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MULTIPLE ZEROS OF POLYNOMIALS

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by

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A National Aeronautics and Space Administration

Research Grant

Grant Number NASA NGR 37-002-084

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Various classical methods exist for extracting the zeros of a polynomial

$$P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_{N+1}$$

where $a_1 \neq 0$ and $a_1, a_2, \ldots, a_{N+1}$ are complex numbers, when $N=1,2,3,4$. For polynomials of higher degree, iterative numerical methods must be used. In this material four iterative methods are presented for approximating the zeros of a polynomial using a digital computer. Newton's method and Muller's method are two well known iterative methods which are presented. They extract the zeros of a polynomial by generating a sequence of approximations converging to each zero. However, both of these methods are very unstable when used on a polynomial which has multiple zeros. That is, either they fail to converge to some or all of the zeros, or they converge to very bad approximations of the polynomial's zeros.

This material introduces two new methods, the greates common divisor (G.C.D.) method and the repeated greatest common divisor (repeated G.C.D.) method, which are superior methods for numerically approximating the zeros of a polynomial having multiple zeros.

The above methods were all programmed in FORTRAN IV and comparisons in time and accuracy are given. These programs were executed on the

IBM 360/50 computer as well as the UNIVAC 1108 and the CDC 6600 computer.

This material also contains complete documentations for six FORTRAN IV programs. Flow charts, program listings, definition of variables used in the program, and instructions for use of each program are included.

PREFACE

Four iterative methods for approximating the zeros of a polynomial using a digital computer are presented in this material. Chapter I is an introduction. Chapters II and III contain Newton's and Muller's methods, respectively. Chapters IV and V present two new methods which depend upon finding the greatest common divisor of two polynomials. Chapter VI contains a comparison of the four methods. Flow charts, FORTRAN IV programs, and complete program documentations for these four methods are presented in appendicies A through H.

I would like to express my appreciation to the National Aeronautics and Space Administration, specifically the Manned Spacecraft Center in Houston, Texas, for their financial support in making this work possible under grant number NASA NGR 37-002-084. I would also like to thank Randy Snider, a graduate assistant supported by this grant, for the great deal of work he put in on the FORTRAN programs. In particular, the material on Newton's and Muller's Methods included in this paper is part of his masters thesis at Oklahoma State University.

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CHAPTER I

INTRODUCTION

Frequently in scientific work it becomes necessary to find the zeros, real or complex, of the polynomial of degree N

$$P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_NX + a_{N+1}$$

where $a_1 \neq 0$ and the coefficients $a_1, a_2, \ldots, a_{N+1}$ are complex numbers. Various classical methods calculate the exact roots of polynomials of degree 1,2,3, or 4. For polynomials of higher degree, no such methods exist. Thus, to solve for the zeros of such polynomials, numerical methods of iteration based on successive approximations must be employed. In the following material four such methods are given which are particularly suited for modern high speed computers.

Newton's method is an iterative procedure which generates a sequence of successive approximations of a zero of P(X) by using the iteration formula

$$X_{n+1} = X_n - P(X_n)/P'(X_n).$$

An initial approximation to the zero is required to start the iterative process. Under certain conditions this sequence will converge quadratically to the desired root. It is, however, necessary to compute the value of the polynomial and its derivative for each step in the iterative procedure. Once a zero of P(X) has been found, it is divided out of P(X), giving a deflated polynomial of lower degree. P(X) is replaced by the deflated polynomial and the iterative process is applied to extract another zero of P(X). This procedure is repeated until all zeros of P(X) have been found. The zeros may then be rechecked and their accuracy possibly improved by using them as initial approximations with Newton's process applied to the full (undeflated) polynomial.

Muller's method is also an iterative procedure generating a sequence $X_1, X_2, \ldots, X_n, \ldots$ of successive approximations of a root of P(X). This method converges almost quadratically near a zero and does not require the evaluation of the derivative of the polynomial. Muller's method requires three distinct approximations of a root to start the process of iteration. A quadratic equation is constructed through the three given points as an approximation of P(X). The root of the quadratic closest to X_n is taken as X_{n+1}, the next approximation to the zero. This process is then repeated on the last three points of the sequence. After a root of P(X) has been found, P(X) is deflated, and replaced in the above procedure by the deflated polynomial. After all zeros of P(X) are found from successive deflations, they are improved as in Newton's method.

The greatest common divisor method reduces the problem of finding all zeros (possibly multiple zeros) of P(X) to one of extracting the zeros of a polynomial $P_1(X) = P(X)/D(X)$, all of whose zeros are simple. D(X), the greatest common divisor of P(X) and its derivative, $P'(X)$, is obtained by repeated application of the division algorithm. Once $P_1(X)$ is obtained, some suitable method such as Newton's or Muller's method is used to find the zeros of $P_1(X)$. By finding all the zeros of $P_1(X)$, all the zeros of $P(X)$ are obtained. The multiplicity of each zero may then be determined.

The repeated greatest common divisor method repeatedly uses the greatest common divisor method to extract the zeros of $P(X)$ and their multiplicities at the same time. That is, the repeated greatest common divisor method reduces the problem of finding the zeros of $P(X)$, which possibly has multiple zeros, to one of finding the zeros of a polynomial which has only simple zeros and the zeros of this polynomial are all the zeros of $P(X)$ of a given multiplicity. The repeated greatest common divisor method must also use a supporting method such as Newton's method or Muller's method.

Chapters II-V contain the examinations of these methods. Each examination includes a development of the method together with the conditions necessary for convergence of the method. Chapter VI contains a comparison of the methods giving advantages and disadvantages of each method.

A complete set of documentations is given for six FORTRAN IV programs in Appendices A-H. Flow charts, program listings, definition of variables used in the program, and instructions for use of each program are included.

It should also be noted that the expressions "zero of a polynomial" and "root of a polynomial" and the words "zero" and "root" are used interchangeably in this material.

CHAPTER II

NEWTON'S METHOD

1. Derivation of the Algorithm

Newton's method is probably the most popular iterative procedure for finding the zeros of a polynomial. This fact is due to the excellent results obtained, the simplicity of the computational routine, and the fast rate of convergence obtained provided the initial approximation of a zero is close enough. Also, the method can be applied to the extraction of complex as well as real zeros.

Consider the polynomial

$$P(X) = a_1X^N + a_2X^{N-1} + \dots + a_NX + a_{N+1} \tag{2-1}$$

where $a_1 \neq 0$ and the coefficients $a_1, a_2, \dots, a_{N+1}$ are complex. The algorithm for Newton's method can be derived by approximating $P(X)$ by a Taylor series expansion about an approximation, X_0, of a zero, α, of $P(X)$. Using only the first two terms of the expansion, the expression

$$P(X) \doteq P(X_0) + P'(X_0)(X - X_0)$$

is obtained. If this equation is solved for $P(X) = 0$, then

$$0 \doteq P(X_0) + P'(X_0)(X - X_0)$$

results. Rearranging terms produces

$$0 \doteq P(X_0) + P'(X_0)\,X - P'(X_0)\,X_0$$

followed by

$$P'(X_0)\,X_0 - P(X_0) \doteq P'(X_0)\,X$$

from which division by $P'(X_0)$ produces

$$X_0 - P(X_0)/P'(X_0) \doteq X$$

which is the basic formula for Newton's method. Thus, in general, we obtain the $(n+1)^{th}$ approximation, X_{n+1}, of α from the n^{th} approximation, X_n, by

$$X_{n+1} = X_n - P(X_n)/P'(X_n). \qquad (2\text{-}2)$$

As a result of repeated use of this algorithm, we obtain the sequence

$$X_0, X_1, X_2, \ldots, X_n, \ldots \qquad (2\text{-}3)$$

of successive approximations of the root, α. It should be noted that an initial approximation is necessary to start the iterative process for each new zero; that is, a polynomial of degree N may require N initial approximations.

In order to use equation (2-2), it is necessary to compute, for each X_n, the value of the polynomial, $P(X_n)$, and its derivative, $P'(X_n)$. The division algorithm states that if P(X) and G(X) are polynomials, then there exists polynomials H(X) and K(X) such that $P(X) = H(X)\,G(X) + K(X)$ where $K(X) = 0$ or deg. K(X) < deg. G(X). From this expression of P(X), the following remainder theorem is obtained:

<u>Theorem 2.1</u>. If $P(X)$ is a polynomial and c is a complex number, then the remainder obtained from dividing $P(X)$ by $(X - c)$ is $P(c)$.

The proof of Theorem 2.1 is given in [3, P. 102]. Thus, $P(X)$ can be written as $P(X) = (X - c)\, H(X) + R$ where $P(c) = R$. $P'(X)$ is then obtained by the following theorem, the proof of which can be found in [3, PP. 105-106].

<u>Theorem 2.2</u>. If $P(X)$ and $H(X)$ are polynomials and c is a complex number such that $P(X) = (X - c)\, H(X) + R$ where $P(c) = R$, then the remainder obtained from dividing $H(X)$ by $(X - c)$ is $P'(c)$.

From synthetic division, an algorithm known as Horner's Method is acquired for computing $P(X_n)$ and $P'(X_n)$.

<u>Theorem 2.3</u>. Let $P(X)$ be defined as in equation (2-1) and let d be a complex number. Define a sequence $b_1, b_2, \ldots, b_{N+1}$ by

$$b_1 = a_1$$

$$b_i = a_i + db_{i-1} \quad (i = 2,3,\ldots,N+1).$$

Define another sequence $c_1, c_2, \ldots, c_N$ by

$$c_1 = b_1$$

$$c_j = b_j + dc_{j-1} \quad (j = 2,3,\ldots,N).$$

Then $P(d) = b_{N+1}$ and $P'(d) = c_N$. The elements $b_1, b_2, \ldots, b_N$ are the coefficients of the polynomial $H(X)$ in Theorem 2.2 when $P(X)$ is divided by $(X - d)$.

These formulas are derived in [3, PP. 106-107]. Thus with equation (2-2) and the iteration formulas of the previous theorem, Newton's method can now be applied to generate the sequence (2-3) which will converge to the root, α, if the convergence conditions given in Theorem 2.4 are satisfied.

A criterion is needed to determine when to terminate the sequence (2-3); that is, when has a zero been found? For convergence of the sequence, there must exist a term in the sequence beyond which the difference between any two successive terms is arbitrarily small. Therefore, it is desirable to make the quotient $|X_n/X_{n+1}|$ sufficiently near 1. From equation (2-2)

$$1 = \left| \frac{X_n}{X_{n+1}} - \frac{\frac{P(X_n)}{P'(X_n)}}{X_{n+1}} \right|$$

$$\geq \left| \frac{X_n}{X_{n+1}} \right| - \left| \frac{\frac{P(X_n)}{P'(X_n)}}{X_{n+1}} \right| .$$

Thus

$$1 + \frac{\left|\frac{P(X_n)}{P'(X_n)}\right|}{\left| X_{n+1} \right|} \geq \left| \frac{X_n}{X_{n+1}} \right|$$

where $P'(X_n)$ and $X_{n+1} \neq 0$. Thus, iterations are continued until an X_n is obtained such that $|P(X_n)/P'(X_n)| / |X_{n+1}|$ is as small as desired.

After a zero, α, of $P(X)$ has been found, the term $(X - \alpha)$ is synthetically divided out of $P(X)$ by deflation using Theorem 2.3 obtaining a polynomial, $P_1(X)$, of degree N-1. The root finding process is then repeated to extract a zero, α_1, of $P_1(X)$. $P(X)$ can be written as

$$P(X) = (X - \alpha)\ P_1(X) + R$$

where $R = P(\alpha)$. But $P(\alpha) = 0$. Therefore, substitution produces

$$P(X) = (X - \alpha)\ P_1(X).$$

Now $P_1(\alpha_1) = 0$ implies that $P(\alpha_1) = 0$. Hence, α_1 is a zero of $P(X)$.

By the process of root finding and successive deflations, zeros $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ of the deflated polynomials

$$P(X) = P_0(X), P_1(X), \ldots, P_{N-1}(X),$$

respectively, are extracted. Each α_i $(i = 0,1,2,\ldots,N-1)$ is a zero of $P(X)$ since each α_i is a zero of $P_{i-1}(X), P_{i-2}(X), \ldots, P_1(X), P(X)$.

After all zeros of $P(X)$ have been found, it may be possible to improve their accuracy by using them as initial approximations with Newton's method applied to the full (undeflated) polynomial, $P(X)$. This should correct any loss of accuracy which may have resulted from the successive deflations.

2. Convergence of Newton's Method

The following theorem from [2, PP. 79-81] gives sufficient conditions for the convergence of sequence (2-3).

Theorem 2.4. Let $P(X)$ be a polynomial and let the following conditions be satisfied on the closed interval $[a,b]$:

1. $P(a)\ P(b) < 0$

2. $P'(X) \neq 0,\ X \varepsilon\ [a,b]$

3. $P''(X)$ is either ≥ 0 or ≤ 0 for all $X \varepsilon\ [a,b]$

4. If c denotes the endpoint of $[a,b]$ at which $|P'(X)|$ is smaller, then $|P(c)/P'(c)| \leq b - a$.

Then Newton's method converges to the (only) solution, s, of $P(X) = 0$ for any choice of X_0 in $[a,b]$.

When convergence is obtained, it is quadratic; that is,

$$e_{i+1} = \frac{1}{2} F''(\eta_i)\ e_i^2$$

where $F(X_i) = X_i - P(X_i)/P'(X_i)$, η_i is between X_i and the zero, α, and e_i is the error in X_i. This means that the error obtained in the $(i+1)^{th}$ iteration of Newton's algorithm is proportional to the square of the error obtained in the i^{th} iteration. A proof of quadratic convergence can be found in [1, PP. 31-33].

3. Procedure for Newton's Method

The general procedure for applying Newton's method is enumerated sequentially as follows, starting with initial approximation X_0:

1. Calculate a new approximation X_{n+1} by
$$X_{n+1} = X_n - P(X_n)/P'(X_n).$$

2. Test for convergence; that is, test
$$|P(X_n)/P'(X_n)| \Big/ |X_{n+1}| < \varepsilon$$
for some ε chosen as small as desired.

3. If convergence is obtained, perform the following:

a. Save X_{n+1} as the desired approximation to a zero of P(X).

b. Deflate P(X) using X_{n+1}.

c. Replace P(X) by the deflated polynomial.

d. Return to step 1 with a new initial approximation.

4. If no convergence is obtained, increase n by 1 and return to step 1.

In order to prevent an unending iteration process in case the method does not produce convergence, a maximum number of iterations should be specified. If convergence is not obtained within this number of iterations, change the initial approximation and return to step 1 above.

4. Geometrical Interpretation of Newton's Method

A geometrical interpretation of Newton's method is given in Figure 2.1. X_i is an approximation to the zero, α. $P'(X_i)$ is the slope of the line tangent to P(X) at X_i. X_{i+1} is the intersection of the tangent line with the x axis.

5. Determining Multiple Roots

If P(X) has m distinct zeros, then P(X) can be written as

$$P(X) = a_1(X - \alpha_1)^{e_1} (X - \alpha_2)^{e_2} \ldots (X - \alpha_m)^{e_m}, \quad (m \le N)$$

where α_i is a zero of P(X) and e_i is the multiplicity of α_i $(i = 1,2,\ldots,m)$. Consider the root α_j. Dividing out the term

$(X - \alpha_j)$ by deflating $P(X)$ gives $P_1(X)$ of degree N-1 which can be written as

$$P_1(X) = (X - \alpha_1)^{e_1} (X - \alpha_2)^{e_2} \ldots (X - \alpha_j)^{e_j - 1} \ldots (X - \alpha_m)^{e_m}.$$

Evaluating $P_1(X)$ at the zero, α_j, gives $P_1(\alpha_j) = 0$ if $e_j > 1$. Thus, after a zero, α, of $P(X)$ is determined by Newton's iterative process and the current polynomial is deflated giving $P_1(X)$, then $P_1(\alpha)$ is evaluated. If $P_1(\alpha) \leq \varepsilon$ for some small number ε, α is a root of $P_1(X)$ and thus has multiplicity at least equal to two. $P_1(X)$ is then deflated giving $P_2(X)$. If $P_2(\alpha) \leq \varepsilon$, α is of multiplicity at least three. This process is continued until a deflated polynomial $P_k(X)$ is encountered such that either deg. $P_k(X) = 0$ or $P_k(\alpha) > \varepsilon$. α is then a zero of multiplicity k+1.

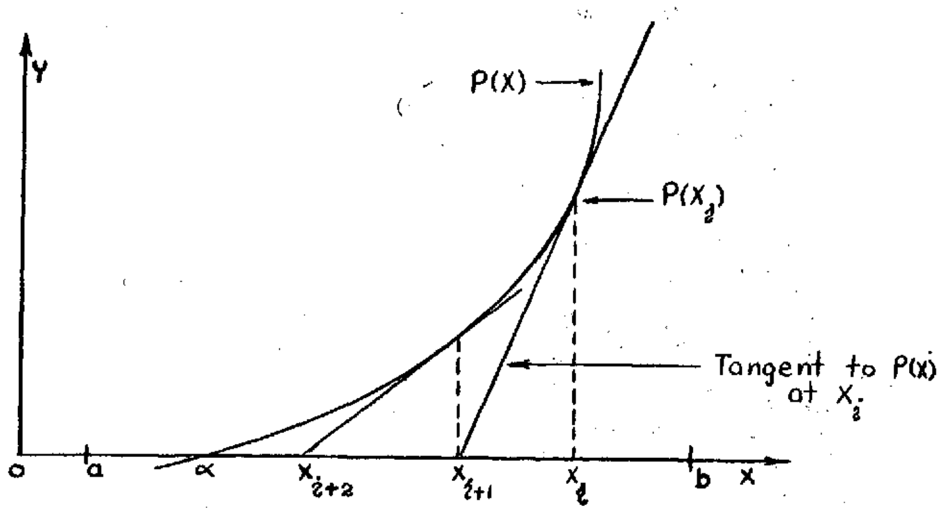


Figure 2.1. Geometrical Interpretation of Newton's Method

CHAPTER III

MULLER'S METHOD

1. Derivation of the Algorithm

Muller's method in [4] is an iterative procedure designed to find any prescribed number of zeros, real or complex, of a polynomial. The method does not require the evaluation of the derivative and near a zero the convergence is almost quadratic.

Consider the polynomial

$$P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_NX + a_{N+1} \tag{3-1}$$

with complex coefficients such that $a_1 \neq 0$. Given three distinct approximations, X_{n-2}, X_{n-1}, X_n, to a root, α, of $P(X)$, the problem is to determine X_{n+1} in such a way as to generate a sequence

$$X_1, X_2, X_3, \ldots, X_n, X_{n+1}, \ldots \tag{3-2}$$

of approximations converging to α. The points $(X_{n-2}, P(X_{n-2}))$, $(X_{n-1}, P(X_{n-1}))$, and $(X_n, P(X_n))$ determine a unique quadratic polynomial, $Q(X)$, approximating $P(X)$ in the vicinity of X_{n-2}, X_{n-1}, X_n. A general proof of this can be found in [2, PP. 133-134]. Thus, the zeros of $Q(X)$ will be approximations of the zeros of $P(X)$ in this region of approximation. From the general representation in [2, P. 184] of the Legrangian interpolating polynomial, the representation of $Q(X)$ is given by

$$Q(X) = \frac{(X - X_{n-1})(X - X_{n-2})}{(X_n - X_{n-1})(X_n - X_{n-2})} P(X_n)$$

$$+ \frac{(X - X_n)(X - X_{n-2})}{(X_{n-1} - X_n)(X_{n-1} - X_{n-2})} P(X_{n-1})$$

$$+ \frac{(X - X_n)(X - X_{n-1})}{(X_{n-2} - X_n)(X_{n-2} - X_{n-1})} P(X_{n-2})$$

which can be rewritten as

$$Q(X) = Q(X - X_n + X_n)$$

$$= \frac{(X - X_n + X_n - X_{n-1})(X - X_n + X_n - X_{n-1} + X_{n-1} - X_{n-2})}{(X_n - X_{n-1})(X_n - X_{n-1} + X_{n-1} - X_{n-2})} P(X_n)$$

$$- \frac{(X - X_n)(X - X_n + X_n - X_{n-1} + X_{n-1} - X_{n-2})}{(X_n - X_{n-1})(X_{n-1} - X_{n-2})} P(X_{n-1})$$

$$+ \frac{(X - X_n)(X - X_n + X_n - X_{n-1})}{(X_n - X_{n-1} + X_{n-1} - X_{n-2})(X_{n-1} - X_{n-2})} P(X_{n-2}).$$

In order to simplify this expression, introduce the quantities

$$h_n = X_n - X_{n-1}, \quad h = X - X_n.$$

Then

$$Q(X) = Q(X_n + h)$$

$$= \frac{(h + h_n)(h + h_n + h_{n-1})}{h_n(h_n + h_{n-1})} P(X_n)$$

$$- \frac{h(h + h_n + h_{n-1})}{h_n h_{n-1}} P(X_{n-1})$$

$$+ \frac{h(h + h_n)}{(h_n + h_{n-1})h_{n-1}} P(X_{n-2})$$

$$= \frac{h^2 + 2hh_n + hh_{n-1} + h_n^2 + h_n h_{n-1}}{h_n^2 + h_n h_{n-1}} P(X_n)$$

$$- \frac{h^2 + hh_n + hh_{n-1}}{h_n h_{n-1}} P(X_{n-1})$$

$$+ \frac{h^2 + hh_n}{h_n h_{n-1} + h_{n-1}^2} P(X_{n-2}).$$

Collecting terms containing like powers of h produces

$$Q(X) = Q(X_n + h)$$

$$= \left(\frac{P(X_n)}{h_n^2 + h_n h_{n-1}} - \frac{P(X_{n-1})}{h_n h_{n-1}} + \frac{P(X_{n-2})}{h_n h_{n-1} + h_{n-1}^2} \right) h^2$$

$$+ \left(\frac{(2h_n + h_{n-1})\ P(X_n)}{h_n^2 + h_n h_{n-1}} - \frac{(h_n + h_{n-1})\ P(X_{n-1})}{h_n h_{n-1}} + \frac{h_n\ P(X_{n-2})}{h_n h_{n-1} + h_{n-1}^2} \right) h$$

$$+ \frac{h_n (h_n + h_{n-1})\ P(X_n)}{h_n^2 + h_n h_{n-1}}$$

$$= \left(\frac{P(X_n)\ h_{n-1}}{h_n^2 h_{n-1} + h_n h_{n-1}^2} - \frac{P(X_{n-1})}{h_n h_{n-1}} + \frac{P(X_{n-2})}{h_n h_{n-1} + h_{n-1}^2} \right) h^2$$

$$+ \left(\frac{(2h_n h_{n-1} + h_{n-1}^2)\ P(X_n)}{h_n^2 h_{n-1} + h_n h_{n-1}^2} - \frac{(h_n + h_{n-1})P(X_{n-1})}{h_n h_{n-1}} + \frac{h_n\ P(X_{n-2})}{h_n h_{n-1} + h_{n-1}^2} \right) h$$

$$+ \frac{(h_n^2 h_{n-1} + h_{n-1}^2 h_n)\ P(X_n)}{h_n^2 h_{n-1} + h_n h_{n-1}^2} \quad .$$

Using the common denominator, $h_n^2 h_{n-1} + h_n h_{n-1}^2$, and combining terms yields

$$Q(X_n + h) = \left(\frac{P(X_n)\ h_{n-1} - P(X_{n-1})(h_n + h_{n-1}) + P(X_{n-2})\ h_n}{h_n^2 h_{n-1} + h_n h_{n-1}^2} \right) h^2$$

$$+ \left(\frac{(2h_n h_{n-1} + h_{n-1}^2)\ P(X_n) - (h_n + h_{n-1})^2\ P(X_{n-1}) + h_n^2 P(X_{n-2})}{h_n^2 h_{n-1} + h_n h_{n-1}^2} \right) h$$

$$+ \frac{(h_n^2 h_{n-1} + h_{n-1}^2 h_n)\ P(X_n)}{h_n^2 h_{n-1} + h_n h_{n-1}^2} \quad .$$

Multiplying by h_n/h_{n-1}^2 results in

$$Q(X_n + h) = \left[\frac{P(X_n)\ \frac{h_n}{h_{n-1}} - P(X_{n-1}) \left(\left(\frac{h_n}{h_{n-1}} \right)^2 + \frac{h_n}{h_{n-1}} \right) + P(X_{n-2}) \left(\frac{h_n}{h_{n-1}} \right)^2}{\frac{h_n^3}{h_{n-1}} + h_n^2} \right] h^2$$

$$+ \left[\frac{\left(2\frac{h_n^2}{h_{n-1}} + h_n \right) P(X_n) - h_n \left[\left(\frac{h_n}{h_{n-1}} \right) + \left(\frac{h_{n-1}}{h_{n-1}} \right) \right]^2 P(X_{n-1}) + \frac{h_n^3}{h_{n-1}^2}\ P(X_{n-2})}{\frac{h_n^3}{h_{n-1}} + h_n^2} \right] h$$

$$+ \left[\frac{\left(\frac{h_n^{\,3}}{h_{n-1}} + h_n^{\,2} \right) P(X_n)}{\frac{h_n^{\,3}}{h_{n-1}} + h_n^{\,2}} \right] .$$

Let $q_n = \frac{h_n}{h_{n-1}}$ and $q = \frac{h}{h_n}$. Then

$$Q(X_n + h) = \left(\frac{P(X_n)\ q_n - P(X_{n-1})(q_n^{\,2} + q_n) + P(X_{n-2})\ q_n^{\,2}}{q_n+1} \right) q^2$$

$$+ \left(\frac{(2\ q_n+1)\ P(X_n) - (q_n+1)^2\ P(X_{n-1}) + q_n^{\,2}\ P(X_{n-2})}{q_n+1} \right) q$$

$$+ \frac{(q_n+1)\ P(X_n)}{q_n+1} .$$

Now let

$$A_n = q_n\ P(X_n) - q_n(q_n+1)\ P(X_{n-1}) + q_n^{\,2}\ P(X_{n-2})$$

$$B_n = (2q_n+1)\ P(X_n) - (q_n+1)^2\ P(X_{n-1}) + q_n^{\,2}\ P(X_{n-2})$$

$$C_n = (q_n+1)\ P(X_n).$$

Then

$$Q(X_n + h) = Q(X_n + qh_n)$$

and

$$Q(X_n + qh_n) = \frac{A_n\ q^2 + B_n\ q + C_n}{q_n+1} .$$

Solving the quadratic equation $Q(X_n + qh_n) = 0$ and denoting the result by q_{n+1} gives:

$$q_{n+1} = \frac{-B_n \pm \sqrt{B_n^2 - 4A_nC_n}}{2A_n}$$

and the new approximation is found as follows:

$$q_{n+1} = \frac{h_{n+1}}{h_n} = \frac{X_{n+1} - X_n}{h_n} .$$

Thus

$$X_{n+1} = X_n + h_n\, q_{n+1}$$

In order to avoid loss of accuracy, q_{n+1} can be written in a better form as follows:

$$q_{n+1} = \frac{-B_n \pm \sqrt{B_n^2 - 4A_nC_n}}{2A_n} \cdot \frac{B_n \pm \sqrt{B_n^2 - 4A_nC_n}}{B_n \pm \sqrt{B_n^2 - 4A_nC_n}}$$

$$= \frac{-B_n^2 + B_n^2 - 4A_nC_n}{2A_n\,(B_n \pm \sqrt{B_n^2 - 4A_nC_n})}$$

$$q_{n+1} = \frac{-2C_n}{B_n \pm \sqrt{B_n^2 - 4A_nC_n}} . \tag{3-3}$$

The sign in the denominator should be chosen such that the magnitude of the denominator is largest, thus causing $|q_{n+1}|$ to be smallest. This, in turn, will make X_{n+1} closest to X_n.

Note that each iteration of this process requires three approximations, X_{n-2}, X_{n-1}, X_n, in order to compute X_{n+1}. Thus, when X_{n+1} is found, X_{n-1}, X_n, X_{n+1} are used to compute X_{n+2}; that is, the last three terms of the generated sequence are used to compute the next term.

Convergence of the sequence (3-2) to a zero is obtained when the elements X_k and X_{k+1} of the sequence are found such that

$$\frac{|X_{k+1} - X_k|}{|X_{k+1}|} < \varepsilon, \; X_{k+1} \neq 0;$$

that is, the ratio of the change in the approximation to the approximation itself is as small as desired.

In order to use the iterative formulas, it is necessary to compute the value, $P(X_j)$, of the polynomial $P(X)$ at the approximation X_j. The procedure for doing this is discussed in Chapter II, § 1. The iteration formulas are given in Theorem 2.3 of Chapter II.

After a zero, α, of $P(X)$ has been found, $P(X)$ is deflated as described in Chapter II, § 1, and the process repeated to extract a zero, α_1, of $P_1(X)$. By applying Muller's method to successively deflated polynomials, all the zeros of $P(X)$ are obtained. For more detailed discussion of this procedure see Chapter II, § 1, keeping in mind that Muller's instead of Newton's method is used.

Muller's method requires three initial approximations to a zero in order to start the iteration process. If three are not known, the values $X_1 = -1$, $X_2 = 1$, $X_3 = 0$ can be used.

Convergence of Muller's method is almost quadratic provided the three initial approximations are sufficiently close to a zero of $P(X)$. This is natural to expect since $P(X)$ is being approximated by a quadratic polynomial. Quadratic convergence means that the error obtained in the $(n+1)^{th}$ step of the iterative process is proportional to the square of the error obtained in the n^{th} iteration. However, no general proof of convergence has been obtained for Muller's method. It has produced convergence in the majority of the cases tested.

In application of Muller's method, an alteration should be made to handle the case in which the denominator of equation (3-3) is zero (0). This occurs whenever $P(X_n) = P(X_{n-1}) = P(X_{n-2})$. If this happens, set $q_{n+1} = 1$.

Another alteration which should be made in actual practice is to compute the quantity $|P(X_{n+1})| / |P(X_n)|$ whenever the value $P(X_{n+1})$ is calculated. If the former quantity exceeds ten (10), q_{n+1} is halved and h_n, X_{n+1}, and $P(X_{n+1})$ are recomputed accordingly.

2. Procedure for Muller's Method

The basic steps performed by Muller's method are listed sequentially as follows, starting with initial approximations X_1, X_2, and X_3.

1. Compute h_n, q_n, D_n, B_n, C_n, q_{n+1} as defined previously.
2. Compute the next approximation X_{n+1} by

$$X_{n+1} = X_n + h_n q_{n+1}.$$

3. Test for convergence; that is, test

$$|X_{n+1} - X_n| / |X_{n+1}| < \varepsilon$$

for some suitably small number ε.

4. If the test fails, return to step 1 with the last three approximations X_{n+1}, X_n, X_{n-1}.

5. If the test passes, do the following:

   a. Save X_{n+1} as the desired approximation to a zero.

   b. Deflate the current polynomial using X_{n+1}.

   c. Replace the current polynomial by the deflated polynomial.

   d. Return to step 1 with a new set of initial approximations.

In order to avoid an unending iteration process in case the method does not produce convergence, a maximum number of iterations should be specified. If convergence is not obtained within this number of iterations, the initial approximations should be altered.

3. Geometrical Interpretation of Muller's Method

Figure 3.1. shows the geometrical interpretation of Muller's method for real roots of P(X) and the quadratic Q(X). The root of Q(X) closest to X_i is chosen as the next approximation X_{i+1}.

4. Determining Multiple Roots

For a discussion concerning multiple roots see Chapter II, § 5.

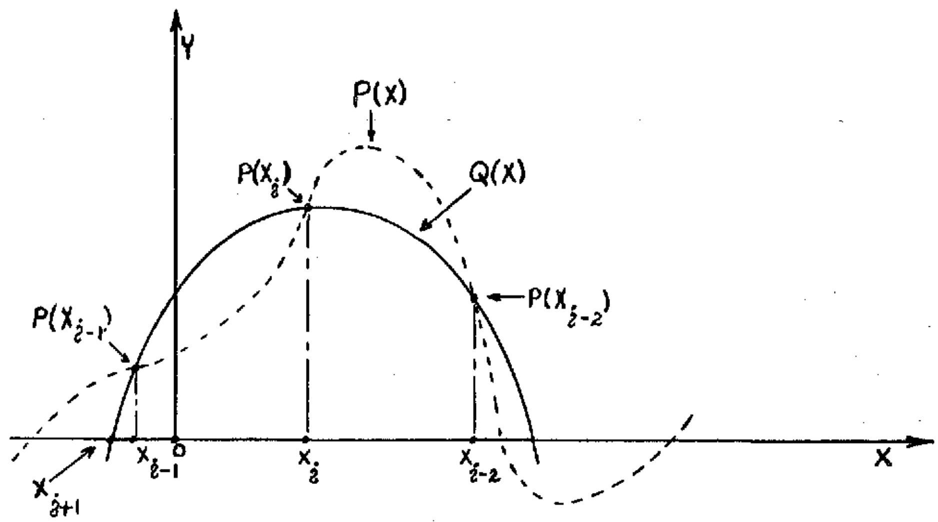


Figure 3.1. Geometrical Interpretation of Muller's Method

CHAPTER IV

GREATEST COMMON DIVISOR METHOD

1. Derivation of the Algorithm

The greatest common divisor (g.c.d.) method reduces the problem of finding all the zeros of a polynomial, possibly having multiple zeros, to one of solving for zeros of a polynomial all of whose zeros are simple.

Consider the N^{th} degree polynomial

$$P(X) = a_1 X^N + a_2 X^{N-1} + \dots + a_N X + a_{N+1}$$

where $a_1 \neq 0$ and $a_1, a_2, \dots, a_{N+1}$ are complex numbers. If P(X) has m distinct zeros, $\alpha_1, \alpha_2, \dots, \alpha_m$, then P(X) can be expressed in the form

$$P(X) = a_1 (X - \alpha_1)^{e_1} (X - \alpha_2)^{e_2} \dots (X - \alpha_m)^{e_m} \tag{4-1}$$

where e_i is the multiplicity of α_i, $i = 1,2,\dots,m$. The derivative of P(X) is

$$P'(X) = N\, a_1 X^{N-1} + (N-1)\, a_2 X^{N-2} + \dots + a_N$$

which can also be expressed as

$$P'(X) = a_1(X - \alpha_i)^{e_1-1} (X - \alpha_2)^{e_2-1} \ldots (X - \alpha_m)^{e_m-1} \sum_{i=1}^{m} e_i \prod_{\substack{j=1 \\ j\neq i}}^{m} (X - \alpha_j). \tag{4-2}$$

The greatest common divisor of P(X) and $P'(X)$ is obtained from the following theorem.

Theorem 4.1. Let P(X) be an N^{th} degree polynomial having m distinct zeros $\alpha_1, \alpha_2, \ldots, \alpha_m$ of multiplicity $e_1, e_2, \ldots, e_m$ respectively. Then the polynomial

$$D(X) = (X - \alpha_1)^{e_1-1} (X - \alpha_2)^{e_2-1} \ldots (X - \alpha_m)^{e_m-1}$$

is the unique monic greatest common divisor of P(X) and its derivative $P'(X)$.

Proof. Since the set of all polynomials over the complex number field is a unique factorization domain and since each factor $X - \alpha_i$ is irreducible, it follows from (4-1) and (4-2) that D(X) is the unique monic greatest common divisor of P(X) and $P'(X)$.

It follows from Theorem 4.1 that each zero of D(X) is also a zero of P(X) and $P'(X)$. Hence we have the following result.

Theorem 4.2. If P(X) is a polynomial, then P(X) and $P'(X)$ are relatively prime if and only if P(X) has no multiple zeros.

