), (60), (61), and (64) form a complete set of equations for the problem. To insure that the payoff function has been minimized, the inequality (65) must also be satisfied.

VII. COMPUTATIONAL PROCEDURE

The problem formulated is of a general nature and the equations involved are quite complex. It is highly improbable that a closed form solution can be found. Therefore, no time has been spent in search of this type solution. A computational scheme is suggested in order that trajectories may be generated on a digital computer. For convenience in the discussion of the computational scheme, the principle equations are written in functional notation.

<u>Gimbal Set 1</u>

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66a}$$

$$\dot{\phi}_r = \pm \dot{\phi}_r (\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67a}$$

$$\phi_p = \phi_p(\phi_r, \overline{X}, \dot{\overline{X}}, \alpha, \alpha_y) \tag{68a}$$

$$\phi_y = \phi_y(\phi_p, \overline{X}, \dot{\overline{X}}, \alpha, \alpha_y) \tag{69a}$$

$$\ddot{X} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y) \tag{70a}$$

$$F_r = F_r(\alpha, \alpha_y) \tag{71a}$$

$$H_1 = H_1(\overline{X}, \dot{\overline{X}}, \overline{\phi}, \alpha, \alpha_y, \lambda_i) \tag{72a}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{73a}$$

$$\frac{\partial H_1}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_1(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74a}$$

Gimbal Set 2

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66b}$$

$$\dot{\emptyset}_y = \pm \dot{\emptyset}_y(\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67b}$$

$$\emptyset_r = \emptyset_r(\emptyset_y, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{68b}$$

$$\emptyset_p = \emptyset_p(\emptyset_r, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{69b}$$

$$\ddot{\overline{X}} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y) \tag{70b}$$

$$F_r = F_r(\emptyset_y, \alpha, \alpha_y) \tag{71b}$$

$$H_2 = H_2(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{72b}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{73b}$$

$$\frac{\partial H_2}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_2(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74b}$$

Gimbal Set 3

The important equations expressed in functional notation are:

$$\alpha = \alpha(\alpha_y) \tag{66c}$$

$$\dot{\emptyset}_p = \pm \dot{\emptyset}_p(\alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{67c}$$

$$\emptyset_r = \emptyset_r(\emptyset_p, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{68c}$$

$$\emptyset_y = \emptyset_y(\emptyset_r, \alpha, \alpha_y, \overline{X}, \dot{\overline{X}}) \tag{69c}$$

$$\ddot{\overline{X}} = \ddot{\overline{X}}(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y) \tag{70c}$$

$$F_r = F_r(\emptyset_y, \alpha, \alpha_y) \tag{71c}$$

$$H_3 = H_3(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{72c}$$

$$\dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) \tag{73c}$$

$$\frac{\partial H_3}{\partial \alpha_y} = \frac{\partial}{\partial \alpha_y} H_3(\overline{X}, \dot{\overline{X}}, \overline{\emptyset}, \alpha, \alpha_y, \lambda_i) = 0 \tag{74c}$$

A complete set of equations has been developed for each gimbal set. Therefore, three independent, but similar, computational procedures are written. All three computational procedures require the following initial data:

Atmospheric tables for ρ as a function of position

Atmospheric tables for $\overline{W}$ as a function of position

Aerodynamic tables for $C_x(\alpha^*)$ and $C_z(\alpha^*)$ as a function of α^*

Values for:

A	R_o
G	$\overline{x}_{cp}$
m	$\overline{z}_r$
M	$[\mu]$

Plumbline position, $\overline{X}_o$, and velocity, $\dot{\overline{X}}_o$, vectors at the initial point on the optimum trajectory

Computational procedure for Gimbal Set 1

Initial values for the auxiliary variables, λ_i, and the roll angle, ϕ_r, are required. It is assumed that these values are known. These initial data are referred to as:

$$\overline{\lambda}_{I_0} = \begin{bmatrix} \lambda_{1_0} \\ \lambda_{2_0} \\ \lambda_{3_0} \end{bmatrix} , \qquad \overline{\lambda}_{II_0} = \begin{bmatrix} \lambda_{4_0} \\ \lambda_{5_0} \\ \lambda_{6_0} \end{bmatrix} ,$$

$$\lambda_{7_0} ,$$

$$\lambda_{8_0} = 1 ,$$

$$\phi_{r_0} .$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180°$
2. Choose the positive sign in equation (67a) and compute:

   a. α from (66a); iterate (11c) for α^*

   b. $\dot{\phi}_r$ from (67a)

   c. ϕ_p from (68a)

d. ρ_y from (69a)

e. $\ddot{\overline{X}}$ from (70a)

f. H_1 from (72a)

g. $\frac{\partial H_1}{\partial \alpha_y}$ from (74a)

3. Choose $\alpha_y = \alpha_y + 5°$ and repeat step 2. Continue until $\alpha_y = +180°$.
4. Repeat steps 1 through 3 using the negative sign in equation (67a).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67a)	α_y	H_1	$\frac{\partial H_1}{\partial \alpha_y}$	Eqn. - (67a)	α_y	H_1	$\frac{\partial H_1}{\partial \alpha_y}$

A plot of H_1 vs α_y should yield insight as to the number of solutions that exist. In addition, this plot should yield a starting value of α_y for the iteration of equation (74a).

Preload Computation II

5. Use the positive sign in equation (67a) and the results of Preload Computation I to iterate equation (74a) for α_y.

6. Use the α_y computed in step 5 to compute

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} .$$

7. If the inequality

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} > 0$$

is satisfied, a minimum exists. Proceed to step 12. Use the positive sign in equation (67a) in all remaining calculations. If the inequality is not satisfied, proceed to step 8.

8. Use the negative sign in equation (67a) and the results from Preload Computation I to iterate equation (74a) for α_y.

9. Use the α_y found in step 8 to compute

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_1}{\partial \alpha_y^2} > 0 \ .$$

11. Proceed to step 12. Use the negative sign in equation (67a) in all remaining calculations.

<u>"N" line computation</u>

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67a) to iterate (74a) for α_y.

13. Use the α_y computed in step 12 and the initial data to compute:

a. α from equation (66a); iterate (11c) for α^*

b. $\dot{\phi}_r$ from equation (67a)

c. ϕ_p from equation (68a)

d. ϕ_y from equation (69a)

e. $\ddot{X}$ from equation (70a)

f. F_r from equation (71a)

g. H_1 from equation (72a)

h. $\dot{\lambda}_I$ from equation (73a)

i. $\dot{\overline{\lambda}}_{II}$ from equation (75a)

j. $\dot{\lambda}_7$ from equation (75a)

14. Use a numerical integration technique to integrate

$$\ddot{\overline{X}} \text{ for } \dot{\overline{X}} \text{ for } \overline{X} \quad ,$$

$$\dot{\phi}_r \text{ for } \phi_r \quad ,$$

$$\dot{\overline{\lambda}}_I \text{ for } \overline{\lambda}_I \quad ,$$

$$\dot{\overline{\lambda}}_{II} \text{ for } \overline{\lambda}_{II} \quad ,$$

$$\dot{\lambda}_7 \text{ for } \lambda_7 \quad .$$

15. Use the integrated values from step 14 for the new initial values in the "N + 1" line computation.

Computational procedure for Gimbal Set 2

Initial values for the auxiliary varibles and the yaw angle are required. It is assumed that these values a known. These initial data re referred to as:

$$\overline{\lambda}_{III_0} = \begin{bmatrix} \lambda_{9_0} \\ \lambda_{10_0} \\ \lambda_{11_0} \end{bmatrix} , \qquad \overline{\lambda}_{IV_0} = \begin{bmatrix} \lambda_{12_0} \\ \lambda_{13_0} \\ \lambda_{14_0} \end{bmatrix} ,$$

4. Repeat steps 1 through 3 but use the negative sign in equation (67b).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67b)	α_y	H_2	$\frac{\partial H_2}{\partial \alpha_y}$	Eqn. - (67b)	α_y	H_2	$\frac{\partial H_2}{\partial \alpha_y}$

A plot of H_2 vs α_y should yield insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for α_y to be used in the iteration of equation (74b).

Preload Computation II

5. Use the positive sign in equation (67b) and the results of Preload Computation I to iterate equation (74b) for α_y.
6. Use the value of α_y found in step 5 to compute

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} .$$

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7. If the inequality

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} > 0$$

is satisfied a minimum exists. Proceed to step 12. Use the positive sign in equation (67b) in all remaining calculations. If the inequality is not satisfied, proceed to step 8.

8. Use the negative sign in equation (67b) and the results from Preload Computation I to iterate equation (74b) for α_y.

9. Use the value of α_y found in step 8 to compute

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_2}{\partial \alpha_y^2} > 0 .$$

11. Proceed to step 12. Use the negative sign in equation (67b) in all remaining calculations.

<u>"N" line computation</u>

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67b) to iterate equation (74b) for α_y.

13. Use the value of α_y computed in step 12 and the initial data to compute:

a. α from equation (66b); iterate (11c) for α^*

b. $\dot{\phi}_y$ from equation (67b)

c. ϕ_r from equation (68b)

d. ϕ_p from equation (69b)

e. $\ddot{\bar{X}}$ from equation (70b)

f. F_r from equation (71b)

g. H_2 from equation (72b)

h. $\dot{\bar{\lambda}}_{III}$ from equation (73b)

i. $\dot{\bar{\lambda}}_{IV}$ from equation (73b)

j. $\dot{\lambda}_{15}$ from equation (73b)

14. Use a numerical integration technique to integrate

$$\ddot{\bar{X}} \text{ for } \dot{\bar{X}} \text{ for } \bar{X} \quad ,$$

$$\dot{\phi}_y \text{ for } \phi_y \quad ,$$

$$\dot{\bar{\lambda}}_{III} \text{ for } \bar{\lambda}_{III} \quad ,$$

$$\dot{\bar{\lambda}}_{IV} \text{ for } \bar{\lambda}_{IV} \quad ,$$

$$\dot{\lambda}_{15} \text{ for } \lambda_{15} \quad .$$

15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.

Computational procedure for Gimbal Set 3

Initial values for the auxiliary variables and the pitch angle are required. It is assumed that these values are known. These initial data are referred to as:

$$\bar{\lambda}_{V_0} = \begin{bmatrix} \lambda_{17_0} \\ \lambda_{18_0} \\ \lambda_{19_0} \end{bmatrix} , \qquad \bar{\lambda}_{VI_0} = \begin{bmatrix} \lambda_{20_0} \\ \lambda_{21_0} \\ \lambda_{22_0} \end{bmatrix} ,$$

$$\lambda_{23_0} ,$$

$$\lambda_{24_0} = 1 ,$$

$$\phi_{r_0} .$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180°$
2. Choose the positive sign in equation (67c) and compute:

   a. α from (66c); iterate (11c) for α^*

   b. $\ddot{\phi}_p$ from (67c)

   c. ϕ_r from (68c)

   d. ϕ_y from (69c)

   e. $\ddot{\bar{X}}$ from (70c)

   f. H_3 from (72c)

   g. $\dfrac{\partial H_3}{\partial \alpha_y}$ from (74c)

3. Choose $\alpha_y = \alpha_y + 5°$ and repeat step 2. Continue until $\alpha_y = +180°$.
4. Repeat steps 1 through 3, but use the negative sign in equation (67c).

The results of Preload Computation I should be tabulated as follows:

Eqn. + (67c)	α_y	H_3	$\frac{\partial H_3}{\partial \alpha_y}$	Eqn. - (67c)	α_y	H_3	$\frac{\partial H_3}{\partial \alpha_y}$

A plot of H_3 vs α_y should give insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for α_y to be used in the iteration of equation (74c).

Preload Computation II

5. Use the positive sign in equation (67c) and the results of Preload Computation I to iterate equation (74c) for α_y.
6. Use the value of α_y found in step 5 to compute

$$\frac{\partial^2 H_3}{\partial \alpha_y^2}$$

7. If the inequality

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} > 0$$

is satisfied, a minimum exists. Proceed to step 12. Use the positive sign in equation (67c) in all remaining calculations. If the inequality is not satisfied proceed to step 8.

8. Use the negative sign in equation (67c) and the results from Preload Computation I to iterate equation (74c) for α_y.

9. Use the value of α_y found in step 8 to compute

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} .$$

10. Check to assure that

$$\frac{\partial^2 H_3}{\partial \alpha_y^2} > 0 .$$

11. Proceed to step 12. Use the negative sign in equation (74c) in all remaining calculations.

"N" line computation

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67c) to iterate equation (74c) for α_y.

13. Use the value of α_y computed in step 12 and the initial data to compute:

   a. α from equation (66c); iterate (11c) for α^*

   b. $\dot{\phi}_p$ from equation (67c)

c. $\emptyset_r$ from equation (68c)

d. $\emptyset_y$ from equation (69c)

e. $\ddot{\overline{X}}$ from equation (70c)

f. F_r from equation (71c)

g. H_3 from equation (72c)

h. $\dot{\overline{\lambda}}_V$ from equation (73c)

i. $\dot{\overline{\lambda}}_{VI}$ from equation (73c)

j. $\dot{\lambda}_{23}$ from equation (73c)

14. Use a numerical integration technique to integrate

$$\ddot{\overline{X}} \text{ for } \dot{\overline{X}} \text{ for } \overline{X} \quad ,$$

$$\dot{\emptyset}_p \text{ for } \emptyset_p \quad ,$$

$$\dot{\overline{\lambda}}_V \text{ for } \overline{\lambda}_V \quad ,$$

$$\dot{\overline{\lambda}}_{VI} \text{ for } \overline{\lambda}_{VI} \quad ,$$

$$\dot{\lambda}_{23} \text{ for } \lambda_{23}$$

15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.

VIII. DISCUSSION

The problem studied has application to cases involving the flight of any "unpowered" vehicle through any atmosphere--subject to the assumptions given in the problem statement. For example any space vehicle returning to the earth's surface must pass through the earth's atmosphere. This paper provides a method for determining an optimum trajectory for the transfer of the vehicle through the atmosphere. The pay-off function to be minimized over the atmospheric trajectory is a function of the state and control variables. For example, it may be desirable to minimize quantities such as the accumulative aerodynamic drag or the aerodynamic heating.

In order to solve the rotational equations of motion for three unknowns, it was necessary to introduce particular definitions for the angular acceleration, $\ddot{\bar{\phi}}$, and the angular velocity, $\dot{\bar{\phi}}$, of the vehicle. The definitions essentially eliminate all angular acceleration and two of the three components of the angular velocity for any given gimbal set. Thus, response of equipment and/or crew on the vehicle to a particular angular velocity may dictate choice of gimbal sets.

In the numerical generation of a trajectory, it is possible that an Euler angle will be computed that produces gimbal lock. A trajectory that produces gimbal lock is not admissible since gimbals will not function when in the gimbal lock orientation. Should the situation of gimbal lock arise, a new set of initial values for the auxiliary variables may be selected and a new trajectory generated. A particular set of auxiliary variables will yield an optimum trajectory for each gimbal set. The trajectory generated will not be the same for each gimbal set even though the same initial values of the auxiliary variables are chosen. No attempt has been made in this paper to determine the initial values of the auxiliary variables for any of the gimbal sets.

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APPENDIX A

__Experimentally Determined Values of C_x and C_z for a Typical Space Vehicle__

α^* Degrees	C_z	C_x
± 0	± 0	+ 1.828
± 5	± .014	+ 1.812
± 10	± .028	+ 1.772
± 15	± .040	+ 1.710
± 20	± .052	+ 1.626
± 25	± .063	+ 1.520
± 30	± .074	+ 1.388
± 35	± .084	+ 1.246
± 40	± .096	+ 1.092
± 45	± .118	+ .932
± 50	± .146	+ .768
± 55	± .182	+ .588
± 60	± .224	+ .416
± 65	± .268	+ .256
± 70	± .318	+ .112
± 75	± .372	- .020
± 80	± .426	- .134
± 85	± .486	- .236
± 90	± .546	- .322
± 95	± .596	- .394
± 100	± .628	- .444
± 105	± .690	- .486
± 110	± .728	- .516
± 115	± .756	- .542
± 120	± .772	- .566
± 125	± .776	- .582
± 130	± .772	- .586
± 135	± .756	- .584
± 140	± .730	- .576
± 145	± .686	- .560
± 150	± .628	- .544
± 155	± .554	- .524
± 160	± .468	- 510
± 165	± .366	- .498
± 170	± .248	- .490
± 175	± .130	- .484
± 180	± 0	- .480

APPENDIX B

Uniqueness of Solution for the Euler Angles

The relative velocity constraint equations are solved for two Euler angles in each gimbal set. The identity

$$\sin^2 \phi + \cos^2 \phi = 1$$

is used. Thus, the question arises as to which sign should be used with the radical that appears. This question is answered for each gimbal set by considering the way in which the coordinate systems are defined.

Gimbal Set 1

A first algebraic solution of equations (45) for ϕ_p and ϕ_y yields

$$SP = \frac{J\,V_{RY} \pm V_{RX}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)}, \qquad (B1)$$

and

$$CP = \frac{J\,V_{RX} \pm V_{RY}\sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RY}^2 + V_{RY}^2)} \qquad (B2)$$

where

$$J = CR\ V_{rmx} - SR\ V_{rmz} .$$

$$SY = \frac{V_{rmy} V_{RZ} \pm K \sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} , \quad \text{(B3)}$$

and

$$CY = \frac{V_{rmy} K \pm V_{RZ} \sqrt{V_{RZ}^2 - V_{rmy}^2 + K^2}}{(V_{RZ}^2 + K^2)} \quad \text{(B4)}$$

where

$$K = CP\ V_{RY} - SP\ V_{RX} .$$

The identity

$$SP^2 + CP^2 = 1$$

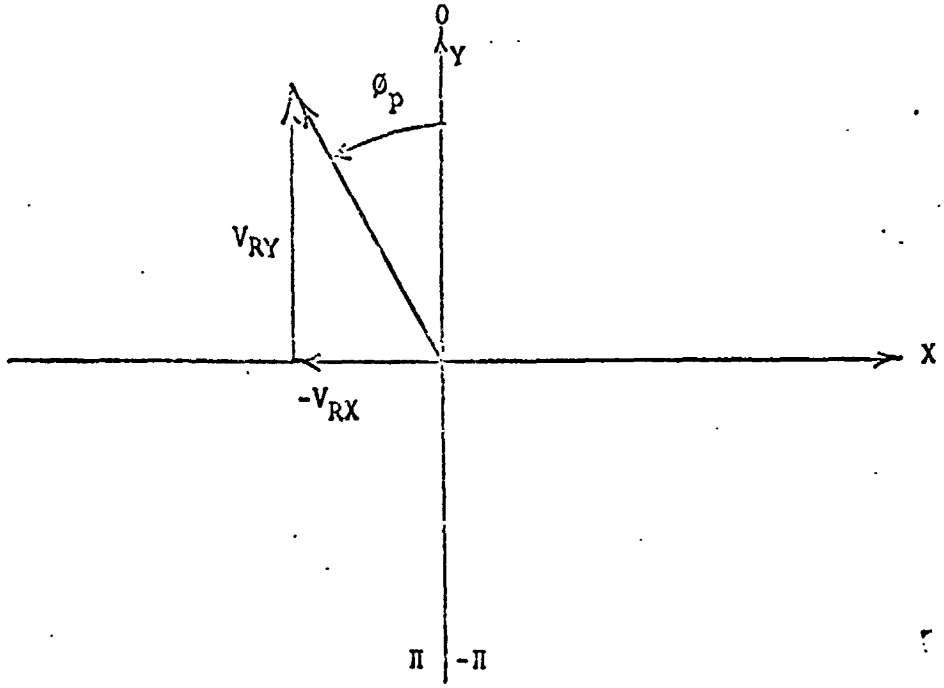
is satisfied only if opposite signs appear with the radical in (B1) and (B2). Let $\phi_r = \alpha = 0$. Then equations (B1) and (B2) reduce to

$$SP = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} , \quad \text{(B5)}$$

and

$$CP = \frac{\pm V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} . \quad \text{(B6)}$$

Consider the positive pitch angle, $\emptyset_p$, shown in Appendix Figure 1. Now restrict $\emptyset_p$, $-\Pi \leqq \emptyset_p \leqq \Pi$.



Coordinate System Showing A Positive Pitch Angle

Appendix Figure 1

Thus, the correct signs for the sine and cosine are

$$SP = \frac{-\ V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} , \qquad (B7)$$

and

$$CP = \frac{+\ V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} . \qquad (B8)$$

The identity

$$SY^2 + CY^2 + 1$$

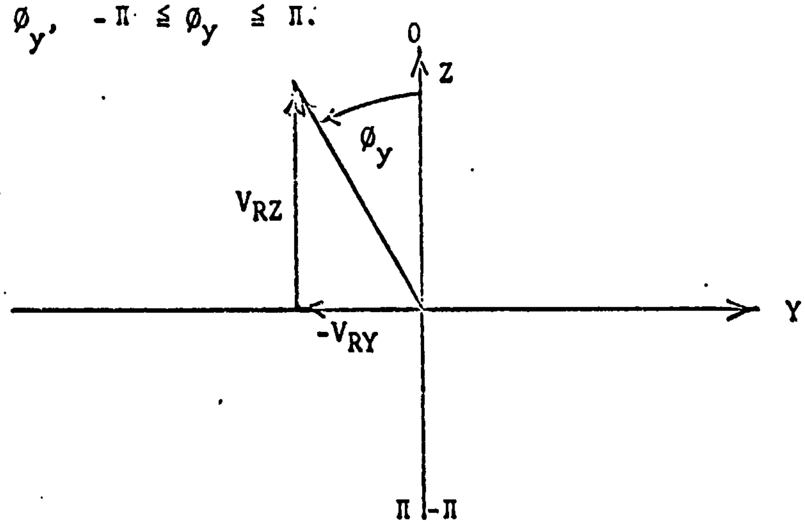
is satisfied only if opposte signs appear with the radical in (B3) and (B4). Let $\alpha_y = 90$ and $\phi_p = 0$. Then equations (B3) and (B4) reduce to

$$SY = \frac{\pm V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad (B9)$$

and

$$CY = \frac{\pm V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad (10)$$

Consider the positive yaw angle, ϕ_y, shown in Appendix Figure 2. Now restrict ϕ_y, $-\Pi \leq \phi_y \leq \Pi$.



Coordinate System Showing A Positive Yaw Angle ϕ_y

Appendix Figure 2

Thus, the correct signs for the sine and cosine are

$$SY = \frac{-V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad \text{(B11)}$$

and

$$CY = \frac{+V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad \text{(B12)}$$

<u>Gimbal Set 2</u>

A first algebraic solution of equations (48) for ϕ_p and ϕ_r yields

$$SP = \frac{F\ V_{RX} \pm V_{RY} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} , \quad \text{(3)}$$

and

$$CP = \frac{-F\ V_{RY} \pm V_{RX} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad \text{(B14)}$$

where

$$F = -V_{rmy}\ CY + V_{rmz}\ SY .$$

$$SR = \frac{V_{RZ}\, V_{rmx} \pm G \sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \quad , \qquad \text{(B15)}$$

and

$$CR = \frac{G\, V_{rmx} \pm V_{RZ} \sqrt{V_{RZ}^2 - V_{rmx}^2 + G^2}}{(V_{RZ}^2 + G^2)} \qquad \text{(B16)}$$

where

$$G = V_{RX}\, CP + V_{RY}\, SP \quad .$$

The identity

$$SP^2 + CP^2 = 1$$

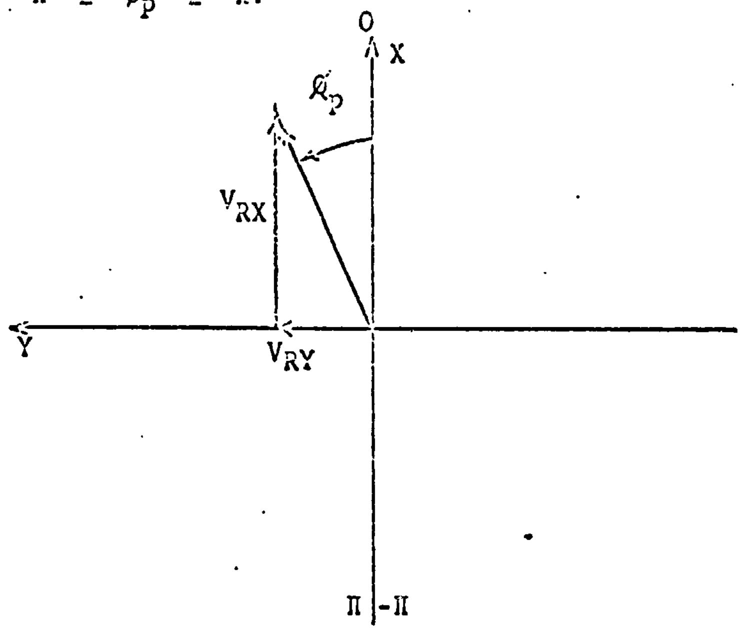
is satisfied only if the same sign appears with the radical in (B13) and (B14). Let $\alpha = 0$ and $\emptyset_y = 90°$. Then equations (B13) and (B14) reduce to

$$SP = \frac{\pm V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad , \qquad \text{(B17)}$$

and

$$CP = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}} \quad . \qquad \text{(B18)}$$

Consider the positive pitch angle, ϕ_p, shown in Appendix Figure 3. Now restrict ϕ_p, $-\Pi \leq \phi_p \leq \Pi$.



Coordinate System Showing A Positive Pitch Angle

Appendix Figure 3

Thus, the positive sign is chosen for both the sine and cosine.

$$SP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RX}^2}} , \qquad (B19)$$

and

$$CP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RX}^2}} . \qquad (B20)$$

The identity

$$SR^2 + CR^2 = 1$$

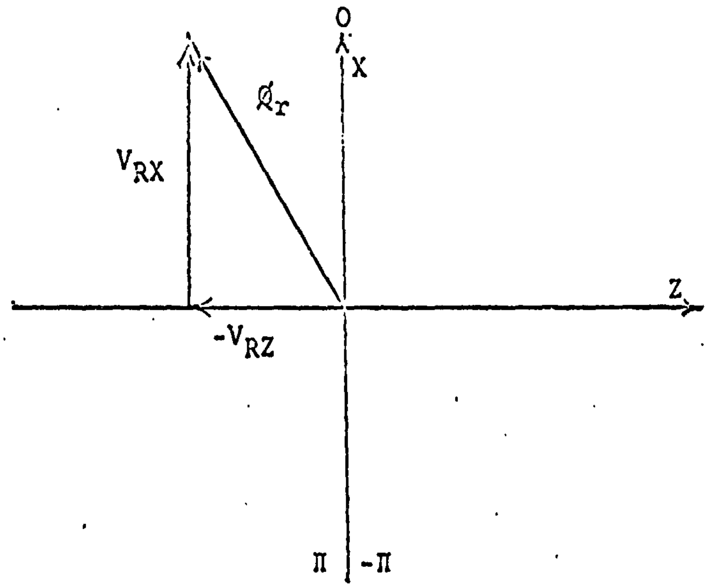
is satisfied only if opposite signs appear with the radical in (B15) and (B16). Let $\alpha = \phi_p = 0$. Then equations (B15) and (B16) reduce to

$$SR = \frac{\pm V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} , \quad (B21)$$

and

$$CR = \frac{\pm V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} . \quad (B22)$$

Consider the positive roll angle, ϕ_r, shown in Appendix Figure 4. Now restrict ϕ_r, $-\Pi \leqq \phi_r \leqq \Pi$.



Coordinate System Showing A Positive Ro.. Angle

Appendix Figure 4

Thus, the correct signs for the sine and cosine are

$$SR = \frac{-V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} , \quad (B23)$$

and

$$CR = \frac{+V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} . \quad (B24)$$

Gimbal Set 3

A first algebraic solutic . of equations (51) for ϕ_y and ϕ_r yields

$$SR = \frac{V_{RZ}\,A \pm V_{RX}\sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} , \quad (B25)$$

where

$$CR = \frac{V_{RX}\,A \pm V_{RZ}\sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} \quad (B26)$$

where

$$A = CP\,V_{rmx} - SP\,V_{rmy} .$$

$$SY = \frac{-V_{RY}\, V_{rmz} \pm B\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \quad , \qquad (B27)$$

and

$$CY = \frac{B\, V_{rmz} \pm V_{RY}\sqrt{V_{RY}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \quad . \qquad (B28)$$

where

$$B = V_{RZ}\, CR - V_{RX}\, SR \quad .$$

The identity

$$SR^2 + CR^2 = 1$$

is satisfied only if opposite signs appear with the radical in (B25) and (B26). Let $\alpha = \phi_p = 0$. Then equations (B25) and (B26) reduce to

$$SR = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RZ}^2}} \quad , \qquad (B29)$$

and

$$CR = \frac{\pm V_{RZ}}{\sqrt{V_{RX}^2 + V_{RZ}^2}} \quad . \qquad (B30)$$

Note, equations (B29) and (B30) are the same as (B21) and (B22). The identity

$$SR^2 + CR^2 = 1$$

is satisfied in the same way in each case. Hence, the signs for the sine and cosine are chosen the same as in equations (B23) and (B24). The identity

$$SY^2 + CY^2 = 1$$

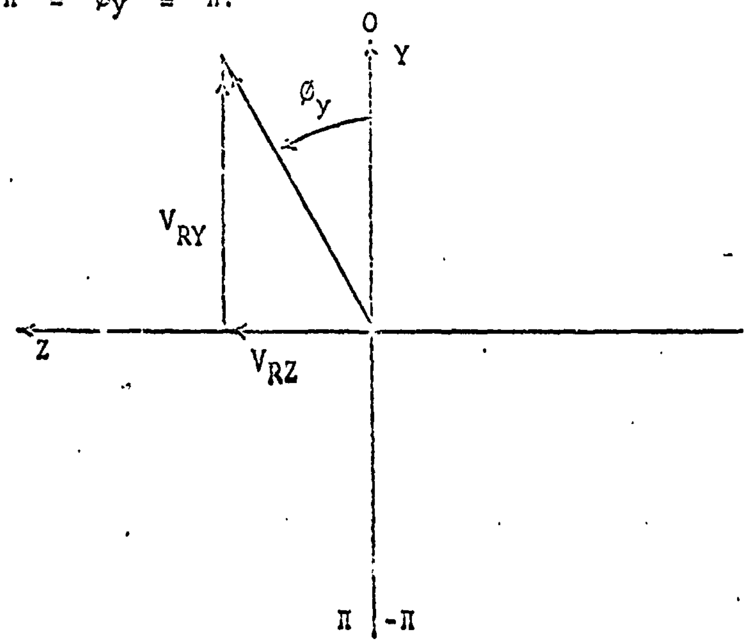
is satisfied only if the same sign appears with the radical in (B27) and (B28). Let $\alpha = \phi_r = 0$. Then equations (B27) and (B28) reduce to

$$SY = \frac{\pm V_{RZ}}{\sqrt{V_{RY}^2 + V_{RZ}^2}} \quad , \qquad \text{(B31)}$$

and

$$CY = \frac{\pm V_{RY}}{\sqrt{V_{RY}^2 + V_{RZ}^2}} \quad . \qquad \text{(B32)}$$

Consider the positive yaw angle, ϕ_y, shown in Appendix Figure 5. Now restrict ϕ_y, $-\Pi \leqq \phi_y \leqq \Pi$.



Coordinate System Showing A Positive Yaw Angle

Appendix Figure 5

Thus, the positive sign is chosen for both the sine and cosine.

$$SY = \frac{+ V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} , \quad \text{(B33)}$$

and

$$CY = \frac{+ V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} . \quad \text{(B34)}$$

N74-31301

Semi-Annual Report on NASA Grant NGR-01-003-008

RESEARCH ON

DEVELOPMENT OF EQUATIONS FOR PERFORMANCE TRAJECTORY COMPUTATIONS

During the period November 1, 1967, to May 1, 1968, at the suggestion of Mr. W. E. Miner of NASA, ERC, Cambridge, Massachusets, major emphasis was placed on investigating the analytical foundation of the Hamilton-Jacobi theory from the standpoint of its possible applications of space flight. Several references were obtained, as listed in the back of this report, and a study of previous work by several authors was undertaken.

As of May 1, 1968, a specific problem area had been defined as follows.

To attempt to utilize the first order perturbatin theory, which has been developed for the motion of a uniaxial satellite in a gravitational field (reference 8) in studying the motion of a triaxial satellite in a gravity field. Also to expand the theory for the uniaxial case to higher order.

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NGR-01-003-008,

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VEHICLE CONTROL FOR FUEL OPTIMIZATION

By

Klaus D. Dannenberg and Grady R. Harmon
Department of Mechanical Engineering

NASA Grant NGR-01-003-008-S-1

Prepared For

GUIDANCE LABORATORY
ELECTRONICS RESEARCH CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CAMBRIDGE, MASSACHUSETTS

November 1967

AUBURN, ALABAMA

N74-31302

VEHICLE CONTROL FOR FUEL OPTIMIZATION

Klaus D. Dannenberg and Grady R. Harmon

Department of Mechanical Engineering
Auburn University

ABSTRACT

The problem considered in this report is that of predicting a minimum fuel trajectory for a six degree of freedom vehicle which has a motion characterized by the first order differential equations of translational and rotational dynamics. The thrust direction and center of gravity of the vehicle are assumed to be fixed with respect to the vehicle. Thrust magnitude and the control moment are used as control variables and appear linearly in the equations of motion.

Pontryagin's Maximum Principle is used to solve the variational problem. With this formulation, the extremal controls are bang-bang with the exception of the singular case. A unique feature of this problem is a combination of nonlinear state and linear control will allow the computation of the initial values of the Lagrange multipliers by an appropriate choice of some of the initial states. Initial values of the multipliers are always necessary for the complete solution, but no process is generally available for their determination.

NOMENCLATURE

A_D	Matrix transformation from plumbline system to vehicle system
A_j	Exit area of vehicle engines
A_ω	Matrix transformatic or $\overline{\psi}$ vector into vehicle system
c	Abbreviation for cosine
$\overline{C}$	Control variable vector
C_D	Coefficient of drag
C_L	Coefficient of lift
$\overline{F}$	Force vector
G	Gravitational constant
H	Hamiltonian
m	Total mass of vehicle
$\dot{m}_a$	Mass flow rate of air through vehicle engines
m_f	Mass of vehicle's fuel
M	Mass of earth
$\overline{M}$	Moment vector
p_j	Exit pressure of vehicle engines
p_o	Freestream pressure
$\overline{r}$	Arbitrary vector
s	Abbreviation for sine
t	Time
T	Rotational kinetic energy of vehicle

$\overline{u}$	Translational velocity of vehicle
v_j	Exit velocity of air and fuel of vehicle engines
v_o	Freestream velocity
$\overline{x}$	Translational position of vehicle
$\overline{x}_{cp}$	Position vector of center of pressure in vehicle system
$\overline{X}$	State variable vector
α	Angle between y-axis and relative velocity vector
α_y	Angle measured in xz-plane from x-axis, locating plane containing relative velocity vector and y-axis
$\overline{\Lambda}$	Lagrange multiplier vector
μ	Inertia tensor of vehicle in vehicle coordinate system
ρ	Freestream air density
$\overline{\phi}$	Eulerian angular position of vehicle
$\overline{\psi}$	Time rate of change of $\overline{\phi}$
$\overline{\omega}$	Angular velocity vector in vehicle coordinate system

Subscripts

a	Relating to aerodynamic force
p	Relating to pitching motion about vehicle's z-axis
r	Relating to rolling motion about vehicle's y-axis
t	Relating to thrust force
v	Relating to vehicle coordinate system
y	Relating to yawing motion about vehicle's x-axis

INTRODUCTION

The Maximum Principle is a mathematical optimization process, yielding a continuous set of controls, as contrasted with the computer search technique of optimization. One of the primary drawbacks of the Maximum Principle is the necessity for determining the initial values of the Lagrange multipliers. Since no physical significance is attached to the Lagrange multipliers, a system of assumed initial values is commonly used with the hope that a maximum can be found.

In the problem formulated in this paper, a unique situation arises: the Hamiltonian is linear in the control variables and nonlinear in the state variables. If these nonlinearities are used with appropriate nonrestrictive initial values for some of the states, a set of equations is produced which can be solved for the initial Lagrange multipliers. Thus, a complete extremal solution can be found for the optimization problem presented in this report.

COORDINATE SYSTEMS

Two coordinate systems are used to describe the motion of the vehicle. One of these, the plumbline system, is fixed to the earth's center and is assumed to be a primary inertial system. The other is fixed to the vehicle at the center of gravity and moves with the vehicle. The directions of the vehicle axes are shown in Figure 1. The position of the center of gravity of the vehicle is given by its Cartesian coordinates relative to the plumbline system. The angular orientation is given by a series of three consecutive rotations, which are illustrated in Figure 2. From an initial position in which all axes of the vehicle and plumbline systems are parallel, the following rotations are made about the vehicle's center of gravity:

1) Yawing rotation ϕ_y about the x axis
2) Pitching rotation ϕ_p about the z axis
3) Rolling rotation $-\phi_r$ about the -y axis

Consequently,

$$\overline{r}_v = [-\phi_r][\phi_p][\phi_y]\overline{r} = [A_D]\overline{r}$$

or

$$\overline{r}_v = \begin{bmatrix} CRCP & CRSPCY - SRSY & CRSPSY + SRCY \\ -SP & CPCY & CPSY \\ -CPSR & -SPSRCY - SYCR & -SRSPSY + CYCR \end{bmatrix}\overline{r}$$

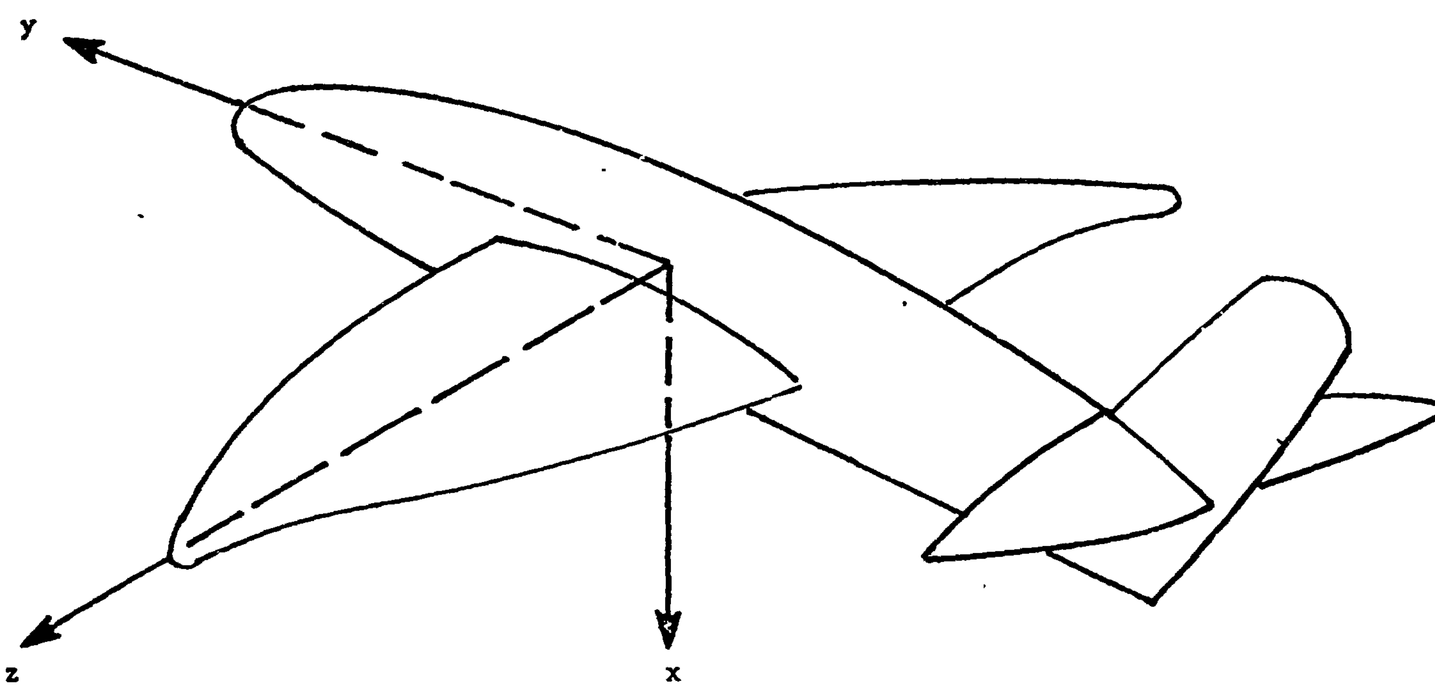

Figure 1. Vehicle Axes Orientation

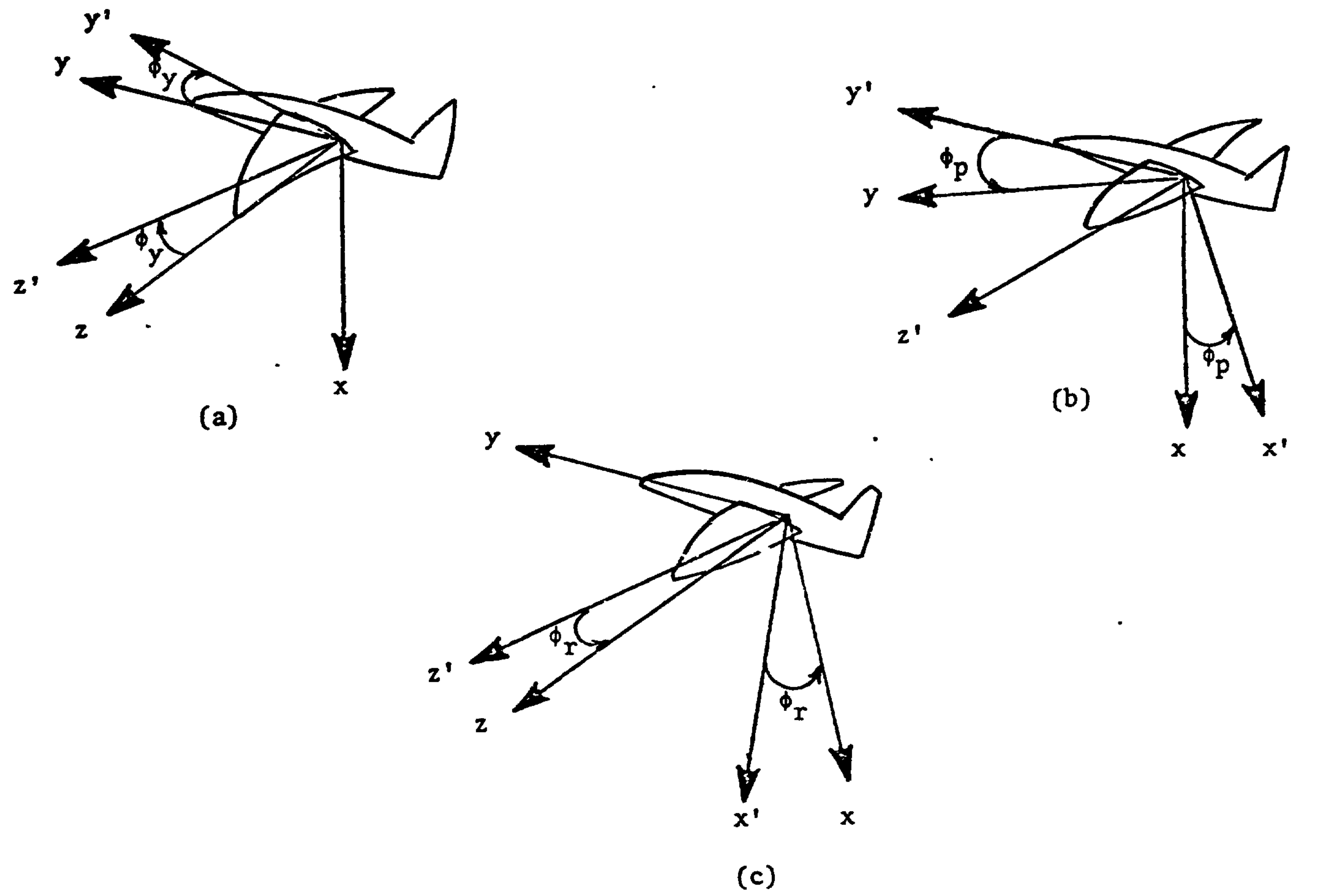

Figure 2. Euler Angles

PROBLEM FORMULATION

The minimization of the performance index

$$\int_0^t \dot{m}_f d\_$$

will be accomplished through utilization of the Maximum Principle. Thus, for a minimum of

$$\int_0^t \dot{m}_f dt,$$

a maximum of the Hamiltonian H is desired, where H is defined as

$$H \equiv \overline{\Lambda} \cdot \dot{\overline{X}}$$

where $\overline{X}$ is the state variable vector and $\overline{\Lambda}$ is the Lagrange multiplier vector.

The state variables chosen for this problem are the translational and rotational position and velocity $\overline{x}$, $\overline{u}$, $\overline{\phi}$, and $\overline{\psi}$, respectively. From a knowledge of mechanics, the state equations are as follows:

$$\dot{\overline{x}} = \overline{u}$$

$$\dot{\overline{u}} = \overline{F}/m - \dot{m}\overline{u}/m$$

$$\dot{\overline{\phi}} = \overline{\psi}$$

$$\dot{\overline{\psi}} = [B]\overline{M} + [C]\overline{\psi} + [F]\overline{D}$$

Thus, the Hamiltonian becomes

$$H = \lambda_o \dot{m}_f + \overline{\lambda}_I \cdot \overline{u} + \overline{\lambda}_{II} \cdot \left(\frac{\dot{\overline{F}}}{m} - \frac{\dot{m}\overline{u}}{m}\right) + \overline{\lambda}_{III}$$

$$\cdot \, \overline{\psi} + \overline{\lambda}_{IV} \cdot \left([B]\overline{M} + [C]\overline{\psi} + [F]\overline{D}\right)$$

After substitution of the forces and moments discussed in the appendix, the Hamiltonian takes the following form:

$$H = \lambda_o\left[\frac{F_t - \dot{m}_a(v_j - v_o) - A_j(p_j - p_o)}{v_j}\right] + \overline{\lambda}_I \cdot \overline{u} + \overline{\lambda}_{II}$$

$$\cdot \left\{\overline{E}\,\frac{F_a}{m} + [A_D]^T\,\frac{\overline{F}_t}{m} - \frac{GM}{|\overline{x}|^3}\,\overline{x}\right.$$

$$\left. + \frac{\dot{m}_a(v_j - v_o) + A_j(p_j - \dot{p}_o) - F_t}{v_j m}\,\overline{u}\right\}$$

$$+ \quad \overline{\lambda}_{III} \cdot \overline{\psi} + \overline{\lambda}_{IV} \cdot \{[B]\overline{M} + [C]\overline{\psi} + [F]\overline{D}\}$$

From the Hamiltonian, the necessary conditions can be obtained as

$$\dot{\overline{\Lambda}} = -\frac{\partial H}{\partial \overline{X}}$$

Expanding into scalar form, these equations become:

$$\dot{\lambda}_0 = \overline{\lambda}_{II} \cdot \left\{\overline{E}\,\frac{F_a}{m^2} + [A_D]^T\,\frac{\overline{F}_t}{m^2}\right.$$

$$\left. + \frac{\dot{m}_a(v_j - v_o) + A_j(p_j - p_o) - F_t}{v_j m^2}\,\overline{u}\right\}$$

$$\dot{\lambda}_1 = -\frac{1}{m}\left(\overline{\lambda}_{II} \cdot \overline{E}\right)\frac{\partial F_a}{\partial x} + GM\left(\frac{1}{|\overline{x}|^3} - \frac{3x^2}{|\overline{x}|^5}\right)$$

$$\dot{\lambda}_2 = -\frac{1}{m}\left(\overline{\lambda}_{II} \cdot \overline{E}\right)\frac{\partial F_a}{\partial y} + GM\left(\frac{1}{|\overline{x}|^3} - \frac{3y^2}{|\overline{x}|^5}\right)$$

$$\dot{\lambda}_3 = -\frac{1}{m}\left(\overline{\lambda}_{II}\cdot\overline{E}\right)\frac{\partial F_a}{\partial z} + GM\left(\frac{1}{|\overline{x}|^3} - \frac{3z^2}{|\overline{x}|^5}\right)$$

$$\dot{\lambda}_4 = -\frac{\lambda_0}{v_j}\frac{\partial v_0}{\partial u} - \lambda_1 - \frac{\overline{\lambda}_{II}\cdot\overline{E}}{m}\frac{\partial \overline{F}_a}{\partial u} + \frac{\lambda_4}{v_j m}\left(v_o + u\frac{\partial v_0}{\partial u}\right)$$

$$\dot{\lambda}_5 = -\frac{\lambda_0}{v_j}\frac{\partial v_0}{\partial v} - \lambda_2 - \frac{\overline{\lambda}_{II}\cdot\overline{E}}{m}\frac{\partial \overline{F}_a}{\partial v} + \frac{\lambda_4}{v_j m}\left(v_o + v\frac{\partial v_0}{\partial v}\right)$$

$$\dot{\lambda}_6 = -\frac{\lambda_0}{v_j}\frac{\partial v_0}{\partial w} - \lambda_3 - \frac{\overline{\lambda}_{II}\cdot\overline{E}}{m}\frac{\partial \overline{F}_a}{\partial w} + \frac{\lambda_4}{v_j m}\left(v_o + w\frac{\partial v_0}{\partial w}\right)$$

$$\dot{\lambda}_7 = -\overline{\lambda}_{II}\cdot\left\{\frac{\partial\overline{E}}{\partial\phi_y}\frac{F_a}{m} + \frac{\partial[A_D]^T}{\partial\phi_y}\frac{\overline{F}_t}{m}\right\} - \overline{\lambda}_{IV}$$
$$\cdot\left\{\frac{\partial}{\partial\phi_y}[B]\overline{M} + \frac{\partial}{\partial\phi_y}[C]\overline{\psi} + \frac{\partial}{\partial\phi_y}([F]\overline{D})\right\}$$

$$\dot{\lambda}_8 = -\overline{\lambda}_{II}\cdot\left\{\frac{\partial\overline{E}}{\partial\phi_r}\frac{F_a}{m} + \frac{\partial[A_D]^T}{\partial\phi_r}\frac{\overline{F}_t}{m}\right\} - \overline{\lambda}_{IV}$$
$$\cdot\left\{\frac{\partial}{\partial\phi_r}[B]\overline{M} + \frac{\partial}{\partial\phi_r}[C]\overline{\psi} + \frac{\partial}{\partial\phi_r}([F]\overline{D})\right\}$$

$$\dot{\lambda}_9 = -\overline{\lambda}_{II}\cdot\left\{\frac{\partial\overline{E}}{\partial\phi_p}\frac{F_a}{m} + \frac{\partial[A_D]^T}{\partial\phi_p}\frac{\overline{F}_t}{m}\right\} - \overline{\lambda}_{IV}$$
$$\cdot\left\{\frac{\partial}{\partial\phi_p}[B]\overline{M} + \frac{\partial}{\partial\phi_p}[C]\overline{\psi} + \frac{\partial}{\partial\phi_p}([F]\overline{D})\right\}$$

$$\dot{\lambda}_{10} = -\lambda - \overline{\lambda}_{IV}\cdot\left(\sum_{i=1}^{3} C_{i1} + \frac{\partial}{\partial\psi_y}([F]\overline{D})\right)$$

$$\dot{\lambda}_{11} = -\lambda - \overline{\lambda}_{IV}\cdot\left(\sum_{i=1}^{3} C_{i2} + \frac{\partial}{\partial\psi_r}([F]\overline{D})\right)$$

$$\dot{\lambda}_{12} = -\lambda_9 - \overline{\lambda}_{IV} \cdot \left(\sum_{i=1}^{3} C_{i3} + \frac{\partial}{\partial \psi_p} \left([F]\overline{D} \right) \right)$$

The solution of these equations for $\overline{\Lambda}$ depends on the initial values of $\overline{\Lambda}$. Since no physical significance can be given to the Lagrange multipliers, some method must be developed to determine their initial values. When one realizes that the Hamiltonian is of the form

$$H = f(\overline{X}, \overline{\Lambda}) + \frac{\partial H}{\partial F_t} F_t + \frac{\partial H}{\partial \overline{M}} \cdot \overline{M},$$

the possibility arises that the nonlinear function of state can be made zero at the initial time by an appropriate choice of initial state without the necessity of all states being zero. Consequently, since on an optimal path H = 0, the remainder of the Hamiltonian must be zero; i.e.,

$$\frac{\partial H}{\partial F_t} F_t + \frac{\partial H}{\partial \overline{M}} \cdot \overline{M} = 0$$

Since F_t and $\overline{M}$, in general, are not zero,

$$\frac{\partial H}{\partial F_t} = 0 \qquad \text{and} \qquad \frac{\partial H}{\partial \overline{M}} = \overline{0}$$

This is the normal necessary condition used for the case of nonlinear controls.

If one chooses the initial state to be a position of rest, i.e., $\overline{\psi} = 0$ and $\overline{u} = 0$, and if one selects an initial thrust which satisfies the equation

$$-(\lambda_4 x + \lambda_5 y + \lambda_6 z) \frac{GM}{|\overline{x}|^3} + \lambda_0 \left(\frac{m_a(v_j - v_7) + A_j(p_j - p_0)}{v_j} \right) = 0$$

the coefficients of the controls are zero at the initial time step, allowing an analytic solution for the unknown initial values of the twelve variable Lagrange multipliers. If one uses these initial values, the given differential equations can be solved for the time history of $\overline{\Lambda}$. Similarly, the state equations can be solved for a time history of the state variables.

Extremal control is determined by the coefficients of the control variables. Since the Hamiltonian is linear in all controls, the extremal control is bang-bang unless the control coefficient is zero; i.e., if

$$\frac{\partial H}{\partial C_i} > 0, \qquad C_i = C_{i_{MAX}} \qquad i = F_t, \overline{M}$$

$$\frac{\partial H}{\partial C_i} < 0, \qquad C_i = C_{i_{MIN}} \qquad i = F_t, \overline{M}$$

For the singular control case of a zero coefficient over a nonzero time interval, the equation(s)

$$\frac{\partial H}{\partial C_i} = 0 \qquad i = F_t \text{ or } M_x \text{ or } M_y \text{ or } M_z$$

can be added to the differential multiplier equations over the appropriate time period to solve for the extremal control.

CONCLUSIONS

A set of initial values of the Lagrange multipliers for the state problem can be found analytically through a choice of appropriate initial velocities. This is by no means a unique solution to the problem, but it is a method of making a feasible choice of initial multipliers for a certain realizable initial state. The actual numerical solution of the equations should present no major difficulties if the intial values are no longer a problem.

This method of solving for the initial Lagrange multipliers will not be applicable to most problems. With the selection of an appropriate number of initial states, the problem becomes too restrictive to be of any great general value.

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APPENDIX: MATHEMATICAL MODEL

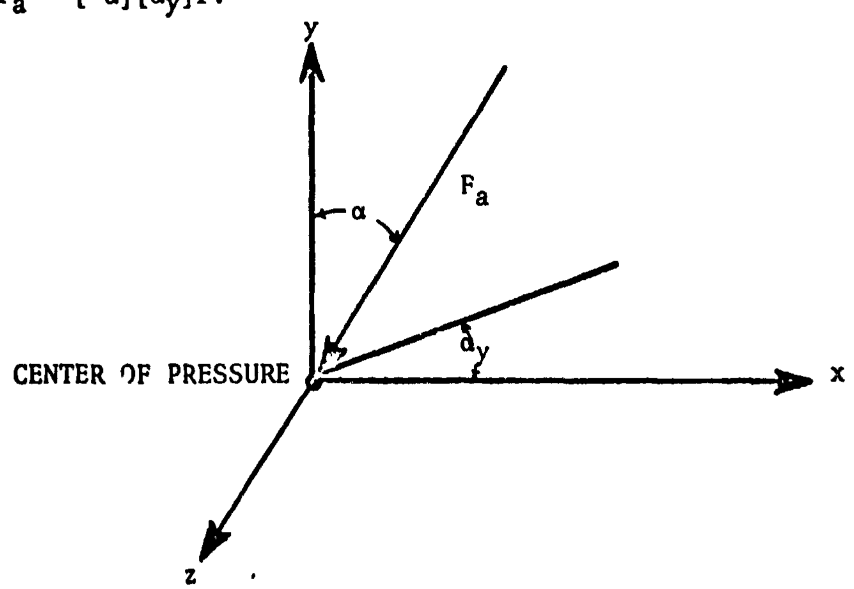
A mathematical model for the basic mechanics of the problem will be deduced using the separability of the rotational and translational motions of a rigid body. The forces and moments will be discussed first.

A. Forces

An *aerodynamic force* $\overline{F}_a$ is assumed to act at the vehicle's center of pressure. The orientation of the aerodynamic force is determined by two rotations from the vehicle system to a new coordinate system denoted by $\overline{r}_a$. The rotations align the aerodynamic force with the $-y_a$ axis. The maneuvers necessary for this alignment (Appendix Figure 1) are:

1) Roll α_y about the y axis.
2) Pitch α about the z axis to align the y axis with the relative velocity vector.

Thus, $\overline{r}_a = [-\alpha][\alpha_y]\overline{r}$.



Appendix Figure 1. Aerodynamic Force System

The magnitude of $\overline{F}_a$ is given by

$$|\overline{F}_a| = \tfrac{1}{2} v_o^2 A(C_D^2 + C_L^2)^{1/2}$$

A *thrust force* $\overline{F}_T$ is assumed to act along the longitudinal axis of the aircraft. The magnitude of this force is given by

$$|\overline{F}_T| = \dot{m}_a(v_j - v_o) + \dot{m}_f v_j + A_j(p_j - p_o)$$

where $\dot{m}_a$, v_j, and p_j are known functions of $|\overline{F}_T|$ for a given engine.

The *gravitational force* of a spherical earth acting at the center of gravity of the vehicle is

$$\overline{F} = -\frac{GMm}{|\overline{x}|^3}\,\overline{x} .$$

B. Moments

An *aerodynamic moment* and a *thrust moment* are present as a result of the nonconcurrency of the center of pressure and the center of gravity. Collectively, the moments are

$$\overline{x}_{cp} \times \left\{ [-\alpha_y]\,[\alpha] \begin{bmatrix} 0 \\ F_a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_T \\ 0 \end{bmatrix} \right\}$$

where $\overline{x}_{cp}$ is the position vector of the center of pressure in the vehicle system.

The *control surface moment* $\overline{M}_F$ is a control of the optimization problem. These are the collective moments resulting from the flaps, ailerons, and all other vehicle control surfaces.

Clasle's theorem for rigid body motion states that the motion may be divided into a pure translation of the center of gravity and a pure rotation about the center of gravity. Therefore, for the translational motion, the following equation results from Newton's law:

$$\dot{\overline{u}} = \frac{\overline{F}}{m} - \frac{\dot{m}\overline{u}}{m}$$

or

$$\dot{\bar{u}} = \bar{E}\,\frac{F_a}{m} + [A_D]^T\,\frac{\bar{F}_T}{m} - \frac{GM}{|\bar{x}|^3}\,\bar{x}$$

$$+ \frac{m_a(v_j - v_o) + A_j(p_j - p_o) - F_T}{v_j m}\,\bar{u}$$

where

$$\bar{E} = [\phi_y]^T\,[\phi_p]^T\,[-\phi_r]^T\,[\cdot\alpha_y]\,[\alpha]\,\frac{\bar{F}_A}{|\bar{F}_A|}$$

and

$$[A_D] = [-\phi_r]\,[\phi_p]\,[+\phi_y]$$

The rotational motion equation is obtained from energy considerations. The rotational kinetic energy in matrix form is

$$T = \frac{1}{2}\,\bar{\omega}^T\,[\mu]\bar{\omega}$$

where $\bar{\omega}$ is the vehicle-fixed angular velocity vector and $[\mu]$ is the inertia tensor for motion about the vehicle axes. The Lagrangian form for generalized coordinates of angular character is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) - \frac{\partial T}{\partial \phi_i} = M\phi_i$$

When one carries out the indicated operations, the Lagrangian equations become:

$$\frac{d}{dt}\left(\frac{\partial \bar{\omega}^T}{\partial \dot{\phi}_i}\right)[\mu]\bar{\omega} + \frac{\partial \bar{\omega}^T}{\partial \dot{\phi}_i}\,[\mu]\,\frac{d\bar{\omega}}{dt} - \frac{\partial \bar{\omega}^T}{\partial \phi_i}\,[\mu]\bar{\omega} = M_{\phi_i}$$

After substitution of the angular velocity components of $\dot{\bar{\phi}}_p$, $\dot{\bar{\phi}}_y$, and $\dot{\bar{\phi}}_r$ for $\bar{\omega}$ in the vehicle system and simplification, the resulting equation is

$$\dot{\bar{\Psi}} = [B]\bar{M} + [C]\bar{\Psi} + [F]\bar{D}$$

111171917101 Report

NGR-01-003-008

N74-31303

Semi-Annual Report on NASA Grant NGR-01-003-008

RESEARCH ON

DEVELOPMENT OF EQUATIONS FOR PERFORMANCE TRAJECTORY COMPUTATION

SUMMARY

During the second six months of the original one-year period of the grant work has progressed on two projects:

1. Development of a computer program for the study formulated earlier, as discussed in the last report, and
2. An analytical study of a minimum fuel flight for high speed aircraft.

Included in this report are a listing of the program to compute a minimum time re-entry into the atmosphere for an Apollo-type capsule, and a technical summary of the minimum fuel problem. A detailed report on item two is to be presented to the Guidance Laboratory of Electronics Research Center in Cambridge, Massachusetts, on April 19 an 20. A full report will be forwarded to you after this presentation.