Consider the polynomial H(X) obtained by dividing P(X) by its monic g.c.d., D(X).

$$H(X) = P(X)/D(X)$$

$$= a_1 \prod_{i=1}^{m} (X - \alpha_i)^{e_i} \Big/ \prod_{i=1}^{m} (X - \alpha_i)^{e_i - 1}$$

$$= a_1 \prod_{i=1}^{m} (X - \alpha_i).$$

The zeros of $H(X)$ are all simple zeros and are also all the distinct zeros of $P(X)$. Use of the g.c.d. method involves computation of $H(X)$ when given $P(X)$.

In order to obtain $H(X)$, a computational algorithm is necessary to find the g.c.d. of $P(X)$ and $P'(X)$. The general method for computing the g.c.d. of two polynomials is as follows: Let $R_0(X)$ and $R_1(X)$ be two polynomials having degrees N_0 and N_1 respectively such that $N_1 \leq N_0$. The g.c.d. of $R_0(X)$ and $R_1(X)$ is desired. By the division algorithm, there exists polynomials $S_1(X)$ and $R_2(X)$ such that

$$R_0(X) = R_1(X)\ S_1(X) + R_2(X)$$

where either $R_2(X) = 0$ or deg. $R_2(X)$ < deg. $R_1(X)$. Similarly if $R_2(X) \neq 0$, there exists polynomials $S_2(X)$ and $R_3(X)$ such that

$$R_1(X) = S_2(X)\ R_2(X) + R_3(X)$$

where either $R_3(X) = 0$ or deg. $R_3(X)$ < deg. $R_2(X)$. Continuing in the above manner, suppose $R_i(X)$ and $R_{i+1}(X)$ have been found where deg. $R_{i+1}(X)$ < deg. $R_i(X)$. Then there exists polynomials $R_{i+2}(X)$ and $S_{i+1}(X)$ such that

$$R_i(X) = R_{i+1}(X)\ S_{i+1}(X) + R_{i+2}(X)$$

where either $R_{i+2}(X) = 0$ or deg. $R_{i+2}(X) <$ deg. $R_{i+1}(X)$. Then we obtain a sequence $R_0(X), R_1(X), \ldots, R_K(X), R_{K+1}(X)$ such that deg. $R_i(X) <$ deg. $R_{i-1}(X)$, $i = 1,2,\ldots,K+1$. Since a polynomial cannot have degree less than zero, the above process, in a finite number of steps (at most N_1), results in polynomials $R_{K-1}(X)$, $S_K(X)$ and $R_K(X)$ with deg. $R_K(X) <$ deg. $R_{K-1}(X)$ such that

$$R_{K-1}(X) = R_K(X)\ S_K(X) + R_{K+1}(X)$$

and $R_{K+1}(X) = 0$.

<u>Theorem 4.3</u>. Let the sequence $R_0(X), R_1(X), \ldots, R_K(X), R_{K+1}(X)$ be defined as above. Then $R_K(X)$ is the greatest common divisor of $R_0(X)$ and $R_1(X)$.

<u>Proof</u>. It is clear that $R_K(X)$ divides $R_{K-1}(X)$. If $R_K(X)$ divides $R_i(X)$ for $0 \leq j < i \leq k$, then $R_j(X) = R_{j+1}(X)\ S_{j+1}(X) + R_{j+2}(X)$. Thus, $R_K(X)$ divides $R_j(X)$ and it follows by induction that $R_K(X)$ divides both $R_0(X)$ and $R_1(X)$. By reversing the inductive argument given above, it is easy to see that if $L(X)$ divides $R_0(X)$ and $R_1(X)$, the $L(X)$ divides $R_i(X)$ for $i = 0,1,\ldots,K$. Therefore, $L(X)$ divides $R_K(X)$ which shows that $R_K(X)$ is the greatest common divisor of $R_0(X)$ and $R_1(X)$.

The above theorem tells how to obtain the greatest common divisor of two polynomials. A machine oriented method is now developed for computing the sequence of $R_j(X)$'s. Beginning the sequence with $R_0(X)$ and $R_1(X)$, the polynomial $R_{i+1}(X)$ of the sequence is derived from $R_i(X)$

and $R_{i-1}(X)$ as follows: Let $R_{i-1}(X)$ of degree N_{i-1} be given by

$$R_{i-1}(X) = r_{i-1,1} X^{N_{i-1}} + r_{i-1,2} X^{N_{i-1}-1} + \dots + r_{i-1,N_{i-1}} X + r_{i-1,N_{i-1}+1}$$

and $R_i(X)$ of degree N_i be given by

$$R_i(X) = r_{i,1} X^{N_i} + r_{i,2} X^{N_i-1} + \dots + r_{i,N_i} X + r_{i,N_i+1}$$

where $N_i \leq N_{i-1}$. Define $U_1(X)$ by

$$U_1(X) = (r_{i-1,1} / r_{i,1}) X^{N_{i-1}-N_i} .$$

Then define $T_1(X)$ by

$$\begin{aligned} T_1(X) &= R_{i-1}(X) - U_1(X) R_i(X) \\ &= [r_{i-1,1} - r_{i,1} (r_{i-1,1} / r_{i,1})] X^{N_{i-1}} \\ &\quad + [r_{i-1,2} - r_{i,2} (r_{i-1,1} / r_{i,1})] X^{N_{i-1}-1} \\ &\quad + \dots \\ &\quad + [r_{i-1,N_{i-1}+1} - r_{i,N_{i-1}+1} (r_{i-1,1} / r_{i,1})] \end{aligned}$$

where $r_{i,j} = 0$ for $j > N_i+1$.

We consider three cases.

(1) If $T_1(X) = 0$, then $R_i(X) = R_K(X)$; that is, $R_i(X)$ is the g.c.d. of $R_0(X)$ and $R_1(X)$.

(2) If $T_1(X) \neq 0$ and deg. $T_1(X) < N_i$, then $R_{i+1}(X) = T_1(X)$.

(3) If $T_1(X) \neq 0$ and deg. $T_1(X) = M_1 \geq N_i$, then define $U_2(X)$ by

$$U_2(X) = (t_{1,1} / r_{i,1})\, X^{M_1 - N_i}$$

where

$$T_1(X) = t_{1,1} X^{M_1} + t_{1,2} X^{M_1-1} + \dots + t_{1,M_1} X + t_{1,M_1+1}.$$

Define $T_2(X) = T_1(X) - U_2(X)\, R_i(X)$ which can be expressed by

$$\begin{aligned} T_2(X) = {} & [t_{1,1} - (t_{1,1} / r_{i,1})\, r_{i,1}]\, X^{M_1-1} \\ & + [t_{1,2} - (t_{1,1} / r_{i,1})\, r_{i,2}]\, X^{M_1-2} \\ & + \dots \\ & + [t_{1,M_1+1} - (t_{1,1} / r_{i,1})\, r_{i,M_1+1}] \end{aligned}$$

where $r_{i,j} = 0$ for $j > N_i+1$. We again consider the following three cases.

(1) If $T_2(X) = 0$, then $R_i(X)$ is the g.c.d. of $R_0(X)$ and $R_1(X)$.

(2) If $T_2(X) \neq 0$ and deg. $T_2(X) <$ deg. $R_i(X)$, then $R_{i+1}(X) = T_2(X)$.

(3) If $T_2(X) \neq 0$ and deg. $T_2(X) = M_2 \geq N_i$, then define $U_3(X)$ by

$$U_3(X) = (t_{2,1} / r_{i,1})\, X^{M_2 - N_i}$$

where

$$T_2(X) = t_{2,1} X^{M_2} + t_{2,2} X^{M_2-1} + \ldots + t_{2,M_2} X + t_{2,M_2+1} .$$

Since deg. $T_{i+1}(X)$ < deg. $T_i(X)$, then this process is finite (not to exceed N_{i-1}) ending, for some integer S, in $T_S(X)$ such that

(1) $T_S(X) = 0$ and $R_i(X)$ is the g.c.d. of $R_0(X)$ and $R_1(X)$ or

(2) $T_S(X) \neq 0$ but deg. $T_S(X)$ < deg. $R_i(X)$, in which case $T_S(X) = R_{i+1}(X)$.

Thus, using this algorithm and given $R_0(X)$ and $R_1(X)$, the sequence $R_0(X), R_1(X), R_2(X), \ldots, R_i(X), R_{i+1}(X)$ can be generated such that either

(1) $R_{i+1}(X) = 0$ and $R_i(X)$ is the g.c.d. of $R_0(X)$ and $R_1(X)$ or

(2) $R_{i+1}(X) \neq 0$ and $N_{i+1} < N_i$. In a finite number of iterations, $R_K(X)$, the g.c.d. of $R_0(X)$ and $R_1(X)$, can be obtained.

Recall that we wanted to obtain the polynomial $H(X) = P(X)/D(X)$ where $D(X)$ is the g.c.d. of $P(X)$ and $P'(X)$. Thus, after obtaining $D(X)$ by the above algorithm, it is necessary to divide $P(X)$ by $D(X)$ obtaining $H(X)$ all whose zeros are simple.

Once $H(X)$ is obtained, an appropriate method such as Newton's method or Muller's method is applied to extract the zeros of $H(X)$. This gives all the zeros of $P(X)$.

As in Newton's or Muller's method, the zeros may be checked for accuracy and possibly improved by using them as initial approximations with the particular method applied to the full (undeflated) polynomial, $P(X)$.

2. Determining Multiplicities

After all zeros of P(X) are found, the multiplicity of each zero can be determined by the process outlined in Chapter II, § 5.

3. Procedure for the G.C.D. Method

The basic steps performed by the greatest common divisor method are listed sequentially as follows:

1. Given a polynomial, P(X), in the form
   $P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_NX + a_{N+1}$.
2. Calculate the derivative, $P'(X)$, of P(X) in the form
   $P'(X) = b_1X^{N-1} + b_2X^{N-2} + \ldots + b_N$ where $b_1 = Na_1$, $b_2 = (N-1)a_2, \ldots, b_N = a_N$.
3. Find D(X), the g.c.d. of P(X) and $P'(X)$ using the algorithms developed above.
4. Calculate $H(X) = P(X)/D(X)$, the polynomial having only simple zeros.
5. Use some appropriate method to extract the zeros of H(X).
6. Determine the multiplicity of each of the zeros obtained in step 5.

CHAPTER V

REPEATED GREATEST COMMON DIVISOR METHOD

1. Derivation of the Algorithm

The repeated greatest common divisor (repeated g.c.d.) method makes repeated use of the g.c.d. method to extract the zeros and their multiplicities of a polynomial with complex coefficients. That is, the repeated g.c.d. method reduces the problem of finding the zeros of a polynomial, P(X), which possibly has multiple zeros, to one of finding the zeros of a polynomial which has only simple zeros and the zeros of this polynomial are all the zeros of P(X) of a given multiplicity.

Let

$$\begin{aligned} P(X) &= a_1 X^N + a_2 X^{N-1} + \dots + a_N X + a_{N+1} \\ &= a_1 (X - \alpha_1)^{e_1} (X - \alpha_2)^{e_2} \dots (X - \alpha_m)^{e_m} \end{aligned}$$

where $a_1 \neq 0$, each a_i is a complex number, and $\alpha_1, \alpha_2, \dots, \alpha_m$ are the distinct zeros of P(X) having multiplicity $e_1, e_2, \dots, e_m$, respectively. If $D_1(X)$ is the monic greatest common divisor of P(X) and P'(X), then Theorem 4.1 shows that

$$D_1(X) = (X - \alpha_1)^{e_1 - 1} (X - \alpha_2)^{e_2 - 1} \dots (X - \alpha_m)^{e_m - 1}$$

where we assume that if $e_j = 1$, then $X - \alpha_j$ does not appear in the representation. Let $D_2(X)$ be the monic greatest common divisor of $D_1(X)$ and $D_1'(X)$. Then

$$D_2(X) = (X - \alpha_1)^{e_1-2} (X - \alpha_2)^{e_2-2} \dots (X - \alpha_m)^{e_m-2}$$

where we assume that if $e_j \leq 2$, then $X - \alpha_j$ does not appear in the representation. From the above it is clear that the zeros of $D_1(X)$ are just the multiple zeros of $P(X)$ to one lower power. The zeros of $D_2(X)$ are just the multiple zeros of $D_1(X)$ to one lower power. Thus, the zeros of $D_2(X)$ are just the zeros of $P(X)$ which have multiplicity greater than two, and their multiplicity in $D_2(X)$ is reduced by two. Therefore, it follows that

$$G_1(X) = [P(X)/D_1(X)]/[D_1(X)/D_2(X)] = P(X)D_2(X)/[D_1(X)]^2$$

has only simple zeros and they are just the simple zeros of $P(X)$. In general if $D_j(X)$ has been defined for $1 \leq j \leq i$ and if $D_{i+1}(X)$ is the monic greatest common divisor of $D_i(X)$ and $D_i'(X)$, then the zeros of $D_{i+1}(X)$ are the multiple zeros of $D_i(X)$ to one lower power. Thus, the zeros of $D_{i+1}(X)$ are just the zeros of $P(X)$ which have multiplicity greater than i+1 and their multiplicity in $D_{i+1}(X)$ is reduced by i+1. It follows that

$$\begin{aligned} G_i(X) &= [D_{i-1}(X)/D_i(X)]/[D_i(X)/D_{i+1}(X)] \\ &= D_{i-1}(X)\ D_{i+1}(X)/[D_i(X)]^2 \end{aligned}$$

has simple zeros and they are just the zeros of $P(X)$ that have multiplicity i. Thus, we have proven the following theorem.

Theorem 5.1. Let $P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_NX + a_{N+1}$ where $a_1 \neq 0$ and $a_1, a_2, \ldots, a_{N+1}$ are complex numbers. If $D_0(X) = P(X)$ and if $D_{i+1}(X)$ is the monic greatest common divisor of $D_i(X)$ and $D_i'(X)$ for $i \geq 0$, then

$$G_i(X) = D_{i-1}(X)\ D_{i+1}(X)/[D_i(X)]^2$$

has only simple zeros and they are just the zeros of P(X) that have multiplicity i.

Thus, by the above theorem we can generate a sequence of polynomials $G_1(X), G_2(X), \ldots, G_K(X)$ where the set of zeros of P(X) is the same as the set of zeros of this sequence and the multiplicity of each zero in P(X) is given by the corresponding subscript on G(X). Therefore, by using a method such as Newton's method of Muller's method to calculate the zeros of each $G_i(X)$, we will have the zeros of P(X) along with their multiplicities.

2. Procedure for the Repeated G.C.D. Method

The basic steps performed by the greatest common divisor method are listed sequentially as follows:

1. Given a polynomial, P(X), in the form $P(X) = a_1X^N + a_2X^{N-1} + \ldots + a_NX + a_{N+1}$.
2. Set $D_0(X) = P(X)$.
3. Calculate the derivative, $D_0'(X)$, of $D_0(X)$ in the form

$$D_0'(X) = b_1X^{M-1} + b_2X^{M-2} + \ldots + b_M$$

where deg. $D_0(X) = M$, $D_0(X) = d_1X^M + \ldots + d_{M+1}$,

and $b_1 = Md_1$, $b_2 = (M-1)d_2$, $\ldots$, $b_M = d_M$.

4. Find $D_1(X)$, the g.c.d. of $D_0(X)$ and $D_0'(X)$ using the algorithms developed in Chapter IV.
5. Similar to 3., calculate $D_1'(X)$.
6. Find $D_2(X)$, the g.c.d. of $D_1(X)$ and $D_1'(X)$ using the algorithms developed in Chapter IV.
7. Calculate $G(X) = D_0(X)\ D_2(X)/[D_1(X)]^2$.
8. Use some appropriate method to extract the zeros of $G(X)$ and assign these zeros the correct multiplicity as zeros of $P(X)$.
9. Set $D_0(X) = D_1(X)$, $D_0'(X) = D_1'(X)$, and $D_1(X) = D_2(X)$. Then repeat 5.-8. above until all the zeros of $P(X)$ are found.

CHAPTER VI

CONCLUSION

In order to compare Newton's, Muller's, the greatest common divisor, and the repeated greatest common divisor methods, we consider the polynomials as being divided into the following classes:

1. polynomials with all distinct zeros.
2. polynomials with multiple zeros.

The comparisons in the following material are results of tests made on the IBM 360/50 computer which has a 32 bit word. The programs were successfully run on the CDC 6600 and the UNIVAC 1108 which have a 60 bit word and a 36 bit word respectively. It was noted that the UNIVAC 1108 is about 15 times faster than the IBM 360/50. The CDC 6600 is faster than the UNIVAC 1108 but the difference is not as great as that between the UNIVAC 1108 and the IBM 360/50.

1. Polynomials With all Distinct Zeros

First we consider the class of polynomials having distinct zeros. Newton's method is particularly suited for this class of polynomials. Its quadratic convergence is very fast which can save time and money to the user. The accuracy obtained is excellent as shown in Exhibit 6.1 which presents the zeros of a 15^{th} degree polynomial in double precision. In most cases, the method produces convergence for almost any initial approximation given.

Muller's method also produces good results on this class of polynomials. The rate of convergence is, however, somewhat slower than Newton's method. This fact is especially significant when working with polynomials of high degree. The accuracy obtained by Muller's method is comparable to, but does not exceed that of Newton's method. In most cases, the accuracy of the two methods does not differ by more than one or two decimal places. Exhibit 6.2 shows results of Muller's method for the polynomial of Exhibit 6.1. As in Newton's method, convergence is produced for almost any initial approximation given.

The g.c.d. method, whether used with Newton's or Muller's method as a supporting method on this class of polynomials, is no better than Newton's or Muller's method alone. The reason for this is that the greatest common divisor of the polynomial, P(X), and its derivative is 1. Thus, H(X) = P(X)/g.c.d. P(X) = P(X); that is, the polynomial solved by the supporting method is the same as the original polynomial. Thus, in this case the g.c.d. method will not produce better results than the supporting method used alone. The above comments also hold for the repeated g.c.d. method.

Thus, this class of polynomials presents no difficulty to any of these four methods. Newton's method, because of its speed, is therefore recommended.

2. Polynomials With Multiple Zeros

Next consider the class of polynomials containing multiple zeros. Exhibits 6.3 - 6.26 illustrate output from six different programs using the methods described in Chapters II - V. Four polynomials are used where the zeros of these polynomials are listed below. The number in parentheses indicates the multiplicity of that zero.

Polynomial #1	Polynomial #2	Polynomial #3	Polynomial #4
2+2i (3)	-2.33 (1)	2+2i (3)	1+i (6)
1+2i (2)	.003 (2)	1+2i (2)	1-i (6)
-1+.5i (1)	i (2)	-1+.5i (3)	
	1.5i (2)		
	-1.5i (2)		
	3i (3)		
	-1-i (3)		

Note the relationship between polynomials #1 and #3.

This class presents considerable difficulty for Newton's method, especially those polynomials containing zeros of high multiplicity or containing a considerable number of multiple zeros. The iteration formula for Newton's method is

$$X_{n+1} = X_n - P(X_n)/P'(X_n).$$

If c is a multiple zero then $P(c) = P'(c) = 0$. Hence, as $X_n \to c$, $P(X_n) \to 0$ and $P'(X_n) \to 0$ and the iteration formula may be unstable, resulting in no convergence or bad accuracy. As the number of multiple zeros increases, the polynomial becomes more ill-conditioned, convergence becomes more difficult, and accuracy is lost. Thus, the possibility of convergence decreases. This also holds true if the multiplicities of the zeros are increased. The rate of convergence of Newton's method is much slower for multiple zeros than for distinct zeros. Exhibit 6.3 shows a polynomial (#1) containing two multiple zeros solved in double precision. Note the following from Exhibit 6.3.

1. Roots #2 and #3 are greatly improved by iterating on the original polynomial. Distinct roots are usually improved in this manner.

2. The time taken to solve this 6^{th} degree equation with multiple roots is greater than the time taken by the same program to solve a 15^{th} degree polynomial with all distinct roots (Exhibit 6.1).
3. Root #2 did not pass the convergence test after 200 iterations even though it was improved. This is probably due to the fact that the polynomial from which root 2 was extracted had only one multiple root but the original polynomial from which it was extracted the second time had two multiple roots; that is, the original polynomial is more ill-conditioned.
4. The accuracy of the roots before the attempt to improve accuracy is very poor. Root #2 is accurate to only three decimal places as compared to the 15 decimal places in Exhibit 6.1 for distinct roots. Root #3 is especially bad, the imaginary part being accurate to only one decimal place.

Exhibit 6.4 uses polynomial #2. Note the poor results obtained before the attempt to improve accuracy and the improvement afterward. Also note that after the attempt to improve accuracy, one of the zeros, namely 3i, is lost and an extra zero, namely 1.5, is included in the list. (See Appendix A, § 4.) A convergence requirement of 10^{-5} was used on this polynomial to get it to converge to all of the zeros in a maximum number of 200 iterations.

In many cases, Newton's method fails to converge altogether. Polynomial #3 could not be solved using Newton's method with a maximum number of 200 iterations and a convergence requirement of 10^{-9}. Exhibit 6.5 illustrates the bad results for a convergence requirement of 10^{-5} which was needed in order to get convergence. In Exhibit 6.6 a convergence requirement of 10^{-3} was needed in order to get convergence to the zeros of polynomial #4.

Muller's method also encounters difficulty, although to a lesser degree than Newton's method, on this class of polynomials. In most cases, Muller's method produces convergence even when Newton's method completely fails. Newton's method completely failed for polynomials #3 and #4 with a convergence requirement of 10^{-9} but convergence was obtained using Muller's method as shown in Exhibits 6.9 and 6.10. The accuracy obtained by Muller's method is not good but usually better than Newton's method using the same convergence requirement. The rate of convergence of Muller's method is considerably slower for multiple zeros than for distinct zeros. However, for multiple zeros, Muller's method is as fast or faster than Newton's.

The g.c.d. method is perfectly suited for polynomials with multiple zeros. All multiple zeros are removed leaving only a polynomial of class 1 (all distinct roots) to be solved. This indicates that best results should be obtained by using Newton's method as the supporting method, since Newton's method enjoys the advantage of speed over Muller's method for distinct zeros. This has indeed proved to be true. The accuracy of the zeros obtained decreases, somewhat, when the number of multiple zeros is increased. This is due to accuracy lost in computing the g.c.d. and the quotient polynomial and not as a result of the supporting method. It is easy to see that the accuracy of the g.c.d. method is best when the degree of the greatest common divisor of

$P(X)$ and $P'(X)$ is maximum. This is due to the fact that the error in the greatest common divisor is minimized in this case. The accuracy obtained using Newton's method and Muller's method as supporting methods is about the same. This is verified by Exhibits 6.11 - 6.14 (g.c.d. method with Newton) and Exhibits 6.15 - 6.18 (g.c.d. method with Muller).

Multiplicities are determined with excellent accuracy. The g.c.d. method is not as sensitive to zeros of high multiplicity or polynomials containing a large number of multiple zeros as are both Newton's and Muller's methods. A quick comparison of Exhibits 6.11 - 6.14 and 6.15 - 6.18 with Exhibits 6.3 - 6.6 and 6.7 - 6.10 show that the g.c.d. method with either supporting method is much more accurate than either Newton's or Muller's method. For example, Exhibits 6.5 and 6.9 show polynomial #3 for which Newton's method and Muller's method both gave poor convergence. But Exhibits 6.13 and 6.17 show very accurate results for polynomial #3.

The repeated g.c.d. method is also suited very well for polynomials with multiple zeros. Exhibits 6.19 - 6.22 and Exhibits 6.23 - 6.26 are results of the repeated g.c.d. method with Newton's method and Muller's method as supporting methods, respectively. However, the results of the repeated g.c.d. method are not as good as those obtained from the g.c.d. method. Since the repeated g.c.d. method repeated uses the g.c.d. algorithm, the error tends to build up in this method when a polynomial has several zeros of different multiplicities. This can be observed by comparing Exhibits 6.20 and 6.24 with Exhibits 6.12 and 6.16 on polynomial #2 and by comparing Exhibits 6.21 and 6.25 with Exhibits 6.13 and 6.17 on polynomial #3. As was the case of the g.c.d.

method, there is little difference between the repeated g.c.d. method with Newton's method or Muller's method as a supporting method. This can be observed by comparing Exhibits 6.19 - 6.22 (Newton) with Exhibits 6.23 - 6.26 (Muller). Even though the results of the repeated g.c.d. method are not quite as good as the results of the g.c.d. method, they are far superior to the results of both Newton's method and Muller's method.

Table 6.I gives a comparison of the execution times of the six methods for polynomials #1 - #4.

TABLE 6.I

METHOD	EXECUTION TIME*
Newton	104.16 seconds
Muller	96.79 seconds
G.C.D. with Newton	7.51 seconds
G.C.D. with Muller	8.91 seconds
Repeated G.C.D. with Newton	7.71 seconds
Repeated G.C.D. with Muller	15.16 seconds

It is clear from Table 6.I that the g.c.d. and the repeated g.c.d. methods are much faster than both Newton's and Muller's method on

*These times are from execution runs on the IBM 360/50 WATFOR system.

polynomials with multiple zeros. Therefore, for polynomials with multiple zeros, the order in which the methods are recommended is as follows.

1. G.C.D. with Newton.
2. G.C.D. with Muller.
3. Repeated G.C.D. with Newton.
4. Repeated G.C.D. with Muller.
5. Muller.
6. Newton.

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  7 OF DEGREE 15


THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.30000000000000000D 01 +  0.00000000000000000D 00 I
 P( 2) = -0.17900000000000000D 02 + -0.00000000000000000D 00 I
 P( 3) =  0.20100000000000000D 02 + -0.65760000000000001D 02 I
 P( 4) =  0.17450000000000000D 03 +  0.28436800000000000D 03 I
 P( 5) = -0.75942000000000000D 03 +  0.31118080000000000D 03 I
 P( 6) =  0.82743600000000000D 03 + -0.30690400000000000D 04 I
 P( 7) =  0.13290840000000000D 04 +  0.47106240000000000D 04 I
 P( 8) = -0.56118600000000000D 04 + -0.16745760000000000D 04 I
 P( 9) =  0.72247560000000000D 04 + -0.15482880000000000D 04 I
 P(10) = -0.22769920000000000D 04 +  0.30463200000000000D 04 I
 P(11) = -0.12414720000000000D 04 + -0.40972800000000000D 04 I
 P(12) =  0.54028000000000001D 04 +  0.12634880000000000D 04 I
 P(13) = -0.64683360000000001D 04 +  0.12364800000000000D 04 I
 P(14) =  0.14674560000000000D 04 +  0.22720000000000000D 02 I
 P(15) = -0.10771200000000000D 03 + -0.54758400000000000D 03 I
 P(16) =  0.34560000000000000D 02 +  0.12672000000000000D 03 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                   MULTIPLICITIES              INITIAL APPROXIMATION


 ROOT( 1) =  0.30000000000000000D 00 +  0.16093581151665310D-16 I      1       0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.20000000000000000D 00 + -0.20000000000000000D 00 I      1       0.70710675530463460D 00 +  0.70710680706845950D 00 I
 ROOT( 3) = -0.20000000000000000D 00 +  0.20000000000000000D 00 I      1       0.38822847926540560D 00 +  0.14488887631171930D 01 I
 ROOT( 4) =  0.24850261096558030D-17 +  0.10000000000000000D 01 I      1      -0.51763825519667240D 00 +  0.19318516083687550D 01 I
 ROOT( 5) = -0.10000000000000000D 01 + -0.53219537064599460D-16 I      1      -0.17677671470807010D 01 +  0.17677667588520150D 01 I
 ROOT( 6) = -0.10000000000000000D 01 + -0.99999999999999980D 00 I      1      -0.28977775830749900D 01 +  0.77645674639870700D 00 I
 ROOT( 7) = -0.31069187592603490D-15 + -0.99999999999999940D 00 I      1      -0.33807402483312290D 01 + -0.90586719408161600D 00 I
 ROOT( 8) = -0.20000000000000000D 01 + -0.30000000000000000D 01 I      1      -0.28284266071078960D 01 + -0.28284276423843900D 01 I
 ROOT( 9) =  0.10000000000000160D 01 + -0.10097649963521000D-12 I      1      -0.11646848013998990D 01 + -0.43466664598733680D 01 I
 ROOT(10) =  0.20000000000000040D 01 + -0.10000000000000032D 01 I      1       0.12940963450986450D 01 + -0.48296288314530270D 01 I
 ROOT(11) =  0.19999999999999980D 01 +  0.13855078472222020D-12 I      1       0.38890882929795090D 01 + -0.38890863000722580D 01 I
 ROOT(12) =  0.30000000000000010D 01 + -0.14175491574492190D-13 I      1       0.57955553935123030D 01 + -0.15529126442689740D 01 I
 ROOT(13) =  0.39999999999999980D 01 +  0.40000000000000030D 01 I      1       0.62785173577341250D 01 +  0.16823257082477520D 01 I
 ROOT(14) =  0.99999999999996060D 00 +  0.10000000000000050D 01 I      1       SOLVED BY DIRECT METHOD
 ROOT(15) = -0.33333333333333340D 01 + -0.18503717077085940D-15 I      1       SOLVED BY DIRECT METHOD
```

Exhibit 6.1.

AFTER THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE

ROOTS OF P(X)	MULTIPLICITIES	INITIAL APPROXIMATION
ROOT(1) = 0.30000000000000000D 00 + 0.33289435377059130D-16 I	1	0.48296291156562790D 00 + 0.12940952844381870D 00 I
ROOT(2) = 0.20000000000000000D 00 + -0.20000000000000000D 00 I	1	0.70710675530463460D 00 + 0.70710680706845950D 00 I
ROOT(3) = -0.20000000000000000D 00 + 0.20000000000000000D 00 I	1	0.38822847926540560D 00 + 0.14488887631171930D 01 I
ROOT(4) = -0.11260430437627530D-17 + 0.10000000000000000D 01 I	1	-0.51763825519667240D 00 + 0.19318516083687550D 01 I
ROOT(5) = -0.10000000000000000D 01 + -0.18824780305507960D-16 I	1	-0.17677671470807010D 01 + 0.17677667588520150D 01 I
ROOT(6) = -0.10000000000000000D 01 + -0.10000000000000000D 01 I	1	-0.28977775830749900D 01 + 0.77645674639870700D 00 I
ROOT(7) = 0.61318693193791400D-16 + -0.10000000000000000D 01 I	1	-0.33807402483312290D 01 + -0.90586719408161600D 00 I
ROOT(8) = -0.20000000000000000D 01 + -0.30000000000000000D 01 I	1	-0.28284266071078960D 01 + -0.28284276423843900D 01 I
ROOT(9) = 0.10000000000000000D 01 + 0.72877654808950640D-16 I	1	-0.11646848013998990D 01 + -0.43466664598733680D 01 I
ROOT(10) = 0.20000000000000001D 01 + -0.99999999999999900D 00 I	1	0.12940963450986450D 01 + -0.48296288314530270D 01 I
ROOT(11) = 0.20000000000000003D 01 + -0.39680396208113570D-14 I	1	0.38890882929795090D 01 + -0.38890863000722580D 01 I
ROOT(12) = 0.29999999999999997D 01 + 0.28966896304378480D-14 I	1	0.57955553935123030D 01 + -0.15529126442689740D 01 I
ROOT(13) = 0.40000000000000000D 01 + 0.40000000000000001D 01 I	1	0.62785173577341250D 01 + 0.16823257082477520D 01 I
ROOT(14) = 0.99999999999999995D 00 + 0.10000000000000000D 01 I	1	SOLVED BY DIRECT METHOD
ROOT(15) = -0.33333333333333334D 01 + -0.89980636705176380D-16 I	1	SOLVED BY DIRECT METHOD

Exhibit 6.1. Roots Are: $-1 - i$, $1 + i$, $-2 - 3i$, $2 - i$, 3, 2, i, $-i$, $-10/3$, .3, -1, 1, $4 + 4i$, $-.2 + .2i$, $.2 - .2i$.