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A MAXIMUM PRINCIPLE RE-ENTRY STUDY BY GRADY R. HARMON
AS EDITED BY FRANK J. WALKER, JR. - FEBRUARY 1967


INPUT CARD 5     JUMP = 1     PARTS 1 AND 2
COL NO 35             = 2     PART 1 ONLY
IS JUMP               = 3     PART 2 ONLY

SYMBOLS USED IN PROGRAM

PLANET DATA

GM       = GRAVITATIONAL CONSTANT OF PLANET
RO       = RADIUS OF PLANET
RHO      = DENSITY OF PLANET ATMOSPHERE
 RHOD     =PARTIAL OF RHO W.R.T. ALTITUDE
OMEGA    = ANGULAR VELOCITY OF PLANET ABOUT ROTATION AXIS
(WEX,WEY,WEZ) = ANGULAR VELOCITY COMPONENTS OF THE PLANET IN THE
INERTIAL FRAME

VEHICLE DATA

CX       = AERODYNAMIC COEFFICIENT (LONGITUDINAL AXIS)
 CXMD     = PARTIAL OF CX W.R.T. ALPHA
CZ       = AERODYNAMIC COEFFICIENT (PERPENDICULAR TO LONGITUDINAL
AXIS)
 CZMD     = PARTIAL OF CZ W.R.T. ALPHA
A        = CROSS-SECTION OF VEHICLE
VM       = MASS OF VEHICLE

GENERAL DATA

(X,Y,Z) = CARTESIAN COORDINATES (INERTIAL FRAME)
(U,V,W) = VELOCITY COMPONENTS (INERTIAL FRAME)
R        = MAGNITUDE OF RADIUS VECTOR TO VEHICLE
HGT      = ALTITUDE
(VRX,VRY,VRZ) = RELATIVE WIND VELOCITY COMPONENTS(INERTIAL FRAME)
(VRMX,VRMY,VRMZ) = RELATIVE WIND VELOCITY COMPONENTS (MISSILE-
FIXED FRAME)
VR       = MAGNITUDE OF VEHICLE VELOCITY RELATIVE TO AIR

FPA      = AERODYNAMIC ACCELERATION
GGG      = GRAVITATIONAL ACCELERATION

H        = PONTRYAGIN H FUNCTION
PHA      = PARTIAL OF H W.R.T. ALPHA
PHAY     = PARTIAL OF H W.R.T. ALPHA Y

XLAM(1) = LAGRANGE MULTIPLIER  (1)
XLAM(2) = LAGRANGE MULTIPLIER  (2)
XLAM(3) = LAGRANGE MULTIPLIER  (3)
XLAM(4) = LAGRANGE MULTIPLIER  (4)
XLAM(5) = LAGRANGE MULTIPLIER  (5)
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XLAM(6) = LAGRANGE MULTIPLIER  (6)
XLAM(7) = CONSTANT = +1

PREFIX OF R INDICATES ANGLE IS IN RADIANS. OTHERWISE IT IS ASSUMED
TO BE IN DEGREES.
PHIO     = INERTIAL FRAME ORIENTATION ANGLE
AO       = INERTIAL FRAME ORIENTATION ANGLE
A09      = (90 - AO )
PHIR     = ROLL  ANGLE
PHIY     = YAW   ANGLE
PHIP     = PITCH ANGLE
ALFY     = ROLL ANGLE OF VEHICLE (AERODYNAMIC FRAME)
ALF      = ANGLE OF ATTACK OF THE VEHICLE
CRALF    = COS(RALF)
SRALF    = SIN(RALF)
CRALFY   = COS(RALFY)
SRALFY   = SIN(RALFY)

CPHIO    = COS(RPHIO)
SPHIO    = SIN(RPHIO)
CRA09 = COS(RA09)
CPHIR    = COS(RPHIR)
SPHIR    = SIN(RPHIR)
CPHIP    = COS(RPHIP)
SPHIP    = SIN(RPHIP)
CPHIY    = COS(RPHIY)
SPHIY    = SIN(RPHIY)



 EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
1,VEX(1))
 EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
3),AO),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
9 (37),VRMPD1)
 EQUIVALENCE   (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
 EQUIVALENCE (ODDS(57),FA)
 EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
3(TABS(495),TCXPP(1))
 EQUIVALENCE (VEX(1),XN(1)),(VEX(4),UBDOT(1)),(VEX(7),XBAR(1)),
1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
 EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
 EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
 EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
 EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
 EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
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      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88),K(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38),J(38)
      DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UB(3)
      DIMENSION OUTD(4,100)
      DOUBLE PRECISION FA,AST
       DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,AO,            PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
     4 VM,GM,GOG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
     5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
     6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
     7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
     8SRALFP,CRALFP
      DOUBLE PRECISION ARCOS,COLAT,CRIT,DEL,TESP,TIREC,TLIMIT,TPRINT,
     1TSTEP,TY,UO,
     2 VO,VLAT,VLONG,WO,XO,XLAM1O,XLAM2O,XLAM3O,XLAM4O,XLAM5O,XLAM6O,
     3ZO
      DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
      DIMENSION STX(3),STY(3),STAY(3),RD(7)
      DOUBLE PRECISION STX,STY,STAY,SLOPE,STALF
      DOUBLE PRECISION CONA,CONB,CONC,YO,Y2,YO2,Y22,DEL2
      DOUBLE PRECISION HH
      DIMENSION HH(2,4)
      DIMENSION OE(115),OUTA(45),OUTC(205)
      DATA       OUTC(2) ,OUTC(3) ,OUTC(4) ,OUTC(5) ,OUTC(7) ,OUTC(8) ,
     1 OUTC(9) ,OUTC(10),OUTC(12),OUTC(13),OUTC(14),OUTC(15),OUTC(17),
     2 OUTC(18),OUTC(19),OUTC(20),OUTC(22),OUTC(23),OUTC(24),OUTC(25),
     3 OUTC(27),OUTC(28),OUTC(29),OUTC(30),OUTC(32),OUTC(33),OUTC(34),
     4 OUTC(35),OUTC(37),OUTC(38),OUTC(39),OUTC(40),OUTC(42),OUTC(43),
     5 OUTC(44),OUTC(45),OUTC(47),OUTC(48),OUTC(49),OUTC(50),OUTC(52),
     6 OUTC(53),OUTC(54),OUTC(55),OUTC(57),OUTC(58),OUTC(59),OUTC(60),
     7 OUTC(62),OUTC(63),OUTC(64),OUTC(65),OUTC(67),OUTC(68),OUTC(69),
     8 OUTC(70),OUTC(72),OUTC(73),OUTC(74),OUTC(75),OUTC(77),OUTC(78),
     9 OUTC(79),OUTC(80),OUTC(82),OUTC(83),OUTC(84)/ 67*6HBLANKS/
      DATA          OUTC(85) ,OUTC(87) ,OUTC(88) ,OUTC(89) ,OUTC(90) ,
     1 OUTC(92) ,OUTC(93) ,OUTC(94) ,OUTC(95) ,OUTC(97) ,OUTC(98) ,
     2 OUTC(99) ,OUTC(100),OUTC(102),OUTC(103),OUTC(104),OUTC(105),
     3 OUTC(107),OUTC(108),OUTC(109),OUTC(110),OUTC(112),OUTC(113),
     4 OUTC(114),OUTC(115),OUTC(117),OUTC(118),OUTC(119),OUTC(120),
     5 OUTC(122),OUTC(123),OUTC(124),OUTC(125),OUTC(127),OUTC(128),
     6 OUTC(129),OUTC(130),OUTC(132),OUTC(133),OUTC(134),OUTC(135),
     7 OUTC(137),OUTC(138),OUTC(139),OUTC(140),OUTC(142),OUTC(143),
     8 OUTC(144),OUTC(145),OUTC(147),OUTC(148),OUTC(149),OUTC(150),
     9 OUTC(152),OUTC(153),OUTC(154),OUTC(155)/ 57*6HBLANKS/
      DATA       OUTC(157),OUTC(158),OUTC(159),OUTC(160),OUTC(162),
     1 OUTC(163),OUTC(164),OUTC(165),OUTC(167),OUTC(168),OUTC(169),
     2 OUTC(170),OUTC(172),OUTC(173),OUTC(174),OUTC(175),OUTC(177),
     3 OUTC(178),OUTC(179),OUTC(180),OUTC(182),OUTC(183),OUTC(184),
     4 OUTC(185),OUTC(187),OUTC(188),OUTC(189),OUTC(190),OUTC(192),
     5 OUTC(193),OUTC(194),OUTC(195),OUTC(197),OUTC(198),OUTC(199),
```

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6 OUTC(200),OUTC(202),OUTC(203),OUTC(204),OUTC(205) /40*6HBLANKS/
 DATA    OF(2) ,OE(3) ,OF(4) ,OF(5) ,OF(7) ,OE(8) ,OF(9) ,OF(10),
1 OF(12),OF(13),OF(14),OF(15),OE(17),OF(18),OE(19),OE(20),OF(22),
2 OF(23),OF(24),OF(25),OF(27),OF(28),OF(29),OF(30),OF(32),OF(33),
3 OF(34),OF(35),OF(37),OF(38),OF(39),OF(40),OF(42),OF(43),OF(44),
4  F(45),OF(47),OF(48),OF(49),OF(50),OF(52),OF(53),OF(54),OF(55),
5 OF(57),OF(58),OF(59),OE(60),OE(62),OF(63),OE(64),OE(65),OE(67),
6 OF(68),OF(69),OF(70),OF(72),OF(73),OF(74),OF(75),OF(77),OE(78),
7 OF(79),OF(80),OF(82),OF(83),OF(84),OF(85),OF(87),OF(88),OF(89),
8 OF(90),OF(92),OF(93),OE(94),OF(95),OE(97),OF(98),OE(99),
9 OE(100),OF(102),OF(103),OF(104),OE(105)/ 84*6HBLANKS/
 DATA       OUTC(1)   ,OUTC(6)   ,OUTC(11)  ,OUTC(16)  ,OUTC(21)  ,
1 OUTC(26)  ,OUTC(31)  ,OUTC(36)  ,OUTC(41)  ,OUTC(46)  ,OUTC(51)  ,
2 OUTC(56)  ,OUTC(61)  ,OUTC(56)  ,OUTC(71)  ,OUTC(76)  ,OUTC(81)  ,
3 OUTC(86)  ,OUTC(91)  ,OUTC(96)  ,OUTC(101),OUTC(106),OUTC(111),
4 OUTC(116),OUTC(121),OUTC(126),OUTC(131),OUTC(136)/
56HTIME   ,6HX      ,6HY      ,6HZ      ,6HU      ,6HV      ,6HW      ,
66HLAM1   ,6HLAM2   ,6HLAM3   ,6HLAM4   ,6HLAM5   ,6HLAM6   ,6HALF    ,
76HALFY   ,6HPHIR   ,6HPHIY   ,6HPHIP   ,6HGGG    ,6HVR     ,6HM      ,
86HRHO    ,6HO      ,6HFA     ,6HN      ,6HP      ,6HQ      ,6HH      /
 DATA       OUTC(141),OUTC(146),OUTC(151),OUTC(156),OUTC(161),
1 OUTC(166),OUTC(171),OUTC(176),OUTC(181),OUTC(186),OUTC(191),
2 OUTC(196),OUTC(201),OE(1),OE(6),OE(11),OF(16),OF(21),OE(26),
3 OE(31),OF(36),OF(41),OE(46),OE(51),OE(56),OE(61),OE(66),OE(71),
4 OF(76),OE(81),OE(86),OF(91),OE(96),OE(101),OE(106)/
56HPHAY   ,6HPHA    ,6HU DOT  ,6HV DOT  ,6HW DOT  ,6HLAM1D  ,6HLAM2D  ,
66HLAM3D  ,6HLAM4D  ,6HLAM5D  ,6HLAM6D  ,6HDRAG   ,6HHGT    ,6HVM     ,
76HA      ,6HGM     ,6HRO     ,6HXO     ,6HYO     ,6HZO     ,6HUO     ,
86HVO     ,6HWO     ,6HLAM10  ,6HLAM20  ,6HLAM30  ,6HLAM40  ,6HLAM50  ,
96HLAM60  ,6HLAM7   ,6HTSTEP  ,6HTPRINT,6HTLIMIT,6HALF    ,6HALFY   /
 DATA    OE(107),OE(108),OE(109),OE(110),OE(112),OE(113),OE(114),
1 OF(115),OF(111) / 8*6HBLANKS,6HOMEGA /
```

~~ARCOS(X) = DATAN2(DSQRT(1. - X*X),X)~~ ARCOS(X) = DATAN2(DSQRT(1. - X*X),X.)

```
  TFSP = 666.0

 READ IN DATA

  READ IN  HYPERSONIC DATA TABLE
 DO 8001 I=1,38
 READ(5,1000) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),RD(7),J(I)
 ALPHAT(I) = DBLE(RD(1))
 TCZ(I) = DBLE(RD(2))
 TCZP(I) = DBLE(RD(3))
 TCZPP(I) = DBLE(RD(4))
 TCX(I) = DBLE(RD(5))
 TCXP(I) = DBLE(RD(6))
01 TCXPP(I) = DBLE(RD(7))
00 FORMAT(F10.0,F10.3,F10.5,F10.6,F10.3,F10.5,F10.6,I2)
 DO 120  I=1,38
 IF(J(I)-I)   101,120,101
20 CONTINUE
  GO TO 100
01 WRITE(6,199)
09 FORMAT(1H1,15X,23HDATA CARDS OUT OF ORDER)
 GO TO 888
 READ IN ALTITUDE VS DENSITY TABLE
00 DO 8002 I =1,88
 READ(5,1001) RD(1),RD(2),K(I)
 ALT(I) = DBLE(RD(1))
02 PRESS(I) = DBLE(RD(2))
```

PRECEDING PAGE BLANK NOT FILMED

```
01 FORMAT(F10.0,E11.4,48X,I3)
   DO 122 I=1,88
   IF(K(I)-I)    101,122,101
22 CONTINUE
    READ INPUT PARMETERS
99 CONTINUE
   READ(5,1002) RD(1),RD(2),RD(3),RD(4),RD(5),L1
   GM = DBLE(RD(1))
   R0 = DBLE(RD(2))
   OMEGA = DBLE(RD(3))
   A0 = DBLE(RD(4))
   PHI0 = DBLE(RD(5))
02 FORMAT(3F20.8,2F4.0,2X,I2)
    IF(L1-1)101,124,101
24 CONTINUE
    GO TO 102
02 READ(5,1003) RD(1),RD(2),RD(3),RD(4),L2
   ALF = DBLE(RD(1))
   ALFY = DBLE(RD(2))
   VM = DBLE(RD(3))
   A =    DBLE(RD(4))
03 FORMAT(4F10.4,30X,I2)
    IF(L2-2)101,126,101
26 CONTINUE
    GO TO 103
03 READ(5,1004) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),L3
   X0 = DBLE(RD(1))
   Y0 = DBLE(RD(2))
   Z0 = DBLE(RD(3))
   U0 = DBLE(RD(4))
   V0 = DBLE(RD(5))
   W0 = DBLE(RD(6))
04 FORMAT(3F10.0,3F10.3,10X,I2)
    IF(L3-3)101,128,101
28 CONTINUE
   READ(5,1005) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),RD(7),L4
   XLAM10 = DBLE(RD(1))
   XLAM20 = DBLE(RD(2))
   XLAM30 = DBLE(RD(3))
   XLAM40 = DBLE(RD(4))
   XLAM50 = DBLE(RD(5))
   XLAM60 = DBLE(RD(6))
   XLAM7  = DBLE(RD(7))
05 FORMAT(7F10.3,I2)
    IF(L4-4)101,130,101
30 CONTINUE
   GO TO 135
35 READ(5,1010) RD(1),RD(2),RD(3),JUMP,IFF,L5
   TPRINT = DBLE(RD(1))
   TLIMIT = DBLE(RD(2))
   TSTEP  = DBLE(RD(3))
   TPRINT MUST BE GREATER THAN OR EQUAL TO TSTEP
10 FORMAT(3F10.0,2I5,30X,I2)
   IF(L5-5) 101,137,101
37 CONTINUE
   IF((JUMP.EQ.2).OR.(JUMP.EQ.3)) GO TO 138
   JUMP=1
38 CONTINUE
   WRITE(6,1493)
93 FORMAT(1H1)
```

```
 PRINT INPUT

OUTA(1) = SNGL(VM)
OUTA(2) = SNGL(A)
OUTA(3) = SNGL(GM)
OUTA(4) = SNGL(RO)
OUTA(5) = SNGL(XO)
OUTA(6) = SNGL(YO)
OUTA(7) = SNGL(ZO)
OUTA(8) = SNGL(UO)
OUTA(9) = SNGL(VO)
OUTA(10) = SNGL(WO)
OUTA(11) = SNGL(XLAM10)
OUTA(12) = SNGL(XLAM20)
OUTA(13) = SNGL(XLAM30)
OUTA(14) = SNGL(XLAM40)
OUTA(15) = SNGL(XLAM50)
OUTA(16) = SNGL(XLAM60)
OUTA(17) = SNGL(XLAM7)
OUTA(18)=SNGL(TSTEP)
OUTA(19)=SNGL(TPRINT)
OUTA(20)=SNGL(TLIMIT)
OUTA(21)=SNGL(ALF)
OUTA(22)=SNGL(ALFY)
OUTA(23)=SNGL(OMEGA)
CALL CONV(OE,OUTA,23)
WRITE(6,6854)
54 FORMAT(1X,27HINPUT VALUES ARE AS FOLLOWS)
WRITE(6,6855) (OE(LL),LL=1,115)


INITIALIZE PROGRAM


01 CONTINUE
CALL TRAP
U=UO
V=VO
W=WO
X=XO
Y=YO
Z=ZO
XLAM1=XLAM10
XLAM2=XLAM20
XLAM3=XLAM30
XLAM4=XLAM40
XLAM5=XLAM50
XLAM6=XLAM60
CALL JACOB(HH,-5.0,-5.0)
CALL INVERS(HH,INDX,IORD,2,4,KERR)
DO 200 I=1,2
DO 200 N=1,2
00 HH(I,N)=-HH(I,N)
WRITE(6,299)
99 FORMAT(1HO,31HTHE FOLLOWING ARE VALUES FOR HH)
WRITE(6,31)  ((HH(N,I),I=1,2),N=1,2)
31 FORMAT(2F20.5)
IF(JUMP.EQ.3) GO TO 4002
```

```
  THE FOLLOWING IS PART I AS CALLED FOR BY JUMP

  AST = ALF
  AST1=ALFY
  ALFY=-180.0
  DO 4522 NN=1,71
  ALFY=ALFY+5.0
  ALF=-180.0
  DO 4500 JX=1,71
   ALF = ALF + 5.
  CALL HCALC
  OUTD(1,JX) = SNGL(ALF)
  OUTD(2,JX) = SNGL(H)
  OUTD(3,JX) = SNGL(PHAY)
00 OUTD(4,JX) = SNGL(PHA)
  IF(IFF)4522,4533,4522
33 WRITE(6,4511) ALFY
11 FORMAT(79H1      ALF                 H                        PHAY
 1     PHA             ALFY=,F6.1//)
  WRITE(6,4510)  ((OUTD(KKK,LLL),KKK=1,4),LLL=1,71)
10 FORMAT(1H0,F10.2,3E20.8)
22 CONTINUE
  WRITE(6,1493)
  IF(JUMP.EQ.2) GO TO 888
  JUMP = 3
  ALF = AST
  ALFY=AST1
  GO TO 4001
02 CONTINUE

  THE FOLLOWING IS PART II AS CALLED FOR BY JUMP

  WRITE(6,4534)
34 FORMAT(1H0//,16HITERATIONS BEGIN/)
30 TIREC=0.
  TY = TPRINT
31 JZ=0

  ITERATE FOR ALPHAY

00 CONTINUE
  IF(TIREC.GE.TLIMIT) GO TO 888
  COLAT = ARCOS(DABS(Z)/DSQRT(X*X + Y*Y +Z*Z))
  VLAT =  DSIGN((1.570796 - COLAT),Z)*57.2958
  VLONG = (DATAN2(Y,X)- OMEGA*TIREC)*57.2958
  CALL SLVNL(ALF,ALFY,HH,PHA,PHAY,1.E-14,28,TIREC)
88  CONTINUE
  CALL PDHY
  IF(TIREC.EQ.0.0) GO TO 8008
  TY=TY+TSTEP
  IF(TY.LT.TPRINT)  GO TO     6848
 5 CONTINUE
  TY=0.

        PRINT OUTPUT
```

```
     IKK = 1
     GO TO 8006
  05 CONTINUE
  48 CONTINUE

     EXIT IF PHASE II IS COMPLETE

     CHECK FOR SPEED LESS THAN TESP

     IF(VR.LE.TESP)   GO TO 6900
     TIREC = TIREC + TSTEP
  02 JZ=JZ+1
     CALL TGRATE (JZ,TSTEP)
     GO TO 6800
  00 IKK = 2
     GO TO 8006
  07 CONTINUE
     GO TO 888
  08 TY = 0.0
     IKK = 1
  06 CONTINUE
     OUTA(1)   = SNGL(TIREC)
     OUTA(2)   = SNGL(X)
     OUTA(3)   = SNGL(Y)
     OUTA(4)   = SNGL(Z)
     OUTA(5)   = SNGL(U)
     OUTA(6)   = SNGL(V)
     OUTA(7)   = SNGL(W)
     OUTA(8)   = SNGL(XLAM1)
     OUTA(9)   = SNGL(XLAM2)
     OUTA(10)  = SNGL(XLAM3)
     OUTA(11)  = SNGL(XLAM4)
     OUTA(12)  = SNGL(XLAM5)
     OUTA(13)  = SNGL(XLAM6)
     OUTA(14)  = SNGL(ALF)
     OUTA(15)  = SNGL(ALFY)
     OUTA(16)  = SNGL(PHIR)
     OUTA(17)  = SNGL(PHIY)
     OUTA(18)  = SNGL(PHIP)
     OUTA(19)  = SNGL(GGG)
     OUTA(20)  = SNGL(VR)
     OUTA(21)  = 0.
     OUTA(22)  = SNGL(RHO)
     OUTA(23)  = 0.
     OUTA(24)  = SNGL(FA)
     OUTA(25)  = SNGL(XN(1))
     OUTA(26)  = SNGL(XN(2))
     OUTA(27)  = SNGL(XN(3))
     OUTA(28)  = SNGL(H)
     OUTA(29)  = SNGL(PHAY)
     OUTA(30)  = SNGL(PHA)
     OUTA(31)  = SNGL(UBDOT(1))
     OUTA(32)  = SNGL(UBDOT(2))
     OUTA(33)  = SNGL(UBDOT(3))
     OUTA(34)  = SNGL(XLMID(1))
     OUTA(35)  = SNGL(XLMID(2))
     OUTA(36)  = SNGL(XLMID(3))
     OUTA(37)  = SNGL(XLMIID(1))
     OUTA(38)  = SNGL(XLMIID(2))
```

```
      OUTA(39) = SNGL(XLMIID(3))
      OUTA(40 ) = 0.
      OUTA(41)=SNGL(HGT)
      CALL CONV(OUTC,OUTA,41)
      WRITE(6,6855)  (OUTC(I),I=1,205)
6855 FORMAT(5(2X,A6,1X,A1,F10.8,A1,I2))
      WRITE(6,6856)
6856 FORMAT(1H0//)
      GO TO (8005,8007),IKK
8008 STOP
      END
$IBFTC PRELOD
      SUBROUTINE PRELOD
      EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
     1,VEX(1))
      EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
     1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
     2 SRALFY),(ODDS(9),PHI0),(ODDS(10),CPHI0),(ODDS(11),SPHI0),(ODDS(12
     3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
     4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
     5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
     6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
     7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
     8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
     9 (37),VRMRD1)
      EQUIVALENCE   (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
     1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
     2 (ODDS(45),R),(ODDS(46),R0),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
     3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
     4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
      EQUIVALENCE (ODDS(57),FA)
      EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
     1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
     2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
     3(TABS(495),TCXPP(1))
      EQUIVALENCE (VEX(1),XN(1)),(VEX(4),URDOT(1)),(VEX(7),XBAR(1)),
     1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
     2,XLMIID(1)),(VEX(22),UR(1)),(VEX(25),XLAM7)
      EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
      EQUIVALENCE (URDOT(1),UD),(URDOT(2),VD),(URDOT(3),WD)
      EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
      EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
      EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UR(1),U),(UR(2),V),(UR(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38)
      DIMENSION URDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UR(3)
      DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHI0,CPHI0,SPHI0,A0,          PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD  ,CZ,CZMD  ,CC,RHO,RHOD,R,R0,HGT,A,
     4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
```

```
5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,URDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
6 UR,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,J,V,W,
8SRALFP,CRALFP
 DOUBLE PRECISION RA09
 DOUBLE PRECISION FA
 DOUBLE PRECISION STALF,ALPHA1, C,CXMD1,CZMD1,DALF,B
 DOUBLE PRECISION ARCOS,COLAT,CRIT,DEL,TESP,TIREC,TLIMIT,TPRINT,
1TSTEP,TY,U0,
2 V0,VLAT,VLONG,W0,X0,XLAM10,XLAM20,XLAM30,XLAM40,XLAM50,XLAM60,
3Z0
 DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
 DOUBLE PRECISION CONA,CONB,CONC,Y0,Y2,Y02,Y22,DEL2
 RAD = 3.1415926535897932 /180.    ARCOS (X) = DATAN2(DSQRT(1. - X*X),X.)
 RPHIO = PHIO * RAD
 RA09 = (90.-A0)*RAD
  RALF =ALF*RAD
  RALFY=ALFY*RAD
  CALCULATE SINES AND COSINES FOR ALPHA,ALPHA Y,AND PHI
 CRALF = DCOS(RALF)
 SRALF = DSIN(RALF)
 CRALFY = DCOS(RALFY)
 SRALFY = DSIN(RALFY)
 CPHIO = DCOS(RPHIO)
 SPHIO = DSIN(RPHIO)
 CRA09 = DCOS(RA09)
 SRA09 = DSIN(RA09)
  CALCULATE OMEGA-E BAR
 WEX = CPHIO*SRA09*OMEGA
 WEY = SPHIO*OMEGA
 WEZ = -CPHIO*CRA09*OMEGA
  CALCULATE VR
  VRX =Y*WEZ-Z*WEY+U
  VRY =Z*WEX-X*WEZ+V
  VRZ =X*WEY-Y*WEX+W
 VR = DSQRT(VRX*VRX + VRY*VRY + VRZ*VRZ)
  B=VRY
 R = DSQRT(X*X + Y*Y + Z*Z)
 C = DSQRT(VRX*VRX + VRZ*VRZ)
  CALCULATE ALTITUDE
  HGT=R-R0
  GGG=-GM/R**3
  CALCULATE VRM-BAR
 VRMX=VR*SRALF*CRALFY
 VRMY= VR*CRALF
 VRMZ = -VR*SRALF*SRALFY
 STALF=ALF
80 IF(ALF.LT.0.)    ALF=-ALF
 IF(ALF.LT.180.)    GO TO 1390
 ALF=ALF-360.
 GO TO 1380
0 CONTINUE
 DO 140 I=1,36
 J=I+2
  IF(ALPHAT(J).GE.ALF)  GO TO 141
0 CONTINUE
 WRITE(6,143)
3 FORMAT(1H1,15X,42HPROGRAM DUMPED BECAUSE ALPHAT IS LESS THAN/16X,
137HALF AS COMPUTED BY SUBROUTINE PRELOD.)
 STOP
```

```
41 DEL=(ALF-ALPHAT(J-1))/5.
   DEL2=DEL*DEL
   CX=TCX(J-1)+.5*(TCX(J)-TCX(J-2))*DEL+.5*(TCX(J)-2.*TCX(J-1)+TCX(J-
  12))*DEL2
   CZ=TCZ(J-1)+.5*(TCZ(J)-TCZ(J-2))*DEL+.5*(TCZ(J)-2.*TCZ(J-1)+TCZ(J-
  12))*DEL2
   CXMD=TCXP(J-1)+.5*(TCXP(J)-TCXP(J-2))*DEL+.5*(TCXP(J)-2.*TCXP(J-1)
  1+TCZP(J-2))*DEL2
   CZMD=TCZP(J-1)+.5*(TCZP(J)-TCZP(J-2))*DEL+.5*(TCZP(J)-2.*TCZP(J-1)
  1+TCXP(J-2))*DEL2
   DO 202 I=1,86
   J=I+2
   IF(ALT(J).GE.HGT)  GO TO 203
02 CONTINUE
   WRITE(6,204)
04 FORMAT(1H1,15X,39HPROGRAM DUMPED BECAUSE ALT IS LESS THAN/16X,
  137HHGT AS COMPUTED BY SUBROUTINE PRELOD.)
   STOP
03 Y0 = ALT(J-2)-ALT(J-1)
   Y2=ALT(J)-ALT(J-1)
   CONA=Y0*Y2*(Y2-Y0)
   Y02=Y0*Y0
   Y22=Y2*Y2
   CONB=Y22*PRESS(J-2)+(Y02-Y22)*PRESS(J-1)-Y02*PRESS(J)
   CONC=-Y2*PRESS(J-2)+(Y2-Y0)*PRESS(J-1)+Y0*PRESS(J) .
   CONB=CONB/CONA
   CONC=CONC/CONA
   DEL=HGT-ALT(J-1)
   RHO=PRESS(J-1)+CONB*DEL+CONC*DEL*DEL
   RHOD=CONB+2.*CONC*DEL
   ALF=STALF
   CC = DSQRT(CX* CX + CZ*CZ)
00 FPA = (A/(2.0*VM))*RHO*VR*VR*CC
   FA = FPA * VM
    XMDOT=FPA**2
   SPALFP=CZ/CC
   CRALFP=CX/CC
    CALCULATE PHI-P
   VRMPD1 = DSQRT(VRMX*VRMX + VRMY*VRMY)
    SP=VRMX/VRMPD1
    CP=VRMY/VRMPD1
   PHIP=DATAN2(SP,CP)
    CALCULATE PHI-Y
   VRMYD1 = DSQRT(VR*VR-VRMZ*VRMZ)
    SY=(-B*VRMZ+C*VRMYD1)/VR**2
    CY=(C*VRMZ+B*VRMYD1)/VR**2
   PHIY = DATAN2(SY,CY)
    CALCULATE PHI-R
   VRMRD1 = DSQRT(VRX*VRX + VRZ*VRZ)
    SR=VRX/VRMRD1
    CR=VRZ/VRMRD1
   PHIR = DATAN2 (SR,CR)
   XNX=-(CP*CR+SP*SY*SR)*SRALFP*CRALFY+(SP*CR-CP*SY*SR)*CRALFP+CY*SR*
  1SRALFP*SRALFY
   XNY=-(SP*CY*CRALFY*SRALFP)-(CP*CY*CRALFP)-(SY*SRALFP*SRALFY)
   XNZ=(CP*SR-SP*SY*CR)*CRALFY*SRALFP-(SP*SR+CP*SY*CR)*CRALFP
  1+CY*CR*SRALFP*SRALFY
    RETURN
    END
FTC GETH
```

```
      SUBROUTINE GETH
      EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
     1,VEX(1))
      EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
     1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
     2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
     3),A0),(ODDS(13),CRA09),(ODDS(14),SPA09),(ODDS(15),PHIP),(ODDS(16),
     4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
     5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
     6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
     7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
     8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
     9 (37),VRMRD1)
      EQUIVALENCE    (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
     1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
     2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
     3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
     4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
      EQUIVALENCE (ODDS(57),FA)
      EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
     1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
     2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
     3(TABS(495),TCXPP(1))
      EQUIVALENCE (VEX(1),XN(1)),(VEX(4),UBDOT(1)),(VEX(7),XBAR(1)),
     1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
     2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
      EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
      EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
      EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
      EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
      EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38)
      DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UB(3)
       DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,A0,           PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
     4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
     5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
     6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
     7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
     8SRALFP,CRALFP
      DOUBLE PRECISION FA
      DOUBLE PRECISION SRA09,CPA09,RPHIO,RAD
      DO 1000 I=1,3
   00 UBDOT(I)=FPA*XN(I)+GGG*XBAR(I)
      H=XLAM7*FPA**2
      DO 1100 I=1,3
   00 H=H+XLAMI(I)*UB(I)+XLAMII(I)*UBDOT(I)
      RETURN
```

```
      END
$FTC HCALC
      SUBROUTINE HCALC
      EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
     1,VEX(1))
      EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
     1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
     2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
     3),AO),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
     4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
     5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
     6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
     7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
     8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
     9 (37),VRMRD1)
      EQUIVALENCE   (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
     1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
     2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
     3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
     4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
      EQUIVALENCE (ODDS(57),FA)
      EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
     1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
     2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
     3(TABS(495),TCXPP(1))
      EQUIVALENCE (VEX(1),XN(1)),(VEX(4),UBDOT(1)),(VEX(7),XBAR(1)),
     1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
     2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
      EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
      EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
      EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
      EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
      EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38)
      DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UB(3)
      DOUBLE PRECISION DELA, DELA2,H2
      DOUBLE PRECISION FA
      DOUBLE PRECISION DEL,STORE,DC,DA,DB
       DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,AO,          PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
     4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
     5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
     6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
     7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,  ,V,W,
     8SRALFP,CRALFP
      DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
      DEL=.1
      STORE = ALF
```

```
      ALF = STORE + DEL
      CALL PRELOD
      CALL GETH
      DA=H
      ALF = ALF + DEL
      CALL PRELOD
      CALL GETH
      DB=H
      ALF = ALF + DEL
      CALL PRELOD
      CALL GETH
      DC=H
      ALF = STORE - DEL
      CALL PRELOD
      CALL GETH
      DA=DA-H
      ALF = ALF-DEL
      CALL PRELOD
      CALL GETH
      DB=DB-H
      ALF = ALF-DEL
      CALL PRELOD
      CALL GETH
      DC=DC-H
      PHA=(.75*DA-.15*DB+DC/60.)/DEL
      SDA = SNGL(DA)
      SDB = SNGL(DB)
      SDC = SNGL(DC)
      WRITE(6,1000) SDA,SDB,SDC
   00 FORMAT(36H VALUES DA,DB,DC USED TO COMPUTE PHA/10X,3(E14.8,5X))
      ALF = STORE
      STORE=ALFY
      ALFY=STORE+DEL
      CALL PRELOD
      CALL GETH
      DA=H
      ALFY=ALFY+DEL
      CALL PRELOD
      CALL GETH
      DB=H
      ALFY=ALFY+DEL
      CALL PRELOD
      CALL GETH
      DC=H
      ALFY=STORE-DEL
      CALL PRELOD
      CALL GETH
      DA=DA-H
      ALFY=ALFY-DEL
      CALL PRELOD
      CALL GETH
      DB=DB-H
      ALFY=ALFY-DEL
      CALL PRELOD
      CALL GETH
      DC=DC-H
      PHAY=(.75*DA-.15*DB+DC/60.)/DEL
      SDA = SNGL(DA)
      SDB = SNGL(DB)
      SDC = SNGL(DC)
```

```
      WRITE(6,2000) SDA,SDB,SDC
 00   FORMAT(37H VALUES DA,DB,DC USED TO COMPUTE PHAY/10X,3(E14.8,5X))
      ALFY=STORE
      CALL PRELOD
      CALL GETH
      RETURN
      END
 FTC PDHY
      SUBROUTINE PDHY
      EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
     1,VEX(1))
      EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
     1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
     2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
     3),AO),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
     4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
     5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
     6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
     7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
     8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
     9 (37),VRMRD1)
      EQUIVALENCE    (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
     1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
     2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
     3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
     4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
      EQUIVALENCE (ODDS(57),FA)
      EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
     1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
     2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
     3(TABS(495),TCXPP(1))
      EQUIVALENCE (VEX(1),XN(1)),(VEX(4),UBDOT(1)),(VEX(7),XBAR(1)),
     1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
     2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
      EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
      EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
      EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
      EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
      EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38)
      DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UB(3)
      DIMENSION STORE(3)
      DOUBLE PRECISION DELU2,DELU,DELX,DELX2,H2,STORE
       DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,AO,          PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
     4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
     5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
     6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
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7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
8SRALFP,CRALFP
 DOUBLE PRECISION FA
 DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
 DELX=100.
 DELU=10.
 DELX2=200.
 DELU2=20.
 DO 1000 I=1,3
 STORE(I)=XBAR(I)
 XBAR(I)=STORE(I)+DELX
 CALL PRELOD
 CALL GETH
 H2=H
 XBAR(I)=STORE(I)-DELX
 CALL PRELOD
 CALL GETH
 XLMID(I)=(H-H2)/DELX2
 XBAR(I)=STORE(I)
 STORE(I)=UB(I)
 UB(I)=STORE(I)+DELU
 CALL PRELOD
 CALL GETH
 H2=H
 UB(I)=STORE(I)-DELU
 CALL PRELOD
 CALL GETH
 XLMIID(I)=(H-H2)/DELU2
 UB(I)=STORE(I)
00 CONTINUE
 CALL PRELOD
 CALL GETH
 RETURN
 END
TC STF2
 SUBROUTINE STF2(DERIV)
 EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
1,VEX(1))
 EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
9 (37),VRMRD1)
 EQUIVALENCE   (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
1 (ODDS(41),CZMD ),(ODDS(42),GC),(ODDS(43),RHO),(ODDS(44),RHOD),
2 (ODDS(45),R),(ODDS(46),R0),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
 EQUIVALENCE (ODDS(57),FA)
 EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
3(TABS(495),TCXPP(1))
 EQUIVALENCE (VEX(1),XN(1)),(VEX(4),UBDOT(1)),(VEX(7),XBAR(1)),
1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
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 2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
  EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
  EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
  EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
  EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
  EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
  EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
  EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
 1D)
  EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
  COMMON MASCOM
  DIMENSION MASCOM(693)
  DIMENSION ODDS(100),TABS(568),VEX(25)
  DIMENSION ALT(88),PRESS(88)
  DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
 1TCXPP(38)
  DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
 1XLMIID(3),UB(3)
  DIMENSION DERIV(14)
  DOUBLE PRECISION DERIV
   DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
 1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,A0,SA09,CA09,PHIP,CP,SP,PHIY
 2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
 3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
 4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
 5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
 6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
 7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
 8SRALFP,CRALFP
  DOUBLE PRECISION FA
  DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
  DO 1000  I=1,3
  DERIV(I)=UB(I)
  DERIV(I+3)=UBDOT(I)
  DERIV(I+6)=XLMID(I)
1000 DERIV(I+9)=XLMIID(I)
  RETURN
  END
TC STF1
  SUBROUTINE STF1(XVAL)
  EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
 1,VEX(1))
  EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
 1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
 2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
 3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
 4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
 5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
 6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
 7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
 8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
 9 (37),VRMPD1)
  EQUIVALENCE    (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
 1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
 2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
 3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
 4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
  EQUIVALENCE (ODDS(57),FA)
  EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
 1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
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2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
3(TABS(495),TCXPP(1))
 EQUIVALENCE (VFX(1),XN(1)),(VFX(4),UBDOT(1)),(VFX(7),XBAR(1)),
1(VFX(10),XLAMI(1)),(VFX(13),XLAMII(1)),(VFX(16),XLMID(1)),(VFX(19)
2,XLMIID(1)),(VFX(22),UB(1)),(VFX(25),XLAM7)
 EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
 EQUIVALENCE (UBDOT(1),UD),(UBDOT(2),VD),(UBDOT(3),WD)
 EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
 EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
 EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
 EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
 EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
1D)
 EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
 COMMON MASCOM
 DIMENSION MASCOM(693)
 DIMENSION ODDS(100),TABS(568),VFX(25)
 DIMENSION ALT(88),PRESS(88)
 DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
1TCXPP(38)
 DIMENSION UBDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
1XLMIID(3),UB(3)
 DIMENSION XVAL(14)
 DOUBLE PRECISION  XVAL
  DOUBLE PRECISION MASCOM,ODDS,TABS,VFX,ALF,RALF,CRALF,SRALF,ALFY,
1 RALFY,CRALFY,SRALFY,PHI0,CPHI0,SPHI0,A0,SA09,CA09,PHIP,CP,SP,PHIY
2 ,CY,SY,PHIR,CR,SR,OMEGA,WFX,WFY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,R0,HGT,A,
4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,UBDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
8SRALFP,CRALFP
 DOUBLE PRECISION FA
 DOUBLE PRECISION SRA09,CRA09,B,RPHIO
 DO 1000 I=1,3
 XVAL(I)=XBAR(I)
 XVAL(I+3)=UB(I)
 XVAL(I+6)=XLAMI(I)
1000 XVAL(I+9)=XLAMII(I)
 RETURN
 END
TC STF3
 SUBROUTINE STF3(XVAL)
 EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
1,VFX(1))
 EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
2 SRALFY),(ODDS(9),PHI0),(ODDS(10),CPHI0),(ODDS(11),SPHI0),(ODDS(12
3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
6 (ODDS(25),WFX),(ODDS(26),WFY),(ODDS(27),WFZ),(ODDS(28),VR),(ODDS
7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
9 (37),VRMRD1)
 EQUIVALENCE    (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
2 (ODDS(45),R),(ODDS(46),R0),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
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 4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
  EQUIVALENCE (ODDS(57),FA)
  EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
 1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
 2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
 3(TABS(495),TCXPP(1))
  EQUIVALENCE (VEX(1),XN(1)),(VEX(4),URDOT(1)),(VEX(7),XBAR(1)),
 1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
 2,XLMIID(1)),(VEX(22),UB(1)),(VEX(25),XLAM7)
  EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
  EQUIVALENCE (URDOT(1),UD),(URDOT(2),VD),(URDOT(3),WD)
  EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
  EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
  EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
  EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
  EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
 1D)
  EQUIVALENCE (UB(1),U),(UB(2),V),(UB(3),W)
  COMMON MASCOM
  DIMENSION MASCOM(693)
  DIMENSION ODDS(100),TABS(568),VEX(25)
  DIMENSION ALT(88),PRESS(88)
  DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
 1TCXPP(38)
  DIMENSION URDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
 1XLMIID(3),UB(3)
  DIMENSION XVAL(14)
  DOUBLE PRECISION  XVAL
   DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
 1 RALFY,CRALFY,SRALFY,PHI0,CPHI0,SPHI0,A0,SA09,CA09,PHIP,CP,SP,PHIY
 2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
 3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,R0,HGT,A,
 4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
 5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,URDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
 6 UB,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
 7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
 8SRALFP,CRALFP
  DOUBLE PRECISION FA
  DOUBLE PRECISION SRA09,CRA09,RPHI0,RAD
  DO 1000 I=1,3
  XBAR(I)=XVAL(I)
  UB(I)=XVAL(I+3)
  XLAMI(I)=XVAL(I+6)
0 XLAMII(I)=XVAL(I+9)
  RETURN
  END
TC SHIFT
  SUBROUTINE SHIFT (A,B,K)
  DIMENSION A(14),B(14)
  DOUBLE PRECISION  A,B
  DO 1000 I=1,  K
0 A(I)=B(I)
  RETURN
  END
TC TGRATE
  SUBROUTINE TGRATE(N,DT)
  DIMENSION XVAL(14),STDRV(14,4),STORX(14),PCN(14)
  DOUBLE PRECISION STDRV,STORX,PCN,XVAL
  IF(N.GT.3)   GO TO 1000
  CALL PDHY
```

```
      CALL GETH
      CALL RKINT(DT)
      CALL STF2(STDRV(1,N))
      RETURN
 00   CONTINUE
      CALL STF1(XVAL)
      CALL PDHY
      CALL GETH
      CALL STF2 (STDRV(1,4))
      CALL SHIFT(STORX,XVAL,14)

                PREDICTOR - ADAMS BASHFORTH

      DO 1100  I=1,14
      XVAL(I)=STORX(I)+DT*(-9.*STDRV(I,1)+37.*STDRV(I,2)-59.*STDRV(I,3)
     1+55.*STDRV(I,4))/24.
 00   CONTINUE
      DO 1200  J=1,3
 00   CALL SHIFT(STDRV(1,J),STDRV(1,J+1),14)

                CORRECTOR - ADAMS BASHFORTH

      CALL STF3(XVAL)
      CALL PDHY
      CALL GETH
      CALL STF2(STDRV(1,4))
      DO 1300 I=1,14
      STORX(I)=STORX(I)+DT*(STDRV(I,1)-5.*STDRV(I,2)+19.*STDRV(I,3)
     1+9.*STDRV(I,4))/24.
 00   PCN(I)=STORX(I)-XVAL(I)
      CALL STF3(STORX)
 00   RETURN
      END
 ETC CONV
      SUBROUTINE CONV(B,A,K)
      DIMENSION A(2),B(2)
      REAL MINUS
      DATA PLUS/1H+/,MINUS/1H-/
      EQUIVALENCE(EX,IEX)
      DO 1000 I=1,K
      IEX=0
      JJA = 5*I-3
      X=A(I)
      IF(X) 1005,1001,1010
 01   B(JJA)=PLUS
      B(JJA+2) = PLUS
      B(JJA+1) = 0.0
      GO TO 1000
 05   B(JJA)=MINUS
      X= -X
      GO TO 1020
 10   B(JJA) = PLUS
 20   CONTINUE
      B(JJA+2) = MINUS
      IF(X.LT.1.) GO TO 1035
      B(JJA+2)= PLUS
 25   IF(X.LT.1.) GO TO 1040
      IEX=IEX+1
      X=X/10.
      GO TO 1025
```

```
5 IF(X.GE..1) GO TO 1040
  IEX=IEX+1
  X=X*10.
  GO TO 1035
0 B(JJA+1) = X
0 B(JJA+3) = EX
  RETURN
  END
  RKINT
  SUBROUTINE RKINT(DT)
  DIMENSION XK(14,4),STORX(14),XVAL(14),DERIV(14),C(3),D(4)
  DOUBLE PRECISION  C,DERIV,D,STORX,XK,XVAL
  C(1)=.5
  C(2)=.5
  C(3)=1.
  D(1)=.166666667
  D(2)=.333333333
  D(3)=.333333333
  D(4)=.166666667
  DO 1000 J=1,4
  CALL STF1(XVAL)
  CALL SHIFT(STORX,XVAL,14)
  CALL PDHY
  CALL GETH
  CALL STF2(DERIV)
  DO 1000 I=1,14
  XK(I,J)=DT*DERIV(I)
  IF(J.NE.1)  XVAL(I)=STORX(I)+C(J-1)*XK(I,J)
0 CONTINUE
  DO 1100 I=1,14
  XVAL(I)=STORX(I)
  DO 1100 J=1,4
0 XVAL(I)=XVAL(I)+XK(I,J)*D(J)
  CALL STF3(XVAL)
  RETURN
  END
TC JACOB
  SUBROUTINE JACOB(HH,DEL1,DEL22)
  EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
 1,VEX(1))
  EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
 1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
 2 SRALFY),(ODDS(9),PHI0),(ODDS(10),CPHI0),(ODDS(11),SPHI0),(ODDS(12
 3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
 4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
 5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
 6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
 7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
 8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
 9 (37),VRMRD1)
  EQUIVALENCE    (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
 1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
 2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
 3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
 4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
  EQUIVALENCE (ODDS(57),FA)
  EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
 1    S(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
 2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
 3(TABS(495),TCXPP(1))
```

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      EQUIVALENCE (VEX(1),XN(1)),(VEX(4),URDOT(1)),(VEX(7),XBAR(1)),
     1(VEX(10),XLAMI(1)),(VEX(13),XLAMII(1)),(VEX(16),XLMID(1)),(VEX(19)
     2,XLMIID(1)),(VEX(22),UR(1)),(VEX(25),XLAM7)
      EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
      EQUIVALENCE (URDOT(1),UD),(URDOT(2),VD),(URDOT(3),WD)
      EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
      EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
      EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
      EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
      EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
     1D)
      EQUIVALENCE (UR(1),U),(UR(2),V),(UR(3),W)
      COMMON MASCOM
      DIMENSION MASCOM(693)
      DIMENSION ODDS(100),TABS(568),VEX(25)
      DIMENSION ALT(88),PRESS(88),K(88)
      DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
     1TCXPP(38),J(38)
      DIMENSION URDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
     1XLMIID(3),UR(3)
      DIMENSION OUTD(4,100)
      DOUBLE PRECISION EA,AST
       DOUBLE PRECISION MASCOM,ODDS,TABS,VEX,ALF,RALF,CRALF,SRALF,ALFY,
     1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,AC,         PHIP,CP,SP,PHIY
     2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
     3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RC,HGT,A,
     4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
     5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,URDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
     6 UR,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
     7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
     8SRALFP,CRALFP
      DOUBLE PRECISION ARCOS,COLAT,CRIT,DEL,TESP,TIREC,TLIMIT,TPRINT,
     1TSTEP,TY,UO,
     2 VO,VLAT,VLONG,WO,XO,XLAM10,XLAM20,XLAM30,XLAM40,XLAM50,XLAM60,
     3ZO
      DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
      DIMENSION STX(3),STY(3),STAY(3)
      DOUBLE PRECISION STX,STY,STAY,SLOPE
      DOUBLE PRECISION STALF
      DOUBLE PRECISION CONA,CONB,CONC,YO,Y2,YO2,Y22,DEL2
      DIMENSION RD(7)
      DIMENSION OF(110),OUTA(40),OUTC(200)
      DIMENSION HH(2,4),PA(3),PAY(3)
      DOUBLE PRECISION PA,PAY,HH,DEL1,DEL22,SAVE1,SAVE2
      SAVE1=ALF
      SAVE2=ALFY
      CALL HCALC
      PA(1)=PHA
      PAY(1)=PHAY
      ALF=SAVE1+DEL1
      CALL HCALC
      PA(2)=PHA
      PAY(2)=PHAY
      HH(1,1)=(PA(2)-PA(1))/DEL1
      HH(2,1)=(PAY(2)-PAY(1))/DEL1
      ALF=SAVE1
      ALFY=SAVE2+DEL22
      CALL HCALC
      PA(3) =PHA
      PAY(3)=PHAY
```

```
      WRITE(6,299)
 99   FORMAT(1H0,50HTHE FOLLOWING VALUES ARE PHA AND PHAY RESPECTIVELY)
      WRITE(6,300) (PA(I),PAY(I),I=1,3)
 00   FORMAT(F20.8,5X,F20.8)
      HH(1,2)=(PA(3)-PA(1))/DEL22
      HH(2,2)=(PAY(3)-PAY(1))/DEL22
      ALFY=SAVE2
      RETURN
      END
ETC SLVNL
      SUBROUTINE SLVNL(XX1,XX2,H,FF,FFY,CRIT,NTOP,TIREC)
         PURPOSE
            SOLVE A SYSTEM OF N NONLINEAR EQUATIONS

         DESCRIPTION OF INPUT PARAMETERS
            X - INITIAL VALUE OF VECTOR X
            H - APPROXIMATION OF THE INVERZ JACOBIAN MATRIX, H=1/A
            N - NUMBER OF VARIABLES AND EQUATIONS
            EVALF - FORTRAN SUBROUTINE TO COMPUTE VECTOR F
            CRIT - PRESCRIBED ACCURACY LIMIT OF NORM(F)

         DESCRIPTION OF OUTPUT PARAMETERS
            X -  FINAL  VALUE OF VECTOR X
            H - APPROXIMATION OF THE INVERZ JACOBIAN MATRIX, H=1/A
            F - VECTOR OF N FUNCTIONS
            ITER - NUMBER OF ITERATIONS

         SUBROUTINES REQUIRED
            MATMPY
            LINCOM
            FNORM
            GETT
            SHIFT
            EVALF

      DIMENSION X(2),H(2,2),F(2)
      DIMENSION P(10),Y(10),FN(10),XN(10)
      DOUBLE PRECISION   P,Y,FN,XN,H,F,VAL,VALO,X,XX1,XX2,FF,FFY,SCALF
      N=2
      N2=N*N
      ITER=0
      X(1)=XX1
      X(2)=XX2
      F(1)=FF
      F(2)=FFY
 10   CALL EVALF(X,F,N)

      START NEW ITERATION
      EVALUATE VECTOR P=H*F

 0    CALL MATMPY(H,F,P,N,N,1)
      ITER=ITER+1
      CALL FNORM(F,N,VALO)
      CALL GETT(0,T,VALO)

      FIND A VALUE OF T SUCH THAT THE NORM OF F(X+T*P) IS LESS
      THAN THE NORM OF F(X)
      XN IS THE NEW TRIAL VALUE OF X, OBTAINED AS  XN=X+T*P
      -VAL- AND -VALO- ARE THE NORM OF F(XN) AND F(X) RESPECTIVELY
```

```
      DO 1075  I=1,10
      CALL LINCOM(1.,X,T,P,XN,N,1)
      CALL EVALF(XN,FN,N)
      CALL FNORM(FN,N,VAL)
      IF(VAL.LT.VALO) GO TO 1080
      CALL GETT(I,T,VAL)
 75   CONTINUE
      GO TO 2000

      ONCE A SATSFACTORY T WAS FOUND, X IS REPLACD BY XN,
      F IS REPLACED BY FN
      IF REQUIRED ACCURACY IS OBTAINED OR ALLOWED NUMBER OF INTERATIONS
      EXHAUSTED RETURN TO CALLING PROGRAM
      A NEW APPROXIMATION OF MATRIX H IS COMPUTED
      NEW H IS OBTAINED AS  H=H-(H*Y+T*P)*(P*H/SCALE)

 30   CONTINUE
      IF(ITER.GT.NTOP) GO TO 2000
      CALL SHIFT1(X,XN,N)

      COMPUTE  Y=FN-F

      CALL LINCOM(1.,FN,-1.,F,Y,N,1)

      REPLACE F BY FN

      CALL SHIFT1(F,FN,N)

      COMPUTE  H*Y

      CALL MATMPY(H,Y,FN,N,N,1)

      COMPUTE  H*Y+T*P

      CALL LINCOM(1.,FN,T,P,FN,N,1)

      COMPUTE  P*H

      CALL MATMPY(P,H,XN,1,N,N)

      COMPUTE  SCALE=(P*H)*Y

      CALL MATMPY(XN,Y,SCALE,1,N,1)

      COMPUTE  P*H/SCALE

      DO 1100 I=1,N
 00   XN(I)=XN(I)/SCALE
      DO 1200 I=1,N
      DO 1200 J=1,N
 0    H(I,J)=H(I,J)-FN(I)*XN(J)
      IF(VAL.LE.CRIT) GO TO 2000
      GO TO 1050
 00   WRITE(6,3300) ITER,TIREC
 0    FORMAT(12H ITERATIONS=,I6,5X,2HT=,E14.8)
      XX1=X(1)
      XX2=X(2)
      FF=F(1)
      FFY=F(2)
      RETURN
```

```
      END
C MATMPY
      SUBROUTINE MATMPY(A,B,C,N1,N2,N3)
      DIMENSION A(N1,N2),B(N2,N3),C(N1,N3)
      DOUBLE PRECISION A,B,C,TEMP
      DO 3 I=1,N1
      DO 2 K=1,N3
      TEMP=0.
      DO 1 J=1,N2
      TEMP=TEMP+A(I,J)*B(J,K)
      C(I,K)=TEMP
    2 CONTINUE
    3 CONTINUE
      RETURN
      END
C LINCOM
      SUBROUTINE LINCOM(S,A,T,B,C,M,N)
      DIMENSION A(M,N),B(M,N),C(M,N)
      DOUBLE PRECISION A,B,C
      DO 2 I=1,M
      DO 1 J=1,N
      C(I,J)=S*A(I,J)+T*B(I,J)
      CONTINUE
    2 CONTINUE
      RETURN
      END
C FNORM
      SUBROUTINE FNORM(F,N,VAL)
      DIMENSION F(N)
      DOUBLE PRECISION F,VAL
      VAL=0.
      DO 1 I=1,N
      VAL=VAL+F(I)*F(I)
      SVAL = SNGL(VAL)
      WRITE(6,1000) SVAL
      FORMAT(5H VAL=,E14.8)
      RETURN
      END
C GETT
      SUBROUTINE GETT(IT,T,F)
      IF(IT.NE.0)   GO TO 1
      T=1.
      F0=F
      RETURN
      IF(IT.NE.1) GO TO 2
      F1=F
      TH=F1/F0
      T=(SQRT(1.+6.*TH)-1.)/3./TH
      RETURN
      T=-T/2.
      RETURN
      END
C SHIFT1
      SUBROUTINE SHIFT1(A,B,K)
      DIMENSION A(1), B(1)
      DOUBLE PRECISION A,B
      DO 1000 I=1,  K
      A(I)=B(I)
      RETURN
      END
```

```
TC EVALF
  SUBROUTINE EVALF(XX,F,N1)
  EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(101),TABS(1)),(MASCOM(669)
 1,VFX(1))
  EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4),
 1 SRALF),(ODDS(5),ALFY),(ODDS(6),RALFY),(ODDS(7),CRALFY),(ODDS(8),
 2 SRALFY),(ODDS(9),PHIO),(ODDS(10),CPHIO),(ODDS(11),SPHIO),(ODDS(12
 3),A0),(ODDS(13),CRA09),(ODDS(14),SRA09),(ODDS(15),PHIP),(ODDS(16),
 4 CP),(ODDS(17),SP),(ODDS(18),PHIY),(ODDS(19),CY),(ODDS(20),SY),
 5 (ODDS(21),PHIR),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA),
 6 (ODDS(25),WEX),(ODDS(26),WEY),(ODDS(27),WEZ),(ODDS(28),VR),(ODDS
 7 (29),VRX),(ODDS(30),VRY),(ODDS(31),VRZ),(ODDS(32),VRMX),(ODDS(33)
 8 ,VRMY),(ODDS(34),VRMZ),(ODDS(35),VRMPD1),(ODDS(36),VRMYD1),(ODDS
 9 (37),VRMRD1)
  EQUIVALENCE   (ODDS(38),CX),(ODDS(39),CXMD ),(ODDS(40),CZ),
 1 (ODDS(41),CZMD ),(ODDS(42),CC),(ODDS(43),RHO),(ODDS(44),RHOD),
 2 (ODDS(45),R),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49),
 3 VM),(ODDS(50),GM),(ODDS(51),GGG),(ODDS(52),FPA),(ODDS(53),XMDOT),
 4 (ODDS(54),H),(ODDS(55),PHA),(ODDS(56),PHAY)
  EQUIVALENCE (ODDS(57),FA)
  EQUIVALENCE (TABS(1),ALT(1)),(TABS(89),PRESS(1)),
 1(TABS(265),ALPHAT(1)),(TABS(303),TCZ(1)),(TABS(341),TCZP(1)),
 2(TABS(379),TCZPP(1)),(TABS(417),TCX(1)),(TABS(457),TCXP(1)),
 3(TABS(495),TCXPP(1)))
  EQUIVALENCE (VFX(1),XN(1)),(VFX(4),URDOT(1)),(VFX(7),XBAR(1)),
 1(VFX(10),XLAMI(1)),(VFX(13),XLAMII(1)),(VFX(16),XLMID(1)),(VEX(19)
 2,XLMIID(1)),(VFX(22),UR(1)),(VEX(25),XLAM7)
  EQUIVALENCE (XN(1),XNX),(XN(2),XNY),(XN(3),XNZ)
  EQUIVALENCE (URDOT(1),UD),(URDOT(2),VD),(URDOT(3),WD)
  EQUIVALENCE (XBAR(1),X),(XBAR(2),Y),(XBAR(3),Z)
  EQUIVALENCE (XLAMI(1),XLAM1),(XLAMI(2),XLAM2),(XLAMI(3),XLAM3)
  EQUIVALENCE (XLAMII(1),XLAM4),(XLAMII(2),XLAM5),(XLAMII(3),XLAM6)
  EQUIVALENCE (XLMID(1),XLAM1D),(XLMID(2),XLAM2D),(XLMID(3),XLAM3D)
  EQUIVALENCE (XLMIID(1),XLAM4D),(XLMIID(2),XLAM5D),(XLMIID(3),XLAM6
 1D)
  EQUIVALENCE (UR(1),U),(UR(2),V),(UR(3),W)
  COMMON MASCOM
  DIMENSION MASCOM(693)
  DIMENSION ODDS(100),TABS(568),VFX(25)
  DIMENSION ALT(88),PRESS(88),K(88)
  DIMENSION ALPHAT(38),TCZ(38),TCZP(38),TCZPP(38),TCX(38),TCXP(38),
 1TCXPP(38),J(38)
  DIMENSION URDOT(3),XN(3),XBAR(3),XLAMI(3),XLAMII(3),XLMID(3),
 1XLMIID(3),UR(3)
  DIMENSION OUTD(4,100)
  DOUBLE PRECISION FA,AST
   DOUBLE PRECISION MASCOM,ODDS,TABS,VFX,ALF,RALF,CRALF,SRALF,ALFY,
 1 RALFY,CRALFY,SRALFY,PHIO,CPHIO,SPHIO,A0,          PHIP,CP,SP,PHIY
 2 ,CY,SY,PHIR,CR,SR,OMEGA,WEX,WEY,WEZ,VR,VRX,VRY,VRZ,VRMX,VRMY,VRMZ
 3 ,VRMPD1,VRMYD1,VRMRD1,CX,CXMD ,CZ,CZMD ,CC,RHO,RHOD,R,RO,HGT,A,
 4 VM,GM,GGG,FPA,XMDOT,H,PHA,PHAY,ALT,PRESS,ALPHAT,TCZ,
 5TCZP,TCZPP,TCX,TCXP,TCXPP,XN,URDOT,XBAR,XLAMI,XLAMII,XLMID,XLMIID,
 6 UR,XLAM7,XNX,XNY,XNZ,UD,VD,WD,X,Y,Z,XLAM1,XLAM2,XLAM3,XLAM4,
 7 XLAM5,XLAM6,XLAM1D,XLAM2D,XLAM3D,XLAM4D,XLAM5D,XLAM6D,U,V,W,
 8SRALFP,CRALFP
  DOUBLE PRECISION ARCOS,COLAT,CRIT,DEL,TESP,TIREC,TLIMIT,TPRINT,
 1TSTEP,TY,U0,
 2 V0,VLAT,VLONG,W0,X0,XLAM10,XLAM20,XLAM30,XLAM40,XLAM50,XLAM60,
 3Z0
  DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
```

```
DIMENSION STX(3),STY(3),STAY(3)
DOUBLE PRECISION STX,STY,STAY,SLOPE
DOUBLE PRECISION STALF
DOUBLE PRECISION CONA,CONB,CONC,Y0,Y2,Y02,Y22,DEL2
DOUBLE PRECISION XX,F
DIMENSION RD(7)
DIMENSION OF(110),OUTA(40),OUTC(200)
DIMENSION XX(N1),F(N1)
ALF=XX(1)
ALFY=XX(2)
CALL HCALC
F(1)=PHA
F(2)=PHAY
RETURN
END
TC INVERS
SUBROUTINE INVERS(A,INDX,IORD,N,NN,KERR)
DOUBLE PRECISION A,R
DIMENSION A(N,NN)
DIMENSION INDX(N),IORD(N)
KERR=0
J1=N+1
J2=2*N
DO 23 I=1,N
J3=I+N
DO 24 J=J1,J2
A(I,J)=0.0
A(I,J3)=1.0
DO 10 I=1,N
INDX(I)=0
N2=N-1
DO 11 J=1,N
DO 12 I=1,N2
IF(INDX(I).EQ.0) GO TO 13
CONTINUE
R=-.1D+36
K=I
IL=1
IT=1
DO 14 M=K,N
IF(INDX(M).NE.0) GO TO 14
IL=IL+1
IF(A(M,J).NE.0.0) GO TO 17
IT=IT+1
GO TO 14
IF((R-DABS(A(M,J))).GT.0.0) GO TO 14
R=A(M,J)
MM=M
CONTINUE
IF((IT-IL).NE.0) GO TO 19
WRITE(6,21)
FORMAT(///30H MATRIX INVERSION NOT POSSIBLE///)
KERR=1
RETURN
INDX(MM)=1
IORD(J)=MM
IP=J+1
DO 25 JJ=IP,J2
A(MM,JJ)=A(MM,JJ)/A(MM,J)
DO 26 K=1,N
```

```
      IF((MM-K).EQ.0) GO TO 26
      DO 28 JJ=IP,J2
      A(K,JJ)=A(K,JJ)-A(K,J)*A(MM,JJ)
      CONTINUE
      CONTINUE
      DO 29 I=1,N
      IRE=IORD(I)
      DO 29 J=1,N
      L=J+N
      A(I,J)=A(IRE,L)
      RETURN
      END
AP TRAP    DECK
       ENTRY    TRAP
       AXT      **,4
       TRA      **
       SXA      TRAP-1,4
       CLA      8
       STA      RESET+1
       CLA      FIX
       TSX      S.SCCR,4
       STO      8
       TRA      TRAP-1
       CLA      0
       TMI      **
       ARS      20
       LBT
       TRA      *+2
       TRA*     RESET+1
       SXA      OUT,4
       TSX      S.WRIT,4
       PZE      3,,MES
       AXT      **,4
       ZAC
       LRS      35
       TRA*     0
       TRA      RESET
       BCI      3, **** UNDERFLOW
       END
PY
```

0	0.00	288	000000	1828	00000	0 1
5	014	00282	-000014	1812	-00122	-000084 2
10	028	00270	-000042	1772	-00940	-002448 3
15	040	00252	-000054	1710	-02432	-000996 4
20	052	00234	-000048	1626	-02786	-000476 5
25	063	00214	-000026	1520	-02880	-000328 6
30	074	00208	000023	1388	-03014	-000252 7
35	084	00244	000124	1246	-03136	-000208 8
40	096	00336	000238	1092	-03234	-000184 9
45	118	00490	000274	0932	-03320	-00015210
50	146	00642	000270	0768	-03380	-00008811
55	182	00768	000254	0588	-03406	00003412
60	224	00874	000230	0416	-03352	00022213
65	268	00966	000200	0256	-03128	00041614
70	318	01052	000168	0112	-02922	00056815
75	372	01124	000108	-0020	-02600	00064616
80	426	01174	000044	-0134	-02264	00067017
85	486	01178	-000032	-0236	-01934	00064618
90	546	01138	-000108	-0322	-01622	00060019
95	596	01056	-000182	-0394	-01340	00056020

100	628	00944	-000246	-0444	-01078	00051621
105	690	00814	-000306	-0486	-00834	00046622
110	728	00648	-000360	-0516	-00620	00041623
115	756	00466	-000408	-0542	-00432	00036824
120	772	00256	-000454	-0566	-00262	00031825
125	776	00036	-000494	-0582	-00108	00027026
130	772	-00202	-000526	-0586	00028	00026827
135	756	-00460	-000522	-0584	00154	00016428
140	730	-00736	-000568	-0576	00264	00011029
145	686	-01022	-000576	-0560	00336	00004830
150	628	-01308	-000574	-0544	00368	-00001031
155	554	-01588	-000568	-0524	00348	-00006632
160	468	-01872	-000550	-0510	00290	-00010833
165	366	-02140	-000508	-0498	00216	-00014034
170	248	-02346	-000260	-0490	00142	-00014235
175	130	-02414	-000056	-0484	00068	-00014036
180	000	-02428	000000	-0480	00000	-0 012837
						38

0000	12250	1
2001	10065	2
4003	81916E-01	3
6006	65973E-01	4
8010	52519E-01	5
10016	41262E-01	6
11019	36392E-01	7
12023	31082E-01	8
14031	22674E-01	9
16040	16541E-01	10
18051	12067E-01	11
20063	88033E-02	12
22076	63726E-02	13
24091	46266E-02	14
26107	33687E-02	15
28124	24598E-02	16
30142	18011E-02	17
32162	13225E-02	18
34183	96086E-03	19
36205	70342E-03	20
38229	51867E-03	21
40253	38510E-03	22
42279	28780E-03	23
44307	21645E-03	24
46335	16376E-03	25
47350	14275E-03	26
48365	12582E-03	27
50396	97747E-04	28
52429	75939E-04	29
54463	59769E-04	30
56498	46873E-04	31
58534	36622E-04	32
60572	28503E-04	33
61591	25108E-04	34
62611	22261E-04	35
64651	17396E-04	36
66692	13482E-04	37
68735	10355E-04	38
70779	78782E-05	39
72825	59322E-05	40
74872	44176E-05	41
76920	32504E-05	42