```
MULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOMIAL
POLYNOMIAL NUMBER  7 OF DEGREE 15

THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.3000000000000000D 01 +  0.0000000000000000D 00 I
 P( 2) = -0.1790000000000000D 02 + -0.0000000000000000D 00 I
 P( 3) =  0.2010000000000000D 02 + -0.6576000000000001D 02 I
 P( 4) =  0.1745000000000000D 03 +  0.2843680000000000D 03 I
 P( 5) = -0.7594200000000000D 03 +  0.3118080000000000D 03 I
 P( 6) =  0.8274360000000000D 03 + -0.3069040000000000D 04 I
 P( 7) =  0.1329084000000000D 04 +  0.4710624000000000D 04 I
 P( 8) = -0.5611860000000000D 04 + -0.1674576000000000D 04 I
 P( 9) =  0.7224756000000000D 04 + -0.1548288000000000D 04 I
 P(10) = -0.2276992000000000D 04 +  0.3046320000000000D 04 I
 P(11) = -0.1241472000000000D 04 + -0.4097280000000000D 04 I
 P(12) =  0.5402800000000001D 04 +  0.1263488000000000D 04 I
 P(13) = -0.6468336000000001D 04 +  0.1236480000000000D 04 I
 P(14) =  0.1467456000000000D 04 +  0.2272000000000000D 02 I
 P(15) = -0.1077120000000000D 03 + -0.5475840000000000D 03 I
 P(16) =  0.3456000000000000D 02 +  0.1267200000000000D 03 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.              0.100-09
TEST FOR MULTIPLICITIES.           0.100-01
RADIUS TO START SEARCH.            0.000 00
RADIUS TO END SEARCH.              0.000 00


BEFORE ATTEMPT TO IMPROVE ACCURACY


ROOTS OF P(X)                                                    MULTIPLICITIES                INITIAL APPROXIMATION

 ROOT( 1) =  0.3000000000000000D 00 +  0.3383998751751866D-16 I        1          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.2000000000000000D 00 + -0.2000000000000000 00 I         1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.9999999999999995D 00 +  0.9999999999999997D 00 I        1          0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) = -0.2389082408313382D-16 +  0.1000000000000000D 01 I        1         -0.5176382551966724D 00 +  0.1931851608368755D 01 I
 ROOT( 5) = -0.1000000000000000 01 +  0.2531629277617640D-15 I         1         -0.1767767147080701D 01 +  0.1767766758852015D 01 I
 ROOT( 6) = -0.3333333333333334D 01 + -0.2144722669087737D-15 I        1         -0.2897777583074990D 01 +  0.7764567463987070D 00 I
 ROOT( 7) = -0.1000000000000008D 01 + -0.9999999999999661D 00 I        1         -0.3380740248331229D 01 + -0.9058671940816160D 00 I
 ROOT( 8) = -0.2000000000000001D 01 + -0.3000000000000000D 01 I        1         -0.2828426607107896D 01 + -0.2828427642384390D 01 I
 ROOT( 9) = -0.1138816730936650D-12 + -0.1000000000000109D 01 I        1         -0.1164684801399899D 01 + -0.4346666459873368D 01 I
 ROOT(10) =  0.1999999999999879D 01 + -0.9999999999999954D 00 I        1          0.1294096345098645D 01 + -0.4829628831453027D 01 I
 ROOT(11) =  0.2999999999999946D 01 +  0.1921591204439124D-14 I        1          0.3889088292979509D 01 + -0.3889086300072258D 01 I
 ROOT(12) =  0.2000000000000122D 01 + -0.2254154879580170D-12 I        1          0.5795555393512303D 01 + -0.1552912644268974D 01 I
 ROOT(13) =  0.3999999999999999D 01 +  0.4000000000000005D 01 I        1          0.6278517357734125D 01 +  0.1682325708247752D 01 I
 ROOT(14) =  0.1000000000000208D 01 +  0.2297444641937337D-12 I        1          SOLVED BY DIRECT METHOD
 ROOT(15) = -0.2000000000000366D 00 +  0.2000000000000595D 00 I        1          SOLVED BY DIRECT METHOD
```

Exhibit 6.2.

AFTER THE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)	MULTIPLICITIES	INITIAL APPROXIMATION
ROOT(1) = 0.30000000000000000D 00 + 0.33767394620832440D-16 I	1	0.48296291156562790 00 + 0.12940952844381870D 00 I
ROOT(2) = 0.20000000000000000D 00 + -0.20000000000000000D 00 I	1	0.70710675530463460 00 + 0.70710680706845950 00 I
ROOT(3) = 0.99999999999999950D 00 + 0.10000000000000000D 01 I	1	0.38822847926540560 00 + 0.14488887631171930 01 I
ROOT(4) = 0.11537757751536690D-16 + 0.10000000000000000D 01 I	1	-0.51763825519667240 00 + 0.19318516083667550 01 I
ROOT(5) = -0.10000000000000000D 01 + -0.99356260300168480D-17 I	1	-0.17677671470807010 01 + 0.17677667588570150 01 I
ROOT(6) = -0.33333333333333340D 01 + -0.19191253971536640D-15 I	1	-0.28977775830749900 01 + 0.77645674639870700 00 I
ROOT(7) = -0.10000000000000000D 01 + -0.99999999999999980 00 I	1	-0.33807407483717290 01 + -0.90586719408161600 00 I
ROOT(8) = -0.20000000000000000D 01 + -0.30000000000000000D 01 I	1	-0.28284266071078960 01 + -0.28284276423843900 01 I
ROOT(9) = 0.55291352338277240D-16 + -0.10000000000000000D 01 I	1	-0.11646848013998990 01 + -0.43466664598733680 01 I
ROOT(10) = 0.20000000000000020 01 + -0.99999999999999860 00 I	1	0.12940963450986450 01 + -0.48296288314530270 01 I
ROOT(11) = 0.29999999999999960 01 + 0.24394773729111440D-14 I	1	0.38890882929795090 01 + -0.38890863000722580 01 I
ROOT(12) = 0.20000000000000020 01 + -0.41020504900166570-14 I	1	0.57955553935123030 01 + -0.15529126442689740 01 I
ROOT(13) = 0.40000000000000000D 01 + 0.40000000000000010 01 I	1	0.62785173577341250 01 + 0.16823257082477520 01 I
ROOT(14) = 0.10000000000000000D 01 + 0.76824382838570700-16 I	1	SOLVED BY DIRECT METHOD
ROOT(15) = -0.20000000000000000D 00 + 0.20000000000000000D 00 I	1	SOLVED BY DIRECT METHOD

Exhibit 6.2. Roots Are: -1 - i, 1 + i, -2 - 3i, 2 - i, 3, 2, i, -i, -10/3, .3, -1, 1, 4 + 4i, -.2 +.2i, .2 - .2i.

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  1 OF DEGREE  6


THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P( 2) = -0.70000000000000001D 01 + -0.10500000000000000D 02 I
 P( 3) = -0.28000000000000000D 02 +  0.58000000000000001D 02 I
 P( 4) =  0.17100000000000000D 03 +  0.15000000000000000D 01 I
 P( 5) = -0.73000000000000000D 02 + -0.25100000000000000D 03 I
 P( 6) = -0.22800000000000000D 03 +  0.10400000000000000D 03 I
 P( 7) =  0.72000000000000001D 02 +  0.10400000000000000D 03 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                    MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) =  0.99999988361250190D 00 +  0.20000000521382840D 01 I      2        0.48296291156562790D 00 +  0.12940952844381870D 00 I
  ROOT( 2) =  0.19967378102574860D 01 +  0.19952538211436840D 01 I      3        0.70710675530463460D 00 +  0.70710680706845950D 00 I
  ROOT( 3) = -0.99021319799746240D 00 +  0.51423843229238120D 00 I      1        SOLVED BY DIRECT METHOD

IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT( 2) =  0.19967378102574860D 01 +  0.19952538211436840D 01 I DID NOT CONVERGE.
THE PRESENT APPROXIMATION AFTER 200 ITERATIONS IS PRINTED BELOW.


AFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                    MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) =  0.99999988361250190D 00 +  0.20000000521382840D 01 I      2        0.48296291156562790D 00 +  0.12940952844381870D 00 I
  ROOT( 2) =  0.19999929075033090D 01 +  0.19999594740016890D 01 I      3        0.70710675530463460D 00 +  0.70710680706845950D 00 I
  ROOT( 3) = -0.99999999999999998D 00 +  0.50000000000000000D 00 I      1        SOLVED BY DIRECT METHOD
```

Exhibit 6.3. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  2 OF DEGREE 15


THE COEFFICIENTS OF P(X) ARE

   P( 1) =  0.4800000000000000D 02 +  0.0000000000000000D 00 I
   P( 2) =  0.2557120000000000D 03 + -0.3840000000000000D 03 I
   P( 3) = -0.7353556800000000D 02 + -0.2189696000000000D 04 I
   P( 4) = -0.3855565696000000D 04 + -0.6946851456000001D 04 I
   P( 5) = -0.1733386464800000D 05 + -0.1420625972800000D 05 I
   P( 6) = -0.4967989270400001D 05 + -0.1765857464000000D 05 I
   P( 7) = -0.1022394522130000D 06 + -0.6030664232000001D 04 I
   P( 8) = -0.1642742200560000D 06 +  0.4137366230400000D 05 I
   P( 9) = -0.2036625888420000D 06 +  0.1093899227670000D 06 I
   P(10) = -0.1871255780010000D 06 +  0.1929865440330000D 06 I
   P(11) = -0.1274997298590000D 06 +  0.2171341227420000D 06 I
   P(12) = -0.2814692716800000D 05 +  0.1928489727960000D 06 I
   P(13) =  0.1329434434800000D 05 +  0.1038130226550000D 06 I
   P(14) =  0.3053900774700000D 05 +  0.2998989141300000D 05 I
   P(15) = -0.1835899020000000D 03 + -0.1827632160000000D 03 I
   P(16) =  0.2755620000000000D 00 +  0.2755620000000000D 00 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.               0.10D-05
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                          MULTIPLICITIES                 INITIAL APPROXIMATION

  ROOT( 1) =  0.3000000736398184D-02 +  0.2609394906552981D-08 I           2          0.4829629115656279D 00 +  0.1294095284438187D 00 I
  ROOT( 2) =  0.8626623124099139D-04 +  0.1000080173722988D 01 I           2          0.7071067559304634D 00 +  0.7071068070684595D 00 I
  ROOT( 3) =  0.5768342464927569D-02 +  0.1508145174316570D 01 I           1          0.3882284792654056D 00 +  0.1448868763117193D 01 I
  ROOT( 4) = -0.5926634471058762D-32 +  0.1491722999922961D 01 I           1         -0.5176382551966724D 00 +  0.1931851608368755D 01 I
  ROOT( 5) = -0.1014904754401601D 01 + -0.1028420931384370D 01 I           1         -0.1767767147080701D 01 +  0.1767766758852015D 01 I
  ROOT( 6) = -0.2333333617834863D 01 +  0.2168624494945200D-05 I           1         -0.2897777583074990D 01 +  0.7764567463987070D 00 I
  ROOT( 7) = -0.1018508435678558D 01 + -0.9728595602756189D 00 I           1         -0.3380740248331229D 01 + -0.9058671940816160D 00 I
  ROOT( 8) = -0.9665441989282835D 00 + -0.9987774789527914D 00 I           1         -0.2828426607107896D 01 + -0.2828427642384390D 01 I
  ROOT( 9) = -0.2481483709937032D-02 + -0.1497879862710794D 01 I           2         -0.1164684801399899D 01 + -0.4346666459873368D 01 I
  ROOT(10) =  0.1560514417537318D 00 +  0.2495207685212091D 01 I           1          0.1294096345098645D 01 + -0.4829628831453027D 01 I
  ROOT(11) =  0.3325675653760710D 00 +  0.3365993627402638D 01 I           1          SOLVED BY DIRECT METHOD
  ROOT(12) = -0.4837126081291009D 00 +  0.3134585687890842D 01 I           1          SOLVED BY DIRECT METHOD


AFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE
```

Exhibit 6.4.

```
ROOTS OF P(X)                                                MULTIPLICITIES         INITIAL APPROXIMATION

ROOT( 1) =  0.30000003683722620-02 +  0.13040676163986510-08 I       2         0.48296291156562790 00 +  0.12940952844381870 00 I
ROOT( 2) =  0.67420350824634080-06 +  0.10000006276927750 01 I       2         0.70710675530463460 00 +  0.70710680706845950 00 I
ROOT( 3) =  0.76247249328106050-06 +  0.15000010300088330 01 I       1         0.38822847926540560 00 +  0.14488887631171930 01 I
ROOT( 4) = -0.66718756627305410-06 +  0.14999990111650930 01 I       1        -0.51763825519667240 00 +  0.19318516083687550 01 I
ROOT( 5) = -0.99999601110180270 00 + -0.10000074938436410 01 I       1        -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) = -0.23333333333117100 01 +  0.64990590582774950-11 I       1        -0.28977775830749900 01 +  0.77645674639870700 00 I
ROOT( 7) = -0.10000076929704720 01 + -0.99999836986193700 00 I       1        -0.33807402483312290 01 + -0.90586719408161600 00 I
ROOT( 8) = -0.99999517401584100 00 + -0.99999502636776190 00 I       1        -0.28284266071078960 01 + -0.28284276423843900 01 I
ROOT( 9) = -0.59248785305784770-06 + -0.14999994871877380 01 I       2        -0.11646848013998990 01 + -0.43466664598733680 01 I
ROOT(10) = -0.60936544515891930-06 +  0.15000010898249810 01 I       1         0.12940963450986450 01 + -0.48296288314530270 01 I
ROOT(11) =  0.26195991030834680-04 +  0.30000078386717850 01 I       1         SOLVED BY DIRECT METHOD
ROOT(12) = -0.98192911809282650-05 +  0.29999741215895750 01 I       1         SOLVED BY DIRECT METHOD
```

Exhibit 6.4. Roots Are: -2.33, .003 (2), i (2), 1.5i (2), -1.5i (2) 3i (3), -1-i (3)

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  3 OF DEGREE  8


THE COEFFICIENTS OF P(X) ARE

  P( 1) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
  P( 2) = -0.5000000000000001D 01 + -0.1150000000000000D 02 I
  P( 3) = -0.5175000000000001D 02 +  0.4300000000000001D 02 I
  P( 4) =  0.1572500000000000D 03 +  0.1446250000000000D 03 I
  P( 5) =  0.3075000000000000D 03 + -0.3475000000000000D 03 I
  P( 6) = -0.4952500000000000D 03 + -0.4948750000000000D 03 I
  P( 7) = -0.5857500000000001D 03 +  0.4247500000000001D 03 I
  P( 8) =  0.1810000000000000D 03 +  0.4420000000000001D 03 I
  P( 9) =  0.1580000000000000D 03 +  0.6000000000000001D 01 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.               0.10D-05
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                       MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) =  0.9999992498044742D 00 +  0.1999998467904090D 01 I          2         0.4829629115656279D 00 +  0.1294095284438187D 00 I
  ROOT( 2) = -0.9921687016776328D 00 +  0.5027396405227564D 00 I          2         0.7071067553046346D 00 +  0.7071068070684595D 00 I
  ROOT( 3) =  0.1916689684272422D 01 +  0.1966959787044516D 01 I          1         0.3882284792654056D 00 +  0.1448888763117193D 01 I
  ROOT( 4) =  0.2013726806167612D 01 +  0.2086055012105922D 01 I          1        -0.5176382551966724D 00 +  0.1931851608368755D 01 I
  ROOT( 5) =  0.2069406641003821D 01 +  0.1946950095885711D 01 I          1          SOLVED BY DIRECT METHOD
  ROOT( 6) = -0.1015484227697537D 01 +  0.4945588881101592D 00 I          1          SOLVED BY DIRECT METHOD


AFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                       MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) =  0.9999996239397971D 00 +  0.1999999233348969D 01 I          2         0.4829629115656279D 00 +  0.1294095284438187D 00 I
  ROOT( 2) = -0.9999926539148827D 00 +  0.4999975557955272D 00 I          2         0.7071067553046346D 00 +  0.7071068070684595D 00 I
  ROOT( 3) =  0.1999956221578001D 01 +  0.2000020836036100D 01 I          1         0.3882284792654056D 00 +  0.1448888763117193D 01 I
  ROOT( 4) =  0.2000005829322798D 01 +  0.1999952499501282D 01 I          1        -0.5176382551966724D 00 +  0.1931851608368755D 01 I
  ROOT( 5) =  0.2000008738509706D 01 +  0.1999950053877640D 01 I          1          SOLVED BY DIRECT METHOD
  ROOT( 6) = -0.1000005373981587D 01 +  0.4999950920121687D 00 I          1          SOLVED BY DIRECT METHOD
```

Exhibit 6.5. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  4 OF DEGREE 12


THE COEFFICIENTS OF P(X) ARE

  P( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
  P( 2) = -0.12000000000000000D 02 + -0.00000000000000000D 00 I
  P( 3) =  0.72000000000000001D 02 +  0.00000000000000000D 00 I
  P( 4) = -0.28000000000000000D 03 + -0.00000000000000000D 00 I
  P( 5) =  0.78000000000000000D 03 +  0.00000000000000000D 00 I
  P( 6) = -0.16320000000000000D 04 + -0.00000000000000000D 00 I
  P( 7) =  0.26240000000000000D 04 +  0.00000000000000000D 00 I
  P( 8) = -0.32640000000000000D 04 + -0.00000000000000000D 00 I
  P( 9) =  0.31200000000000000D 04 +  0.00000000000000000D 00 I
  P(10) = -0.22400000000000000D 04 + -0.00000000000000000D 00 I
  P(11) =  0.11520000000000000D 04 +  0.00000000000000000D 00 I
  P(12) = -0.38400000000000000D 03 + -0.00000000000000000D 00 I
  P(13) =  0.64000000000000001D 02 +  0.00000000000000000D 00 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR CONVERGENCE.             0.10D-03
TEST FOR MULTIPLICITIES.          0.10D-01
RADIUS TO START SEARCH.           0.00D 00
RADIUS TO END SEARCH.             0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                      MULTIPLICITIES              INITIAL APPROXIMATION


  ROOT( 1) =  0.99940798033691520 00 +  0.99513914896439410 00 I        5         0.48296291156562790 00 +  0.12940952844381870 00 I
  ROOT( 2) =  0.10200968856097200 01 + -0.54206461303878890 00 I        1         0.70710675530463460 00 +  0.70710680706845950 00 I
  ROOT( 3) =  0.10027211066816580 01 +  0.10233106615266340 01 I        1         0.38822847926540560 00 +  0.14488887631171930 01 I
  ROOT( 4) =  0.61899689593614000 00 + -0.79497020137361440 00 I        1        -0.51763825519667240 00 +  0.19318516083687550 01 I
  ROOT( 5) =  0.98514220474234140 00 + -0.13956979981330380 01 I        1        -0.17677671470807010 01 +  0.17677667588520150 01 I
  ROOT( 6) =  0.65158282151654210 00 + -0.12069876664549360 01 I        1        -0.28977775630749900 01 +  0.77645674639870700 00 I
  ROOT( 7) =  0.13937999844005080 01 + -0.82719243111768510 00 I        1         SOLVED BY DIRECT METHOD
  ROOT( 8) =  0.13306201994285140 01 + -0.12320934962305410 01 I        1         SOLVED BY DIRECT METHOD

IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT( 2) =  0.10200968856097200 01 + -0.54206461303878890 00 I DID NOT CONVERGE.
THE PRESENT APPROXIMATION AFTER 200 ITERATIONS IS PRINTED BELOW.

IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT( 5) =  0.98514220474234140 00 + -0.13956979981330380 01 I DID NOT CONVERGE.
THE PRESENT APPROXIMATION AFTER 200 ITERATIONS IS PRINTED BELOW.


AFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE
```

Exhibit 6.6.

ROOTS OF P(X)	MULTIPLICITIES	INITIAL APPROXIMATION
ROOT(1) = 0.99948113795146260 00 + 0.99560775860617460 00 I	5	0.48296291156562790 00 + 0.12940952844381870 00 I
ROOT(2) = 0.99592313317176770 00 + -0.10017819876754680 01 I	1	0.70710675530463460 00 + 0.70710680706845950 00 I
ROOT(3) = 0.99585780717884340 00 + 0.10031671344932110 01 I	1	0.38822847926540560 00 + 0.14488887631171930 01 I
ROOT(4) = 0.99991569963089460 00 + -0.99503023164852320 00 I	1	-0.51763825519667240 00 + 0.19318516083687550 01 I
ROOT(5) = 0.99391042189066210 00 + -0.99411359917024930 00 I	1	-0.17677671470807010 01 + 0.17677667588520150 01 I
ROOT(6) = 0.99605385885658690 00 + -0.10027599236645720 01 I	1	-0.28977775830749900 01 + 0.77645674639870700 00 I
ROOT(7) = 0.10037198385540710 01 + -0.99684656745825120 00 I	1	SOLVED BY DIRECT METHOD
ROOT(8) = 0.10001682256999940 01 + -0.99537592890632930 00 I	1	SOLVED BY DIRECT METHOD

Exhibit 6.6. Roots Are: 1+i(6), 1-i(6)

```
MULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOMIAL
POLYNOMIAL NUMBER  1 OF DEGREE  6

THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P( 2) = -0.70000000000000010D 01 + -0.10500000000000000D 02 I
 P( 3) = -0.28000000000000000D 02 +  0.58000000000000001D 02 I
 P( 4) =  0.17100000000000000D 03 +  0.15000000000000000D 01 I
 P( 5) = -0.73000000000000000D 02 + -0.25100000000000000D 03 I
 P( 6) = -0.22800000000000000D 03 +  0.10400000000000000D 03 I
 P( 7) =  0.72000000000000001D 02 +  0.10400000000000000D 03 I

NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00

BEFORE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)                                                      MULTIPLICITIES            INITIAL APPROXIMATION

 ROOT( 1) =  0.1999954888749810D 01 +  0.2000017733363912D 01 I         3          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1013502627672869D 01 +  0.1996163083160882D 01 I         1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.9866348122507864D 00 +  0.2003806882325521D 01 I         1          SOLVED BY DIRECT METHOD
 ROOT( 4) = -0.1000002106173084D 01 +  0.4999768344218605D 00 I         1          SOLVED BY DIRECT METHOD

AFTER THE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)                                                      MULTIPLICITIES            INITIAL APPROXIMATION

 ROOT( 1) =  0.1999954835419787D 01 +  0.2000017627898503D 01 I         3          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1000000159779278D 01 +  0.1999999979600045D 01 I         1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.9999998306711213D 00 +  0.2000000071685089D 01 I         1          SOLVED BY DIRECT METHOD
 ROOT( 4) = -0.9999999999999998D 00 +  0.5000000000000000D 00 I         1          SOLVED BY DIRECT METHOD
```

Exhibit 6.7. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
MULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOMIAL
POLYNOMIAL NUMBER  2 OF DEGREE 15

THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.4800000000000000D 02 +  0.0000000000000000D 00 I
 P( 2) =  0.2557120000000000D 03 + -0.3840000000000000D 03 I
 P( 3) = -0.7353556800000000D 02 + -0.2189696000000000D 04 I
 P( 4) = -0.3855565696000000D 04 + -0.6946851456000001D 04 I
 P( 5) = -0.1733386464800000D 05 + -0.1420625972800000D 05 I
 P( 6) = -0.4967989270400001D 05 + -0.1765857464000000D 05 I
 P( 7) = -0.1022394522130000D 06 + -0.6030664232000001D 04 I
 P( 8) = -0.1642742200560000D 06 +  0.4137366230400000D 05 I
 P( 9) = -0.2036625888420000D 06 +  0.1093899227670000D 06 I
 P(10) = -0.1871255780010000D 06 +  0.1929865440330000D 06 I
 P(11) = -0.1274997298590000D 06 +  0.2171341227420000D 06 I
 P(12) = -0.2814692716800000D 05 +  0.1928489727960000D 06 I
 P(13) =  0.1329434434800000D 05 +  0.1038130226550000D 06 I
 P(14) =  0.3053900774700000D 05 +  0.2998989141300000D 05 I
 P(15) = -0.1835899020000000D 03 + -0.1827632160000000D 03 I
 P(16) =  0.2755620000000000D 00 +  0.2755620000000000D 00 I

NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.              0.100-09
TEST FOR MULTIPLICITIES.           0.10D-01
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00

BEFORE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)                                                        MULTIPLICITIES               INITIAL APPROXIMATION

 ROOT( 1) =  0.3000000056259372D-02 +  0.8277931932119355D-12 I         2          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1686472961057889D-04 +  0.1000001835704009D 01 I         1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.1280227161896996D-04 +  0.1499998729963919D 01 I         1          0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) =  0.6019678782146761D-04 +  0.2999936661658190D 01 I         1         -0.5176382551966724D 00 +  0.1931851608368755D 01 I
 ROOT( 5) = -0.1686485393117943D-04 +  0.9999981635776160D 00 I         1         -0.1767767147080701D 01 +  0.1767766758852015D 01 I
 ROOT( 6) = -0.2333333333333649D 01 +  0.3249409455271494D-12 I         1         -0.2897777583074990D 01 +  0.7764567463987070D 00 I
 ROOT( 7) = -0.1000163769022359D 01 + -0.1000173771459914D 01 I         2         -0.3380740248331229D 01 + -0.9058671940816160D 00 I
 ROOT( 8) = -0.3334905054308976D-03 + -0.1500133373398480D 01 I         1         -0.2828426607107896D 01 + -0.2828427642384390D 01 I
 ROOT( 9) = -0.9996724783532904D 00 + -0.9996521399325563D 00 I         1         -0.1164684801399899D 01 + -0.4346666459873368D 01 I
 ROOT(10) =  0.3334811914008498D-03 + -0.1499866916248616D 01 I         1          0.1294096345098645D 01 + -0.4829628831453027D 01 I
 ROOT(11) = -0.1715220973482156D-03 +  0.2999990216209916D 01 I         1          0.3889088292979509D 01 + -0.3889086300072258D 01 I
 ROOT(12) =  0.1113136898246577D-03 +  0.3000073149878609D 01 I         1          SOLVED BY DIRECT METHOD
 ROOT(13) = -0.1276492775835444D-04 +  0.1500001215505238D 01 I         1          SOLVED BY DIRECT METHOD

AFTER THE ATTEMPT TO IMPROVE ACCURACY
```

Exhibit 6.8.

```
ROOTS OF P(X)                                                    MULTIPLICITIES              INITIAL APPROXIMATION

ROOT( 1) =  0.30000000562581190-02 +  0.82793704892536880-12 I        2          0.48296291156562790 00 +  0.12940952844381870 00 I
ROOT( 2) =  0.30010861721562150-07 +  0.10000000438390740 01 I        1          0.70710675530463460 00 +  0.70710680706845950 00 I
ROOT( 3) =  0.61890537903191320-07 +  0.15000001407680000 01 I        1          0.38822847926540560 00 +  0.14488887631171930 01 I
ROOT( 4) =  0.23973274100738230-04 +  0.30000087372252940 01 I        1         -0.51763825519667240 00 +  0.19318516083687550 01 I
ROOT( 5) =  0.39739845404284450-07 +  0.10000000424546840 01 I        1         -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) = -0.23333333333333340 01 + -0.30279561954291830-15 I        1         -0.28977775830749900 01 +  0.77645674639870700 00 I
ROOT( 7) = -0.10000069904049780 01 + -0.99999891224724300 00 I        2         -0.33807402483312290 01 + -0.90586719408161600 00 I
ROOT( 8) =  0.92844225443363060-08 + -0.15000000131286020 01 I        1         -0.28284266071078960 01 + -0.28284276423843900 01 I
ROOT( 9) = -0.10000062510065850 01 + -0.10000015374992320 01 I        1         -0.11646848013998990 01 + -0.43466664598733680 01 I
ROOT(10) =  0.16717142793063800-07 + -0.15000000185263290 01 I        1          0.12940963450986450 01 + -0.48296288314530270 01 I
ROOT(11) = -0.25587686230373010-04 +  0.30000216537917870 01 I        1          0.38890882929795090 01 + -0.38890863000722580 01 I
ROOT(12) =  0.21109261089828320-04 +  0.30000149073606530 01 I        1          SOLVED BY DIRECT METHOD
ROOT(13) = -0.54537993846922740-07 +  0.14999998637957010 01 I        1          SOLVED BY DIRECT METHOD
```

Exhibit 6.8. Roots Are: -2.33, .003 (2), i(2), 1.5i (2),
-1.5i (2) 3i (3), -1-i(3)

```
MULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOMIAL
POLYNOMIAL NUMBER  3 OF DEGREE  8

THE COEFFICIENTS OF P(X) ARE

 P( 1) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 P( 2) = -0.5000000000000001D 01 + -0.1150000000000000D 02 I
 P( 3) = -0.5175000000000001D 02 +  0.4300000000000001D 02 I
 P( 4) =  0.1572500000000000D 03 +  0.1446250000000000D 03 I
 P( 5) =  0.3075000000000000D 03 + -0.3475000000000000D 03 I
 P( 6) = -0.4952500000000000D 03 + -0.4948750000000000D 03 I
 P( 7) = -0.5857500000000001D 03 +  0.4247500000000001D 03 I
 P( 8) =  0.1810000000000000D 03 +  0.4420000000000001D 03 I
 P( 9) =  0.1580000000000000D 03 +  0.6000000000000001D 01 I

NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.             0.10D-09
TEST FOR MULTIPLICITIES.          0.10D-01
RADIUS TO START SEARCH.           0.00D 00
RADIUS TO END SEARCH.             0.00D 00

BEFORE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)                                                       MULTIPLICITIES                 INITIAL APPROXIMATION

 ROOT( 1) =  0.2000042202873018D 01 +  0.2000028743998294D 01 I         3            0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1000563806908988D 01 +  0.1980638952568216D 01 I         1            0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) = -0.1063528418749844D 01 +  0.4942372379199120D 00 I         1            0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) = -0.9722122516869824D 00 +  0.5603797457973125D 00 I         1           -0.5176382551966724D 00 +  0.1931851608368755D 01 I
 ROOT( 5) =  0.9994370680506410D 00 +  0.2019094286928692D 01 I         1            SOLVED BY DIRECT METHOD
 ROOT( 6) = -0.9643868131418565D 00 +  0.4455635447909865D 00 I         1            SOLVED BY DIRECT METHOD

AFTER THE ATTEMPT TO IMPROVE ACCURACY

ROOTS OF P(X)                                                       MULTIPLICITIES                 INITIAL APPROXIMATION

 ROOT( 1) =  0.2000042175889591D 01 +  0.2000028750713895D 01 I         3            0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1000000023165255D 01 +  0.1999999805777700D 01 I         1            0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) = -0.1000006594122314D 01 +  0.4999952821126052D 00 I         1            0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) = -0.1000000972571865D 01 +  0.5000084175726651D 00 I         1           -0.5176382551966724D 00 +  0.1931851608368755D 01 I
 ROOT( 5) =  0.9999999168409400D 00 +  0.2000000100132659D 01 I         1            SOLVED BY DIRECT METHOD
 ROOT( 6) = -0.9999925738129557D 00 +  0.4999962818337703D 00 I         1            SOLVED BY DIRECT METHOD
```

Exhibit 6.9. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
MULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOMIAL
POLYNOMIAL NUMBER  4 OF DEGREE 12


THE COEFFICIENTS OF P(X) ARE


 P( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P( 2) = -0.12000000000000000D 02 + -0.00000000000000000D 00 I
 P( 3) =  0.72000000000000001D 02 +  0.00000000000000000D 00 I
 P( 4) = -0.28000000000000000D 03 + -0.00000000000000000D 00 I
 P( 5) =  0.78000000000000000D 03 +  0.00000000000000000D 00 I
 P( 6) = -0.16320000000000000D 04 + -0.00000000000000000D 00 I
 P( 7) =  0.26240000000000000D 04 +  0.00000000000000000D 00 I
 P( 8) = -0.32640000000000000D 04 + -0.00000000000000000D 00 I
 P( 9) =  0.31200000000000000D 04 +  0.00000000000000000D 00 I
 P(10) = -0.22400000000000000D 04 + -0.00000000000000000D 00 I
 P(11) =  0.11520000000000000D 04 +  0.00000000000000000D 00 I
 P(12) = -0.38400000000000000D 03 + -0.00000000000000000D 00 I
 P(13) =  0.64000000000000001D 02 +  0.00000000000000000D 00 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR CONVERGENCE.              0.10D-09
TEST FOR MULTIPLICITIES.           0.10D-01
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00


BEFORE ATTEMPT TO IMPROVE ACCURACY


ROOTS OF P(X)                                                   MULTIPLICITIES              INITIAL APPROXIMATION


 ROOT( 1) =  0.1004468192948739D 01 +  0.1002186117532751D 01 I       4          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.9858978428495002D 00 +  0.1006804518996020D 01 I       1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.9962262248473947D 00 +  0.9844382357895168D 00 I       1          0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) =  0.1007069490520364D 01 + -0.8278426247752856D 00 I       2         -0.5176382551966724D 00 +  0.1931851608368755D 01 I
 ROOT( 5) =  0.7584518107423498D 00 + -0.9410320205052365D 00 I       1         -0.1767767147080701D 01 +  0.1767766758852015D 01 I
 ROOT( 6) =  0.8439013544164170D 00 + -0.1215158277290118D 01 I       1         -0.2897777583074990D 01 +  0.7764567463987070D 00 I
 ROOT( 7) =  0.1245479586151551D 01 + -0.9609731096770448D 00 I       1          SOLVED BY DIRECT METHOD
 ROOT( 8) =  0.1138031428157103D 01 + -0.1227138567893571D 01 I       1          SOLVED BY DIRECT METHOD


AFTER THE ATTEMPT TO IMPROVE ACCURACY


ROOTS OF P(X)                                                   MULTIPLICITIES              INITIAL APPROXIMATION


 ROOT( 1) =  0.1003576740125457D 01 +  0.1001742634790606D 01 I       4          0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.9990524788681438D 00 +  0.1004512114384271D 01 I       1          0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.1003596793567675D 01 +  0.9965600317041306D 00 I       1          0.3882284792654056D 00 +  0.1448888763117193D 01 I
```

Exhibit 6.10.

```
ROOT( 4) =  0.10007324697602040 01 + -0.99564295721310670 00 I        2        -0.51763825519667240 00 +  0.19318516083687550 01 I
ROOT( 5) =  0.10047654833534890 01 + -0.10006957183589740 01 I        1        -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) =  0.99947825662743310 00 + -0.99607660789158700 00 I        1        -0.28977775830749900 01 +  0.77645674639870700 00 I
ROOT( 7) =  0.99541663095228990 00 + -0.10001898299218090 01 I        1         SOLVED BY DIRECT METHOD
ROOT( 8) =  0.10000745905941600 01 + -0.99512853288497680 00 I        1         SOLVED BY DIRECT METHOD
```

Exhibit 6.10. Roots Are: 1+i (6), 1-i (6)

```
GREATEST COMMON DIVISOR METHOD USED WITH NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  1


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS  6  THE COEFFICIENTS ARE

 P(7 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(6 ) = -0.70000000000000001D 01 + -0.10500000000000000D 02 I
 P(5 ) = -0.28000000000000000D 02 +  0.58000000000000001D 02 I
 P(4 ) =  0.17100000000000000D 03 +  0.15000000000000000D 01 I
 P(3 ) = -0.73000000000000000D 02 + -0.25100000000000000D 03 I
 P(2 ) = -0.22800000000000000D 03 +  0.10400000000000000D 03 I
 P(1 ) =  0.72000000000000001D 02 +  0.10400000000000000D 03 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  3  THE COEFFICIENTS ARE

 Q(4 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 Q(3 ) = -0.20000000000000150D 01 + -0.45000000000000118D 01 I
 Q(2 ) = -0.70000000000000423D 01 +  0.35000000000000060D 01 I
 Q(1 ) =  0.99999999999997762D 00 +  0.69999999999999812D 01 I


ROOTS OF Q(X)                                                                                     INITIAL APPROXIMATION

 ROOT( 1) =  0.9999999999997565D 00 +  0.1999999999999574D 01 I                 0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.2000000000003347D 01 +  0.2000000000000529D 01 I                   RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.9999999999999539D 00 +  0.5000000000001490D 00 I                   RESULTS OF SUBROUTINE QUAD


ROOTS OF P(X)                                                     MULTIPLICITIES                 INITIAL APPROXIMATION

 ROOT( 1) =  0.9999999999997565D 00 +  0.1999999999999574D 01 I         2        0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.2000000000003470D 01 +  0.2000000000000530D 01 I         3          RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.9999999999999541D 00 +  0.5000000000001470D 00 I         1          RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.11. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
GREATEST COMMON DIVISOR METHOD USED WITH NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  2


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS 15  THE COEFFICIENTS ARE

 P(16) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 P(15) =  0.25571200000000000D 03 + -0.38400000000000000D 03 I
 P(14) = -0.73535568000000000D 02 + -0.21896960000000000D 04 I
 P(13) = -0.38555656960000000D 04 + -0.69468514560000010D 04 I
 P(12) = -0.17333864648000000D 05 + -0.14206259728000000D 05 I
 P(11) = -0.49679892704000001D 05 + -0.17658574640000000D 05 I
 P(10) = -0.10223945221300000D 06 + -0.60306642320000001D 04 I
 P(9 ) = -0.16427422005600000D 06 +  0.41373662304000000D 05 I
 P(8 ) = -0.20366258884200000D 06 +  0.10938992276700000D 06 I
 P(7 ) = -0.18712557800100000D 06 +  0.19298654403300000D 06 I
 P(6 ) = -0.12749972985900000D 06 +  0.21713412274200000D 06 I
 P(5 ) = -0.28146927168000000D 05 +  0.19284897279600000D 06 I
 P(4 ) =  0.13294344348000000D 05 +  0.10381302265500000D 06 I
 P(3 ) =  0.30539007747000000D 05 +  0.29989891413000000D 05 I
 P(2 ) = -0.18358990200000000D 03 + -0.18276321600000000D 03 I
 P(1 ) =  0.27556200000000000D 00 +  0.27556200000000000D 00 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  7  THE COEFFICIENTS ARE

 Q(8 ) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 Q(7 ) =  0.15985599999590830D 03 + -0.14400000000184630D 03 I
 Q(6 ) =  0.26751999998662870D 03 + -0.52756800000682380D 03 I
 Q(5 ) =  0.32719599997819040D 03 + -0.91441600000759420D 03 I
 Q(4 ) =  0.23015999960445530D 02 + -0.15212520000037680D 04 I
 Q(3 ) = -0.72072000047996880D 02 + -0.13274279999966590D 04 I
 Q(2 ) = -0.75578400005415630D 03 + -0.75200400003332210D 03 I
 Q(1 ) =  0.22680001335502310D 01 +  0.22679999258286330D 01 I


ROOTS OF Q(X)                                                                        INITIAL APPROXIMATION

 ROOT( 1) =  0.3000000038645997D-02 + -0.1370543176638413D-09 I         0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.1466635543509054D-09 +  0.1000000000302556D 01 I         0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) = -0.1626206088095917D-09 +  0.1499999999807067D 01 I         0.3882284792654056D 00 +  0.1448888763117193D 01 I
 ROOT( 4) = -0.1167767202194363D-10 + -0.1500000000006763D 01 I        -0.5176382551966724D 00 +  0.1931851608368755D 01 I
```

Exhibit 6.12.