```
                                                                  32
 78069   23606E-05                                                43
 79994   20009E-05                                                44
 81020   16562E-05                                                45
 83072   11346E-05                                                46
 85125   77729E-06                                                47
 87179   53250E-06                                                48
 89235   35480E-06                                                49
 90000   31698E-06                                                50
 92      21368E-06                                                51
 94      14589E-06                                                52
 96      10080E-06                                                53
 98      70428E-07                                                54
100      49731E-07                                                55
102      34924E-07                                                56
104      24918E-07                                                57
106      18038E-07                                                58
108      13233E-07                                                59
110      98277E-08                                                60
112      71512E-08                                                61
114      53196E-08                                                62
116      40338E-08                                                63
118      31109E-08                                                64
120      24352E-08                                                65
122      18435E-08                                                66
124      14336E-08                                                67
126      11395E-08                                                68
128      92254E-09                                                69
130      75873E-09                                                70
132      63252E-09                                                71
134      53357E-09                                                72
136      45478E-09                                                73
138      39121E-09                                                74
140      33929E-09                                                75
142      29643E-09                                                76
144      26071E-09                                                77
146      23067E-09                                                78
148      20522E-09                                                79
150      18350E-09                                                80
152      16644E-09                                                81
154      15141E-09                                                82
156      13812E-09                                                83
158      12633E-09                                                84
160      11584E-09                                                85
162      10738E-09                                                86
164      99669E-10                                                87
                                                                  88
   3.986016E+14        6.378145E+06        7.292116E-05 90.   0.   1
 -95.        65.      4350.       12.02                            2
   0.   6490000.        0.   8878.284   -620.864         0.        3
 .001        .001      .001     10000.     10000.    10000.     1. 4
   2.        100.        1.       3    1                           5
YS
```

The procedure for solving the problem of flying a minimum fuel point-to-point transfer with a high speed aircraft is as follows.

Minimize the integral

$$I = \int_{to}^{t} \dot{m}_f(t)dt$$

where $\dot{m}_f(t)$ is the time variable fuel burning rate, subject to the differential constraints

$$\dot{\overline{x}} = \overline{u} \tag{1}$$

$$\dot{\overline{u}} = \frac{[T]\overline{F}_T}{m} + \frac{[T][\alpha,\alpha_y]}{m}\overline{F}_a + \frac{\overline{F}_g}{m} - \frac{F_T\overline{u}}{V_e m} \tag{2}$$

$$\dot{\overline{\phi}} = \overline{\Psi} \tag{3}$$

and

$$\dot{\overline{\Psi}} = [B_2]^{-1} \{\overline{M}_\phi + (A)\overline{\Psi} + \overline{B}_3\} \tag{4}$$

which are the equations of motion, and the algebraic constraints

$$\overline{V}_R = [\alpha,\alpha_y]\overline{V}_r. \tag{5}$$

The Hamiltonian is formed as

$$H = \overline{\lambda}_I \cdot \dot{\overline{x}} + \overline{\lambda}_{II} \cdot \dot{\overline{u}} + \overline{\lambda}_{III} \cdot \dot{\overline{\phi}} + \overline{\lambda}_{IV} \cdot \dot{\overline{\Psi}} + \lambda_{13}\dot{m}_f \tag{6}$$

The control variables will be F_T and $\overline{M}_F$ where F_T is the thrust magnitude and $\overline{M}_F$ is the moment generated about the C.P. by the aircraft flaps.

The canonical and control equations become

$$\dot{\bar{\lambda}}_I = -\frac{\partial H}{\partial \bar{\bar{x}}} \qquad (7)$$

$$\dot{\bar{\lambda}}_{II} = -\frac{\partial H}{\partial \bar{u}} \qquad (8)$$

$$\dot{\bar{\lambda}}_{III} = -\frac{\partial H}{\partial \bar{\phi}} \qquad (9)$$

$$\dot{\bar{\lambda}}_{IV} = -\frac{\partial H}{\partial \bar{\Psi}} \qquad (10)$$

$$\dot{\lambda}_{13} = -\frac{\partial H}{\partial m_f} \qquad (11)$$

$$\frac{\partial H}{\partial F_T} = 0 \qquad (12)$$

$$\frac{\partial H}{\partial \bar{M}_F} = 0 \qquad (13)$$

There are four control variables F_T, $\bar{M}_F$, fourteen state variables, $\bar{x}$, $\dot{\bar{x}}$ $\bar{\phi}$, $\dot{\bar{\phi}}$, α, α_y, and thirteen multipliers. Equations (1-13) provide thirty-one scalar equations from which to determine thirty-one unknowns.

From Equation (13)

$$\frac{\partial H}{\partial \bar{M}_F} = 0 \rightarrow \bar{\lambda}_{IV} = 0 \qquad (14)$$

$$\dot{\bar{\lambda}}_{IV} = 0 = \bar{\lambda}_{III} \qquad (15)$$

$$\dot{\bar{\lambda}}_{III} = -\frac{\partial H}{\partial \bar{\phi}} = f_2(\bar{\lambda}_{II}, F_T, \bar{\phi}, \alpha, \alpha_y, \bar{x}, \dot{\bar{x}}) \qquad (16)$$

$$\frac{\partial H}{\partial F_T} = 0 = f_3(\bar{\phi}, \bar{\lambda}_{II}) \qquad (17)$$

Solve Equations (16) and (17) simultaneously for $\bar{\lambda}_{II}$ and F_T.

$$\overline{\lambda}_{II} = \overline{\lambda}_{II}(\overline{\phi}, \alpha, \alpha_y, \overline{x}, \dot{\overline{x}})$$

$$F_T = F_T(\phi, \alpha, \alpha_y, \overline{x}, \dot{\overline{x}})$$

Compute $\dot{\overline{\lambda}}_{II} = f_5(\overline{\phi}, \dot{\overline{\phi}}, \overline{x}, \overline{u}, \dot{\overline{u}})$

From (8) $\dot{\overline{\lambda}}_{II} = -\frac{\partial H}{\partial \overline{u}} = -\overline{\lambda}_I - \frac{\partial(\overline{\lambda}_{II} \cdot \overline{u})}{\partial \overline{u}}$

Solve for $\overline{\lambda}_I = -\dot{\overline{\lambda}}_{II} - \frac{\partial(\overline{\lambda}_{II} \cdot \overline{u})}{\partial \overline{u}}$

Compute $\dot{\overline{\lambda}}_I = f_7(\overline{\phi}, \overline{\Psi}, \dot{\overline{\Psi}}, \overline{x}, \overline{u}, \dot{\overline{u}}, \ddot{\overline{u}})$

From (7) $\dot{\overline{\lambda}}_I = -\frac{\partial H}{\partial \overline{x}} = f_8(\overline{\phi}, \overline{\Psi}, \dot{\overline{\Psi}}, \overline{x}, \overline{u}, \dot{\overline{u}}, \ddot{\overline{u}})$

Solve for $\dot{\overline{\Psi}}$.

Plug $\dot{\overline{\Psi}}$ into Equation (4) and solve for $\overline{M}_F$.

Technical Report ME–(NGR-01-003-0(

N74 31304

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Prepared by

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Six Months Report to

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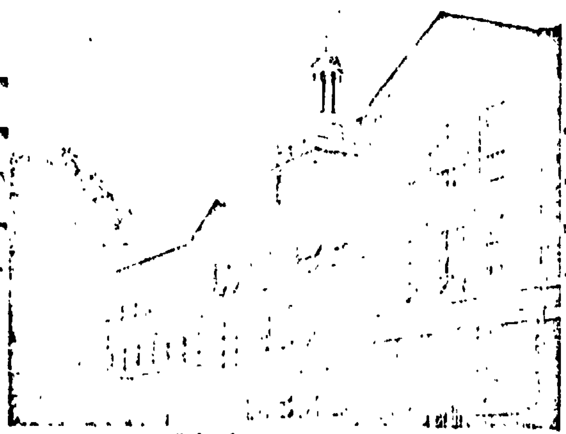
National Aeronautics and Space Administration

On

NASA Grant NGR-01-003-008-S-2
(May 1 — November 1, 1968)



School of Engineering

January 1969

ENGINEERING EXPERIMENT STATION

AUBURN UNIVERSITY

Auburn, Alabama

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Technical Report ME-(NGR-01-003-008)-5

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ABSTRACT

Some methods of approaching a solution to the Hamilton-Jacobi equation are outlined and examples are given to illustrate particular methods. These methods may be used for cases where the Hamilton-Jacobi equation is not separable and have been particularly useful in solving the rigid body motion of an earth satellite subjected to gravity torques. It is felt that these general methods may also have applications in studying the motion of satellites with aerodynamic torque and in studying space vehicle motion where thrusting is involved.

TABLE OF CONTENTS

INTRODUCTION

During the six months included in this reporting period (May to November 1968), work has continued on an investigation of the analytical foundation of the Hamilton-Jacobi theory and its application to space flight problems.

In studying the literature, many questions arose. An attempt was made to formulate these questions and then find satisfactory answers to them. The first work during this reporting period was directed toward comparing the different methods available for solving the Hamilton-Jacobi partial differential equation. Five different methods for obtaining a generator S were studied:

1. $S = \int - L dt$, Where L Is the Lagrangian
2. Liouville's Theorem for Obtaining S
3. Jacobi's Method of Integration of Partial Differential Equations
4. Separation of Variables
5. Method of Characteristics

The following questions arose during the discussions of the different methods available for solving the Hamilton-Jacobi equation.

1. Can a solution be obtained by Jacobi's method; i.e., by obtaining half the integrals for p_i and then building S from

   $$dS = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$$

   that cannot be obtained by quadratures

   $$\frac{dt}{1} = \frac{dq_1}{\partial H/\partial p_1} = \cdots = \frac{dq_n}{\partial H/\partial p_n}$$
   $$= -\frac{dp_1}{\partial H/\partial q_1} = \cdots = -\frac{dp_n}{\partial H/\partial q_n}$$

   which result from Hamilton's equations?

2. Same question as one except separation of variables versus Jacobi's method?

3. Same question except quadratures versus separation of variables?

4. All three of the above questions with the Hamiltonian given as an explicit function of time?

In discussing Jacobi's method, the following question and answer was developed. Given one complete integral, is there *any* technique for constructing another distinct complete integral? Yes, an infinite number of other distinct complete integrals can be constructed. Given a complete integral containing two arbitrary constants α and β, another complete integral can be constructed by replacing α and β as arbitrary functions of two other arbitrary constants A and B. Thus, the integration constants associated with each distinct complete integral of the Hamilton-Jacobi equation can be functionally related. There is a question as to whether any of these constants are canonical. Also, if the same problem were solved by integrating Hamilton's equations by quadratures, then there would be other constants of integration. One would want to know how these constants are related to those obtained from the Hamilton-Jacobi equation. Also, are they canonical?

Some of these questions are answered in subsequent sections of this report. One paper (see Appendix) has grown out of this work and has been submitted to the *American Journal of Physics* for possible publication.

DEFINITIONS OF ANGLES

The angles θ, ϕ-ϕ', θ', ϕ^*, θ^*, and ψ-ψ^* are defined by their geometry in the spherical triangle (see Figure 1):

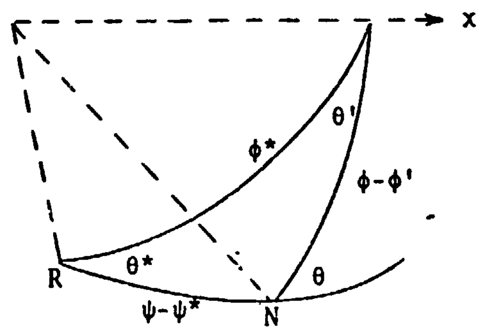


Figure 1

$$\cos\theta = \frac{\alpha_2\alpha_3}{h^2} - \frac{\sqrt{(h^2-\alpha_2^2)(h^2-\alpha_3^2)}}{h^2}\cos\frac{h}{A}(t-\beta_1)$$

$$= \cos\theta'\cos\theta^* - \sin\theta'\sin\theta^*\cos\phi^*$$

$$\cos(\phi-\phi') = \frac{\alpha_3 - \alpha_2\cos\theta}{\sqrt{h^2-\alpha_2^2}\sin\theta} = \frac{\cos\theta^* - \cos\theta'\cos\theta}{\sin\theta'\sin\theta}$$

$$\sin(\phi-\phi') = \frac{\sin\phi^*\sin\theta^*}{\sin\theta}$$

$$\cos\theta' = \frac{\alpha_2}{h}$$

$$\sin\theta' = \frac{\sqrt{h^2-\alpha_2^2}}{h}$$

$$\cos \phi^* = \frac{\alpha_2\alpha_3 - h^2 \cos \theta}{\sqrt{(h^2 - \alpha_2^2)(h^2 - \alpha_3^2)}} = \cos \frac{h}{A}(t - \beta_1)$$

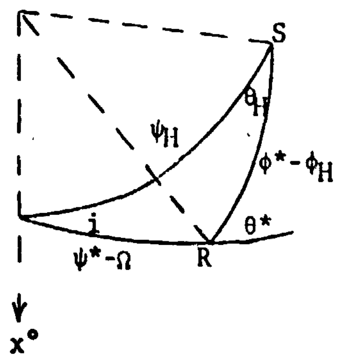
$$\cos \theta^* = \frac{\alpha_3}{h}$$

$$\sin \theta^* = \frac{\sqrt{h^2 - \alpha_3^2}}{h}$$

$$\cos(\psi-\psi^*) = \frac{\alpha_2 - \alpha_3 \cos \theta}{\sqrt{h^2 - \alpha_3^2} \sin \theta} = \frac{\cos \theta' - \cos \theta^* \cos \theta}{\sin \theta^* \sin \theta}$$

$$\sin(\psi-\psi^*) = \frac{\sin \phi^* \sin \theta}{\sin \theta}$$

The angles θ_H, ϕ_H, and ψ_H are defined by their geometry.



$$\cos \theta_H = \cos i \cos \theta^* + \sin i \sin \theta^* \cos(\psi^* - \Omega)$$

$$\cot(\phi^*-\phi_H) = \frac{\cos i \sin \theta^* - \sin i \cos \theta^* \cos(\psi^* - \Omega)}{\sin i \sin(\psi^* - \Omega)}$$

$$\cot \psi_H = \frac{\cos i \sin \theta^* \cos(\psi^* - \Omega) - \sin i \cos \theta^*}{\sin \theta^* \sin(\psi^* - \Omega)}$$

Functional Relations

$$h = h(\alpha_1,\alpha_2)$$

$$\psi = \psi(\alpha_1,\alpha_2,\alpha_3,\beta_3,\theta) = \psi(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_3,t)$$

$$\psi^* = \psi^*(\beta_3)$$

$$\psi_H = \psi_H(\psi^*,\theta^*;i,\Omega) = \psi_H(\alpha_1,\alpha_2,\alpha_3,\beta_3;i,\Omega)$$

$$\theta = \theta(\alpha_1,\alpha_2,\alpha_3,\beta_1,t)$$

$$\theta' = \theta'(\alpha_1,\alpha_2)$$

$$\theta^* = \theta^*(\alpha_1,\alpha_2,\alpha_3)$$

$$\theta_H = \theta_H(\psi^*,\theta^*;i,\Omega) = \theta_H(\alpha_1,\alpha_2,\alpha_3,\beta_3;i,\Omega)$$

$$\phi = \phi(\alpha_1,\alpha_2,\alpha_3,\beta_2,\theta) = \phi(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,t)$$

$$\phi' = \phi'(\alpha_1,\alpha_2,\alpha_3,\beta_2,\theta) = \phi'(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,t)$$

$$\phi^* = \phi^*(\alpha_1,\alpha_2,\alpha_3,\theta) = \phi^*(\alpha_1,\alpha_2,\alpha_3,\beta_1,t)$$

$$\phi_H = \phi_H(\psi^*,\theta^*,\phi^*;i,\Omega) = \phi_H(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_3,t;i,\Omega)$$

Canonical Transformations

The motion of the body is such that θ oscillates between θ_0 and θ_1, where $\theta_0 \equiv \theta' + \theta^*$ and $\theta_1 \equiv |\theta' - \theta^*|$. Let t_0 denote an instant at which $\theta = \theta_0$. Let θ_{01} refer to θ at the instant t_0, θ_{11} refer to θ at the first instant after t_0 that $\theta = \theta_1$, θ_{02} refer to θ at the first instant after θ_{11} that $\theta = \theta_0$, θ_{12} refer to θ at the first instant after θ_{02} that $\theta = \theta_1$, and so forth.

A generator, S, of a canonical transformation is given by

$$S = -\alpha_1 t + \alpha_2\phi + \alpha_3\psi - \int_{\theta_{01}}^{\theta} Q(\theta)d\theta,$$

where

$$Q(\theta) \equiv \begin{cases} \sqrt{f(\theta)}, & \theta_{0n} \rightarrow \theta \rightarrow \theta_{1n}, \\ -\sqrt{f(\theta)}, & \theta_{1n} \rightarrow \theta \rightarrow \theta_{0(n+1)}, \end{cases}$$

and

$$f(\theta) \equiv 2A\alpha_1 - \frac{A}{C}\alpha_0^2 - \csc^2\theta\left(\alpha_3 - \alpha_2 \cos\theta\right)^2.$$

The symbol $\theta_{0n} \rightarrow \theta \rightarrow \theta_{1n}$ means that θ has passed through θ_{0n} and is going toward θ_{1n}.

In terms of the variables ϕ', ϕ^*, and ψ^*,

$$S' = -\alpha_1 t + \alpha_2\phi' + \alpha_3\psi^* + h\phi^*$$

$$Q(\theta) = h \sin\theta^* \sin(\psi - \psi^*)$$

When $\theta = \theta_0$, it can be shown that ϕ^*, $\phi-\phi'$, and $\psi-\psi^*$ must be multiples of 2π. To avoid ambiguity, $\phi^* = \phi - \phi' = \psi - \psi^* = 0$ is defined when $\theta = \theta_{01}$.

$$p_\psi = \alpha_3$$

$$p_\phi = \alpha_2$$

$$p_\theta = -Q(\theta) = -h \sin \theta^* \sin(\psi - \psi^*)$$

$$\beta_1 = t - \frac{A}{h} \phi^*$$

$$\beta_2 = -\phi' - \frac{\alpha_2}{h}\left(\frac{C - A}{C}\right)\phi^*$$

$$\beta_3 = -\psi^*$$

$$\phi^* = \frac{h}{A}(t - \beta_1)$$

$$\phi' = -\beta_2 - \frac{\alpha_2}{A}\left(\frac{C - A}{C}\right)\left(t - \beta_1\right)$$

$$\psi^* = -\beta_3$$

Alternatively,

$$\beta_1 = t + A\int_{\theta_{01}}^{\theta} \frac{d\theta}{Q(\theta)}$$

$$= t - \frac{A}{h} \cos^{-1} \frac{\alpha_2\alpha_3 - h^2 \cos \theta}{\sqrt{(h^2 - \alpha_2{}^2)(h^2 - \alpha_3{}^2)}}\Bigg|_{\theta_{01}}^{\theta}$$

$$= t - \frac{A}{h} \cos^{-1} \frac{\alpha_2\alpha_3 - h^2 \cos \theta}{\sqrt{(h \quad - \alpha_2{}^2)(h \quad - \alpha_3{}^2)}}$$

$$\beta_2 = -\phi + \int_{\theta_{01}}^{\theta} \left\{\frac{(\alpha_3 - \alpha_2 \cos \theta)\cos \theta}{\sin^2\theta} - \frac{A\alpha_2}{C}\right\} \frac{d\theta}{Q(\theta)} =$$

$$= -\phi + \cos^{-1} \frac{\alpha_3 - \alpha_2 \cos\theta}{\sqrt{h^2 - \alpha_3^2}\sin\theta}\Bigg|_{\theta_{01}}^{\theta}$$

$$- \frac{\alpha_2}{h}\left(\frac{C - A}{C}\right)\cos^{-1} \frac{\alpha_2\alpha_3 - h^2\cos\theta}{\sqrt{(h^2 - \alpha_2^2)(h^2 - \alpha_3^2)}}\Bigg|_{\theta_{01}}^{\theta}$$

$$= -\phi + \cos^{-} \frac{\alpha_3 - \alpha_2 \cos\theta}{\sqrt{h^2 - \alpha_2^2}\sin\theta}$$

$$- \frac{\alpha_2}{h}\left(\frac{C - A}{C}\right)\cos^{-1} \frac{\alpha_2\alpha_3 - h^2\cos\theta}{\sqrt{(h^2 - \alpha_2^2)(h^2 - \alpha_3^2)}}$$

$$\beta_3 = -\psi - \int_{\theta_{01}}^{\theta}\left(\frac{\alpha_3 - \alpha_2\cos\theta}{\sin^2\theta}\right)\frac{d\theta}{Q(\theta)}$$

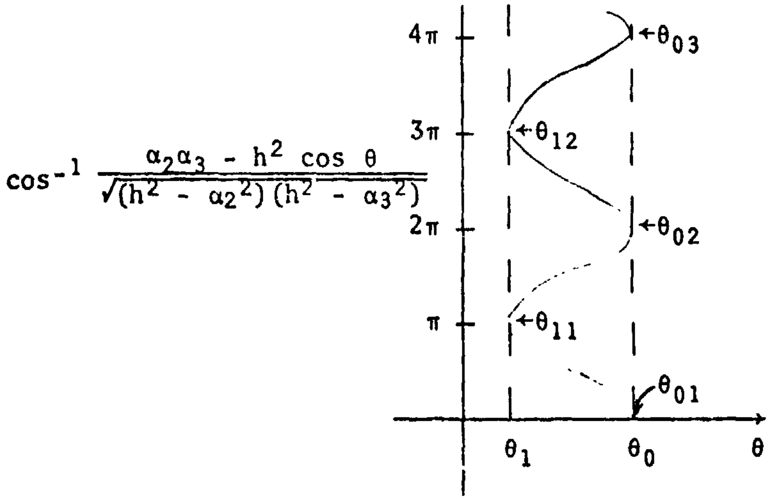
$$= -\psi + \cos^{-1} \frac{\alpha_2 - \alpha_3\cos\theta}{\sqrt{(h^2 - \alpha_3^2)}\sin\theta}\Bigg|_{\theta_{01}}^{\theta}$$

$$= -\psi + \cos^{-1} \frac{\alpha_2 - \alpha_3\cos\theta}{\sqrt{(h^2 - \alpha_3^2)}\sin\theta}$$

The multi-valued $\cos^{-1}$ functions appearing above are to be interpreted as follows:

$$\cos^{-1}g(\theta) \equiv \begin{cases} 2(n-1)\pi + \mathrm{Cos}^{-1}g(\theta), & \theta_{0n} \to \theta \to \theta_{1n} \\ 2n\pi \quad - \mathrm{Cos}^{-1}g(\theta), & \theta_{1n} \to \theta \to \theta_{0(n+1)} \end{cases}$$

where Cos^{-1} denotes the principal value (that is, the value between 0 and π) of the $\cos^{-1}$ function.



Miscellaneous

$$h^2 = 2A\alpha_1 + \left(\frac{C - A}{C}\right)\alpha_2^2$$

$$\cos \theta = \frac{\alpha_2\alpha_3 - \sqrt{(h^2 - \alpha_2^2)(h^2 - \alpha_3^2)} \cos \frac{h}{A}(t - \beta_1)}{h^2}$$

$$= \cos \theta' \cos \theta^* - \sin \theta' \sin \theta^* \cos \phi^*$$

$$\phi = -\beta_2 + \cos^{-1} \frac{\alpha_3 - \alpha_2 \cos \theta}{\sin \theta \sqrt{h^2 - \alpha_2^2}}$$

$$- \frac{\alpha_2}{h}\left(\frac{C - A}{C}\right)\cos^{-1} \frac{\alpha_2\alpha_3 - h^2 \cos \theta}{\sqrt{(h^2 - \alpha_2^2)(h^2 - \alpha_3^2)}}$$

$$\psi = -\beta_3 + \cos^{-1} \frac{\alpha_2 - \alpha_3 \cos \theta}{\sin \theta \sqrt{h^2 - \alpha_3^2}}$$

A NOTE ON DISTINCT COMPLETE INTEGRALS*

Problem: Show that the differential equation

$$4XZQ^2 + P = 0, \quad P = \partial Z/\partial X, \quad Q = \partial Z/\partial Y$$

possesses the distinct complete integrals

$$Z^2 = \alpha Y - \alpha^2 X^2 + \beta$$

and

$$Z^2(4X^2 + a) = (Y + b)^2$$

Find a functional relation between α, β, a, and b; hence, find the second solution as a particular case of the general integral obtained from the first.

Solution: First, transform to new variables according to the scheme

$$X \rightarrow x_1, \qquad Y \rightarrow x_2, \qquad Z \rightarrow x_3$$

$$P = -\frac{p_1}{p_3}, \qquad Q = -\frac{p_2}{p_3}, \qquad p_3 = \frac{\partial u}{\partial x_3}$$

See Frederic H. Miller, *Partial Differential Equations* (New York: John Wiley & Sons, 1949), Chapter V, for details on transformation. The differential equation

$$F(X,Z,Q,P) = 4XZQ^2 + P = 0 \qquad (1)$$

now becomes

$$F(x_1,x_3,p_1,p_2,p_3) = 4x_1x_3p_2^2 - p_1p_3 = 0 \qquad (1')$$

Jacobi's method will be used to solve Eq (1'). First, write

$$\frac{dp_1}{\frac{\partial F}{\partial x_1}} = \frac{dp_2}{\frac{\partial F}{\partial x_2}} = \frac{dp_3}{\frac{\partial F}{\partial x_3}} = -\frac{dx_1}{\frac{\partial F}{\partial p_1}} = -\frac{dx_2}{\frac{\partial F}{\partial p_2}} = -\frac{dx_3}{\frac{\partial F}{\partial p_3}} \qquad (2)$$

Explicitly,

$$\frac{dp_1}{4x_3p_2^2} = \frac{dp_2}{0} = \frac{dp_3}{4x_1p_2^2} = \frac{dx_1}{\ldots}$$

$$= -\frac{dx_2}{8x_1x_3p_2} = \frac{dx_3}{p_1} \qquad (2')$$

Using the second ratio,

$$F_1 = p_2 = a_1 = \text{constant} \qquad (3)$$

Using the first and sixth ratios,

$$p_1dp_1 = 4a_1^2x_3dx_3$$

and

$$F_2 = p_1^2 - 4a_1^2x_3^2 = a_2 = \text{constant} \qquad (4)$$

Using the third and fourth ratios,

$$p_3dp_3 = 4a_1^2x_1dx_1$$

and

$$F_2^* = p_3^2 - 4a_1^2x_1^2 = a_2^* = \text{constant} \qquad (5)$$

$(F_1,F_2) = 0$; also, $(F_1,F_2^*) = 0$, as is readily verified. Using $F_1 = p_2 = a_1$, and $F_2 = p_1^2 - 4a_1^2x_3^2 = a_2$, take

$$p_1 = \sqrt{a_2 + 4a_1^2x_3^2}$$

Substitute into Eq (1') and solve for p_3

$$4x_1x_3a_1^2 - \sqrt{a_2 + 4a_1^2x_3^2}\, p_3 = 0$$

$$p_3 = \frac{4x_1x_3a_1^2}{\sqrt{a_2 + 4a_1^2x_3^2}}$$

Use F_1 and F_2 in conjunction with F to obtain a complete integral. One has $(F_1,F_2) = 0$

$$du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$\frac{\partial u}{\partial x_1} = p_1$$

$$\frac{\partial u}{\partial x_2} = p_2$$

$$\frac{\partial u}{\partial x_3} = p_3$$

$$\frac{\partial u}{\partial x_2} = a_1 \rightarrow u = a_1 x_2 + f(x_1, x_3)$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial f}{\partial x_1} = p_1 \rightarrow f = p_1 x_1 + g(x_3)$$

$$\frac{\partial u}{\partial x_3} = \frac{\partial f}{\partial x_3} = x_1 \frac{\partial p_1}{\partial x_3} + g'(x_3)$$

$$= p_3 \rightarrow g'(x_3) = p_3 - x_1 \frac{\partial p_1}{\partial x_3}$$

But,

$$p_3 - x_1 \frac{\partial p_1}{\partial x_3} = p_3 - x_1 (\tfrac{1}{2})(a_2 + 4a_1^2 x_3^2)^{-1/2}(8a_1^2 x_3)$$

$$= p_3 - p_3 = 0 \rightarrow g(x_3) = a_3 = \text{constant}$$

$$u = a_1 x_2 + x_1 p_1 + a_3 = 0$$

$$p_1 = \sqrt{a_2 + 4a_1^2 x_3^2}$$

$$x_1 p_1 = -a_3 - a_1 x_2$$

$$p_1^2 = a_2 + 4a_1^2 x_3^2 = \frac{(a_3 + a_1 x_2)^2}{x_1^2}$$

$$4a_1^2 Z^2 + a_2 = \frac{(a_3 + a_1 Y)^2}{X^2} = \frac{a_1^2}{X^2}\left(\frac{a_3}{a_1} + Y\right)^2$$

$$4X^2 Z^2 + \frac{a_2}{a_1^2} X^2 = \left(\frac{a_3}{a_1} + Y\right)^2$$

Set A equal to a_2/a_1^2 and B equal to a_3/a_1. Then,

$$X^2(4Z^2 + A) = (Y + B)^2 \tag{6}$$

and Eq (6) is a complete integral of Eq (1).

If Eqs (3) and (5) are used in conjunction with Eq (1'), observing that Eq (1') is unchanged if $p_1 \rightarrow p_3$ and $x_1 \rightarrow x_3$ are interchanged, one has

$$u = a_1 x_2 + x_3 p_3 + a_3^* = 0$$

$$p_3^2 = \frac{(a_3 + a_1 x_2)^2}{x_3^2}$$

$$x_3^2\left(4a_1^2 x_1^2 + a_2^*\right) = \left(a_3 + a_1 x_2\right)^2 = a_1^2\left(\frac{a_3}{a_1} + x_2\right)^2$$

$$Z^2\left(4X^2 + \frac{a_2^*}{a_1^2}\right) = \left(\frac{a_3}{a_1} + Y\right)^2$$

Set a equal to a_2^*/a_1^2 and b equal to a_3/a_1

$$Z^2(4X^2 + a) = (Y + b)^2 \tag{7}$$

' Eq (7) is a complete integral of Eq (1).

Still another distinct complete integral of Eq (1) can be obtained by separating the variables in Eq (1'). Since $4x_1x_3p_2^2 - p_1p_3 = 0$ is free of x_2, $p_2 = \partial u/\partial x_2 = a_1$, a constant, and

$$4x_1x_3a_1^2 - p_1p_3 = 0 = 4x_1x_3a_1^2 - \frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_3} \tag{8}$$

Assume a solution of Eq (8) of the form

$$\begin{aligned} u &= f_1(x_1) + f_2(x_2) + f_3(x_3) \\ &= a_1x_2 + f_1(x_1) + f_3(x_3) \end{aligned} \tag{9}$$

Substitute Eq (9) into Eq (8) to obtain

$$\frac{4x_1a_1^2}{df_1/dx_1} = \frac{1}{x_3}\frac{df_3}{dx_3} = C_1 \tag{10}$$

$$\rightarrow \quad f_1 = \frac{2a_1^2x_1^2}{C_1} + C_2$$

$$f_3 = \frac{C_1}{2}x_3^2 + C_3$$

$$u = \frac{2a_1^2}{C_1}x_1^2 + \frac{C_1}{2}x_3^2 + C_2 + C_3 + a_1x_2 = 0$$

where $x_3 = Z$, $x_1 = X$, and $x_2 = Y$.

$$\frac{C_1}{2}Z^2 = -(C_2 + C_3) - a_1Y - \frac{2a_1^2}{C_1}X^2$$

$$Z^2 = -\frac{2}{C_1}(C_2 + C_3) - \frac{2a_1}{C_1}Y - \frac{4a_1^2}{C_1^2}X^2$$

Set α equal to $-2a_1/C_1$ and β equal to $-2(C_2 + C_3)/C_1$

$$Z^2 = \alpha Y - \alpha^2X^2 + \beta \tag{11}$$

and Eq (11) is a complete integral of Eq (1).

Consider the distinct integrals Eqs (7) and (11), and renumber them I and II

$$Z^2(4X^2 + a) = (Y + b)^2 \tag{I}$$

$$Z^2 = \alpha Y - \alpha^2X^2 + \beta \tag{II}$$

$$\frac{\partial I}{\partial X}; \quad ZP(4X^2 + a) + 4XZ^2 = 0 \tag{III}$$

$$\frac{\partial I}{\partial Y}; \quad ZQ(4X^2 + a) - (Y + b) = 0 \tag{IV}$$

$$\frac{\partial II}{\partial X}; \quad ZP + \alpha^2X = 0 \tag{V}$$

$$\frac{\partial II}{\partial Y}; \quad 2ZQ - \alpha = 0 \tag{VI}$$

X, Y, Z, P, and Q must be eliminated from the six equations above. From Eq (III) and (V), eliminate P:

$$4Z^2 = \alpha^2(4X^2 + a)$$

Use Eq (II):

$$4\alpha Y - 4\alpha^2 X^2 + 4\beta = \alpha^2(4X^2 + a)$$

Solve for Y to obtain

$$Y = 2\alpha X^2 + \frac{a\alpha}{4} - \frac{\beta}{\alpha} \qquad \text{(VII)}$$

From Eqs (IV) and (VI), eliminate ZQ:

$$Y + b = \frac{\alpha}{2}(4X^2 + a)$$

Solve for Y to obtain

$$Y = 2\alpha X^2 + \frac{a\alpha}{2} - b \qquad \text{(VIII)}$$

Equating Eqs (VII) and (VIII), one obtains

$$\beta = \alpha b - \frac{a\alpha^2}{4} \qquad \text{(IX)}$$

Substitute Eq (IX) into Eq (II) to obtain

$$Z^2 = \alpha Y - \alpha^2 X^2 + \alpha b - \frac{a\alpha^2}{4} \qquad \text{(X)}$$

$$Y - 2\alpha X^2 + b - \frac{2a\alpha}{4} = 0$$

$$\alpha\left(\frac{a}{2} + 2X^2\right) = b + Y$$

$$\alpha = \frac{2(b + Y)}{a + 4X^2} \qquad \text{(XI)}$$

Substitute Eq (XI) into Eq (X):

$$Z^2 = \alpha(b + Y) - \alpha^2\left(\frac{a}{4} + X^2\right)$$

$$= \frac{2(b + Y)^2}{a + 4X^2} - \frac{4(b + Y)^2}{(a + 4X^2)^2} - \left(\frac{a + 4X^2}{4}\right)$$

$$= \frac{(b + Y)^2}{a + 4X^2}$$

$$Z^2(4X^2 + a) = (b + Y)^2$$

---

*A Further Note:

$$\left.\begin{aligned} F &= Z^2 - \alpha Y + \alpha^2 X^2 - \beta = 0 \\ G &= Z^2(4X^2 + a) - (Y + b)^2 = 0 \end{aligned}\right\} \quad \left\{\begin{aligned} &\beta = f(\alpha), \quad \frac{\partial F}{\partial \alpha} = 0 \\ &\rightarrow \text{General Integral} \end{aligned}\right\}$$

F and G are two distinct complete integrals. Let

$$\beta_1 = \alpha B - \frac{A\alpha^2}{4}$$

Note that β_1 is one possible functional form of $\beta = f(\alpha)$. For all possible choices A and B in β_1, only a subset of the elements for the arbitrary choice $\beta = f(\alpha)$ is obtained. Better said: Let H be the set of functions of α

$$\beta = B\alpha - \frac{A\alpha^2}{4}$$

for fixed A and B. H is a proper subset of the set Q of all possible functions $\beta = f(\alpha)$. By inserting $\beta = B\alpha - A\alpha^2/4$ into F and forming $\partial F/\partial\alpha$ for fixed A and B, α can be eliminated, and the two-parameter family of surfaces G can be obtained. Thus, the surfaces G are part of the totality of envelopes which go to make up the general integral.

A NOTE ON OBTAINING A COMPLETE INTEGRAL

OF THE HAMILTON-JACOBI EQUATION

On page 324 of *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: The University Press, 1937), E. T. Whittaker states--without proof--the following lemma.

Lemma: If dW is the perfect differential of some function $W(q_i,\alpha_i,t)$, then the first Pfaff's system of the differential form

$$dW - \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i$$

is

$$d\left(\frac{\partial W}{\partial \alpha_i}\right) = 0, \qquad d\alpha_i = 0 \qquad (i=1,2,\ldots,n)$$

Let $W(q_i,\alpha_i,t)$ be in C_1 but otherwise arbitrary, and consider the differential form

$$\sum_{i=1}^{n} p_i dq_i - H(q_i,p_i,t)\,dt \tag{1}$$

and the transformation

$$p_i = f_i(q_i,\alpha_i,t), \qquad q_i = q_n \qquad (i=1,2,\ldots,n) \tag{2}$$

The following theorem is established.

Theorem 1: If the transformation Eq (2) transforms Eq (1) into the differential form

$$dW - \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i$$

where

$$dW = \sum_{i=1}^{n} f_i dq_i - H_1(q_i, f_i, t)dt + \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i$$

$$= \sum_{i=1}^{n} \frac{\partial W}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i + \frac{\partial W}{\partial t} dt$$

is a perfect differential of some function $W(q_i, \alpha_i, t)$ of the variables (q_i, α_i, t), which contains n independent constants α_i, then W is a complete integral of the Hamilton-Jacobi equation.

Proof: By equating coefficients, the necessary conditions can be obtained

$$f_i = \frac{\partial W}{\partial q_i}$$

$$H_1(q_i, f_i, t) + \frac{\partial W}{\partial t}(q_i, \alpha_i, t) = 0$$

Thus,

$$H_1\left(q_i, \frac{\partial W}{\partial q_i}, t\right) + \frac{\partial W}{\partial t}\left(q_i, \alpha_i, t\right) = 0$$

which establishes the theorem.

Note: This result agrees with a statement in Pars, p. 450, if it is assumed that a typographical error has been made there and that he means equation 16.5-4 rather than 16.5-6. This would be consistent with his earlier reference to 16.5-4 as "the modified partial differential equation."

Example--Central Orbit, Polar Coordinates:

(a) $H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2} p_\theta^2\right) + V(r) = h$

(b) $p_\theta = \alpha = \text{constant}$

Solve (a) for p_r. One has*

*See next page.

$$p_z^{2} = 2h - 2V - \frac{\alpha^2}{r^2} \qquad\qquad r = r$$

$$p_r = \pm\sqrt{2h - 2V - \frac{\alpha^2}{r^2}} = \pm\sqrt{f(r)} \qquad\qquad \theta = \theta$$

where

$$f(r) = 2h - 2V - \frac{\alpha^2}{r^2}.$$

$$dW = p_r dr + p_\theta d\theta - hdt$$

$$\text{(c)} \quad W = -ht + \alpha\theta \pm \int_{r_1}^{r} \sqrt{f(r)}\, dr$$

Either $\pm$ yields a complete integral of the Hamilton-Jacobi equation. The Hamilton-Jacobi equation is:

$$\text{(d)} \quad \frac{\partial W}{\partial t} + \frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + V = 0$$

$$\frac{\partial W}{\partial \theta} = \alpha$$

$$\frac{\partial W}{\partial t} = -h$$

$$\frac{\partial W}{\partial r} = \pm\sqrt{f(r)}$$

Substituting into (d), one finds

$$-h + \frac{1}{2} f(r) + \frac{1}{2r^2}\alpha^2 + V = -h + \frac{1}{2}\left(2h - 2V - \frac{\alpha^2}{r^2}\right) + \frac{1}{2r^2}\alpha^2 + V = 0$$

*A theorem on page 323 of Whittaker's *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* assures the reader that the transformation

$$p_\theta = \alpha \qquad\qquad r = r$$

$$p_r = \pm\sqrt{f(r)} \qquad\qquad \theta = \theta$$

transforms

$$\sum_{i=1}^{n} p_i dq_i - H(q_i, p_i, t)dt$$

into the differential form

$$dW - \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i$$

It is a simple matter to show that the functions

$$\begin{array}{ll} \text{(a)} & \left\{ \begin{array}{lll} \frac{1}{2} p_r^2 + \frac{1}{r^2} p_\theta^2 + V(r) & = & \phi_1 \\ \\ p_\theta = \alpha & = & \phi_2 \end{array} \right. \\ \text{(b)} & \end{array}$$

are in involution; i.e., $[\phi_1, \phi_2] = 0$. Poisson brackets are zero, so that the theorem just cited may be applied.

It may be that there are n distinct integrals (in involution)

$$\phi_i(q_i, p_i, t) = \alpha_i \qquad (i=1,2,\ldots,n) \qquad (3)$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are arbitrary constants, for the dynamical system

$$\left.\begin{array}{lll} \dot{q}_i = \frac{\partial H}{\partial p_i}(q_i, p_i, t) & & (i=1,2,\ldots,n) \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}(q_i, p_i, t) & & (i=1,2,\ldots,n) \end{array}\right\} \qquad (4)$$

It may be that all of these integrals cannot be solved for $(p_1, p_2, \ldots, p_n)$ so that they can be obtained in the form

$$p_i = f_i(q_i, \alpha_i, t) \qquad (i=1,2,\ldots,n) \quad (5)$$

However, it may be possible to solve $\ell (\ell \le n)$ of these integrals for the p's and express them in the form of Eq (5). Suppose further that the remaining can be solved in integrals $(m \le n,\ \ell + m = n)$ for the q's and express them in terms of p_i, α_i, and t. Relabel the coordinates, setting P_j, $(j=1,2,\ldots,\ell)$, in one-to-one correspondence with the ℓp's which have been solved for explicitly, taken in any order, and Q_k, $(k=1,2,\ldots,m)$, in one-to-one correspondence with the mq's which have been solved for explicitly, taken in any order. Thus, the n integrals may be written in the form

$$\left.\begin{aligned} P_j &= F_j(q_i, \alpha_i, t) & (j=1,2,\ldots,\ell) \\ Q_k &= H_k(p_i, \alpha_i, t) & (k=1,2,\ldots,m) \end{aligned}\right\} (6)$$

Suppose now that in Eqs (6) none of the P_j appear in the right-hand sides of the expressions for Q_k and that none of the Q_k appear in the right-hand sides of the expressions for P_j.

Introduce the following canonical transformation of coordinates

$$\left.\begin{aligned} p_j^* &= P_j, \quad (j=1,2,\ldots,\ell) & p_{\ell+k}^* &= -Q_k, \quad (k=1,2,\ldots,m) \\ q_k^* &= P_{\ell+k}, \quad (k=1,2,\ldots,m) & q_{m+j}^* &= Q_{m+j}, \quad (j=1,2,\ldots,\ell) \end{aligned}\right\} (7)$$

Equations (6) may now be written in the form

$$p_i^* = f_i(q_i^*, \alpha_i, t) \qquad (i=1,2,\ldots,n) \quad (8)$$

Since Eq (8) is in the form of Eq (2), Theorem 1 may be applied, in conjunction with Whittaker's theorem, to obtain a complete integral of the Hamilton-Jacobi equation expressed in the *starred* coordinates.

A NOTE ON DISTINCT INTEGRALS FOR A PARTICLE

IN A UNIFORM GRAVITY FIELD IN A PLANE

$$H = \tfrac{1}{2}\left(p_x{}^2 + p_h{}^2\right) + gh$$

$$\dot{x} = \frac{\partial H_o}{\partial p_x} = p_x$$

$$\dot{h} = \frac{\partial H_o}{\partial p_h} = p_h$$

$$H_o = \tfrac{1}{2}\left(p_x{}^2 + p_h{}^2\right)$$

$$\dot{p}_x = 0 \rightarrow p_x = a$$

$$\dot{p}_h = -\frac{\partial H_o}{\partial h} = 0 \rightarrow p_h = b$$

Direct integration of canonical equations:

$$\left.\begin{aligned} \dot{x} &= a, & x &= at + c \\ \dot{h} &= b, & h &= bt + d \end{aligned}\right\} \qquad (1)$$

Unperturbed problem (Hamilton-Jacobi Equation):

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial h}\right)^2\right) = 0$$

Assume

$$S = -\alpha_1 t + S_1(x) + S_2(h)$$

$$\left(\frac{dS_1}{dx}\right)^2 + \left(\frac{dS_2}{dh}\right)^2 = 2\alpha_1$$

$$\left(\frac{dS_1}{dx}\right)^2 = 2\alpha_1 - \left(\frac{dS_2}{dh}\right)^2 = \alpha_2{}^2$$

$$S_1 = \alpha_2 x$$

$$\left(\frac{dS_2}{dh}\right)^2 = 2\alpha_1 - \alpha_2{}^2$$

$$S_2 = \sqrt{2\alpha_1 - \alpha_2{}^2}\, h$$

$$S = -\alpha_1 t + \alpha_2 x + \sqrt{2\alpha_1 - \alpha_2{}^2}\, h \tag{2}$$

$$\left.\begin{aligned} p_x &= \frac{\partial S}{\partial x} = \alpha_2 \\ p_h &= \frac{\partial S}{\partial h} = \sqrt{2\alpha_1 - \alpha_2{}^2}\, h \\ \beta_1 &= -\frac{\partial S}{\partial \alpha_1} = t - \frac{h}{\sqrt{2\alpha_1 - \alpha_2{}^2}} \\ \beta_2 &= -\frac{\partial S}{\partial \alpha_2} = \frac{h\alpha_2}{\sqrt{2\alpha_1 - \alpha_2{}^2}} - x \end{aligned}\right\} \tag{3}$$

A comparison of solution (3) with Eq (1) yields

$$\left.\begin{aligned} \alpha_2 &= a, & \beta_1 &= t - \frac{h}{b} = -\frac{d}{b} \\ \alpha_1 &= \frac{a^2 + b^2}{2}, & \beta_2 &= -c + \frac{ad}{b} \end{aligned}\right\} \tag{4}$$

Substitute into Eq (2) to get

$$S^* = -\frac{(a^2 + b^2)}{2} t + ax + bh \tag{5}$$

$$\left.\begin{aligned} p_x &= \frac{\partial S^*}{\partial x} = a, & \beta_1{}^* &= -\frac{\partial S^*}{\partial a} = at - x \\ p_h &= \frac{\partial S^*}{\partial h} = b, & \beta_2{}^* &= -\frac{\partial S^*}{\partial b} = bt - h \end{aligned}\right\} \tag{6}$$

It appears that $\beta_1{}^* = -c$ and $\beta_2{}^* = -d$.

Perturbation:

$$\left.\begin{aligned} \dot{\alpha}_1 &= \frac{\partial H_1}{\partial \beta_1}, & \dot{\beta}_1 &= -\frac{\partial H_1}{\partial \alpha_1} \\ \dot{\alpha}_2 &= \frac{\partial H_1}{\partial \beta_2}, & \dot{\beta}_2 &= -\frac{\partial H_1}{\partial \alpha_2} \end{aligned}\right\} \quad (7)$$

where, since $h = \sqrt{2\alpha_1 - \alpha_2^2}\,(t - \beta_1)$, $H_1 = H_1(\alpha_1, \alpha_2, \beta_1, t)$.

$$\left.\begin{aligned} \dot{a} &= \frac{\partial H_1}{\partial \beta_1^*}, & \dot{\beta}_1^* &= -\frac{\partial H_1}{\partial a} \\ \dot{b} &= \frac{\partial H_1}{\partial \beta_2^*}, & \dot{\beta}_2^* &= -\frac{\partial H_1}{\partial b} \end{aligned}\right\} \quad (8)$$

where, since $h = bt - \beta_2^*$, $H_1 = H_1(b, \beta_2^*, t)$.

Variation of Parameters:

Assume

$$\begin{aligned} x &= at + c, & p_x &= a \\ h &= bt + d, & p_h &= b \end{aligned}$$

is a transformation of variables from the canonical set of equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p}_x &= -\frac{\partial H}{\partial x} \\ \dot{h} &= \frac{\partial H}{\partial p_h}, & \dot{p}_h &= -\frac{\partial H}{\partial h} \end{aligned}$$

to new coordinates a, b, c, and d. Thus,

$$\left.\begin{aligned} \dot{x} &= a + \dot{a}t + \dot{c} \\ \dot{h} &= b + \dot{b}t + \dot{d} \\ \dot{p}_x &= \dot{a} \\ \dot{p}_h &= \dot{b} \end{aligned}\right\} \left.\begin{aligned} &(1) & \dot{c} + \dot{a}t &= 0 \\ &(2) & \dot{d} + \dot{b}t &= 0 \\ &(3) & \dot{a} &= 0 \\ &(4) & \dot{b} &= g \end{aligned}\right\} \begin{aligned} c &= c_1 \\ a &= a_1 \\ b &= -gt + b_1 \\ d &= \frac{gt^2}{2} + d_1 \end{aligned}$$

$$x = a_1 t + c_1$$

$$h = -\frac{gt^2}{2} + b_1 t + d_1$$

Return to Eq (8) $\quad H_1 = g\left(bt - \beta_2^*\right)$

$$\dot{a} = \frac{\partial H_1}{\partial \beta_1^*} = 0, \qquad \dot{\beta}_1^* = -\frac{\partial H_1}{\partial a} = 0$$

$$\dot{b} = \frac{\partial H_1}{\partial \beta_2^*} = -g, \qquad \dot{\beta}_2^* = -\frac{\partial H_1}{\partial b} = -gt$$

$$a = a^*, \qquad \beta_1^* = \beta_1^{*\prime}$$

$$b = -gt + b^*, \qquad \beta_2^* = -\frac{gt^2}{2} + \beta_2^{*\prime}$$

$$x = a't - \beta_1^{*\prime}$$

$$h = -\frac{gt^2}{2} + b_2't - \beta_2^{*\prime}$$

Return to Eq (7) $\quad H_1 = g\sqrt{2\alpha_1 - \alpha_2^2}\,\left(t - \beta_1\right)$

$$\dot{\alpha}_1 = \frac{\partial H_1}{\partial \beta_1} = -g\sqrt{2\alpha_1 - \alpha_2^2}, \qquad \dot{\beta}_1 = -\frac{\partial H_1}{\partial \alpha_1} = \frac{g\left(t - \beta_1\right)}{\sqrt{2\alpha_1 - \alpha_2^2}}$$

$$\dot{\alpha}_2 = 0, \qquad \dot{\beta}_2 = -\frac{\partial H_1}{\partial \alpha_2} = \frac{g\alpha_2\left(t - \beta_2\right)}{\sqrt{2\alpha_1 - \alpha_2^2}}$$

Conclusion: The constants which appear in the solution of Hamilton's equations obtained by quadratures are not in general canonical even though in some problems it appears so.

HAMILTON FUNCTION FOR TRIAXIAL BODY (NO FORCES)

Let

$$f(\phi) \equiv \left(\frac{\sin^2\phi}{2A} + \frac{\cos^2\phi}{2B}\right)$$

$$g(\phi) \equiv \left(\frac{1}{A} - \frac{1}{B}\right) \sin\phi \cos\phi$$

$$H = \frac{f}{\sin^2\theta} p_\psi^2 + \left(\frac{\cos^2\phi}{2A} + \frac{\sin^2\phi}{2B}\right) p_\theta^2$$

$$+ \frac{f}{\sin^2\theta} \cos^2\theta \, p_\phi^2 + \frac{1}{2C} p_\phi^2$$

$$+ \frac{g}{\sin\theta} p_\psi p_\theta - \frac{g}{\sin\theta} p_\theta p_\phi \cos\theta$$

$$- \frac{f}{\sin^2\theta} 2 p_\psi p_\phi \cos\theta$$

$$H = \frac{f}{\sin^2\theta}\left[p_\psi - p_\phi \cos\theta\right]^2 + \frac{1}{2C} p_\phi^2$$

$$+ \frac{g}{\sin\theta} p_\theta \left(p_\psi - p_\phi \cos\theta\right)$$

$$+ \left(\frac{\cos^2\phi}{2A} + \frac{\sin^2\phi}{2B}\right) p_\theta^2$$

If $A = B$, $f(\phi) = 1/2A$ and $g(\phi) = 0$, and one has

$$H = \frac{1}{2A \sin^2\theta} \left(p_\psi - p_\phi \cos\theta\right)^2 + \frac{1}{2C} p_\phi^2 + \frac{1}{2A} p_\theta^2$$

Let

$$q(\phi) = \frac{\cos^2\phi}{2A} + \frac{\sin^2\phi}{2B}$$

and rewrite the general form of H. Note that $q(\phi) = A+B/2AB - f(\phi)$.

$$H = \frac{f(\phi)}{\sin^2\theta}\left[p_\psi - p_\phi \cos\theta\right]^2 + \frac{1}{2C}p_\phi^2$$

$$+ \frac{g(\phi)}{\sin\theta} p_\theta\left(p_\psi - p_\phi \cos\theta\right) + q(\phi)p_\theta^2$$

The Hamilton-Jacobi equation may be written: $\partial S/\partial t + H = 0$, where $H = \alpha_1$, a constant.

$$\frac{\partial S}{\partial t} = -\alpha_1$$

$$\alpha_1 = \frac{f}{\sin^2\theta} z^2 + \frac{1}{2C}\left(\frac{\partial S}{\partial \phi}\right)^2 + \frac{g}{\sin\theta}\frac{\partial S}{\partial \theta} z + q\left(\frac{\partial S}{\partial \theta}\right)^2$$

where

$$z = p_\psi - p_\phi \cos\theta = \frac{\partial S}{\partial \psi} - \frac{\partial S}{\partial \phi}\cos\theta$$

Assume

$$S = S_1(t) + S_2(\phi) + S_3(\psi) + S_4(\theta)$$

Then

$$\frac{\partial S}{\partial t} = \frac{dS_1}{dt} = -\alpha_1$$

and

$$S_1 = -\alpha_1 t$$

$$\alpha_1 = \frac{f}{\sin^2\theta} z^2 + \frac{1}{2C}\left(\frac{dS_2}{d\phi}\right)^2 + \frac{g}{\sin\theta}\left(\frac{dS_4}{d\theta}\right)z + q\left(\frac{dS_4}{d\theta}\right)^2$$

where

$$z = \frac{dS_3}{d\psi} - \frac{dS_2}{d\phi}\cos\theta$$

Using the quadratic formula, one may write:

$$z^2 + \frac{\sin^2\theta}{2Cf}\left(\frac{dS_2}{d\phi}\right)^2 + \frac{\sin\theta\ g}{f}\left(\frac{dS_4}{d\theta}\right)z + \frac{q\sin^2\theta}{f}\left(\frac{dS_4}{d\theta}\right)^2 = \frac{\sin^2\theta\ \alpha_1}{f}$$

$$z^2 + \frac{g}{f} \sin \theta \left(\frac{dS_4}{d\theta}\right) z + \frac{\sin^2\theta}{f}\left[\frac{1}{2C}\left(\frac{dS_2}{d\phi}\right)^2 + q\left(\frac{dS_4}{d\theta}\right)^2 - \alpha_1\right] = 0$$

$$z = -\frac{g}{2f} \sin \theta \left(\frac{dS_4}{d\theta}\right) \pm \frac{1}{2}\left(\frac{g^2}{f^2} \sin^2\theta \left(\frac{dS_4}{d\theta}\right)^2 - \frac{4 \sin^2\theta}{f}\left[\frac{1}{2C}\left(\frac{dS_2}{d\phi}\right)^2 + q\left(\frac{dS_4}{d\theta}\right)^2 - \alpha_1\right]\right)^{1/2}$$

$$\frac{dS_3}{d\psi} = \frac{dS_2}{d\phi} \cos \theta + \frac{\sin \theta}{2}\left[-\frac{g}{f}\left(\frac{dS_4}{d\theta}\right) + \sqrt{\frac{g^2}{f^2}\left(\frac{dS_4}{d\theta}\right)^2 + \frac{4}{f}\left[\alpha_1 - \frac{1}{2C}\left(\frac{dS_2}{d\phi}\right)^2 + q\left(\frac{dS_4}{d\theta}\right)^2\right]}\right]$$

$$= \alpha_3$$

$$S_3 = \alpha_3\psi$$

$$\frac{g^2}{f^2}\left(\frac{dS_4}{d\theta}\right)^2 - \frac{4q}{f}\left(\frac{dS_4}{d\theta}\right)^2 = \left(\frac{dS_4}{d\theta}\right)^2 \frac{1}{f^2}(g^2 - 4qf)$$

$$\alpha_3 = \frac{dS_3}{d\psi} = \frac{dS_2}{d\phi} \cos \theta + \frac{\sin \theta}{2}\left[-\frac{g}{f}\left(\frac{dS_4}{d\theta}\right) + \sqrt{\left(\frac{g^2 - 4qf}{f^2}\right)\left(\frac{dS_4}{d\theta}\right)^2 + \frac{4\alpha_1}{f} - \frac{4}{2Cf}\left(\frac{dS_2}{d\phi}\right)}\right]$$

$$g^2 = \left(\frac{1}{A^2} - \frac{2}{AB} + \frac{1}{B^2}\right)\sin^2\phi \cos^2\phi$$

$$4qf = 4\left(\frac{\cos^2\phi}{2A} + \frac{\sin^2\phi}{2B}\right)\left(\frac{\sin^2\phi}{2A} + \frac{\cos^2\phi}{2B}\right)$$

$$= \frac{\sin^2\phi \cos^2\phi}{A^2} + \frac{\sin^2\phi \cos^2\phi}{B^2} + \frac{\sin^4\phi + \cos^4\phi}{AB}$$

$$g^2 - 4qf = -\frac{\left(\sin^4\phi + 2\sin^2\phi\cos^2\phi + \cos^4\phi\right)}{AB}$$

$$= -\left(\frac{\sin^2\phi + \cos^2\phi}{AB}\right)^2 = -\frac{1}{AB}$$

$$\alpha_3 = \frac{dS_2}{d\phi}\cos\theta + \frac{\sin\theta}{2}\left[-\frac{g}{f}\left(\frac{dS_4}{d\theta}\right) + \sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2}\left(\frac{dS_4}{d\theta}\right)^2 - \frac{2}{Cf}\left(\frac{dS_2}{d\phi}\right)^2}\,\right]$$

There does not appear to be any way to separate the right-hand side of the preceding equation.

$$2\csc\theta\left(\alpha_3 - p_\phi\cos\theta\right) = -\frac{g}{f}p_\theta + \sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2}p_\theta^{\,2} - \frac{2}{Cf}p_\phi^{\,2}}$$

The explicit dependence on θ can be eliminated by using the relationship

$$p_\theta^{\,2} = h^2 - p_\phi^{\,2} - \csc^2\theta\left(\alpha_3 - p_\phi\cos\theta\right)^2$$

which is valid for the triaxial problem with no forces if h is constant.

$$4\csc^2\theta\left(\alpha_3 - p_\phi\cos\theta\right)^2 = \frac{4\alpha_1}{f} - \frac{1}{ABf^2}p_\theta^{\,2} - \frac{2}{Cf}p_\phi^{\,2} + \frac{g^2}{f^2}p_\theta^{\,2} - \frac{2g}{f}p_\theta\sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2}p_\theta^{\,2} - \frac{2}{Cf}p_\phi^{\,2}}$$