```
ROOT( 5) = -0.23333333333340885D 01 +  0.42923013959255540-11 I          -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) =  0.11404521771396500-09 +  0.30000000000539440 01 I            RESULTS OF SUBROUTINE QUAD
ROOT( 7) = -0.10000000000322600 01 + -0.99999999998557810 00 I            RESULTS OF SUBROUTINE QUAD


ROOTS OF P(X)                                         MULTIPLICITIES                INITIAL APPROXIMATION

ROOT( 1) =  0.30000000386459970-02 + -0.13705431766384130-09 I     2        0.48296291156562790 00 +  0.12940952844381870 00 I
ROOT( 2) =  0.14666370652311620-09 +  0.10000000003025560 01 I     2        0.70710675530463460 00 +  0.70710680706845950 00 I
ROOT( 3) = -0.16262085454662920-09 +  0.14999999998070660 01 I     2        0.38822847926540560 00 +  0.14488887631171930 01 I
ROOT( 4) = -0.11677692475724520-10 + -0.15000000000067630 01 I     2       -0.51763825519667240 00 +  0.19318516083687550 01 I
ROOT( 5) = -0.23333333333408850 01 +  0.42915512506970990-11 I     1       -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) =  0.11404556756092100-09 +  0.30000000000539440 01 I     3          RESULTS OF SUBROUTINE QUAD
ROOT( 7) = -0.10000000000322600 01 + -0.99999999998557790 00 I     3          RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.12. Roots Are: -2.33, .003 (2), i(2), 1.5i (2), -1.5i (2) 3i (3), -1-i (3)

```
GREATEST COMMON DIVISOR METHOD USED WITH NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  3


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS  8  THE COEFFICIENTS ARE

 P(9 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(8 ) = -0.50000000000000001D 01 + -0.11500000000000000D 02 I
 P(7 ) = -0.51750000000000001D 02 +  0.43000000000000001D 02 I
 P(6 ) =  0.15725000000000000D 03 +  0.14462500000000000D 03 I
 P(5 ) =  0.30750000000000000D 03 + -0.34750000000000000D 03 I
 P(4 ) = -0.49525000000000000D 03 + -0.49487500000000000D 03 I
 P(3 ) = -0.58575000000000001D 03 +  0.42475000000000001D 03 I
 P(2 ) =  0.18100000000000000D 03 +  0.44200000000000001D 03 I
 P(1 ) =  0.15800000000000000D 03 +  0.60000000000000001D 01 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  3  THE COEFFICIENTS ARE

 Q(4 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 Q(3 ) = -0.20000000000002273D 01 + -0.45000000000002275D 01 I
 Q(2 ) = -0.70000000000009350D 01 +  0.35000000000004690D 01 I
 Q(1 ) =  0.99999999999941730D 00 +  0.70000000000004980D 01 I


ROOTS OF Q(X)                                                          INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999995483D 00 +  0.19999999999997755D 01 I      0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.20000000000006650D 01 +  0.20000000000005390D 01 I        RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.99999999999994080D 00 +  0.49999999999998140D 00 I        RESULTS OF SUBROUTINE QUAD


ROOTS OF P(X)                                        MULTIPLICITIES         INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999995483D 00 +  0.19999999999997755D 01 I    2     0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.20000000000006650D 01 +  0.20000000000005390D 01 I    3       RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.99999999999994080D 00 +  0.49999999999998120D 00 I    3       RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.13. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
GREATEST COMMON DIVISOR METHOD USED WITH NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  4


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.   0.100-02
TEST FOR CONVERGENCE.              0.100-09
TEST FOR ZERO IN SUBROUTINE QUAD.  0.100-19
TEST FOR MULTIPLICITIES.           0.100-01
RADIUS TO START SEARCH.            0.000 00
RADIUS TO END SEARCH.              0.000 00


THE DEGREE OF P(X) IS 12  THE COEFFICIENTS ARE

 P(13) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(12) = -0.12000000000000000D 02 + -0.00000000000000000D 00 I
 P(11) =  0.72000000000000001D 02 +  0.00000000000000000D 00 I
 P(10) = -0.28000000000000000D 03 + -0.00000000000000000D 00 I
 P(9 ) =  0.78000000000000000D 03 +  0.00000000000000000D 00 I
 P(8 ) = -0.16320000000000000D 04 + -0.00000000000000000D 00 I
 P(7 ) =  0.26240000000000000D 04 +  0.00000000000000000D 00 I
 P(6 ) = -0.32640000000000000D 04 + -0.00000000000000000D 00 I
 P(5 ) =  0.31200000000000000D 04 +  0.00000000000000000D 00 I
 P(4 ) = -0.22400000000000000D 04 + -0.00000000000000000D 00 I
 P(3 ) =  0.11520000000000000D 04 +  0.00000000000000000D 00 I
 P(2 ) = -0.38400000000000000D 03 + -0.00000000000000000D 00 I
 P(1 ) =  0.64000000000000001D 02 +  0.00000000000000000D 00 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  2  THE COEFFICIENTS ARE

 Q(3 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 Q(2 ) = -0.20000000000000150D 01 + -0.00000000000000000D 00 I
 Q(1 ) =  0.19999999999998830D 01 +  0.00000000000000000D 00 I


ROOTS OF P(X)                                                          MULTIPLICITIES

 ROOT( 1) =  0.10000000000000007D 01 +  0.99999999999990740D 00 I         6           RESULTS OF SUBROUTINE QUAD
 ROOT( 2) =  0.10000000000000007D 01 + -0.99999999999990740D 00 I         6           RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.14. Roots Are: 1+i (6), 1-i (6)

```
GREATEST COMMON DIVISOR METHOD USED WITH MULLERS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  1


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS  6  THE COEFFICIENTS ARE

 P(7 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 P(6 ) = -0.7000000000000001D 01 + -0.1050000000000000D 02 I
 P(5 ) = -0.2800000000000000D 02 +  0.5800000000000001D 02 I
 P(4 ) =  0.1710000000000000D 03 +  0.1500000000000000D 01 I
 P(3 ) = -0.7300000000000000D 02 + -0.2510000000000000D 03 I
 P(2 ) = -0.2280000000000000D 03 +  0.1040000000000000D 03 I
 P(1 ) =  0.7200000000000001D 02 +  0.1040000000000000D 03 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  3  THE COEFFICIENTS ARE

 Q(4 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 Q(3 ) = -0.2000000000000150D 01 + -0.4500000000000118D 01 I
 Q(2 ) = -0.7000000000000423D 01 +  0.3500000000000060D 01 I
 Q(1 ) =  0.9999999999997762D 00 +  0.6999999999998812D 01 I


ROOTS OF Q(X)                                                                        INITIAL APPROXIMATION

 ROOT( 1) =  0.9999999999997565D 00 +  0.1999999999999574D 01 I             0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.2000000000000347D 01 +  0.2000000000000529D 01 I             SOLVED BY DIRECT METHOD
 ROOT( 3) = -0.9999999999999539D 00 +  0.5000000000000149D 00 I             SOLVED BY DIRECT METHOD


ROOTS OF P(X)                                                  MULTIPLICITIES        INITIAL APPROXIMATION

 ROOT( 1) =  0.9999999999997568D 00 +  0.1999999999999574D 01 I      2        0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.2000000000000347D 01 +  0.2000000000000530D 01 I      3        RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.9999999999999541D 00 +  0.5000000000000148D 00 I      1        RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.15. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
GREATEST COMMON DIVISOR METHOD USED WITH MULLERS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  2


NUMBER OF INITIAL APPROXIMATIONS GIVEN.      0
MAXIMUM NUMBER OF ITERATIONS.              200
TEST FOR ZERO IN SUBROUTINE GCD.     0.10D-02
TEST FOR CONVERGENCE.                0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.    0.10D-19
TEST FOR MULTIPLICITIES.             0.10D-01
RADIUS TO START SEARCH.              0.00D 00
RADIUS TO END SEARCH.                0.00D 00


THE DEGREE OF P(X) IS 15  THE COEFFICIENTS ARE

 P(16) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 P(15) =  0.25571200000000000D 03 + -0.38400000000000000D 03 I
 P(14) = -0.73535568000000000D 02 + -0.21896960000000000D 04 I
 P(13) = -0.38555656960000000D 04 + -0.69468514560000001D 04 I
 P(12) = -0.17333864648000000D 05 + -0.14206259728000000D 05 I
 P(11) = -0.49679892704000001D 05 + -0.17658574640000000D 05 I
 P(10) = -0.10223945221300000D 06 + -0.60306642320000001D 04 I
 P(9 ) = -0.16427422005600000D 06 +  0.41373662304000000D 05 I
 P(8 ) = -0.20366258884200000D 06 +  0.10938992276700000D 06 I
 P(7 ) = -0.18712557800100000D 06 +  0.19298654403300000D 06 I
 P(6 ) = -0.12749972985900000D 06 +  0.21713412274200000D 06 I
 P(5 ) = -0.28146927168000000D 05 +  0.19284897279600000D 06 I
 P(4 ) =  0.13294344348000000D 05 +  0.10381302265500000D 06 I
 P(3 ) =  0.30539007747000000D 05 +  0.29989891413000000D 05 I
 P(2 ) = -0.18358990200000000D 03 + -0.18276321600000000D 03 I
 P(1 ) =  0.27556200000000000D 00 +  0.27556200000000000D 00 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  7  THE COEFFICIENTS ARE

 Q(8 ) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 Q(7 ) =  0.15985599999590083D 03 + -0.14400000000184630D 03 I
 Q(6 ) =  0.26751999999866287D 03 + -0.52756800000682380D 03 I
 Q(5 ) =  0.32719599997819040D 03 + -0.91441600000759420D 03 I
 Q(4 ) =  0.23015999960044553D 02 + -0.15212520000037680D 04 I
 Q(3 ) = -0.72072000047996880D 02 + -0.13274279999966590D 04 I
 Q(2 ) = -0.75578400005415630D 03 + -0.75200400003332210D 03 I
 Q(1 ) =  0.22680001335502310D 01 +  0.22679999258286330D 01 I


ROOTS OF Q(X)                                                                                     INITIAL APPROXIMATION

 ROOT( 1) =  0.30000000386459970D-02 + -0.13705431778101480D-09 I      0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.14666372491173310D-09 +  0.10000000003025560D 01 I      0.70710675530463460D 00 +  0.70710680706845950D 00 I
 ROOT( 3) = -0.16262101465322160D-09 +  0.14999999998070670D 01 I      0.38822847926540560D 00 +  0.14488887631171930D 01 I
 ROOT( 4) =  0.11404545919875120D-09 +  0.30000000000539440D 01 I     -0.51763825519667240D 00 +  0.19318516083687550D 01 I
```

Exhibit 6.16.

```
ROOT( 5) = -0.23333333333408840 01 +  0.42926697510937080-11 I          -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) = -0.11677251758139090-10 + -0.15000000000067630 01 I           SOLVED BY DIRECT METHOD
ROOT( 7) = -0.10000000000032261D 01 + -0.99999999998557870 00 I          SOLVED BY DIRECT METHOD


ROOTS OF P(X)                                                  MULTIPLICITIES              INITIAL APPROXIMATION


ROOT( 1) =  0.30000000386459970-02 + -0.13705431778187450-09 I      2          0.48296291156562790 00 +  0.12940952844381870 00 I
ROOT( 2) =  0.14666360907717090-09 +  0.10000000003025560 01 I      2          0.70710675530463460 00 +  0.70710680706845950 00 I
ROOT( 3) = -0.16262091348697400-09 +  0.14999999998070660 01 I      2          0.38822847926540560 00 +  0.14488887631171930 01 I
ROOT( 4) =  0.11404556580843080-09 +  0.30000000000539440 01 I      3         -0.51763825519667240 00 +  0.19318516083687550 01 I
ROOT( 5) = -0.23333333333408850 01 +  0.42914027148801480-11 I      1         -0.17677671470807010 01 +  0.17677667588520150 01 I
ROOT( 6) = -0.11677735717066360-10 + -0.15000000000067630 01 I      2          RESULTS OF SUBROUTINE QUAD
ROOT( 7) = -0.10000000000322600 01 + -0.99999999998557790 00 I      3          RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.16. Roots Are: -2.33, .003 (2), i (2), 1.5i (2), -1.5i (2) 3i (3), -1-i (3)

```
GREATEST COMMON DIVISOR METHOD USED WITH MULLERS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  3


NUMBER OF INITIAL APPROXIMATIONS GIVEN.     0
MAXIMUM NUMBER OF ITERATIONS.             200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.000 00
RADIUS TO END SEARCH.               0.000 00


THE DEGREE OF P(X) IS  8  THE COEFFICIENTS ARE


 P(9 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(8 ) = -0.50000000000000001D 01 + -0.11500000000000000D 02 I
 P(7 ) = -0.51750000000000001D 02 +  0.43000000000000001D 02 I
 P(6 ) =  0.15725000000000000D 03 +  0.14462500000000000D 03 I
 P(5 ) =  0.30750000000000000D 03 + -0.34750000000000000D 03 I
 P(4 ) = -0.49525000000000000D 03 + -0.49487500000000000D 03 I
 P(3 ) = -0.58575000000000001D 03 +  0.42475000000000001D 03 I
 P(2 ) =  0.18100000000000000D 03 +  0.44200000000000001D 03 I
 P(1 ) =  0.15800000000000000D 03 +  0.60000000000000001D 01 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  3  THE COEFFICIENTS ARE


 Q(4 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 Q(3 ) = -0.20000000000002730D 01 + -0.45000000000002750D 01 I
 Q(2 ) = -0.70000000000009350D 01 +  0.35000000000004690D 01 I
 Q(1 ) =  0.99999999999994173D 00 +  0.70000000000004980D 01 I


ROOTS OF Q(X)                                                                            INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999995483D 00 +  0.19999999999997755D 01 I                   0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.20000000000006650D 01 +  0.20000000000005390D 01 I                   SOLVED BY DIRECT METHOD
 ROOT( 3) = -0.99999999999999408D 00 +  0.49999999999999814D 00 I                   SOLVED BY DIRECT METHOD


ROOTS OF P(X)                                                    MULTIPLICITIES          INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999995483D 00 +  0.19999999999997755D 01 I          2            0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.20000000000006650D 01 +  0.20000000000005390D 01 I          3            RESULTS OF SUBROUTINE QUAD
 ROOT( 3) = -0.99999999999994100D 00 +  0.49999999999998120D 00 I          3            RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.17. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
GREATEST COMMON DIVISOR METHOD USED WITH MULLERS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  4


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.   0.10D-02
TEST FOR CONVERGENCE.              0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.  0.10D-19
TEST FOR MULTIPLICITIES.           0.10D-01
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00


THE DEGREE OF P(X) IS 12  THE COEFFICIENTS ARE

 P(13) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 P(12) = -0.1200000000000000D 02 + -0.0000000000000000D 00 I
 P(11) =  0.7200000000000001D 02 +  0.0000000000000000D 00 I
 P(10) = -0.2800000000000000D 03 + -0.0000000000000000D 00 I
 P(9 ) =  0.7800000000000000D 03 +  0.0000000000000000D 00 I
 P(8 ) = -0.1632000000000000D 04 + -0.0000000000000000D 00 I
 P(7 ) =  0.2624000000000000D 04 +  0.0000000000000000D 00 I
 P(6 ) = -0.3264000000000000D 04 + -0.0000000000000000D 00 I
 P(5 ) =  0.3120000000000000D 04 +  0.0000000000000000D 00 I
 P(4 ) = -0.2240000000000000D 04 + -0.0000000000000000D 00 I
 P(3 ) =  0.1152000000000000D 04 +  0.0000000000000000D 00 I
 P(2 ) = -0.3840000000000000D 03 + -0.0000000000000000D 00 I
 P(1 ) =  0.6400000000000001D 02 +  0.0000000000000000D 00 I


Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X).
THE DEGREE OF Q(X) IS  2  THE COEFFICIENTS ARE

 Q(3 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 Q(2 ) = -0.2000000000000015D 01 + -0.0000000000000000D 00 I
 Q(1 ) =  0.1999999999999830D 01 +  0.0000000000000000D 00 I


ROOTS OF P(X)                                                      MULTIPLICITIES

 ROOT( 1) =  0.1000000000000007D 01 +  0.9999999999999074D 00 I        6          RESULTS OF SUBROUTINE QUAD
 ROOT( 2) =  0.1000000000000007D 01 + -0.9999999999999074D 00 I        6          RESULTS OF SUBROUTINE QUAD
```

Exhibit 6.18. Roots Are: 1+i(6), 1-i (6)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND NEWTONS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  1


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS  6  THE COEFFICIENTS ARE


 P(7 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 P(6 ) = -0.7000000000000001D 01 + -0.1050000000000000D 02 I
 P(5 ) = -0.2800000000000000D 02 +  0.5800000000000001D 02 I
 P(4 ) =  0.1710000000000000D 03 +  0.1500000000000000D 01 I
 P(3 ) = -0.7300000000000000D 02 + -0.2510000000000000D 03 I
 P(2 ) = -0.2280000000000000D 03 +  0.1040000000000000D 03 I
 P(1 ) =  0.7200000000000001D 02 +  0.1040000000000000D 03 I


****************************************************************************************************************************


THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  1

 G(2 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 G(1 ) =  0.9999999999997350D 00 + -0.5000000000017542D 00 I


ROOTS OF P(X)                                                     MULTIPLICITIES                 INITIAL APPROXIMATION

 ROOT( 1) = -0.9999999999997350D 00 +  0.5000000000017542D 00 I          1                  NO INITIAL APPROXIMATIONS


****************************************************************************************************************************


THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(2 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 G(1 ) = -0.9999999999999179D 00 + -0.1999999999996845D 01 I
```

Exhibit 6.19.

```
ROOTS OF P(X)                                            MULTIPLICITIES          INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999991790 00 +  0.19999999999968450 01 I        2              NO INITIAL APPROXIMATIONS


**************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

    2 ) =  0.10000000000000000 01 +  0.00000000000000000 00 I
    1 ) = -0.19999999999999670 01 + -0.20000000000015190 01 I

ROOTS OF P(X)                                            MULTIPLICITIES          INITIAL APPROXIMATION

 ROOT( 3) =  0.19999999999999670 01 +  0.20000000000015190 01 I        3              NO INITIAL APPROXIMATIONS
```

Exhibit 6.19. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
      REPEATED USE OF THE GREATEST COMMON DIVISOR AND NEWTONS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
      POLYNOMIAL NUMBER  2


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.   0.10D-02
TEST FOR CONVERGENCE.              0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.  0.10D-19
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00


THE DEGREE OF P(X) IS 15  THE COEFFICIENTS ARE

 P(16) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 P(15) =  0.25571200000000000D 03 + -0.38400000000000000D 03 I
 P(14) = -0.73535568000000000D 02 + -0.21896960000000000D 04 I
 P(13) = -0.38555656960000000D 04 + -0.69468514560000001D 04 I
 P(12) = -0.17333864648000000D 05 + -0.14206259728000000D 05 I
 P(11) = -0.49679892704000001D 05 + -0.17658574640000000D 05 I
 P(10) = -0.10223945221300000D 06 + -0.60306642320000001D 04 I
 P(9 ) = -0.16427422005600000D 06 +  0.41373662304000000D 05 I
 P(8 ) = -0.20366258884200000D 06 +  0.10938992276700000D 06 I
 P(7 ) = -0.18712557800100000D 06 +  0.19298654403300000D 06 I
 P(6 ) = -0.12749972985900000D 06 +  0.21713412274200000D 06 I
 P(5 ) = -0.28146927168000000D 05 +  0.19284897279600000D 06 I
 P(4 ) =  0.13294344348000000D 05 +  0.10381302265500000D 06 I
 P(3 ) =  0.30539007747000000D 05 +  0.29989891413000000D 05 I
 P(2 ) = -0.18358990200000000D 03 + -0.18276321600000000D 03 I
 P(1 ) =  0.27556200000000000D 00 +  0.27556200000000000D 00 I


******************************************************************************************************************************


THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  1

 G(2 ) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 G(1 ) =  0.11199999928960310D 03 + -0.82749886587407680D-06 I


ROOTS OF P(X)                                              MULTIPLICITIES                  INITIAL APPROXIMATION

 ROOT( 1) = -0.23333333185333397D 01 +  0.17239559705709930D-07 I        1                NO INITIAL APPROXIMATIONS
```

Exhibit 6.20.

```
**************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(5 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 G(4 ) = -0.2999970655863438D-02 + -0.9999999656362725D 00 I
 G(3 ) =  0.2249999931178885D 01 +  0.3000058598117361D-02 I
 G(2 ) = -0.6750109902460278D-02 + -0.2250000129067665D 01 I
 G(1 ) =  0.4643479465382683D-06 +  0.6749603982170172D-02 I

ROOTS OF G(X)                                                                                     INITIAL APPROXIMATION

 ROOT( 1) =  0.2999823203921417D-02 + -0.2056988264688649D-06 I                 0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.4767581845696651D-06 +  0.1000000555850184D 01 I                 0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.2331721616933697D-07 + -0.1499999972431725D 01 I                     RESULTS OF SUBROUTINE QUAD
 ROOT( 4) = -0.3526234587180479D-06 +  0.1499999587916640D 01 I                     RESULTS OF SUBROUTINE QUAD

ROOTS OF P(X)                                                     MULTIPLICITIES                INITIAL APPROXIMATION

 ROOT( 1) =  0.2999823203921417D-02 + -0.2056988264688661D-06 I         2           0.4829629115656279D 00 +  0.1294095284438187D 00 I
 ROOT( 2) =  0.4767581845685002D-06 +  0.1000000555850184D 01 I         2           0.7071067553046346D 00 +  0.7071068070684595D 00 I
 ROOT( 3) =  0.2340917191042791D-07 + -0.1499999972431725D 01 I         2                 NO INITIAL APPROXIMATIONS
 ROOT( 4) = -0.3527154144585488D-06 +  0.1499999587916640D 01 I         2                 NO INITIAL APPROXIMATIONS

**************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

 G(3 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 G(2 ) =  0.9999999853705541D 00 + -0.2000000017162632D 01 I
 G(1 ) =  0.3000000002533335D 01 + -0.3000000031791514D 01 I

ROOTS OF P(X)                                                     MULTIPLICITIES                INITIAL APPROXIMATION

 ROOT( 1) =  0.1462930432349907D-07 +  0.3000000017162633D 01 I         3                 NO INITIAL APPROXIMATIONS
 ROOT( 2) = -0.9999999999998584D 00 + -0.1000000000000002D 01 I         3                 NO INITIAL APPROXIMATIONS
```

Exhibit 6.20 Roots Are: -2.33, .003 (2), i (2), 1.5i (2), -1.5i (2), 3i (3), -1-i (3)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND NEWTONS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  3


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.   0.100-02
TEST FOR CONVERGENCE.              0.100-09
TEST FOR ZERO IN SUBROUTINE QUAD.  0.100-19
RADIUS TO START SEARCH.            0.000 00
RADIUS TO END SEARCH.              0.000 00


THE DEGREE OF P(X) IS  8  THE COEFFICIENTS ARE

 P(9 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 P(8 ) = -0.5000000000000001D 01 + -0.1150000000000000D 02 I
 P(7 ) = -0.5175000000000001D 02 +  0.4300000000000001D 02 I
 P(6 ) =  0.1572500000000000D 03 +  0.1446250000000000D 03 I
 P(5 ) =  0.3075000000000000D 03 + -0.3475000000000000D 03 I
 P(4 ) = -0.4952500000000000D 03 + -0.4948750000000000D 03 I
 P(3 ) = -0.5857500000000001D 03 +  0.4247500000000001D 03 I
 P(2 ) =  0.2810000000000000D 03 +  0.4420000000000001D 03 I
 P(1 ) =  0.1580000000000000D 03 +  0.6000000000000001D 01 I


**************************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  1


**************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(2 ) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
 G(1 ) = -0.9999999999946498D 00 + -0.1999999999996468D 01 I


ROOTS OF P(X)                                                          MULTIPLICITIES                 INITIAL APPROXIMATION

 ROOT( 1) =  0.9999999999946498D 00 +  0.1999999999996468D 01 I          2                   NO INITIAL APPROXIMATIONS
```

Exhibit 6.21.

```
****************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

 G(3 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(2 ) = -0.10000000000025390D 01 + -0.25000000000016290D 01 I
 G(1 ) = -0.30000000000032960D 01 + -0.10000000000004210D 01 I


ROOTS OF P(X)                                                       MULTIPLICITIES            INITIAL APPROXIMATION

 ROOT( 1) =  0.20000000000025320D 01 +  0.20000000000016530D 01 I         3                NO INITIAL APPROXIMATIONS
 ROOT( 2) = -0.99999999999999260D 00 +  0.49999999999997590D 00 I         3                NO INITIAL APPROXIMATIONS
```

Exhibit 6.21. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND NEWTONS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  4


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.    0.100-02
TEST FOR CONVERGENCE.               0.100-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.100-19
RADIUS TO START SEARCH.             0.000 00
RADIUS TO END SEARCH.               0.000 00


THE DEGREE OF P(X) IS 12  THE COEFFICIENTS ARE


 P(13) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(12) = -0.12000000000000000D 02 + -0.00000000000000000D 00 I
 P(11) =  0.72000000000000001D 02 +  0.00000000000000000D 00 I
 P(10) = -0.28000000000000000D 03 + -0.00000000000000000D 00 I
 P(9 ) =  0.78000000000000000D 03 +  0.00000000000000000D 00 I
 P(8 ) = -0.16320000000000000D 04 + -0.00000000000000000D 00 I
 P(7 ) =  0.26240000000000000D 04 +  0.00000000000000000D 00 I
 P(6 ) = -0.32640000000000000D 04 + -0.00000000000000000D 00 I
 P(5 ) =  0.31200000000000000D 04 +  0.00000000000000000D 00 I
 P(4 ) = -0.22400000000000000D 04 + -0.00000000000000000D 00 I
 P(3 ) =  0.11520000000000000D 04 +  0.00000000000000000D 00 I
 P(2 ) = -0.38400000000000000D 03 + -0.00000000000000000D 00 I
 P(1 ) =  0.64000000000000001D 02 +  0.00000000000000000D 00 I


***************************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  1


***************************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  2
```

Exhibit 6.22.

```
****************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  3


****************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  4


****************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  5


****************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  6

 G(3 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(2 ) = -0.20000000000000066D 01 + -0.00000000000000000D 00 I
 G(1 ) =  0.20000000000000007D 01 +  0.00000000000000000D 00 I

ROOTS OF P(X)                                                    MULTIPLICITIES              INITIAL APPROXIMATION

 ROOT( 1) =  0.10000000000000033D 01 +  0.99999999999997060D 00 I       6             NO INITIAL APPROXIMATIONS
 ROOT( 2) =  0.10000000000000033D 01 + -0.99999999999997060D 00 I       6             NO INITIAL APPROXIMATIONS
```

Exhibit 6.22. Roots Are: 1+i (6), 1-i (6)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND MULLERS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  1


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR ZERO IN SUBROUTINE GCD.   0.10D-07
TEST FOR CONVERGENCE.              0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.  0.10D-19
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00


THE DEGREE OF P(X) IS  6  THE COEFFICIENTS ARE

 P(7 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(6 ) = -0.70000000000000010D 01 + -0.10500000000000000D 02 I
 P(5 ) = -0.28000000000000000D 02 +  0.58000000000000001D 02 I
 P(4 ) =  0.17100000000000000D 03 +  0.15000000000000000D 01 I
 P(3 ) = -0.73000000000000000D 02 + -0.25100000000000000D 03 I
 P(2 ) = -0.22800000000000000D 03 +  0.10400000000000000D 03 I
 P(1 ) =  0.72000000000000010D 02 +  0.10400000000000000D 03 I


*****************************************************************************************************************************


THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  1

 G(2 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(1 ) =  0.99999999999973500D 00 + -0.50000000000175420D 00 I


ROOTS OF G(X)                                                                           INITIAL APPROXIMATION

 ROOT( 1) = -0.99999999999973500D 00 +  0.50000000000175410D 00 I                 0.48296291156562790D 00 +  0.12940952844381870D 00 I


ROOTS OF P(X)                                                       MULTIPLICITIES      INITIAL APPROXIMATION

 ROOT( 1) = -0.99999999999973500D 00 +  0.50000000000175410D 00 I         1        0.48296291156562790D 00 +  0.12940952844381870D 00 I


*****************************************************************************************************************************
```

Exhibit 6.23.

```
THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(2 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(1 ) = -0.99999999999991790D 00 + -0.19999999999968450 01 I


ROOTS OF G(X)                                                                                   INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999991790 00 +  0.19999999999968450 01 I                   0.48296291156562790 00 +  0.12940952844381870 00 I


ROOTS OF P(X)                                                  MULTIPLICITIES                  INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999991790 00 +  0.19999999999968450 01 I        2          0.48296291156562790 00 +  0.1294095284438187D 00 I


**************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

    2 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
    1 ) = -0.19999999999999670 01 + -0.20000000000015190 01 I


ROOTS OF P(X)                                                  MULTIPLICITIES                  INITIAL APPROXIMATION

 ROOT( 3) =  0.1999999999999967D 01 +  0.2000000000001519D 01 I        3          NO INITIAL APPROXIMATIONS
```

Exhibit 6.23. Roots Are: 2+2i (3), 1+2i (2), -1+.5i

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND MULLERS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  2

NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS 15  THE COEFFICIENTS ARE

 P(16) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 P(15) =  0.25571200000000000D 03 + -0.38400000000000000D 03 I
 P(14) = -0.73535568000000000D 02 + -0.21896960000000000D 04 I
 P(13) = -0.38555656960000000D 04 + -0.69468514560000001D 04 I
 P(12) = -0.17333866464800000D 05 + -0.14206259728000000D 05 I
 P(11) = -0.49679892704000001D 05 + -0.17658574640000000D 05 I
 P(10) = -0.10223945221300000D 06 + -0.60306642320000001D 04 I
 P(9 ) = -0.16427422005600000D 06 +  0.41373662304000000D 05 I
 P(8 ) = -0.20366258884200000D 06 +  0.10938992276700000D 06 I
 P(7 ) = -0.18712557800100000D 06 +  0.19298654403300000D 06 I
 P(6 ) = -0.12749972985900000D 06 +  0.21713412274200000D 06 I
 P(5 ) = -0.28146927168000000D 05 +  0.19284897279600000D 06 I
 P(4 ) =  0.13294344348000000D 05 +  0.10381302265500000D 06 I
 P(3 ) =  0.30539007747000000D 05 +  0.29989891413000000D 05 I
 P(2 ) = -0.18358990200000000D 03 + -0.18276321600000000D 03 I
 P(1 ) =  0.27556200000000000D 00 +  0.27556200000000000D 00 I


*************************************************************************************************************************************


THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  1

 G(2 ) =  0.48000000000000000D 02 +  0.00000000000000000D 00 I
 G(1 ) =  0.11199999928960310D 03 + -0.82749886587407680D-06 I

ROOTS OF G(X)                                                                          INITIAL APPROXIMATION

 ROOT( 1) = -0.23333333185333397D 01 +  0.17239559705709930D-07 I              0.48296291156562790D 00 +  0.12940952844381870D 00 I

ROOTS OF P(X)                                                 MULTIPLICITIES           INITIAL APPROXIMATION
```

Exhibit 6.24.

```
ROOT( 1) = -0.23333333185333397D 01 +  0.17239559705709930-07 I          1          0.48296291156562790 00 +  0.12940952844381870 00 I

*******************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(5 ) =  0.10000000000000000 01 +  0.00000000000000000 00 I
 G(4 ) = -0.29999706558634360-02 + -0.99999996563627250 00 I
 G(3 ) =  0.22499999311788850 01 +  0.30000585981173610-02 I
 G(2 ) = -0.67501099024602780-02 + -0.22500001290676650 01 I
 G(1 ) =  0.46434794653826830-06 +  0.67496039821701720-02 I

ROOTS OF G(X)                                                                        INITIAL APPROXIMATION

 ROOT( 1) =  0.29998232039214170-02 + -0.20569882646886600-06 I                     0.48296291156562790 00 +  0.12940952844381870 00 I
 ROOT( 2) =  0.47675818456954060-06 +  0.10000005558501840 01 I                     0.70710675530463460 00 +  0.70710680706845950 00 I
 ROOT( 3) =  0.23317216169372120-07 + -0.14999999724317250 01 I                     SOLVED BY DIRECT METHOD
 ROOT( 4) = -0.35262345871801270-06 +  0.14999995879166400 01 I                     SOLVED BY DIRECT METHOD

ROOTS OF P(X)                                                 MULTIPLICITIES        INITIAL APPROXIMATION

 ROOT( 1) =  0.29998232039214170-02 + -0.20569882646886650-06 I        2           0.48296291156562790 00 +  0.12940952844381870 00 I
 ROOT( 2) =  0.47675818456831650-06 +  0.10000005558501840 01 I        2           0.70710675530463460 00 +  0.70710680706845950 00 I
 ROOT( 3) =  0.23409171910410740-07 + -0.14999999724317250 01 I        2           NO INITIAL APPROXIMATIONS
 ROOT( 4) = -0.35271541445867050-06 +  0.14999995879166400 01 I        2           NO INITIAL APPROXIMATIONS

*******************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

 G(3 ) =  0.10000000000000000 01 +  0.00000000000000000 00 I
 G(2 ) =  0.99999998537055410 00 + -0.20000000171626320 01 I
 G(1 ) =  0.30000000025333350 01 + -0.30000000317915140 01 I

ROOTS OF G(X)                                                                        INITIAL APPROXIMATION
```

Exhibit 6.24.

```
ROOT( 1) = -0.99999999999985870 00 + -0.10000000000000002D 01 I          0.48296291156562790 00 +  0.12940952844381870 00 I
ROOT( 2) =  0.1462930461493261D-07 +  0.3000000017162634D 01 I          SOLVED BY DIRECT METHOD


ROOTS OF P(X)                                          MULTIPLICITIES            INITIAL APPROXIMATION

ROOT( 1) = -0.9999999999998586D 00 + -0.10000000000000002D 01 I     3         0.4829629115656279D 00 +  0.1294095284438187D 00 I
ROOT( 2) =  0.1462930456963442D-07 +  0.3000000017162634D 01 I     3         NO INITIAL APPROXIMATIONS
```

Exhibit 6.24. Roots Are: -2.33, .003 (2), i (2), 1.5i (2), -1.5i (2) 3i (3), -1-i (3)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND MULLERS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  3


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS  8  THE COEFFICIENTS ARE

 P(9 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(8 ) = -0.50000000000000001D 01 + -0.11500000000000000D 02 I
 P(7 ) = -0.51750000000000001D 02 +  0.43000000000000001D 02 I
 P(6 ) =  0.15725000000000000D 03 +  0.14462500000000000D 03 I
 P(5 ) =  0.30750000000000000D 03 + -0.34750000000000000D 03 I
 P(4 ) = -0.49525000000000000D 03 + -0.49487500000000000D 03 I
 P(3 ) = -0.58575000000000001D 03 +  0.42475000000000001D 03 I
 P(2 ) =  0.18100000000000000D 03 +  0.44200000000000001D 03 I
 P(1 ) =  0.15800000000000000D 03 +  0.60000000000000001D 01 I

****************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  1

****************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  2

 G(2 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(1 ) = -0.99999999999464498D 00 + -0.19999999999964468D 01 I


ROOTS OF G(X)                                                                              INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999464498D 00 +  0.19999999999964468D 01 I        0.48296291156562790D 00 +  0.12940952844381870D 00 I
```

Exhibit 6.25.