$$4\left(h^2 - p_\phi^{\,2} - p_\theta^{\,2}\right) = \frac{4\alpha_1}{f} + \frac{1}{f^2}\left(g^2 - \frac{1}{AB}\right)p_\theta^{\,2} - \frac{2}{Cf}p_\phi^{\,2} - \frac{2g}{f}p_\theta\sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2}p_\theta^{\,2} - \frac{2}{Cf}p_\phi^{\,2}}$$

$$\frac{2}{Cf} p_\phi^2 + 4\left(h^2 - p_\phi^2 - \frac{\alpha_1}{f}\right) = \left[4 + \frac{1}{f^2}\left(g^2 - \frac{1}{AB}\right)\right] p_\theta^2$$

$$- \frac{2g}{f} p_\theta \sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{Cf} p_\phi^2}$$

Let

$$\tau \equiv \left[\frac{2}{Cf} p_\phi^2 + 4\left(h^2 - p_\phi^2 - \frac{\alpha_1}{f}\right)\right]$$

$$\nu \equiv 4 + \frac{1}{f^2}\left(g^2 - \frac{1}{AB}\right)$$

$$\xi \equiv \frac{\tau}{\nu}$$

$$\eta = - \frac{2g}{f\nu}$$

$$\eta \sqrt{\frac{4\alpha_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{Cf} p_\phi^2}\, p_\theta = \xi - p_\theta^2$$

$$\eta^2 \left(\frac{4\alpha_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{Cf} p_\phi^2\right) = \xi^2 - 2\xi p_\theta^2 + p_\theta^4$$

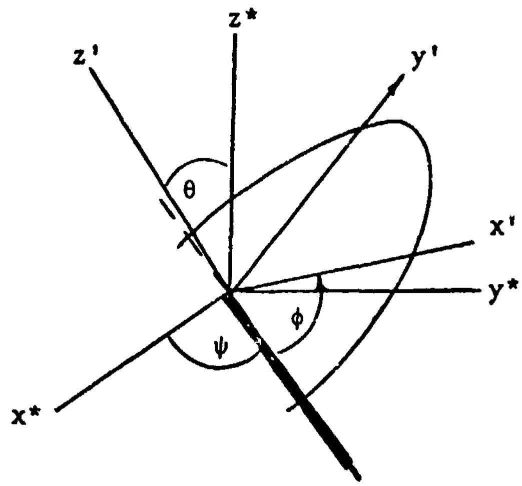
$$p_\theta^4 + \left(\frac{\eta^2}{ABf^2} - 2\xi\right) p_\theta^2 = \xi^2 - \eta^2\left(\frac{4\alpha_1}{f} - \frac{2}{Cf} p_\phi^2\right)$$

RELATIONSHIP BETWEEN CONJUGATE MOMENTA AND ANGULAR MOMENTUM

Let

$$\tilde{P} = \begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix}, \qquad \underline{H} = \begin{pmatrix} h_{x*} \\ h_{y*} \\ h_{z*} \end{pmatrix}, \qquad \underline{H}' = \begin{pmatrix} h_{x'} \\ h_{y'} \\ h_{z'} \end{pmatrix}$$

where the Euler angles ψ, θ, and ϕ are shown below, relating the body-fixed axes $\theta x'y'z'$ to the space-fixed axes $0x^*y^*z^*$.



The matrix

$$\tilde{\xi} = \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

and $\tilde{P}$ represents the conjugate momenta matrix while $\underline{H}$ and $\underline{H}'$ represent the angular momentum vector referenced to space-fixed and body-fixed axes, respectively.

$$\tilde{P} = \tilde{x}\tilde{\xi} \tag{1}$$

Explicitly,

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} =$$

$$\begin{bmatrix} (A\sin^2\phi + B\cos^2\phi)\sin^2\theta + C\cos^2\theta, & (A-B)\sin\phi\cos\phi\sin\theta, & C\cos\theta \\ (A-B)\sin\phi\cos\phi\sin\theta, & A\cos^2\phi + B\sin^2\phi, & 0 \\ C\cos\theta, & 0, & C \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

The kinetic energy for a force free triaxial rigid body is given by

$$T = \frac{1}{2A} h_{x'}^2 + \frac{1}{2B} h_{y'}^2 + \frac{1}{2C} h_{z'}^2 \tag{2}$$

where

$$\left.\begin{aligned} h_{x'} &= A\omega_{x'} = A\dot{\theta}\cos\phi + A\dot{\psi}\sin\theta\sin\phi \\ h_{y'} &= B\omega_{y'} = -B\dot{\theta}\sin\phi + B\dot{\psi}\sin\theta\cos\phi \\ h_{z'} &= C\omega_{z'} = C\dot{\phi} + C\dot{\psi}\cos\theta \end{aligned}\right\} \tag{3}$$

Also,

$$\underline{H} = \tilde{T}\underline{H}' \tag{4}$$

Explicitly,

$$\begin{pmatrix} h_{x^*} \\ h_{y^*} \\ h_{z^*} \end{pmatrix} =$$

$$\begin{pmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, & -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta, & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta, & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, & -\cos\psi\sin\theta \\ \sin\phi\sin\theta, & \cos\phi\sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} h_{x'} \\ h_{y'} \\ h_{z'} \end{pmatrix}$$

$$\tilde{p} = \tilde{N}^T \underline{H}' \qquad (5)$$

Explicitly,

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \sin\phi \sin\theta & \cos\phi \sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_{x'} \\ h_{y'} \\ h_{z'} \end{pmatrix}$$

$$\tilde{p} = \tilde{M}^T \underline{H} \qquad (6)$$

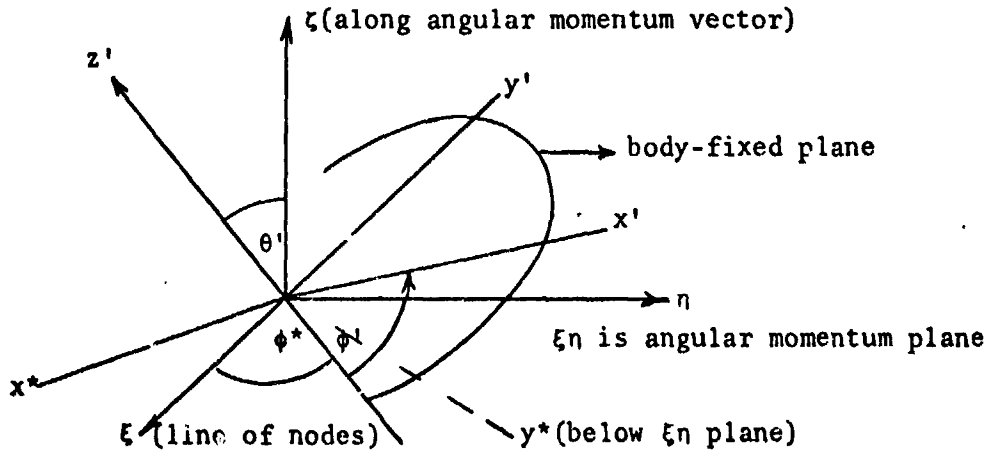
Explicitly,

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\psi & \sin\psi & 0 \\ \sin\psi \sin\theta & -\cos\psi \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_{x^*} \\ h_{y^*} \\ h_{z^*} \end{pmatrix}$$

Also,

$$\left.\begin{aligned} h_{x'} &= p_\theta \cos\phi + \frac{\sin\phi}{\sin\theta}\left(p_\psi - p_\phi \cos\theta\right) \\ h_{y'} &= -p_\theta \sin\phi + \frac{\cos\phi}{\sin\theta}\left(p_\psi - p_\theta \cos\theta\right) \\ h_{z'} &= p_\phi = h\cos\theta' \end{aligned}\right\} \qquad (7)$$

Consider the case in which the direction of the angular momentum vector is fixed in space. Choose this direction as an axis and redesignate it by the letter ζ. Let the line of nodes of the angular momentum plane (a plane through the center of mass of the body perpendicular to the ζ axis) with the space-fixed plane x^*y^* be designated by ξ. Consider the figure below.



$\underline{H}$ may be represented in the form

$$\underline{H} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{8}$$

If Eq (6) is used with ψ and θ replaced by ϕ^* and θ', respectively,

$$\begin{pmatrix} p_{\phi^*} \\ p_{\theta'} \\ p_{\phi'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi^* & \sin\phi^* & 0 \\ \sin\phi^*\sin\theta' & -\cos\phi^*\sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{9}$$

$$p_{\phi^*} = h$$

$$p_{\theta'} = 0$$

$$p_{\phi'} = h\cos\theta'$$

Since $p_{\theta'} = 0$,

$$\left.\begin{aligned} h_{x'} &= \frac{\sin\phi'}{\sin\theta'}\left(p_{\phi^*} - p_{\phi'}\cos\theta'\right) \\ h_{y'} &= \frac{\cos\phi'}{\sin\theta'}\left(p_{\phi^*} - p_{\phi'}\cos\theta'\right) \\ h_{z'} &= p_{\phi'} \end{aligned}\right\} \tag{10}$$

By utilizing the fact that $p_{\phi'} = h\cos\theta'$, one can write T in the form

$$T = \frac{1}{2A}h_{x'}^2 + \frac{1}{2B}h_{y'}^2 + \frac{1}{2C}h_{z'}^2$$

$$T = \left(\frac{\sin^2\phi'}{2A} + \frac{\cos^2\phi'}{2B}\right)\left(p_{\phi^*}^2 - p_{\phi'}^2\right) + \frac{1}{2C}p_{\phi'}^2$$

$$= \text{Hamilton Function} \tag{11}$$

Designate H = Hamilton function = α_1, a constant of motion.

$$\left.\begin{aligned}
\dot{\phi}' &= \frac{\partial H}{\partial p_{\phi'}} \\
\dot{\phi}^* &= \frac{\partial H}{\partial p_{\phi^*}} \\
\dot{p}_{\phi^*} &= -\frac{\partial H}{\partial \phi^*} = 0 \rightarrow h = \text{constant} = p_{\phi^*} \\
\dot{p}_{\phi'} &= -\frac{\partial H}{\partial \phi'}
\end{aligned}\right\} \quad (12)$$

Also,

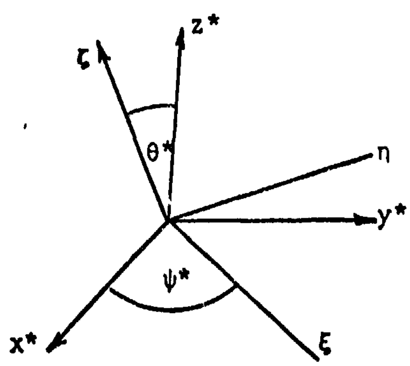
$$\cos\theta' = \frac{p_{\phi'}}{h}$$

and

$$p_{\theta'} = 0$$

Suppose ϕ', ϕ^*, p_{ϕ^*}, $p_{\phi'}$, and $p_{\theta'}$ are known. Are θ, ψ, ϕ, p_θ, p_ψ, and p_ϕ referenced to some space-fixed system $0x^*y^*z^*$ known?

$$\left.\begin{aligned}
\theta &= \theta(\theta', \theta^*, \phi^*) \\
\psi &= \psi(\theta', \theta^*, \phi^*, \psi^*) \\
\phi &= \phi(\theta', \theta^*, \phi^*, \phi') \\
p_\phi &= h\cos\theta' = p_{\phi'} \\
p_\psi &= h\cos\theta^* = p_{\phi^*}\cos\theta^* \\
p_\theta &= \cos\psi\, h_{x^*} + \sin\psi\, h_{y^*}
\end{aligned}\right\} \quad (13)$$



Using Eq (4) with ψ, θ, and ϕ replaced by ϕ^*, θ^*, and 0,

$$\begin{pmatrix} h_{x^*} \\ h_{y^*} \\ h_{z^*} \end{pmatrix} = \begin{pmatrix} \cos\psi^* & -\sin\psi^*\cos\theta^* & \sin\psi^*\sin\theta^* \\ \sin\psi^* & \cos\psi^*\cos\theta^* & -\cos\psi^*\sin\theta^* \\ 0 & \sin\theta^* & \cos\theta^* \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

$$\left.\begin{aligned} h_{x^*} &= h\sin\psi^*\sin\theta^* \\ h_{y^*} &= h\cos\psi^*\sin\theta^* \\ h_{z^*} &= h\cos\theta^* \end{aligned}\right\} \tag{14}$$

$$\begin{aligned} p_\theta &= \cos\psi\left(h\sin\psi^*\sin\theta^*\right) + \sin\psi\left(-\cos\psi^*\sin\theta^*\right) \\ &= -h\sin\theta^*\cdot\sin\left(\psi - \psi^*\right) \end{aligned} \tag{15}$$

If $p_{\phi'}$ is known, θ' is also known. Also, θ^* and ψ^* are prescribed constants, independent of each other and independent of ϕ', ϕ^*, $p_{\phi'}$, and p_{ϕ^*}. Hence, ψ, θ, ϕ, p_ψ, p_θ, and p_ϕ are known. Thus, the independent variables ϕ', ϕ^*, $p_{\phi'}$, p_{ϕ^*}, θ^*, and ψ^* serve to describe the motion of the triaxial body with respect to the space-fixed system $0x^*y^*z^*$.

If Eqs (6) and (14) are used, $p_\psi = h\cos\theta^*$ is obtained. Since p_ψ does not depend on ψ, $p_{\psi^*} = h\cos\theta^*$ may be written where ψ and ψ^* lie in the same plane. If p_{ψ^*} is known, θ^* is known; hence, the independent variables (ϕ', ϕ^*, ψ^*, $p_{\phi'}$, p_{ϕ^*}, and p_{ψ^*}) will serve to describe the triaxial motion with respect to the space-fixed system $0x^*y^*z^*$. Kinetic energy $T = H$ (for this extended problem) is expressed in terms of (ϕ', ϕ^*, ψ^*, $p_{\phi'}$, p_{ϕ^*}, and p_{ψ^*}) and still given by Eq (11). The canonical equations can be extended to include

$$\left.\begin{aligned} \dot{p}_{\psi^*} &= -\frac{\partial H}{\partial \psi^*} = 0 \\ \dot{\psi}^* &= \frac{\partial H}{\partial p_{\psi^*}} = 0 \end{aligned}\right\} \tag{16}$$

since Eqs (16) are consistent with the facts that p_{ψ^*} and ψ^* are constants of motion.

Thus, H can be interpreted, as given by Eq (11), as the Hamilton function for the motion of a triaxial body with respect to the space-fixed system under no forces. The corresponding canonical equations are

$$\left.\begin{aligned}
\dot{p}_{\phi'} &= -\frac{\partial H}{\partial \phi'} & \dot{\phi}' &= \frac{\partial H}{\partial p_{\phi'}} \\
\dot{p}_{\phi*} &= -\frac{\partial H}{\partial \phi*} & \dot{\phi}* &= \frac{\partial H}{\partial p_{\phi*}} \\
\dot{p}_{\psi*} &= -\frac{H}{\partial \psi*} & \dot{\psi}* &= \frac{H}{\partial p_{\psi*}}
\end{aligned}\right\} \tag{17}$$

The differential equations are explicitly:

$$\left.\begin{aligned}
\dot{p}_{\phi'} &= \left(p_{\phi*}^2 - p_{\phi'}^2\right)\frac{(A - B)}{AB} \sin \phi' \cos \phi' \\
\dot{p}_{\phi*} &= 0 \\
\dot{p}_{\psi*} &= 0 \\
\dot{\phi}' &= \left[\frac{1}{C} - \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)\right]p_{\phi'} \\
\dot{\phi}* &= \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)p_{\phi*} \\
\dot{\psi}* &= 0
\end{aligned}\right\} \tag{18}$$

Then,

$$p_{\phi*} = \alpha_2, \text{ a constant} \tag{19a}$$

$$p_{\psi*} = \alpha_3, \text{ a constant} \tag{19b}$$

$$\psi* = -\beta_3, \text{ a constant} \tag{19c}$$

The differential equations for $p_{\phi'}$, ϕ', and ϕ^* may now be written:

$$\dot{p}_{\phi'} = \left(\alpha_2{}^2 - p_{\phi'}{}^2\right)\left(\frac{A - B}{AB}\right)\sin\phi' \cos\phi' \tag{20a}$$

$$\dot{\phi}' = \left[\frac{1}{C} - \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)\right]p_{\phi'} \tag{20b}$$

$$\dot{\phi}^* = \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)\alpha_2 \tag{20c}$$

From Eqs (20a) and (20b),

$$-\frac{dw}{w} = \frac{du}{u} \tag{21}$$

where

$$w = \alpha_2{}^2 - p_{\phi'}{}^2.$$

$$u = \frac{1}{C} - \left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)$$

Integration of Eq (21) yields

$$wu = \kappa, \text{ a constant} \tag{22}$$

To evaluate κ, it is noted from Eq (11) that

$$\alpha_1 = \frac{p_{\phi'}{}^2}{2} u + \frac{\alpha_2{}^2}{2}\left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)$$

and from Eq (22), it is found that

$$wu = \kappa = \frac{\alpha_2{}^2}{C} - 2\alpha_1 \tag{23}$$

Equation (23) permits the expression of $p_{\phi'}$ in terms of ϕ'. One first writes

$$u = \frac{c' + d' \sin^2\phi'}{ABC}$$

$$c' \equiv A(B - C)$$

$$d' \equiv C(A - B)$$

Then,

$$p_{\phi'}{}^2 = \alpha_2{}^2 - \frac{(\alpha_2{}^2 - aC\alpha_1)}{Cu}$$

This last equation reduces to

$$p_{\phi'}{}^2 = \frac{C[a' + b' \sin^2\phi']}{[c' + d' \sin^2\phi']}$$

$$\left.\begin{aligned} a' &= A\left(2B\alpha_1 - \alpha_2{}^2\right) \\ b' &= \alpha_2{}^2(A - B) \\ c' &\equiv A(B - C) \\ d' &= C(A - B) \end{aligned}\right\} \qquad (24)$$

Whence,

$$p_{\phi'} = \pm \sqrt{\frac{C(a' + b' \sin^2\phi')}{c' + d' \sin^2\phi'}} \qquad (25)$$

APPENDIX

ON A METHOD OF OBTAINING A COMPLETE INTEGRAL OF THE HAMILTON-JACOBI EQUATION ASSOCIATED WITH A DYNAMICAL SYSTEM

Philip M. Fitzpatrick and John E. Cochran

Consider a dynamical system whose equations of motion are

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H(q_j;p_j;t)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q_j;p_j;t)}{\partial q_i}. \end{aligned}\right\} \quad i=1,2,\ldots,n;\ j=1,2,\ldots,n \qquad (1)$$

where the Hamiltonian, $H(q_j;p_j;t)$, is understood to be a function of the generalized coordinates, q_j, and their conjugate momenta, p_j, $j=1,2,\ldots,n$, and possibly the time, t. If one-half of the integrals of Eq (1) have been obtained in a suitable form, there is a well-known theorem, due to Liouville,[1] which may be used to find the remaining integrals. The purpose of this note is to point up the related, but perhaps not so well-known fact that a method of obtaining a complete integral of the Hamilton-Jacobi partial differential equation associated with (1) is implicitly contained in the theorem. Since a complete integral of (1) will permit us to express the solution of (1) in terms of canonical constants of integration, recognition of this fact is of importance in studying perturbations of the original system. The method will be discussed and applied in what follows.

Suppose that n integrals of a dynamical system with 2n degrees of freedom are known in the form

$$\Phi_i(q_j;p_j;t) = \alpha_i, \qquad i=1,2,\ldots,n;\ j=1,2,\ldots,n \qquad (2)$$

[1]E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (New York: Cambridge University Press, 1959), pp. 323-325.

where the α_i form a set of n independent constants of integration. If the Poisson bracket expression, (ϕ_i,ϕ_j), vanishes for each i and j and if the ϕ_i are solvable for the p_i in the form

$$p_i = f_i(q_j;\alpha_j;t), \qquad i=1,2,\ldots,n;\ j=1,2,\ldots,n \tag{3}$$

the Liouville theorem states that the difference between

$$\sum_{i=1}^{n} f_i\,dq_i$$

and $H(q_j;\alpha_j;t)dt$ is the perfect differential of a function $W(q_j;\alpha_j;t)$ and that the remaining n integrals of the system are given by

$$\frac{\partial W}{\partial \alpha_i} = \beta_i, \qquad i=1,2,\ldots,n \tag{4}$$

where the β_i form a set of n constants of integration which are independent of each other and of the set formed by the α_i.

To say that

$$\sum_{i=1}^{n} f_i\,dq_i - H(q_j;\alpha_j;t)dt, \quad j=1,2,\ldots,n \tag{5}$$

is the perfect differential of a function $W(q_j;\alpha_j;t)$ means that

$$\frac{\partial W}{\partial q_i} = f_i = p_i, \qquad i=1,2,\ldots,n \tag{6}$$

$$\frac{\partial W}{\partial t} = -H \tag{7}$$

Thus, implicit in the Liouville theorem is the fact that the function W is a complete integral of (7) which is the Hamilton-Jacobi partial differential equation associated with the system.

When the n integrals of (2) can be solved for the q_i instead of the p_i, $i=1,2,\ldots,n$, the theorem may also be applied, if the canonical transformation

$$\left.\begin{aligned} Q_i &= p_i \\ P_i &= -q_i \end{aligned}\right\} \quad (8)$$

to new variables (Q_i, P_i) is first introduced. Even if we are not able to solve the n integrals (2) explicitly for the p_i, or for the q_i, a complete integral may still be obtained in certain important cases now to be discussed.

Suppose we are able to solve the integrals (2) explicitly for $\ell(\ell < n)$ momenta and $n-\ell$ coordinates. Suppose further that, after reordering the subscripts, the expressions for the ℓ momenta and $n-\ell$ coordinates can be written in the restricted form

$$\left.\begin{aligned} p_i &= f_i(q_k;p_m;\alpha_j;t), \quad && i=1,2,\ldots, \; ; \; k\leq\ell; \; m>\ell; \; j=1,2,\ldots,n \\ q_i &= h_i(q_m;p_k;\alpha_j;t), \quad && i=\ell+1,\ell+2,\ldots,n; \; k>\ell; \; m\leq\ell; \; j=1,2,\ldots,n \end{aligned}\right\} \quad (9)$$

By introducing the canonical transformation

$$\left.\begin{aligned} p_i^* &= p_i, \quad q_i^* = q_i, \quad && i=1,2,\ldots,\ell \\ p_i^* &= -q_i, \quad q_i^* = p_i, \quad && i=\ell+1,\ell+2,\ldots,n \end{aligned}\right\} \quad (10)$$

Eqs (9) may be written in the form

$$p_i^* = f_i^*(q_j^*;\alpha_j;t), \quad i=1,2,\ldots,n; \; j=1,2,\ldots,n \quad (11)$$

Equations (11) are in the form (3) and the theorem may be applied.

Example 1: Central Orbit in the Plane, Polar Coordinates

For a particle moving in a plane under a central force derivable from the potential V(r), the Hamiltonian function is a constant α_1. If we designate by (p_r, p_θ), the momenta conjugate to the polar coordinates (r, θ), respectively, see Figure 1, the system has the well-known integrals

$$p_\theta = \alpha_2, \text{ a constant} \quad (12)$$

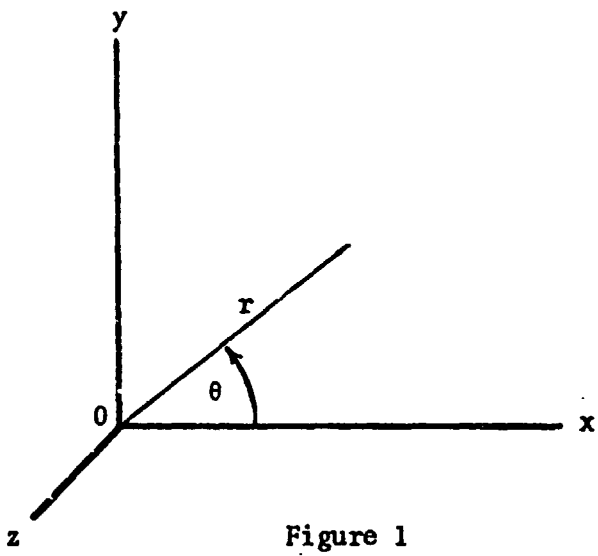


Figure 1

$$p_r = \pm\sqrt{2[\alpha_1 + V(r)] - \frac{\alpha_2^2}{r^2}} \tag{13}$$

From (5), we write

$$dW = p_r dr + p_\theta d\theta - \alpha_1 dt \tag{14}$$

If r_0 is chosen so that no new independent constant is introduced, the function

$$W = \int_{r_0}^{r} p_r dr + \alpha_2 \theta - \alpha_1 t \tag{15}$$

obtained by integrating (14), satisfies (7). Also, W is a complete integral of (7) since it contains two non-additive independent constants α_1 and α_2.

Example 2: Free Motion of a Triaxial Rigid Body

For the free rotation of a triaxial, rigid body about a fixed point 0, the Hamiltonian function, which is a constant of the motion, α_1, may be written in terms of the Euler angles (θ,ϕ,ψ), which specify the position of principal axes at 0 relative to space-fixed axes $0\xi\eta\zeta$ and their conjugate momenta (p_θ,p_ϕ,p_ψ). See Figure 2.

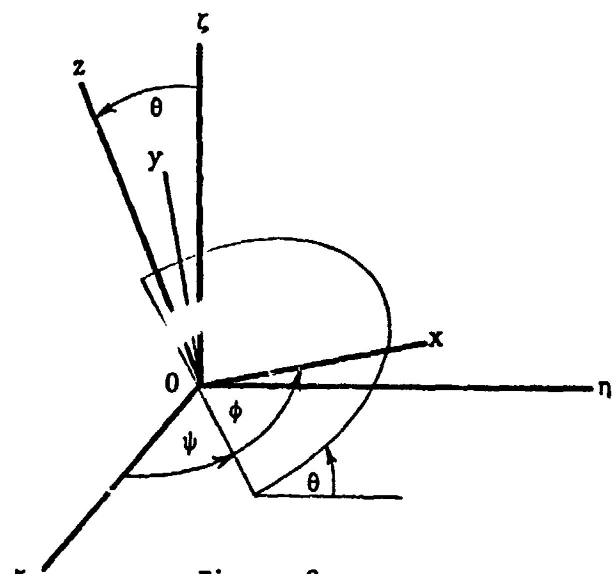


Figure 2

Three known integrals for this dynamical system are[2]

$$p_\psi = \alpha_3, \text{ a constant} \tag{16}$$

$$\theta = \tan^{-1}\left\{\frac{\sqrt{\alpha_2^2 - \alpha_3^2 - p_\theta^2}}{\alpha_3}\right\} - \tan^{-1}\left\{\frac{\sqrt{\alpha_2^2 - p_\phi^2 - p_\theta^2}}{p_\phi}\right\} \tag{17}$$

$$\phi = \tan^{-1}\left\{\frac{p_\theta}{\sqrt{\alpha_2^2 - p_\phi^2 - p_\theta^2}}\right\} + \tan^{-1}\left\{-\left(\frac{A}{B}\right)\frac{(2B\alpha_1 - \alpha_2^2)C + (C - B)p_\phi^2}{(2A\alpha_1 - \alpha_2^2)C + (C - A)p_\phi^2}\right\}^{\frac{1}{2}} \tag{18}$$

where A, B, and C are the principal moments of inertia at 0 and α_2 is the constant magnitude of the angular momentum about 0.

[2]See Whittaker, p. 325.

Although it is not possible to solve (17) and (18) so that p_ϕ and p_θ are expressed in the form (3), the set of equations (16), (17), and (18) is of the form (9); hence, the canonical transformation

$$\left.\begin{aligned} p_1 &= -\phi & \qquad q_1 &= p_\phi \\ p_2 &= -\theta & \qquad q_2 &= p_\theta \\ p_3 &= p_\psi & \qquad q_3 &= \psi \end{aligned}\right\} \tag{19}$$

allows us to write (16), (17), and (18) in the form (11). Then, from (5), we write

$$dW = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - \alpha_1 dt \tag{20}$$

If q_{10} and q_{20} are chosen in a manner so that no new independent constants are introduced, the function

$$\begin{aligned} W = -\alpha_1 t + \alpha_3 q_3 &+ \int_{q_{20}}^{q_2} \tan^{-1}\left\{\frac{\sqrt{\alpha_2^2 - \alpha_3^2 - x^2}}{\alpha_3}\right\} dx \\ &- \int_{q_{20}}^{q_2} \tan^{-1}\left\{\frac{\sqrt{\alpha_2^2 - q_1^2 - x^2}}{q_1}\right\} dx \\ &+ \int_{q_{10}}^{q_1} \tan^{-1}\left\{-\left(\frac{A}{B}\right)\frac{(2B\alpha_1 - \alpha_2^2)C + (C - B)x^2}{(2A\alpha_1 - \alpha_2^2)C + (C - A)x^2}\right\}^{\frac{1}{2}} dx \end{aligned} \tag{21}$$

obtained by integrating (20), is a complete integral of (7).

Technical Report ME—(NGR-01-003-(

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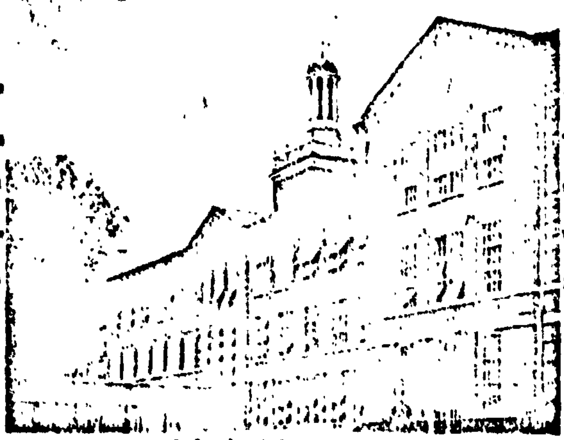

School of Engineering

October 1969

ENGINEERING EXPERIMENT STATION

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Technical Report ME-(NGR-01-003-008)-6

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N74-31305

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ABSTRACT

The formalism for studying perturbations of a triaxial rigid body within the Hamilton-Jacobi framework is set up. In particular, the motion of a triaxial artificial earth satellite about its center of mass is studied. Variables are found which permit separation, and the Euler angles and associated conjugate momenta are obtained as functions of canonical constants and time.

TABLE OF CONTENTS

INTRODUCTION

This report summarizes the results obtained on Grant NGR-01-003-008-S-2(ME) between November 1968 and May 1969.* These studies are aimed at applying the perturbation methods of celestial mechanics to the rigid body problem with particular emphasis on the problem of motion of an artificial earth satellite about its center of mass. During this reporting period, the investigators were able to express the Hamiltonian for the triaxial rigid body in terms of variables in which it is readily separable. This, in turn, permits introduction of a canonical transformation to new parameters which are constants in the torque-free motion. The equations of transformation are then inverted to allow the investigators to express the original Euler angles and associated conjugate momenta in terms of the canonical constants and the time. Thus, they are able to set up the formalism for studying perturbations of a triaxial rigid body within the Hamilton-Jacobi framework.

RECTANGULAR COORDINATE SYSTEM AND EULER ANGLES

Let O represent the center of mass of the rigid body. Choose a space-fixed rectangular system $O\xi\eta\zeta$ such that the positive ζ axis lies along the angular momentum vector $\underline{H}$ and in the sense of $\underline{H}$. Consider a plane through the center of mass and perpendicular to the ζ axis. This plane intersects the fundamental plane of the space-fixed, but otherwise arbitrary, rectangular frame $Ox^*y^*z^*$ in a line of nodes ON, shown in the figure. The ξ axis is chosen to lie along the line of nodes, its positive sense being arbitrarily chosen. Then, the η axis is chosen to form a right-handed system.

Let $Ox'y'z'$ be a body-fixed (principal axes) rectangular frame and let ϕ^*, θ', and ϕ' represent the Euler angles relating the $Ox'y'z'$ and $O\xi\eta\zeta$ systems. The $x'y'$ plane will be called the body-fixed plane. The angle ψ^* is the angle between the x^* and the ξ axes, measured in the x^*y^* plane while the angle θ^* is the angle between the positive z^* and ζ axes.

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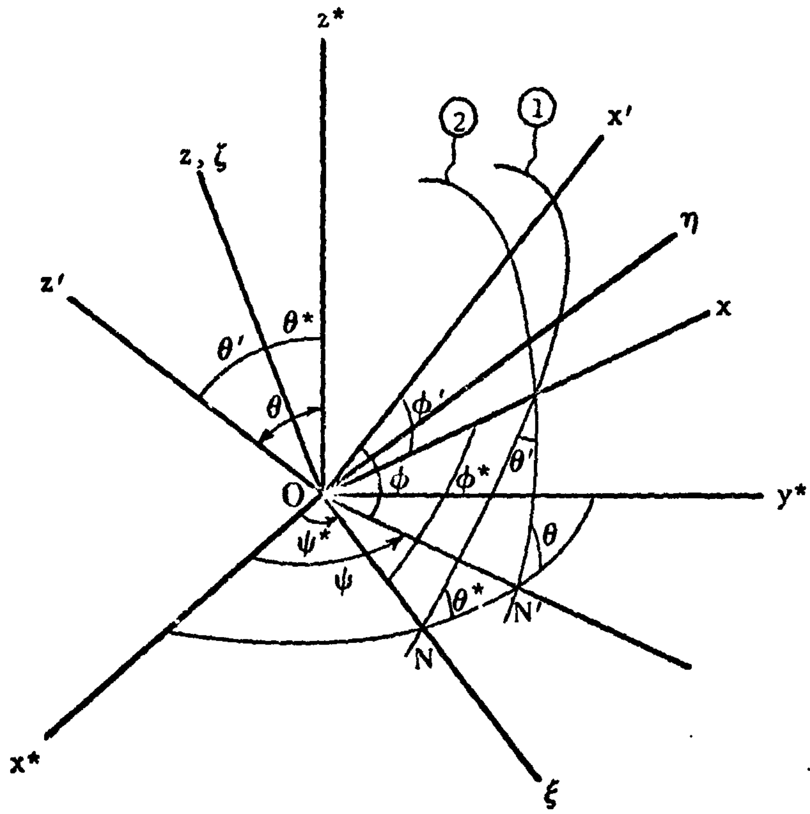
*Work co-sponsored by Contract NAS8-20175 with the George C. Marshall Space Flight Center.



1	Angular-Momentum Plane
2	Body-Fixed Plane
Ox*y*z*	Space-Fixed Axes
Ox'y'z'	Body-Fixed Axes
Oxyz	Angular-Momentum Axes

SOLUTION OF THE HAMILTON/JACOBI EQUATION

ASSOCIATED WITH A TRIAXIAL BODY PROBLEM WITH NO EXTERNAL FORCES

Hamilton Function and Canonical Equations

Although the eventual goal is to give a complete discription of the motion in the Ox*y*z* system, the description of the motion will first be given in the Oξηζ system. In this manner, a straightforward, coherent approach to the prolem and its solution can be presented.

Let

$$\underline{P} = \begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} \tag{1a}$$

$$\underline{\xi} = \begin{pmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{pmatrix} \tag{1b}$$

$$\underline{H}^* = \begin{pmatrix} h_x{}^* \\ h_y{}^* \\ h_z{}^* \end{pmatrix} \tag{1c}$$

$$\underline{H}' = \begin{pmatrix} h_x{}' \\ h_y{}' \\ h_z{}' \end{pmatrix} \tag{1d}$$

where $\underline{P}$ represents the conjugate momenta matrix and $\underline{H}$ and $\underline{H}'$ represent the angular momentum w.r.t. space-fixed and body-fixed axes, respectively. A recapitualation of some of the formulas from an earlier report (*Some Suggested Approaches to Solving the Hamilton-Jacobi Equation Associated with Constrained Rigid Body Motion*, January 1969, pp. 31-35) is given below to help the reader follow the subsequent discussion. It should be pointed out for anyone who has a copy of the referenced report that $\underline{H}$ should read $\underline{H}^*$ through Eq (6); the other notation is correct.

One has

$$\underline{P} = X\underline{\xi} \tag{2}$$

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} (A\sin^2\phi + B\cos^2\phi)\sin^2\theta + C\cdot\cos^2\theta & (A-B)\sin\phi\cos\phi\sin\theta & C\cos\phi \\ (A-B)\sin\phi\cos\phi\sin\theta & A\cos^2\phi + B\sin^2\phi & 0 \\ C\cos\theta & 0 & C \end{pmatrix} \begin{pmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{pmatrix}$$

$$\underline{H}^* = T\underline{H}' \tag{3}$$

or

$$\begin{pmatrix} h_x{}^* \\ h_y{}^* \\ h_z{}^* \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta & -\cos\psi\sin\phi - \sin\psi\cos\phi\cos\theta & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta & -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta & -\cos\psi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_x{}' \\ h_y{}' \\ h_z{}' \end{pmatrix}$$

$$\underline{P} = N^T\underline{H}' \tag{4}$$

or

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_{x'} \\ h_{y'} \\ h_{z'} \end{pmatrix}$$

$$\underline{P} = M^T \underline{H}^* \tag{5}$$

or

$$\begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\psi & \sin\psi & 0 \\ \sin\psi\sin\theta & -\cos\psi\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_{x^*} \\ h_{y^*} \\ h_{z^*} \end{pmatrix}$$

and from Eq (3),

$$\underline{H}' = T^I \underline{H}^* \tag{6}$$

or explicitly,

$$\begin{pmatrix} h_{x'} \\ h_{y'} \\ h_{z'} \end{pmatrix} \begin{aligned} &= p_\theta \cos\phi + \frac{\sin\phi}{\sin\theta}(p_\psi - p_\phi\cos\theta) \\ &= -p_\theta \sin\phi + \frac{\cos\phi}{\sin\theta}(p_\psi - p_\phi\cos\theta) \\ &= p_\phi = h\cos\theta' \end{aligned}$$

In the $O\xi\eta\zeta$ system the angular momentum can be written as

$$\underline{H} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{7}$$

If Eq (5) is used and ψ, θ, and ϕ are replaced by ϕ^*, θ', and ϕ', respectively, then

$$\begin{pmatrix} p_{\phi^*} \\ p_{\theta'} \\ p_{\phi'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi^* & \sin\phi^* & 0 \\ \sin\phi^*\sin\theta' & -\cos\phi^*\sin\theta' & \cos\phi' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{8}$$

or

$$\begin{aligned} p_{\phi^*} &= h \\ p_{\theta'} &= 0 \\ p_{\phi'} &= h\cos\theta' \end{aligned}$$

Similarly, after Eq (8) is used and with ψ, θ, and ϕ replaced by ϕ^*, θ', and ϕ', respectively, from Eq (6) one has

$$\begin{pmatrix} h_{x'} = \dfrac{\sin\phi'}{\cos\theta'}\left(p_{\phi^*} - p_{\phi'}\cos\theta'\right) \\ h_{y'} = \dfrac{\cos\phi'}{\sin\theta'}\left(p_{\phi^*} - p_{\phi'}\cos\theta'\right) \\ h_{z'} = p_{\phi'} \end{pmatrix} \tag{9}$$

Using Eq (9), coupled with (8), the kinetic energy (the Hamiltonian function) of the rigid body can be written in the form

$$T = \frac{1}{2A}h_x^2 + \frac{1}{2B}h_y^2 + \frac{1}{2C}h_z^2 = H$$

or

$$H = T = \left(\frac{\sin^2\phi'}{2A} + \frac{\cos^2\phi'}{2B}\right)\left(p_{\phi^*}^2 - p_\phi^2\right) + \frac{p_{\phi'}^2}{2C} \tag{10}$$

where A, B, and C are the moments of inertia of the rigid body referenced to the principal axes $Ox'y'z'$. It is assumed that

$$A > B > C$$

The associated canonical equations are

$$\dot{\phi}^* = \frac{\partial H}{\partial p_{\phi^*}} = h\left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right) \tag{11a}$$

$$\dot{\phi}' = \frac{\partial H}{\partial p_{\phi'}} = -h \cos \theta' \left(\frac{\sin^2 \phi'}{A} + \frac{\cos^2 \phi'}{B}\right) + \frac{p_{\phi'}}{C} \tag{11b}$$

$$\dot{p}_{\phi^*} = -\frac{\partial H}{\partial \phi^*} = 0 \tag{11c}$$

$$\dot{p}_{\phi'} = -\frac{\partial H}{\partial \phi'} = h^2\left(\frac{1}{B} - \frac{1}{A}\right)\sin \phi' \cos \phi' \sin^2 \theta' = -h \sin \theta' \dot{\theta}' \tag{11d}$$

$$p_{\theta'} = 0 \tag{11e}$$

$$\cos \theta' = \frac{p_{\phi'}}{h} \tag{11f}$$

Description of the Motion in the Ox*y*z* System

A set of relationships is given which allows the description of the motion in the space-fixed system (ψ, θ, ϕ, p_ψ, p_θ, and p_ϕ) to be obtained completely from the description of the motion in the body-fixed system (ϕ^*, θ', ϕ', p_{ϕ^*}, $p_{\theta'}$, and $p_{\phi'}$).

From elementary trigonometry,

$$\begin{aligned} \cos \theta &= \cos \theta' \cos \theta^* - \sin \theta' \sin \theta^* \cos \phi^* \\ \sin \theta &= \sqrt{1 - \cos^2\theta} \end{aligned} \tag{12a}$$

$$\begin{aligned} \sin \theta^* \sin \theta \cos(\psi - \psi^*) &= \cos \theta' - \cos \theta^* \cos \theta \\ \sin \theta \sin(\psi - \psi^*) &= \sin \phi^* \sin \theta' \end{aligned} \tag{12b}$$

$$\begin{aligned} \sin \theta' \cos \theta' \cos(\phi - \phi') &= \cos \theta^* - \cos \theta' \cos \theta \\ \sin \theta \sin(\phi - \phi') &= \sin \phi^* \sin \theta^* \end{aligned} \tag{12c}$$

With Eqs (3), (4), and (5), the variables p_ψ, p_θ, and p_ϕ can be related to the variables $p_{\phi'}$, p_{ϕ^*}, θ^*, ψ^*, and ϕ^*. Explicitly, these relationships can be written as

$$p_\theta = -p_{\phi^*} \sin \theta' \sin(\phi - \phi') \tag{13a}$$

$$p_\psi = h \cos \theta^* \tag{13b}$$

$$p_\phi = p_{\phi'}$$

Since θ^* and ψ^* are prescribed constants, independent of each other and independent of ϕ^*, ϕ', p_{ϕ^*}, and $p_{\phi'}$, the independent quantities (ϕ^*, ϕ', θ^*, ψ^*, p_{ϕ^*}, and $p_{\phi'}$) serve to describe the motion of the triaxial body in the $Ox^*y^*z^*$ system.

Generator and Equations of Transformation*

The Hamilton-Jacobi equation associated with Eq (10) is

$$\frac{1}{2}\left(\frac{\sin^2\phi'}{A} + \frac{\cos^2\phi'}{B}\right)\left[\left(\frac{\partial S}{\partial\phi^*}\right)^2 - \left(\frac{\partial S}{\partial\phi'}\right)^2\right] + \frac{1}{2C}\left(\frac{\partial S}{\partial\phi'}\right)^2 + \frac{\partial S}{\partial t} = 0 \tag{14}$$

from which the generator S of a canonical transformation is to be determined. A complete integral S of Eq (14) can be obtained by separation of variables. It is found that

$$S = -\alpha_1 t + h\phi^* + \alpha_3\psi^* + S_1(\phi') \tag{15}$$

where

$$\begin{aligned} \alpha_1 &= H \\ h &= p_{\phi^*} = \frac{\partial S}{\partial\phi^*} \\ \alpha_3 &= p_{\psi^*} = \frac{\partial S}{\partial\psi^*} \end{aligned} \tag{16}$$

are independent canonical variables. The function $S_1(\phi')$ is related to α_1 and h through the expression

$$S_1(\phi') = \int_{\phi_0'}^{\phi'} p_{\phi'}\,d\phi' \tag{17}$$

where

$$p_{\phi'} = \pm\sqrt{C\left(\frac{a' + b'\sin^2\phi'}{c' + d'\sin^2\phi'}\right)}$$

---

*The variables (ϕ^*, ϕ', θ^*, p_{ϕ^*}, $p_{\phi'}$, p_{ψ^*}) in which θ^* is replaced by $\cos^{-1}(p_{\psi^*}/h)$ are introduced here (see "Perturbation of the Force Free Motion of the Triaxial Rigid Body, page 20, for justification).

and

$$
\begin{aligned}
a' &= A(2B\alpha_1 - h^2) \\
b' &= h^2(A - B) \\
c' &= A(B - C) \\
d' &= C(A - B)
\end{aligned}
\tag{19}
$$

The complete set of transformation equations from (ψ^*, ϕ^*, ϕ', p_{ψ^*}, p_{ϕ^*}, $p_{\phi'}$) to (α_1, h, α_3, β_1, β_2, β_3) is obtained from Eq (15). The equations are:

$$\beta_1 = -\frac{\partial S}{\partial \alpha_1} = t - L(\phi') \tag{20a}$$

$$\beta_2 = -\frac{\partial S}{\partial h} = M(\phi') - \phi^* \tag{20b}$$

$$\beta_3 = -\frac{\partial S}{\partial \alpha_3} = -\psi^* \tag{20c}$$

$$p_{\psi^*} = \frac{\partial S}{\partial \psi^*} = \alpha_3 \tag{20d}$$

$$p_{\phi^*} = \frac{\partial S}{\partial \phi^*} = h \tag{20e}$$

$$p_{\phi'} = \frac{\partial S}{\partial \phi'} = \pm\sqrt{C\left(\frac{a' + b' \sin^2\phi'}{c' + d' \sin^2\phi'}\right)} \tag{20f}$$

where

$$
\begin{aligned}
L(\phi') &= \pm AB\sqrt{C}\, I_2(\phi') \\
M(\phi') &= \pm\sqrt{C}\, \alpha_2 I_3(\phi')
\end{aligned}
\tag{21}
$$

and

$$I_2(\phi') = \int_{\phi'_o}^{\phi'} \frac{d\phi'}{\sqrt{(a' + b' \sin^2\phi')(c' + d' \sin^2\phi')}} \tag{22a}$$

$$I_3(\phi') = \int_{\phi'_o}^{\phi'} \frac{[(A - B)\sin^2\phi' - A]d\phi'}{\sqrt{(a' + b' \sin^2\phi')(c' + d' \sin^2\phi')}} \tag{22b}$$

In three of the six Eqs (20), the right-hand sides are preceded by ± symbols. The choice of the sign in these equations is determined by the choice of sign for $p_{\phi'}$. Also,

$$\underline{P} = M^T\underline{H}$$

or

$$\begin{pmatrix} p_{\phi^*} \\ p_{\theta'} \\ p_{\phi'} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\phi^* & \sin\phi^* & 0 \\ \sin\phi^* \sin\theta' & -\cos\phi^* \sin\theta' & \cos\theta' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{23}$$

hence,

$$p_{\phi'} = h \cos\theta'$$

Thus, the sign of $p_{\phi'}$ depends upon whether $\cos\theta'$ is positive or negative. It is assumed that $0 < \theta' < \pi/2$. Therefore, Eqs (20) and (21) become

$$t - \beta_1 = L(\phi') \tag{24a}$$

$$\phi^* + \beta_2 = M(\phi') \tag{24b}$$

$$\beta_3 = -\psi^* \tag{24c}$$

$$p_{\phi'} = \sqrt{C\left(\frac{a' + b' \sin^2\phi'}{c' + d' \sin^2\phi'}\right)} \tag{24d}$$

$$p_{\phi^*} = h \tag{24e}$$

$$p_{\psi^*} = \alpha_3 \tag{24f}$$

where

$$L(\phi') = AB\sqrt{C}\, I_2(\phi') \tag{25a}$$

$$M(\phi') = -h\sqrt{C}\, I_3(\phi') \tag{25b}$$

INVERSION OF THE SOLUTION FOR THE TRIAXIAL RIGID BODY PROBLEM WITH NO EXTERNAL FORCES

Equation (24) must be inverted to express the variables (ϕ^*, ϕ', ψ^*, p_{ϕ^*}, $p_{\phi'}$, p_{ψ^*}) in terms of the canonical constants (α_1, h, α_3, β_1, β_2, β_3) and time t. The inversion is shown below.

Inversion of the Equation $t - \beta_1 = L(\phi')$

Since it is assumed that A > B > C, the quantities b', c', and d', given in Eqs (19), are positive. The quantity a' may be either positive or negative. In what follows, it is assumed that $a' \geq 0$.

From Eq (8), it is noted that

$$\frac{a'}{b'} < \frac{c'}{d'} \tag{26}$$

since

$$\frac{2C\alpha_1}{h^2} = \frac{AC\omega_{x'}^2 + BC\omega_{y'}^2 + C^2\omega_{z'}^2}{A^2\omega_{x'}^2 + B^2\omega_{y'}^2 + C^2\omega_{z'}^2}$$

where $\omega_{x'}$, $\omega_{y'}$, and $\omega_{z'}$ are components of the angular velocity w.r.t. the primed system.

For convenience the following parameters are defined:

$$n_1^2 \equiv \frac{b'}{a' + b'} \tag{27a}$$

$$n_2^2 \equiv \frac{d'}{c' + d'} \tag{27b}$$

$$\xi \equiv \frac{1}{\sqrt{(a' + b')(c' + d')}} = \frac{1}{B\sqrt{(A - C)(2A\alpha_1 - h^2)}} \tag{27c}$$

$$k \equiv \sqrt{\frac{n_1^2 - n_2^2}{1 - n_2^2}} = \sqrt{\frac{(A - B)(h^2 - 2C\alpha_1)}{(B - C)(2A\alpha_1 - h^2)}} \tag{27d}$$

$$g \equiv \frac{1}{\sqrt{1 - n_2^2}} = \sqrt{\frac{B(A - C)}{A(B - C)}} \tag{27e}$$

$$k' \equiv \sqrt{1 - k^2} = \sqrt{\frac{(A - C)(2B\alpha_1 - h^2)}{(B - C)(2A\alpha_1 - h^2)}} \tag{27f}$$

Clearly, $1 \geq n_1^2 > n_2^2 > 0$; thus, $0 < k < 1$, and k' is real since $a'/b' < c'/d'$.

To cast Eq (22a) into a more convenient form, a new variable is introduced by the equation

$$\alpha = \phi' + \pi/2 \tag{28}$$

It follows immediately, by substituting α and the parameters in Eq (16) into (22a), that

$$I_2(\phi') = \xi \int_{\alpha_0}^{\alpha} r(\alpha)\,d\alpha \tag{29}$$

where

$$\alpha_0 = \phi_0' + \pi/2$$

$$r(\alpha) = \frac{1}{\sqrt{(1 - n_1^2 \sin^2\alpha)(1 - n_2^2 \sin^2\alpha)}} \tag{30}$$

Since the lower limit of integration of Eq (29) may be taken to be an absolute constant, $\phi_0' = -\pi/2$ is chosen; hence, $\alpha_0 = 0$. Therefore,

$$I_2(\phi') = \xi \int_0^{\alpha} r(\alpha)\,d\alpha \tag{31}$$

In what follows, the formulas which appear in Byrd and Friedman [1] will be referenced. Such formula numbers will be indicated by prefixing the numbers with the designation B-F.

Using B-F (284.00) and Eq (24a),

$$\int_0^{\alpha} r(\alpha)\,d\alpha = gu = \frac{1}{\xi AB\sqrt{C}}(t - \beta_1) \tag{32}$$

or

$$u = \lambda t + \varepsilon \tag{33}$$

where

$$\lambda \equiv \frac{1}{g\xi AB\sqrt{C}} = \sqrt{\frac{(2A\alpha_1 - h^2)(B - C)}{ABC}} \tag{34a}$$

$$\xi \equiv -\lambda\beta_1 \tag{34b}$$

Also from Byrd-Friedman,

$$\mathrm{sn}^2u \equiv [\sin(\mathrm{am}u)]^2 = \frac{(1 - n_2^2)\sin^2\alpha}{1 - n_2^2 \sin^2\alpha} \tag{35}$$

Solving the above equation for sin α, one writes

$$\sin\alpha = \frac{\mathrm{sn}u}{\sqrt{1 - n_2^2\mathrm{cn}^2u}} \tag{36a}$$

and

$$\cos\alpha = \frac{\sqrt{1 - n_2^2}\ \mathrm{cn}u}{\sqrt{1 - n_2^2\mathrm{cn}^2u}} \tag{36b}$$

where

$$\mathrm{cn}u \equiv \cos(\mathrm{am}u)$$

and

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1$$

Since $\alpha = \phi' + \pi/2$,

$$\sin\phi' = \frac{\sqrt{1 - n_2^2}\ \mathrm{cn}u}{\sqrt{1 - n_2^2\mathrm{cn}^2u}} \tag{37a}$$

$$\cos\phi' = \frac{\mathrm{sn}u}{\sqrt{1 - n_2^2\mathrm{cn}^2u}} \tag{37b}$$

$$\tan\phi' = -\frac{\mathrm{cn}u}{g\ \mathrm{sn}u} \tag{37c}$$

The quadrant of ϕ' is uniquely determined by studying the signs of cnu and snu.

Equation (37) is not in a convenient form for calculation since powers of t appear in the expressions for cnu and snu. This difficulty can be avoided by introducing theta functions. From B-F (907.01), (907.02), (907.03), (900.04), and (901.01), for $|u| < K'$,

$$\mathrm{sn}u = u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!}$$

$$- (1 + 135k^2 + 135k^4 + k^6)\frac{u^7}{7!} + \cdots \tag{38a}$$

$$\mathrm{cnu} = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - (1 + 44k^2 + 16k^4)\frac{u^6}{6!}$$

$$+ (1 + 408k^2 + 912k^4 + 64k^6)\frac{u^8}{8!} - \cdots \quad (38b)$$

$$\mathrm{dnu} = 1 - k^2\frac{u^2}{2!} + (4 + k^2)k^2\frac{u^4}{4!}$$

$$- (16 + 44k^2 + k^4)k^2\frac{u^6}{6!} + \cdots \quad (38c)$$

where

$$K' \equiv K(k') \quad (39a)$$

$$K \equiv \frac{\pi}{2}\left(1 + 4\sum_{m=1}^{\infty}\frac{q^m}{1 + q^{2m}}\right) = \int_0^{\pi/2}\frac{d\nu}{\sqrt{1 - k^2\sin^2\nu}} \quad (39b)$$

$$q \equiv \tfrac{1}{2}k\left[1 + 2\left(\frac{k_1}{2}\right)^4 + 15\left(\frac{k_1}{2}\right)^8 + 150\left(\frac{k_1}{2}\right)^{12}\right.$$

$$\left. + 1707\left(\frac{k_1}{2}\right)^{16} + \cdots\right] \quad (39c)$$

$$k' \equiv \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}\left(k_1^2 < k^2 < 1\right) \quad (39d)$$

If B-F (1050.01), (1052.01), and (1052.02) are used, then

$$\tan\phi' = -\frac{\sqrt{k'}}{g}\,\frac{\cos\nu + q^2\cos 3\nu + q^6\cos 5\nu + \cdots}{\sin\nu - q^2\sin 3\nu + q^6\sin 5\nu - \cdots} \quad (40)$$

where

$$\nu \equiv \frac{\pi}{2K}u$$

The series (c) of (39) for computing q converges rapidly. Hence, the angle ϕ' can be expressed in terms of canonical parameters and time through Eq (37) and can be computed by using expression (40).

Inversion of the Equation $\phi^* + \beta_1 = M(\phi')$

By using Eqs (27), (28), and (30) and recalling that $\alpha_0 = 0$, Eq (24b) can be rewritten as

$$\phi^* + \beta_2 = -h\sqrt{C}\xi \int_0^\alpha [(A - B)\cos^2\alpha - A]r(\alpha)d\alpha \qquad (41)$$

From Eq (32),

$$\phi^* + \beta_2 = \frac{h}{B}(t - \beta_1) - h\sqrt{C}\xi(A - B)\int_0^\alpha \cos^2\alpha \; r(\alpha)d\alpha \qquad (42)$$

Using B-F (284.08) and (432.03), Eq (42) becomes

$$\phi^* + \beta_2 = \frac{h}{B}(t - \beta_1) - \left(\frac{\pi}{2K}\right)[\Omega_5 - u\Lambda_0(\beta,k)] \qquad (43)$$

where

$$\beta \equiv \sin^{-1}\frac{1}{\sqrt{1 - \gamma^2}} \qquad (44a)$$

$$\gamma^2 \equiv -\frac{n_2^2}{1 - n_2^2} \qquad (1 < -\gamma^2 < \infty) \qquad (44b)$$?

and the functions Ω_5 and Λ_0 are defined in B-F, Sections 430 and 150, respectively. Since $u = \lambda(t - \beta_1)$, it can be written

$$\phi^* + \beta_2 = M^*(t - \beta_1) - \frac{\pi}{2K}(\Omega_5 - u) \qquad (45)$$

where

$$M^* = \frac{h}{B} - \frac{\pi}{2K}[1 - \Lambda_0(\beta,k)]\lambda \qquad (46)$$

Expressions for $p_{\phi'}$ and θ'

By applying Eqs (27) and (37), Eq (24d) can be written as

$$p_{\phi'} = \sqrt{\frac{C(a' + b')(1 - n_2^2)}{\sqrt{C'}}}\,(k'^2 + k^2 cn^2 u)^{\frac{1}{2}} \qquad (47)$$

From B-F (121.00), one has

$$dn^2 u = k'^2 + k^2 cn^2 u \qquad (48)$$

Hence. Eq (47) takes the form

$$p_{\phi'} = \frac{\sqrt{C(a' + b')(1 - n_2^2)}}{\sqrt{C'}} \, \mathrm{dnu}$$

$$= \sqrt{\frac{C(2A\alpha_1 - h^2)}{A - C}} \, \mathrm{dnu} \qquad (49)$$

and since $p_{\phi'} = h \cos \theta'$,

$$\cos \theta' = \frac{p_{\phi'}}{h} = \sqrt{\frac{C(2A\alpha_1 - h^2)}{h^2(A - C)}} \, \mathrm{dnu} \qquad (50)$$

Inverted Solution for the Triaxial Rigid Body Problem with No External Forces

The general solution for the triaxial rigid body problem with no external forces can then be summarized as follows:

$$\tan \phi' = -\frac{\mathrm{cnu}}{g \, \mathrm{snu}}$$

$$= -\frac{\sqrt{k'}}{g} \frac{\cos \nu + q^2 \cos 3\nu + q^6 \cos 5\nu + \cdots}{\sin \nu - q^2 \sin 3\nu + q^6 \sin 5\nu - \cdots} \qquad (51a)$$

$$\phi^* + \beta_2 = M^*\left(t - \beta_1\right) - \frac{\pi}{2K}\left(\Omega_5 - u\right) \qquad (51b)$$

$$\psi^* = \beta_3 \qquad (51c)$$

$$p_{\phi'} = \sqrt{\frac{C(2A\alpha_1 - h^2)}{A - C}} \, \mathrm{dnu} \qquad (51d)$$

$$p_{\phi^*} = h \qquad (51e)$$

$$p_{\psi^*} = \alpha_3 \qquad (51f)$$

This solution coupled with Eqs (12) and (13) gives a complete description of the motion of the triaxial body in the space-fixed system $Ox^*y^*z^*$ in terms of the canonical constants and time.

UNIAXIAL SOLUTION

By letting A equal B, the triaxial solution (51) can be reduced to the corresponding uniaxial solution. To distinguish between the canonical parameters which appear in the triaxial solution and the reduced solution, the latter will be labeled with the subscript u; that is, α_{1u}, h_u, α_{3u}, β_{1u}, β_{2u}, and β_{3u}.

For the case A = B, one has

$$n_2^2 = n_2^2 = 0, \qquad k = 0,$$

$$k' = 1, \qquad g = 1,$$

$$\text{snu} = \sin u, \qquad \text{cnu} = \cos u,$$

and

$$\lambda = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A - C)}{A^2C}}$$

Thus, from Eq (37e), one obtains

$$\phi' - \phi_0 = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A - C)}{A^2C}}\left(t - \beta_{1u}\right) \tag{52}$$

When A = B,

$$\beta = \frac{\pi}{2}, \qquad \Lambda_0\left(\frac{\pi}{2},0\right) = 1,$$

$$M^* = \frac{h_u}{A}, \qquad \Omega_5 = u;$$

therefore, Eq (45) reduces to

$$\phi^* = -\beta_{2u} + \frac{h_u}{A}\left(t - \beta_{1u}\right) \tag{53}$$

Furthermore, for A = B, dn(u,0) = 1, and Eq (49) reduces to

$$p_\phi = \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}} \tag{54}$$

In summary, the uniaxial solution is given as follows:

$$\phi' - \phi'_o = \sqrt{\frac{(2A\alpha_{1u} - h_u^2)(A - C)}{A^2C}}\left(t - \beta_{1u}\right) \tag{55a}$$

$$\phi^* = -\beta_{2u} + \frac{h_u}{A}\left(t - \beta_{1u}\right) \tag{55b}$$

$$\psi^* = -\beta_{3u} \tag{55c}$$

$$p_{\phi'} = \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}} \tag{55d}$$

$$p_{\phi^*} = h_u \tag{55e}$$

$$p_{\psi^*} = \alpha_{3u} \tag{55f}$$

and the corresponding generator is

$$S_u = -\alpha_{1u}t + h_u\phi^* + \alpha_{3u}\psi^* + \sqrt{\frac{C(2A\alpha_{1u} - h_u^2)}{A - C}}\left(\phi' - \phi'_o\right) \tag{56}$$

Through the use of Eqs (12) and (13), the complete solution of the force-free uniaxial motion can be obtained in the space-fixed system $Ox^*y^*z^*$.

The parameters which appear in the treatment of the force-free uniaxial problem, given in [2], will be labeled with superscript asterisks; that is, α_{1u}^*, α_{2u}^*, α_{3u}^*, β_{1u}^*, β_{2u}^*, and β_{3u}^*. It has been shown that

$$h_u^2 = 2A\alpha_{1u}^* + \left(\frac{C - A}{C}\right)\alpha_{2u}^{*2} \tag{57}$$

The corresponding generator, in which h_u is interpreted as a function of α_{1u}^* and α_{2u}^* through Eq (57), takes the form

$$S_u^* = -\alpha_{1u}^*t + \sqrt{2A\alpha_{1u}^* + \left(\frac{C - A}{C}\right)\alpha_{2u}^{*2}}\ \phi^* + \alpha_{3u}^*\psi^*$$

$$+ \alpha_{2u}^*\left(\phi' - \phi'_o\right) \tag{58}$$

After inversion, the associated equations of transformation are

$$\phi' - \phi'_o = -\beta^*_{2u} + \frac{\alpha^*_{2u}}{A}\left(\frac{A - C}{C}\right)\left(t - \beta^*_{1u}\right) \tag{59a}$$

$$\phi^* = \sqrt{2A\alpha^*_{1u} + \left(\frac{A - C}{C}\right)\alpha^{*2}_{2u}}\ \left(t - \beta^*_{1u}\right) \tag{59b}$$

$$\psi^* = -\beta^*_{3u} \tag{59c}$$

$$p_{\phi'} = \alpha^*_{2u} \tag{59d}$$

$$p_{\phi^*} = \sqrt{2A\alpha^*_{1u} + \left(\frac{C - A}{C}\right)\alpha^*_{2u}} \tag{59e}$$

$$p_{\psi^*} = \alpha^*_{3u} \tag{59f}$$

If Eqs (59) and (60) are compared, the parameters (α^*_{1u}, α^*_{2u}, α^*_{3u}, β^*_{1u}, β^*_{2u}, and β^*_{3u}) and (α_{1u}, h_u, α_{3u}, β_{1u}, β_{2u}, and β_{3u}) are related as follows