```
ROOTS OF P(X)                                                        MULTIPLICITIES              INITIAL APPROXIMATION

 ROOT( 1) =  0.99999999999464980 00 +  0.19999999999964680 01 I          2         0.48296291156562790 00 +  0.12940952844381870 00 I


****************************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  3

 G(3 ) =  0.10000000000000000 01 +  0.00000000000000000 00 I
 G(2 ) = -0.10000000000025390 01 + -0.25000000000016290 01 I
 G(1 ) = -0.30000000000032960 01 + -0.10000000000004210 01 I


ROOTS OF G(X)                                                                                    INITIAL APPROXIMATION

 ROOT( 1) = -0.99999999999999280 00 +  0.49999999999997590 00 I                    0.48296291156562790 00 +  0.12940952844381870 00 I
 ROOT( 2) =  0.20000000000025320 01 +  0.20000000000016530 01 I                    SOLVED BY DIRECT METHOD


IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT( 2) =  0.20000000000025320 01 +  0.20000000000016530 01 I
DID NOT CONVERGE AFTER 200 ITERATIONS
THE PRESENT APPROXIMATION IS  0.20019999742533280 01 +  0.20019999742524480 01 I


ROOTS OF P(X)                                                        MULTIPLICITIES              INITIAL APPROXIMATION

 ROOT( 1) = -0.99999999999999300 00 +  0.49999999999997590 00 I          3         0.48296291156562790 00 +  0.12940952844381870 00 I

NOT ALL ROOTS OF THE ABOVE POLYNOMIAL,G, WERE FOUND
```

Exhibit 6.25. Roots Are: 2+2i (3), 1+2i (2), -1+.5i (3)

```
REPEATED USE OF THE GREATEST COMMON DIVISOR AND MULLERS  METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIALS
POLYNOMIAL NUMBER  4


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    0
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR ZERO IN SUBROUTINE GCD.    0.10D-02
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR ZERO IN SUBROUTINE QUAD.   0.10D-19
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


THE DEGREE OF P(X) IS 12  THE COEFFICIENTS ARE


 P(13) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 P(12) = -0.12000000000000000D 02 + -0.00000000000000000D 00 I
 P(11) =  0.72000000000000001D 02 +  0.00000000000000000D 00 I
 P(10) = -0.28000000000000000D 03 + -0.00000000000000000D 00 I
 P(9 ) =  0.78000000000000000D 03 +  0.00000000000000000D 00 I
 P(8 ) = -0.16320000000000000D 04 + -0.00000000000000000D 00 I
 P(7 ) =  0.26240000000000000D 04 +  0.00000000000000000D 00 I
 P(6 ) = -0.32640000000000000D 04 + -0.00000000000000000D 00 I
 P(5 ) =  0.31200000000000000D 04 +  0.00000000000000000D 00 I
 P(4 ) = -0.22400000000000000D 04 + -0.00000000000000000D 00 I
 P(3 ) =  0.11520000000000000D 04 +  0.00000000000000000D 00 I
 P(2 ) = -0.38400000000000000D 03 + -0.00000000000000000D 00 I
 P(1 ) =  0.64000000000000001D 02 +  0.00000000000000000D 00 I


*************************************************************************************************************************************


NO ROOTS OF MULTIPLICITY  1


*************************************************************************************************************************************


NO ROOTS OF MULTIPLICITY  2
```

Exhibit 6.26.

```
**********************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  3


**********************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  4


**********************************************************************************************************************************

NO ROOTS OF MULTIPLICITY  5


**********************************************************************************************************************************

THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY  6

 G(3 ) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
 G(2 ) = -0.20000000000000066D 01 + -0.00000000000000000D 00 I
 G(1 ) =  0.20000000000000007D 01 +  0.00000000000000000D 00 I

ROOTS OF G(X)                                                                                 INITIAL APPROXIMATION

 ROOT( 1) =  0.10000000000000033D 01 +  0.99999999999999707D 00 I              0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.10000000000000033D 01 + -0.99999999999999707D 00 I              SOLVED BY DIRECT METHOD

ROOTS OF P(X)                                                    MULTIPLICITIES                INITIAL APPROXIMATION

 ROOT( 1) =  0.10000000000000033D 01 +  0.99999999999999707D 00 I         6         0.48296291156562790D 00 +  0.12940952844381870D 00 I
 ROOT( 2) =  0.10000000000000033D 01 + -0.99999999999999707D 00 I         6         NO INITIAL APPROXIMATIONS
```

Exhibit 6.26. Roots Are: 1+i (6), 1-i (6)

REFERENCES

1. S. D. Conte, Elementary Numerical Analysis, McGraw Hill, New York, 1965.

2. Peter Henrici, Elements of Numerical Analysis, John Wiley and Sons, Inc., New York, 1964.

3. Thomas R. McCalla, Introduction to Numerical Methods and FORTRAN Programming, John Wiley and Sons, Inc., New York, 1967.

4. David E. Muller, A method for solving algebraic equations using an automatic computer, Math. Tables and Aids to Comp., 10 (1956), 208-215.

APPENDIX A

SPECIAL FEATURES OF NEWTON'S AND MULLER'S PROGRAMS

Several special features have been provided in each program as an aid to the user and to improve accuracy of the results. These are explained and illustrated below.*

1. Generating Approximations

If the user does not have initial approximations available, subroutine GENAPP can systematically generate, for an N^{th} degree polynomial, N initial approximations of increasing magnitude, beginning with the magnitude specified by XSTART. If XSTART is 0., XSTART is automatically initialized to 0.5 to avoid the approximation 0. + 0.i. The approximations are generated according to the formula:

$$X_K = (\text{XSTART} + 0.5K)\ (\text{Cos}\ \beta + i\ \text{Sin}\ \beta)$$

where

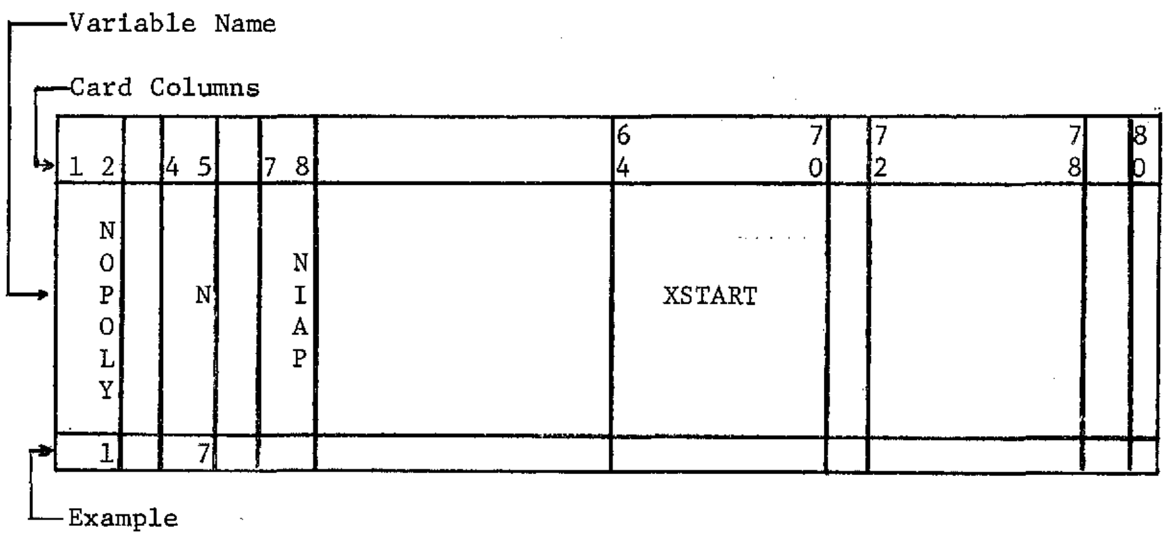
$$\beta = \frac{\Pi}{12} + K\frac{\Pi}{6}\ ,\ K = 0,1,2,\ldots$$

To accomplish this, the user defined the number of initial approximations to be read (NIAP) on the control card to be zero (0) or these

*These illustrations are representative of Newton's method in double precision. The control cards for Muller's method are similarly prepared.

columns (7-8) may be left blank. If XSTART is left blank, it is interpreted as 0.

For example, a portion of a control card which generates initial approximations beginning at the origin for a seventh degree polynomial is shown in Example A.1.



Example A.1

The approximations are generated in a spiral configuration as illustrated in Figure A.1. Exhibit 6.1 is an example of output resulting from generated approximations.

Example A.2 shows a portion of a control card which generated initial approximations beginning at a magnitude of 25.0 for a sixth degree polynomial.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		N		NIAP		XSTART				
1		6				2.5D+01				

Example A.2

Note that if the approximations are generated beginning at the origin, the order in which the roots are found will probably be of increasing magnitude. Roots obtained in this way are usually more accurate.

2. Altering Approximations

If an initial approximation, X_0, does not produce convergence to a zero within the maximum number of iterations, it is systematically altered a maximum of five times until convergence is possibly obtained according to the following formulas:

If the number of the alteration is odd: $(j = 1,3)$

$$X_{j+1} = |X_0| \, (\text{Cos } \beta + i \text{ Sin } \beta) \text{ where}$$

$$\beta = \text{Tan}^{-1} \frac{\text{Im } X_0}{\text{Re } X_0} + K \frac{\Pi}{3} \,; \quad K = 1 \text{ if } j = 1, \; 2 \text{ if } j = 3.$$

If the number of the alteration is even: $(j = 0,2,4)$

$$X_{j+1} = -X_j .$$

Each altered approximation is then taken as a starting approximation. Each initial or altered approximation which does not produce convergence is printed as in Exhibit A.1. If none of the six starting approximations produce convergence, the next initial approximation is taken, and the process repeated. The six approximations are spaced 60 degrees apart on a circle of radius $|X_0|$ centered at the origin as illustrated in Figure A.2.

3. Searching the Complex Plane

By use of initial approximations and the altering technique, any region of the complex plane in the form of an annulus centered at the origin can be searched for roots. This procedure can be accomplished in two ways.

The first way is more versatile but requires more effort on the part of the user. Specifically selected initial approximations can be used to define particular regions to be searched. For example, if the roots of a particular polynomial are known to have magnitudes between 20 and 40, an annulus of inner radius 20 and outer radius 40 could be searched by using the initial approximations 20. + i, 23. + i, 26. + i, 29. + i, 32. + i, 35. + i, 38. + i, 40. + i.

By generating initial approximations internally, the program can search an annulus centered at the origin of inner radius XSTART and outer radius XEND. Values for XSTART and XEND are supplied on the control card by the user. Example A.3 shows a portion of a control card to search the above annulus of inner radius 20.0 and outer radius 40.0.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		N		NIAP		XSTART		XEND		
1		7				2.0D+01		4.0D+01		

Example A.3

Note that since not less than N initial approximations can be generated at one time, the outer radius of the annulus actually searched may be greater than XEND but not greater than XEND + .5N.

Example A.4 shows a control card to search a circle of radius 15.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		N		NIAP		XSTART		XEND		
2		7						1.5D+01		

Example A.4

Figure A.3 shows the distribution of initial and altered approximations for an annulus of width 2 and inner radius a.

4. Improving Zeros Found

After the zeros of a polynomial are found, they are printed under the heading "Before the Attempt to Improve Accuracy." They are then used as initial approximations with Newton's (Muller's) method applied each time to the full (undeflated) polynomial. In most cases, zeros that have lost accuracy due to roundoff error in the deflation process are improved. The improved zeros are then printed under the heading "After the Attempt to Improve Accuracy." Since each root is used as an approximation to the original (undeflated) polynomial, it is possible that the root may converge to an entirely different root. This is especially true where several zeros are close together. Therefore, the user should check both lists of zeros to determine whether or not this has occurred. See Exhibit 6.4.

5. Solving Quadratic Polynomial

After N-2 roots of an N^{th} degree polynomial have been extracted, the remaining quadratic, $aX^2 + bX + c$, is solved using the quadratic formula

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the two remaining roots. These are indicated by the words "Solved By Direct Method" in the initial approximation column. If only a polynomial of degree 1 is to be solved, the solution is found directly as $(X - C) = 0$ implies $X - C$.

6. Missing Roots

If not all N roots of an N^{th} degree polynomial are found, the coefficients of the remaining deflated polynomial are printed under the heading "Coefficients of Deflated Polynomial For Which No Zeros Were Found." The user may then work with this polynomial in an attempt to find the remaining roots. The coefficient of the highest degree term will be printed first (Exhibit A.2).

7. Miscellaneous

By using various combinations of values for NIAP, XSTART, and XEND, the user has several options available as illustrated below.

Example A.5 shows the control card for a seventh degree polynomial. Three initial approximations are supplied by the user. At most three distinct roots will be found and the remaining deflated polynomial will be printed (Exhibit A.2).

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		N		NIAP		XSTART		XEND		
1		7		3						

Example A.5

Note that if several roots are known to the user, they may be "divided out" of the original polynomial by using this procedure.

Example A.6 indicates that 2 initial approximations are supplied by the user to a 7^{th} degree polynomial. After these approximations are used the circle of radius 15 will be searched for the remaining roots.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		N		NIAP		XSTART		XEND		
1		7		2				1.5D+01		

Example A.6

By defining XSTART between 0. and 15. an annulus instead of the circle will be searched (Exhibit A.3).

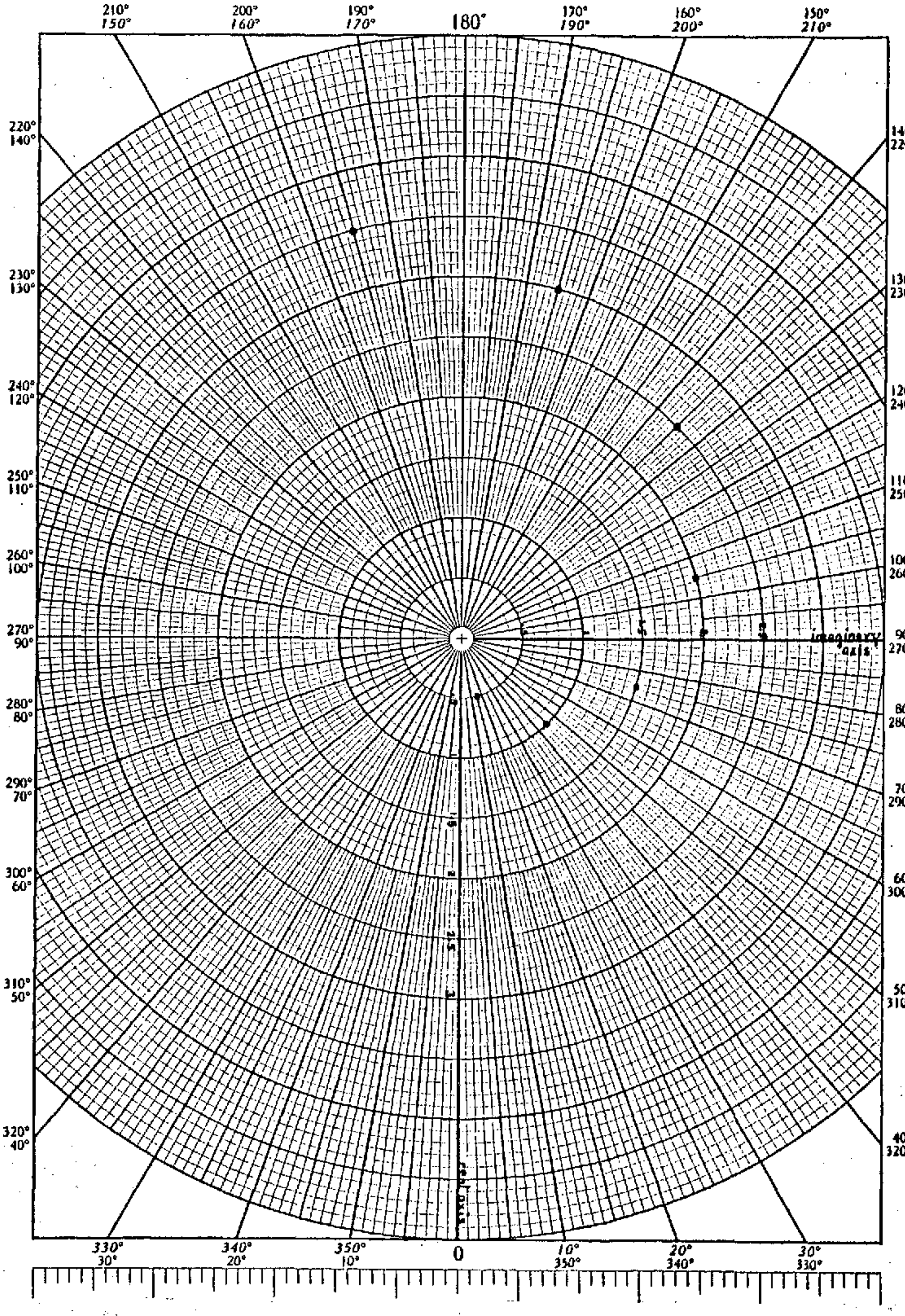


Figure A.1. Generating Initial Approximations

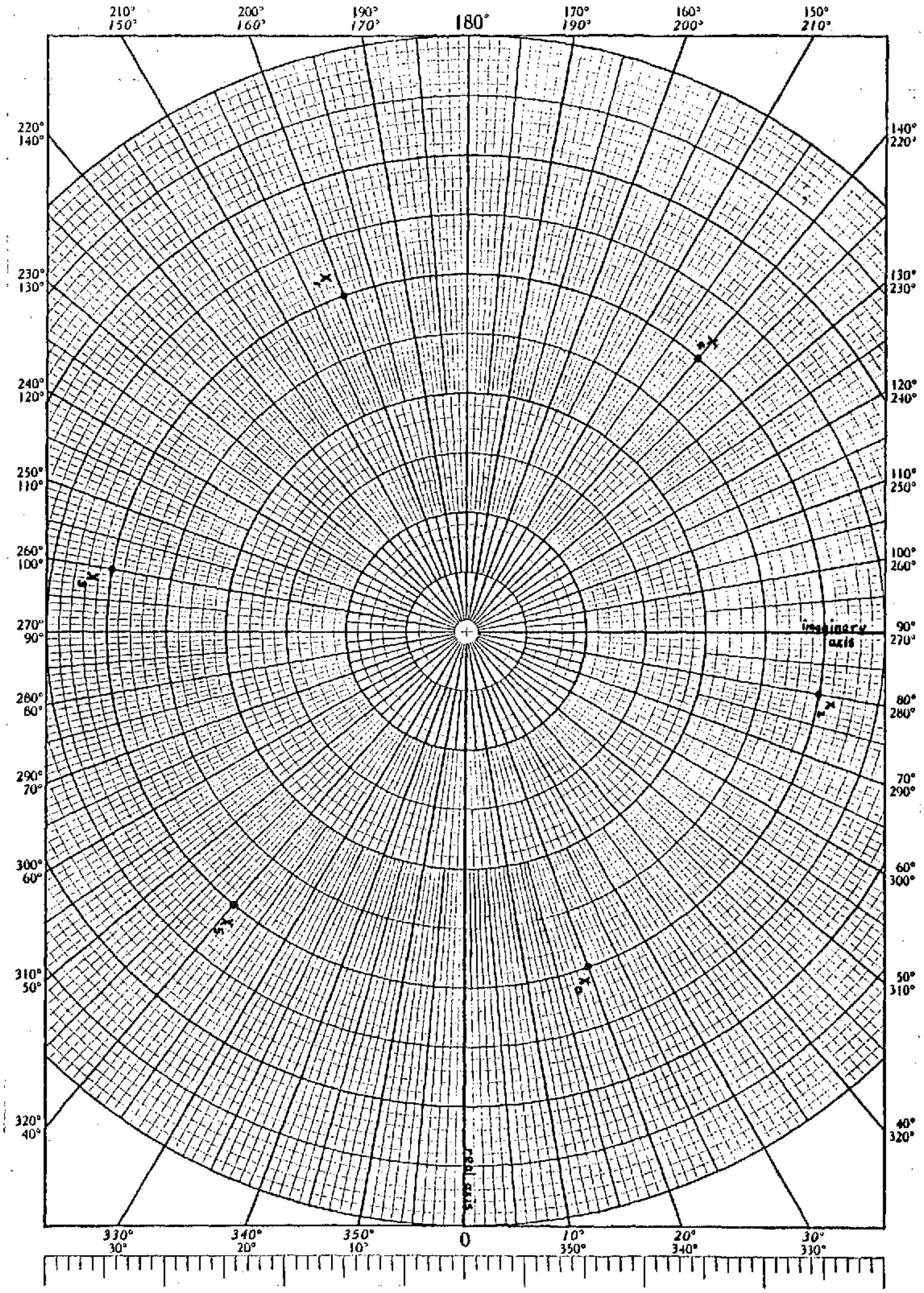


Figure A.2. Altering Approximations

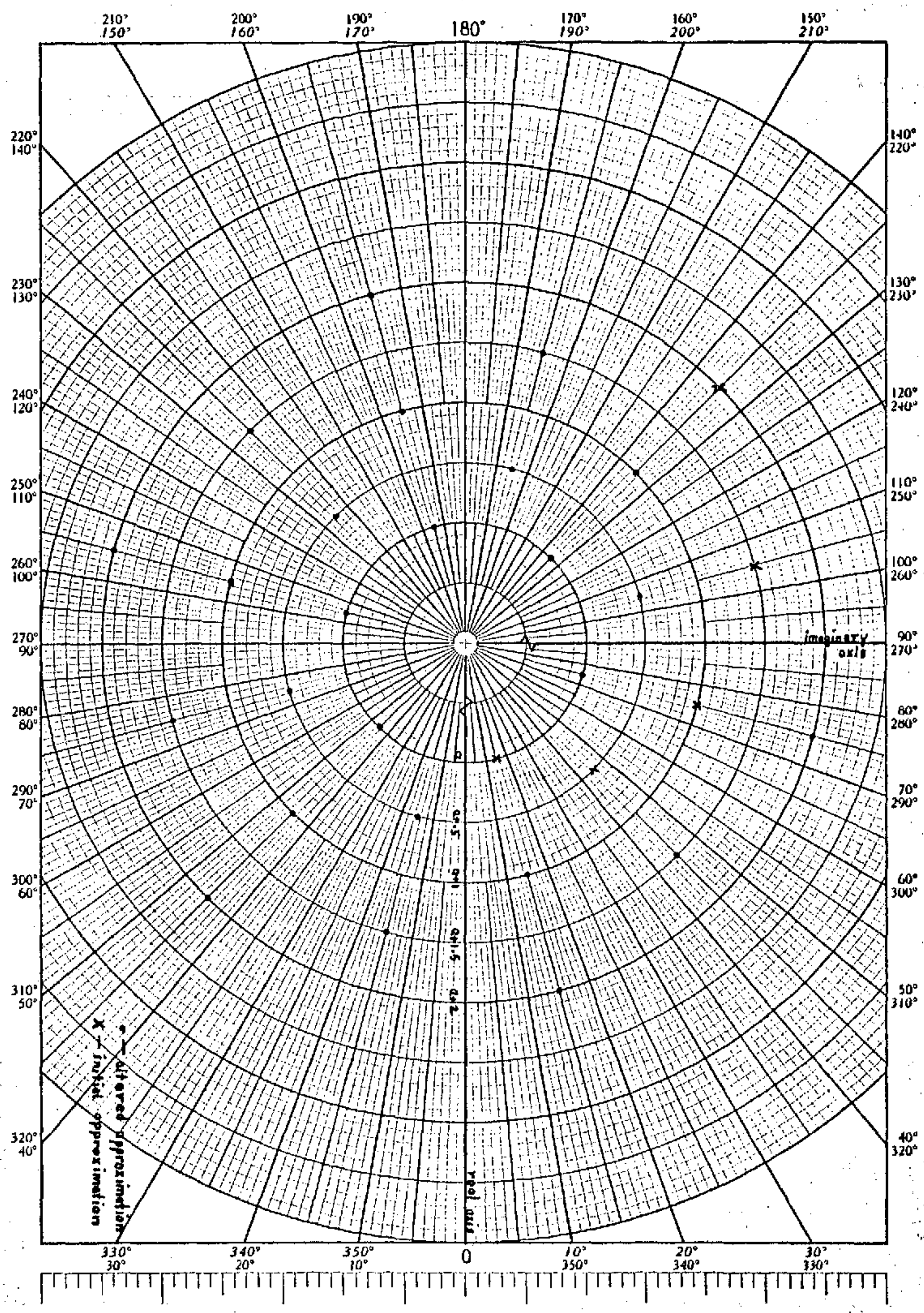


Figure A.3. Distribution of Approximations

```
        NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
        POLYNOMIAL NUMBER  2 OF DEGREE  3


THE COEFFICIENTS OF P(X) ARE

  P( 1) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
  P( 2) =  0.2000000000000000D 01 +  0.0000000000000000D 00 I
  P( 3) = -0.1000000000000000D 01 + -0.0000000000000000D 00 I
  P( 4) = -0.2000000000000000D 01 + -0.0000000000000000D 00 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   0
MAXIMUM NUMBER OF ITERATIONS.             3
TEST FOR CONVERGENCE.              0.10D-03
TEST FOR MULTIPLICITIES.           0.10D-01
RADIUS TO START SEARCH.            0.00D 00
RADIUS TO END SEARCH.              0.00D 00


NO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AFTER   3 ITERATIONS.


 0.4829629115656279D 00 +  0.1294095284438187D 00 I          INITIAL APPROXIMATION
-0.4829629115656279D 00 + -0.1294095284438187D 00 I          ALTERED APPROXIMATION
 0.1294094930884686D 00 +  0.4829629210390644D 00 I          ALTERED APPROXIMATION
-0.1294094930884686D 00 + -0.4829629210390644D 00 I          ALTERED APPROXIMATION
-0.3535534294161402D 00 +  0.3535533517704030D 00 I          ALTERED APPROXIMATION
 0.3535534294161402D 00 + -0.3535533517704030D 00 I          ALTERED APPROXIMATION

 0.7071067553046346D 00 +  0.7071068070684595D 00 I          INITIAL APPROXIMATION
-0.7071067553046346D 00 + -0.7071068070684595D 00 I          ALTERED APPROXIMATION
-0.2588191275983359D 00 +  0.9659258041843774D 00 I          ALTERED APPROXIMATION
 0.2588191275983359D 00 + -0.9659258041843774D 00 I          ALTERED APPROXIMATION
-0.9659258610249968D 00 +  0.2588189154662357D 00 I          ALTERED APPROXIMATION
 0.9659258610249968D 00 + -0.2588189154662357D 00 I          ALTERED APPROXIMATION

 0.3882284792654056D 00 +  0.1448888763117193D 01 I          INITIAL APPROXIMATION
-0.3882284792654056D 00 + -0.1448888763117193D 01 I          ALTERED APPROXIMATION
-0.1060660288248421D 01 +  0.1060660055311209D 01 I          ALTERED APPROXIMATION
 0.1060660288248421D 01 + -0.1060660055311209D 01 I          ALTERED APPROXIMATION
-0.1448888677856240D 01 + -0.3882287974635502D 00 I          ALTERED APPROXIMATION
 0.1448888677856240D 01 +  0.3882287974635502D 00 I          ALTERED APPROXIMATION


COEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO ZEROS WERE FOUND

  D( 1) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
  D( 2) =  0.2000000000000000D 01 +  0.0000000000000000D 00 I
  D( 3) = -0.1000000000000000D 01 + -0.0000000000000000D 00 I
  D( 4) = -0.2000000000000000D 01 + -0.0000000000000000D 00 I
```

Exhibit A.1.

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  1 OF DEGREE  7


THE COEFFICIENTS OF P(X) ARE

  P( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
  P( 2) = -0.10000000000000000D 01 +  0.11000000000000000D 02 I
  P( 3) = -0.59000000000000001D 02 + -0.29000000000000000D 02 I
  P( 4) =  0.19500000000000000D 03 + -0.16900000000000000D 03 I
  P( 5) =  0.70000000000000001D 02 +  0.72300000000000000D 03 I
  P( 6) = -0.16240000000000000D 04 + -0.69600000000000001D 03 I
  P( 7) =  0.19220000000000000D 04 + -0.18320000000000000D 04 I
  P( 8) =  0.15960000000000000D 04 +  0.16920000000000000D 04 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.   3
MAXIMUM NUMBER OF ITERATIONS.           200
TEST FOR CONVERGENCE.               0.10D-09
TEST FOR MULTIPLICITIES.            0.10D-01
RADIUS TO START SEARCH.             0.00D 00
RADIUS TO END SEARCH.               0.00D 00


BEFORE THE ATTEMPT TO IMPROVE ACCURACY. THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                      MULTIPLICITIES                INITIAL APPROXIMATION

  ROOT( 1) = -0.29999999999999997D 01 + -0.30000000000000002D 01 I        1       -0.35000000000000000D 01 + -0.35000000000000000D 01 I
  ROOT( 2) =  0.20000000000000000D 01 +  0.20000000000000000D 01 I        1        0.25000000000000000D 01 +  0.25000000000000000D 01 I
  ROOT( 3) = -0.99999999999999962D 00 + -0.39999999999999994D 01 I        1       -0.15000000000000000D 01 + -0.45000000000000001D 01 I


AFTER  THE ATTEMPT TO IMPROVE ACCURACY. THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                      MULTIPLICITIES                INITIAL APPROXIMATION

  ROOT( 1) = -0.29999999999999996D 01 + -0.30000000000000001D 01 I        1       -0.35000000000000000D 01 + -0.35000000000000000D 01 I
  ROOT( 2) =  0.20000000000000000D 01 +  0.20000000000000000D 01 -I       1        0.25000000000000000D 01 +  0.25000000000000000D 01 I
  ROOT( 3) = -0.99999999999999974D 00 + -0.39999999999999996D 01 I        1       -0.15000000000000000D 01 + -0.45000000000000001D 01 I


COEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO ZEROS WERE FOUND

  D( 1) =  0.10000000000000000D 01 +  0.00000000000000000D 00 I
  D( 2) = -0.29999999999999993D 01 +  0.60000000000000005D 01 I
  D( 3) = -0.20000000000000005D 02 + -0.18999999999999999D 02 I
  D( 4) =  0.41000000000000003D 02 + -0.22000000000000008D 02 I
  D( 5) =  0.23000000000000000D 02 +  0.41000000000000009D 02 I
```

Exhibit A.2. Roots Are: $-1 - 4i$, $-2 - 3i$, $-3 - 3i$, $-1 - i$, $2 + 2i$, $4 - i$, $2 - i$.

```
NEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS
POLYNOMIAL NUMBER  1 OF DEGREE  7


THE COEFFICIENTS OF P(X) ARE

  P( 1) =  0.1000000000000000D 01 +  0.0000000000000000D 00 I
  P( 2) = -0.1000000000000000D 01 +  0.1100000000000000D 02 I
  P( 3) = -0.5900000000000001D 02 + -0.2900000000000000D 02 I
  P( 4) =  0.1950000000000000D 03 + -0.1690000000000000D 03 I
  P( 5) =  0.7000000000000001D 02 +  0.7230000000000000D 03 I
  P( 6) = -0.1624000000000000D 04 + -0.6960000000000001D 03 I
  P( 7) =  0.1922000000000000D 04 + -0.1832000000000000D 04 I
  P( 8) =  0.1596000000000000D 04 +  0.1692000000000000D 04 I


NUMBER OF INITIAL APPROXIMATIONS GIVEN.    2
MAXIMUM NUMBER OF ITERATIONS.            200
TEST FOR CONVERGENCE.                0.10D-09
TEST FOR MULTIPLICITIES.             0.10D-01
RADIUS TO START SEARCH.              0.70D 01
RADIUS TO END SEARCH.                0.15D 02


BEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                   MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) = -0.2999999999999997D 01 + -0.3000000000000002D 01 I        1        -0.3500000000000000D 01 + -0.3500000000000000D 01 I
  ROOT( 2) =  0.2000000000000000D 01 +  0.2000000000000000D 01 I        1         0.2500000000000000D 01 +  0.2500000000000000D 01 I
  ROOT( 3) =  0.4000000000000001D 01 + -0.1000000000000001D 01 I        1         0.6761480761918791D 01 +  0.1811733398213462D 01 I
  ROOT( 4) =  0.1999999999999997D 01 + -0.9999999999999976D 00 I        1         0.5303300664784760D 01 +  0.5303301053013447D 01 I
  ROOT( 5) = -0.9999999999999998D 00 + -0.1000000000000004D 01 I        1         0.2070551889415497D 01 +  0.7727406736625032D 01 I
  ROOT( 6) = -0.9999999999999976D 00 + -0.3999999999999995D 01 I        1         SOLVED BY DIRECT METHOD
  ROOT( 7) = -0.2000000000000004D 01 + -0.3000000000000001D 01 I        1         SOLVED BY DIRECT METHOD


AFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS OF P(X) ARE


 ROOTS OF P(X)                                                   MULTIPLICITIES              INITIAL APPROXIMATION

  ROOT( 1) = -0.2999999999999996D 01 + -0.3000000000000001D 01 I        1        -0.3500000000000000D 01 + -0.3500000000000000D 01 I
  ROOT( 2) =  0.2000000000000000D 01 +  0.2000000000000000D 01 I        1         0.2500000000000000D 01 +  0.2500000000000000D 01 I
  ROOT( 3) =  0.4000000000000001D 01 + -0.1000000000000001D 01 I        1         0.6761480761918791D 01 +  0.1811733398213462D 01 I
  ROOT( 4) =  0.2000000000000000D 01 + -0.9999999999999998D 00 I        1         0.5303300664784760D 01 +  0.5303301053013447D 01 I
  ROOT( 5) = -0.9999999999999997D 00 + -0.1000000000000000D 01 I        1         0.2070551889415497D 01 +  0.7727406736625032D 01 I
  ROOT( 6) = -0.9999999999999982D 00 + -0.4000000000000000D 01 I        1         SOLVED BY DIRECT METHOD
  ROOT( 7) = -0.2000000000000003D 01 + -0.3000000000000007D 01 I        1         SOLVED BY DIRECT METHOD
```

Exhibit A.3. Roots Are: $-1 - 4i$, $-2 - 3i$, $-3 - 3i$, $-1 - i$, $2 + 2i$, $4 - i$, $2 - i$.

APPENDIX B

NEWTON'S METHOD

1. Use of the Program

A double precision FORTRAN IV program using Newton's method is presented here. Flow charts for this program are given in Figure B.6 while Table B.VIII gives a FORTRAN IV listing of this program. Single precision variables are listed in some of the tables. The single precision variables are used in the flow charts and the corresponding double precision variables can be obtained from the appropriate tables.

The program is designed to solve polynomials of degree 25 or less. Both the coefficient of the highest degree term and the constant coefficient should be non-zero. In order to solve polynomials of degree N, where N > 25, certain array dimensions must be changed. These are listed in Table B.I for the main program and subprograms in double precision.

TABLE B.I

PROGRAM CHANGES FOR SOLVING POLYNOMIALS
OF DEGREE GREATER THAN 25
BY NEWTON'S METHOD

Double Precision

Main Program

RA(N+1), VA(N+1)
RB(N+1), VB(N+1)
RC(N+1), VC(N+1)
RD(N+1), VD(N+1)
RCOEF(N+1), VCOEF(N+1)
MULT(N)
RXZERO(N), VXZERO(N)
RX(N), VX(N)
RXINIT(N), VXINIT(N)

Subroutine HORNER

RA(N+1), VA(N+1)
RB(N+1), VB(N+1)
RC(N+1), VC(N+1)

Subroutine BETTER

RXZERO(N), VXZERO(N)
RX(N), VX(N)
RA(N+1), VA(N+1)
RCOEF(N+1), VCOEF(N+1)
RC(N+1), VC(N+1)
RB(N+1), VB(N+1)

Subroutine GENAPP

APPR(N), APPI(N)

Subroutine QUAD

UA(N+1), VA(N+1)
UROOT(N), VROOT(N)
MULTI(N)

Table B.II lists the system functions used in the program of Newton's method. In the table "d" denotes a double precision variable name.

TABLE B.II

SYSTEM FUNCTIONS USED IN NEWTON'S METHOD

Double Precision	
DABS(d)	- obtain absolute value
DCOS(d)	- obtain cosine of angle
DSIN(d)	- obtain sine of angle
DATAN2(d_1,d_2)	- arctangent of d_1/d_2
DSQRT(d)	- square root

2. Input Data for Newton's Method

The input data for Newton's method is grouped into polynomial data sets. Each polynomial data set consists of the data for one and only one polynomial. As many polynomials as the user desires may be solved by placing the polynomial data sets one behind the other. Each polynomial data set consists of three kinds of information placed in the following order:

1. Control information.
2. Coefficients of the polynomial.
3. Initial approximations. These may be omitted as described in Appendix A, § 1.

An end card follows the entire collection of data sets. It indicates that there is no more data to follow and terminates execution of the program. This information is displayed in Figure B.1 and described below. For the double precision data, the D-type specification should be used. All data should be right justified. The recommendations given in Table B.III are those found to give best results on the IBM 360/50 computer which has a 32 bit word.

Control Information

The control card is the first card of the polynomial data set and contains the information given in Table B.III. See Figure B.2.

TABLE B.III

CONTROL DATA FOR NEWTON'S METHOD

Variable Name	Card Columns	Description
NOPOLY	c.c. 1-2	Number of the polynomial. Integer. Right justified.
N	c.c. 4-5	Degree of the polynomial. Integer. Right justified.
NIAP	c.c. 7-8	Number of initial approximations to be read. Integer. If no approximations are given, this should be left blank.
MAX	c.c. 19-21	Maximum number of iterations. Integer. Right justified. 200 is recommended.
EPSCNV	c.c. 30-35	Convergence requirement. Double precision. 1.D-10 is recommended.

TABLE B.III (Continued)

Variable Name	Card Columns	Description
EPSQ	c.c. 37-42	Tolerance check for zero (0) in subroutine QUAD. Double precision. Right justify. 1.D-20 is recommended.
EPSMUL	c.c. 44-49	Multiplicity requirement. Double precision. Right justify. 1.D-02 is recommended.
XSTART	c.c. 64-70	Magnitude at which to begin generating initial approximations. Double precision. Right justify. This is a special feature of the program and may be omitted.
XEND	c.c. 72-78	Magnitude at which to end the generating of initial approximations. Double precision. Right justify. This is a special feature of the program and may be omitted.
KCHECK	c.c. 80	This should be left blank.

Coefficients of the Polynomial

The coefficient cards follow the control card. For an N^{th} degree polynomial, N+1 coefficients must be entered one per card. The coefficient of the highest degree term is entered first. For example, if the polynomial $X^5 + 3X^4 + 2X + 5$ were to be solved, the order in which the coefficients would be entered is: 1, 3, 0, 0, 2, 5. Each coefficient is entered, one per card, as described in Table B.IV and illustrated in Figure B.3.

TABLE B.IV

COEFFICIENT DATA FOR NEWTON'S METHOD

Variable Name	Card Columns	Description
RA (A in single precision)	c.c. 1-30	Real part of complex coefficient. Double precision. Right justify. If none, leave blank or enter 0.0D00.
VA (A in single precision)	c.c. 31-60	Imaginary part of complex coefficient. Double precision. Right justify. If none, leave blank or enter 0.0D00.

Initial Approximations

The initial approximation cards follow the set of coefficient cards. The number of initial approximations read must be the number specified on the control card and are entered, one per card, as given in Table B.V and illustrated in Figure B.4.

TABLE B.V

INITIAL APPROXIMATION DATA FOR NEWTON'S METHOD

Variable Name	Card Columns	Description
RXZERO (XZERO in single precision)	c.c. 1-30	Real part of complex number. Double precision. Right justify. If none, leave blank or enter 0.0D00.
VXZERO (XZERO in single precision)	c.c. 31-60	Imaginary part of complex number. Double precision. Right justify. If none, leave blank or enter 0.0D00.

End Card

The end card is the last card of the input data to the program. It indicates that there is no more data to be read. When this card is read, program execution is terminated. This card is described in Table B.VI and illustrated in Figure B.5.

TABLE B.VI

DATA TO END EXECUTION OF NEWTON'S METHOD

Variable Name	Card Columns	Description
KCHECK	c.c. 80	Must contain the number 1. Integer.

3. Variables Used in Newton's Method

The definitions of the major variables used in Newton's method are given in Table B.VII. The symbols used to indicate type are:

R - real variable

I - integer variable

C - complex variable

D - double precision

L - logical variable

A - alphanumeric variable

When two variables are listed, the one on the left is the real part of the corresponding single precision complex variable; the one on the right is the imaginary part. The symbols used to indicate disposition are:

E - entered

R - returned

ECR - entered, changed, and returned

C - variable in common

4. Description of Program Output

The output from Newton's method programs consist of the following information.

The number and degree of the polynomial are printed in the heading (Exhibit 6.1).

The coefficients are printed under the heading "THE COEFFICIENTS OF P(X) ARE." The coefficient of the highest degree term is listed first (Exhibit 6.1).

As an aid to ensure the control information is correct, the number of initial approximations given, maximum number of iterations, test for convergence, test for multiplicities, radius to start search, and radius to end search are printed as read from the control card (Exhibit 6.1).

The zeros found before and after the attempt to improve accuracy are printed. See Appendix A, § 4 for further explanation (Exhibit 6.1).

If not all zeros of the polynomial are found, the coefficients of the remaining unsolved polynomial will be printed, with coefficient of highest degree term first, under the heading "COEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO ZEROS WERE FOUND." See Appendix A, § 6. This is illustrated in Exhibit A.2.

The multiplicity of each zero is given under the title "MULTIPLICITIES" (Exhibit 6.1).

The initial approximation producing convergence to a root is printed to the right of the corresponding root and headed by "INITIAL APPROXIMATION." The initial approximations may be those supplied by the user, or generated by the program, or a combination of both (Exhibit A.3). See Appendix A, § 1 and § 2 for discussion of approximations. The message "SOLVED BY DIRECT METHOD" indicates that the corresponding root or roots was obtained by Subroutine QUAD. See Appendix A, § 5.

If an approximation does not produce convergence within the maximum number of iterations, it is printed under the heading "NO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AFTER XXX ITERATIONS." XXX is replaced by the maximum number of iterations. The type of the approximation, that is, initial approximation or altered approximations is given (Exhibit A.1). See Appendix A, § 1 and § 2 for discussion of approximations.

5. Informative and Error Messages

The output may contain informative or error messages. These are intended as an aid to the user and are described as follows:

"IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(X) = YYY DID NOT CONVERGE THE PRESENT APPROXIMATION AFTER ZZZ ITERATIONS IS PRINTED BELOW." X is the number of the zero, YYY is the value of the zero before the attempt to improve accuracy, ZZZ is the maximum number of iterations. This message indicates that a zero found before attempting to improve accuracy did not converge sufficiently when being used as an initial approximation on the full (undeflated) polynomial. The current approximation is printed in the list of improved zeros. In many cases, this failure to converge is a result of an ill-conditioned polynomial and this current approximation of the root may be better than its approximation before the attempt to improve accuracy. In most cases, the polynomial from which this root was first extracted had fewer multiple roots, due to deflations, than the original polynomial.

"THE VALUE OF THE DERIVATIVE AT XO = XXX IS ZERO."

This message is printed as a result of the value of the derivative of the original polynomial at an approximation, XXX, being zero (0). It occurred in the attempt to improve the accuracy of a zero. The previous message is then printed.

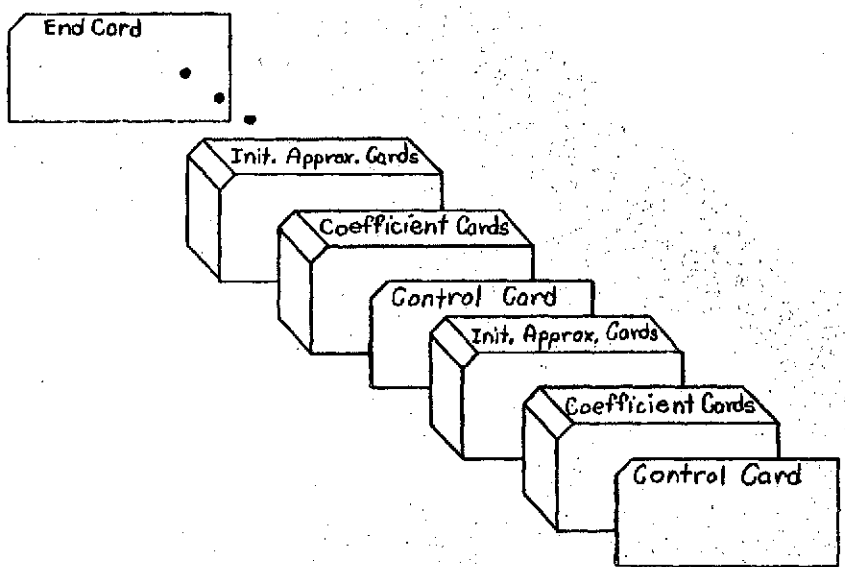


Figure B.1. Sequence of Input Data for Newton's Method

Card Columns	01–02	03	04–05	06	07–08	09–18	19–21	22–29	30–35	36	37–42	43	44–49	50–63	64–70	71	72–79	80
Variable Name	NOPOLY		N		NIAP		MAX		EPSCNV		ESPQ		EPSMUL		XSTART		XEND	KCHECK
Example	1		7		7		200		1.D-10		1.D-20		1.D-02		1.0D+01		5.0D+02	

Figure B.2. Control Card for Newton's Method

000000000111111111122222222223 123456789012345678901234567890	333333333444444444455555555556 123456789012345678901234567890	66666666677777777778 12345678901234567890
A (RA)	A (VA)	
0.621735D+01	-0.132714D-02	

Figure B.3. Coefficient Card for Newton's Method

000000000111111111122222222223 123456789012345678901234567890	333333333444444444455555555556 123456789012345678901234567890	66666666677777777778 12345678901234567890
XZERO (RXZERO)	XZERO (VXZERO)	
0.15D+01	-0.25D-00	

Figure B.4. Initial Approximation Card for Newton's Method

0000000001111111111222222222233333333334444444444555555555566666666667777777777 1234567890123456789012345678901234567890123456789012345678901234567890123456789	8 0
	KCHECK
	1

Figure B.5. End Card for Newton's Method

TABLE B. VII

VARIABLES USED IN NEWTON'S METHOD

Single Precision Variable	Single Precision Type	Double Precision Variable	Double Precision Type	Disposition of Argument	Description
					Main Program
NOPOLY	I	NOPOLY	I		Number of the polynomial
N	I	N	I		Degree of the polynomial
NIAP	I	NIAP	I		Number of initial approximations to be read
MAX	I	MAX	I		Maximum number of iterations to be performed
EPSCNV	R	EPSCNV	D		Tolerance check for convergence
EPSMUL	R	EPSMUL	D		Tolerance check for multiplicities
EPSQ	R	EPSQ	D		Tolerance check for zero in subroutine QUAD
XSTART	R	XSTART	D		Magnitude from which to begin the search for zeros
XEND	R	XEND	D		Magnitude to end the search for zeros
KCHECK	I	KCHECK	I		Program Control. When KCHECK = 1, program will terminate execution.
NA	I	NA	I		Number of coefficients or original polynomial
A	C	RA,VA	D		Array containing the coefficients of original polynomial P(X)
NDEF	I	NDEF	I		Degree of current deflated polynomial
L	I	L	I		Counter for number of initial approximations used
ITER	I	ITER	I		Counter for number of iterations
NROOT	I	NROOT	I		Counter for number of roots found (counting multiplicities)
IALTER	I	IALTER	I		Counter for number of alterations of each initial approximation
ITIME	I	ITIME	I		Program control
K	I	K	I		Counter for number of distinct roots found
ND	I	ND	I		Program control & number of coefficient of deflated polynomial for which no zeros were found

TABLE B. VII (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
XO	C	RXO,VXO	D		Current approximation (X_n) to root
COEF	C	RCOEF,VCOEF	D		Working array containing coefficients of current deflated polynomial
DPX	C	RDPX,VDPX	D		Derivative of P(X) at some value X
PX	C	RPX,VPX	D		Value of P(X) at some point X
XZERO	C	RXZERO, VXZERO	D		Array containing the initial approximations
XNEW	C	RXNEW,VXNEW	D		New approximation (X_{n+1}) obtained from old approximation (X_n) by Newton's Algorithm
KANS	I	KANS	I		KANS = 1 implies convergence, KANS = 0 implies no convergence
MULT	I	MULT	I		Array containing the number of multiplicities of each root
X	C	RX,VX	D		Array containing the zeros of P(X)
XINIT	C	RXINIT, VXINIT	D		Array containing the initial or altered approximations which produced convergence to each root
NUM	I	NUM	I		Number of coefficients of current deflated polynomial
B	C	RB,VB	D		Array containing the coefficients of newly deflated polynomial
IROOT	I	IROOT	I		Number of distinct roots found by Newton's method, i.e. not solved for directly by subroutine QUAD
D	C	RD,VD	D		Array containing the coefficients of deflated polynomial for which no zeros were found
IO1	I	IO1	I		Unit number of input device
IO2	I	IO2	I		Unit number of output device
C	C	RC,VC	D		Array containing sequence of values leading to the derivative
EPSCHK	R	EPSCHK	D		Current tolerance for checking convergence or multiplicity

TABLE B. VII (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine HORNER
A	C	RA,VA	D	E	Array of coefficients of polynomial
B	C	RB,VB	D	R	Array of coefficients of deflated polynomial
NDEF	I	NDEF	I	E	Degree of polynomial
NUM	I	NUM	I		Number of coefficients of polynomial
XO	C	RXO,VXO	D	E	Point (X_n) at which to evaluate the polynomial and its derivative. Also current approximation (X_{n+1}) used to deflate the polynomial
PX	C	RPX,VPX	D	R	Value of polynomial at X_n
DPX	C	RDPX,VDPX	D	R	Value of the derivative of polynomial at X_n
C	C	RC,VC	D	R	Array of containing sequence of values leading to the derivative
					Subroutine NEWTON
PX	C	RPX,VPX	D	E	Value of polynomial at X_n
DPX	C	RDPX,VDPX	D	E	Derivative of polynomial at X_n
XO	C	RXO,VXO	D	E	Current approximation (X_n) to root
XNEW	C	RXNEW,VXNEW	D	R	New approximation (X_{n+1}) to root
					Subroutine CHECK
EPSLON	R	EPS	D	C	Tolerance for convergence or multiplicity check
PX	C	RPX,VPX	D	E	Value of P(X) at X_n
DPX	C	RDPX,VDPX	D	E	Derivative of P(X) at X_n
XO	C	RXO,VXO	D	E	Current approximations (X_{n+1}) to root
IO2	I	IO2	I	C	Unit number of output device
KANS	I	KANS	I	R	KANS = 1 implies convergence, KANS = 0 implies no convergence

TABLE B. VII (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine BETTER
IO2	I	IO2	I	C	Unit number of output device
XZERO	C	RXZERO, VXZERO	D	E	Array of approximations
X	C	RX,VX	D	ECR	Array of roots
A	C	RA,VA	D	E	Coefficients of original (undeflated) polynomial, P(X)
COEF	C	RCOEF,VCOEF	D	E	Working array for coefficients of polynomial
NA	I	NA	I	E	Number of coefficients of original polynomial
XO	C	RXO,VXO	D		Current approximation (X_n) to root
DPX	C	RDPX,VDPX	D		Derivative of P(X) at X_n
PX	C	RPX,VPX	D		Value of P(X) at X_n
KANS	I	KANS	I		KANS = 1 implies convergence; KANS = 0 implies no convergence
ITER	I	ITER	I		Counter for number of iterations
XNEW	C	RXNEW,VXNEW	D		New approximation (X_{n+1}) to root
NN	I	NN	I		Degree of polynomial
C	C	RC,VC	D	E	Array containing the sequence of values leading to the derivative
K	I	K	I	E	Number of distinct roots of P(X) found
N	I	N	I	E	Degree of polynomial P(X)
B	C	RB,VB	D	E	Array of coefficients of deflated polynomial
MAX	I	MAX	I	C	Maximum number of iterations permitted
EPSCHK	R	EPS	D	C	Tolerance for checking convergence
					Subroutine GENAPP
APP	C	APPR,APPI	D	R	Array containing initial approximations
NAPP	I	NAPP	I	E	Number of initial approximations to be generated

TABLE B. VII (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
XSTART	R	XSTART	D	ECR	Magnitude at which to begin generating approximations; also magnitude of the approximation being generated
BETA	R	BETA	D		Argument of the complex approximation being generated
U	R	APPR(I)	D		Real part of complex approximation
V	R	APPI(I)	D		Imaginary part. of complex approximation
					Subroutine ALTER
XOLD	C	XOLDR,XOLDI	D	ECR	Old approximation to be altered to new approximation
NALTER	I	NALTER	I	ECR	Number of alterations performed on an initial approximation
ITIME	I	ITIME	I	E	Program control
MAX	I	MAX	I	C	Maximum number of iterations permitted
Y	R	XOLDI	D		Imaginary part of original initial approximation (unaltered)
X	R	XOLDR	D		Real part of original unaltered initial approximation
R	R	R	D		Magnitude of original unaltered initial approximation
BETA	R	BETA	D		Argument of new approximation
XOLDR	R	XOLDR	D		Real part of new approximation
XOLDI	R	XOLDI	D		Imaginary part of new approximation
IO2	I	IO2	I	C	Unit number of output device
					Subroutine QUAD
A	C	UA,VA	D	E	Coefficients of polynomial to be solved
NA	I	NA	I	E	Degree of polynomial
ROOT	C	UROOT,VROOT	D	ECR	Array of roots of P(X) (original polynomial)
NROOT	I	NROOT	I	ECR	Number of distinct roots of P(X) (the original polynomial)

TABLE B. VII (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
MULTI	I	MULTI	I	ECR	Array containing multiplicities of each root
EPST	R	EPST	D	E	Tolerance check for the number zero
DISC	C	UDISC,VDISC	D		Value of the discriminate ($b^2 - 4ac$) of Quadratic

Subroutine COMSQT

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
		UX,VX	D	E	Complex number for which the square root is desired
		UY,VY	D	R	Square root of the complex number

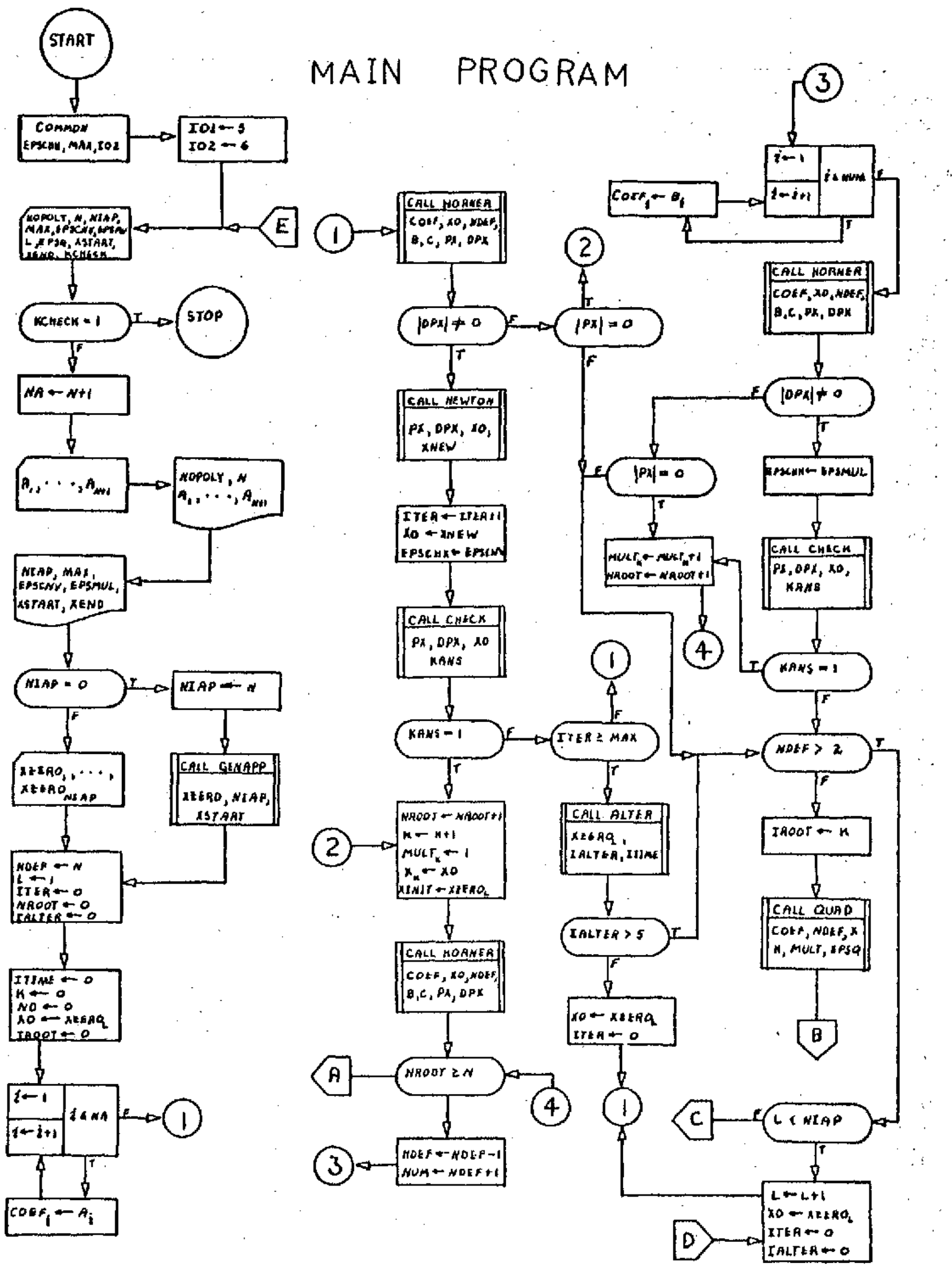


Figure B.6. Flow Charts for Newton's Method

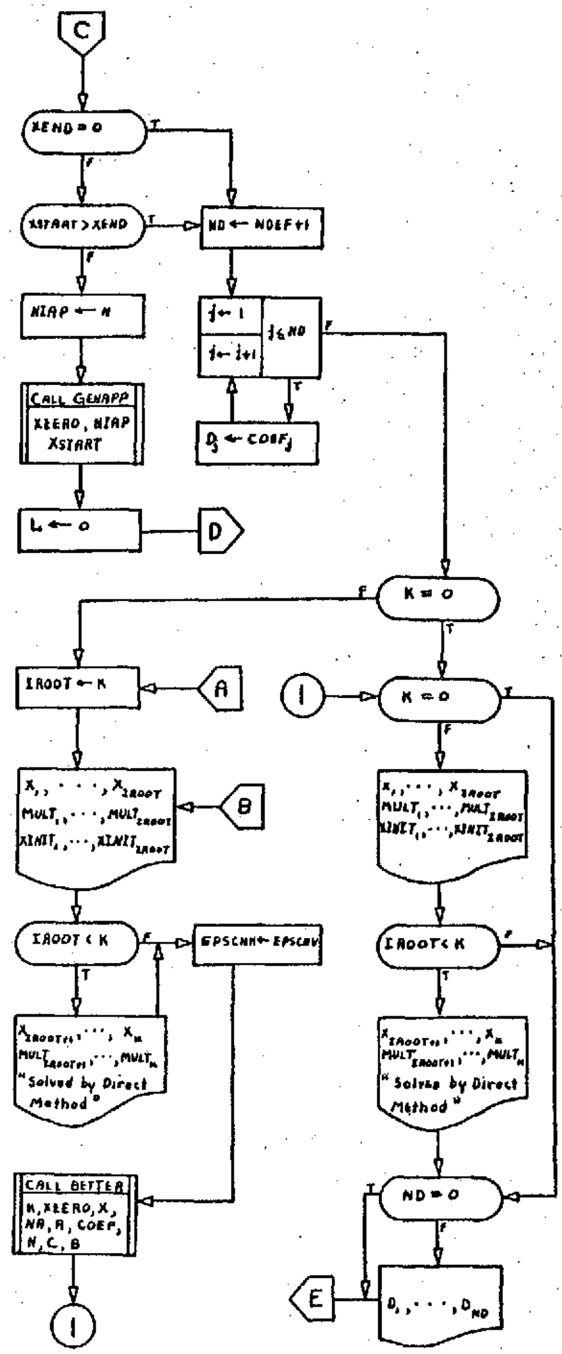


Figure B.6. (Continued)

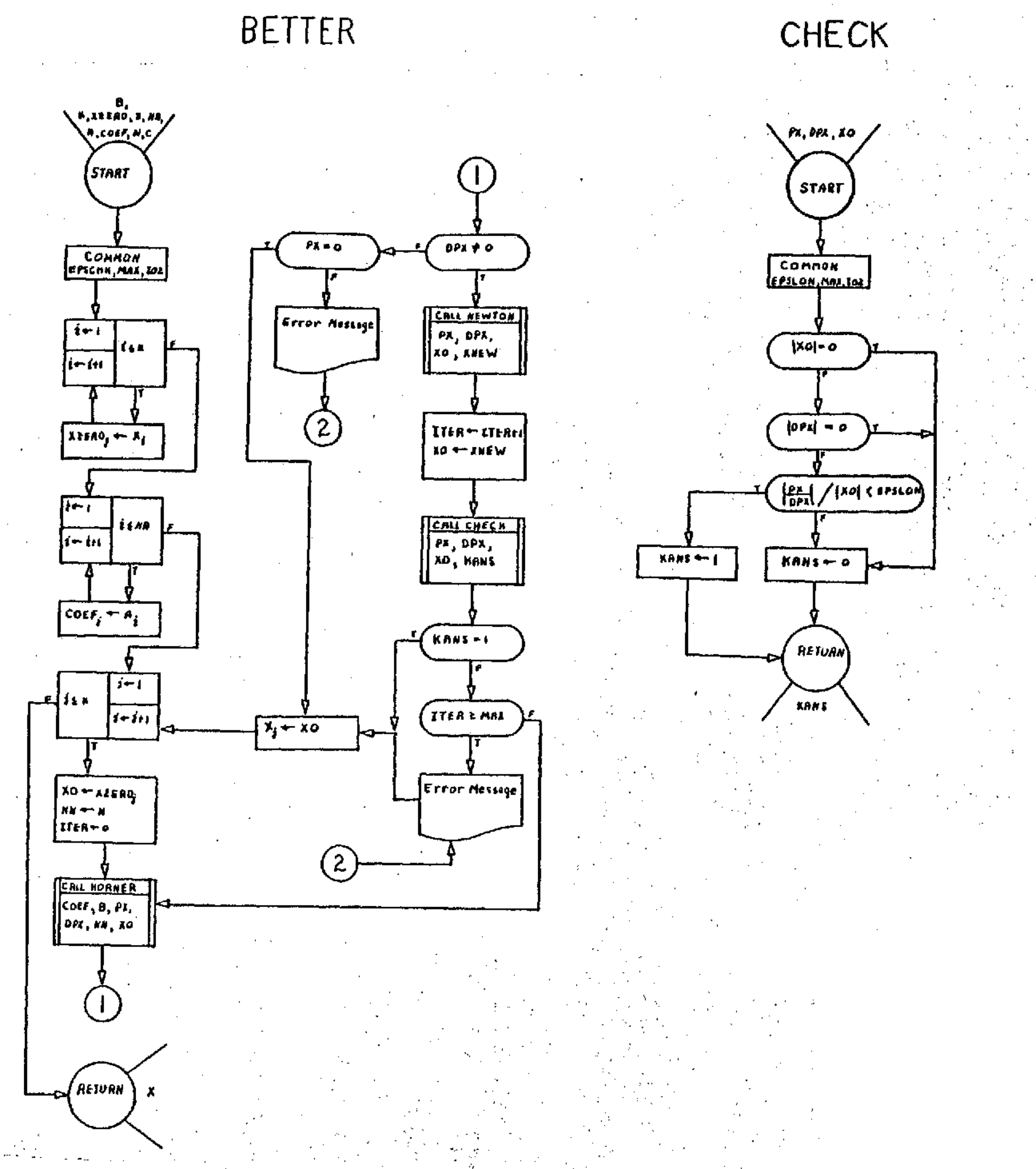
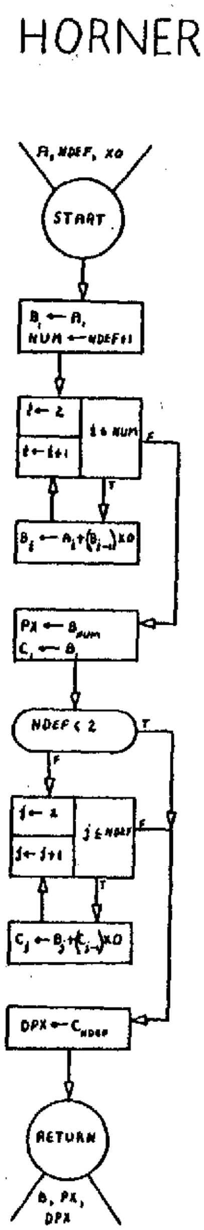


Figure B.6. (Continued)



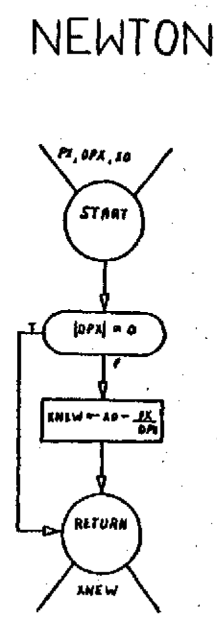
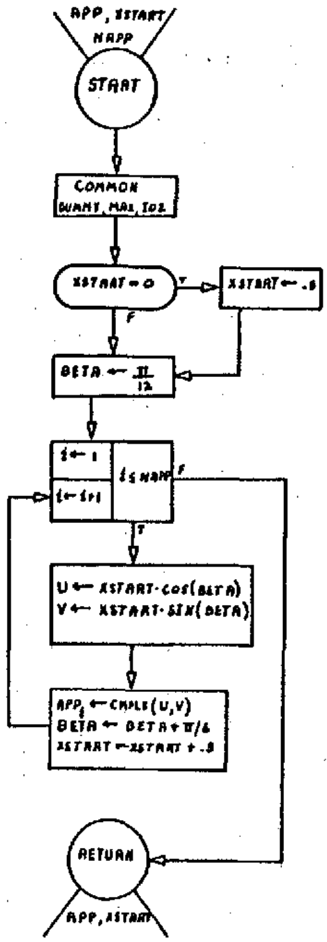


Figure B.6. (Continued)

GENAPP



ALTER

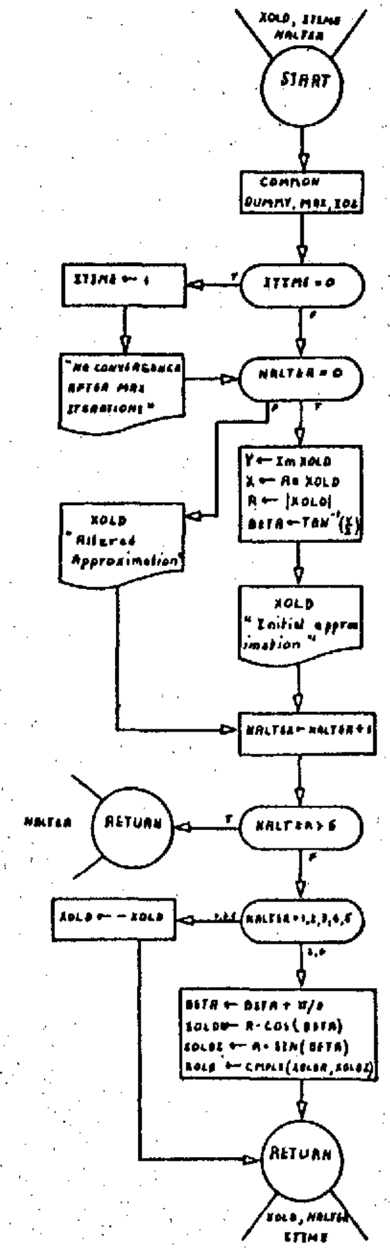
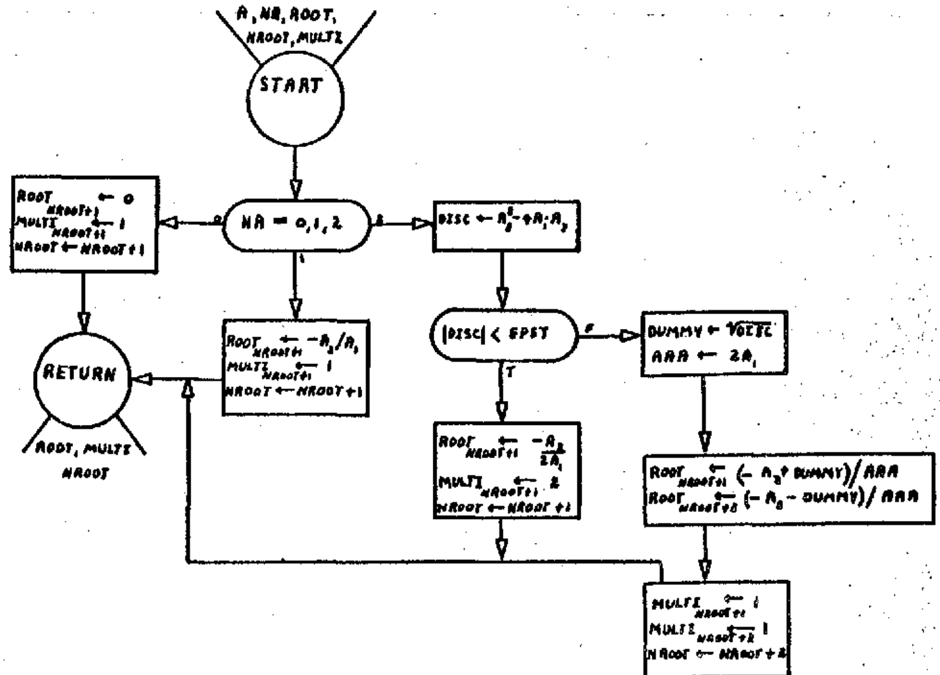


Figure B.6. (Continued)

QUAD



COMSQT

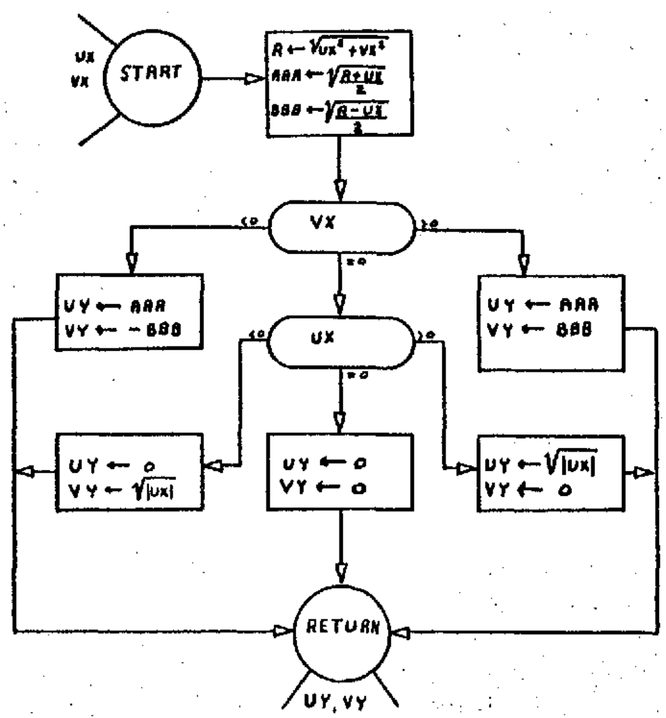


Figure B.6. (Continued)