$$\alpha^*_{1u} = \alpha_{1u} \tag{60a}$$

$$\alpha^*_{2u} = \sqrt{\frac{C(2A\alpha_{1u} - h^2_u)}{A - C}} \tag{60b}$$

$$\alpha^*_{3u} = \alpha_{3u} \tag{60c}$$

$$\beta^*_{1u} = \beta_{1u} + \frac{A}{h_u}\beta_{2u} \tag{60d}$$

$$\beta^*_{2u} = -\frac{1}{h_u}\sqrt{\frac{(2A\alpha_{1u} - h^2_u)(A - C)}{C}}\ \beta_{2u} \tag{60e}$$

$$\beta^*_{3u} = \beta_{3u} \tag{60f}$$

PERTURBATION OF THE FORCE FREE MOTION OF THE TRIAXIAL RIGID BODY

Recalling the section entitled "Generator and Equations of Transformation," page eight, θ^* must be replaced with an equivalent parameter p_{ψ^*}, the momentum conjugate to ψ^*, to use the canonical perturbation equations of Hamilton-Jacobi theory in studying the perturbations of the force free motion of the triaxial rigid body. It follows from Eq (13b) that either θ^* or p_{ψ^*} will give an equivalent description of the motion. It is clear from this equation that the momentum conjugate to any angle ψ which lies in the x^*y^* plane is independent of the angle ψ and depends only on h and θ^*. Therefore,

$$p_\psi = p_{\psi^*} = h \cos\theta^* = p_{\phi^*} \cos\theta^* \tag{61}$$

Thus, the six independent quantities (ϕ^*, ϕ', ψ^*, p_{ϕ^*}, $p_{\phi'}$, and p_{ψ^*}) will completely describe the motion of the triaxial rigid body with respect to the $Ox^*y^*z^*$ system. The Hamilton function from which ϕ^*, ϕ', p_{ϕ^*}, and $p_{\phi'}$ are to be obtained is, of course, still given by Eq (10). Furthermore, H, as given in Eq (10), can be considered to be the Hamilton function of an extended system of variables (ϕ^*, ϕ', ψ^*, p_{ϕ^*}, $p_{\phi'}$, p_{ψ^*}, which satisfy the canonical equations of motion.

$$\dot{\phi}^* = \frac{\partial H}{\partial p_{\phi^*}} \tag{62a}$$

$$\dot{\phi}' = \frac{\partial H}{\partial p_{\phi'}} \tag{62b}$$

$$\dot{\psi}^* = \frac{\partial H}{\partial p_{\psi^*}} \tag{62c}$$

$$\dot{p}_{\phi^*} = -\frac{\partial H}{\partial \phi^*} \tag{62d}$$

$$\dot{p}_{\phi'} = -\frac{\partial H}{\partial \phi'} \tag{62e}$$

$$\dot{p}_{\psi^*} = -\frac{\partial H}{\partial \psi^*} \tag{62f}$$

subject to the constraints

$$\psi^* = \text{constant} \tag{63a}$$

$$p_{\psi^*} = h \cos\theta^* = \text{constant} \tag{63b}$$

This follows from the fact that the two differential equations (62c) and (62f), which have been added to the system, are entirely consistent with (63), the equations of constraint.

GRAVITY GRADIENT POTENTIAL FOR THE TRIAXIAL BODY

The gravity gradient potential V for the triaxial body is given by

$$V = -\frac{3}{2}\kappa[(A - C)\cos^2\chi + (A - B)\cos^2\beta] \tag{64}$$

where $\kappa = n'^2$ and n' is the mean motion of the Earth about the triaxial body. A circular orbit will be considered for which κ is a constant. The angles α, β, and χ are the direction angles of the line segment from the center of mass of the body to the center of mass of the Earth with respect to $0x'y'z'$, the principal axes of the body. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, Eq (64) can be rewritten as

$$V = \frac{3}{2}\kappa(B - A) + W \tag{65}$$

where

$$W = \left(\frac{B - C}{A - C}\right)W_1 + W_2 \tag{66}$$

and

$$\begin{aligned} W_1 &= -\frac{3}{2}\kappa(B - C)\cos^2\chi \\ W_2 &= \frac{3}{2}\kappa(A - B)\cos^2\alpha \end{aligned} \tag{67}$$

The expression for $\cos\chi$ in terms of canonical constants and time t is deduced in [1] and can be written in the form

$$\cos\chi = D_1 \cos\theta' + D_2 \sin\theta' \sin\phi^* + D_3 \sin\theta' \cos\phi^* \tag{68}$$

where

$$\begin{aligned} D_1 &= \sin i \sin \ell \cos\theta^* - [\cos i \sin \ell \cos(\Omega + \beta_3) \\ &\qquad + \cos \ell \sin(\Omega + \beta_3)]\sin\theta^* \\ D_2 &= -\cos i \sin \ell \sin(\Omega + \beta_3) + \cos \ell \cos(\Omega + \beta_3) \\ D_3 &= -\sin i \sin \ell \sin\theta^* - [\cos i \sin \ell \cos(\Omega + \beta_3) \\ &\qquad + \cos \ell \sin(\Omega + \beta_3)]\cos\theta^* \end{aligned} \tag{69}$$

Note that D_1, D_2, and D_3 are functions of three canonical constants only, namely, $\alpha_2 = h$, α_3, and β_3 and contain t explicitly only through ℓ and Ω, which are both linear in t.

A suitable expresion for $\cos \alpha$ can be derived similarly. From spherical trigonometry,

→ not complete

$$\cos \alpha = \cos \phi' \cos \phi_H - \sin \phi' \sin \phi_H \cos \theta' \tag{70}$$

and

$$\cos \theta_H = \cos i \cos \theta' - \sin i \sin \theta^* \cos(\Omega + \beta_3) \tag{71}$$

Introducing

$$E_1 \equiv \cos(\phi^* - \phi_H) = \frac{\cos i - \cos \theta_H \cos \theta^*}{\sin \theta_H \sin \theta^*} \tag{72}$$

$$E_2 \equiv \sin(\phi^* - \phi_H) = -\frac{\sin i \sin(\Omega + \beta_3)}{\sin \theta_H} \tag{73}$$

Equation (7) can be written in the form

$$\begin{aligned}\cos \alpha = {} & E_1(\cos \phi' \cos \phi^* - \cos \theta' \sin \phi' \sin \phi^*) \\ & - E_2(\cos \phi' \sin \phi^* \\ & + \cos \theta' \sin \phi' \cos \phi^*)\end{aligned} \tag{74}$$

Note that E_1 and E_2 are functions of only three canonical constants, namely, α_2, α_3, and β_3 and contain t explicitly only through ℓ and Ω. It is important to note that D_1, D_2, D_3, E_1, and E_2 do not contain the moments of intertia A, B, and C. Thus, these coefficients can be treated as constants when $\cos \chi$ and $\cos \alpha$ are expanded in Taylor's series about their values at B = A. The reason for the expansion is the angles ϕ^*, ϕ', and θ' for the unperturbed triaxial body are no longer either constant or simple linear functions of time (as was the case in the uniaxial problem). Thus, since difficulties are anticipated in the integration of the perturbation equations, attempts are made to linearize the arguments of the trigonometric functions which will appear in the integration.

Introducing the notation

$$\begin{aligned} f(\chi) &\equiv \cos \chi \\ g(\alpha) &\equiv \cos \alpha \end{aligned} \tag{75}$$

$f(\chi)$ and $g(\alpha)$ are treated as functions of B and are expanded about the value B = A. Using prime notation to indicate derivatives with respect to B, one has

$$f(\chi) = f(B) - f'(B)(A - B) + \tfrac{1}{2}f''(B)(A - B)^2 + O(A - B)^3] \tag{76}$$

where

$$f(B) = D_1[\cos\theta']_{B=A} + D_2[\sin\theta' \sin\phi^*]_{B=A} + D_3[\sin\theta' \cos\phi^*]_{B=A}$$

$$f'(B) = D_1\left[\frac{\partial}{\partial B}\cos\theta'\right]_{B=A} + D_2\left[\frac{\partial}{\partial B}(\sin\theta' \sin\phi^*)\right]_{B=A} + D_3\left[\frac{\partial}{\partial B}(\sin\theta' \cos\phi^*)\right]_{B=A} \tag{77}$$

$$f''(B) = D_1\left[\frac{\partial^2}{\partial B^2}\cos\theta'\right]_{B=A} + D_2\left[\frac{\partial^2}{\partial B^2}(\sin\theta' \sin\phi^*)\right]_{B=A} + D_3\left[\frac{\partial^2}{\partial B^2}(\sin\theta' \cos\phi^*)\right]_{B=A}$$

and

$$g(\alpha) = g(B) - g'(B)(A - B) + O[(A - B)^2] \tag{78}$$

where

$$g(B) = E_1[\cos\theta' \cos\phi^* - \cos\theta' \sin\phi' \sin\phi^*]_{B=A} + E_2[\cos\phi' \sin\phi^* + \cos\theta' \sin\phi' \cos\phi^*]_{B=A}$$

$$g'(B) = E_1\left[\frac{\partial}{\partial B}(\cos\phi' \cos\phi^* - \cos\phi' \sin\phi' \sin\phi^*)\right]_{B=A} + E_2\left[\frac{\partial}{\partial B}(\cos\phi' \sin\phi^* + \cos\theta' \sin\phi' \sin\phi^*)\right]_{B=A} \tag{79}$$

In Eq (78), only two terms are carried since $g(\alpha)$ is multiplied by the fact $r(A - B)$ in W.

Equations (66), (67), (76), and (78) yield

$$W = \left(\frac{B - C}{A - C}\right)W_{1u} + W_{2t} + O[(A - B)^3] \tag{80}$$

where

$$W_{1u} = -\frac{3\kappa}{2}(A - C)[f(B)]^2$$

$$W_{2t} = \frac{3\kappa}{2}(A - B)\left\{2(B - C)f(B)f'(B) + [g(B)^2]\right\}$$
$$- \frac{3\kappa}{2}(A - B)^2\left\{(B - C)[f'(B)]^2 + f(B)f''(B)] + 2g(B)g'(B)\right\}$$

These expressions for W_{1u} and W_{2t} can be used to study the perturbations of the variables (α_1, h, α_3, β_1, β_2, and β_3) which are given by the following relations

$$\begin{aligned} \dot{\alpha}_i &= \left(\frac{B - C}{A - C}\right)\frac{\partial W_{1u}}{\partial \beta_i} + \frac{\partial W_{2t}}{\partial \beta_i} \\ \dot{\beta}_i &= -\left(\frac{B - C}{A - C}\right)\frac{\partial W_{1u}}{\partial \alpha_i} - \frac{\partial W_{2t}}{\partial \alpha_i} \end{aligned} \qquad (i=1,2,3), \quad (\alpha_2 = h) \qquad (81)$$

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N74-31306

STRESSES IN DOME-SHAPED SHELLS OF REVOLUTION WITH DISCONTINUITIES AT THE APEX

C. H. Chen, J. C. M. Yu, W. A. Shaw

ABSTRACT

Asymptotic solutions to Novozhilov's equations of shells of revolution are derived for axisymmetric and first harmonic loadings. The solutions obtained are valid throughout the shallow and nonshallow regions.

Stresses in dome-shaped shells of revolution with a discontinuity in the form of a circular hole; or a circular rigid insert; or a nozzle, at the apex have been investigated. Numerical results are obtained for spheres, ellipsoids, and paraboloids, containing a discontinuity under an internal pressure and a moment. Good correlation between theoretical and experimental stresses is obtained for the spherical shell. Curves depicting stress distributions are given. The influence of three types of discontinuity on the stresses of the shells is also investigated.

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LIST OF SYMBOLS

Upper-case Letters

A_1, A_2	Lame' parameters.
D	Extensional stiffness, $Eh/(1-\mu^2)$
E	Young's modulus.
$G(\ )$	Differential operator defined by Eq.(2-26).
$G_1(\ )$	Differential operator defined by Eq.(2-29).
H_1	$= A_1(1 + z/R_1)$
H_2	$= A_2(1 + z/R_2)$
I_0, I_1	Modified Bessel functions of the first kind of order zero and one, respectively.
J_0, J_1	Bessel functions of the first kind of order zero and one, respectively.
K	Bending rigidity, $Eh^3/(12(1-\mu^2))$.
K_0, K_1	Modified Bessel functions of the second kind of order zero and one, respectively.
$M_1, M_2; M_{12}, M_{21}$	Bending moments and twisting moments defined by Eqs.(A-13).
M	$= M_1 + M_2$, or applied external moment.
N_1, N_2	Transverse shears defined by Eqs.(A-13).
R_1, R_2	Principal radii of curvature.
R_o	Minimum radius of curvature of a shell of revolution.
R^*	Radius of curvature at the apex of a shell of revolution.
S	$= T_{12} - M_{21}/R_2 = T_{21} - M_{12}/R_1$
$\bar{S}$	A complex force

T_1, T_2, T_{12}, T_{21}	Normal forces and in plane shearing forces defined by Eqs.(A-13).
$\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}$	Complex forces defined by Eqs.(A-26).
T	$= T_1 + T_2$
T_1^*, T_2^*	Membrane direct forces.
$\bar{U}$, $\bar{V}$	Complex auxiliary functions defined by Eqs.(2-24).
$\bar{W}$	Complex function defined by Eq.(2-30).
X_2	Rotation about the line $\alpha_1 =$ constant.
Y_0, Y_1	Bessel functions of the second kind of order zero and one, respectively.

Lower-case Letters

a, ℓ	Major and minor semiaxes of an ellipsoid.
b^2	$= \sqrt{3(1-\mu^2)}\; r_0/h^*$
b_1	$= b(1 + 1/(2b^2))$
b_2	$= b(1 - 1/(2b^2))$
c	$= h/\sqrt{12(1-u^2)}$
c^*	$= h^*/\sqrt{12(1-u^2)}$
$\vec{e}_1$, $\vec{e}_2$, $\vec{e}_n$	Unit vectors in the directions of α_1, α_2 and along the normal to the middle surface.
e_{11}, e_{22}, e_{12} e_{31}, e_{32}, e_{33}	Strain components.
h, h^*	Thicknesses of a shell of revolution and a cylindrical shell, respectively.
h_1	$= (R_0/R_2)^{3/4}(\eta/\lambda \sin\theta)^{\frac{1}{2}}$
ker, kei, ker_1, kei_1	Thompson functions of order zero and one.
p	Internal pressure.
$p(\theta)$	A function of θ.

q_1, q_2, q_n	Components of a surface loading in the directions of $\vec{e}_1, \vec{e}_2, \vec{e}_n$, respectively.
$q(\theta)$	A function of θ.
r_o	Radius of a circular cylindrical shell.
u, v, w	Displacements of the middle surface in the directions of $\vec{e}_1, \vec{e}_2, \vec{e}_n$, respectively.
u_z, v_z, w_z	Displacements of a point at a distance z from the middle surface, in the directions of $\vec{e}_1, \vec{e}_2, \vec{e}_n$, respectively.
$\bar{u}, \bar{v}, \bar{w}$	Complex displacements.
z	Distance along the normal from the middle surface.

<u>Greek Letters</u>

α_1, α_2	Coordinates of a middle surface.
θ, ϕ	Coordinates of the middle surface of a shell of revolution, see Eq.(2-2).
α, β	Coordinates of the middle surface of a cylindrical shell, see Eq.(2-41).
$\epsilon_1, \epsilon_2, \omega, \kappa_1, \kappa_2, \tau$	Deformation parameters of a middle surface, see Eqs.(A-6) and (A-9).
$\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{31}, \sigma_{32}, \sigma_{33}$	Stress components.
μ	Poisson ratio.
λ^2	$= \sqrt{12(1-\mu^2)}\, R_o/h$.
η	$= \int_0^\theta \frac{R_1}{\sqrt{R_2 R_o}}\, d\theta$
ϑ, ψ	Angles of rotation defined by Eqs.(A-3d).
$\Delta(\ )$	Differential operator defined by Eqs.(A-3a), (2-5), (2-7) and (2-43).

I. INTRODUCTION

Background

(A) General Theory

The basic equations of the linear theory of thin elastic shells have been well developed (1-5)*. These equations involve the deformation-displacement relations, the equations of equilibrium and relations between forces, moments and the deformation parameters. The derivation of these equations and methods for effecting solutions, especially the deduction by means of complex transformations to a fourth order system of differential equations, are briefly outlined in the Appendix.

(B) Shells of Revolution

The basic equations for shells of revolution can be deduced from those of the general theory of thin elastic shells by proper choice of a coordinate system and Lame' parameters. There are several methods of reduction of the basic equations to a system of equations from which solutions can be readily derived.

Axisymmetric Doformation

Reduction of the basic equations to a set of two equations which relate the rotation ϑ and the transverse shear N_1 was first obtained by Reissner (6) for spherical shells and generalized by Meissner (7) for shells of revolution. The procedure of this derivation is also given in reference (3). By similar procedure, Naghdi and DeSilva (8) extended the work of

* Numbers inside the brackets refer to the references.

Meissner and obtained two equations which, for uniform thickness and for some classes of variable thickness, can be combined to give a single complex differential equation. Novozhilov[2], from the approach of the complex transformation, obtained a single differential equation, which is valid only for shells of uniform thickness.

Exact solutions to these equations have been derived for some particular classes of shells of revolution, such as circular cylindrical shells, conical shells and spherical shells[2,3,5], of which the curvatures of the generators of the middle surfaces are either zero or a constant. When the curvatures of the generators are functions of position, as is the case in ellipsoidal, paraboloidal and other shells of revolution, the exact solution becomes prohibitively difficult. Development of approximate solutions such as asymptotic solutions is indispensable to the analysis.

The method of asymptotic integration has been widely applied to obtaining approximate solutions for shell equations, which for shells of revolution may be transformed into the form

$$\frac{d^2w}{d\theta^2} = \{\lambda p(\theta) + q(\theta)\} w \qquad (1\text{-}1)$$

where λ is a large parameter. The asymptotic character of the solutions of Eq.(1-1) as λ approaches infinity can take many different forms depending on the properties of $p(\theta)$ and $q(\theta)$. However, three cases are usually encountered in the equations of shells of revolution. The first case, the so-called classical type, is an asymptotic solution of Eq.(1-1) in which, on some interval $\theta_1 \leq \theta \leq \theta_2$, $p(\theta)$ and $q(\theta)$ are both bounded and $p(\theta)$ is also bounded from zero. The second case is an asymptotic solution of Eq.(1-1) containing a turning point. In this case, $p(\theta)$ vanishes at a point θ_o within the interval $\theta_1 \leq \theta \leq \theta_2$; such a point is called a turning point. The third case is an asymptotic solution of Eq.(1-1) containing a singular point. In such a case, there exists a point θ_o in the interval $\theta_1 \leq \theta \leq \theta_2$ at which $q(\theta)$ may have a pole of first or second order and $p(\theta)$ contains as a factor $(\theta - \theta_o)^a$ where a is a real nonnegative constant, and $p(\theta)$ and $q(\theta)$ are both bounded in the rest of the interval. The solutions of these three cases have been investigated extensively by Langer [9] and Olver [10, 11]. The first case occurs in the differential equation for shells of revolution with two open edges where the region of interest lies in the nonshallow region (large values of θ). The second case occurs in the differential equation for toroidal shells, and the third case is encountered in the differential equation for dome-shaped shells of revolution where the region of interest lies in the shallow region (small values of θ) including a singular point at the apex.

Asymptotic integration of the third case has been applied to the investigation of ellipsoidal, paraboloidal and other dome-shaped shells of revolution. Naghdi and DeSilva [12] applied this method to the study of deformations of ellipsoidal shells of revolution of uniform thickness under axisymmetric loading. Solutions valid in the shallow region were obtained in terms of Kelvin functions. Clark and Reissner [13] obtained the solution based on the bending theory for complete ellipsoidal shells of revolution subjected to internal pressure by the use of small-parameter expansion. Deformation of paraboloidal shells of uniform thickness subjected to a load uniformly distributed over a small region about the apex and clamped at the open edge was studied by DeSilva and Arbor [14]. Study of dome-shaped shells of revolution subjected to axisymmetric loading was made by Baker and Cline [15], and Steele and Hartung [16].

Application of the first case of asymptotic solution which is valid only in the nonshallow region was made by Novozhilov in the study of nonshallow shells of revolution under axisymmetric loads.

Nonsymmetric Deformation

There are three basic procedures in reducing the basic equations of shells of revolution subjected to arbitrary loads. In the first of these, the basic equations are reduced to three differential equations which relate the displacements u, v and w. This procedure was employed by Vlasov [1] and Donnell [17] in deriving the governing equations for circular cylindrical shells. Stelle [18] also used the same procedure for reduction of the basic equations of shells of revolution under nonsymmetric edge loads, and obtained, by neglecting the transverse shear terms in the first two equations of equilibrium, three differential equations which relate the displacements u, v and w. The membrane and bending solutions that are valid throughout the shallow and nonshallow regions were obtained by means of asymptotic integration. In the second, a stress function is introduced and the governing equations are reduced to two differential equations which relate the stress function F and the normal displacement w. Reissner [19] employed this method and obtained a set of two equations for small deformation of shallow spherical shells. In the third, the basic equations are reduced by means of complex transformation developed by Novozhilov [2] to two differential equation which relate to two complex functions. The procedure of derivation is given in the Appendix and in Chapter II. Asymptotic solutions to Novozhilov's equations valid in in the nonshallow region were derived by Schile [20] for external loads including (a) sinusoidal loading and (b) higher harmonic load distribution. No literature on solutions to Novozhilov's equations that are valid in the shallow region is known to the author.

(C) Application

Numerous investigations have been made on the application of the solutions mentioned previously to engineering structures. Attention here will be limited to dome-shaped shells of revolution having a discontinuity of the types: (a) a hole; (b) a rigid insert; (c) a nozzle attachment.

The problem of the stress distribution around holes in shells has been investigated by a number of workers. Hemispherical shells with a circular opening at the vertex subjected to axisymmetric self-equilibrating forces were studied by Galletly (21) . An elliptical opening in a spherical shell under internal pressure was investigated by Leckie and Payne (22) who expressed the equation in elliptical coordinates and obtained the solution in terms of Mathieu functions. For a more general case, Savin (23) investigated the stress distribution around an arbitrary hole with smooth contour in thin shells and obtained solutions to the shell equations which had been transformed by the use of conformal mapping into a coordinate system such that along the contour of the hole one of the coordinates is constant. The general method was described and applied to a cylindrical shell with a circular hole and to a spherical shell with either a circular or an elliptical hole. Further studies (24) were made of a spherical shell under internal pressure weakened by an elliptical hole, square and triangular holes with rounded corners.

Spherical shells with a circular rigid insert have been considered by Bijlaard (25) and, with an elliptical rigid insert, by Leckie and Payne

[22], and Foster [26].

The effect of local loading on spherical shells in which external loads are transmitted from a nozzle radially attached to the shells has been investigated extensively by Bijlaard [27] and Leckie and Payne [28]. Studies of the case in which the nozzle is obliquely attached to a spherical shell were made by Johnson [29] and Yu, Chen and Shaw [30].

All the investigations mentioned here are restricted to shallow shells with a hole or a rigid insert, the size of which is small compared to the radii of curvatures of the shells so that shallow shell equations hold for the problems under consideration. The case of a nozzle attachment has been also limited to nozzle-to-spherical shells.

As far as the author knows, little attention has been given to systematic studies of stress distribution in nonshallow shells of dome shape around a discontinuity of a size which is not necessarily small compared to the radius of curvature of the shells.

Statement of the Problems

Investigation of the following problems is suggested upon the review made in the preceding sections:

(1) Derivation of solutions to Novozhilov's equations for shells of revolution which are valid in the shallow and nonshallow regions under axisymmetric and first harmonic loads. This extends the work of Novozhilov who derived the equations and obtained solutions valid only in the non-shallow region. The development here also differs from that of Steele in that it does not neglect the transverse shear terms in the first two equations of equilibrium.

(2) Application of the solution derived in (1) to the study of the stresses in the vicinity of a discontinuity at the apex of dome-shaped shells of revolution under external loads including (a) internal pressure and (b) a couple applied to the discontinuity. The discontinuity is in the form of a circular hole, or a circular rigid insert, or a nozzle. No restriction is placed on the size of the discontinuity in relation to the radius of the shells. Application to discontinuities is embedded in a uniform treatment and includes discontinities in geometries on which little information is available.

(3) Analysis of the influence of the different types of discontinuity on the stresses of the shells.

For systematic study of these problems, the procedures for the reduction of the basic shell equations to a fourth order system of three equations are briefly outlined in the Appendix. Further reductions to a second order differential equation in terms of a complex force are derived in Chapter II. Solutions to this equation valid in the shallow region are derived in Chapter III using the method of asymptotic integration.

Applications of these solutions to the study of problem (2) are investigated in Chapter IV in which the boundary conditions for each of the appropriate cases are derived. The study of problem (3) is given in Chapter V.

II. GOVERNIG DIFFERENTIAL EQUATIONS

A second order differential equation governing the deformation of dome-shaped shells of revolution and of circular cylindrical shells will be deduced from the system of differential equations (A-27) for both axisymmetric and first harmonic loads.

Shells of Revolution

The coordinate system chosen for shells of revolution will be θ and ϕ, which determine the position of a point on the middle surface (Fig. 2-1a). Let R_1 be the radius of curvature of the meridian (ϕ = constant) and R_2 be the length along the normal to the middle surface between the axis of revolution and the middle surface. R_2 is sometimes referred to as the second radius of curvature. Thus, the first fundamental form of the surface is (Fig. 2-1b)

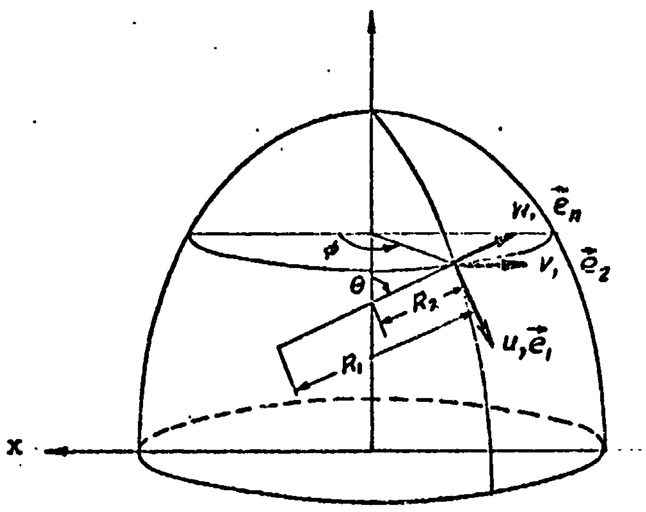
$$(ds)^2 = (R_1 d\theta)^2 + (R_2 \sin\theta \, d\phi)^2 \qquad (2\text{-}1)$$

By comparison of Eq.(2-1) with Eq.(A-1) for shells of arbitrary shape one sees that

$$\begin{aligned} \alpha_1 &= \theta, & \alpha_2 &= \phi \\ A_1 &= R_1 & A_2 &= R_2 \sin\theta \end{aligned} \qquad (2\text{-}2)$$

The last two of the conditions of Gauss-Codazzi, Eq.(A-2), are identically satisfied, since R_1 and R_2 are functions of θ only. The first condition reduces to

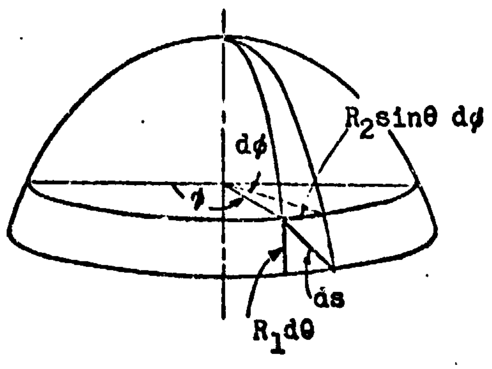
$$(R_2 \sin\theta)' = R_1 \cos\theta \qquad (2\text{-}3)$$



R_1: radius of curvature of the line ϕ = constant.

R_2: length between the axis of revolution and the middle surface.

(a)



(b)

Fig. 2-1: Coordinate system of a shell of revolution where the prime indicates differentiation with respect to θ. By use of these relations, Eqs.(A-27) and (A-29) as given in the Appendix are expresses by

$$\frac{1}{R_1}\frac{\partial \bar{T}_1}{\partial \theta} - \frac{\cot\theta}{R_2}(\bar{T}_1 - \bar{T}_2) + \frac{1}{R_2\sin\theta}\frac{\partial \bar{S}}{\partial \phi} + i\frac{c}{R_2}\frac{\partial \bar{T}}{\partial \theta} = -q_1$$

$$\frac{1}{R_1}\frac{\partial \bar{S}}{\partial \theta} + 2\frac{\cot\theta}{R_2}\bar{S} + \frac{1}{R_2\sin\theta}\frac{\partial \bar{T}_2}{\partial \phi} + \frac{c}{R_2^2\sin\theta}\frac{\partial \bar{T}}{\partial \phi} = -q_2 \qquad (2\text{-}4a)$$

$$\frac{\bar{T}_1}{R_1} + \frac{\bar{T}_2}{R_2} - ic\,\Delta(\bar{T}) = q_n$$

and

$$R_1\bar{\epsilon}_1 = \frac{\partial \bar{u}}{\partial \theta} + \bar{w} = \frac{R_1}{Eh}(\bar{T}_1 - \mu\bar{T}_2)$$

$$R_2\bar{\epsilon}_2 = \frac{1}{\sin\theta}\frac{\partial \bar{v}}{\partial \phi} + \bar{u}\cot\theta + \bar{w} = \frac{R_2}{Eh}(\bar{T}_2 - \mu\bar{T}_1)$$

$$\bar{\kappa}_1 = -\frac{1}{R_1}\frac{\partial}{\partial \theta}\left[\frac{1}{R_1}\left(\frac{\partial \bar{w}}{\partial \theta} - \bar{u}\right)\right] = \frac{i}{cEh}(\bar{T}_2 - T_2^*)$$

$$\bar{\kappa}_2 = -\frac{1}{R_2^2\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \bar{w}}{\partial \phi} - \bar{v}\right) - \frac{\cot\theta}{R_1R_2}\left(\frac{\partial \bar{w}}{\partial \theta} - \bar{u}\right) = \frac{i}{cEh}(\bar{T}_1 - T_1^*) \qquad (2\text{-}4b)$$

$$R_2\bar{\omega} = \frac{R_2}{R_1}\frac{\partial \bar{v}}{\partial \theta} - \bar{v}\cot\theta + \frac{1}{\sin\theta}\frac{\partial \bar{u}}{\partial \phi} = \frac{2(1+\mu)R_2}{Eh}\bar{S}$$

$$\bar{\tau} = -\frac{1}{R_1}\frac{\partial}{\partial \theta}\left[\frac{1}{R_2}\left(\frac{1}{\sin\theta}\frac{\partial \bar{w}}{\partial \phi} - \bar{v}\right)\right] + \frac{1}{R_1R_2\sin\theta}\left(\frac{\partial \bar{u}}{\partial \phi} - \bar{v}\cos\theta\right) = -\frac{i}{cEh}(\bar{S} - S^*)$$

where $\bar{T}$, $\bar{T}_1$, $\bar{T}_2$, and $\bar{S}$ are complex forces defined by Eqs.(A-26); q_1, q_2, and q_n are components of surface loading in the directions of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_n$, respectively; $\bar{\epsilon}_1$, $\bar{\epsilon}_2$ $\bar{\omega}$, $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\tau}$ are the complex deformation parameters of the middle surface and $\bar{u}$, $\bar{v}$, and $\bar{w}$ are the complex displacements; and

$$\Delta(\;) = \frac{1}{R_1^2}\frac{\partial^2(\;)}{\partial\theta^2} + \left(\frac{\cot\theta}{R_1R_2} - \frac{1}{R_1^3}\frac{dR_1}{d\theta}\right)\frac{\partial(\;)}{\partial\theta} + \frac{1}{R_2^2\sin^2\theta}\frac{\partial^2(\;)}{\partial\phi^2} \qquad (2\text{-}5)$$

(A) <u>Axisymmetric Deformation</u>

Because of the assumed symmetry all quantities are independent of ϕ. If, in addition,

$$q_1 = q_2 = 0 \quad \text{and} \quad q_n = p = \text{constant}$$

then, Eqs.(2-4a) reduce to the form

$$\frac{1}{R_1}\bar{T}_1' + \frac{\cot\theta}{R_2}(\bar{T}_1 - \bar{T}_2) + i\frac{c}{R_1^2}\bar{T}' = 0$$

$$\frac{1}{R_1}\bar{S}' + 2\frac{\cot\theta}{R_2}\bar{S} = 0 \tag{2-6}$$

$$\frac{\bar{T}_1}{R_1} + \frac{\bar{T}_2}{R_2} - ic\Delta(\bar{T}) = p$$

where

$$\Delta(\ ) = \frac{1}{R_1^2}(\ )'' + \left(\frac{\cot\theta}{R_1 R_2} - \frac{1}{R_1^3}R_1'\right)(\ )' \tag{2-7}$$

and the prime indicates differentiation with respect to θ. By use of the first Gauss-Codazzi condition, Eq.(2-3), the second of Eqs.(2-6) may be written in the form

$$\frac{d\bar{S}}{\bar{S}} + 2\frac{d(R_2\sin\theta)}{R_2\sin\theta} = 0$$

which, upon integration, has the solution

$$\bar{S} = \frac{\bar{C}_1}{R_2^2\sin^2\theta} \tag{2-8}$$

where $\bar{C}_1$ is a complex constant of integration. Since , due to symmetry, $\bar{S}$ vanishes on an edge of θ = constant $\bar{C}_1$ must be set to zero.

Next, the solution for the auxiliary functions $\bar{T}_1$ and $\bar{T}_2$ (see Eqs. (A-26) in the Appendix) will be obtained from the first and the third of Eqs.(2-6). By use of Eq.(2-3) the first of Eqs.(2-6) may be written in the form

$$\frac{1}{R_1 R_2\sin\theta}(R_2\sin\theta\,\bar{T}_1)' - \frac{\cot\theta}{R_2}\bar{T}_2 + i\frac{c}{R_1^2}\bar{T}' = 0 \tag{2-9}$$

Eliminating $\bar{T}_2$ from Eq.(2-9) by multiplying the third of Eqs.(2-6) by $\cot\theta$ and adding it to Eq.(2-9), and multiplying the result by $R_1R_2\sin\theta$, there results

$$(R_2\sin\theta\,\bar{T}_1)' + R_2\cos\theta\,\bar{T}_1 + icR_1R_2\sin\theta\left(\frac{1}{R_1^2}\bar{T}'' - \cot\theta\,\Delta(\bar{T})\right) = R_1R_2\cos\theta\,p \tag{2-10}$$

The first two terms of Eq.(2-10) may be combined to give

$$(R_2\sin\theta\,\bar{T}_1)' + R_2\cos\theta\,\bar{T}_1 = \frac{1}{\sin\theta}(R_2\sin^2\theta\,\bar{T}_1)'$$

and the third term of Eq.(2-10) can be shown to be equal to

$$-\frac{ic}{\sin\theta}\left(\frac{R_2}{R_1}\sin\theta\cos\theta\,\bar{T}'\right)'$$

Thus, Eq.(2-10) reduces to

$$(R_2\sin^2\theta\,\bar{T}_1)' - ic\left(\frac{R_2}{R_1}\sin\theta\cos\theta\,\bar{T}'\right)' = pR_1R_2\sin\theta\cos\theta \tag{2-11}$$

Now, introduce a function $\bar{U}$ defined by

$$\bar{U} = R_2\sin^2\theta\,\bar{T}_1 - ic\frac{R_2}{R_1}\sin\theta\cos\theta\,\bar{T}' \tag{2-12}$$

Eq.(2-11) becomes

$$\bar{U}' = pR_1R_2\sin\theta\cos\theta$$

It follows upon integration that

$$\bar{U} = \bar{C}_2 + \frac{p}{2}R_2^2\sin^2\theta \tag{2-13}$$

The fourth of Eqs.(A-26), i.e.,

$$\bar{T} = \bar{T}_1 + \bar{T}_2 \tag{2-14}$$

can be substituted into the third of Eqs.(2-6) to eliminate $\bar{T}_2$. Also, $\bar{T}_1$ can be eliminated by using Eq.(2-12). The final result of this manipulation is a second order differential equation on $\bar{T}$ which can be written as

$$\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{\bar{U}}{R_2\sin^2\theta}+ic\frac{1}{R_1}\cot\theta\,\bar{T}'\right)+\frac{\bar{T}}{R_2}-ic\Delta(\bar{T})=p$$

or after rearrangement

$$\bar{T}''+\left[\left(2\frac{R_1}{R_2}-1\right)\cot\theta-\frac{R_1'}{R_1}\right]\bar{T}'+i\frac{R_1^2}{R_2c}\bar{T}=i\frac{R_1^2}{R_2c}F(\theta) \qquad (2\text{-}15)$$

where

$$F(\theta)=pR_2-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{\bar{U}}{\sin^2\theta} \qquad (2\text{-}16)$$

Thus, the axisymmetric deformation of shells of revolution under internal pressure reduces to the solution of the second order differential equation (2-15). Eqs.(2-12) and (2-14) can be written as

$$\bar{T}_1=i\frac{c}{R_1}\cot\theta\,\bar{T}'+\frac{\bar{U}}{R_2\sin^2\theta} \qquad (2\text{-}17)$$

$$\bar{T} = \bar{T} - \bar{T} \ldots$$

$$\frac{1}{R_1}(\bar{U}' + \bar{w}) = \frac{1}{Eh}(\bar{T}_1 - \mu \bar{T}_2)$$

$$\frac{1}{R_2}(\bar{u}\cot\theta + \bar{w}) = \frac{1}{Eh}(\bar{T}_2 - \mu \bar{T}_1)$$

$$-\frac{1}{R_1}\left[\frac{1}{R_1}(\bar{w}' - \bar{u}')\right]' = \frac{i}{cEh}(\bar{T}_2 - T_2^*) \qquad (2\text{-}19)$$

$$-\frac{\cot\theta}{R_1 R_2}(\bar{w}' - \bar{u}) = \frac{i}{cEh}(\bar{T}_1 - T_1^*)$$

The last one of Eqs.(2-19) may be written in the form

$$-\frac{1}{R_1}(\bar{w}' - \bar{u}) = \frac{iR_2}{cEh}\tan\theta\,(\bar{T}_1 - T_1^*) \qquad (2\text{-}20)$$

Comparing this equation with the third of Eqs.(2-19), one observes that these two equations are compatible only if

$$\frac{1}{R_1}\left[R_2\tan\theta\,(\bar{T}_1 - T_1^*)\right]' \equiv \bar{T}_2 - T_2^* \qquad (2\text{-}21)$$

is identically satisfied. Eq.(2-21), upon substitution for $\bar{T}_1$ and $\bar{T}_2$ by their expressions from Eqs.(2-17) and with the consideration of Eq.(2-15), becomes

$$\bar{T} - F - \frac{ic}{R_1}\cot\theta\,\bar{T}' + \frac{1}{R_1}\left[R_2\tan\theta\left(\frac{\bar{U}}{R_2\sin^2\theta} - T_1^*\right)\right]' \equiv \bar{T} - \frac{\bar{U}}{R_2\sin^2\theta} - T_2^* - \frac{ic}{R_1}\cot\theta\,\bar{T}'$$

This equation is satisfied if

$$T_1^* = \frac{\bar{U}}{R_2\sin^2\theta}$$

and

$$F = T_2^* + \frac{\bar{U}}{R_2\sin^2\theta} = T_2^* + T_1^* \qquad (2\text{-}22)$$

Comparison of the first of Eqs.(2-22) with the third of Eqs.(2-18) yields

$$\bar{U} = U$$

from which it follows that

$$\bar{C}_2 = C_2 = \text{real constant}$$

(B) Non-symmetric Deformation - Edge Loads only

In that which follows, equations will be developed for the non-symmetric deformation of shells of revolution due to edge effects only. In addition, deduction to a single second-order differential equation will be obtained for the special case where the resultant edge loads consist only of moment.

Since the surface loads q_1, q_2, and q_n are zero, Eqs.(2-4) become

$$\frac{1}{R_1}\frac{\partial \bar{T}_1}{\partial \theta} + \frac{\cot\theta}{R_2}(\bar{T}_1 - \bar{T}_2) + \frac{1}{R_2 \sin\theta}\frac{\partial \bar{S}}{\partial \phi} + i\frac{c}{R_1^2}\frac{\partial \bar{T}}{\partial \theta} = 0$$

$$\frac{1}{R_1}\frac{\partial \bar{S}}{\partial \theta} + 2\frac{\cot\theta}{R_2}\bar{S} + \frac{1}{R_2 \sin\theta}\frac{\partial \bar{T}_2}{\partial \phi} + i\frac{c}{R_2^2 \sin\theta}\frac{\partial \bar{T}}{\partial \phi} = 0 \qquad (2\text{-}23)$$

$$\frac{\bar{T}_1}{R_1} + \frac{\bar{T}_2}{R_2} - ic\,\Delta(\bar{T}) = 0$$

Following the procedure of reduction to a single second-order differential equation for symmetric deformation, one may introduce, on the basis of Eqs.(2-8) and (2-12), two auxiliary functions

$$\bar{U} = R_2 \sin^2\theta\, \bar{T}_1 - ic\frac{R_2}{R_1}\sin\theta\cos\theta\, \bar{T}'$$

$$\bar{V} = R_2^2 \sin^2\theta\, \bar{S} \qquad (2\text{-}24)$$

Eqs.(2-23), through certain manipulations with the help of Eqs.(2-3) and (2-24), may be reduced to the following system of three partial differential equations[2] of which the first two involve two unknowns $\bar{U}$ and $\bar{T}$.

$$G(\bar{U}) - \left[1 - ic\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\frac{\partial^2 \bar{T}}{\partial \phi^2} = 0$$

$$-icG(\bar{T}) + \bar{T} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\bar{U} = 0 \qquad (2\text{-}25)$$

$$\frac{\partial \bar{V}}{\partial \phi} + \frac{R_2^2 \sin\theta}{R_1}\frac{\partial \bar{U}}{\partial \theta} - icR_2\cos\theta\frac{\partial^2 \bar{T}}{\partial \phi^2} = 0$$

where

$$G(\ ) = \frac{1}{R_1 R_2 \sin\theta}\frac{\partial}{\partial\theta}\left[\frac{R_2^2 \sin\theta}{R_1}\frac{\partial(\ )}{\partial\theta}\right] + \frac{1}{R_2 \sin^2\theta}\frac{\partial^2(\ )}{\partial\phi^2} \tag{2-26}$$

Thus, the analysis of shells of revolution subject to any type of edge loading has been reduced to the solution of the system of Eqs.(2-25). However, the following will be restricted to the case where the resultant edge load at $\theta = \theta_0$ (near the apex) of a shell of revolution is equivalent to a moment. For this particular case the auxiliary functions $\bar{T}$, $\bar{U}$ and $\bar{V}$ may be expressed as

$$\begin{aligned} \bar{T} &= \bar{T}^\circ(\theta)\cos\phi \\ \bar{U} &= \bar{U}^\circ(\theta)\cos\phi \\ \bar{V} &= \bar{V}^\circ(\theta)\sin\phi \end{aligned} \tag{2-27}$$

Substitution of Eqs.(2-27) into Eqs.(2-25) yields

$$\begin{aligned} G_1(\bar{U}^\circ) + \left[1 - ic\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\right]\bar{T}^\circ &= 0 \\ -icG_1(\bar{T}^\circ) + \bar{T}^\circ + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\bar{U}^\circ &= 0 \\ \bar{V}^\circ + \frac{R_2^2\sin\theta}{R_1}\bar{U}^{\circ\prime} + icR_2\cos\theta\,\bar{T}^\circ &= 0 \end{aligned} \tag{2-28}$$

where

$$G_1(\ ) = \frac{1}{R_1R_2\sin\theta}\left[\frac{R_2^2\sin\theta}{R_1}(\ )'\right]' - \frac{1}{R_2\sin^2\theta}(\ ) \tag{2-29}$$

The first two of Eqs.(2-28) may be uncoupled by subtracting the second equation from the first and then introducing the new function

$$\bar{W} = \bar{U}^\circ + ic\,\bar{T}^\circ \tag{2-30}$$

into the result. In this way there results

$$G_1(\bar{W}) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta}\bar{W} = 0 \tag{2-31}$$

Expanding this equation with the help of Eq.(2-29) one arrives at

$$\left(\frac{R_2^2 \sin\theta}{R_1} \bar{W}' \right)' - \frac{R_1}{\sin\theta} \bar{W} = 0 \tag{2-32}$$

It may be verified that one of the solutions of Eq.(2-32) is

$$\bar{W} = \frac{1}{R_2 \sin\theta}$$

The second solution may be obtained by assuming

$$\bar{W} = \frac{A}{R_2 \sin\theta} \tag{2-33}$$

where A is a function of θ. Eq.(2-32) upon substitution for $\bar{W}$ from Eq. (2-33) reduces to the form

$$\left(\frac{1}{R_1 \sin\theta} A' \right)' = 0$$

from which it follows

$$A = \bar{B}_1 + \bar{B}_2 \int R_1 \sin\theta \, d\theta$$

Thus

$$\bar{W} = \frac{\bar{B}_1}{R_2 \sin\theta} + \frac{\bar{B}_2}{R_2 \sin\theta} \int R_1 \sin\theta \, d\theta \tag{2-34}$$

Eliminating $\bar{U}^0$ in the second of Eqs.(2-28) by its expression from Eq.(2-30), one arrives at the following differential equation in a single unknown $\bar{T}^0$

$$G_1(\bar{T}^0) + \frac{i}{c} \bar{T}^0 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \frac{1}{\sin^2\theta} \bar{T}^0 = -\frac{i}{c}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \frac{\bar{W}}{\sin^2\theta}$$

which may be written in expanded form

$$\begin{aligned} \bar{T}^{0''} + \left[\left(2\frac{R_2}{R_1} - 1\right)\cot\theta - \frac{R_1'}{R_1} \right] \bar{T}^{0'} + \frac{R_1}{R_2}\left(1 - 2\frac{R_1}{R_2}\right) \frac{1}{\sin^2\theta} \bar{T}^0 \\ + i \frac{R_1^2}{R_2 c} \bar{T}^0 = i \frac{R_1^2}{R_2 c} F_1(\theta) \end{aligned} \tag{2-35}$$

where

$$F_1(\theta) = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \frac{1}{\sin^2\theta} \bar{W} \tag{2-36}$$

Once $\bar{U}^0$ and $\bar{T}^0$ have been found the auxiliary function $\bar{V}^0$ can be found from the third of Eqs.(2-28). It is noted that, by letting $c = 0$ in the third of Eqs.(2-23), the first of Eqs.(2-24) and Eq.(2-30), one arrives at the membrane theory solution.

$$\bar{U}^{\circ} = W, \qquad T_1^{\circ *} = \frac{W}{R_2 \sin^2\theta}$$
$$T_2^{\circ *} = -\frac{W}{R_1 \sin^2\theta}, \qquad S^{\circ *} = -\frac{W'}{R_1 \sin\theta} \tag{2-37}$$

And Eq.(2-36) becomes

$$F_1(\theta) = T_1^{\circ *} + T_2^{\circ *}$$

There remains the evaluation of the displacements which for this particular problem are assumed to be

$$\bar{U} = \bar{u}_1(\theta)\cos\phi$$
$$\bar{V} = \bar{V}_1(\theta)\sin\phi \tag{2-38}$$
$$\bar{W} = \bar{w}_1(\theta)\cos\phi$$

On substituting these expressions into Eqs.(2-4), a system of equations relating the complex displacements $\bar{u}_1$, $\bar{v}_1$, $\bar{w}_1$ to the complex forces is obtained

$$\bar{u}_1' + \bar{w}_1 = \frac{R_1}{Eh}(\bar{T}_1^{\circ} - \mu\bar{T}_2^{\circ})$$
$$\frac{1}{\sin\theta}\bar{V}_1 + \bar{u}_1\cot\theta + \bar{w}_1 = \frac{R_2}{Eh}(\bar{T}_2^{\circ} - \mu\bar{T}_1^{\circ})$$
$$\frac{R_2}{R_1}\bar{V}_1' - \bar{V}_1\cot\theta - \frac{1}{\sin\theta}\bar{u}_1 = \frac{2(1+\mu)R_2}{Eh}\bar{S}^{\circ}$$
$$-\frac{1}{R_1}\left(\frac{\bar{w}_1' - \bar{u}_1}{R_1}\right)' = \frac{i}{cEh}(\bar{T}_2^{\circ} - T_2^{\circ *}) \tag{2-39}$$
$$\frac{1}{R_2^2\sin\theta}\left(\frac{1}{\sin\theta}\bar{w}_1 + \bar{V}_1\right) - \frac{\cot\theta}{R_1R_2}(\bar{w}_1' - \bar{u}_1) = \frac{i}{cEh}(\bar{T}_1^{\circ} - T_1^{\circ *})$$
$$\frac{1}{R_1}\left[\frac{1}{R_1}\left(\frac{\bar{w}_1}{\sin\theta} + \bar{V}_1\right)\right]' - \frac{1}{R_1R_2\sin\theta}(\bar{u}_1 + \bar{V}_1\cos\theta) = -\frac{i}{cEh}(\bar{S}^{\circ} - S^{\circ *})$$

This completes the reduction of the basic equations of the general theory to the governing equations for shells of revolution pertaining to investigation stated in Chapter I.

Circular Cylindrical Shells

The coordinates identifying the position of points on the middle surface are α and β (Fig.2-2) and r_0 is the radius of a circular cross section. Thus, the first fundamental form of the surface is

$$(ds)^2 = (r_0 d\alpha)^2 + (r_0 d\beta)^2 \tag{2-40}$$

from this one may verify that

$$\begin{aligned} \alpha_1 &= \alpha, \qquad \alpha_2 = \beta \\ A_1 &= A_2 = r_0 \end{aligned} \tag{2-41}$$

Hence, the differential equations (A-27) for complex forces may be written in the form

$$\begin{aligned} \frac{\partial \bar{\bar{T}}_1}{\partial \alpha} + \frac{\partial \bar{\bar{S}}}{\partial \beta} &= -q_1 r_0 \\ \frac{\partial \bar{\bar{S}}}{\partial \alpha} + \frac{\partial \bar{\bar{T}}_2}{\partial \beta} + \frac{i}{2b^2}\frac{\partial \bar{\bar{T}}}{\partial \beta} &= -q_2 r_0 \\ \bar{\bar{T}}_2 - \frac{i}{2b^2}\Delta(\bar{\bar{T}}) &= q_n r_0 \end{aligned} \tag{2-42}$$

where

$$\begin{aligned} 4b^4 &= 12(1-\mu^2)\left(\frac{r_0}{h}\right)^2 \\ \Delta(\ ) &= \frac{\partial^2(\ )}{\partial \alpha^2} + \frac{\partial^2(\ )}{\partial \beta^2} \\ \bar{\bar{T}} &= \bar{\bar{T}}_1 + \bar{\bar{T}}_2 \end{aligned} \tag{2-43}$$

The complex forces in these equations are related to the forces and

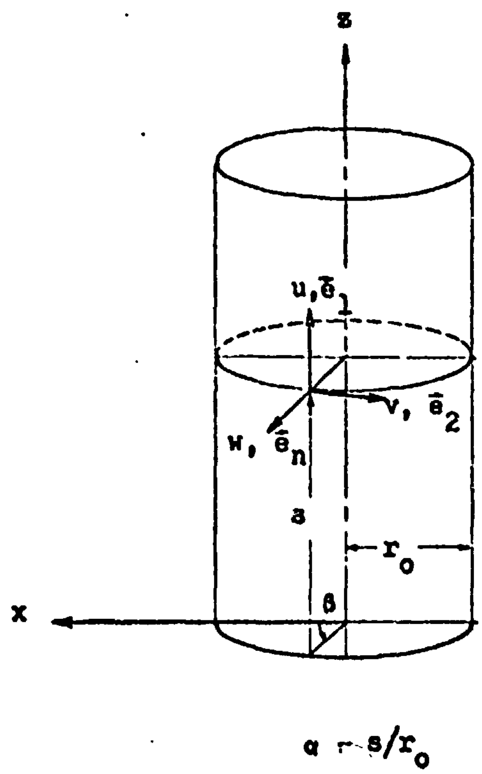


Fig. 2-2: Cylindrical Coordinate system

moments by the following expressions

$$
\begin{aligned}
\bar{T}_1 &= T_1 - i\frac{2b^2}{r_0}\frac{M_2-\mu M_1}{1-\mu^2} \\
\bar{T}_2 &= T_2 - i\frac{2b^2}{r_0}\frac{M_1-\mu M_2}{1-\mu^2} \qquad (2\text{-}44) \\
\bar{S} &= S + i\frac{2b^2}{r_0}\frac{H}{1-\mu}
\end{aligned}
$$

Also q_1, q_2 and q_n are the components of the surface loading in the directions of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_n$, respectively.

The relations (A-29) between complex displacements and complex forces for circular cylindrical shells (taking into account Eq.(2-41) and $R_1 = \infty$, $R_2 = r_0$) may be expressed in the form

$$
\begin{aligned}
\bar{\epsilon}_1 &= \frac{1}{r_0}\left(\frac{\partial \bar{U}}{\partial \alpha}\right) = \frac{1}{Eh}\left(\bar{T}_1 - \mu \bar{T}_2\right) \\
\bar{\epsilon}_2 &= \frac{1}{r_0}\left(\frac{\partial \bar{V}}{\partial \beta} + \bar{W}\right) = \frac{1}{Eh}\left(\bar{T}_2 - \mu \bar{T}_1\right) \\
\bar{\omega} &= \frac{1}{r_0}\left(\frac{\partial \bar{U}}{\partial \beta} + \frac{\partial \bar{V}}{\partial \alpha}\right) = \frac{2(1+\mu)}{Eh}\bar{S} \qquad (2\text{-}45) \\
\bar{\kappa}_1 &= -\frac{1}{r_0^2}\frac{\partial^2 \bar{W}}{\partial \alpha^2} = i\frac{2b^2}{Ehr_0}\left(\bar{T}_2 - T_2^*\right) \\
\bar{\kappa}_2 &= -\frac{1}{r_0^2}\frac{\partial}{\partial \beta}\left(\frac{\partial \bar{W}}{\partial \beta} - \bar{V}\right) = i\frac{2b^2}{Ehr_0}\left(\bar{T}_1 - T_1^*\right) \\
\bar{\tau} &= -\frac{1}{r_0^2}\frac{\partial}{\partial \alpha}\left(\frac{\partial \bar{W}}{\partial \beta} - \bar{V}\right) = -\frac{i2b^2}{Ehr_0}\left(\bar{S} - S^*\right)
\end{aligned}
$$

(A) <u>Axisymmetric Deformation</u>

On account of the assumed symmetry all quantities are independent of β, and $q_2 = 0$. Thus, Eqs.(2-42) reduce to

$$
\begin{aligned}
\bar{T}_1' &= -q_1 r_0 \\
\bar{S}' &= 0 \qquad (2\text{-}46) \\
\bar{T}_2 - \frac{i}{2b^2}\bar{T}^* &= q_n r_0
\end{aligned}
$$

The prime indicates the derivative with respect to α. From the second of

Eqs.(2-46) one obtains, in view of symmetry

$$\bar{S} = \bar{C}_1 = 0 \tag{2-47}$$

The first of Eqs.(2-46) gives

$$\bar{T}_1 = \bar{C}_2 - r_0 \int q_1 \, d\alpha \tag{2-48}$$

Eliminating $\bar{T}$ in the third of Eqs.(2-46) by its expression in terms of $\bar{T}_1$ and taking into consideration Eq.(2-48) one arrives at a second order differential equation for a single unknown $\bar{T}_2$

$$\bar{T}_2'' + i\,2b^2\,\bar{T}_2 = i\,2b^2\,q_n r_0 + r_0\,q_1' \tag{2-49}$$

The displacements can be obtained from Eqs.(2-45) which, for this case, reduce to the form

$$\begin{aligned} \frac{1}{r_0}\,\bar{U}' &= \frac{1}{Eh}\,(\bar{T}_1 - \mu\bar{T}_2) \\ \frac{1}{r_0}\,\bar{w} &= \frac{1}{Eh}\,(\bar{T}_2 - \mu\bar{T}_1) \end{aligned} \tag{2-50}$$

(B) <u>Non-symmetric Deformation</u>

Eliminating $\bar{S}$ from the first two of Eqs.(2-42), there results

$$\frac{\partial^2\bar{T}}{\partial\alpha^2} - \Delta(\bar{T}_2) - \frac{i}{2b^2}\,\frac{\partial^2\bar{T}}{\partial\beta^2} = r_0\left(\frac{\partial q_2}{\partial\beta} - \frac{\partial q_1}{\partial\alpha}\right) \tag{2-51}$$

Substitution in Eq.(2-51) for $\bar{T}_2$ by its expression from the third of Eqs. (2-42) yields a fourth order partial differential equation in a single unknown $\bar{T}$

$$\Delta\Delta(\bar{T}) + \frac{\partial^2\bar{T}}{\partial\beta^2} + i\,2b^2\,\frac{\partial^2\bar{T}}{\partial\alpha^2} = i\,2b^2 r_0\left[\frac{\partial q_2}{\partial\beta} - \frac{\partial q_1}{\partial\alpha} + \Delta(q_n)\right] \tag{2-52}$$

Thus, the analysis of the non-symmetric deformation of a circular cylindrical shell has been reduced to the solution of this equation. Once $\bar{T}$ has been obtained the complex forces may be found from the following equations:

$$\begin{aligned}
\bar{T}_2 &= r_0 q_n + \frac{i}{2b^2}\Delta(\bar{T}) \\
\bar{T}_1 &= \bar{T} - \bar{T}_2 = \bar{T} - \frac{i}{2b^2}\Delta(\bar{T}) - r_0 q_n \\
\frac{\partial \bar{S}}{\partial \alpha} &= -\frac{i}{2b^2}\left[\frac{\partial}{\partial \beta}\Delta(\bar{T}) + \frac{\partial \bar{T}}{\partial \beta}\right] - r_0\left[\frac{\partial q_n}{\partial \beta} + q_2\right] \\
\frac{\partial \bar{S}}{\partial \beta} &= -\frac{\partial \bar{T}}{\partial \alpha} + \frac{i}{2b^2}\frac{\partial}{\partial \alpha}\Delta(\bar{T}) + r_0\left(\frac{\partial q_n}{\partial \alpha} - q_1\right)
\end{aligned} \tag{2-53}$$

As was done in the non-symmetric deformation for shells of revolution the problem will be restricted to that of pure bending. For such a case

$$q_1 = q_2 = q_n = 0$$

and the complex forces can be assumed to be

$$\begin{aligned}
\bar{T} &= \bar{T}^\circ(\alpha)\cos\beta, & \bar{T}_1 &= \bar{T}_1^\circ(\alpha)\cos\beta \\
\bar{T}_2 &= \bar{T}_2^\circ(\alpha)\cos\beta, & \bar{S} &= \bar{S}^\circ(\alpha)\sin\beta
\end{aligned} \tag{2-54}$$

On substitution in Eq.(2-52) for $\bar{T}$ by its expression from the first of Eqs.(2-54), there results an ordinary differential equation for $\bar{T}^\circ$

$$\bar{T}^{\circ\prime\prime\prime\prime} + (i2b^2 - 2)\,\bar{T}^{\circ\prime\prime} = 0 \tag{2-55}$$

where the prime denotes differentiation with respect to α.

The complex displacements for the given case are assumed to be of the form

$$\bar{U} = \bar{u}_1(\alpha)\cos\beta, \quad \bar{V} = \bar{v}_1(\alpha)\sin\beta, \quad \bar{W} = w_1(\alpha)\cos\beta \tag{2-56}$$

On substituting these expressions into the first three of Eqs.(2-45) the following equations are obtained for the determination of the complex displacements $\bar{u}_1$, $\bar{v}_1$, and $\bar{w}_1$.

$$\begin{aligned}
\bar{u}_1' &= \frac{r_0}{Eh}(\bar{T}_1^\circ - \mu\bar{T}_2^\circ) \\
\bar{v}_1 + \bar{w}_1 &= \frac{r_0}{Eh}(\bar{T}_2^\circ - \mu\bar{T}_1^\circ) \\
-\bar{u}_1 + \bar{v}_1' &= \frac{2(1+\mu)r_0}{Eh}\bar{S}^\circ
\end{aligned} \tag{2-57}$$

III. SOLUTIONS OF THE GOVERNING DIFFERENTIAL EQUATIONS

In this Chapter solutions are obtained to the governing differential equations derived in Chapter II. In addition, formulas for forces, moments and displacements are listed in tables.

Shells of Revolution

(A) Axisymmetric Deformation - Internal Pressure

The analysis of shells of revolution under internal pressure has been reduced to the solution of the second order differential equation (2-15)

$$\bar{T}'' + \left[\left(2\frac{R_1}{R_2} - 1 \right)\cot\theta - \frac{R_1'}{R_1} \right] \bar{T}' + i\frac{R_1^2}{R_2 c}\bar{T} = i\frac{R_1^2}{R_2 c}F(\theta) \qquad (2\text{-}15)$$

where

$$F(\theta) = R_2 p - \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \frac{\bar{U}}{\sin^2\theta} = T_1^* + T_2^* \qquad (2\text{-}16)$$

$$\bar{U} = C_2 + \frac{p}{2} R_2^2 \sin^2\theta \qquad (2\text{-}13)$$

It is noticed that the coefficient iR_1^2/R_2c of $\bar{T}$ is a magnitude of order R/h. For convenience of analysis this coefficient will be expressed in terms of a parameter λ

$$i\frac{R_1^2}{R_2 c} = i\lambda^2 \frac{R_1^2}{R_2 R_0} \qquad (3\text{-}1)$$

where

$$\lambda^2 = \frac{R_0}{c} = \sqrt{12(1-\mu^2)}\,\frac{R_0}{h} \qquad (3\text{-}2)$$

and R_0 is the minimum radius of curvature of a shell of revolution. For thin shells λ^2 is a large parameter.

(a) Homogeneous Solution

It is well known that a second order differential equation of the type

$$y'' + p(x)y' + q(x)y = 0$$

may be reduced to the form

$$\xi'' + m(x)\xi = 0 \tag{3-3}$$

by the transformation

$$y = \xi(x)\eta(x)$$

where

$$\begin{aligned} \xi &= y\, e^{\int p/2\, dx} \\ m(x) &= q - \frac{p'}{2} - \frac{p^2}{4} \end{aligned} \tag{3-4}$$

Now, let $\bar{\mathcal{T}}$ be the solution of the homogeneous equation

$$\bar{\mathcal{T}}'' + \left[\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right]\bar{\mathcal{T}}' + i\lambda^2 \frac{R_1^2}{R_2 R_0}\bar{\mathcal{T}} = 0 \tag{3-5}$$

Applying the results shown above to Eq.(3-5) one obtains

$$\xi'' + m(\theta)\xi = 0 \tag{3-6}$$

where

$$\begin{aligned} \xi &= \bar{\mathcal{T}} \exp\left\{\frac{1}{2}\int\left[\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right] d\theta\right\} = \bar{\mathcal{T}} R_2\left(\frac{\sin\theta}{R_1}\right)^{1/2} \\ m(\theta) &= i\lambda^2\frac{R_1^2}{R_2 R_0} - \frac{2+\cos^2\theta}{4\sin^2\theta} + \frac{R_1}{R_2\sin^2\theta} - \frac{R_1'}{2R_1}\cot\theta + \frac{R_1''}{2R_1} - \frac{3R_1'^2}{4R_1^2} \end{aligned} \tag{3-7}$$

The condition of Codazzi has been used in the above transformation.

It is noted that the coefficient $m(\theta)$ contains a singular point at $\theta = 0$, which characterizes solutions of Eq.(3-6) as two completely different types. The first type is an asymptotic solution of classic type which is valid only in the nonshallow region, i.e. large values of θ.

The other type is an asymptotic solution valid in all regions including the singular point $\theta = 0$. Attention here will be directed more to the second than the first, since the problem of interest is that of the stress distribution near the apex which belongs to the second type.

As regards the first type of solution, comparison of the magnitude of each term in $m(\theta)$ shows that, if the region of interest lies in the non-shallow region, the first term is $O(\lambda^2)$ and the remaining terms are $O(1)$, provided the shell is sufficiently smooth so that the derivatives of R_1 and R_2 have the same order of magnitudes as R_1 and R_2. Thus, Eq.(3-6) may be written in the form

$$\xi'' + \left[i\lambda^2 \frac{R_1^2}{R_2 R_0} \left(1 + O(\lambda^{-2}) \right) \right] \xi = 0 \qquad (3\text{-}8)$$

Through the use of the transformations

$$\xi_1 = \left(i\lambda^2 \frac{R_1^2}{R_2 R_0} \right)^{\frac{1}{4}} \xi , \qquad d\theta_1 = \left(i\lambda^2 \frac{R_1^2}{R_2 R_0} \right)^{\frac{1}{2}} d\theta$$

and by neglecting terms of order λ^{-2} in comparison with unity, Eq.(3-8) may be reduced to a familiar form

$$\frac{d^2 \xi_1}{d\theta_1^2} + \xi_1 = 0$$

which has the solution

$$\xi_1 = e^{\pm i\theta_1}$$

Expressing this solution in terms of the original variables ξ and θ, one obtains the solution of Eq.(3-8)

$$\xi = \left(\frac{R_0 R_2}{R_1^2} \right)^{1/4} e^{\pm i^{3/2} \eta} \qquad (3\text{-}9)$$

where

$$\eta(\theta) = \lambda \int_0^{\theta} \frac{R_1}{\sqrt{R_2 R_0}} \, d\theta$$

Hence

$$\bar{J} = \frac{1}{R_2^{3/4}\sin^{1/2}\theta}\left[\bar{c}_1 e^{\frac{1}{\sqrt{2}}(1-i)\eta} + \bar{c}_2 e^{-\frac{1}{\sqrt{2}}(1-i)\eta}\right] \quad (3\text{-}10)$$

in which $\bar{C}_1$ and $\bar{C}_2$ are complex constants of integration.

As regards the second type of solution which is valid in the entire region including a singular point $\theta = 0$, it is necessary to rewrite Eq.(3-6) in the form

$$\xi'' + [\,i\lambda^2\psi^2(\theta) + \Lambda(\theta)]\,\xi = 0 \quad (3\text{-}11)$$

where

$$\psi^2 = \frac{R_1^2}{R_0 R_2}$$

$$\Lambda = \frac{1}{\sin^2\theta}\left(\frac{R_1}{R_2} - \frac{2+\cos^2\theta}{4}\right) - \frac{R_1'}{2R_1}\cot\theta + \frac{R_1''}{2R_1} - \frac{3}{4}\left(\frac{R_1'}{R_1}\right)^2 \quad (3\text{-}12)$$

It was shown by Langer [9] that there exists, corresponding to Eq.(3-11), a related differential equation whose solution is asymptotic with respect to the solution of Eq.(3-11). The domain of validity of this asymptotic solution depends on the function in the cofficient of ξ, i.e., $\psi^2(\theta)$ and Λ, which meet the following requirements:

(i) Within the interval I_θ which includes a singular point θ_0, $\psi^2(\theta)$ is of the form

$$\psi^2(\theta) = (\theta-\theta_0)^{d-2}\psi_1^2(\theta) = (\theta-\theta_0)^{d-2}\{1 + a_1(\theta-\theta_0) + a_2(\theta-\theta_0)^2 + \cdots\}$$

with d being any real positive constant.

(ii) Within I_θ, $\Lambda(\theta)$ is of the form

$$\Lambda(\theta) = \frac{A_1}{(\theta-\theta_0)^2} + \frac{B_1}{(\theta-\theta_0)} + C_1(\theta)$$

with A_1 and B_1 any constants and $C_1(\theta)$ is analytic and bounded uniformly with respect to λ in I_θ.

If the constants α_1 and B_1 are both zero, the differential equation will be defined to be normal. Thus the normal form of the differential equation which reflects the foregoing requirements can be represented by

$$\xi'' + \left[i\lambda^2(\theta-\theta_0)^{d-2}\psi_1^2 + \frac{A_1}{(\theta-\theta_0)^2} + C_1(\theta)\right]\xi = 0 \qquad (3\text{-}13)$$

If α_1 and B_1 are not zero, the differential equation may always be normalized by substitution

$$\theta-\theta_0 = z^2/4, \qquad \xi = z^{\frac{1}{2}}u$$

Then, according to Langer the functions

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \psi^{-\frac{1}{2}}(\theta)\ \sigma^{\frac{1}{2}} \begin{bmatrix} J_\rho(\sigma) \\ Y_\rho(\sigma) \end{bmatrix} \qquad (3\text{-}14)$$

are the solutions of the related differential equation

$$z'' + \left[i\lambda^2\psi^2 + \frac{A_1}{(\theta-\theta_0)^2} + \Omega(\theta)\right] z = 0 \qquad (3\text{-}15)$$

where $\Omega(\theta)$ is analytic and bounded with respect to λ in I_θ; $J_\rho(\sigma)$ and $Y_\rho(\sigma)$ are Bessel functions of the first and second kinds and

$$\rho = c/d, \qquad c = (1-4A_1)^{1/2}$$
$$\sigma = \int_{\theta_0}^{\theta} (i\lambda^2)^{\frac{1}{2}}\,\psi(\theta)\,d\theta \qquad (3\text{-}16)$$

It will be shown that the functions in the coefficient of ξ in Eq. (3-11) satisfy the requirements stipulated above, provided the shells are smooth at the apex, i.e., if

$$R_1,\ R_2 \to R^* \quad \text{as } \theta \to 0$$

or more specifically, if

$$R_2/R_1 = 1 + f(\theta)\sin^2\theta \qquad (3\text{-}17)$$

where $f(\theta)$ is analytic and bounded in I_θ. For such a shell

$$R_2' = R_1'(1 + f \sin^2\theta) + R_1(f \sin^2\theta)'$$

By use of the condition of Codazzi and Eq.(3-17) the preceding equation may be written in the form

$$R_1' = - \frac{R_1}{1 + f \sin^2\theta} [f \sin\theta \cos\theta + (f \sin^2\theta)'] \qquad (3\text{-}18)$$

As an example, shells of revolution generated by rotation of the second order curves

$$R_1 = \frac{R^*}{(1 + \gamma \sin^2\theta)^{3/2}}$$
$$R_2 = \frac{R^*}{(1 + \gamma \sin^2\theta)^{1/2}} \qquad (3\text{-}19)$$

satisfy the condition given by Eq.(3-17). In fact, these curves generate the classes of surfaces including (i) sphere for $\gamma = 0$; (ii) paraboloids for $\gamma = -1$; (iii) ellipsoids for $\gamma > 1$; and (iv) hyperboloids for $\gamma < -1$.

By use of Eqs.(3-17) and (3-18) $\Lambda(\theta)$ in the second of Eqs.(3-12) reduces to

$$\Lambda(\theta) = \frac{1}{4} \frac{1 - \sin^2\theta}{\sin^2\theta} + \Lambda_1(\theta) \qquad (3\text{-}20)$$

where

$$\Lambda_1(\theta) = \frac{1}{2} + \frac{R_1''}{2R_1} - \frac{3}{4}\left(\frac{R_1'}{R_1}\right)^2 + \frac{1}{2(1 + f \sin^2\theta)} (3 f \cos^2\theta + f' \sin\theta \cos\theta)$$

and is bounded in I_θ, and Eq.(3-11) becomes

$$\xi'' + [i \lambda^2 \psi^2 + \frac{1}{4} \frac{1 - \sin^2\theta}{\sin^2\theta} + \Lambda_1(\theta)] \xi = 0 \qquad (3\text{-}21)$$

To make this equation fit the form of Eq.(3-13), a new independent variable x will be introduced

$$x = \sin \theta/2, \qquad dx = \tfrac{1}{2}(1 - x^2)^{\frac{1}{2}} d\theta$$

Thus, Eq.(3-21) becomes

$$\frac{d^2\xi}{dx^2} - \frac{x}{1-x^2}\frac{d\xi}{dx} + \left[4i\lambda^2\frac{\psi^2}{1-x^2} + \frac{1}{4x^2} + \frac{1}{1-x^2}\left(-\frac{3}{4} + \frac{1}{4(1-x^2)} + \Lambda_1\right)\right]\xi = 0 \tag{3-22}$$

Now, by means of the transformation

$$\bar{\xi} = \xi(1-x^2)^{1/4} \tag{3-23}$$

Eq.(3-22) reduces to the desired form

$$\frac{d^2\bar{\xi}}{dx^2} + \left[i\lambda^2\frac{4\psi^2}{1-x^2} + \frac{1}{4x^2} + \Lambda_2\right]\bar{\xi} = 0 \tag{3-24}$$

where

$$\Lambda_2 = \frac{1}{1-x^2}\left(4\Lambda_1 - \frac{9}{4} + \frac{9}{4}\frac{1}{1-x^2}\right) \tag{3-25}$$

is bounded in $|x| < 1$, i.e., $0 \leq \theta < \pi$. From this one finds

$$c = (1-4A_1)^{1/2} = 0, \qquad \rho = c/d = 0$$

$$\sigma = i^{\frac{1}{2}}\lambda\int_0^x \frac{2\psi}{(1-x^2)^{1/2}}\,dx = i^{\frac{1}{2}}\lambda\int_0^\theta \frac{R_1}{\sqrt{R_0R_2}}\,d\theta = i^{\frac{1}{2}}\eta$$

where

$$\eta(\theta) = \lambda\int_0^\theta \frac{R_1}{\sqrt{R_0R_2}}\,d\theta \tag{3-26}$$

Thus the asymptotic solutions of Eq.(3-24) are given by

$$\begin{bmatrix}\bar{\xi}_1\\ \bar{\xi}_2\end{bmatrix} = \left(\frac{4\psi^2}{1-x^2}\right)^{-\frac{1}{4}}\eta^{\frac{1}{2}}\begin{bmatrix}J_0(i^{\frac{1}{2}}\eta)\\ Y_0(i^{\frac{1}{2}}\eta)\end{bmatrix} \tag{3-27}$$

which, in terms of ξ, becomes

$$\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \left(\frac{R_0R_2}{R_1^2}\right)^{\frac{1}{4}}\eta^{\frac{1}{2}}\begin{bmatrix}J_0(i^{\frac{1}{2}}\eta)\\ Y_0(i^{\frac{1}{2}}\eta)\end{bmatrix} \tag{3-28}$$

$J_0(i^{\frac{1}{2}}\eta)$ and $Y_0(i^{\frac{1}{2}}\eta)$ are Bessel functions of the first and second kinds which are the solutions of the differential equation

$$y'' + \frac{1}{\eta}y' + iy = 0 \tag{3-29}$$

Since these solutions are not tabulated for complex arguments, they will be transformed to modified Bessel functions which are well tabulated in terms of Thompson functions. To do this, let

$$\eta = i^{1/2} x$$

Equation (3-29) is thus transformed to

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - y = 0$$

which has solution

$$y = \bar{A}\, I_0(x) + \bar{B}\, K_0(x) = \bar{A}\, I_0(i^{-\frac{1}{2}}\eta) + \bar{B}\, K_0(i^{-\frac{1}{2}}\eta)$$

where I_0 and K_0 are modified Bessel functions of the first and second kinds and are related to Thompson functions by

$$\begin{aligned} I_0(i^{-\frac{1}{2}}\eta) &= Ber\,\eta - i\,Bei\,\eta \\ K_0(i^{-\frac{1}{2}}\eta) &= Ker\,\eta - i\,Kei\,\eta \end{aligned} \qquad (3\text{-}30)$$

Using the relation between ξ and $\bar{\mathcal{J}}$ given by Eq.(3-7) one finally obtains the solution of Eq.(3-5) to be

$$\bar{\mathcal{J}} = \begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix}\left(\frac{R_0}{R_2}\right)^{3/4}\left(\frac{\eta}{\lambda \sin\theta}\right)^{1/2}\begin{bmatrix} I_0(i^{-\frac{1}{2}}\eta) \\ K_0(i^{-\frac{1}{2}}\eta)\end{bmatrix} \qquad (3\text{-}31)$$

It was shown in (9) that for $\lambda \gg 1$, Eq.(3-31) furnishes asymptotic solution of Eq.(3-5) to within terms of relative order $1/\lambda$ uniformly on an interval $0 \leq \theta < \pi$ provided the function $\lambda(\theta) = O(1)$ on the interval.

(b) <u>Remarks on the Characteristics of the Solution</u>

The following observations on the characteristics of the function are of importance.

(i) The coefficient outside the bracket of Eq.(3-31) is a non-zero slowly varying function of θ while the terms in the bracket vary rapidly with respect to θ. In view of this fact, this coefficient may be regarded as a constant in performing differentiation, admitting the same order of error as the asymptotic solution. This consideration results in a great algebraic simplification.

(ii) The order of magnitude between $\bar{\mathcal{T}}$ and its derivatives obeys the relation

$$\bar{\mathcal{T}}'' = \lambda\, O(\bar{\mathcal{T}}') = \lambda^2\, O(\bar{\mathcal{T}})$$

Thus, the differential equation, Eq.(3-5), is essentially equivalent to the following in the non-shallow region.

$$\bar{\mathcal{T}}'' + i\lambda^2\psi^2\,\bar{\mathcal{T}} = 0$$

(iii) Let

$$h_1 = \left(\frac{R_0}{R_2}\right)^{3/4}\left(\frac{\eta}{\lambda\sin\theta}\right)^{1/2}$$

By regarding h_1 as constant in performing differentiation it may be shown from the property of Bessel function that the solution $\bar{\mathcal{T}}$ given by Eq.(3-31) satisfies the differential equation

$$\bar{\mathcal{T}}'' + \frac{1}{\theta}\bar{\mathcal{T}}' + i\lambda^2\psi^2\,\bar{\mathcal{T}} = 0$$

Transition to this equation from Eq.(3-5), i.e.,

$$\bar{\mathcal{T}}'' + \left[\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right]\bar{\mathcal{T}}' + i\lambda^2\psi^2\,\bar{\mathcal{T}} = 0$$

is made possible by the assumption that the shell is smooth near the apex. Thus, in the $\bar{\mathcal{T}}'$ term, one may approximate R_1/R_2 by unity and neglect the terms of O(θ) in comparison with 1/θ, since this term is significant only in the shallow region. However, it should be noted that one can not make the same approximation on the last term, which is of the order $\lambda^2\psi^2 O(\bar{\mathcal{T}})$.

Since in the expression for ψ^2, i.e.,

$$\psi^2 = \frac{R_1}{R_0}\frac{R_1}{R_2}$$

R_1/R_2 may be far removed from unity in the non-shallow region.

(c) Reduction to the Solution of Spherical Shells

The solution for the spherical shell is obtained from Eq.(3-31) by letting

$$R_1 = R_2 = R_0$$

and

$$\eta = \lambda\theta$$

Thus, Eq.(3-31) reduces to

$$\bar{\mathcal{J}} = \begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\begin{bmatrix}I_0(i^{-\frac{1}{2}}\eta)\\ K_0(i^{-\frac{1}{2}}\eta)\end{bmatrix} \qquad (3\text{-}32)$$

If attention is restricted to shallow spherical shells, then, one may write

$$\sin\theta = \theta\left(1 + \frac{1}{3!}\theta^2 + \cdots\right) = \theta\left(1 + O(\lambda^{-1})\right)$$

which may be approximate by θ within an error of $O(\lambda^{-1})$ if θ is restricted to the interval $0 \le \theta \le \theta_1 = O(\frac{1}{\sqrt{\lambda}})$. Thus, the standard solution for shallow spherical shell is obtained [19].

$$\bar{\mathcal{J}} = \begin{pmatrix}\bar{A}\\ \bar{B}\end{pmatrix}\begin{bmatrix}I_0(i^{-\frac{1}{2}}\eta)\\ K_0(i^{-\frac{1}{2}}\eta)\end{bmatrix} \qquad (3\text{-}33)$$

(d) Complex Forces

With the solution for $\bar{\mathcal{J}}$, the complex forces are ready to compute. In the following the manipulation will be performed only for the solution associated with $\bar{B}$. The other solution may be simply obtained from that associated with $\bar{B}$ by replacing K_0 with I_0 and K_0' with I_0'.

$$\bar{\mathcal{J}}_1 = i\frac{c}{R_1}\cot\theta\,\bar{\mathcal{J}}' = \bar{B}\,i\sqrt{\frac{c}{R_2}}\,\cot\theta\,h_1 K_0' \qquad (3\text{-}34)$$

$$\bar{\mathcal{T}}_2 = \bar{\mathcal{T}} - \bar{\mathcal{T}}_1 = \bar{B}\, h_1 \left[K_0 - i\sqrt{\frac{c}{R_2}}\, \cot\theta\, K_0' \right] \qquad (3\text{-}35)$$

Upon separating the real and imaginary part of Eqs.(3-34) and (3-35) and applying the definition of complex forces and also Eq.(3-30), the forces and moments are obtained which are listed in Table 3-1.

(e) Particular Solution

Let $\bar{t}$ be the particular solution of the equation

$$\bar{T}'' + \left[\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right]\bar{T}' + i\lambda^2\psi^2\bar{T} = i\lambda^2\psi^2 F(\theta) \qquad (2\text{-}15)$$

where

$$F(\theta) = R_2 p - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{U}{\sin^2\theta} = T_1^* + T_2^* \qquad (2\text{-}16)$$

$$U = C_2 + \frac{p}{2} R_2^2 \sin^2\theta \qquad (2\text{-}13)$$

$$T_1^* = \frac{U}{R_2 \sin^2\theta} \qquad T_2^* = R_2 p - \frac{U}{R_1 \sin^2\theta} \qquad (3\text{-}36)$$

The constant C_2 will be determined prior to finding the solution $\bar{t}$. The equilibrium of the forces (Fig.3-1) in the vertical direction requires that

$$T_1 \sin\theta - N_1 \cos\theta = \frac{p}{2} R_2 \sin\theta \qquad (3\text{-}37)$$

It may be shown that the left hand side of this equation is the real part of the complex force $\bar{V}_z$

$$\bar{V}_z = \bar{T}_1 \sin\theta - i\frac{c}{R_1}\bar{T}'\cos\theta \qquad (3\text{-}38)$$

The second term on the right hand side is deduced from

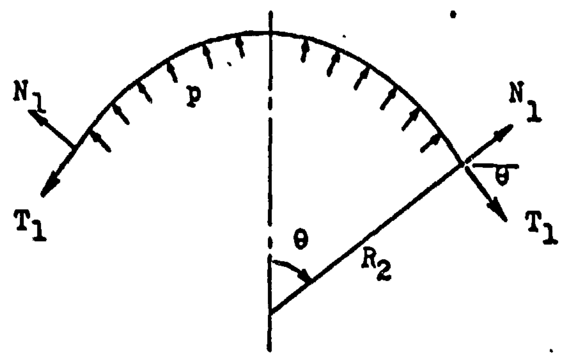


Fig. 3-1: Equilibrium of the forces in a shell of revolution

$$N_1 = \frac{1}{1+\mu}\,\frac{1}{R_1}\,\frac{dM}{d\theta} = R_e\left(i\,\frac{c}{R_1}\,\frac{d\bar{T}}{d\theta}\right)$$

which is the first of Eqs.(A-24). Substituting for $\bar{T}_1$ in Eq.(3-38) by its expression from the first of Eqs.(2-17) one obtains

$$\bar{V}_z = \frac{U}{R_2 \sin\theta} = \frac{C_2}{R_2 \sin\theta} + \frac{p}{2} R_2 \sin\theta$$

It follows from Eq.(3-37) that

$$C_2 = 0$$

$$F(\theta) = R_2 p - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{p}{2} R_2^2$$

If there were a vertical force V applied at the apex, then, $C_2 = V/2\pi$.

Now, return to the particular solution of Eq.(2-15). In view of the assumption that R_1 and R_2 are sufficiently smooth so that $F(\theta)$ is a slowly varying function, the particular solution may be assumed to be

$$\bar{t} = \bar{t}^{(0)} + \frac{1}{\lambda^2}\,\bar{t}^{(1)} + \frac{1}{\lambda^4}\,\bar{t}^{(2)} + \cdots \qquad (3\text{-}39)$$

On substituting this expression into Eq.(2-15) and equating to zero the coefficients of each power of λ, there results the following equations for determination of $\bar{t}^{(0)}$, $\bar{t}^{(1)}$, ...

$$\begin{aligned} &\bar{t}^{(0)} = F(\theta) \\ &\bar{t}^{(n)} = i\,\frac{R_0 R_2}{R_1^2}\left[\bar{t}''^{(n-1)} + \left\{\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right\}\bar{t}'^{(n-1)}\right] \\ &n = 1,\ 2,\ 3,\ \cdots \end{aligned} \qquad (3\text{-}40)$$

Having determined $\bar{t}^{(0)}$ from the first equation, $\bar{t}^{(1)}$, $\bar{t}^{(2)}$,... may be successively found from the second of Eqs.(3-40). However, for consistency in the degree of accuracy with the homogeneous solution only the first term of Eq.(3-39) will be retained as the particular solution of Eq.(2-15). In this way one simply identifies the particular integral as the solution of

TABLE 3-1

BENDING SOLUTIONS OF A SHELL OF REVOLUTION UNDER INTERNAL PRESSURE

u : $-\frac{R_2}{Eh}\left(\frac{R_2}{R_1}+\mu\right)\sqrt{\frac{c}{R_2}}\; h_1\left[B_1 \operatorname{kei}'\eta - B_2 \operatorname{ker}'\eta\right]$

w : $\frac{R_2}{Eh}\; h_1\left[B_1 \operatorname{ker}\eta + B_2 \operatorname{kei}\eta\right]$

χ_2 : $-\frac{1}{Eh}\sqrt{\frac{R_2}{c}}\; h_1\left[B_1 \operatorname{ker}'\eta + B_2 \operatorname{kei}\eta\right]$

T_1 : $\sqrt{\frac{c}{R_2}}\cot\theta\; h_1\left[B_1 \operatorname{kei}'\eta - B_2 \operatorname{ker}'\eta\right]$

T_2 : $h_1\left[B_1\left(\operatorname{ker}\eta - \sqrt{\frac{c}{R_2}}\cot\theta \operatorname{kei}'\eta\right) + B_2\left(\operatorname{kei}\eta + \sqrt{\frac{c}{R_2}}\cot\theta \operatorname{ker}\eta\right)\right]$

M_1 : $c h_1\Big[B_1\left(\operatorname{kei}\eta + (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta \operatorname{ker}'\eta\right)$
$\qquad + B_2\left(-\operatorname{ker}\eta + (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta \operatorname{kei}'\eta\right)\Big]$

M_2 : $c h_1\Big[B_1\left(\mu \operatorname{kei}\eta - (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta \operatorname{ker}'\eta\right)$
$\qquad + B_2\left(-\mu \operatorname{ker}\eta - (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta \operatorname{kei}\eta\right)\Big]$

N_1 : $\sqrt{\frac{c}{R_2}}\; h_1\left[B_1 \operatorname{kei}'\eta - B_2 \operatorname{ker}'\eta\right]$

TABLE 3-2

MEMBRANE SOLUTIONS OF A SHELL OF REVOLUTION UNDER INTERNAL PRESSURE

u : $-\frac{p}{Eh}\sin\theta\int R_2^2\, \frac{1+\frac{R_1}{R_2}\mu}{\sin\theta}\left[1-\frac{1}{2}\left(\frac{R_2}{R_1}-1\right)\right]d\theta$

w : $\frac{p}{Eh}R_2^2\left[\frac{1}{2}-\mu\left(1-\frac{1}{2}\frac{R_2}{R_1}\right)\right]+\frac{p}{Eh}\cos\theta\int R_2^2\,\frac{1+\frac{R_1}{R_2}\mu}{\sin\theta}\left[1-\frac{1}{2}\left(\frac{R_2}{R_1}-1\right)\right]d\theta$

T_1^* : $\frac{1}{2}pR_2$

T_2^* : $pR_2-\frac{p}{2}\frac{R_2^2}{R_1}$

membrane theory, and the homogeneous solution as the solution of bending theory. Thus, the complex forces $\tilde{t}_1$ and $\tilde{t}_2$ are found from the expressions

$$\begin{aligned} \tilde{t}_1 &= \frac{U}{R_2 \sin^2\theta} + i\frac{c}{R_1}\cot\theta\, \tilde{t}' \approx T_1^* \\ \tilde{t}_2 &= \tilde{t} - \tilde{t}_1 \approx T_2^* \end{aligned} \tag{3-41}$$

They are also listed in Table 3-2.

(f) Displacements

The displacements for symmetric deformation may be found from the first two of Eqs.(2-19)

$$\begin{aligned} \tilde{u}' + \tilde{w} &= \frac{R_1}{Eh}(\tilde{T}_1 - \mu\tilde{T}_2) \\ \tilde{u}\cot\theta + \tilde{w} &= \frac{R_2}{Eh}(\tilde{T}_2 - \mu\tilde{T}_1) \end{aligned} \tag{3-42}$$

Eliminating $\tilde{w}$ by subtracting the second from the first equation and taking into consideration the relations between complex forces, one obtains

$$\sin\theta\left(\frac{\tilde{u}}{\sin\theta}\right)' = \frac{ic(1+\mu)}{Eh}\left[\left(1+\frac{R_2}{R_1}\right)\cot\theta\,\tilde{T}' + \frac{i}{c}\,\frac{R_2+\mu R_1}{1+\mu}\,\tilde{T}\right] \tag{3-43}$$

in which

$$\tilde{T} = \tilde{J} + \tilde{t}$$

is the general solution of the governing equation (2-15). Within the admissible error, it has been concluded that this solution is the sum of the solution of membrane theory and bending theory.

(i) Membrane solution

Let $c = 0$, Eq.(3-43) reduces to

$$\sin\theta\left(\frac{U}{\sin\theta}\right)' = -\frac{1}{Eh}(R_2+\mu R_1)\,t$$

$$U = -\frac{\sin\theta}{Eh}\int \frac{R_2+\mu R_1}{\sin\theta}\left[R_2 p - \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{PR_2^2}{2}\right]d\theta \tag{3-44}$$

and

$$w = \frac{p}{Eh} R_2^2 \left[\frac{1}{2} - \mu \left(1 - \frac{1}{2} \frac{R_2}{R_1} \right) \right] - u \cot\theta \tag{3-45}$$

(ii) Bending solution

Upon substitution of $\tilde{T}$ by $\bar{\mathcal{T}}$ Eq.(3-43) becomes

$$\sin\theta \left(\frac{\bar{u}}{\sin\theta} \right)' = \frac{ic(1+\mu)}{Eh} \left[\left(1 + \frac{R_2}{R_1} \right) \cot\theta \, \bar{\mathcal{T}}' + \frac{i}{c} \frac{R_2 + \mu R_1}{1+\mu} \bar{\mathcal{T}} \right] \tag{3-46}$$

Exact integration of this equation is difficult, however, it is possible to determine an approximate solution within the admissible error. Observing the characteristics of the solution mentioned in the previous section one may write this equation in the form

$$\begin{aligned} \sin\theta \left(\frac{\bar{u}}{\sin\theta} \right)' &= \frac{ic(1+\mu)}{Eh} \left[2\cot\theta \, \bar{\mathcal{T}}' + \frac{1}{1+\mu} \frac{R_2}{R_1} \left(-\bar{\mathcal{T}}'' - \cot\theta \, \bar{\mathcal{T}}' \right) \right] \\ &\approx \frac{R_2}{Eh} \frac{ic}{R_1} \left(\frac{R_2}{R_1} + \mu \right) \left[-\bar{\mathcal{T}}'' + \cot\theta \, \bar{\mathcal{T}}' \right] \\ &= -\frac{R_2}{Eh} \frac{ic}{R_1} \left(\frac{R_2}{R_1} + \mu \right) \left(\frac{\bar{\mathcal{T}}'}{\sin\theta} \right)' \sin\theta \end{aligned}$$

It follows that

$$\bar{u} = -\frac{R_2}{Eh} \frac{ic}{R_1} \left(\frac{R_2}{R_1} + \mu \right) \bar{\mathcal{T}}'$$

and

$$\bar{w} = \frac{R_2}{Eh} \left(\bar{\mathcal{T}}_2 - \mu \bar{\mathcal{T}}_1 \right) - \bar{u} \cot\theta \approx \frac{R_2}{Eh} \bar{\mathcal{T}} \tag{3-47}$$

The real parts of Eqs.(3-47) are also listed in Table 3-1.

(B) <u>Non-symmetric Deformation - under a Moment</u>

The analysis of shells of revolution subject to a moment has been reduced to the integration of the second order differential equation (2-35)

$$\bar{T}^{\circ\prime\prime} + \left[\left(2\frac{R_1}{R_2} - 1 \right) \cot\theta - \frac{R_1'}{R_1} \right] \bar{T}^{\circ\prime} + \frac{R_1}{R_2} \left(1 - 2\frac{R_1}{R_2} \right) \frac{1}{\sin^2\theta} \bar{T}^{\circ} + i\lambda^2 \frac{R_1^2}{R_0 R_2} \bar{T}^{\circ} = i\lambda^2 \frac{R_1^2}{R_0 R_2} F_1(\theta) \tag{2-35}$$

where

$$F_1(\theta) = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\tilde{W}}{\sin^2\theta} = T_1^{\circ *} + T_2^{\circ *}$$

$$\tilde{W} = \frac{1}{R_2 \sin\theta}\left[\bar{B}_1 + \bar{B}_2 \int R_1 \sin\theta \, d\theta\right] \tag{3-48}$$

$$T_1^{\circ *} = \frac{\tilde{W}}{R_2 \sin^2\theta} \qquad T_2^{\circ *} = -\frac{\tilde{W}}{R_1 \sin^2\theta}$$

(a) <u>Homogeneous Solution</u>

Let $\bar{\mathcal{T}}^{\circ}$ be the homogeneous solution of Eq.(2-35). By use of the transformation given in Eq.(3-4) the homogeneous part of Eq.(2-35) becomes

$$\xi'' + m(\theta)\,\xi = 0 \tag{3-49}$$

where

$$\xi = \bar{\mathcal{T}}^{\circ} R_2 \left(\frac{\sin\theta}{R_1}\right)^{\frac{1}{2}} \tag{3-50}$$

$$m(\theta) = i\lambda^2 \frac{R_1^2}{R_0 R_2} + \frac{3-\sin^2\theta}{\sin^2\theta}\left[\frac{R_1}{R_2}\left(1 - \frac{R_1}{R_2}\right) - \frac{1}{4}\right] + \cot\theta\left(-\frac{R_1'}{2R_1} + \frac{R_1}{R_2^2}R_2'\right) + \frac{R_1''}{2R_1} - \frac{3}{4}\left(\frac{R_1'}{R_1}\right)^2 \tag{3-51}$$

The assumption that shells are smooth near the apex gives the relation (3-17), from which $m(\theta)$ may be reduced to the form

$$m(\theta) = i\lambda^2 \frac{R_1^2}{R_0 R_2} - \frac{3}{4\sin^2\theta} + \Lambda(\theta) \tag{3-52}$$

and Eq.(3-49) becomes

$$\xi'' + \left[i\lambda^2\psi^2 - \frac{3}{4\sin^2\theta} + \Lambda(\theta)\right]\xi = 0 \tag{3-53}$$

where

$$\psi^2 = \frac{R_1^2}{R_0 R_2} \tag{3-54}$$

and $\Lambda(\theta)$ is analytic and small with respect to λ in $0 \leq \theta < \pi$. Then, with the procedure established in the previous section, it is found that

$$c = (1-4A_1)^{\frac{1}{2}} = ?, \qquad \rho = c/d = 1$$

$$\sigma = i^{\frac{1}{2}} \lambda \int_0^\theta \psi \, d\theta = i^{\frac{1}{2}} \eta \tag{3-55}$$

where

$$\eta = \int_0^\theta \lambda \frac{R_1}{\sqrt{R_0 R_2}} d\theta \tag{3-56}$$

Thus, the asymptotic solutions of Eq.(3-53) are given by

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} = \left(\frac{R_0 R_2}{R'}\right)^{\frac{1}{4}} \eta^{\frac{1}{2}} \begin{Bmatrix} I_1(i^{-\frac{1}{2}}\eta) \\ K_1(i^{-\frac{1}{2}}\eta) \end{Bmatrix} \tag{3-57}$$

where I_1, K_1 are ''' fi st and second kinds of modified Bessel function of order on. Using the relation between ξ and $\bar{T}^*$ given by Eq.(3-50) one finally finds the homogeneous solution of Eq.(2-35)

$$\bar{T}^* = \begin{bmatrix} \bar{A} \\ \bar{B} \end{bmatrix} h_1(\theta) \begin{bmatrix} I_1(i^{-\frac{1}{2}}\eta) \\ K_1(i^{-\frac{1}{2}}\eta) \end{bmatrix} \tag{3-58}$$

For $\lambda \gg$. Eq.(3-58) furnishes asymptoti. solution to within $O(\frac{1}{\lambda})$ on i... ...val $0 \leq \theta < \pi$. The foregoing statements on the characteristics of the solution in Section (A-b) also apply to this solution, which in this case may be regarded as the solution of the differential equation

$$\bar{T}'' + \cot\theta \, \bar{T}' + (i\lambda^2\psi^2 - \frac{1}{\sin^2\theta})\bar{T} = 0 \tag{3-59}$$

The terms with coefficients cotθ and $1/\sin^2\theta$ are significant only in the shallow region.

The complex force $\bar{T}_1^*$ is computed from Eq.(2-24) and $\bar{T}_2^*$ from Eq.(2-14)

$$\bar{T}_1^* = i\bar{B}\sqrt{\frac{c}{R_2}}\, h_1(\cot\theta\, K_1' - \sqrt{\frac{c}{R_2}}\frac{1}{\sin^2\theta} K_1)$$

$$\bar{T}_2^* = \bar{B} h_1 [K_1 - i(\sqrt{\frac{c}{R_2}} \cot\theta\, K_1' - \frac{c}{R_2}\frac{1}{\sin^2\theta} K_1)] \tag{3-60}$$

From the third of Eqs.(2-28) and the second of Eqs.(2-24) one obtains

$$\bar{S}^{\circ} = \frac{ic}{R_1 \sin\theta}\left(-\frac{R_1}{R_2}\cot\theta\,\bar{\mathcal{J}}^{\circ} + \bar{\mathcal{J}}^{\circ\prime}\right)$$

$$= \sqrt{\frac{c}{R_2}}\,h_1 \frac{1}{\sin\theta}\,\bar{\mathcal{E}}\left[K_1' - \sqrt{\frac{c}{R_2}}\cot\theta\,K_1\right] \tag{3-61}$$

Separation of the real and imaginary parts of Eqs.(3-60) and (3-61) yields the expressions for the forces and moments which are listed in Table 3-3.

(b) Particular Solution

Let t be the particular solution of Eq.(2-35). From the assumption given by Eq.(3-17) it may be shown that

$$\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\theta} = \frac{f}{R_2} \tag{3-62}$$

Hence, the expression for $F_1(\theta)$ becomes

$$F_1(\theta) = -\frac{f}{R_2}\bar{W} \tag{3-63}$$

Thus, t will be assumed in the form

$$t = t^{(0)} + \frac{1}{\lambda^2}t^{(1)} + \frac{1}{\lambda^4}t^{(2)} + \cdots \tag{3-64}$$

Substituting this expression into Eq.(2-35) and equating to zero the coefficients of each power of λ one obtains for the determination of $t^{(0)}$, $t^{(1)}$, .. the system of equations

$$\begin{aligned} t^{(0)} &= F_1(\theta) \\ t^{(n)} &= \frac{i}{\psi^2}\left[t''^{(n-1)} + \left\{\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right\}t'^{(n-1)}\right. \\ &\qquad \left. + \frac{R_1}{R_2}\left(1 - 2\frac{R_1}{R_2}\right)\frac{1}{\sin^2\theta}\,t^{(n-1)}\right] \\ n &= 1, 2, 3, \ldots\ldots \end{aligned} \tag{3-65}$$

Notice that $\bar{W}$ satisfies Eq.(2-32), which, in the expanded form, is

$$\bar{W}'' + \left[\left(2\frac{R_1}{R_2} - 1\right)\cot\theta - \frac{R_1'}{R_1}\right]\bar{W}' - \frac{R_1}{R_2}\frac{1}{\sin^2\theta}\bar{W} = 0$$

TABLE 3-3

BENDING SOLUTIONS OF A SHELL OF REVOLUTION UNDER A MOMENT

u:	$-\frac{R_2}{Eh}\left(\frac{R_2}{R_1}+\mu\right)\sqrt{\frac{c}{R_2}}\,h_1\,(D_1\,\mathrm{kei}_1'\eta - D_2\,\mathrm{ker}_1'\eta)$	$\cos\phi$
v:	$\frac{R_2}{Eh}\left(2+\mu-\frac{R_2}{R_1}\right)\frac{c}{R_2}\,h_1\,\frac{1}{\sin\theta}\,(D_1\,\mathrm{kei}_1\eta - D_2\,\mathrm{ker}_1\eta)$	$\sin\phi$
w:	$\frac{R_2}{Eh}\,h_1\,(D_1\,\mathrm{ker}_1\eta + D_2\,\mathrm{kei}_1\eta)$	$\cos\phi$
χ_2:	$-\frac{1}{Eh}\sqrt{\frac{R_2}{c}}\,h_1\,(D_1\,\mathrm{ker}_1'\eta + D_2\,\mathrm{kei}_1'\eta)$	$\cos\phi$
T_1:	$\sqrt{\frac{c}{R_2}}\,h_1\Big[D_1\Big(\cot\theta\,\mathrm{kei}_1'\eta - \sqrt{\frac{c}{R_2}}\frac{1}{\sin^2\theta}\,\mathrm{kei}_1\eta\Big) + D_2\Big(-\cot\theta\,\mathrm{ker}_1'\eta + \sqrt{\frac{c}{R_2}}\frac{1}{\sin^2\theta}\,\mathrm{ker}_1\eta\Big)\Big]$	$\cos\phi$
T_2:	$h_1\Big[D_1\Big(\mathrm{ker}_1\eta - \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{kei}_1'\eta + \frac{c}{R_2}\frac{1}{\sin^2\theta}\,\mathrm{kei}_1\eta\Big) + D_2\Big(\mathrm{kei}_1\eta + \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{ker}_1'\eta - \frac{c}{R_2}\frac{1}{\sin^2\theta}\,\mathrm{ker}_1\eta\Big)\Big]$	$\cos\phi$
T_{12}:	$\sqrt{\frac{c}{R_2}}\,h_1\frac{1}{\sin\theta}\Big[D_1\Big(\mathrm{kei}_1'\eta - \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{kei}_1\eta\Big) + D_2\Big(-\mathrm{ker}_1'\eta + \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{ker}_1\eta\Big)\Big]$	$\sin\phi$
M_1:	$c\,h_1\Big[D_1\Big\{\mathrm{kei}_1\eta + (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{ker}_1'\eta - \frac{c}{R_2}(1-\mu)\frac{1}{\sin^2\theta}\,\mathrm{ker}_1\eta\Big\} + D_2\Big\{-\mathrm{ker}_1\eta + (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{kei}_1'\eta - \frac{c}{R_2}(1-\mu)\frac{1}{\sin^2\theta}\,\mathrm{kei}_1\eta\Big\}\Big]$	$\cos\phi$
M_2:	$c\,h_1\Big[D_1\Big\{\mu\,\mathrm{kei}_1\eta - (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{ker}_1'\eta + \frac{c}{R_2}(1-\mu)\frac{1}{\sin^2\theta}\,\mathrm{ker}_1\eta\Big\} + D_2\Big\{-\mu\,\mathrm{ker}_1\eta - (1-\mu)\sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{kei}_1'\eta + \frac{c}{R_2}(1-\mu)\frac{1}{\sin^2\theta}\,\mathrm{kei}_1\eta\Big\}\Big]$	$\cos\phi$
M_{12}:	$c\,h_1(1-\mu)\sqrt{\frac{c}{R_2}}\frac{1}{\sin\theta}\Big[D_1\Big(\mathrm{ker}_1'\eta - \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{ker}_1\eta\Big) + D_2\Big(\mathrm{kei}_1'\eta - \sqrt{\frac{c}{R_2}}\cot\theta\,\mathrm{kei}_1\eta\Big)\Big]$	$\sin\phi$
N_1:	$h_1\sqrt{\frac{c}{R_2}}\,[D_1\,\mathrm{kei}_1'\eta - D_2\,\mathrm{ker}_1'\eta]$	$\cos\phi$

Through the use of this equation the expression for computing t is obtained in the form

$$t^{(1)} = \frac{i}{\psi^2}\left\{-2\left(\frac{f}{R_2}\right)'\bar{W}' - \left[\left(\frac{f}{R_2}\right)'' + \left(\left(2\frac{R_1}{R_2}-1\right)\cot\theta - \frac{R_1'}{R_1}\right)\left(\frac{f}{R_2}\right)' + 2\frac{R_1^2}{R_2}\left(\frac{f}{R_2}\right)^2\right]\bar{W}\right\}$$

It is noted that the terms containing $\bar{W}'$ and $\cot\theta\,\bar{W}$ in the above equation involve a singularity ($\theta = 0$) of one order higher than $\bar{W}$. For the solution given by Eq.(3-65) to be applicable in the shallow region, a restriction must be imposed on the function f such that the order of magnitude of these terms is at most the order of $\bar{W}$. The condition which is sufficient for this purpose is

$$f/R_2 = k = \text{constant}$$

Thus, the expression for $t^{(1)}$ reduces to

$$t^{(1)} = -2 i k^2 R_0 \bar{W} \tag{3-66}$$

However, this restriction is not necessary if the solution sought is in the non-shallow region.

For consistency in the degree of accuracy with the homogeneous solution, only the first term of Eq.(3-64) will be retained. In doing this, one essentially identifies the particular integral of Eq.(2-35) with the solution of membrane theory. Accordingly, one may write

$$\begin{aligned} t_1 &= T_1^{ov} = \frac{W}{R_2 \sin^2\theta}, \qquad t_2 = T_2^{ov} = -\frac{W}{R_1 \sin^2\theta} \\ S &= S^{ov} = -\frac{W'}{R_1 \sin\theta} \end{aligned} \tag{3-67}$$

which are listed in Table 3-4.

(c) Displacements

With the solution for $\bar{T}$ and complex forces, the displacements may

TABLE 3-4

MEMBRANE SOLUTIONS OF A SHELL OF REVOLUTION UNDER A MOMENT

W:	$\frac{1}{R_2 \sin\theta}\left(D_5 + D_4 \int R_1 \sin\theta \, d\theta \right)$	1
ϵ_1^*:	$\frac{1}{Eh} \frac{W}{\sin^2\theta} \left(\frac{1}{R_2} + \frac{\mu}{R_1} \right)$	$\cos\phi$
ϵ_2^*:	$-\frac{1}{Eh} \frac{W}{\sin^2\theta} \left(\frac{1}{R_1} + \frac{\mu}{R_2} \right)$	$\cos\phi$
ω^*:	$-\frac{2(1+\mu)}{Eh} \frac{W'}{R_1 \sin\theta}$	$\sin\phi$
Φ:	$\frac{1}{Eh}\left[\frac{W}{\sin^2\theta} \left\{ \frac{1}{R_1}\left(1 + \frac{R_1^2}{R_2^2}\right) + \frac{2\mu}{R_2} \right\} - 2(1+\mu)\left\{ \frac{1}{R_1} W'' + \frac{\sin\theta}{R_2}\left(\frac{R_2}{R_1 \sin\theta} \right)' W' \right\}\right]$	1
u:	$R_2 \sin^2\theta \int \Phi/\sin^2\theta \, d\theta - \cos\theta \int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) d\theta - R_2 \sin\theta \, \omega^*/\sin\phi$	$\cos\phi$
v:	$\int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) d\theta$	$\sin\phi$
w:	$R_2 \left(\epsilon_2^*/\cos\phi + \cos\theta \, \omega^*/\sin\phi \right) - R_2 \sin\theta \cos\theta \int \Phi/\sin^2\theta \, d\theta - \sin\theta \int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) d\theta$	$\cos\phi$
T_1^*:	$W/(R_2 \sin^2\theta)$	$\cos\phi$
T_2^*:	$-W/(R_1 \sin^2\theta)$	$\cos\phi$
S^*	$-W'/(R_1 \sin\theta)$	$\sin\phi$

be found from the system of Eqs.(2-39). The first three are

$$\bar{u}_1' + \bar{w}_1 = R_1 \bar{\epsilon}_1^\circ$$
$$\frac{1}{\sin\theta}\bar{v}_1 + \bar{u}_1 \cot\theta + \bar{w}_1 = R_2 \bar{\epsilon}_2^\circ \qquad (3\text{-}68)$$
$$\frac{R_2}{R_1}\bar{v}_1' - \bar{v}_1 \cot\theta - \frac{1}{\sin\theta}\bar{u}_1 = R_2 \bar{\omega}^\circ$$

where $\bar{\epsilon}_1^\circ, \bar{\epsilon}_2^\circ$ and $\bar{\omega}^\circ$ are related to the complex strain components by the relation

$$(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\omega}) = (\bar{\epsilon}_1^\circ(\theta)\cos\phi,\ \bar{\epsilon}_2^\circ(\theta)\cos\phi,\ \bar{\omega}^\circ(\theta)\sin\phi)$$

Elimination of $\bar{w}_1$ from the first two of Eqs.(3-68) by subtracting the second from the first gives

$$\sin\theta\left(\frac{\bar{u}_1}{\sin\theta}\right)' - \frac{\bar{v}_1}{\sin\theta} = R_1 \bar{\epsilon}_1^\circ - R_2 \bar{\epsilon}_2^\circ \qquad (3\text{-}69)$$

The third of Eqs.(3-68), upon using the relation of Codazzi, may be written in the form

$$R_2 \sin\theta\left(\frac{\bar{v}_1}{R_2\sin\theta}\right)' - \frac{R_1}{R_2}\frac{\bar{u}_1}{\sin\theta} = R_1 \bar{\omega}^\circ \qquad (3\text{-}70)$$

Elimination of $\bar{u}_1/\sin\theta$ from Eqs.(3-69) and (3-70) yields after some rearrangement

$$\frac{\sin\theta}{R_2}\left[\frac{R_2^2\sin\theta}{R_1}\left(\frac{\bar{v}_1}{R_2\sin\theta}\right)'\right]' - \frac{\bar{v}_1}{R_2\sin\theta} = \frac{R_1}{R_2}\bar{\epsilon}_1^\circ - \bar{\epsilon}_2^\circ + \frac{\sin\theta}{R_2}(R_2\bar{\omega}^\circ)' \qquad (3\text{-}71)$$

Now, letting

$$z \equiv \frac{\bar{v}_1}{R_2\sin\theta}$$
$$\Phi \equiv \frac{R_1}{R_2}\bar{\epsilon}_1^\circ - \bar{\epsilon}_2^\circ + \frac{\sin\theta}{R_2}(R_2\bar{\omega}^\circ)'$$

equation (3-71) reduces to

$$\frac{\sin\theta}{R_2}\left[\frac{R_2^2\sin\theta}{R_1}z'\right]' - z = \Phi \qquad (3\text{-}72)$$

which takes essentially the same form as Eq.(2-32). Hence, the transformation

$$z = \frac{z_1}{R_2 \sin\theta}$$

reduces Eq.(3-72) to the form

$$\sin^2\theta \left(\frac{z_1'}{R_1 \sin\theta}\right)' = \phi$$

from which it follows that

$$\bar{v}_1 = z_1 = C_1 + C_2 \int R_1 \sin\theta \, d\theta + \int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) d\theta \qquad (3\text{-}73)$$

The solution associated with C_1 and C_2 are the solutions of the homogeneous system of Eqs.(3-68), i.e., solutions of Eqs.(3-68) with $\bar{\epsilon}_1^\circ = \bar{\epsilon}_2^\circ = \bar{\omega}^\circ = 0$. Hence, these two solutions are rigid body displacements and will be discarded in the following computation.

The displacement $\bar{u}_1$, which may be obtained from Eq.(3-70), is

$$\bar{u}_1 = R_2 \sin^2\theta \int \Phi/\sin^2\theta \, d\theta - \cos\theta \int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) - R_2 \sin\theta \, \bar{\omega}^\circ \quad (3\text{-}74)$$

and $\bar{w}_1$, which is found from the first of Eqs.(3-68), takes the form

$$\bar{w}_1 = R_2 (\bar{\epsilon}_2^\circ + \cos\theta \, \bar{\omega}^\circ) - R_2 \sin\theta \cos\theta \int \Phi/\sin^2\theta \, d\theta - \sin\theta \int R_1 \sin\theta \left(\int \Phi/\sin^2\theta \, d\theta \right) d\theta \quad (3\text{-}75)$$