TABLE B. VIII

PROGRAM FOR NEWTON'S METHOD

```
C     ***********************************************************************
C     *                                                                     *
C     * DOUBLE PRECISION PROGRAM FOR NEWTON'S METHOD                        *
C     *                                                                     *
C     *                                                                     *
C     * NEWTONS METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A     *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25 BY COMPUTING A SEQUENCE OF APPROX-  *
C     * IMATIONS CONVERGING TO A ZERO OF THE POLYNOMIAL USING THE ITERATION *
C     * FORMULA                                                             *
C     *              X(N+1) = X(N)-P(X(N))/P'(X(N)).                        *
C     *                                                                     *
C     ***********************************************************************
       DOUBLE PRECISION RA,VA,RXZERO,VXZERO,RB,VB,RCOEF,VCOEF,RX,VX,RXINI
      1T,VXINIT,RC,VC,RD,VD,RPX,VPX,RDPX,VDPX,RXNEW,VXNEW,RXO,VXO,EPSCHK,
      2EPSCNV,EPSQ,EPSMUL,XSTART,XEND,ABPX,ABDPX
       DIMENSION RA(26),VA(26),RB(26),VB(26),RC(26),VC(26),RD(26),VD(26),
      1RCOEF(26),VCOEF(26),MULT(25),RXZERO(25),VXZERO(25),RX(25),VX(25),R
      2XINIT(25),VXINIT(25)
       COMMON EPSCHK,MAX,IO2
       IO1=5                                                              112
       IO2=6                                                              116
     1 READ(IO1,1000) NOPOLY,N,NIAP,MAX,EPSCNV,EPSQ,EPSMUL,XSTART,XEND,KC
      1HECK
       IF(KCHECK.EQ.1) STOP
       NA=N+1                                                             130
       READ(IO1,1010) (RA(I),VA(I),I=1,NA)                                132
       WRITE(IO2,1030) NOPOLY,N
       WRITE(IO2,1040) (I,RA(I),VA(I),I=1,NA)
       WRITE(IO2,2060)
       WRITE(IO2,2000) NIAP
       WRITE(IO2,2010) MAX
       WRITE(IO2,2020) EPSCNV
       WRITE(IO2,2030) EPSMUL
       WRITE(IO2,2040) XSTART
       WRITE(IO2,2050) XEND
       IF(NIAP.NE.0) GO TO 3
       NIAP=N
       CALL GENAPP(RXZERO,VXZERO,NIAP,XSTART)
       GO TO 4
     3 READ(IO1,1020) (RXZERO(I),VXZERO(I),I=1,NIAP)
     4 NDEF=N
       L=1                                                                152
       ITER=0                                                             154
       NROOT=0                                                            160
       IROOT=0
       ITIME=0
       ND=0
       IALTER=0                                                           164
       K=0                                                                168
       RXO=RXZERO(L)                                                      180
       VXO=VXZERO(L)                                                      181
       DO 5 I=1,NA                                                        188
       RCOEF(I)=RA(I)                                                     190
     5 VCOEF(I)=VA(I)                                                     191
    10 CALL HORNER(RCOEF,VCOEF,RXO,VXO,NDEF,RB,VB,RC,VC,RPX,VPX,RDPX,VDPX
      1)
       ABPX=DSQRT(RPX*RPX+VPX*VPX)
       ABDPX=DSQRT(RDPX*RDPX+VDPX*VDPX)
```

TABLE B. VIII (Continued)

```
      IF(ABDPX.NE.0.0) GO TO 20
      IF(ABPX.EQ.0.0) GO TO 70
      GO TO 110
   20 CALL NEWTON(RPX,VPX,RDPX,VDPX,RXO,VXO,RXNEW,VXNEW)
      ITER=ITER+1                                                      200
      RXO=RXNEW                                                        204
      VXO=VXNEW                                                        205
      EPSCHK=EPSCNV
      CALL CHECK(RPX,VPX,RDPX,VDPX,RXO,VXO,KANS)
      IF(KANS.EQ.1) GO TO 70                                           194
      IF(ITER.GE.MAX) GO TO 40
      GO TO 10                                                         208
   40 CALL ALTER(RXZERO(L),VXZERO(L),IALTER,ITIME)
      IF(IALTER.GT.5) GO TO 110
      RXO=RXZERO(L)
      VXO=VXZERO(L)
      ITER=0                                                           244
      GO TO 10                                                         248
   60 ND=NDEF+1
      DO 65 J=1,ND
      RD(J)=RCOEF(J)
   65 VD(J)=VCOEF(J)
      GO TO 140
   70 NROOT=NROOT+1                                                    268
      K=K+1                                                            272
      MULT(K)=1                                                        276
      RX(K)=RXO                                                        280
      VX(K)=VXO                                                        281
      RXINIT(K)=RXZERO(L)                                              288
      VXINIT(K)=VXZERO(L)                                              289
      CALL HORNER(RCOEF,VCOEF,RXO,VXO,NDEF,RB,VB,RC,VC,RPX,VPX,RDPX,VDPX
     1)
   80 IF(NROOT.GE.N) GO TO 147
      NDEF=NDEF-1
      NUM=NDEF+1
      DO 105 I=1,NUM                                                   294
      RCOEF(I)=RB(I)                                                   296
  105 VCOEF(I)=VB(I)                                                   297
      CALL HORNER(RCOEF,VCOEF,RXO,VXO,NDEF,RB,VB,RC,VC,RPX,VPX,RDPX,VDPX
     1)
      ABPX=DSQRT(RPX*RPX+VPX*VPX)
      ABDPX=DSQRT(RDPX*RDPX+VDPX*VDPX)
      IF(ABDPX.NE.0.0) GO TO 107
      IF(ABPX.EQ.0.0) GO TO 130
      GO TO 110
  107 CONTINUE
      EPSCHK=EPSMUL
      CALL CHECK(RPX,VPX,RDPX,VDPX,RXO,VXO,KANS)
      IF(KANS.EQ.1) GO TO 130                                          300
  110 IF(NDEF.GT.2) GO TO 113
      IROOT=K
      CALL QUAD(RCOEF,VCOEF,NDEF,RX,VX,K,MULT,EPSQ)
      GO TO 150
  113 IF(L.LT.NIAP) GO TO 115
      IF(XEND.EQ.0.0) GO TO 60
      IF(XSTART.GT.XEND) GO TO 60
      NIAP=N
      CALL GENAPP(RXZERO,VXZERO,NIAP,XSTART)
```

TABLE B. VIII (Continued)

```
      L=0
  115 L=L+1
      RXO=RXZERO(L)                                                      312
      VXO=VXZERO(L)                                                      313
      ITER=0                                                             316
      IALTER=0                                                           320
      GO TO 10                                                           324
  130 MULT(K)=MULT(K)+1                                                  328
      NROOT=NROOT+1                                                      332
      GO TO 80                                                           336
  140 IF(K.EQ.0) GO TO 160                                               338
  147 IROOT=K
  150 WRITE(IO2,1025)
      WRITE(IO2,1050)
      WRITE(IO2,1060) (I,RX(I),VX(I),MULT(I),RXINIT(I),VXINIT(I),I=1,IRO
     1OT)
      KKK=IROOT+1
      IF(IROOT.LT.K) WRITE(IO2,1062) (I,RX(I),VX(I),MULT(I),I=KKK,K)
      EPSCHK=EPSCNV
      CALL BETTER(K,RXZERO,VXZERO,RX,VX,NA,RA,VA,RCOEF,VCOEF,N,RC,VC,RB,
     1VB)
  160 IF(K.EQ.0) GO TO 170
      WRITE(IO2,1065)
      WRITE(IO2,1050)
      WRITE(IO2,1060) (I,RX(I),VX(I),MULT(I),RXINIT(I),VXINIT(I),I=1,IRO
     1OT)
      KKK=IROOT+1
      IF(IROOT.LT.K) WRITE(IO2,1062) (I,RX(I),VX(I),MULT(I),I=KKK,K)
  170 IF(ND.EQ.0) GO TO 1
      WRITE(IO2,1070)                                                    396
      WRITE(IO2,1075) (J,RD(J),VD(J),J=1,ND)
      GO TO 1                                                            444
 1000 FORMAT(3(I2,1X),9X,I3,8X,3(D6.0,1X),13X,2(D7.0,1X),I1)
 1010 FORMAT(2D30.0)
 1030 FORMAT(1H1,8X,43HNEWTONS METHOD TO FIND ZEROS OF POLYNOMIALS/9X,18
     1HPOLYNOMIAL NUMBER ,I2,11H OF DEGREE ,I2,////1X,28HTHE COEFFICIENT
     2S OF P(X) ARE/)
 1040 FORMAT(3X,2HP(,I2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1020 FORMAT(2D30.0)
 1025 FORMAT(///1X,61HBEFORE THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS
     1OF P(X) ARE)
 1050 FORMAT(///2X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
     1 APPROXIMATION//)
 1060 FORMAT(3X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,7X,D23.
     116,3H + ,D23.16,2H I)
 1062 FORMAT(3X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,8X,23HS
     1OLVED BY DIRECT METHOD)
 1065 FORMAT(///1X,61HAFTER  THE ATTEMPT TO IMPROVE ACCURACY, THE ZEROS
     1OF P(X) ARE)
 1070 FORMAT(///1X,65HCOEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO Z
     1EROS WERE FOUND/)
 1075 FORMAT(3X,2HD(,I2,4H) = ,D23.16,3H + ,D23.16,2H I)
 2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.  ,I2)
 2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
 2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
 2030 FORMAT(1X,24HTEST FOR MULTIPLICITIES.,10X,D9.2)
 2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
 2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
 2060 FORMAT(//1X)
      END                                                                450
```

TABLE B. VIII (Continued)

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE *
C     * DEGREE OF THE ORIGINAL POLYNOMIAL.                                   *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,DUMMY,BETA
      DIMENSION APPR(25),APPI(25)
      COMMON DUMMY,MAX,IO2
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I)=XSTART*DCOS(BETA)
      APPI(I)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      RETURN
      END


      SUBROUTINE ALTER(XOLDR,XOLDI,NALTER,ITIME)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO  *
C     * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION XOLDR,XOLDI,DUMMY,ABXOLD,BETA
      COMMON DUMMY,MAX,IO2
      IF(ITIME.NE.0) GO TO 5
      ITIME =1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) XOLDR,XOLDI
      GO TO 20
   10 ABXOLD=DSQRT(XOLDR*XOLDR+XOLDI*XOLDI)
      BETA=DATAN2(XOLDI,XOLDR)
      WRITE(IO2,1020) XOLDR,XOLDI
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 XOLDR=-XOLDR
      XOLDI=-XOLDI
      GO TO 50
   40 BETA=BETA+1.0471976
      XOLDR=ABXOLD*DCOS(BETA)
      XOLDI=ABXOLD*DSIN(BETA)
   50 RETURN
 1000 FORMAT(1X,D23.16,3H + ,D23.16,2H I,10X,21HALTERED APPROXIMATION)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     1TER ,I3,12H ITERATIONS.//)
 1020 FORMAT(/1X,D23.16,3H + ,D23.16,2H I,10X,21HINITIAL APPROXIMATION)
      END
```

TABLE B. VIII (Continued)

```
      SUBROUTINE QUAD(UA,VA,NA,UROOT,VROOT,NROOT,MULTI,EPST)
C     **********************************************************************
C     *                                                                    *
C     * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES  *
C     * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE   *
C     * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                     *
C     *                                                                    *
C     **********************************************************************
      DOUBLE PRECISION UA,VA,UROOT,VROOT,BBB,UAAA,VAAA,UDISC,VDISC,UDUMM
     1Y,VDUMMY,RDUMMY,SDUMMY,EPST,UBBB,VBBB
      DIMENSION UA(26),VA(26),UROOT(25),VROOT(25),MULTI(25)
      IF(NA.EQ.2) GO TO 7
      IF(NA.EQ.1) GO TO 5
      UROOT(NROOT+1)=0.0
      VROOT(NROOT+1)=0.0
      MULTI(NROOT+1)=1
      NROOT=NROOT+1
      GO TO 50
    5 BBB=UA(1)*UA(1)+VA(1)*VA(1)
      UROOT(NROOT+1)=(-UA(2)*UA(1)-VA(2)*VA(1))/BBB
      VROOT(NROOT+1)=(-VA(2)*UA(1)+UA(2)*VA(1))/BBB
      MULTI(NROOT+1)=1
      NROOT=NROOT+1
      GO TO 50
    7 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(1)*UA(3)-VA(1)*VA(3)))
      VDISC=(VA(2)*UA(2)+UA(2)*VA(2))-(4.0*(VA(1)*UA(3)+UA(1)*VA(3)))
      BBB=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(BBB.LT.EPST) GO TO 10
      CALL COMSQT(UDISC,VDISC,UDUMMY,VDUMMY)
      UBBB=-UA(2)+UDUMMY
      VBBB=-VA(2)+VDUMMY
      RDUMMY=-UA(2)-UDUMMY
      SDUMMY=-VA(2)-VDUMMY
      UAAA=2.0*UA(1)
      VAAA=2.0*VA(1)
      BBB=UAAA*UAAA+VAAA*VAAA
      UROOT(NROOT+1)=(UBBB*UAAA+VBBB*VAAA)/BBB
      VROOT(NROOT+1)=(VBBB*UAAA-UBBB*VAAA)/BBB
      UROOT(NROOT+2)=(RDUMMY*UAAA+SDUMMY*VAAA)/BBB
      VROOT(NROOT+2)=(SDUMMY*UAAA-RDUMMY*VAAA)/BBB
      MULTI(NROOT+1)=1
      MULTI(NROOT+2)=1
      NROOT=NROOT+2
      GO TO 50
   10 UAAA=2.0*UA(1)
      VAAA=2.0*VA(1)
      BBB=UAAA*UAAA+VAAA*VAAA
      UROOT(NROOT+1)=(-UA(2)*UAAA-VA(2)*VAAA)/BBB
      VROOT(NROOT+1)=(-VA(2)*UAAA+UA(2)*VAAA)/BBB
      MULTI(NROOT+1)=2
      NROOT=NROOT+1
   50 RETURN
      END
```

TABLE B. VIII (Continued)

```
      SUBROUTINE COMSQT(UX,VX,UY,VY)
C   ************************************************************************
C   *                                                                      *
C   * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.        *
C   *                                                                      *
C   ************************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

TABLE B. VIII (Continued)

```
      SUBROUTINE HORNER(RA,VA,RXO,VXO,NDEF,RB,VB,RC,VC,RPX,VPX,RDPX,VDPX
C   *********************************************************************
C   *                                                                   *
C   * HORNER'S METHOD COMPUTES THE VALUE OF A POLYNOMIAL P(X) AT A POINT D AND *
C   * ITS DERIVATIVE AT D.  SYNTHETIC DIVISION IS USED TO DEFLATE THE   *
C   * POLYNOMIAL BY DIVIDING OUT THE FACTOR (X-D).                      *
C   *                                                                   *
C   *********************************************************************
     1)
      DOUBLE PRECISION VDPX,RXO,VXO,RB,VB,RC,VC,RPX,VPX,RDPX,RA,VA
      DIMENSION RA(26),VA(26),RB(26),VB(26),RC(26),VC(26)
      RB(1)=RA(1)                                                       516
      VB(1)=VA(1)                                                       517
      NUM=NDEF+1                                                        520
      DO 10 I=2,NUM                                                     524
      RB(I)=RA(I)+(RB(I-1)*RXO-VB(I-1)*VXO)
   10 VB(I)=VA(I)+(VB(I-1)*RXO+RB(I-1)*VXO)
      RPX=RB(NUM)                                                       532
      VPX=VB(NUM)                                                       533
      RC(1)=RB(1)                                                       540
      VC(1)=VB(1)                                                       541
      IF(NDEF.LT.2) GO TO 25
      DO 20 J=2,NDEF                                                    544
      RC(J)=RB(J)+(RC(J-1)*RXO-VC(J-1)*VXO)
   20 VC(J)=VB(J)+(VC(J-1)*RXO+RC(J-1)*VXO)
   25 RDPX=RC(NDEF)
      VDPX=VC(NDEF)                                                     553
      RETURN                                                            572
      END                                                               580
```

```
      SUBROUTINE NEWTON(RPX,VPX,RDPX,VDPX,RXO,VXO,RXNEW,VXNEW)          600
C   *********************************************************************
C   *                                                                   *
C   * THIS SUBROUTINE CALCULATES A NEW APPROXIMATION FROM THE OLD APPROX-  *
C   * IMATION BY USING THE ITERATION FORMULA                            *
C   *                X(N+1) = X(N)-P(X(N))/P'(X(N)).                    *
C   *                                                                   *
C   *********************************************************************
      DOUBLE PRECISION RPX,VPX,RDPX,VDPX,RXO,VXO,RXNEW,VXNEW,ARG
      DOUBLE PRECISION DDD
      ARG=RDPX*RDPX+VDPX*VDPX
      DDD=DSQRT(ARG)
      IF(DDD.EQ.0.0) RETURN
      RXNEW=RXO-((RPX*RDPX+VPX*VDPX)/ARG)
      VXNEW=VXO-((VPX*RDPX-RPX*VDPX)/ARG)
      RETURN                                                            616
      END                                                               620
```

TABLE B. VIII (Continued)

```
      SUBROUTINE CHECK(RPX,VPX,RDPX,VDPX,RXO,VXO,KANS)
C    ***********************************************************************
C    *                                                                     *
C    * THIS SUBROUTINE CHECKS FOR CONVERGENCE OF THE SEQUENCE OF APPROX-   *
C    * IMATIONS BY TESTING THE EXPRESSION                                  *
C    * ABSOLUTE VALUE OF (P(X(N))/P'(X(N)))/ABSOLUTE VALUE OF X(N+1).      *
C    * WHEN IT IS AS SMALL AS DESIRED, CONVERGENCE IS OBTAINED.            *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION RPX,VPX,RDPX,VDPX,RXO,VXO,ABSXO,ABSQUO,RDUMMY,VDU     749
     1MMY,EPS                                                                750
      DOUBLE PRECISION ARG
      DOUBLE PRECISION ODD
      COMMON EPS,MAX,ID2
      ABSXO=DSQRT(RXO*RXO+VXO*VXO)
      IF(ABSXO.EQ.0.) GO TO 25
      ARG=RDPX*RDPX+VDPX*VDPX
      ODD=DSQRT(ARG)
      IF(ODD.EQ.0.0) GO TO 25
      RDUMMY=(RPX*RDPX+VPX*VDPX)/ARG
      VDUMMY=(VPX*RDPX-RPX*VDPX)/ARG
      ABSQUO=DSQRT(RDUMMY*RDUMMY+VDUMMY*VDUMMY)
      IF(ABSQUO/ABSXO.LT.EPS) GO TO 10
      KANS=0                                                                 760
      RETURN                                                                 764
   10 KANS=1                                                                 768
      RETURN                                                                 772
   25 KANS=0
      RETURN
      END                                                                    780
```

TABLE B. VIII (Continued)

```
      SUBROUTINE BETTER(K,RXZERO,VXZERO,RX,VX,NA,RA,VA,RCOEF,VCOEF,N,RC,
     1VC,RB,VB)
C    ***********************************************************************
C    *                                                                     *
C    * SUBROUTINE BETTER ATTEMPTS TO IMPROVE THE ACCURACY OF THE ZEROS FOUND *
C    * BY USING THEM AS INITIAL APPROXIMATIONS WITH NEWTON'S METHOD APPLIED TO *
C    * THE FULL, UNDEFLATED POLYNOMIAL.                                    *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION RXZERO,VXZERO,RX,VX,RA,VA,RCOEF,VCOEF,RC,VC,RB,VB   805
     1,RXO,VXO,RPX,VPX,RDPX,VDPX,RXNEW,VXNEW,EPS
      DIMENSION RXZERO(25),VXZERO(25),RX(25),VX(25),RA(26),VA(26),RCOEF(
     126),VCOEF(26),RC(26),VC(26),RB(26),VB(26)
      DOUBLE PRECISION ABPX,ABDPX
      COMMON EPS,MAX,IO2
      DO 10 I=1,K                                                          812
      RXZERO(I)=RX(I)                                                      815
   10 VXZERO(I)=VX(I)                                                      816
      DO 20 I=1,NA
      RCOEF(I)=RA(I)                                                       824
   20 VCOEF(I)=VA(I)                                                       825
      DO 50 J=1,K                                                          828
      RXO=RXZERO(J)                                                        832
      VXO=VXZERO(J)                                                        833
      NN=N                                                                 834
      ITER=0                                                               836
   30 CALL HORNER(RCOEF,VCOEF,RXO,VXO,NN,RB,VB,RC,VC,RPX,VPX,RDPX,VDPX)
      ABPX=DSQRT(RPX*RPX+VPX*VPX)
      ABDPX=DSQRT(RDPX*RDPX+VDPX*VDPX)
      IF(ABDPX.NE.0.0) GO TO 33
      IF(ABPX.EQ.0.0) GO TO 40
      GO TO 34
   33 CALL NEWTON(RPX,VPX,RDPX,VDPX,RXO,VXO,RXNEW,VXNEW)
      ITER=ITER+1                                                          856
      RXO=RXNEW                                                            860
      VXO=VXNEW                                                            861
      CALL CHECK(RPX,VPX,RDPX,VDPX,RXO,VXO,KANS)
      IF(KANS.EQ.1) GO TO 40                                               844
      IF(ITER.GE.MAX) GO TO 35
      GO TO 30                                                             864
   34 WRITE(IO2,1112) RXO,VXO
   35 WRITE(IO2,100) J,RXZERO(J),VXZERO(J)
      WRITE(IO2,200) MAX
   40 RX(J)=RXO                                                            870
      VX(J)=VXO                                                            871
   50 CONTINUE                                                             872
      RETURN                                                               876
 1112 FORMAT(1H0,36HTHE VALUE OF THE DERIVATIVE AT XO = ,D23.16,3H + ,D2
     13.16,2H I,10H  IS ZERO.)
  100 FORMAT(42H0IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(,I2,4H) = ,D23
     1.16,3H + ,D23.16,2H I,18H DID NOT CONVERGE.)
  200 FORMAT(33H THE PRESENT APPROXIMATION AFTER ,I3,29H ITERATIONS IS P
     1RINTED BELOW.)
      END                                                                  880
```

APPENDIX C

MULLER'S METHOD

2. Use of the Program

A double precision FORTRAN IV program using Muller's method is presented in this appendix. Flow charts for this program are given in Figure C.1 while Table C.V gives a FORTRAN IV listing of this program. Single precision variables are listed in some of the tables. The single precision variables are used in the flow charts and the corresponding double precision variables can be obtained from the appropriate tables.

The program is designed to solve polynomials of degree 25 or less. Both the coefficient of the highest degree term and the constant coefficient should be non-zero. In order to solve polynomials of degree N, where $N > 25$, certain array dimensions must be changed. These are listed in Table C.I for the main program and subprograms in double precision.

TABLE C.I

PROGRAM CHANGES FOR SOLVING POLYNOMIALS OF DEGREE GREATER THAN 25 BY MULLER'S METHOD

Double Precision

Main Program

UROOT(N),VROOT(N)
MULT(N)
UAPP(N,3),VAPP(N,3)
UWORK(N+1),VWORK(N+1)
UB(N+1),VB(N+1)
UA(N+1),VA(N+1)
URAPP(N,3),VRAPP(N,3)

Subroutine BETTER

UROOT(N),VROOT(N)
UA(N+1),VA(N+1)
UBAPP(N,3),VBAPP(N,3)
UB(N+1),VB(N+1)
UROOTS(N),VROOTS(N)
URAPP(N,3),VRAPP(N,3)
MULT(N)

Subroutine GENAPP

APPR(N,3),APPI(N,3)

Subroutine HORNER

UA(N+1),VA(N+1)
UB(N+1),VB(N+1)

Subroutine QUAD

UA(N+1),VA(N+1)
UROOT(N),VROOT(N)
MULTI(N)

Table C.II lists the system functions used in the program of Muller's method. In the table "d" denotes a double precision variable name.

TABLE C.II

SYSTEM FUNCTIONS USED IN MULLER'S METHOD

<u>Double Precision</u>	
DABS(d)	- obtain absolute value
DATAN2(d_1,d_2)	- arctangent of d_1/d_2
DSQRT(d)	- square root
DCOS(d)	- cosine of angle
DSIN(d)	- sine of angle
DSQRT(d)	- square root

2. Input Data for Muller's Method

The input data for Muller's method is identical to the input data for Newton's method as described in Appendix B, § 2 except for the variable names. The correspondence of input variable names is given in Table C.III. Only one (not three) initial approximation, X_0, is given for each root. The other two required by Muller's method are constructed within the program and are $.9X_0$ and $1.1X_0$.

3. Variables Used in Muller's Method

The definitions of the major variables used in Muller's method are given in Table C.IV. For definitions of variables not listed in this table see the definitions of variables for the corresponding subroutine in Table B.VII. The notation and symbols used here are the same as for Table B.VII and are described in Appendix B, § 3.

TABLE C.III

CORRESPONDENCE OF NEWTON'S AND MULLER'S INPUT DATA VARIABLES

Newton's Method	Muller's Method
Control Card	
NOPOLY	NOPOLY
N	NP
NIAP	NAPP
MAX	MAX
EPSCNV	EPS
EPSQ	EPSQ
EPSMUL	EPSM
XSTART	XSTART
XEND	XEND
KCHECK	KCHECK
Coefficient Card	
A (RA)	A (UA)
A (VA)	A (VA)
Initial Approximation Card	
XZERO (RXZERO)	APP (UAPP)
XZERO (VXZERO)	APP (VAPP)
End Card	
KCHECK	KCHECK

4. Description of Program Output

The output from Muller's method is the same as that for Newton's method as described in Apptendix B, § 4. Only one initial approximation, Z, (not three) is printed for each root. It is either that supplied by the user or generated by the program. The other two approximations used were 0.9Z and 1.1Z.

5. Informative and Error Messages

The output may contain informative messages printed as an aid to the user. These are:

"NO ZEROS WERE FOUND FOR POLYNOMIAL NUMBER XX."

XX is the number of the polynomial. This message is printed if no roots of the polynomial were found.

"IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(X) = YYY
DID NOT CONVERGE AFTER ZZZ ITERATIONS
THE PRESENT APPROXIMATION IS AAA"

X is the number of the root before the attempt to improve accuracy, YYY is the value of the root before attempt to improve accuracy, ZZZ is the maximum number of iterations, and AAA is the current approximation after the maximum number of iterations. This message has the same meaning as the corresponding message in Appendix B, § 5.

TABLE C. IV

VARIABLES USED IN MULLER'S METHOD

Single Precision Variable	Single Precision Type	Double Precision Variable	Double Precision Type	Disposition of Argument	Description
					Main Program
NP	I	NP	I		Degree of polynomial P(X)
NROOT	I	NROOT	I		Number of distinct roots found
NOMULT	I	NOMULT	I		Number of roots (counting multiplicities)
ROOT	C	UROOT,VROOT	D		Array containing the roots
NAPP	I	NAPP	I		Number of initial approximations to be read in
APP	C	UAPP,VAPP	D		Array of initial approximations
WORK	C	UWORK,VWORK	D		Working array containing coefficients of current polynomial
B	C	UB,VB	D		Array containing coefficients of deflated polynomial
A	C	UA,VA	D		Array containing coefficients of original polynomial, P(X)
RAPP	C	URAPP,VRAPP	D		Array of initial or altered approximations for which convergence was obtained
X1	C	UX1,VX1	D		One of three current approximations to a root
X2	C	UX2,VX2	D		One of three current approximations to a root
X3	C	UX3,VX3	D		One of three current approximations to a root
PX1	C	UPX1,VPX1	D		Value of polynomial P(X) at X1
PX2	C	UPX2,VPX2	D		Value of polynomial P(X) at X2
PX3	C	UPX3,VPX3	D		Value of polynomial P(X) at X3
X4	C	UX4,VX4	D		Newest approximation (X_{n+1}) to root
PX4	C	UPX4,VPX4	D		Value of polynomial P(X) at X4
MULT	I	MULT	I		Array containing the multiplicities of each root found
ITER	I	ITER	I		Counter for iterations
IO1	I	IO1	I		Unit number of input device
IO2	I	IO2	I		Unit number of output device

TABLE C.IV. (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
EPSRT	R	EPSRT	D		Number used in subroutine BETTER to generate two approximations from the one given
NOPOLY	I	NOPOLY	I		Number of the polynomial
MAX	I	MAX	I		Maximum number of iterations
EPS	R	EPS	D		Tolerance check for convergence
EPSO	R	EPSO	D		Tolerance check for zero (0)
EPSM	R	EPSM	D		Tolerance check for multiplicities
KCHECK	I	KCHECK	I		Program control, KCHECK = 1 stops execution of program
XSTART	R	XSTART	D		Magnitude at which to start generating initial approximations
XEND	R	XEND	D		Magnitude at which to end generating initial approximations
NWORK	I	NWORK	I		Degree of current deflated polynomial whose coefficients are in WORK
ITIME	I	ITIME	I		Program control
NALTER	I	NALTER	I		Number of alterations which have been performed on an initial approximation
IAPP	I	IAPP	I		Counter for number of initial approximations used
CONV	L	CONV	L		When CONV is true, convergence has been obtained
IROOT	I	IROOT	I		Number of distinct roots solved by Muller's method, i.e. not solved directly by subroutine QUAD
					Subroutine HORNER
A	C	UA,VA	D	E	Array of current polynomial coefficients (to be deflated or evaluated)
NA	I	NA	I	E	Degree of polynomial to be deflated or evaluated
X	C	UX,VX	D	E	Approximation at which to evaluate or deflate the polynomial

TABLE C. IV (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
B	C	UB,VB	D	R	Array containing the coefficients of the deflated polynomial
PX	C	UPX,VPX	D	R	Value of the polynomial at X
NUM	I	NUM	I		Number of coefficients of polynomial to be deflated
					Subroutine TEST
X3	C	UX3,VX3	D	E	Approximation to Root (old) (X_n)
X4	C	UX4,VX4	D	E	New approximation to root (X_{n+1})
CONV	L	CONV	L	R	CONV = true implies convergence has been obtained
EPS	R	EPS	D	C	Tolerance for convergence test
EPSO	R	EPSO	D	C	Tolerance check for zero (0)
DENOM	R	DENOM	D		Magnitude of new approximation, (X_{n+1})
					Subroutine BETTER
MULT	I	MULT	I	ECR	Array of multiplicities of each root
A	C	UA,VA	D	E	Array of coefficients of original undeflated polynomial
NP	I	NP	I	E	Degree of original polynomial
ROOT	C	UROOT,VROOT	D	ECR	Array of roots
NROOT	I	NROOT	I	ECR	Number of roots stored in root
BAPP	C	UBAPP,VBAPP	D	E	Array of initial approximations (old roots)
IROOT	I	IROOT	I	ECR	Number of roots solved by the iterative process (Not QUAD)
ROOTS	C	UROOTS,VROOTS	D		Temporary storage for new (better) roots
L	I	L	I		Number of roots found by BETTER
EPSRT	R	EPSRT	D	C	A small number used to generate two of the three approximations when given one
ITER	I	ITER	I	C	Counter for number of iterations

TABLE C.IV (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
B	C	UB,VB	D		Array containing coefficients of deflated polynomial
X1	C	UX1,VX1	D		One of three approximations to the root
X2	C	UX2,VX2	D		One of three approximations to the root
X3	C	UX3,VX3	D		One of three approximations to the root
PX1	C	UPX1,VPX1	D		Value of polynomial (P(X)) at X1
PX2	C	UPX2,VPX2	D		Value of polynomial (P(X)) at X2
PX3	C	UPX3,VPX3	D		Value of polynomial (P(X)) at X3
CONV	L	CONV	L		CONV = true implies convergence has been obtained
X4	C	UX4,VX4	D		Newest approximation to root
J	I	J	I		Program control - counts the number of roots used as initial approximations
MAX	I	MAX	I	C	Maximum number of iterations permitted
IO2	I	IO2	I	C	Unit number of output device
				Subroutine ALTER	
X1	C	X1R,X1I	D	ECR	One of the three approximations to be altered
X2	C	X2R,X2I	D	ECR	One of the three approximations to be altered
X3	C	X3R,X3I	D	ECR	One of the three approximations to be altered
X2R	R	X2R	D		Real part of complex approximation
X2I	R	X2I	D		Imaginary part of complex approximation
				Subroutine QUAD	
EPST	R	EPST	D	E	Tolerance check for zero (0)
				Subroutine CALC	

These variables are dummy variables used for temporary storage and thus, are not defined.