(i) Membrane solution

The strain components are related to the solution $\bar{W}$ by the expressions

$$\begin{aligned} \epsilon_1^\circ &= \frac{1}{Eh} \frac{W}{\sin^2\theta} \left(\frac{1}{R_2} + \frac{\mu}{R_1} \right) \\ \epsilon_2^\circ &= \frac{1}{Eh} \frac{W}{\sin^2\theta} \left(-\frac{1}{R_1} - \frac{\mu}{R_2} \right) \qquad (3\text{-}76) \\ \omega^\circ &= -\frac{2(1+\mu)}{Eh} \frac{W'}{R_1 \sin\theta} \end{aligned}$$

where

$$W = \frac{1}{R_2 \sin\theta} \left[B_1 + B_2 \int R_1 \sin\theta \, d\theta \right]$$

Substitution of these relations into the expression for Φ yields

$$\Phi = \frac{1}{Eh}\left[\frac{W}{\sin^2\theta}\left\{\frac{1}{R_1}\left(1+\frac{R_1^2}{R_2^2}\right)+\frac{2\mu}{R_2}\right\}-2(1+\mu)\left\{\frac{1}{R_1}W''+\frac{\sin\theta}{R_2}\left(\frac{R_2}{R_1\sin\theta}\right)'W'\right]\quad (3\text{-}77)$$

which are then substituted into Eqs.(3-73) to Eqs.(3-75) to obtain the displacements due to the particular solution t. These displacements are also listed in Table 3-4.

(ii) Bending solution

Approximate integration of Eq.(3-71) is possible, however, it involves considerable algbraic manipulation. Only some intermediate steps are shown below. Observing the properties of the function $\bar{\mathcal{T}}^\circ$, and the smoothness of the shell near the apex, one may write the deformation parameters in the following form

$$\begin{aligned}
\bar{\epsilon}_1^\circ &= \frac{1}{Eh}\left[\frac{ic}{R_1}(1+\mu)\left(\cot\theta\,\bar{\mathcal{T}}^{\circ\prime}-\frac{1}{\sin^2\theta}\bar{\mathcal{T}}^\circ\right)-\mu\bar{\mathcal{T}}^\circ\right]\\
\bar{\epsilon}_2^\circ &= \frac{1}{Eh}\left[\bar{\mathcal{T}}^\circ-\frac{ic}{R_1}(1+\mu)\left(\cot\theta\,\bar{\mathcal{T}}^{\circ\prime}-\frac{1}{\sin^2\theta}\bar{\mathcal{T}}^\circ\right)\right] \qquad (3\text{-}78)\\
\bar{\omega}^\circ &= \frac{1}{Eh}\left[2(1+\mu)\frac{ic}{R_1}\left(\frac{1}{\sin\theta}\bar{\mathcal{T}}^{\circ\prime}-\frac{1}{\sin^2\theta}\bar{\mathcal{T}}^\circ\right)\right]
\end{aligned}$$

Substitution of these equations into the expression for Φ yields

$$\Phi = \frac{1}{Eh}\frac{ic}{R_1}(1+\mu)\left[2\bar{\mathcal{T}}^{\circ\prime\prime}-2\cot\theta\,\bar{\mathcal{T}}^{\circ\prime}+\frac{2}{\sin^2\theta}\bar{\mathcal{T}}^\circ\right]-\bar{\mathcal{T}}\left(1+\mu\frac{R_1}{R_2}\right).$$

With the observation that $\bar{\mathcal{T}}^\circ$ satisfies the differential equation

$$\bar{\mathcal{T}}^{\circ\prime\prime}+\cot\theta\,\bar{\mathcal{T}}^{\prime}+\left(i\lambda^2\psi^2-\frac{1}{\sin^2\theta}\right)\bar{\mathcal{T}}=0$$

the expression for Φ reduces to the form

$$\Phi \approx \frac{1}{Eh}\frac{ic}{R_1}\left(2+\mu-\frac{R_2}{R_1}\right)\left[\bar{\mathcal{T}}^{\circ\prime\prime}-3\cot\theta\,\bar{\mathcal{T}}^{\prime}+\frac{3}{\sin^2\theta}\bar{\mathcal{T}}^\circ\right] \qquad (3\text{-}79)$$

from which it follows that

$$\int\frac{\Phi}{\sin^2\theta}d\theta \approx \frac{1}{Eh}\frac{ic}{R_1}\left(2+\mu-\frac{R_2}{R_1}\right)\left[\frac{\bar{\mathcal{T}}^{\circ\prime}}{\sin^2\theta}-\frac{\bar{\mathcal{T}}^\circ}{\sin^3\theta}\right] \qquad (3\text{-}80a)$$

and

$$\int R_1 \sin\theta \left(\int \frac{\Phi}{\sin^2\theta} d\theta \right) d\theta = \frac{1}{Eh} ic \left(2+\mu-\frac{R_2}{R_1}\right) \frac{\bar{\mathcal{T}}^\circ}{\sin\theta} \tag{3-80b}$$

Thus, the displacement $\bar{v}_1$ is obtained

$$\bar{v}_1 = \frac{R_2}{Eh} \frac{ic}{R_2} \left(2+\mu-\frac{R_2}{R_1}\right) \frac{\bar{\mathcal{T}}^\circ}{\sin\theta} \tag{3-81}$$

and $\bar{u}_1$ and $\bar{w}_1$ are found from Eqs.(3-74) and (3-75), respectively.

$$\bar{u}_1 = -\frac{R_2}{Eh} \frac{ic}{R_1} \left(\frac{R_2}{R_1}+\mu\right) \bar{\mathcal{T}}^{\circ\prime} \tag{3-82}$$

$$\bar{w}_1 = \frac{R_2}{Eh} \bar{\mathcal{T}}^\circ \tag{3-83}$$

It is noticed from Eqs.(3-81) to (3-83) that the magnitudes of displacements obey the following order of magnitude relationships

$$\bar{v}_1 = \frac{1}{\lambda^2} O(\bar{w}_1), \qquad \bar{u}_1 = \frac{1}{\lambda} O(\bar{w}_1)$$

and

$$\frac{d^n g}{d\theta^n} = \lambda^n O(g)$$

where g denotes one of $\bar{u}_1$, $\bar{v}_1$, and $\bar{w}_1$. With these relations at the outset, the displacement $\bar{w}_1$ could have been easily obtained from the fourth of Eqs. (2-39), which is

$$-\frac{1}{R_1}\left(\frac{\bar{w}_1'-\bar{u}_1}{R_1}\right)' = \frac{i}{cEh}\left(\bar{T}_2^\circ - T_2^{\circ\nu}\right)$$

Neglecting $\bar{u}_1$ in comparison with $\bar{w}_1'$ from the above equation, there results

$$\left(\frac{\bar{w}_1'}{R_1}\right)' = -\frac{1}{Eh}\frac{iR_1}{c}\bar{T}_2^\circ = -\frac{1}{Eh}\frac{iR_1}{c}\left[\bar{\mathcal{T}}^{\circ\circ} - \frac{ic}{R_1}\left(\cot\theta\, \bar{\mathcal{T}}^{\circ\prime} - \frac{R_1}{R_2 \sin^2\theta}\bar{\mathcal{T}}^\circ\right)\right]$$

By virtue of Eq.(3-59) the preceding equation reduces to

$$\left(\frac{\bar{w}_1'}{R_1}\right)' = \frac{1}{Eh}\frac{R_2}{R_1}\bar{\mathcal{T}}^{\circ\circ}$$

It follows that

$$\bar{w}_1 = \frac{R_2}{Eh}\bar{\mathcal{T}}^\circ$$

Circular Cylindrical Shells

(A) Axisymmetric Deformation - Internal Pressure

The analysis of cylindrical shells under internal pressure has been reduced to the solution of Eqs.(2-48) and (2-49), which after dropping the terms containing q_1 give

$$\begin{aligned} T_1 &= B_5 \\ T_2'' + i2b^2\, T_2 &= i2b^2\, pr_o \end{aligned} \qquad (3\text{-}84)$$

The last equation has the solution

$$\bar{T}_2 = \bar{A}\, e^{(1-i)b\alpha} + \bar{B}\, e^{-(1-i)b\alpha} + pr_o \qquad (3\text{-}85)$$

The displacements may be obtained from Eqs.(2-50), which, upon substitution for $\bar{T}_1$ and $\bar{T}_2$ by their expressions from the first of Eqs.(3-84) and (3-85), yield

$$\bar{u} = \frac{r_o}{Eh}\left[-\mu\frac{1+i}{2b}\left(\bar{A}\, e^{(1-i)b\alpha} - \bar{B}\, e^{-(1-i)b\alpha}\right) - \mu pr_o\alpha + \bar{B}_5\alpha\right] \qquad (3\text{-}86)$$

$$\bar{w} = \frac{r_o}{Eh}\left[\bar{A}\, e^{(1-i)b\alpha} + \bar{B}\, e^{-(1-i)b\alpha} + pr_o - \bar{B}_5\mu\right] \qquad (3\text{-}87)$$

It is noted that the fifth of Eqs.(2-45) is compatible only if

$$\bar{T}_1 = T_1^*$$

from which it follows that

$$\bar{T}_1 = B_5 = \text{real constant}$$

Letting

$$\bar{B} = B_3 + i\, B_4$$

and separating the real and imaginary parts of Eqs.(3-85) to (3-87), then, using the definition of the complex forces given by Eqs.(2-44) one obtains the forces, moments, and displacements as shown in Table 3-5, in which the solution associated with $\bar{A}$ has been dropped by virtue of the property that it becomes unbounded when α increases.

(B) Non-symmetric Deformation - under a Moment

The analysis of circular cylindrical shells due to a moment loading has been reduced to the solution of the differential equation (2-55)

$$\bar{T}^{o''''} + (i2b^2 - 2)\bar{T}^{o''} = 0 \qquad (2\text{-}55)$$

It follows that upon integration

$$\bar{T}^{o''} + (i2b^2-2)\bar{T}^{o} = (i2b^2-2)(\bar{D}_7 + \bar{D}_8\alpha) \qquad (3\text{-}88)$$

which has solution

$$\bar{T}^{o} = \bar{D}e^{-a\alpha} + \bar{E}e^{a\alpha} + \bar{D}_7 + \bar{D}_8 a \qquad (3\text{-}89)$$

where

$$a = i(i2b^2-2)^{\frac{1}{2}} \approx -b\left[(1+\frac{1}{2b^2}) - i(1-\frac{1}{2b^2})\right] \qquad (3\text{-}90)$$

The complex forces obtained from Eqs.(2-53) take the form

$$\bar{T}_2^{o} = (1+\frac{i}{2b^2})\bar{T}^{o} - (1+\frac{i}{b^2})(\bar{D}_7 + \bar{D}_8\alpha) \qquad (3\text{-}91)$$

$$\bar{T}_1^{o} = -\frac{i}{2b^2}\bar{T}^{o} + (1+\frac{i}{b^2})(\bar{D}_7 + \bar{D}_8\alpha) \qquad (3\text{-}92)$$

$$\bar{S}^{o} = \frac{i}{2b^2}\bar{T}^{o\prime} - (1+\frac{i}{b^2})\bar{D}_8 \qquad (3\text{-}93)$$

With the complex forces expressed in terms of $\bar{T}^o$ and its derivatives, the displacements are obtainable from Eqs.(2-58)

$$\begin{aligned}\bar{U}_1' &= \frac{r_0}{Eh}(\bar{T}_1^o - \mu\bar{T}_2^o)\\ \bar{V}_1 + \bar{W}_1 &= \frac{r_0}{Eh}(\bar{T}_2^o - \mu\bar{T}_1^o) \qquad (2\text{-}58)\\ -\bar{U}_1' + \bar{V}_1' &= \frac{r_0}{Eh}2(1+\mu)\bar{S}^o\end{aligned}$$

TABLE 3-5

SOLUTIONS OF A CYLINDRICAL SHELL UNDER INTERNAL PRESSURE

u :	$\frac{r_0}{E h^*}\left[\frac{\mu}{2b} e^{-b\alpha}\{ B_3(\cos b\alpha - \sin b\alpha) - B_4(\sin b\alpha + \cos b\alpha)\} + (B_5 - \mu p r_0)\alpha \right]$
w :	$\frac{r_0}{E h^*}\left[e^{-b\alpha}\{ B_3 \cos b\alpha - B_4 \sin b\alpha \} + p r_0 - \mu B_5 \right]$
T_1:	B_5
T_2:	$e^{-b\alpha}[B_3 \cos b\alpha - B_4 \sin b\alpha] + p r_0$
M_1:	$-c^* e^{-b\alpha}[B_3 \sin b\alpha + B_4 \cos b\alpha]$
M_2:	$-c^* \mu e^{-b\alpha}[B_3 \sin b\alpha + B_4 \cos b\alpha]$
N_1:	$\frac{c^*}{r_0} b e^{-b\alpha}[B_3(\sin b\alpha - \cos b\alpha) + B_4(\cos b\alpha + \sin b\alpha)]$

$c^* = h^*/\sqrt{12(1-\mu^2)}$

Substituting in the first of Eqs.(2-58) for $\bar{T}_1$, $\bar{T}_2$ by their expressions from Eqs.(3-91) and (3-92) and taking into consideration that $\bar{T}^o$ satisfies Eq.(3-88), one has, after neglecting terms of order $1/b^2$ compared with 1

$$\bar{U}_1' = \frac{r_o}{Eh}\left[-\mu\frac{i}{2b^2}\bar{T}^{o''} + (\bar{D}_7 + \bar{D}_8\alpha)\right]$$

Integration of this equation yields

$$\bar{U}_1 = \frac{r_o}{Eh}\left[-\mu\frac{i}{2b^2}\bar{T}^{o'} + \bar{D}_7\alpha + \frac{1}{2}\bar{D}_8\alpha^2\right] \qquad (3\text{-}94)$$

The third of Eqs.(2-58) gives

$$\bar{V}_1' = \frac{r_o}{Eh}\left[(2+\mu)\frac{i}{2b^2}\bar{T}^{o'} + \bar{D}_7\alpha + \bar{D}_8\left(-2(1+\mu) + \frac{1}{2}\alpha^2\right)\right]$$

which yields the solution for $\bar{v}_1$, upon integration

$$\bar{V}_1 = \frac{r_o}{Eh}\left[(2+\mu)\frac{i}{2b}\bar{T}^{o} + \frac{\bar{D}_7}{2}\alpha^2 + \bar{D}_8\left(-2(1+\mu)\alpha + \frac{1}{6}\alpha^3\right)\right] \qquad (3\text{-}95)$$

Finally, $\bar{w}_1$ is obtained from the second of Eqs.(2-58)

$$\bar{W}_1 = \frac{r_o}{Eh}\left[\bar{T}^{o} - \bar{D}_7\left(1+\mu+\frac{1}{2}\alpha^2\right) + \bar{D}_8\left((1+\mu)\alpha - \frac{1}{6}\alpha^3\right)\right] \qquad (3\text{-}96)$$

It may be shown that the constants $\bar{D}_7$ and $\bar{D}_8$ are real. This follows from the fifth of Eqs.(2-45) that it is compatible only if

$$T_1^{o*} = D_7 + D_8\alpha = \text{real value}$$

The forces, moments and displacements are obtained upon substitution for $\bar{T}^o$ into Eq.(3-91) through Eq.(3-96) by its expression from Eq.(3-89) and then separation of the real and imaginary parts. The results of these manipulations are shown in Table 3-6.

TABLE 3-6

SOLUTIONS OF A CYLINDRICAL SHELL UNDER A MOMENT

u:	$\frac{r_0}{E h^*}\left[-\frac{\mu}{2b^2} e^{-b_1\alpha}\{D_5(b_2\sin b_2\alpha - b_1\cos b_2\alpha) + D_6(b_2\cos b_2\alpha + b_1\sin b_2\alpha)\} + D_7\alpha + \frac{D_8}{2}\alpha^2\right]$	$\cos\beta$
v:	$\frac{r_0}{E h^*}\left[\frac{2+\mu}{2b^2} e^{-b_1\alpha}(-D_5\sin b_2\alpha - D_6\cos b_2\alpha) + \frac{D_7}{2}\alpha^2 + D_8(-2(1+\mu)\alpha + \frac{1}{6}\alpha^3)\right]$	$\sin\beta$
w:	$\frac{r_0}{E h^*}\left[e^{-b_1\alpha}(D_5\cos b_2\alpha - D_6\sin b_2\alpha) - D_7(\mu + \frac{1}{2}\alpha^2) + D_8((2+\mu)\alpha - \frac{1}{6}\alpha^3)\right]$	$\cos\beta$
T_1:	$\frac{1}{2b^2} e^{-b_1\alpha}(D_5\sin b_2\alpha + D_6\cos b_2\alpha) + D_7 + D_8\alpha$	$\cos\beta$
T_2:	$e^{-b_1\alpha}(D_5\cos b_2\alpha - D_6\sin b_2\alpha)$	$\cos\beta$
M_1:	$-c^*\left[e^{-b_1\alpha}(D_5\sin b_2\alpha + D_6\cos b_2\alpha) - \frac{1-\mu}{2b^2}(D_7 + D_8\alpha)\right]$	$\cos\beta$
M_2:	$-c^*\left[\mu e^{-b_1\alpha}(D_5\sin b_2\alpha + D_6\cos b_2\alpha) + \frac{1-\mu}{2b^2}(D_7 + D_8\alpha)\right]$	$\cos\beta$
T_{12}:	$\frac{1}{2b^2} e^{-b_1\alpha}\left[D_5(b_1\sin b_2\alpha - b_2\cos b_2\alpha) + D_6(b_1\cos b_2\alpha + b_2\sin b_2\alpha)\right] - D_8$	$\sin\beta$
M_{12}:	$-\frac{c^*(1-\mu)}{2b^2} e^{-b_1\alpha}\left[D_5(b_1\cos b_2\alpha + b_2\sin b_2\alpha) + D_6(-b_1\sin b_2\alpha + b_2\cos b_2\alpha)\right] - \frac{c^*(1-\mu)}{2b^2} D_8$	$\sin\beta$
N_1:	$-\frac{c^*}{r_0} e^{-b_1\alpha}\left[D_5(b_2\cos b_2\alpha - b_1\sin b_2\alpha) + D_6(-b_2\sin b_2\alpha - b_1\cos b_2\alpha)\right]$	$\cos\beta$

$b_1 = b(1 + \frac{1}{2b^2})$, $b_2 = b(1 - \frac{1}{2b^2})$, $c^* = h^*/\sqrt{12(1-\mu^2)}$

IV. BOUNDARY CONDITIONS AND DETERMINATION OF CONSTANTS

As an application of the solutions derived in the previous Chapters, the stresses of a shell of revolution due to the presence of a discontinuity in terms of either a circular hole, a circular rigid insert, or a nozzle will be studied. The external loading is an internal pressure or a moment.

Axisymmetric Deformation - Internal Pressure

Case a: a circular hole at the apex

The discontinuity presented in this case is a small circular hole discribed by $\theta = \theta_o$. The boundary of the hole is free from stresses. However, the internal pressure must be equilibrated with a vertical shear uniformly distributed along $\theta = \theta_o$. The boundary conditions are (Fig.4-1)

$$\left.\begin{array}{l} M_1 = 0 \\ Q_x = 0 \end{array}\right\} \text{ at } \theta = \theta_o \qquad (4\text{-}1)$$

in which Q_x is the component of force in the direction perpendicular to the axis of the shell, i.e.,

$$Q_x = T_1 \cos\theta + N_1 \sin\theta$$

Substitution in Eqs.(4-1) for M_1 and Q_x by their expressions from Table 3-1 yields, for the determination of the

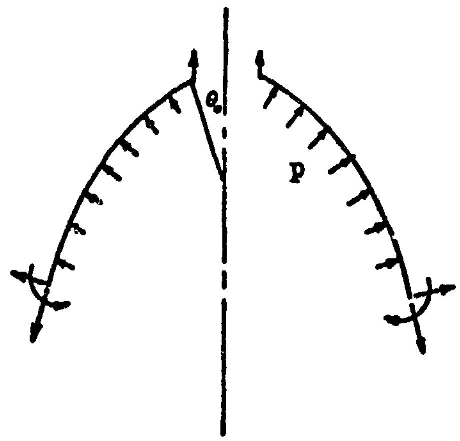


Fig. 4-1: Internal pressure equilibrated with vertical shear acting along the hole

two constants B_1 and B_2, the following equations

$$
\begin{aligned}
A_{11} B_1 + A_{12} B_2 &= 0 \\
A_{21} B_1 + A_{22} B_2 &= H_2
\end{aligned}
\tag{4-2}
$$

where

$$
\begin{aligned}
A_{11} &= h_1 \left[\mathrm{Kei}\,\eta + (1-\mu)\sqrt{\frac{c}{R_2}} \cot\theta \, \mathrm{Ker}'\eta \right]_{\theta_0} \\
A_{12} &= h_1 \left[-\mathrm{Ker}\,\eta + (1-\mu)\sqrt{\frac{c}{R_2}} \cot\theta \, \mathrm{Kei}'\eta \right]_{\theta_0} \\
A_{21} &= h_1 \sqrt{\frac{c}{R_2}} \frac{1}{\sin\theta} \mathrm{Kei}'\eta \Big|_{\theta_0} \\
A_{22} &= -h_1 \sqrt{\frac{c}{R_2}} \frac{1}{\sin\theta} \mathrm{Ker}'\eta \Big|_{\theta_0} \\
H_2 &= -T_1^* \cos\theta \Big|_{\theta_0}
\end{aligned}
\tag{4-3}
$$

Having determined B_1 and B_2 the direct stress σ_D and the bending stress σ_B are obtained by the formulas

$$
\begin{aligned}
\sigma_{jD} &= T_j / h \\
\sigma_{jB} &= 6 M_j / h^2
\end{aligned}
\qquad j = 1, 2.
\tag{4-4}
$$

Case b: <u>a circular rigid insert at the apex</u>

Since the rigid insert, by its definition, does not deform during the deformation of the shell, the rotation X_2 of the shell about the line $\theta = \theta_0$ and the strain ϵ_2 of the shell along the insert $\theta = \theta_0$ should be zero. Thus

$$
\begin{aligned}
\epsilon_2 &= 0 \\
X_2 &= 0
\end{aligned}
\qquad \text{at } \theta = \theta_0
\tag{4-5}
$$

where

$$
\epsilon_2 = (T_2 - \mu T_1)/Eh \tag{4-6}
$$

and X_2 is the real part of Eq.(2-20)

$$X_2 = -\frac{1}{R_1}(w' - u) = Re\left[\frac{iR_2}{cEh}\tan\theta\,\tilde{J}_1\right] = -\frac{R_2}{R_1}\frac{1}{Eh}\mathcal{J}'$$

Substitution for X_2 and ϵ_2 in Eqs.(4-5) by their expressions from Tables 3-1 and 3-2 results in a system of two equations for the determination of the constants B_1 and B_2

$$\begin{aligned} A_{31}\,B_1 + A_{32}\,B_2 &= H_3 \\ A_{41}\,B_1 + A_{42}\,B_2 &= 0 \end{aligned} \qquad (4\text{-}7)$$

where

$$\begin{aligned} A_{31} &= h_1\left[\,Ker\,\eta - \sqrt{\frac{c}{R_2}}\cot\theta\,Kei'\eta\,(1+\mu)\right]_{\theta_0} \\ A_{32} &= h_1\left[\,Kei\,\eta + \sqrt{\frac{c}{R_2}}(1+\mu)\cot\theta\,Ker'\eta\,\right]_{\theta_0} \\ A_{41} &= h_1\sqrt{\frac{R_2}{c}}\,Ker'\eta\,\Big|_{\theta_0} \\ A_{42} &= h_1\sqrt{\frac{R_2}{c}}\,Kei'\eta\,\Big|_{\theta_0} \\ H_3 &= -T_2^{*} + \mu T_1^{*}\Big|_{\theta_0} \end{aligned} \qquad (4\text{-}8)$$

Case c: a nozzle at the apex

The discontinuity in this case is a nozzle attached to the apex of a shell of revolution (Fig.4-2). The conditions of equilibrium and continuity across the junction of the nozzle with the shell of revolution at $e = 0$ and $\theta = \theta_0$ require that the following conditions be satisfied

$$\begin{aligned} M_1 &= M_1^{c}, & \epsilon_2 &= \epsilon_2^{c} \\ Q_x &= -Q_x^{c}, & X_2 &= X_2^{c} \end{aligned} \qquad (4\text{-}9)$$

The quantities on the left of the equal signs of Eqs.(4-9) represent the moment, force, strain, and rotation of the shell of revolution, while those on the right hand side with superscript c denote the corresponding quantities for nozzle (or cylinder), in which ϵ_2^c and X_2^c are given by

$$\epsilon_2^c = (T_2 - \mu T_1)^c / E h^*$$

$$X_2^c = \frac{1}{r_0}\frac{dw_c}{d\alpha}$$

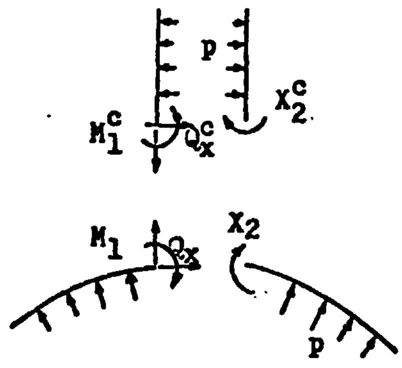


Fig. 4-2:

Forces and deformations at the junction of a nozzle and a shell of revolution

Substitution for those quantities in Eqs.(4-9) by their expressions from Tables 3-1, 3-2 and 3-5, with B_5 set equal to $\frac{1}{2}pr_0$ yields a system of four equations for determination of the four constants B_1, B_2, B_3 and B_4.

$$\begin{aligned} A_{11} B_1 + A_{12} B_2 + A_{14} B_4 &= 0 \\ A_{21} B_1 + A_{22} B_2 + A_{23} B_3 + A_{24} B_4 &= H_2 \\ A_{31} B_1 + A_{32} B_2 + A_{33} B_3 &= G_3 \\ A_{41} B_1 + A_{42} B_2 + A_{43} B_3 + A_{44} B_4 &= 0 \end{aligned} \qquad (4\text{-}10)$$

in which A_{jk} ($k = 1, 2$; $j = 1, 2, 3, 4$) and H_2 have been given by Eqs.(4-8) and (4-3). The rest are defined by

$$\begin{aligned} A_{14} &= c^*/c \\ A_{23} &= -A_{24} - c^* b/r_0 \\ A_{33} &= h/h^* \end{aligned} \qquad (4\text{-}11a,b,c)$$

$$A_{43} = A_{44} = -\,bh/h^{*}$$
$$G_3 = H_3 - h/h^{*}\, pr_0(1 - \mu/2) \qquad (4\text{-}11d,e)$$

<u>Non-symmetric Deformation - under a Moment</u>

A couple is applied in the plane $\phi = 0$ either at the apex of a shell of revolution or at the far end of a nozzle when it is attached to the shell. The constants $_3$, D_4 and D_7, D_8 which associate with the membrane solutions of the shell of revolution and the cylindrical shell, respectively shown in Tables 3-4 and 3-6 will be first determined from the condition of equilibrium. Notice the properties of the functions ker, kei, which diminish rapidly when their argument becomes large. Hence, the bending solutions are insignificant in the range of large values of θ. The state of stress in this region is, in fact, of the membrane type. The equilibrium of moment bout the plane $\phi = \frac{1}{2}\pi$ (Fig.4-3) gives

$$\int_0^{2\pi} T_1 \sin\theta (R_2 \sin\theta \cos\phi) R_2 \sin\theta \, d\phi = M \qquad (4\text{-}12)$$

in which

$$T_1 = T_1^{*} + \mathcal{T}_1 \approx T_1^{*} \quad \text{for large } \theta$$

Equation (4-12) upon introducing the expression for T_1 and performing integration reduces to

$$\pi (D_3 + D_4 \int_0^{2\pi} R_1 \sin\theta \, d\theta) = M \qquad (4\text{-}13)$$

The equilibrium of the forces in the direction of $\phi = 0$ gives

$$\int_0^{2\pi} (T_1 \cos\theta \cos\phi - S \sin\phi) R_2 \sin\theta \, d\phi = 0 \qquad (4\text{-}14)$$

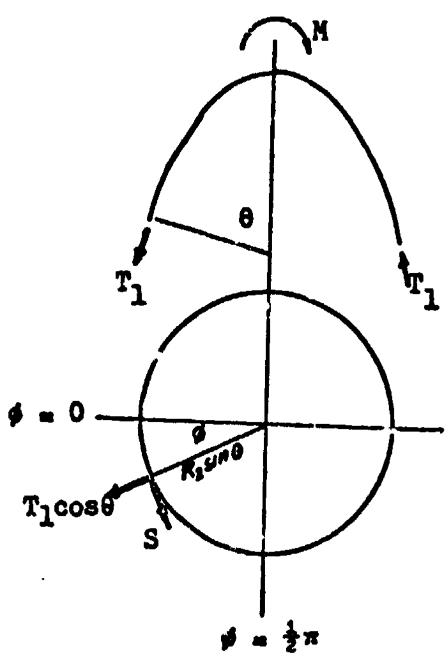


Fig. 4-3: Free body diagram of a shell of revolution under a moment

Substitution for T_1 and S in Eq.(4-14) by their expressions from Table 3-5 and then integration give the result that

$$D_4 = 0$$

from which Eq.(4-13) yields

$$D_3 = M/\pi$$

Similarly, when the moment is applied at the far end of the nozzle (Fig.4-4), the equilibrium of moment about the plane $\beta = \pi/2$ and sum of the forces in the direction $\beta = 0$ gives

$$\begin{aligned} &\int_0^{2\pi} T_1 r_0 \cos\beta \, (r_0 \, d\beta) = M \\ &\int_0^{2\pi} (S \sin\beta - N_1 \cos\beta) \, r_0 \, d\beta = 0 \end{aligned} \qquad (4\text{-}15)$$

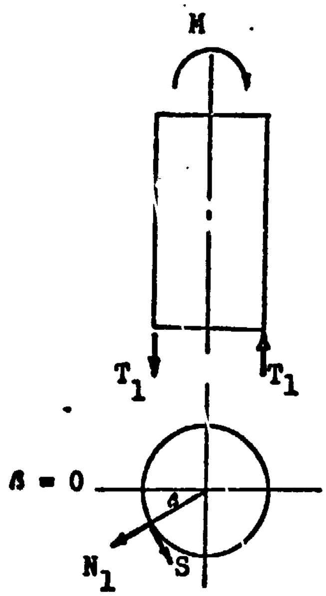


Fig. 4-4:

Free body diagram of a cylindrical shell under a moment

Equations (4-15) upon substitution for T_1 and S from Table 3-6 and then integration reduce to

$$\pi r_0^2 (D_7 + D_8 \alpha) = M$$

$$D_8 = 0$$

Hence,

$$D_7 = M/(\pi r_0^2)$$

Case a: <u>a circular hole at the apex</u>

A couple M is applied by means of a vertical force distribution along the hole $\theta = \theta_0$ with the magnitude of $M \cos\phi / (R_2^2 \sin^2\theta)$. The boundary conditions are

$$M_1 = 0, \quad Q_x = 0 \quad \text{at } \theta = \theta_0 \qquad (4\text{-}16)$$

in which Q_x is the component of force in the direction perpendicular to the axis of the shell, i.e.,

$$Q_x = T_1 \cos\theta + \left(N_1 + \frac{1}{R_2 \sin\theta} \frac{\partial M_{12}}{\partial \phi} \right) \sin\theta$$

Substitution for M_1 and Q_x from Tables 3-3 and 3-4 into Eqs.(4-16) results in a system of two equations for the two constants D_1 and D_2

$$\begin{aligned} E_{11} D_1 + E_{12} D_2 &= 0 \\ E_{21} D_1 + E_{22} D_2 &= F_2 \end{aligned} \qquad (4\text{-}17)$$

where

$$E_{11} = h_1 \left[\mathrm{Kei}_1\eta + (1-\mu)\sqrt{\frac{c}{R_2}} \left\{ \cot\theta \, \mathrm{Ker}_1'\eta - \sqrt{\frac{c}{R_2}} \frac{1}{\sin^2\theta} \mathrm{Ker}_1\eta \right\} \right]_{\theta_0}$$

$$E_{12} = h_1 \left[-\mathrm{Ker}_1\eta + (1-\mu)\sqrt{\frac{c}{R_2}} \left\{ \cot\theta \, \mathrm{Kei}_1'\eta - \sqrt{\frac{c}{R_2}} \frac{1}{\sin^2\theta} \mathrm{Kei}_1\eta \right\} \right]_{\theta_0}$$

$$E_{21} = h_1 \sqrt{\frac{c}{R_2}} \frac{1}{\sin\theta} \left(\mathrm{Kei}_1'\eta - \sqrt{\frac{c}{R_2}} \cot\theta \, \mathrm{Kei}_1\eta \right)_{\theta_0} \qquad (4\text{-}18)$$

$$E_{22} = h_1 \sqrt{\frac{c}{R_2}} \frac{1}{\sin\theta} \left(-\mathrm{Ker}_1'\eta + \sqrt{\frac{c}{R_2}} \cot\theta \, \mathrm{ker}_1\eta \right)\Big|_{\theta_0}$$

$$F_2 = -\,T_1^* \cos\theta / \cos\phi \,\Big|_{\theta_0}$$

Case b: a circular rigid insert at the apex

As shown in Fig.4-5 the rigid insert does not deform but rotates through an angle when the moment M is applied. The shell has to rotate through the same angle to keep its original angle between the insert and the shell. The boundary conditions are

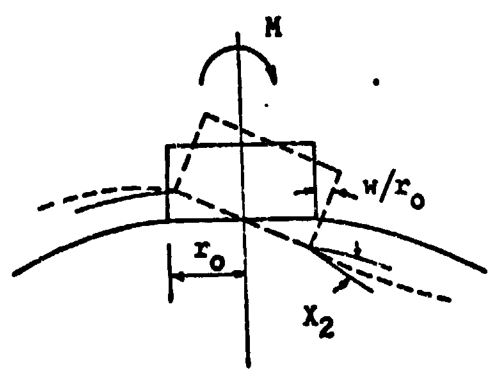


Fig. 4-5:

Deformation of a shell of revolution with a rigid insert under a moment

$$\epsilon_2 = 0, \qquad \chi_2 = -w/r_o \quad \text{at } \theta = \theta_o \qquad (4\text{-}19)$$

Substitution for ϵ_2, X_2 and w from Tables 3-3 and 3-4 into Eqs.(4-19) yields a system of two equations for the determination of the two constants D_1 and D_2

$$\begin{aligned} E_{31} D_1 + E_{32} D_2 &= F_3 \\ E_{41} D_1 + E_{42} D_2 &= 0 \end{aligned} \qquad (4\text{-}20)$$

where

$$E_{31} = h_1 \left[Ker_1 \eta - (1+\mu)\sqrt{\frac{c}{R_2}} \left\{ \cot\theta \, Kei_1' \eta - \sqrt{\frac{c}{R_2}} \frac{1}{\sin^2\theta} Kei_1 \eta \right\} \right]_{\theta_o}$$

$$E_{32} = h_1 \left[Kei_1 \eta + (1+\mu)\sqrt{\frac{c}{R_2}} \left\{ \cot\theta \, Ker_1' \eta - \sqrt{\frac{c}{R_2}} \frac{1}{\sin^2\theta} Ker_1 \eta \right\} \right]_{\theta_o}$$

$$E_{41} = h_1 \left[-\sqrt{\frac{R_2}{c}} Ker_1' \eta + \frac{1}{\sin\theta} Ker_1 \eta \right]_{\theta_o} \qquad (4\text{-}21)$$

$$E_{42} = h_1 \left[-\sqrt{\frac{R_2}{c}} Kei_1' \eta + \frac{1}{\sin\theta} Kei_1 \eta \right]_{\theta_o}$$

$$F_3 = (-T_2^* + \mu T_1^*)/\cos\phi$$

Case c: a nozzle at the apex

The boundary conditions are the same as those in the case c for the axisymmetric deformation, except the rotation which, for this case, is shown in Fig. 4-6

$$M_1 = M_1^c, \qquad \epsilon_2 = \epsilon_2^c$$

$$Q_x = -Q_x^c, \quad X_2 + X_2^c = -w/r_o$$

at $\alpha = 0$ and $\theta = \theta_o$.

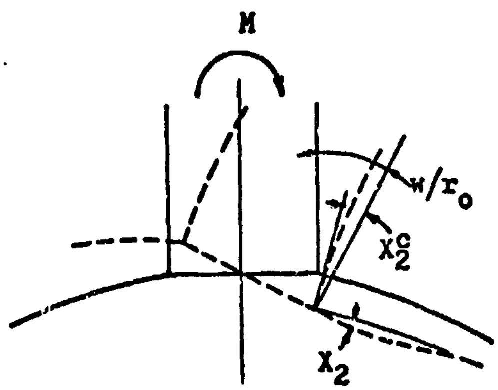


Fig. 4-6:

Deformation of a shell of revolution with a nozzle under a moment

These conditions upon substitution for the quantities M_1, Q_x, ϵ_2 and X_2 by their expressions from Tables 3-3 and 3-6 result in a system of four equations for determination of the four constants D_1, D_2, D_5 and D_6.

$$\begin{aligned} &E_{11} D_1 + E_{12} D_2 + E_{16} D_6 = F_1 \\ &E_{21} D_1 + E_{22} D_2 + E_{25} D_5 + E_{26} D_6 = F_2 \\ &E_{31} D_1 + E_{32} D_2 + E_{35} D_5 + E_{36} D_6 = G_3 \\ &E_{41} D_1 + E_{42} D_2 + E_{45} D_5 + E_{46} D_6 = 0 \end{aligned} \qquad (4\text{-}23)$$

in which E_{jk} for $j = 1, 2, 3, 4$ and $k = 1, 2$ and F_2 have been defined in Eqs.(4-16) and (4-21). The remainder are given by

$$\begin{aligned} E_{16} &= c^*/c \\ E_{25} &= - E_{26} = - b_2\, c^*/r_o \\ E_{35} &= - h/h^* \\ E_{45} &= b_1\, h/h^* \\ E_{36} &= (h/h^*)\, \mu/(2b^2) \\ E_{46} &= b_2\, h/h^* \\ F_1 &= - (1 - \mu)/(2b^2)\, M/(\pi r_o^2) \\ G_3 &= F_3 - (h/h^*)\, \mu M/(\pi r_o^2) \end{aligned} \qquad (4\text{-}24)$$

and

$$r_o = R_2 \sin \theta \Big|_{\theta_o}$$

V. ANALYSIS OF NUMERICAL RESULTS

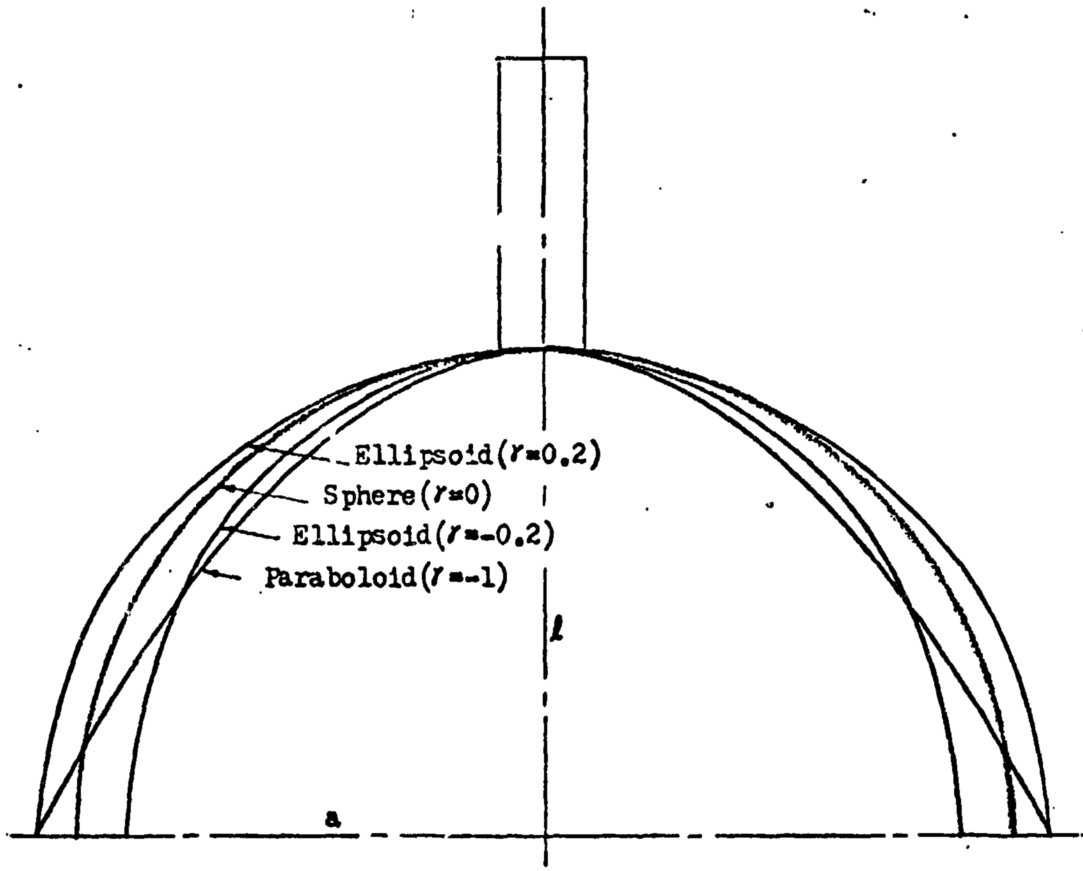
Numerical results are obtained for spherical shells, ellipsoids, and paraboloids, which are of common interest in engineering structures, of which the generating curves (Fig.5-1) are defined by the equations

$$R_1 = R^*/(1 + \gamma \sin^2\theta)^{3/2}$$
$$R_2 = R^*/(1 + \gamma \sin^2\theta)^{\frac{1}{2}} \qquad (5\text{-}1)$$

The results are compared with the limited experimental data which are available only for the spherical shell attached to a cylindrical nozzle. For each class of shells stresses are computed for three different types of discontinuity. Physical interpretation as to the effects on the stresses due to the presence of a discontinuity is given with the spherical shell under internal pressure. A study of the optimum ratio r_o/h^* of the nozzle which makes the stresses of a given spherical shell a minimum has been determined. Determination of a favorable ratio a/l among ellipsoids with a nozzle attachment, which contain the same volume and use the same amount of material, is also studied. A computer program feasible for all these studies has been written in Fortran IV language to accomplish all the necessary computation.

Comparison of Theoretical and Experimental Stresses

Let $\gamma = 0$ and $R^* = R$ in Eqs.(5-1) from which one obtains the equations for the spherical shell



$$R_1 = R^*/(1 + \gamma \sin^2\theta)^{3/2}$$

$$R_2 = R^*/(1 + \gamma \sin^2\theta)^{\frac{1}{2}}$$

Fig. 5-1: Generating curves of shells of revolution

$$R_1 = R_2 = R = \text{constant}$$

The dimensions of the experimental model tested by Maxwell and Holland(31) and the external loads are as follows:

$R = 15.255$ in. $\quad h = 0.38$ in.

$r_0 = 1.281$ in. $\quad h^* = 0.0625$ in.

$p = 200$ psi $\quad M = 2{,}400$ in-lbs.

In all cases Poisson ratio μ is set equal to 0.3. Comparisons of theoretical and experimental stresses are shown in Fig.5-2 for the pressure loading and in Fig. 5-3 for the moment loading. In general, good agreement is obtained except for σ_t of the outer surface of the sphere (Fig. 5-2) which shows a different trend between theoretical and experimental stress near the junction. However, this discrepency is rather insignificant because of its smallness in magnitude in comparison with the magnitude of σ_z. It is seen that better agreement is obtained in the moment loading (Fig. 5-3).

Pressure Loading

(A) Spherical Shells

Effect of a Discontinuity on Stresses and Its Physical Interpretation

To study the effect of the different types of discontinuity on the stresses, the numerical results were obtained for the following set of data

$$R/h = 100, \qquad r_0/h^* = 20$$

and were shown in Fig.5-4 for pressure loading.

Study of Fig.5-4 reveals that the stress concentration in the case

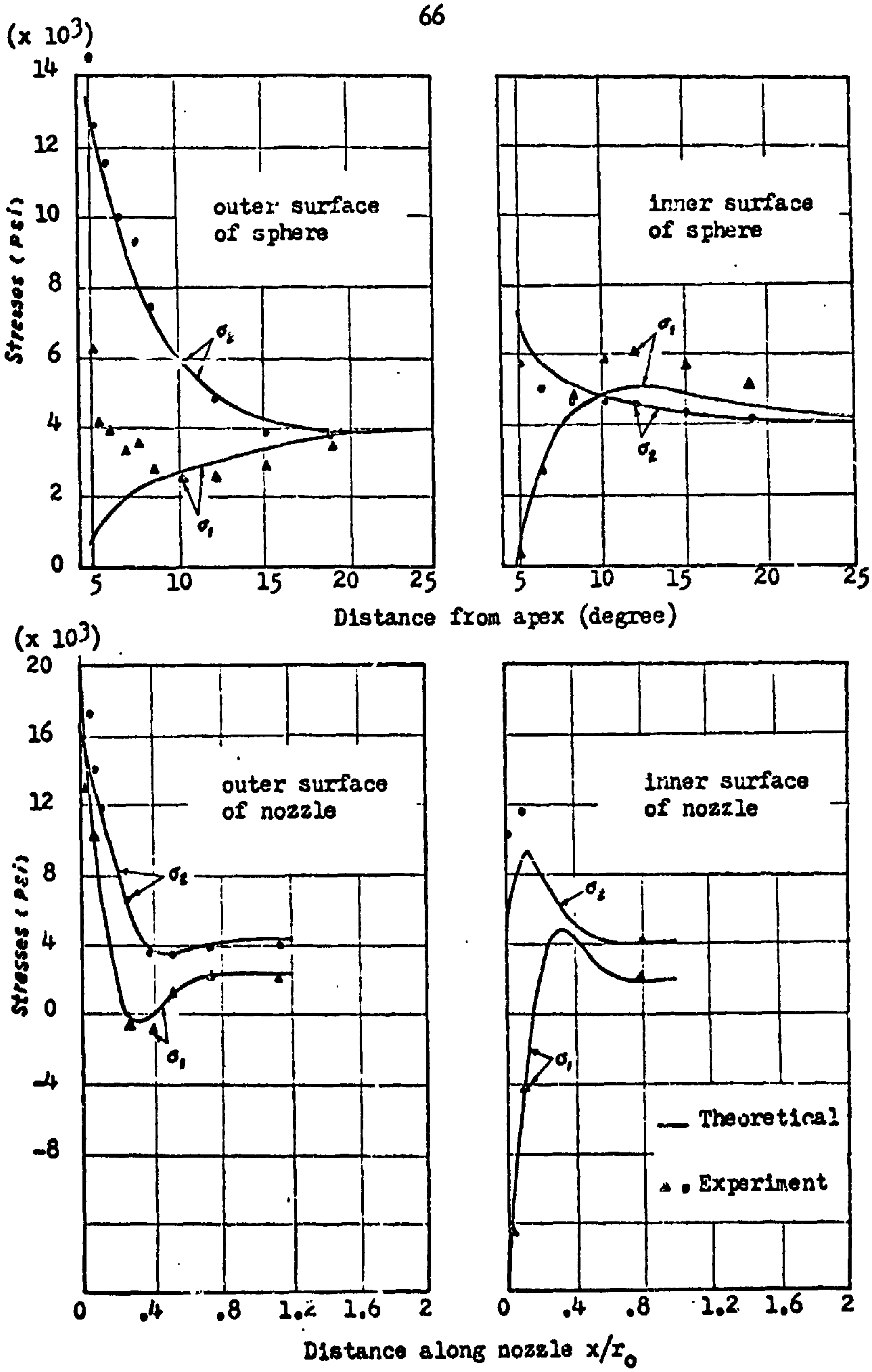


Fig. 5-2: Comparison of theoretical and experimental stresses internal pressure, 200 psi.

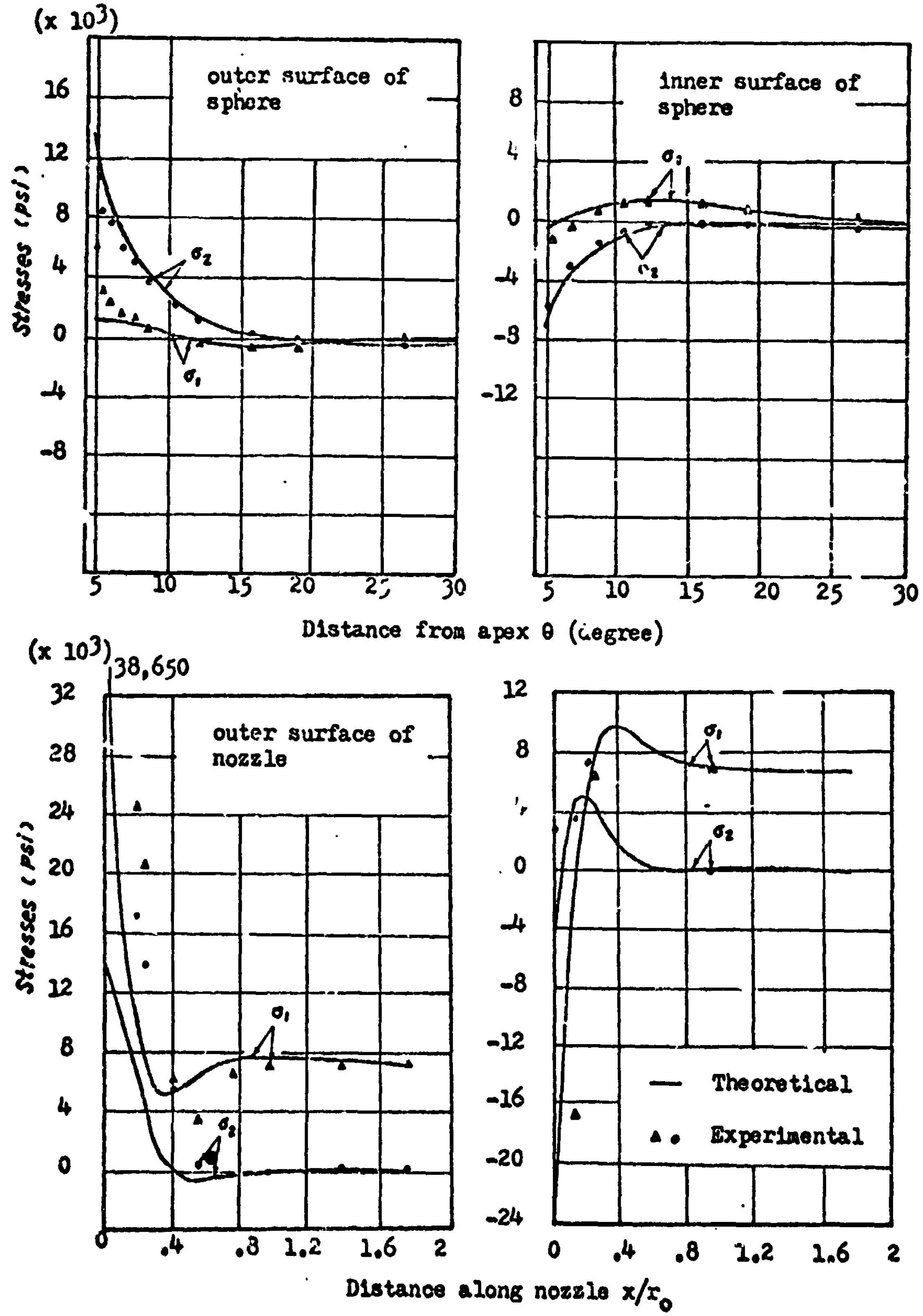


Fig. 5-3: Comparison of theoretical and experimental stresses in the plane of loading - moment 2400 in-lbs.

of the hole is much higher than that in the case of the rigid insert. Presence of the hole causes large values of hoop stress σ_2, while presence of the rigid insert induces significant meridian stress σ_1.

These results can be deduced from the consideration of the deformation. Suppose that the shell does not have any discontinuity, then, due to the application of internal pressure, the shell is essentially in the state of membrane stresses for which $T_1 = T_2 = \frac{1}{2}$ pR. Let Q_x, Q_z be the horizontal and vertical components of T_1, respectively. The radius r_o before deformation is stretched into r_o^* after deformation (Fig.5-5), and the strain ϵ_2 in the circumferential direction is equal to $(1-\mu)pR/2Eh$. When a discontinuity in terms of a circular hole of radius r_o is present the boundary conditions imply that

$$M_{1h} = 0, \qquad Q_{xh} + T_{1h} = 0$$

along the hole (where subscript h is associated with the hole). The hole of radius r_0 deforms into a hole of radius r_{oh}^* (Fig. 5-6), which, because of the zero value of Q_{xh}, will be larger than r_o^*. Consequently, the strain ϵ_{2h} will be also larger than ϵ_2. From this it follows that the hoop tension $T_{2h}(= Eh\,\epsilon_{2h})$ is also larger than T_2.

To show there exists a moment M_{2h} in the circumferential direction, it is noticed that

$$M_{1h} = \frac{Eh^3}{12(1-\mu^2)}(\kappa_1 + \mu\kappa_2) = 0$$

From this it follows that

$$k_1 = -u\,k_2$$

and

$$M_{2h} = \frac{Eh^3}{12(1-\mu^2)}(\kappa_2 + \mu\kappa_1) = -\frac{Eh^3}{12\mu}\kappa_1 \approx -\frac{Eh^3}{12\mu}\frac{\partial X_2}{R\partial\theta}$$

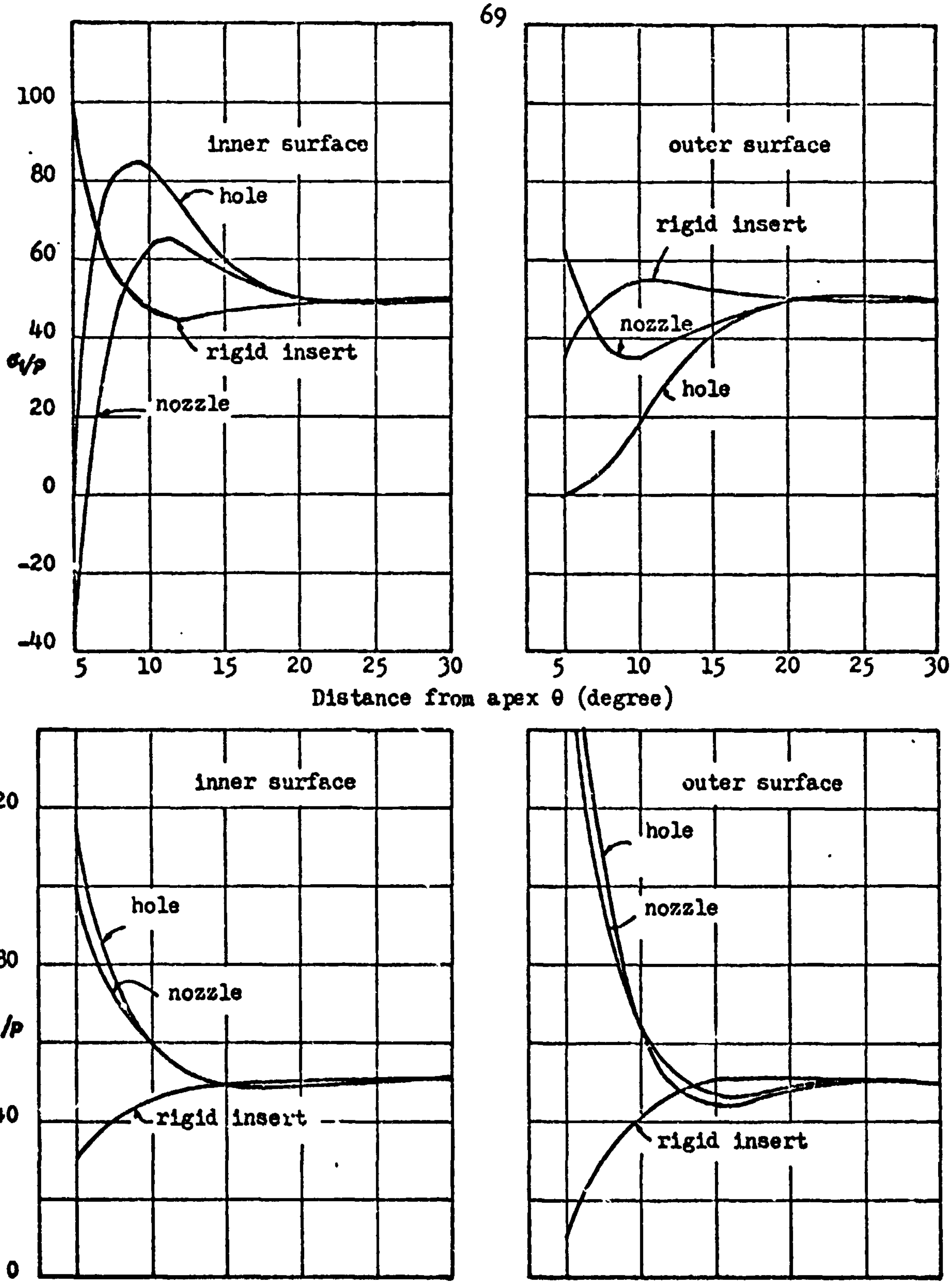


Fig. 5-4: Comparison of stresses among different types of discontinuity for internal pressure. (Sphere, $l/h = 100$, $r_o/h^* = 20$, $\theta_o = 5^o$)

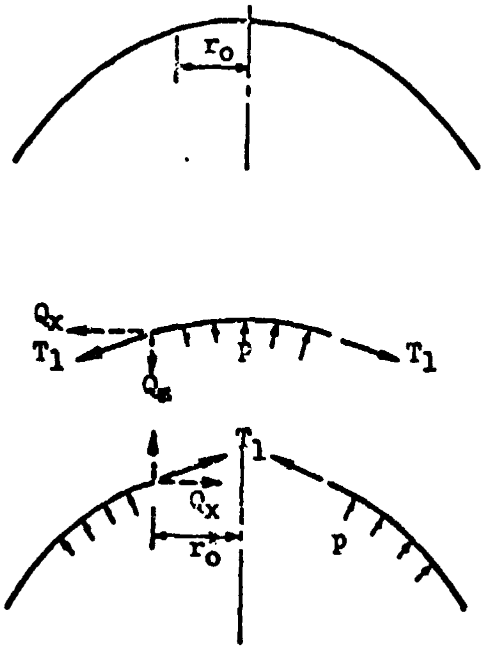


Fig. 5-5:
Deformation of a spherical shell

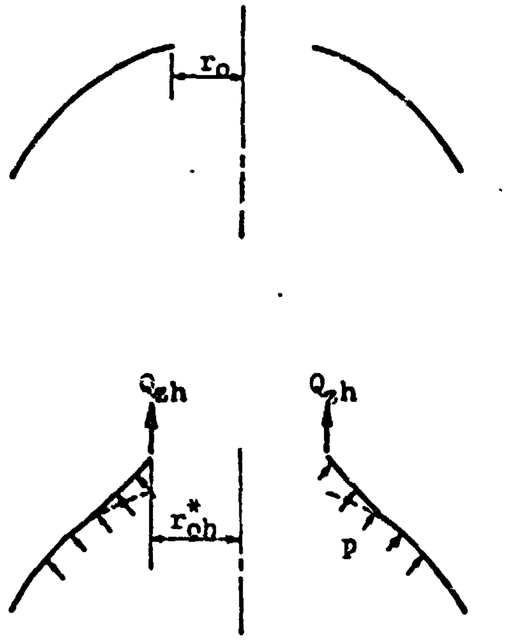


Fig. 5-6:
Deformation of a spherical shell with a circular hole

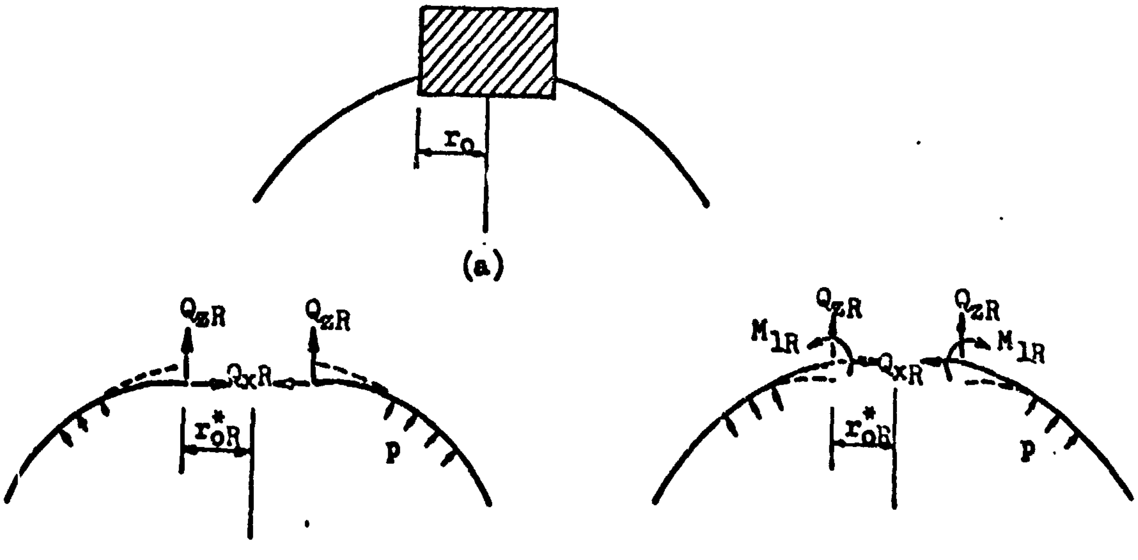


Fig. 5-7: Deformation of a spherical shell with a rigid insert

in which X_2 is the rotation about the line $\theta = \theta_c$. It can be seen from Fig. 5-6 that $\frac{\partial X_2}{\partial \theta}$ is a negative quantity, hence, M_{2h} is a positive value. This agrees with the stress shown in Fig. 5-4. The stress on the surface of the shell is computed using the formula

$$\sigma_2 = \frac{T_2}{h} \pm \frac{6M_2}{h^2}$$

Hence, σ_2 of the outer surface is a significant stress in the case of the circular hole discontinuity.

When a discontinuity in terms of a rigid insert is present in a shell, the strain ϵ_2 and rotation X_2 vanish along the rigid insert. The deformation of the shell is shown in Fig. 5-7 in two steps. Because of the zero strain, r_{oR}^* (the subscript R is associated with rigid insert) must be equal to its original length r_o. To fulfil this condition, the horizontal force Q_{xR} has to be larger than Q_x of the membrane state. As a consequence of this larger Q_{xR} a rotation is produced as shown in Fig. 5-7b. Since the shell has to retain zero rotation along the insert, a negative moment is required to compensate this rotation. The final configuration is shown in Fig. 5-7c. The zero value of strain along the insert implies that

$$T_{2R} = \mu T_{1R}$$

To show the relative magnitude between M_1 and M_2 it is necessary to evaluate the change of curvature k_2.

$$\kappa_2 = -\frac{\cot\theta}{R^2}(w' - u) = \frac{\cot\theta}{R} X_2 = 0$$

Thus,

$$M_{2R} = \mu M_{1R}$$

Notice that M_{1R} is a negative value, hence, M_{2R} is also a negative value. This agree with the stress shown in Fig. 5-4. Both the ratios T_{2R}/T_{1R} and M_{2R}/M_{1R} equal μ, which is less than $\frac{1}{2}$ for most of the materials. Hence, σ_1 of the inner surface is a significant stress in the case of rigid insert.

Next, when the shell is connected by a nozzle, with a rigidity between that of a rigid insert and that of a circular hole, one would anticipate that the stresses of the shell would fall in between these two extreme cases. The rigidity of a nozzle of $r_o/h^* = 20$ being used for computing the numerical results is rather close to the flexibility of a circular hole, in which case σ_2 is of significance. Consequently, the stress σ_2 of the shell should close to that in the case of a circular hole. This result again agrees with the stress σ_2 shown in Fig. 5-4. However, the stress σ_1 does not follow this conclusion at and near the junction. The physical interpretation of this behavior is possible, however, it is complicated by the fact that four conditions are required to be fulfilled across the junction. Besides, the magnitude of σ_1 is less important. No attempt is made to analyze this behavior.

<u>Optimum ratio r_o/h^* of a Nozzle</u>

From the previous analysis it is understood that a discontinuity of a circular hole causes a higher stress concentration than that of a rigid insert. With a nozzle attached to a shell the stress variations of the shell between these two extreme cases can be studied by changing the ratio r_o/h^* of the nozzle. It is believed that a proper choice of a nozzle could minimize the stress concentration in the shell. The stresses of a shell with $R/h = 1{,}000$ have been computed for various values of r_o/h^* of a nozzle and the nondimensional stresses σ_1/p and σ_2/p at the junction ($\theta = 5^o$) are plotted in Fig. 5-8. The stress σ_1 on both the outer and the inner surface attains its maximum values at r_o/h^* around 80, and decreases as r_o/h^* increases and finally approches zero as r_o/h^* goes to infinity (which is the case of the circular hole). σ_2 of the inner and the outer surface increases as the nozzle becomes thinner and thinner and finally approches the values of the stresses for the case of a cicular discontinuity as the ratio r_o/h^* reaches infinity. The stresses of the shell with a discontinuity of rigid insert are shown on the left hand side of the figure. The curves shown in solid lines are terminated at $r_o/h^* = 20$ since below this value the accuracy of thin shell theory is questionable. Nevertheless, the curves showing the stresses in the region between $r_o/h^* = 20$ and rigid insert are connected in a manner with stresses obtained from thin shell theory as a guide. It is quite interesting to see that all curves meet at a point where the stress σ/p is approximately equal to 500, which is the membrane stress. At this point $\sigma_{1\,inner} = \sigma_{1\,outer}$ and the moment $M_1 = 0$. For this optimum value the ratio r_o/h^* is located around 8.

(B) Ellipsoids

When the value of γ is great than -1, Eqs.(5-1) represent generating curves of ellipsoids. γ is related to the ratio of semiaxes by

$$\gamma = \frac{a^2}{l^2} - 1$$

Two ellipsoids with $\gamma = 0.2$ and -0.2 , which are equivalent to having the

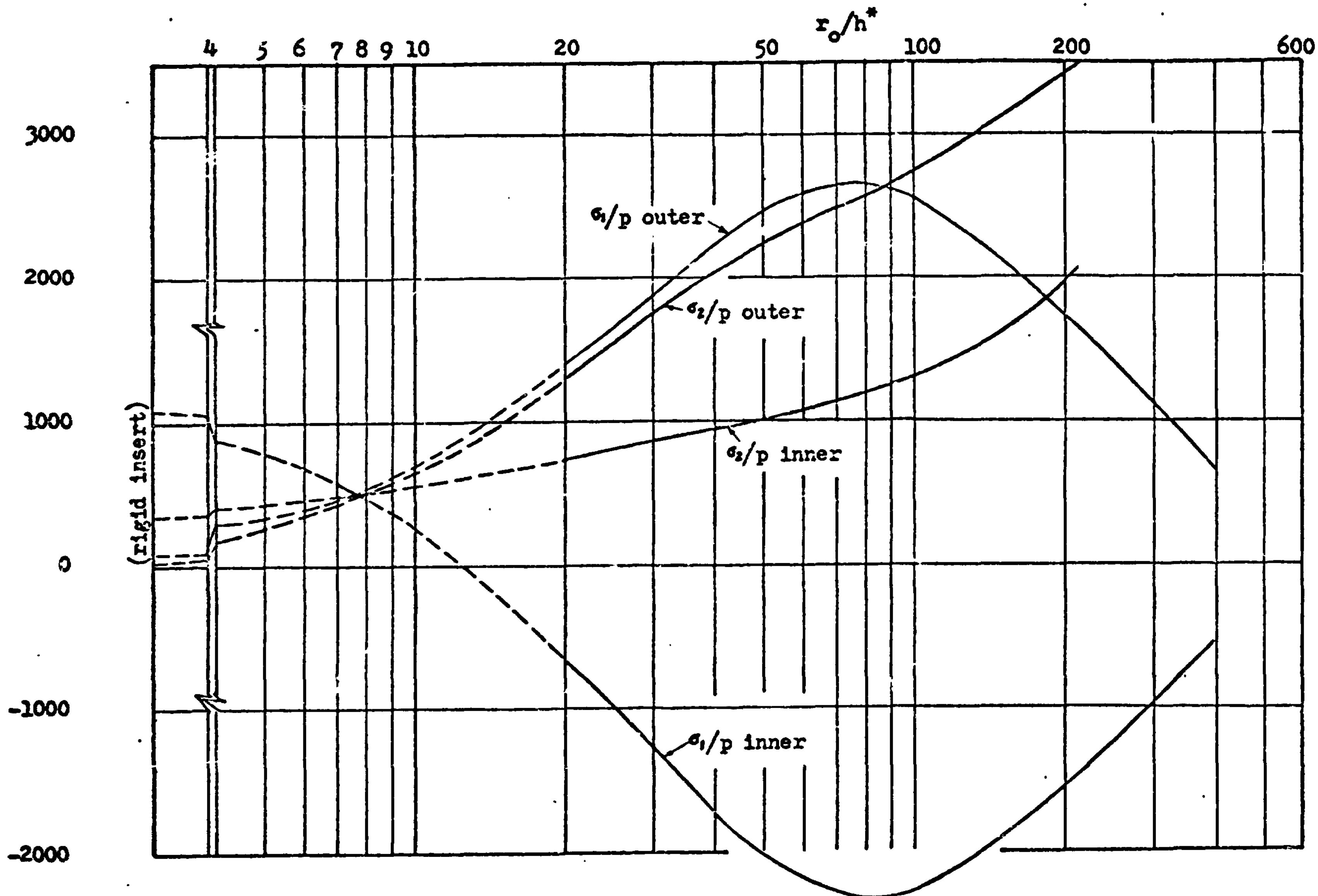


Fig. 5-8: Stresses of a spherical shell at the junction.

square ratio of semiaxes a^2/ℓ^2 equal to 1.2 and 0.8 (Fig. 5-1), respectively, are chosen for computing the stresses. Other parameters used are $\ell/h = 100$, and $r_o/h^* = 20$. The length of semiaxis ℓ remains the same for the two ellipsoids and equals the radius R of the sphere.

Comparison of the stresses due to the effects of three types of discontinuity are shown in Fig. 5-9 and Fig. 5-10. The stress variations along the meridian reveal a similar pattern to those of the spherical shell shown in Fig. 5-4. The ellipsoid with $a^2/\ell^2 = 1.2$ appears to have higher stresses and another one with $a^2/\ell^2 = 0.8$ has lower stresses than the spherical shell. The effects of the discontinuity on the stresses also show that a circular hole type of discontinuity gives higher stresses than a rigid type and that the stresses for the shell with a nozzle fall in between.

(C) Paraboloids

When $\gamma = -1$, Eqs.(5-1) represent generating curves of paraboloids and are reduced to

$$R_1 = R^*/\cos^3\theta$$

$$R_2 = R^*/\cos\theta$$

R^* is chosen to be equal to $a^2/2\ell$ such that the generating curve passes through the end points of the major axis of the ellipsoid with $a^2/\ell^2 = 1.2$ as shown in Fig. 5-1. The stresses are shown in Fig. 5-11 for three types of discontinuity. Similar conclusions to the spherical shell are obtained except that the magnitudes are lower than those of the spherical shell.

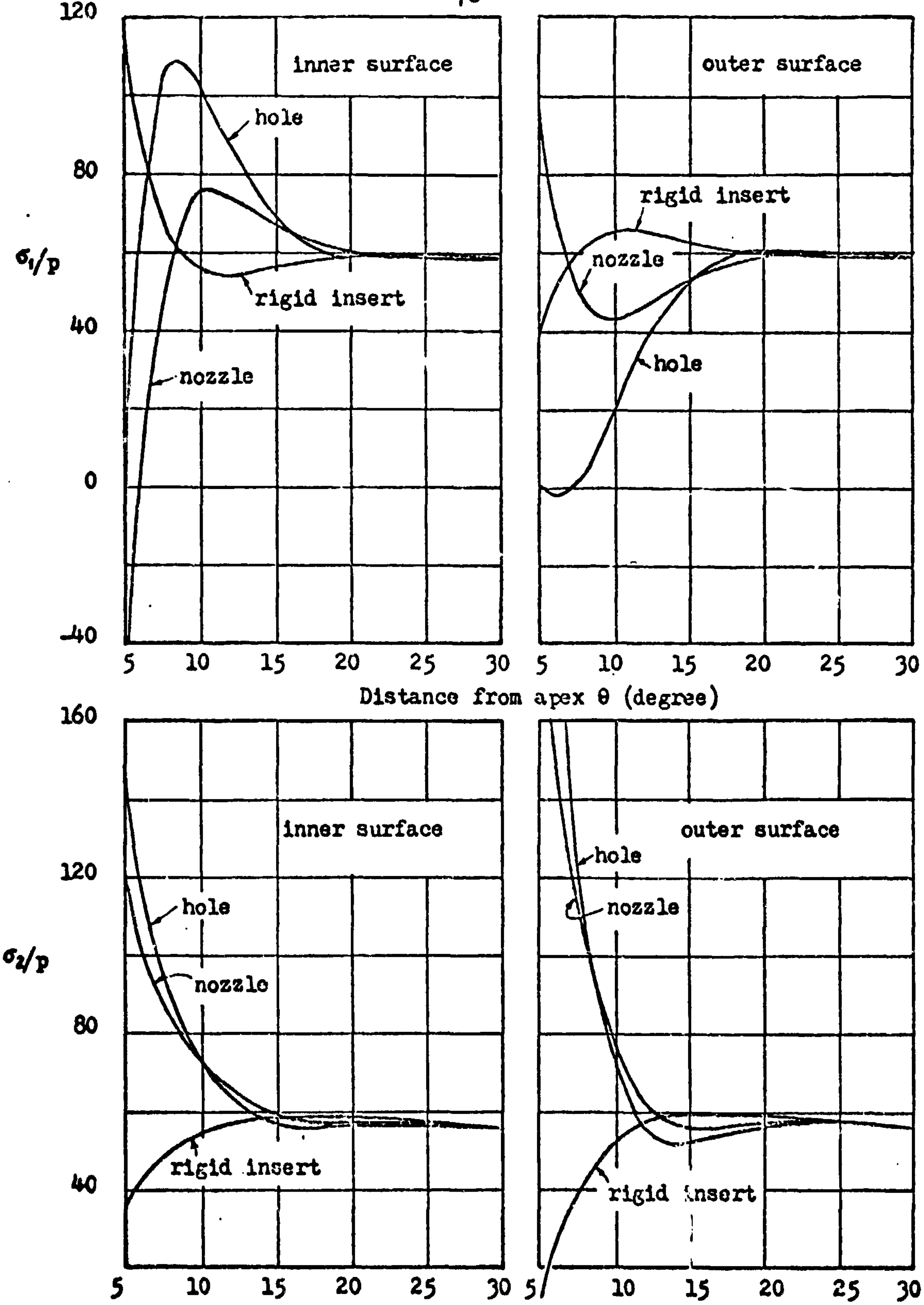


Fig. 5-9: Comparison of stresses among different types of discontinuity, internal pressure, (Ellipsoid, $a^2/\ell^2 = 1.2$, $\ell/h = 100$, $r_o/h^* = 20$, $\theta_o = 5^o$)

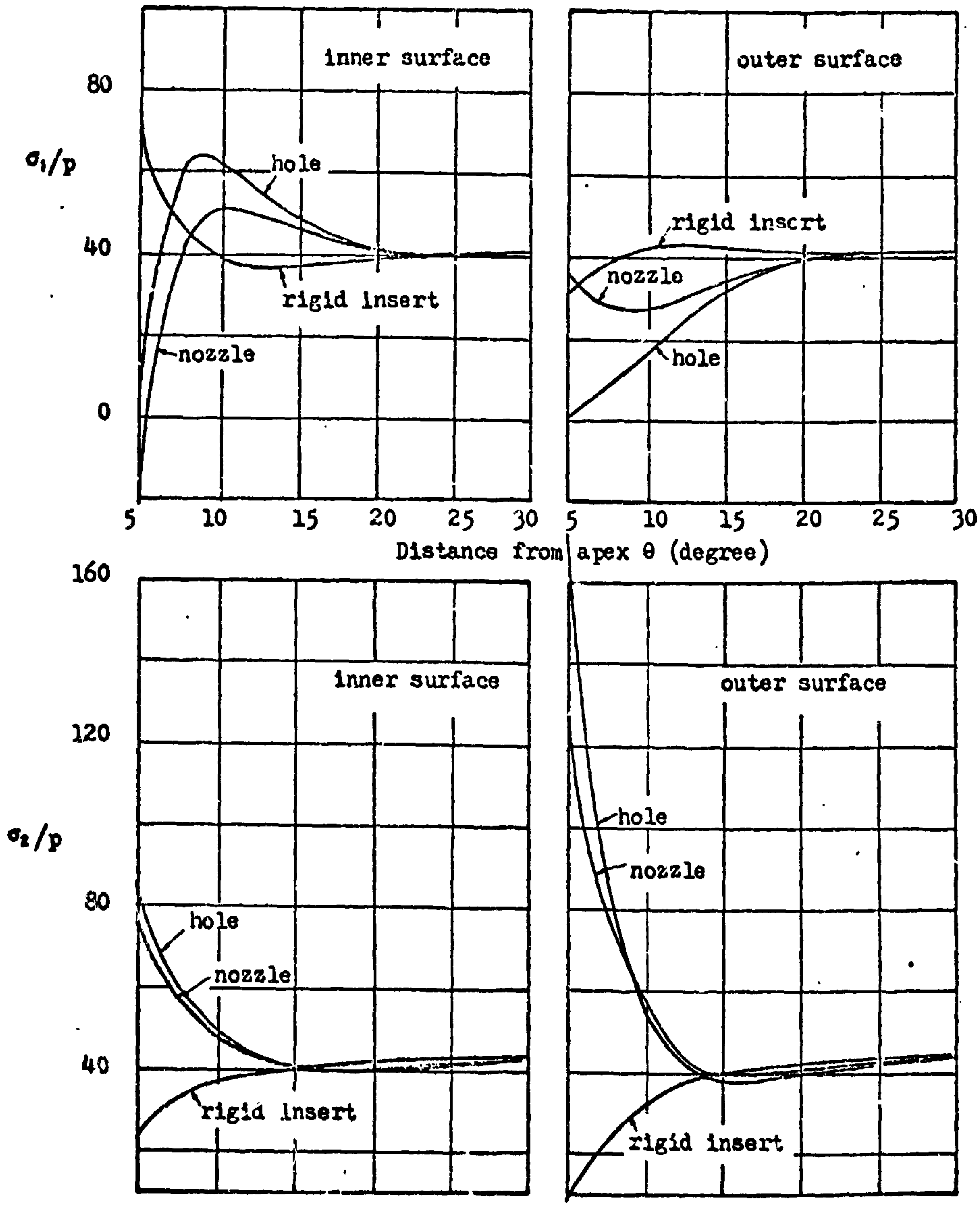


Fig. 5-10: Comparison of stresses among different types of discontinuity, internal pressure. (Ellipsoid, $a^2/\ell^2 = 0.8$, $\ell/h = 100$, $r_o/h^* = 20$, $\theta_o = 5^o$)

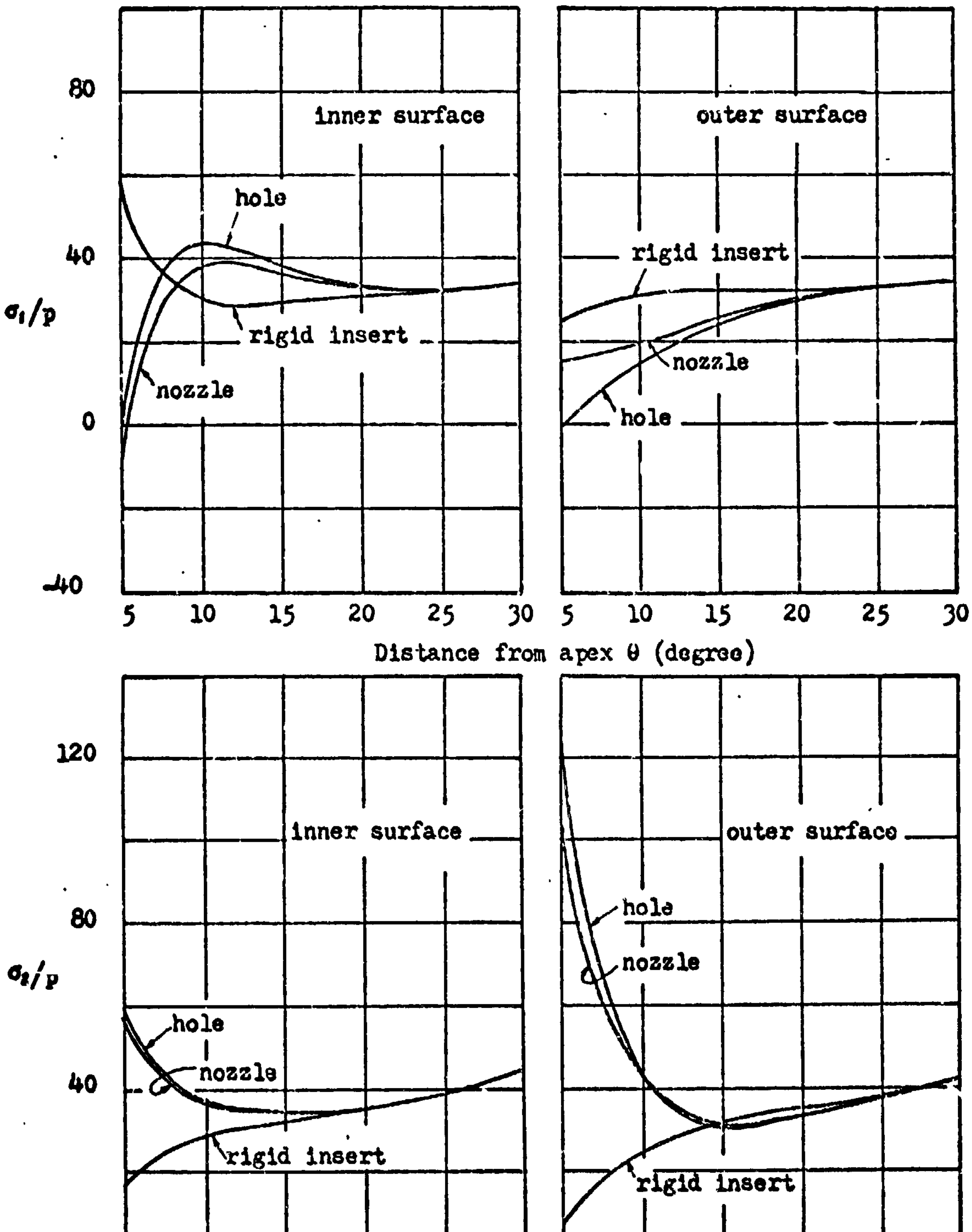


Fig. 5-11: Comparison of stresses among different types of discontinuity, internal pressure. (Paraboloid, $\gamma = -1$, $l/h = 100$, $r_o/h^* = 20$, $\theta_o = 5^o$)

(D) Optimum Ratio a/ℓ of Ellipsoids with a Nozzle under Internal Pressure

It is the attempt of this section to find, among ellipsoids which contain the same volume and use the same amount of material, the one which has minimum stress due to the effect of a nozzle attachment under internal pressure.

Let V and S be the volume and surface area, respectively. For a spherical shell having thickness h, its volume and surface area are given by

$$V = 4\pi R^3 / 3$$

$$S = 4\pi R^2$$

For an ellipsoid with its major axis as the axis of revolution, and its semi-major ℓ, semi-minor a, thickness h_e, the volume and surface area are given by

$$V = 4\pi a^2 \ell / 3$$

$$S = 2\pi a^2 + 2\pi \frac{a\ell}{\epsilon} \sin^{-1}\epsilon$$

where ϵ is the eccentricity defined by

$$\epsilon^2 = 1 - a^2/\ell^2$$

The condition that all ellipsoids have the same volume as the spherical shell of radius R gives

$$\frac{\ell}{R} = 1/(\frac{a}{\ell})^{2/3}, \qquad \frac{a}{R} = (\frac{a}{\ell})^{1/3}$$

Another condition that they use the same amount of material as the spherical shell gives

$$4\pi R^2 h = 2\pi(a^2 + \frac{a\ell}{\epsilon}\sin^{-1}\epsilon)\, h_e$$

from which one obtains, after certain manipulation

$$\frac{\ell}{h_e} = \frac{1}{2}\frac{R}{h}\left[1 + \frac{\sin^{-1}\epsilon}{\epsilon}\frac{\ell}{a}\right]$$

The data chosen for this study are

$$R/h = 100, \quad r_o/h^* = 20, \quad \text{and} \quad \theta_o = 5^o$$

The stresses are computed for various values of the ratio a/ℓ, and are shown in Fig. 5-12 for σ_2 on the outer surface, which gives the maximum stress. For these ellipsoids the corresponding ratios ℓ/h_e are

a/ℓ	ℓ/h_e
1	100
0.8	117.03
0.6	146.59
0.5	170.92

As shown in Fig. 5-12 when the value of a/ℓ decreases the stress σ_2 at the junction ($\theta_o = 5^o$) decreases, however, it increases at $\theta = 90^o$ where the effect due to discontinuity disappears. The ellipsoid which acquires the minimum stress falls somewhere between $a/\ell = 0.6$ and 0.5.

Moment Loading

The stresses of spherical shells due to the effect of three types of discontinuity under moment loading are plotted in Fig. 5-13. It can be seen from this figure that high hoop tensile stress (σ_2) occurs in the discontinuity of a circular hole, while the meridian stress (σ_1) is significant in the discontinuity of a rigid insert, and that σ_2 in the former case is higher than σ_1 in the latter case. In other word, a circular hole causes a higher stress concentration than does a rigid insert in the same spherical shell. The stresses of the sphere with a nozzle

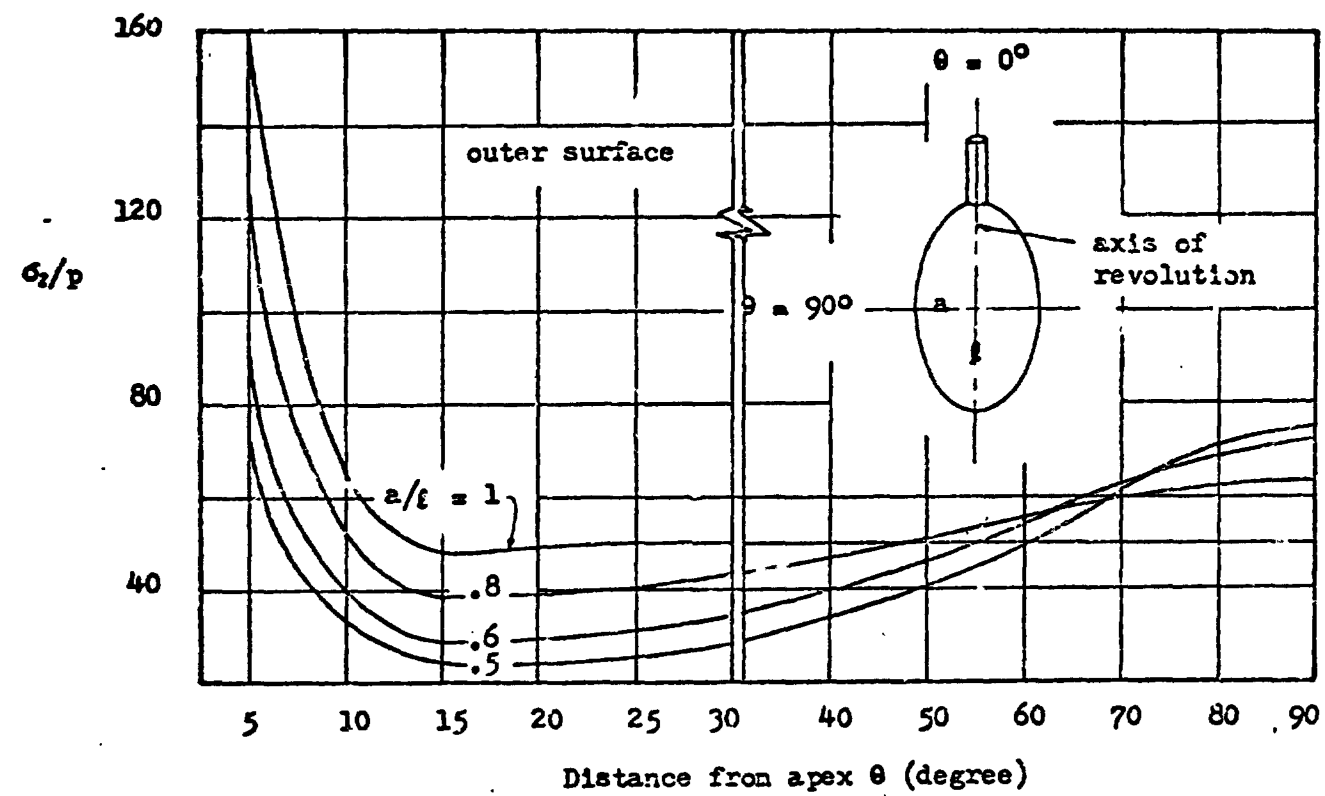


Fig. 5-12: Stress distributions of ellipsoids having the same volume and the same weight.

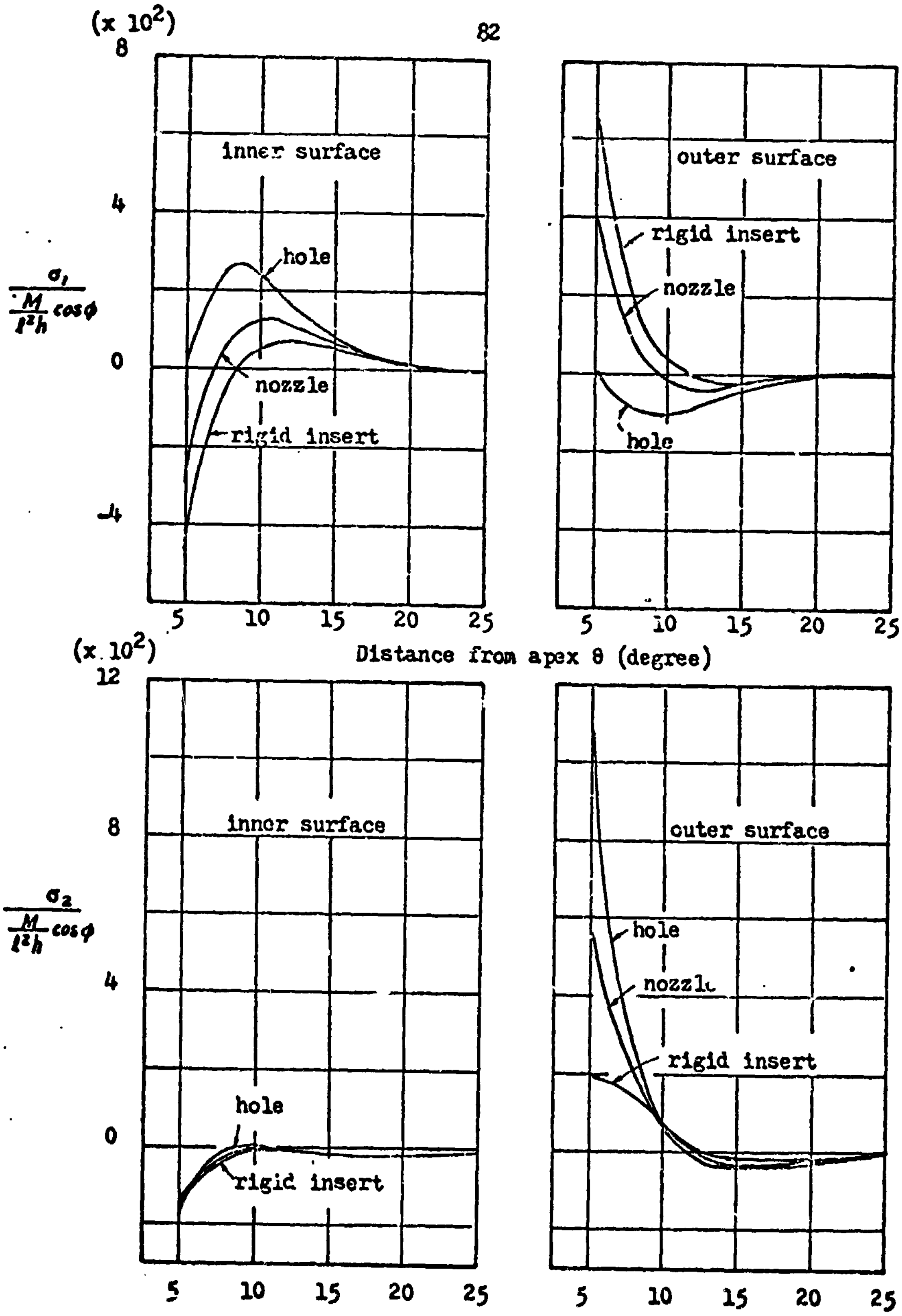


Fig. 5-13: Comparison of stresses among different types of discontinuity, moment loading. (Sphere, $R/h = 100$, $r_o/h^* = 20$, $\theta_o = 5^o$)

attachment fall in between these two extreme cases. The same conclusions were also true in the case of pressure loading.

The stress distribution along the meridian for ellipsoids under moment loading shows a pattern similar to that of a spherical shell except for a slight difference in the magnitude of the stresses. In the case of nozzle attachment, there is little difference in σ_1 among the ellipsoids of the ratio $a^2/\ell^2 = 0.8$, 1, and 1.2 with ℓ remaining constant (Fig. 5-14). However, the difference in σ_2 is pronounced, which increases rapidly as the ratio a^2/ℓ^2 decreases. This result is contrary to the one obtained in the pressure loading, where the stresses decrease as a^2/ℓ^2 decreases.

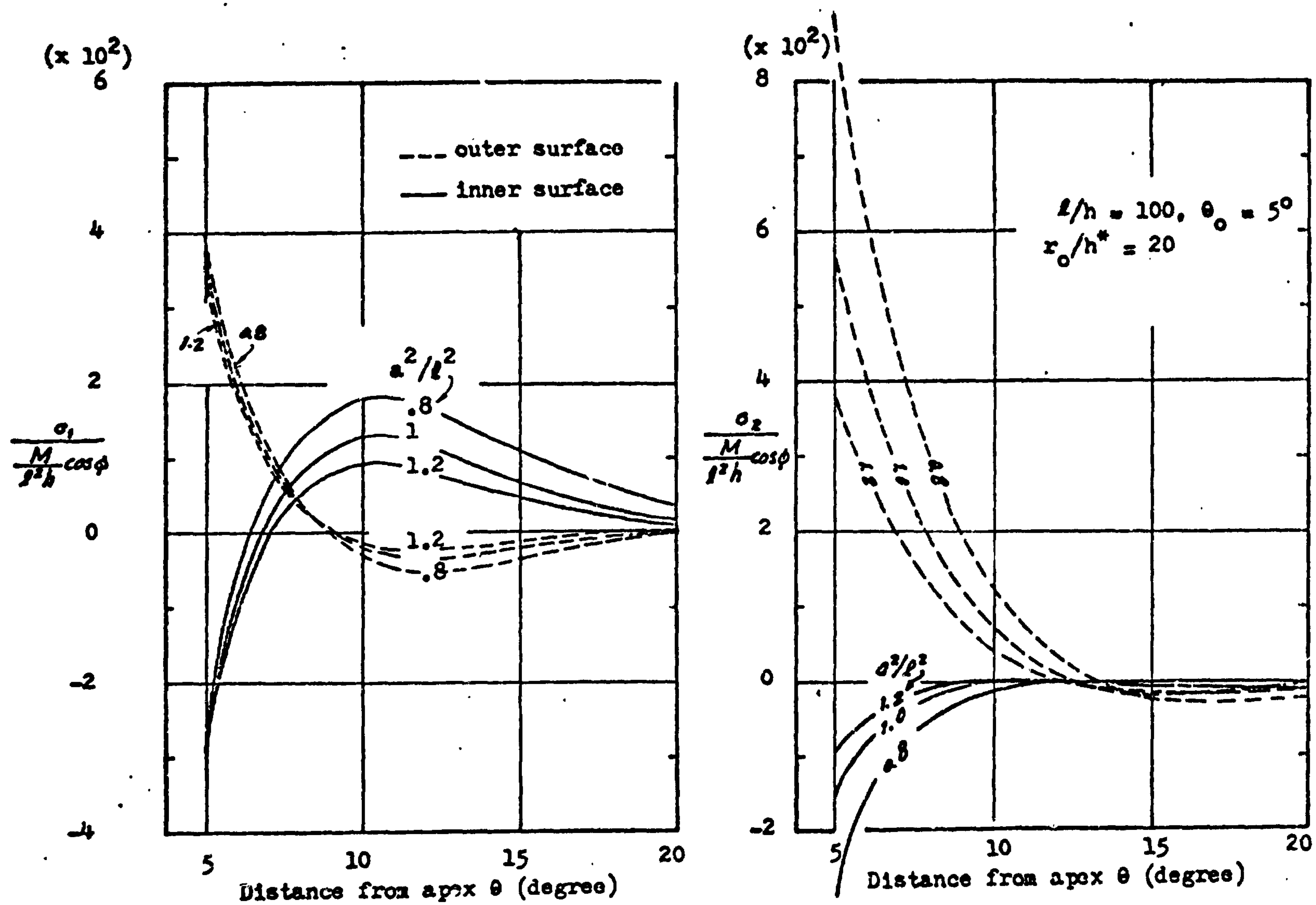


Fig. 5-14: Stresses of ellipsoids with different values of eccentricity, moment loading.

VI. SUMMARY AND CONCLUSIONS

Governing differential equations for shells of revolution pertaining to axisymmetric and moment loadings have been reduced from the basic equations of the general theory of thin shells in terms of complex forces.

For the axisymmetric case, the analysis of shells of revolution has been reduced to the integration of a second order differential equation. Method of asymptotic integration is employed. The solution valid in the region $0 \leq \theta < \pi$ is obtained in terms of Thompson function of order zero, provided the shell is sufficiently smooth near the apex.

For moment loading applied at the apex the problem has been further reduced to the integration of a second order differential equation. Asymptotic solutions valid in the region $0 \leq \theta < \pi$ are also obtained in terms of Thompson function of order one.

Formulas for displacements, forces, and moments for both axisymmetric and moment loadings are also obtained and listed. Side by side with the shell of revolution the governing differential equations for circular cylindrical shells are also derived. Solutions in terms of exponential functions are obtained for both axisymmetric and moment loadings.

As an application of the solutions derived previously, three cases of discontinuity at the apex of shells of revolution have been studied; a circular hole, a circular rigid insert, and a nozzle. The boundary conditions and the determination of the constants for each of the appropriate cases have been derived.

Numerical results in terms of dimensionless stresses are obtained for shells of revolution having the shapes of spheres, ellipsoids, and paraboloids in which each of the three types of discontinuity is present. Good agreement between theoretical and experimental stresses has been obtained for a spherical pressure vessel with a nozzle. Careful studies of these results reveal significant phenomena from which the following conclusions can be drawn:

(1) A circular hole present at the apex of a shell of revolution weakens the shell more than does a rigid insert on the same shell, that is, the stress concentration in the former is higher than that in the latter.

(2) For the case of a circular hole, the hoop stress σ_2 is higher than the meridian stress σ_1, and the maximum stress (σ_2) occurs on the outer surface of the hole. On the other hand, in the case of a rigid insert, σ_1 is larger than σ_2; the maximum stress (σ_1) also occurs on the outer surface of the insert.

(3) The stresses of a shell of revolution with a nozzle attached at the apex fall in between the stresses of the case of a circular hole and the case of a rigid insert. When the radius to thickness ratio r_o/h^* of the nozzle becomes large the stress distribution of the shell tends toward the case of a circular hole.

(4) The stress concentration due to the attachment of a nozzle may be alleviated, to a certain extent, by proper choice of the value r_o/h^* of the nozzle.

(5) By proper adjustment of the ratio of semiaxes of ellipsoid, it is

possible to obtain, among ellipsoidal pressure vessels containing the same volume and using the same amount of materials, the one which has the minimum stress concentration due to the effect of a nozzle attached at the apex.

The solutions obtained in this dissertation can be easily extended to include the study of the problems in which the external loads are one of the followings: (a) a vertical load; (b) a torsion; (c) a horizontal force, applied at the apex of a shell of revolution. The same computer program with a slight modification can be used in obtaining the stresses for these three cases of loadings.

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   Small deformation of shells of revolution under axisymmetric loading was considered. The basic equations were reduced to a single complex equation which is valid for uniform thickness, as well as for a large class of variable thickness.

9. Langer, R. E., "On the Asymptotic Solutions of Ordinary Differential Equations with Reference to Stokes' Phenomenon about a Singular Point," Trans. Amer. Math. Soc., 37(1935), 397-416.

   Asymptotic solutions of the differential equation

$$\frac{d^2 w}{d s^2} + \left\{ \lambda \psi(s) + T(\lambda, s) \right\} w = 0$$

   are investigated, where λ represents a large parameter, and the coefficients $\psi(s)$ and $T(\lambda,s)$ take the following forms:

(i) $\psi(s) \equiv (s-s_0)^{\nu}\,\psi_1(s)$

with $\nu > -2$ and $\psi_1(s)$ a non-vanishing single-valued analytic function.

(ii) $T(\lambda,s) \equiv A_1/(s-s_0)^2 + B_1/(s-s_0) + C_1(\lambda,s)$

with A_1 and B_1 any constants, and $C_1(\lambda, s)$ an analytic function which is bounded uniformly with respect to λ.

10. Olver, F. W. J., "The Asymptotic Solution of Linear Differential Equations of the Second Order for Large Values of a Parameter," Philo. Trans. Roy. Soc., London Ser. A, 247(1955), 307-

Asymptotic solutions of the differential equations

$$\frac{d^2w}{dz^2} = \{uz^n + f(z)\}w \qquad (n = 0, 1)$$

for large positive values of u, have the formal expansions

$$w = P(z)\left\{1 + \sum_{s=1}^{\infty} \frac{A_s(z)}{u^s}\right\} + \frac{P'(z)}{u}\sum_{s=0}^{\infty}\frac{B_s(z)}{u^s}$$

where P is an exponential or Airy function for n = 0 or 1, respectively. The coefficients $A_s(z)$ and $B_s(z)$ are given by recurrence relations. This paper proves that solutions of the differential equations exist whose asymptotic expansions in Poincare's sense are given by these series, and that the expansions are uniformly valid with respect to the complex variable z.

11. Olver, F. W. J., "The Asymptotic Solution of Linear Differential Equations of the Second Order in a Domain Containing One Transition Point," Philo. Trans. Roy. Soc., London Ser. A, 249(1956), 65-97.

The differential equation

$$\frac{d^2w}{dz^2} = \left\{uz^n + \frac{r}{z^2} + g(z)\right\}w$$

where n is an integer(≥ -1), u a parameter and r a constant, has the formal solution

$$w = P(z)\left\{1 + \sum_{s=1}^{\infty}\frac{A_s(z)}{u^s}\right\} + \frac{P'(z)}{u}\sum_{s=0}^{\infty}\frac{B_s(z)}{u^s}$$

where P is a solution of the equation

$$\frac{d^2P}{dz^2} = \left(uz^n + \frac{r}{z^2}\right)P$$

The coefficients $A_s(z)$ and $B_s(z)$ are given by recurrence relations. It is shown that they are analytic at z = 0 if, and only if, the differential equation for w can be transformed into a similar equation with n = 0, r = 0, or n = 1, r = 0, or n = -1. The first two cases have been treated in [10] of this reference. The third case, for which P is a Bessel function of order $\pm(1 + 4r)^{\frac{1}{2}}$, is examined in detail in the present paper.

12. Naghdi, P. M. and DeSilva, C. N., "Deformations of Elastic Ellipsoidal Shells of Revolution," Proc. 2nd U. S. Nat. Cong. Appl. Mech., Ann Arbor, 1964, ASME(1965), 333-343.

Deformations of thin elastic ellipsoidal shells of revolution of uniform thickness under axisymmetric loading are considered in detail. By means of asymptotic integration due to Langer, a solution is obtained which is valid at the apex of the shell and involves Kelvin functions. The stress distribution is obtained for ellipsoidal shells under both uniformly distributed surface and edge loadings.

13. Clark, R. A. and Reissner, E., "On stresses and Deformations of Ellipsoidal Shells Subject to Internal Pressure," J. Mech. Phys. Solids,(1957), 63-70.

Complete ellipsoidal shells of revolution subjected to internal pressure were considered. Stresses and deformations based on bending theory were obtained and compared with those obtained from membrane theory. The range of validity of the membrane solution was established and quantitative corrections to this solution were also determined.

14. DeSilva, C. N. and Arbor, A., "Deformation of Elastic Paraboloidal Shells of Revolution," J. Appl. Mech., Sept. 1957, 397-404.

Asymptotic solution is obtained for paraboloidal shells of uniform thickness subjected to a load uniformly distributed over a small region about the apex and clamped at the open edge. Naghdi's equations for shells of revolution in which the effect of transverse shear deformation on the bending is considered are employed.

15. Baker, B. R. and Cline, G. B., "Influence Coefficients for Thin Smooth Shells of Revolution Subjected to Symmetric Loads," Trans. ASME Series E, 29(1962), 335-339.

The differential equations governing the deformation of shells of revolution of uniform thickness subjected to axisymmetric self-equilibrating edge loads are transformed into a form suitable for asymptotic integration. Asymptotic solutions are obtained for all sufficiently thin shells that possess a smooth meridian curve and that are spherical in the neighborhood of the apex.

16. Steele, C. R. and Hartung, R. F., "Symmetric Loading of Orthotropic Shells of Revolution," Lockheed Missiles and Space Co., Technical Report, March 1964.

Stresses and deformations in thin, homogeneous, orthotropic shells of revolution under action of axisymmetric loads is reduced to solution of a single inhomogeneous second-order differential equation with complex dependent variable; asymptotic solutions are obtained which are uniformly valid in both steep and shallow regions of dome-shaped shell; the solution is equivalent to well-known membrane solution in steep region of shell but in shallow region it gives significant bending stresses.

17. Donnell, L. H., "Stability of Thin Walled Tubes Under Torsion," N. A. C. A. Report No. 479(1933).

18. Steele, C. R., "Nonsymmetric Deformation of Dome-shaped Shells of Revolution," Trans. ASME, Series E, 29(1962), 353-361.

Bending solutions that are uniformly valid in both shallow and non-shallow regions of a dome with arbitrary meridian are determined for edge loads that vary sinusoidally in the circumferential direction. The membrane and inextensional deformation solutions are obtained in terms of a function which satisfies a simple integral equation. For specific application, curve and formulas are obtained for the stresses and deformations of a dome with rigid rings clamped to the edges under the action of axial forces, side force and tilting moment.

19. Reissner, E., "Stresses and Small Displacements of Shallow Spherical Shells, I, II," J. Math. Phys., 25(1946), 80- 85, and 26(1947),297-300.

In the first paper,a set of two equations which relate stress function F and normal displacement w was obtained for small deformation of shallow spherical shells. The assumptions used in the derivation are: (a) shallow shell; (b) the transverse shear terms in the first two equations of equilibrium are neglected; (c) the tangential displacements u and v are neglected in the expressions of bending and twisting deformation parameters.
In the second paper, solutions to these two equations were obtained for the case of axisymmetry. Applications were given to obtain results for the following problems: (a) a shell with no edge restraint carrying a point load at the apex, (b) a shell with no edge restraint carrying a load uniformly distributed over a small area with center at the apex, (c) a shell with edge restraint carrying a point load at the apex.

20. Schile, R. D., "Asymptotic Solution of Nonshallow Shells of Revolution Subjected to Nonsymmetric Loads," J. of the Aerospace Sciences, Nov. 1962, 1375-1379.

Asymptotic solutions of Novozhilov's equations are derived for a constant thickness, nonshallow shell of revolution. Two cases of loadings are considered: (a) sinusoidal loading and (b) higher harmonic load distribution. The asymptotic solutions are obtained by the use of small-parameter expansions and by the use of a standard method for the singular perturbation problem and are valid for a nonshallow shell free of singularities.

21. Galletly, G. D., "Influence Coefficients for Hemispherical Shell with Small Openings at the Vertex," J. Appl. Mech., March 1955, 20-24.

Hemispherical shells of thin, constant thickness with a circular opening at the vertex subjected to axisymmetric self-equilibrating forces were considered. Three methods for obtaining influence coefficients were investigated and utilized in a numerical example.

22. Leckie, F. A., Payne, D. J. and Penny, R. K., "Elliptical Discontinuity in Spherical Shells," J. of Strain Analysis, Vol. 2, No. 1(1967), 34-42.

In part 1, shallow-shell equations expressed in elliptical coordinates have been solved in terms of Mathieu functions. Boundary conditions for the rigid insert and for the unreinforced hole are discussed in some detail. Results for an unreinforced opening are compared with experiment and satisfactory agreement is obtained. In part 2, a parametric study has been made of factors affecting the stresses at rigid inclusions and at unreinforced holes of elliptical shape in spherical shells.

23. Savin, G. N., "The Stress Distribution in a Thin Shell with an Arbitrary Hole," Problem of Continuum Mechanics, Society of Industrial and Applied Mechanics, (1961), 382-405.

The method yields the perturbation of membrane state of stress caused by holes with smooth countours, the size of which is small compared to radii of curvatures of the shell. The perturbation, being confined to a narrow zone, can be established by theory of shallow shells. The isometric system of curvilinear coordinates pertaining to the shell is in this zone replaced by a system such that along the contour one of the coordinates is constant. The deflection and the stress function are combined into one unknown complex function; likewise, the two homogeneous equations for the shallow shell into one complex equation. The general method is described and applied to the cylindrical shell with a circular hole and to a spherical shell with either a circular or an elliptical hole; the solutions are given for the case of the sphere.

24. Savin, G. N., "Concentration of Stresses around a Curvilinear Hole in Plates and Shells," Proceedings of Eleventh International Congress of Applied Mechanics, Springer-Verlag, 1964, 289-298.

25. Bijlaard, P. P., " Computation of the Stresses from Local Loads in Spherical Pressure Vessels or Pressure Vessel Heads," Welding Research Council Bulletin No. 34, March 1957.

As a first approach to the determination of the effect of local loads acting upon attachments to spherical shells, the case of a rigid cylindrical insert, loaded by a radial load or an external moment, is considered. Direct solutions are obtained by using the theory of shallow spherical shells. The numerical results are presented in graphs.

26. Foster, K., "The Hillside Problem: Stresses in a Shallow Spherical Shell due to External Loads on a Nonradial Rigid Cylindrical Insert," Ph. D. Thesis, Cornell University, Ithaca, N. Y., 1959.

Shallow spherical shells with a rigid elliptical insert at the apex subjected to radial and tangential forces, and external moment applied to the insert were investigated. Reissner's equations for shallow spherical shells were expressed in elliptical coordinate system. Solutions were obtained in terms of Mathieu functions.

27. Bijlaard, P. P., "Stresses in a Spherical Vessel from Radial Loads Acting on a Pipe,"
"Stresses in a Spherical Vessel from External Moments Acting on a Pipe," Welding Research Council Bulletin, No. 49, April 1959.

In the first paper, a spherical vessel with a radially inserted tube is investigated for the case where the tube is subjected to a radial load. After deriving the solutions for cylindrical shells and shallow spherical shells the continuity conditions between tube and vessel are established in order to determine the constants involving in the solutions. Graphs showing deflections, forces and moments of the vessel for some values of geometric parameters are presented. In the second paper, the same structure is investigated for the case of external moment acting on an insert pipe. Solutions in the form of the first harmonic are shown to be suitable for this analysis. Curves showing deflections, forces and moments are also presented.

28. Leckie, F. A., and Penny, R. K., "Stress Concentration Factors at Nozzle Intersections in Pressure Vessels," Welding Research Council Bulletin, No. 90, 1963.

The methods of calculation presented apply to the penetration of a spherical pressure vessel by a radial, circular cylindrical nozzle. This nozzle may protrude into the vessel, and allowance is made for the case when local reinforcement in the form of pad is present. Solutions for various loadings given include pressure and nozzle thrust, moment and shear.

29. Johnson, D. E., "Stresses in a Spherical Shell with a Nonradial Nozzle," J. Appl. Mech., Trans. ASME, Vol. 34(1967), 299-307.

An analytical investigation is made of stresses due to external forces and moments acting on an elastic nonradial circular cylindrical nozzle attached to a spherical shell. Results are obtained by combining solutions from shell theory by a Galerkin-type method so as to satisfy boundary conditions at the intersection of the two shells. It is found that, as the nozzle inclination increases, the stresses change gradually from those previously given by Bilaard for the radial nozzle.

30. Yu, J. C. M., Chen, C. H. and Shaw, W. A., "Stress Distribution of a Cylindrical Shell Nonradially Attached to a Spherical Pressure Vessel," ASME paper 71-PVP-42, Contributed by the Pressure Vessels and Piping Division of the ASME for presentation at the First National Congress on Pressure Vessels and Piping, San Francisco, Calif., May 1971.

31. Maxwell, R. L., Holland, R. W. and Stengl G. R., "Experimental Stress Analysis of the Attachment Region of Hemispherical Shells with Attached Nozzles," Technical Report Part 3a, ME-7-69-1, The University of Tennessee.

APPENDIX

A BRIEF REVIEW OF THE GENERAL THEORY OF THIN ELASTIC SHELLS

The derivation of the basic equations for thin elastic shells has been well established and can be found in most of the books on thin shells, for example, in [1,2,3]. For completeness of the text and convenience of application, a general procedures as to the deduction of these basic equations to a system of differential equations which may be readily applied to the problems studied here, will be outlined. The basic assumptions and their consequences will be pointed out wherever they are introduced.

The fundamental assumptions in shell theory are:

(a) Straight fibers normal to the middle surface of a shell before deformation remain so after deformation and do not change their length.

(b) The normal stress acting on surfaces parallel to the middle surface may be neglected in comparison with the other stresses.

(c) The relative thickness of the shell is sufficiently small in comparison with unity.

(d) The displacements are small compared to the thickness of the shell.

In that which follows, the notation and procedures used are those introduced by Novozhilov[2].

Coordinate System and Conditions of Gauss-Codazzi

Let α_1 = constant, α_2 = constant be the coordinate lines of the principal curvature of the middle surface of a shell and R_1 and R_2 be the corresponding radii of curvature (Fig. A-1). Since the lines of principal curvature are orthogonal, the first fundamental form of a surface may be written in the form

$$(ds)^2 = (A_1\, d\alpha_1)^2 + (A_2\, d\alpha_2)^2 \qquad \text{(A-1)}$$

where ds is the length of the differential segment of a line on the middle surface and A_1, A_2 are called Lame' parameters.

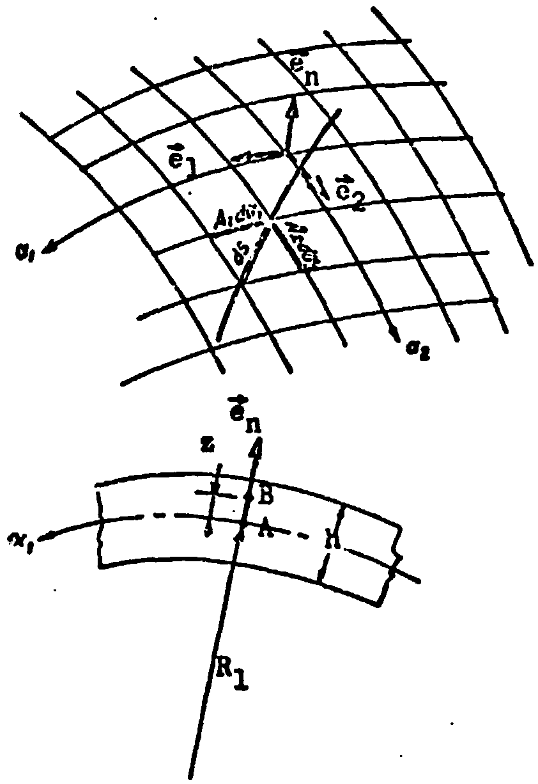

Fig. A-1:

Coordinate lines of a surface

The parameters A_1, A_2, R_1 and R_2 are related by the conditions of Gauss-Codazzi

$$\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right) = \frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1}$$

$$\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right) = \frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2} \qquad \text{(A-2)}$$

$$\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right) = -\frac{A_1A_2}{R_1R_2}$$

The first two conditions may be obtained from the identity

$$\frac{\partial^2\vec{e}_n}{\partial\alpha_1\partial\alpha_2} = \frac{\partial^2\vec{e}_n}{\partial\alpha_2\partial\alpha_1}$$

and the third one from

$$\frac{\partial^2\vec{e}_1}{\partial\alpha_1\partial\alpha_2} = \frac{\partial^2\vec{e}_1}{\partial\alpha_2\partial\alpha_1}$$

where $\vec{e}_1$ is a unit vector tangent to the line α_1 = constant and $\vec{e}_n$ is a unit normal to the middle surface (Fig. A-1). A surface is uniquely defined if the parameters A_1, A_2, R_1 and R_2 satisfy the condition (A-2). Hence, these conditions are usually referred to as the compatibility conditions of a surface.

Strain-Displacement Relations and Compatibility Equations

Let u, v, w be the displacements of a point A on the middle surface in the directions of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_n$, respectively, and u_z, v_z, w_z be the displacements of a point B on the normal through A, at a distance z from the middle surface (Fig. A-1). The assumption (a) implies that

$$e_{31} = e_{32} = e_{33} = 0$$

Expressing these relations in terms of the displacements one obtains

$$\begin{aligned} H_1 \frac{\partial}{\partial z}\left(\frac{u_z}{H_1}\right) + \frac{1}{H_1}\frac{\partial w_z}{\partial \alpha_1} &= 0 \\ H_2 \frac{\partial}{\partial z}\left(\frac{v_z}{H_2}\right) + \frac{1}{H_2}\frac{\partial w_z}{\partial \alpha_2} &= 0 \qquad \text{(A-3a)} \\ \frac{\partial w_z}{\partial z} &= 0 \end{aligned}$$

in which

$$\begin{aligned} H_1 &= A_1 (1 + z/R_1) \\ H_2 &= A_2 (1 + z/R_2) \qquad \text{(A-3b)} \end{aligned}$$

Equations (A-3a) upon integration with respect to z over (0, z) and use of the relation $(u_z, v_z, w_z) = (u, v, w)$ at $z = 0$, yield

$$\begin{aligned} u_z &= u + z\vartheta \\ v_z &= v + z\psi \qquad \text{(A-3c)} \\ w_z &= w \end{aligned}$$

where

$$\vartheta = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u}{R_1}$$

$$\psi = -\frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{v}{R_2} \qquad \text{(A-3d)}$$

Equations (A-3c) show that the variation of the displacements through the thickness is linear and w_z is independent of z.

The remaining three strain components are related to the displacements by

$$e_{11} = \frac{1}{H_1}\left(\frac{\partial u_z}{\partial \alpha_1} + \frac{1}{H_2}\frac{\partial H_1}{\partial \alpha_2} v_z + \frac{\partial H_1}{\partial z} w_z\right)$$

$$e_{22} = \frac{1}{H_2}\left(\frac{\partial v_z}{\partial \alpha_2} + \frac{1}{H_1}\frac{\partial H_2}{\partial \alpha_1} u_z + \frac{\partial H_2}{\partial z} w_z \qquad \text{(A-4)}$$

$$e_{12} = \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v_z}{H_2}\right) + \frac{H_1}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{H_1}\right)$$

Substitution of Eqs.(A-3b) and (A-3c) into Eqs.(A-4) and use of conditions of Codazzi yield, after certain manipulation, the following explicit expressions

$$e_{11} = \frac{1}{1+z/R_1}\left(\epsilon_1 + z\kappa_1\right)$$

$$e_{22} = \frac{1}{1+z/R_2}\left(\epsilon_2 + z\kappa_2\right) \qquad \text{(A-5)}$$

$$e_{12} = \frac{1}{1+z/R_1}\left(\omega_1 + z\tau_1\right) + \frac{1}{1+z/R_2}\left(\omega_2 + z\tau_2\right)$$

where

$$\epsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1}$$

$$\epsilon_2 = \frac{1}{A_2}\frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2}$$

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \vartheta}{\partial \alpha_1} + \frac{\psi}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} \qquad \text{(A-6)}$$

$$\kappa_2 = \frac{1}{A_2}\frac{\partial \psi}{\partial \alpha_2} + \frac{\vartheta}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}$$

and

$$
\begin{aligned}
\omega_1 &= \frac{1}{A_1}\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}u \\
\omega_2 &= \frac{1}{A_2}\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}v \\
\tau_1 &= \frac{1}{A_1}\frac{\partial \psi}{\partial \alpha_1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\vartheta \\
\tau_2 &= \frac{1}{A_2}\frac{\partial \vartheta}{\partial \alpha_2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\psi
\end{aligned}
\qquad \text{(A-7)}
$$

It is possible to reduce the last of Eqs.(A-5) to a form involving only two parameters. In doing this, observing the identity

$$\tau_1 + \frac{\omega_2}{R_1} = \tau_2 + \frac{\omega_1}{R_2}$$

and introducing the new notations

$$\omega = \omega_1 + \omega_2$$

$$\tau = \tau_1 + \frac{\omega_2}{R_1} = \tau_2 + \frac{\omega_1}{R_2}$$

one reduces the last of Eqs.(A-5) to the following form

$$e_{12} = \frac{1}{(1+z/R_1)(1+z/R_2)}\left\{\left(1 - \frac{z^2}{R_1R_2}\right)\omega + 2\left[1 + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{z}{2}\right]z\,\tau\right\} \qquad \text{(A-8)}$$

where

$$
\begin{aligned}
\omega &= \omega_1 + \omega_2 = \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{v}{A_2}\right) \\
\tau &= -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}\right) \\
&\quad + \frac{1}{R_1A_2}\left(\frac{\partial u}{\partial \alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial \alpha_2}u\right) + \frac{1}{R_2A_1}\left(\frac{\partial v}{\partial \alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial \alpha_1}v\right)
\end{aligned}
\qquad \text{(A-9)}
$$

Thus, the deformation of the middle surface is completely described by the six parameters ϵ_1, ϵ_2, ω, κ_1, κ_2 and τ , which are usually referred to as the deformation parameters of a middle surface.

Neglecting the terms z/R_1 and z/R_2 in Eqs.(A-5) in comparison with unity one obtains the expressions given in [3] which differ only in τ

from Novozhilov's expressions, i.e.,

$$\tau' = \tau_1 + \tau_2 = \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\psi}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{\vartheta}{A_1}\right) \tag{A-10}$$

Expanding the factors $(1 + z/R_1)^{-1}$ and $(1 + z/R_2)^{-1}$ in Eqs.(A-5) in the form of a series in the variable z and collecting the terms in the coefficients of z^0 and z^1, one obtains Vlasov's expressions [1] which relate Novozhilov's expressions by

$$\begin{aligned} &\epsilon_1'' = \epsilon_1, \qquad \epsilon_2'' = \epsilon_2, \qquad \omega'' = \omega, \\ &\kappa_1'' = \kappa_1 - \epsilon_1/R_1 \qquad \kappa_2'' = \kappa_2 - \epsilon_2/R_2 \\ &2\tau'' = \tau_1 + \tau_2 - \frac{\omega_1}{R_1} - \frac{\omega_2}{R_2} = 2\tau - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\omega \end{aligned} \tag{A-11}$$

The six parameters relating to the displacements must satisfy the compatibility conditions of the strains, which are given below [2]

$$\begin{aligned} &\frac{\partial}{\partial\alpha_2}(A_1\kappa_1) - \kappa_2\frac{\partial A_1}{\partial\alpha_2} - \frac{\partial A_2\tau}{\partial\alpha_1} - \tau\frac{\partial A_2}{\partial\alpha_1} + \frac{\omega}{R_1}\frac{\partial A_2}{\partial\alpha_1} \\ &\qquad - \frac{1}{R_2}\left(\frac{\partial A_1\epsilon_1}{\partial\alpha_2} - \frac{\partial A_2\omega}{\partial\alpha_1} - \epsilon_2\frac{\partial A_1}{\partial\alpha_2}\right) = 0 \\ &\frac{\partial}{\partial\alpha_1}(A_2\kappa_2) - \kappa_1\frac{\partial A_2}{\partial\alpha_1} - \frac{\partial A_1\tau}{\partial\alpha_2} - \tau\frac{\partial A_1}{\partial\alpha_2} + \frac{\omega}{R_2}\frac{\partial A_1}{\partial\alpha_2} \\ &\qquad - \frac{1}{R_1}\left(\frac{\partial A_2\epsilon_2}{\partial\alpha_1} - \frac{\partial A_1\omega}{\partial\alpha_2} - \epsilon_1\frac{\partial A_2}{\partial\alpha_1}\right) = 0 \\ &\frac{\kappa_1}{R_2} + \frac{\kappa_2}{R_1} + \frac{1}{A_1A_2}\left\{\frac{\partial}{\partial\alpha_1}\frac{1}{A_1}\left[A_2\frac{\partial\epsilon_2}{\partial\alpha_1} + \frac{\partial A_2}{\partial\alpha_1}(\epsilon_2 - \epsilon_1) - \frac{A_1}{2}\frac{\partial\omega}{\partial\alpha_2} - \frac{\partial A_1}{\partial\alpha_2}\omega\right]\right. \\ &\qquad \left. + \frac{\partial}{\partial\alpha_2}\frac{1}{A_2}\left[A_1\frac{\partial\epsilon_1}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}(\epsilon_1 - \epsilon_2) - \frac{A_2}{2}\frac{\partial\omega}{\partial\alpha_1} - \frac{\partial A_2}{\partial\alpha_1}\omega\right]\right\} = 0 \end{aligned} \tag{A-12}$$

The relations (A-12) play in the theory of shells the same role as the compatibility equations in the theory of elasticity, the fulfilment of which ensures the possibility of determining displacements from the given deformation parameters of a shell.

Equations of Equilibrium

The equations of equilibrium of a shell element may be derived in a similar manner as those which are derived in the theory of elasticity, except in the theory of shell, the stresses are replaced by statically equivalent forces and moments (Fig. A-2), which are defined by the following expressions

$$
\begin{aligned}
T_1 &= \int_{-h/2}^{h/2} \sigma_{11}(1 + z/R_2)\,dz \\
T_{12} &= \int_{-h/2}^{h/2} \sigma_{12}(1 + z/R_2)\,dz \\
N_1 &= \int_{-h/2}^{h/2} \sigma_{13}(1 + z/R_2)\,dz \\
T_2 &= \int_{-h/2}^{h/2} \sigma_{22}(1 + z/R_1)\,dz \\
T_{21} &= \int_{-h/2}^{h/2} \sigma_{21}(1 + z/R_1)\,dz \\
N_2 &= \int_{-h/2}^{h/2} \sigma_{23}(1 + z/R_1)\,dz \\
M_1 &= \int_{-h/3}^{h/2} \sigma_{11}\,z(1 + z/R_2)\,dz \\
M_{12} &= \int_{-h/2}^{h/2} \sigma_{12}\,z(1 + z/R_2)\,dz \\
M_2 &= \int_{-h/2}^{h/2} \sigma_{22}\,z(1 + z/R_1)\,dz \\
M_{21} &= \int_{-h/2}^{h/2} \sigma_{21}\,z(1 + z/R_1)\,dz
\end{aligned}
\qquad \text{(A-13)}
$$

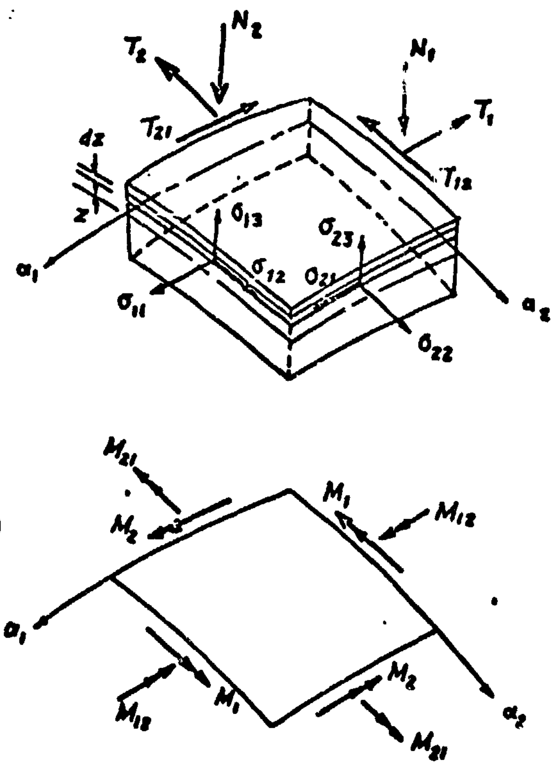

Fig. A-2

Positive directions of forces and moments

The condition that the equilibrium of a shell element requires that the resultant force and moment vanish yields the following equations

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_1}{\partial \alpha_1} + \frac{\partial A_1T_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}T_{12} - \frac{\partial A_2}{\partial \alpha_1}T_2\right] + \frac{N_1}{R_1} + g_1 = 0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_{12}}{\partial \alpha_1} + \frac{\partial A_1T_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1}T_{21} - \frac{\partial A_1}{\partial \alpha_2}T_1\right] + \frac{N_2}{R_2} + g_2 = 0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2N_1}{\partial \alpha_1} + \frac{\partial A_1N_2}{\partial \alpha_2}\right] - \frac{T_1}{R_1} - \frac{T_2}{R_2} + g_n = 0 \qquad \text{(A-14)}$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2M_1}{\partial \alpha_1} + \frac{\partial A_1M_{21}}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}M_{12} - \frac{\partial A_2}{\partial \alpha_1}M_2\right] - N_1 = 0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2M_{12}}{\partial \alpha_1} + \frac{\partial A_1M_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1}M_{21} - \frac{\partial A_1}{\partial \alpha_2}M_1\right] - N_2 = 0$$

$$T_{12} - T_{21} + \frac{M_{12}}{R_1} - \frac{M_{21}}{R_2} = 0$$

The last of Eqs.(A-14) is identically satisfied. This can be verified upon substitution into the equation the forces and moments by their expressions from Eqs.(A-13).

Relations between the Forces, Moments and the Deformation Parameters

The relations between the forces, moments and the deformation parameters (from now on called constitutive equations) can be obtained from Eqs.(A-13). For this purpose, the stress components in these equations are replaced by the strain components through the use of Hooke's law (neglecting σ_{33} in comparison with σ_{11} and σ_{22})

$$\sigma_{11} = \frac{E}{1-\mu^2}(e_{11} + \mu e_{22})$$

$$\sigma_{22} = \frac{E}{1-\mu^2}(e_{22} + \mu e_{11}) \qquad \text{(A-15)}$$

$$\sigma_{12} = \frac{E}{2(1+\mu)} e_{12}$$

and then, the strain components are replaced by the deformation parameters from Eqs.(A-5) and (A-8). On carrying out integration on the result of these manipulations and then, neglecting terms of the order h/R in comparison with unity, Eqs,(A-13) finally yield the following relations

$$\begin{aligned} &T_1 = D(\epsilon_1 + \mu\epsilon_2), && T_2 = D(\epsilon_2 + \mu\epsilon_1) \\ &T_{12} = T_{21} = \frac{D(1-\mu)}{2}\,\omega, && M_1 = K(\kappa_1 + \mu\kappa_2) \\ &M_2 = K(\kappa_2 + \mu\kappa_1), && M_{12} = M_{21} = K(1-\mu)\,\tau \end{aligned} \tag{A-16}$$

where

$$D = \frac{Eh}{1-\mu^2}, \qquad K = \frac{Eh^3}{12(1-\mu^2)}$$

Adopting these relations one is essentially disregarding the differences between T_{12} and T_{21}, and M_{12} and M_{21}. On substituting these relations into the last of Eqs.(A-14) it may be verified that this equation is not satisfied identically. As mentioned previously, the fact that this equation is identically satisfied secures the symmetry of the stress tensor ($\sigma_{12} = \sigma_{21}$) from which it follows that Eqs.(A-16) contradict the symmetric properties of the stress tensor.

This contradiction can be avoided if the constitutive equations are developed from the variational principle of the potential energy by neglecting terms of order h/R in comparison with unity. This approach yield (2)

$$\begin{aligned} &T_1 = D(\epsilon_1 + \mu\epsilon_2), && T_2 = D(\epsilon_1 + \mu\epsilon_1) \\ &T_{12} = \frac{D(1-\mu)}{2}\left(\omega + \frac{h^2}{6R_2}\tau\right), && T_{21} = \frac{D(1-\mu)}{2}\left(\omega + \frac{h^2}{6R_1}\tau\right) \\ &M_1 = K(\kappa_1 + \mu\kappa_2), && M_2 = K(\kappa_2 + \mu\kappa_1) \\ &M_{12} = M_{21} = K(1-\mu)\,\tau \end{aligned} \tag{A-17}$$

Introducing the new notations

$$S = T_{12} - M_{21}/R_2 = T_{21} - M_{12}/R_1$$
$$H = M_{12} = M_{21} \tag{A-18}$$

and substituting from Eqs.(A-17) in Eqs.(A-18), one obtains

$$S = \frac{D(1-\mu)}{2}\omega, \qquad H = K(1-\mu)\tau \tag{A-19}$$

For later use the inverse relation of Eqs.(A-17) is obtained as follows:

$$\epsilon_1 = \frac{1}{Eh}(T_1 - \mu T_2), \qquad \epsilon_2 = \frac{1}{Eh}(T_2 - \mu T_1)$$
$$\omega = \frac{2(1+\mu)}{Eh} S \qquad \kappa_1 = \frac{12}{Eh^3}(M_1 - \mu M_2) \tag{A-20}$$
$$\kappa_2 = \frac{12}{Eh^3}(M_2 - \mu M_1) \qquad \tau = \frac{12(1+\mu)}{Eh^3} H$$

Reduction of the Basic Equations to a Fourth Order System

So far, a system of nineteen equations including six strain-displacement relations, five equations of equilibrium and eight constitutive equations, has been introduced. These equations involve the same number of unknowns, i.e., six forces, four moments, six deformation parameters and three displacements. One now faces the problem of solving these equations subject to appropriate boundary conditions. As in the theory of elasticity, there exist two methods of solving problems of thin elastic shells - in terms of the displacements of the middle surface or in terms of the forces and moments. Before proceeding to further discussion of these methods, the equations of equilibrium will be first simplified. To do this, the forces N_1 and N_2 in the first three of Eqs.(A-14) will be eliminated by substituting for them their expressions as given by the fourth and fifth of Eqs.(A-14). Then, taking into consideration the notations given in Eqs.(A-18) and the conditions of Codazzi, the first three of Eqs.(A-14) may be written in the form

$$\frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 S}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} S - \frac{\partial A_2}{\partial \alpha_1} T_2 + \frac{1}{R_1}\left[\frac{\partial A_2 M_1}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1} M_2 + 2\frac{\partial A_1 H}{\partial \alpha_2} + 2\frac{R_1}{R_2}\frac{\partial A_1}{\partial \alpha_2} H\right] = -A_1 A_2 q_1$$

$$\frac{\partial A_2 S}{\partial \alpha_1} + \frac{\partial A_1 T_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} S - \frac{\partial A_1}{\partial \alpha_2} T_1 + \frac{1}{R_2}\left[\frac{\partial A_1 M_2}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2} M_1 + 2\frac{\partial A_2 H}{\partial \alpha_1} + 2\frac{R_2}{R_1}\frac{\partial A_2}{\partial \alpha_1} H\right] = -A_1 A_2 q_2 \qquad \text{(A-21)}$$

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} - \frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left[\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 H}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} H - \frac{\partial A_2}{\partial \alpha_1} M_2\right] + \frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left[\frac{\partial A_2 H}{\partial \alpha_1} + \frac{\partial A_1 M_2}{\partial \alpha_2} + \frac{\partial A_2}{\partial \alpha_1} H - \frac{\partial A_1}{\partial \alpha_2} M_1\right]\right\} = q_n$$

Now, return to methods of obtaining solutions. The first method involves replacement in Eqs.(A-21) the forces and moments by their expressions in terms of the strains of the middle surface. Then, one obtains, upon substitution for the strains by their expressions in terms of the displacements as given in Eqs.(A-6) and (A-9) a system of three partial differential equations in terms of the three displacements of the middle surface.

The second method consists in supplementing the equations of equilibrium (A-21) by the compatibility equations (A-12), which , for this purpose, must be expressed in terms of the forces and moments. Then, one obtains a system of six partial differential equations for the determination of unknowns T_1, T_2, S, M_1, M_2 and H.

In that which follows, attention will be limited to the second method of solution. Substituting in Eqs.(A-12) the strains from Eqs.(A-20) one obtains the compatibility equations in terms of the forces and moments

$$\frac{\partial A_2(M_2-\mu M_1)}{\partial \alpha_1} - (1+\mu)\left(\frac{\partial A_1 H}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} H\right) - \frac{\partial A_2}{\partial \alpha_1}(M_1 - \mu M_2)$$
$$- \frac{h^2}{12 R_1}\left[\frac{\partial A_2(T_2-\mu T_1)}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}(T_1 - \mu T_2) - 2(1+\mu)\frac{\partial A_1 S}{\partial \alpha_2} - 2(1+\mu)\frac{R_1}{R_2}\frac{\partial A_1}{\partial \alpha_2} S\right] = 0$$

$$\frac{\partial A_1(M_1-\mu M_2)}{\partial \alpha_2} - (1+\mu)\left(\frac{\partial A_2 H}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1} H\right) - \frac{\partial A_1}{\partial \alpha_2}(M_2 - \mu M_1)$$
$$- \frac{h^2}{12 R_2}\left[\frac{\partial A_1(T_1-\mu T_2)}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2}(T_2 - \mu T_1) - 2(1+\mu)\frac{\partial A_2 S}{\partial \alpha_1} - 2(1+\mu)\frac{R_2}{R_1}\frac{\partial A_2}{\partial \alpha_1} S\right] = 0 \qquad \text{(A-22)}$$

$$\frac{M_2-\mu M_1}{R_1} + \frac{M_1-\mu M_2}{R_2} + \frac{h^2}{12}\frac{1}{A_1 A_2}\left\{\frac{\partial}{\partial \alpha_1}\frac{1}{A_1}\left[\frac{\partial A_2(T_2-\mu T_1)}{\partial \alpha_1}\right.\right.$$
$$\left. - (1+\mu)\left(\frac{\partial A_1 S}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2} S\right) - \frac{\partial A_2}{\partial \alpha_1}(T_1 - \mu T_2)\right]$$
$$\left.\frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left[\frac{\partial A_1(T_1-\mu T_2)}{\partial \alpha_2} - (1+\mu)\left(\frac{\partial A_2 S}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1} S\right) - \frac{\partial A_1}{\partial \alpha_2}(T_2 - \mu T_1)\right]\right\} = 0$$

The fulfilment of Eqs.(A-22) ensures the possibility of determining the displacements from the given forces and moments. Eqs.(A-22), after transformation employing the equations of equilibrium and then neglecting a number of terms of the order h/R compared with unity, can be reduced to the form

$$(1+\mu) N_1 = \frac{1}{A_1}\frac{\partial M}{\partial \alpha_1} - \frac{h^2}{12}\frac{1}{R_1 A_1}\frac{\partial T}{\partial \alpha_1}$$
$$(1+\mu) N_2 = \frac{1}{A_2}\frac{\partial M}{\partial \alpha_2} - \frac{h^2}{12}\frac{1}{R_2 A_2}\frac{\partial T}{\partial \alpha_2} \qquad \text{(A-23)}$$
$$\frac{M_1-\mu M_2}{R_2} + \frac{M_2-\mu M_1}{R_1} + \frac{h^2}{12}\Delta(T) = -\frac{h^2}{12}\frac{1+\mu}{A_1 A_2}\left[\frac{\partial A_2 q_1}{\partial \alpha_1} + \frac{\partial A_1 q_2}{\partial \alpha_2}\right]$$

in which

$$M = M_1 + M_2, \qquad T = T_1 + T_2$$
$$\Delta(\ ) = \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{A_1}\frac{\partial (\ )}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{A_2}\frac{\partial (\ )}{\partial \alpha_2}\right)\right] \qquad \text{(A-23a)}$$

The second term on the right hand side of the first two of Eqs.(A-23) is likewise of negligible magnitude. Thus, the first two of the compatibility equations can be written in the following simple form

$$(1+\mu)N_1 \approx \frac{1}{A_1}\frac{\partial M}{\partial \alpha_1}$$
$$(1+\mu)N_2 \approx \frac{1}{A_2}\frac{\partial M}{\partial \alpha_2} \tag{A-24}$$

The compatibility equations have been simplified in the form of Eqs.(A-23) or (A-24), which will be employed to eliminate N_1 and N_2 in the equations of equilibrium. Eliminating from the first three of Eqs.(A-14), letting $T_{12} = T_{21}$, the normal shearing forces N_1 and N_2 by use of Eqs.(A-24), and from the fourth and fifth of Eqs.(A-14) N_1 and N_2 by use of Eqs.(A-23), one obtains a system of six equations with the last one coming from the third of Eqs.(A-23)

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_1}{\partial \alpha_1}+\frac{\partial A_1S}{\partial \alpha_2}+\frac{\partial A_1}{\partial \alpha_2}S-\frac{\partial A_2}{\partial \alpha_1}T_2\right]+\frac{1}{1+\mu}\frac{1}{R_1A_1}\frac{\partial M}{\partial \alpha_1}+q_1=0$$

$$\frac{1}{A_1A_2}\left[-\frac{\partial A_2(M_2-\mu M_1)}{\partial \alpha_1}+(1+\mu)\left(\frac{\partial A_1H}{\partial \alpha_2}+\frac{\partial A_1}{\partial \alpha_2}H\right)+\frac{\partial A_2}{\partial \alpha_1}(M_1-\mu M_2)\right]$$
$$+\frac{h^2}{12R_1A_1}\frac{\partial T}{\partial \alpha_1}=0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2S}{\partial \alpha_1}+\frac{\partial A_1T_2}{\partial \alpha_2}+\frac{\partial A_2}{\partial \alpha_1}S-\frac{\partial A_1}{\partial \alpha_2}T_1\right]+\frac{1}{1+\mu}\frac{1}{R_2A_2}\frac{\partial M}{\partial \alpha_2}+q_2=0$$

$$\frac{1}{A_1A_2}\left[-\frac{\partial A_1(M_1-\mu M_2)}{\partial \alpha_2}+(1+\mu)\left(\frac{\partial A_2H}{\partial \alpha_1}+\frac{\partial A_2}{\partial \alpha_1}H\right)+\frac{\partial A_1}{\partial \alpha_2}(M_2-\mu M_1)\right] \tag{A-25}$$
$$+\frac{h^2}{12R_2A_2}\frac{\partial T}{\partial \alpha_2}=0$$

$$\frac{T_1}{R_1}+\frac{T_2}{R_2}-\frac{1}{1+\mu}\Delta(M)-q_n=0$$

$$\frac{M_2-\mu M_1}{R_1}+\frac{M_1-\mu M_2}{R_2}+\frac{h^2}{12}\Delta(T)=-\frac{h^2}{12}\frac{1+\mu}{A_1A_2}\left[\frac{\partial}{\partial \alpha_2}(A_2q_1)+\frac{\partial}{\partial \alpha_1}(A_1q_2)\right]$$

These six equations constitute an eighth order system and can be reduced to three equations of fourth order system by the use of complex transformation. For this purpose, the auxiliary functions

$$\bar{T}_1 = T_1 - \frac{i}{c}\frac{M_2 - \mu M_1}{1-\mu^2}$$

$$\bar{T}_2 = T_2 - \frac{i}{c}\frac{M_1 - \mu M_2}{1-\mu^2} \qquad \text{(A-26)}$$

$$\bar{S} = S + \frac{i}{c}\frac{H}{1-\mu}$$

$$\bar{T} = \bar{T}_1 + \bar{T}_2$$

will be introduced, where

$$c = \frac{h}{\sqrt{12(1-\mu^2)}}$$

Substituting in Eqs.(A-25) the forces T_1, T_2, S by their expressions in terms of $\bar{T}_1$, $\bar{T}_2$, $\bar{S}$ and M_1, M_2, H as defined in Eqs.(A-26). In this way one obtains a system of six equations from which the quantities M_1, M_2, H may be eliminated. This process leads to the following system of three partial differential equations in terms of three complex forces $\bar{T}_1$, $\bar{T}_2$, and $\bar{S}$.

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2\bar{T}_1}{\partial\alpha_1} + \frac{\partial A_1\bar{S}}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}\bar{S} - \frac{\partial A_2}{\partial\alpha_1}\bar{T}_2\right] + i\frac{c}{R_1A_1}\frac{\partial\bar{T}}{\partial\alpha_1} + g_1 = 0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2\bar{S}}{\partial\alpha_1} + \frac{\partial A_1\bar{T}_2}{\partial\alpha_2} + \frac{\partial A_2}{\partial\alpha_1}\bar{S} - \frac{\partial A_1}{\partial\alpha_2}\bar{T}_1\right] + i\frac{c}{R_2A_2}\frac{\partial\bar{T}}{\partial\alpha_2} + g_2 = 0 \qquad \text{(A-27)}$$

$$\frac{\bar{T}_1}{R_1} + \frac{\bar{T}_2}{R_2} - ic\Delta(\bar{T}) = \bar{g}_n$$

where

$$\bar{g}_n = g_n + ic\frac{1+\mu}{A_1A_2}\left[\frac{\partial A_2 g_1}{\partial\alpha_1} + \frac{\partial A_1 g_2}{\partial\alpha_2}\right]$$

Equations (A-27) include the equations of equilibrium of the shell element and the equations of compatibility for the strains of the middle surface. It is a fourth order system with three unknowns, and is half the number of equations, order and unknowns of the system (A-25).

Letting c = 0 and identifying $\bar{T}_1$, $\bar{T}_2$, $\bar{S}$ by T_1^*, T_2^*, S^*, repectively in Eqs.(A-27), this system reduces to the equations of the membrane theory.

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_1^*}{\partial\alpha_1}+\frac{\partial A_1S^*}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}S^*-\frac{\partial A_2}{\partial\alpha_1}T_2^*\right]+q_1=0$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2S^*}{\partial\alpha_1}+\frac{\partial A_1T_2^*}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}S^*-\frac{\partial A_1}{\partial\alpha_2}T_1^*\right]+q_2=0 \qquad \text{(A-28)}$$

$$\frac{T_1^*}{R_1}+\frac{T_2^*}{R_2}=q_n$$

To get a complete solution, the displacements of the middle surface have to be found. Define the complex displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ which relate to the complex forces by six differential equations

$$\bar{\epsilon}_1=\frac{1}{Eh}(\bar{T}_1-\mu\bar{T}_2), \qquad \bar{\epsilon}_2=\frac{1}{Eh}(\bar{T}_2-\mu\bar{T}_1)$$

$$\bar{\omega}=\frac{2(1+\mu)}{Eh}\bar{S}, \qquad \bar{\kappa}_1=\frac{i}{c}\frac{1}{Eh}(\bar{T}_2-T_2^*) \qquad \text{(A-29)}$$

$$\bar{\kappa}_2=\frac{i}{c}\frac{1}{Eh}(\bar{T}_1-T_1^*), \qquad \bar{\tau}=-\frac{i}{c}\frac{1}{Eh}(\bar{S}-S^*)$$

In these equations $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, $\bar{\omega}$, $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\tau}$ are related to $\bar{u}$, $\bar{v}$, $\bar{w}$ in the same way as the strain-displacement relations given in Eqs.(A-6) and (A-9), and T_1^*, T_2^*, S^* are solutions of the membrane theory, i.e., of the system (A-28). The real parts of $\bar{u}$, $\bar{v}$, $\bar{w}$ are the displacements u, v, w, respectively.

Thus, the solution of problems of a shell reduces to the determination of the complex forces $\bar{T}_1$, $\bar{T}_2$, $\bar{S}$ from Eqs.(A-27) and the complex displacements $\bar{u}$, $\bar{v}$, $\bar{w}$ from Eqs.(A-29) subject to appropriate boundary conditions.

In conclusion it is noted that the error introduced in the system (A-27) is of order h/R compared with unity. Hence, the system of Eqs.(A-29) are only approximately compatible with each other within an error of this order.