MAIN PROGRAM

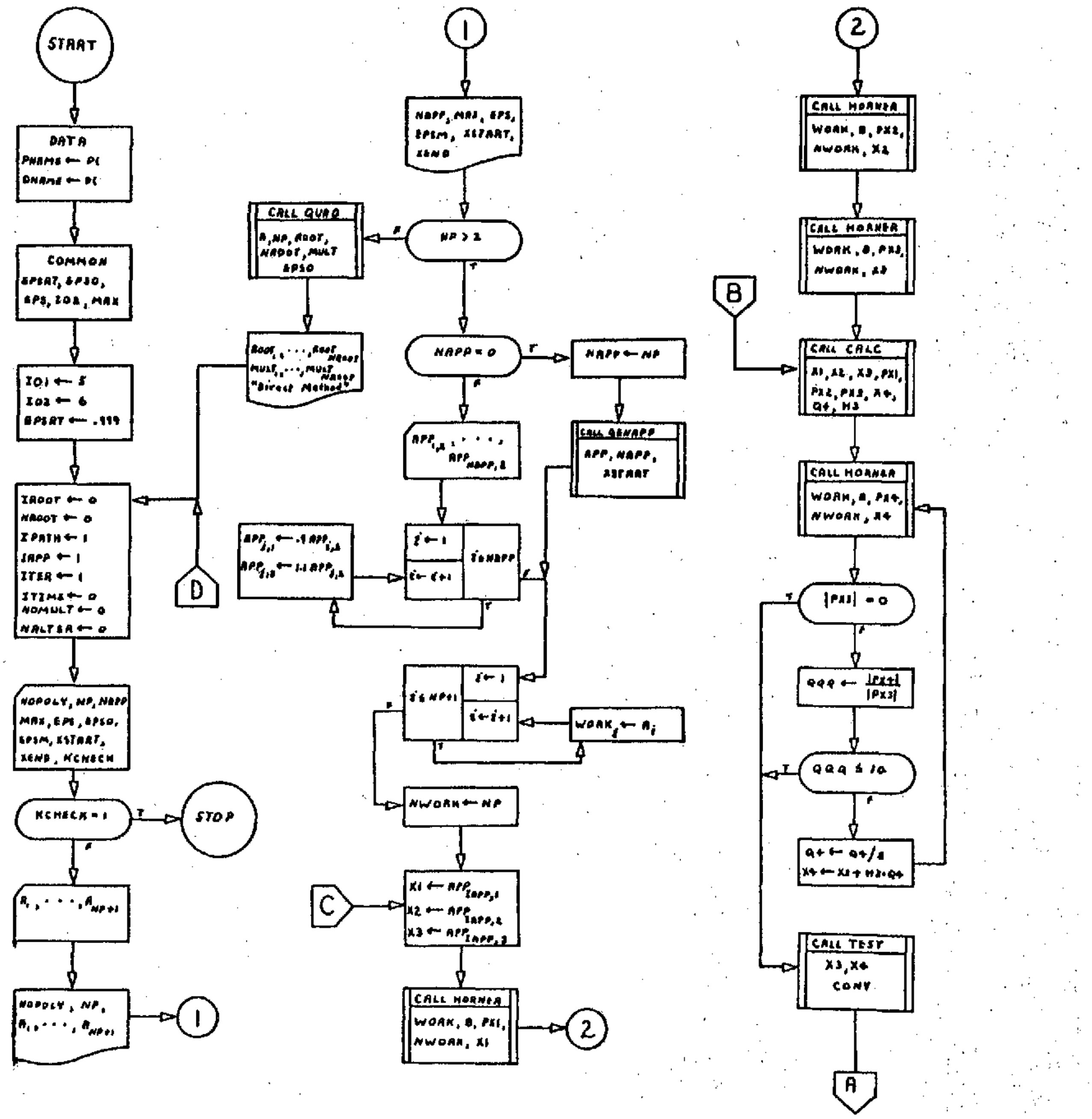


Figure C.1. Flow Charts for Muller's Method

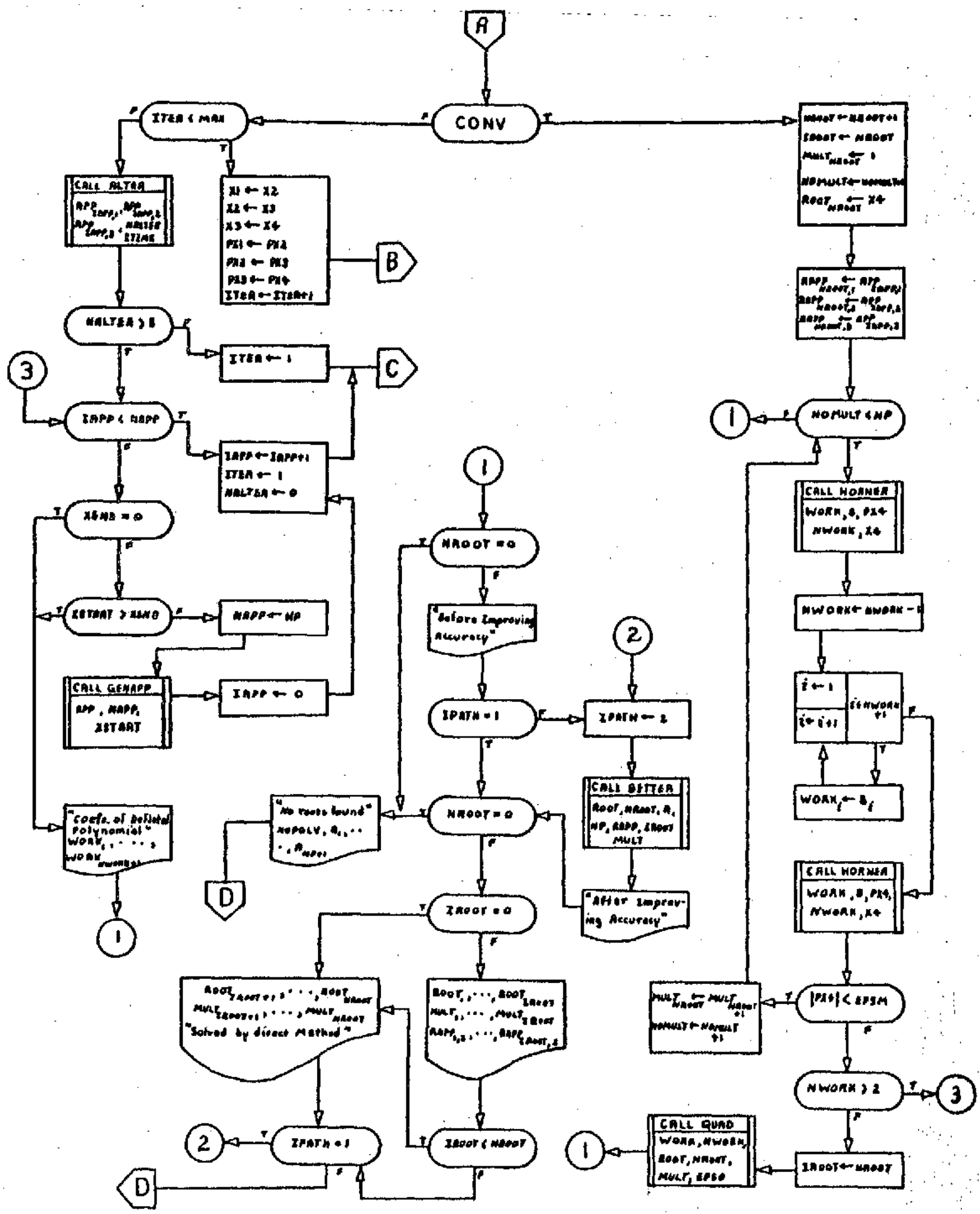


Figure C.1. (Continued)

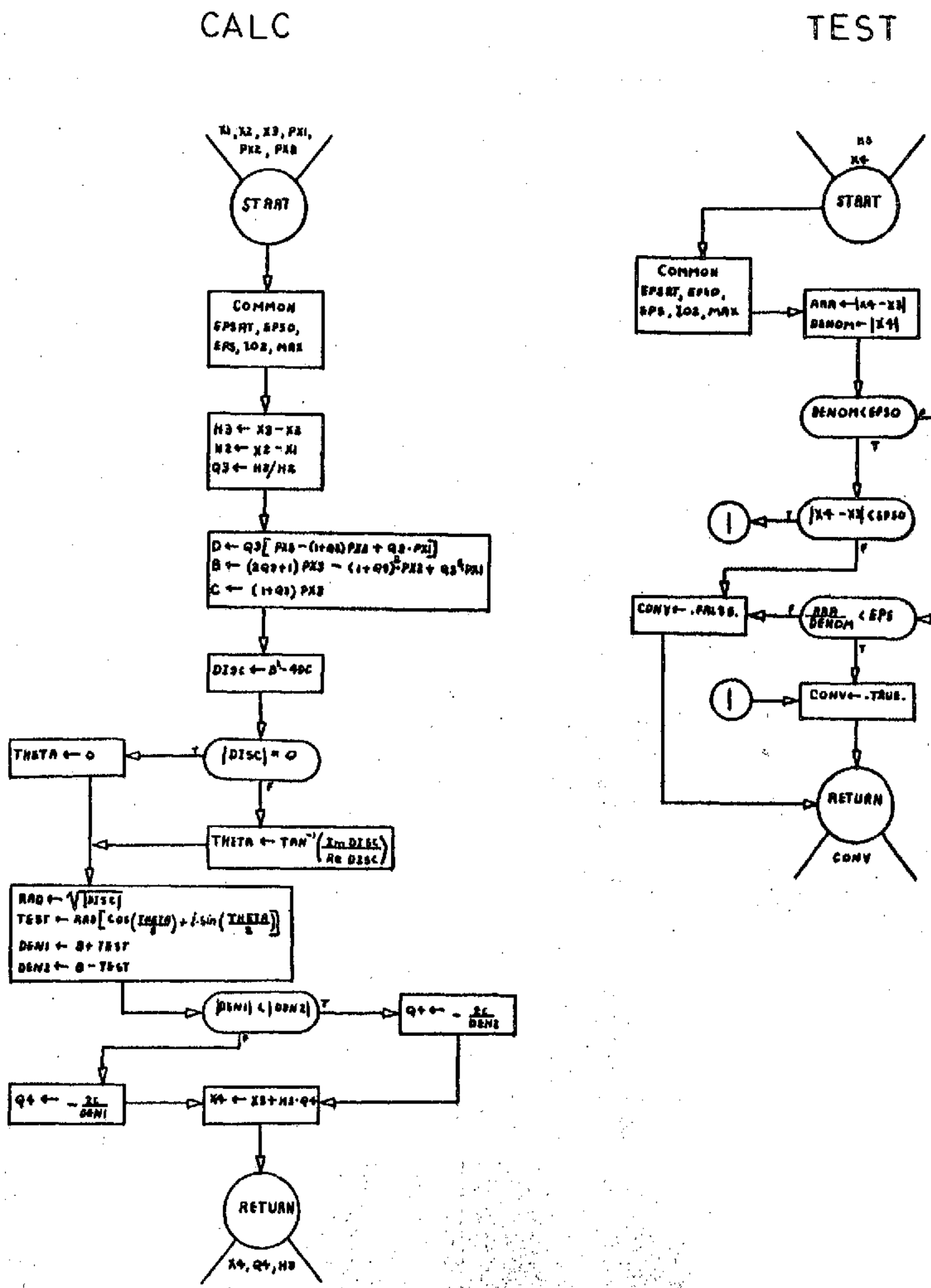


Figure C.1. (Continued)

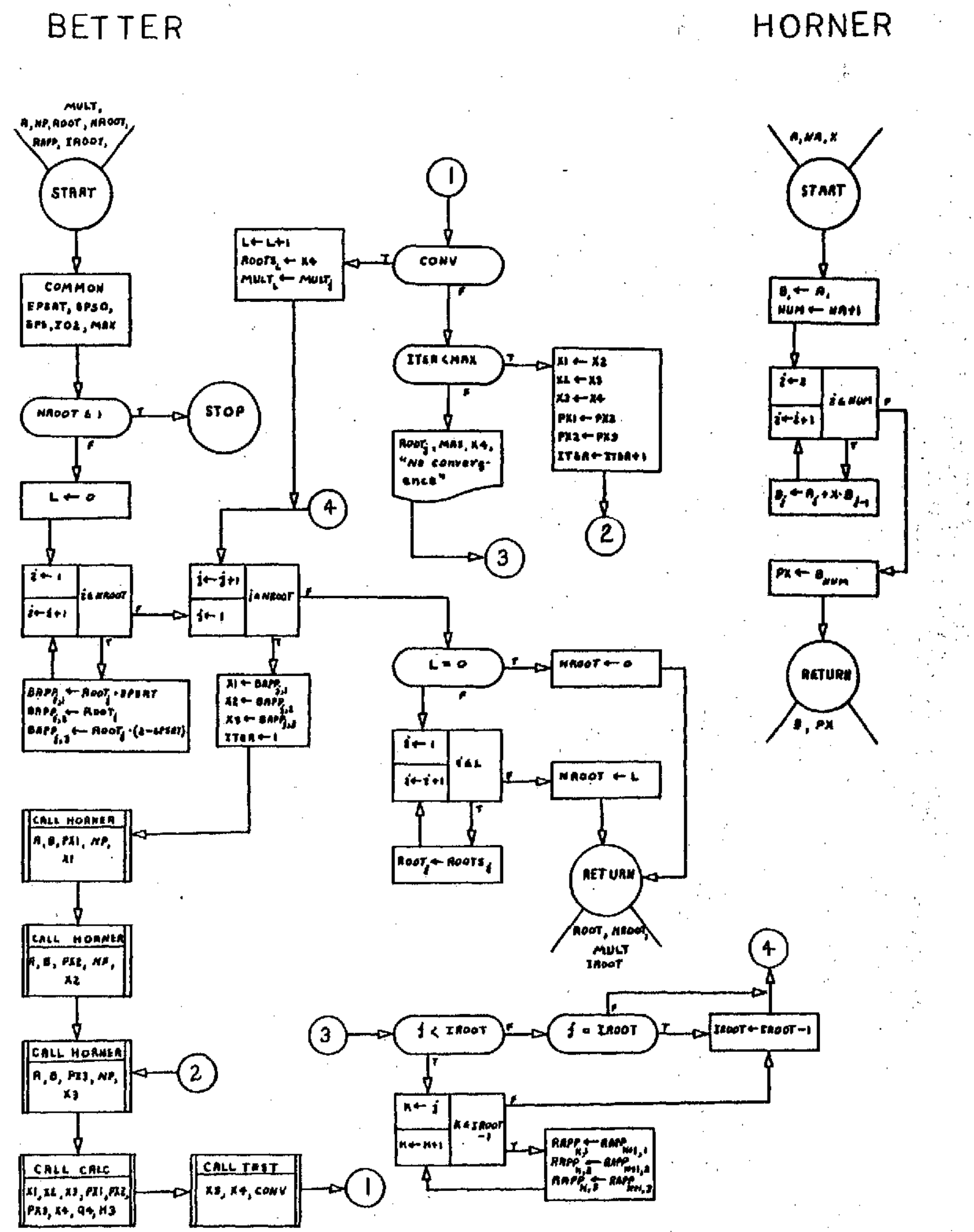
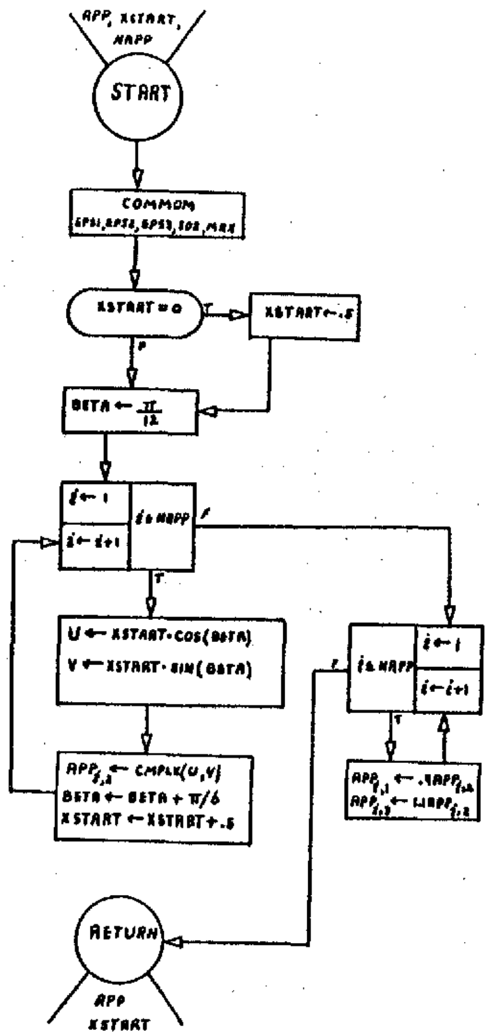


Figure C.1. (Continued)

GENAPP



ALTER

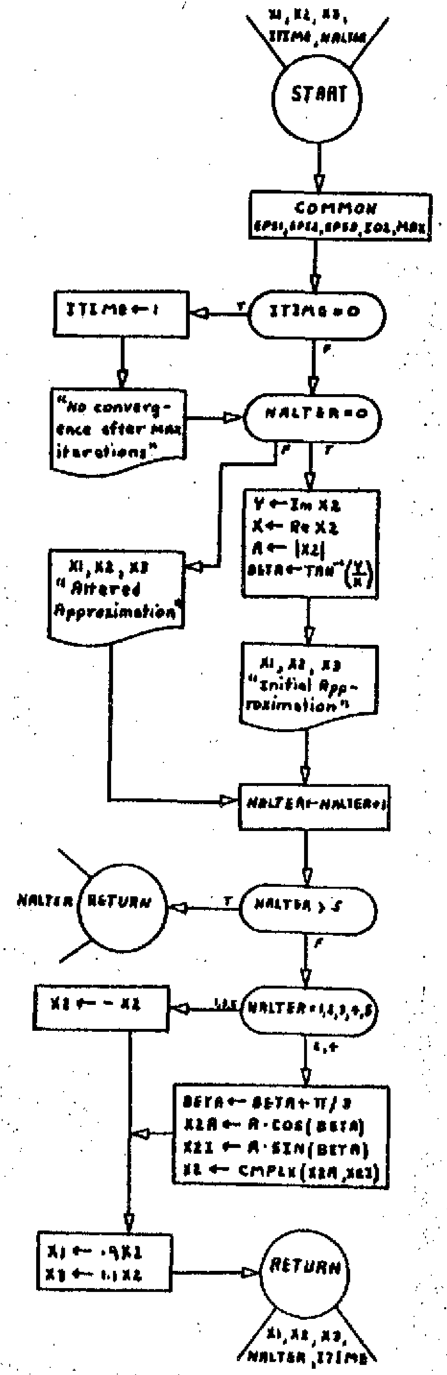
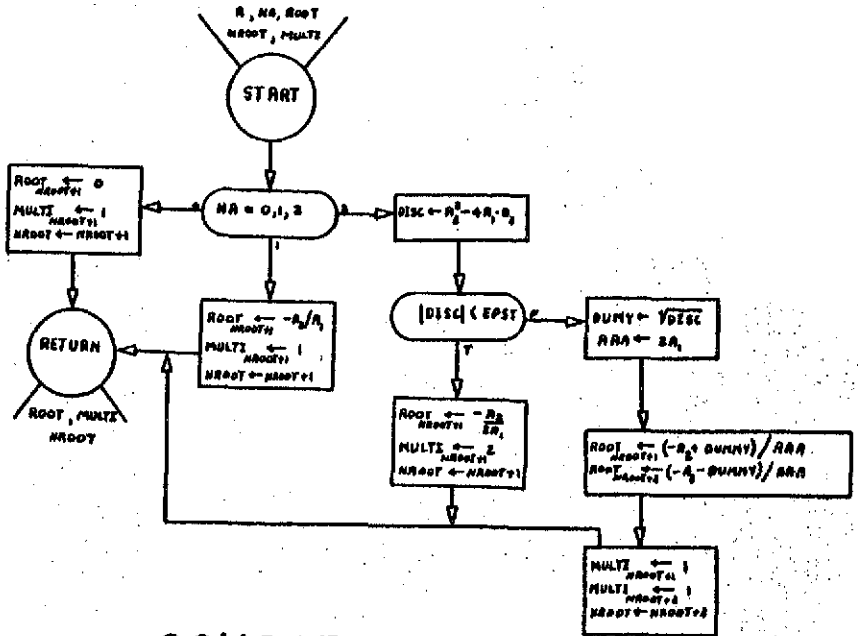


Figure C.1. (Continued)

QUAD



COMSQT

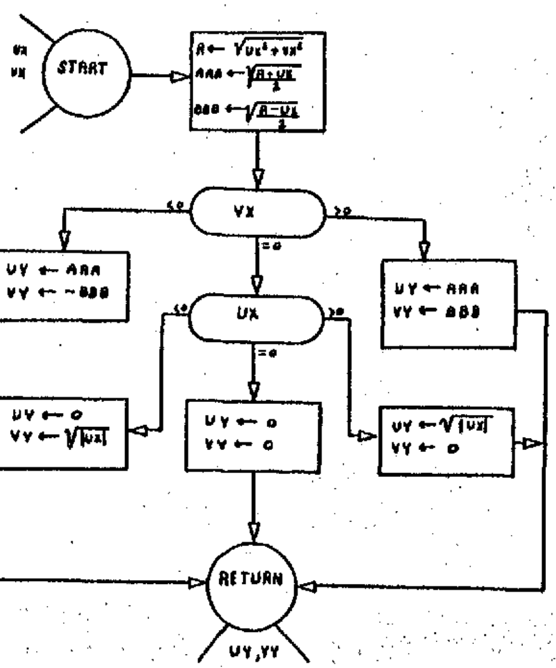


Figure C.1. (Continued)

TABLE C.V.

PROGRAM FOR MULLER'S METHOD

```
C     ************************************************************************
C     *                                                                      *
C     * DOUBLE PRECISION PROGRAM FOR MULLER'S METHOD                         *
C     *                                                                      *
C     *                                                                      *
C     * MULLER'S METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A     *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25.  THROUGH THREE GIVEN POINTS THE     *
C     * POLYNOMIAL IS APPROXIMATED BY A QUADRATIC.  THE ZERO OF THE QUADRATIC *
C     * CLOSEST TO THE OLD APPROXIMATION IS TAKEN AS THE NEW APPROXIMATION.  *
C     * IN THIS MANNER A SEQUENCE IS OBTAINED CONVERGING TO A ZERO.          *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UROOT,VROOT,UX1,VX1,UAPP,VAPP
     1,UX2,VX2,UWORK,VWORK,UX3,VX3,UB,VB,UX4,VX4,UA,VA,UPX1,VPX1,URAPP,V
     2RAPP,UPX4,VPX4,EPSRT,EPSO,EPS,CCC,EPSM,UH3,VH3,UQ4,VQ4,ABPX4,ABPX3
     3,QQQ,XSTART,XEND
      DIMENSION UROOT(25),VROOT(25),MULT(25),UAPP(25,3),VAPP(25,3),UWORK
     1(26),VWORK(26),UB(26),VB(26),UA(26),VA(26),URAPP(25,3),VRAPP(25,3)
      DATA PNAME,DNAME/2HP(,2HD(/
      LOGICAL CONV
      COMMON EPSRT,EPSO,EPS,IO2,MAX
      IO1=5
      IO2=6
      EPSRT=0.999
   10 NROOT=0
      IROOT=0
      IPATH=1
      NOMULT=0
      NALTER=0
      ITIME=0
      IAPP=1
      ITER=1
      READ(IO1,1000) NOPOLY,NP,NAPP,MAX,EPS,EPSO,EPSM,XSTART,XEND,KCHECK
      IF(KCHECK.EQ.1) STOP
      KKK=NP+1
      READ(IO1,1010) (UA(I),VA(I),I=1,KKK)
      WRITE(IO2,1020) NOPOLY,NP
      WRITE(IO2,1035) (PNAME,I,UA(I),VA(I),I=1,KKK)
      WRITE(IO2,2060)
      WRITE(IO2,2000) NAPP
      WRITE(IO2,2010) MAX
      WRITE(IO2,2020) EPS
      WRITE(IO2,2030) EPSM
      WRITE(IO2,2040) XSTART
      WRITE(IO2,2050) XEND
      IF(NP.GT.2) GO TO 15
      CALL QUAD(UA,VA,NP,UROOT,VROOT,NROOT,MULT,EPSO)
      WRITE(IO2,1037)
      WRITE(IO2,1086) (I,UROOT(I),VROOT(I),MULT(I),I=1,NROOT)
      GO TO 10
   15 IF(NAPP.NE.0) GO TO 20
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 27
   20 READ(IO1,1030) (UAPP(I,2),VAPP(I,2),I=1,NAPP)
      DO 25 I=1,NAPP
      UAPP(I,1)=0.9*UAPP(I,2)
      VAPP(I,1)=0.9*VAPP(I,2)
```

TABLE C.V (Continued)

```
      UAPP(I,3)=1.1*UAPP(I,2)
   25 VAPP(I,3)=1.1*VAPP(I,2)
   27 KKK=NP+1
      DO 30 I=1,KKK
      UWORK(I)=UA(I)
   30 VWORK(I)=VA(I)
      NWORK=NP
   40 UX1=UAPP(IAPP,1)
      VX1=VAPP(IAPP,1)
      UX2=UAPP(IAPP,2)
      VX2=VAPP(IAPP,2)
      UX3=UAPP(IAPP,3)
      VX3=VAPP(IAPP,3)
      CALL HORNER(NWORK,UWORK,VWORK,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NWORK,UWORK,VWORK,UX2,VX2,UB,VB,UPX2,VPX2)
      CALL HORNER(NWORK,UWORK,VWORK,UX3,VX3,UB,VB,UPX3,VPX3)
   50 CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   60 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      ABPX4=DSQRT(UPX4*UPX4+VPX4*VPX4)
      ABPX3=DSQRT(UPX3*UPX3+VPX3*VPX3)
      IF(ABPX3.EQ.0.0) GO TO 70
      QQQ=ABPX4/ABPX3
      IF(QQQ.LE.10.) GO TO 70
      UQ4=0.5*UQ4
      VQ4=0.5*VQ4
      UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      GO TO 60
   70 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 120
      IF(ITER.LT.MAX) GO TO 110
      CALL ALTER(UAPP(IAPP,1),VAPP(IAPP,1),UAPP(IAPP,2),VAPP(IAPP,2),UAP
     1P(IAPP,3),VAPP(IAPP,3),NALTER,ITIME)
      IF(NALTER.GT.5) GO TO 75
      ITER=1
      GO TO 40
   75 IF(IAPP.LT.NAPP) GO TO 100
      IF(XEND.EQ.0.0) GO TO 77
      IF(XSTART.GT.XEND) GO TO 77
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      IAPP=0
      GO TO 100
   77 WRITE(IO2,1090)
      KKK=NWORK+1
      WRITE(IO2,1035) (DNAME,J,UWORK(J),VWORK(J),J=1,KKK)
   80 IF(NROOT.EQ.0) GO TO 90
      WRITE(IO2,1060)
      IF(IPATH.EQ.1) GO TO 82
   81 IPATH=2
      CALL BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MULT)
      WRITE(IO2,1200)
   82 IF(NROOT.EQ.0)GO TO 90
      IF(IROOT.EQ.0) GO TO 85
      WRITE(IO2,1080)
      DO 55 I=1,IROOT
   55 WRITE(IO2,1085) I,UROOT(I),VROOT(I),MULT(I),URAPP(I,2),VRAPP(I,2)
```

TABLE C.V (Continued)

```
      IF(IROOT.LT.NROOT) GO TO 85
      GO TO 87
   85 KKK=IROOT+1
      WRITE(IO2,1086) (I,UROOT(I),VROOT(I),MULT(I),I=KKK,NROOT)
   87 IF(IPATH.EQ.1) GO TO 81
      GO TO 10
   90 WRITE(IO2,1070) NOPOLY
      GO TO 10
  100 IAPP=IAPP+1
      ITER=1
      NALTER=0
      GO TO 40
  120 NROOT=NROOT+1
      IROOT=NROOT
      MULT(NROOT)=1
      NOMULT=NOMULT+1
      UROOT(NROOT)=UX4
      VROOT(NROOT)=VX4
      URAPP(NROOT,1)=UAPP(IAPP,1)
      VRAPP(NROOT,1)=VAPP(IAPP,1)
      URAPP(NROOT,2)=UAPP(IAPP,2)
      VRAPP(NROOT,2)=VAPP(IAPP,2)
      URAPP(NROOT,3)=UAPP(IAPP,3)
      VRAPP(NROOT,3)=VAPP(IAPP,3)
  125 IF(NOMULT.LT.NP) GO TO 130
      GO TO 80
  130 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      NWORK=NWORK-1
      KKK=NWORK+1
      DO 140 I=1,KKK
      UWORK(I)=UB(I)
  140 VWORK(I)=VB(I)
      CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      CCC=DSQRT(UPX4*UPX4+VPX4*VPX4)
      IF(CCC.LT.EPSM) GO TO 150
      IF(NWORK.GT.2) GO TO 75
      IROOT=NROOT
      CALL QUAD(UWORK,VWORK,NWORK,UROOT,VROOT,NROOT,MULT,EPSO)
      GO TO 80
  150 MULT(NROOT)=MULT(NROOT)+1
      NOMULT=NOMULT+1
      GO TO 125
  110 UX1=UX2
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      UPX3=UPX4
      VPX3=VPX4
      ITER=ITER+1
      GO TO 50
 1010 FORMAT(2D30.0)
 1020 FORMAT(1H1,1X,52HMULLERS METHOD FOR FINDING THE ZEROS OF A POLYNOM
```

TABLE C.V (Continued)

```
          1IAL/1H ,1X,18HPOLYNOMIAL NUMBER ,I2,11H OF DEGREE ,I2///1H ,1X,28H
          2THE COEFFICIENTS OF P(X) ARE//)
0157      1030 FORMAT(2D30.0)
0158      1090 FORMAT(///,1X,65HCOEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO
          1ZEROS WERE FOUND//)
0159      1080 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
          1 APPROXIMATION//)
0160      1070 FORMAT(//,43H NO ZEROS WERE FOUND FOR POLYNOMIAL NUMBER ,I2)
0161      1086 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,8X,I2,9X,23HS
          1OLVED BY DIRECT METHOD)
0162      1037 FORMAT(///,1X,13HZEROS OF P(X),51X,14HMULTIPLICITIES//)
0163      1035 FORMAT(3X,A2,I2,4H) = ,D23.16,3H + ,D23.16,2H I)
0164      1085 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,8X,I2,8X,D23.
          116,3H + ,D23.16,2H I)
0165      1000 FORMAT(3(I2,1X),9X,I3,8X,3(D6.0,1X),13X,2(D7.0,1X),I1)
0166      1060 FORMAT(///35H BEFORE ATTEMPT TO IMPROVE ACCURACY)
0167      1200 FORMAT(///1X,37HAFTER THE ATTEMPT TO IMPROVE ACCURACY)
0168      2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.   ,I2)
0169      2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
0170      2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
0171      2030 FORMAT(1X,24HTEST FOR MULTIPLICITIES.,10X,D9.2)
0172      2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
0173      2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
0174      2060 FORMAT(//1X)
0175           END
```

TABLE C.V (Continued)

```
      SUBROUTINE ALTER(X1R,X1I,X2R,X2I,X3R,X3I,NALTER,ITIME)
C   ***************************************************************************
C   *                                                                         *
C   * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO     *
C   * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C   *                                                                         *
C   ***************************************************************************
      DOUBLE PRECISION X1R,X1I,X2R,X2I,X3R,X3I,EPS1,EPS2,EPS3,R,BETA
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      IF(ITIME.NE.0) GO TO 5
      ITIME=1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) X1R,X1I,X2R,X2I,X3R,X3I
      GO TO 20
   10 R=DSQRT(X2R*X2R+X2I*X2I)
      BETA=DATAN2(X2I,X2R)
      WRITE(IO2,1020) X1R,X1I,X2R,X2I,X3R,X3I
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 X2R=-X2R
      X2I=-X2I
      GO TO 50
   40 BETA=BETA+1.0471976
      X2R=R*DCOS(BETA)
      X2I=R*DSIN(BETA)
   50 X1R=0.9*X2R
      X1I=0.9*X2I
      X3R=1.1*X2R
      X3I=1.1*X2I
      RETURN
 1000 FORMAT(1X,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HALTERED APPROXIM
     1ATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2,D23.16,2H I/)
 1020 FORMAT(1H0,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HINITIAL APPROXI
     1MATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2 ,D23.16,2H I/)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     1TER ,I3,12H ITERATIONS.//)
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C    ************************************************************************
C    *                                                                      *
C    * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE *
C    * DEGREE OF THE ORIGINAL POLYNOMIAL.                                   *
C    *                                                                      *
C    ************************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,EPS1,EPS2,EPS3,BETA
      DIMENSION APPR(25,3),APPI(25,3)
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I,2)=XSTART*DCOS(BETA)
      APPI(I,2)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      DO 20 I=1,NAPP
      APPR(I,1)=0.9*APPR(I,2)
      APPI(I,1)=0.9*APPI(I,2)
      APPR(I,3)=1.1*APPR(I,2)
   20 APPI(I,3)=1.1*APPI(I,2)
      RETURN
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MUL
     1T)
C    **********************************************************************
C    *                                                                    *
C    * SUBROUTINE BETTER ATTEMPTS TO IMPROVE THE ACCURACY OF THE ZEROS FOUND *
C    * BY USING THEM AS INITIAL APPROXIMATIONS WITH MULLER'S METHOD APPLIED TO *
C    * THE FULL, UNDEFLATED POLYNOMIAL.                                   *
C    *                                                                    *
C    **********************************************************************
      DOUBLE PRECISION UROOT,VROOT,UA,VA,UBAPP,VBAPP,UX1,VX1,UX2,VX2,UX3
     1,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UB,VB,UROOTS,VROOTS,EPSRT,UX4,V
     2X4,URAPP,VRAPP,EPSO,EPS,UQ4,VQ4,UH3,VH3
      LOGICAL CONV
      DIMENSION UROOT(25),VROOT(25),UA(26),VA(26),UBAPP(25,3),VBAPP(25,3
     1),UB(26),VB(26),UROOTS(25),VROOTS(25),URAPP(25,3),VRAPP(25,3),MULT
     3(25)
      COMMON EPSRT,EPSO,EPS,IO2,MAX
      IF(NROOT.LE.1) RETURN
      L=0
      DO 10 I=1,NROOT
      UBAPP(I,1)=UROOT(I)*EPSRT
      VBAPP(I,1)=VROOT(I)*EPSRT
      UBAPP(I,2)=UROOT(I)
      VBAPP(I,2)=VROOT(I)
      UBAPP(I,3)=UROOT(I)*(2.0-EPSRT)
   10 VBAPP(I,3)=VROOT(I)*(2.0-EPSRT)
      DO 100 J=1,NROOT
      UX1=UBAPP(J,1)
      VX1=VBAPP(J,1)
      UX2=UBAPP(J,2)
      VX2=VBAPP(J,2)
      UX3=UBAPP(J,3)
      VX3=VBAPP(J,3)
      ITER=1
      CALL HORNER(NP,UA,VA,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NP,UA,VA,UX2,VX2,UB,VB,UPX2,VPX2)
   20 CALL HORNER(NP,UA,VA,UX3,VX3,UB,VB,UPX3,VPX3)
      CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   30 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 50
      IF(ITER.LT.MAX) GO TO 40
      WRITE(IO2,1000) J,UROOT(J),VROOT(J),MAX
      WRITE(IO2,1010) UX4,VX4
      IF(J.LT.IROOT) GO TO 33
      IF(J.EQ.IROOT) GO TO 35
      GO TO 100
   33 KKK=IROOT-1
      DO 34 K=J,KKK
      URAPP(K,1)=URAPP(K+1,1)
      VRAPP(K,1)=VRAPP(K+1,1)
      URAPP(K,2)=URAPP(K+1,2)
      VRAPP(K,2)=VRAPP(K+1,2)
      URAPP(K,3)=URAPP(K+1,3)
   34 VRAPP(K,3)=VRAPP(K+1,3)
   35 IROOT=IROOT-1
      GO TO 100
   40 UX1=UX2
```

TABLE C.V (Continued)

```
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      ITER=ITER+1
      GO TO 20
   50 L=L+1
      UROOTS(L)=UX4
      VROOTS(L)=VX4
      MULT(L)=MULT(J)
  100 CONTINUE
      IF(L.EQ.0) GO TO 120
      DO 110 I=1,L
      UROOT(I)=UROOTS(I)
  110 VROOT(I)=VROOTS(I)
      NROOT=L
      RETURN
  120 NROOT=0
      RETURN
 1000 FORMAT(///42H IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(,I2,4H) = ,
     1D23.16,3H + ,D23.16,2H I/24H DID NOT CONVERGE AFTER ,I3,11H ITERAT
     2IONS)
 1010 FORMAT(30H THE PRESENT APPROXIMATION IS ,D23.16,3H + ,D23.16,2H I/
     1/)
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,V
     1PX3,UX4,VX4,UQ4,VQ4,UH3,VH3)
C     ***********************************************************************
C     *                                                                     *
C     * GIVEN THREE APPROXIMATIONS X(N-2), X(N-1), AND X(N), SUBROUTINE CALC *
C     * APPROXIMATES THE POLYNOMIAL BY A QUADRATIC AND SOLVES FOR THE ZERO OF *
C     * THE QUADRATIC CLOSEST TO X(N). THIS ZERO IS THE NEW APPROXIMATION   *
C     * X(N+1) TO THE ZERO OF THE POLYNOMIAL.                               *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION ARG1,ARG2
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UX1,VX1,UX2,VX2,UX3,VX3,UPX1,
     1VPX1,UH3,VH3,UH2,VH2,UQ3,VQ3,UD,VD,UB,VB,UC,VC,UDISC,VDISC,UCCC,VC
     2CC,UDEN1,VDEN1,UDEN2,VDEN2,UQ4,VQ4,UX4,VX4,EPSRT,EPSO,EPS,UDDD,VDD
     3D,AAA,BBB,RAD,UAAA,VAAA,UBBB,VBBB
      DOUBLE PRECISION THETA,ANGLE,UTEST,VTEST
      COMMON EPSRT,EPSO,EPS,IO2,MAX
      UH3=UX3-UX2
      VH3=VX3-VX2
      UH2=UX2-UX1
      VH2=VX2-VX1
      BBB=UH2*UH2+VH2*VH2
      UQ3=(UH3*UH2+VH3*VH2)/BBB
      VQ3=(VH3*UH2-UH3*VH2)/BBB
      UDDD=1.0+UQ3
      VDDD=VQ3
      UD=(UPX3-(UDDD*UPX2-VDDD*VPX2))+(UQ3*UPX1-VQ3*VPX1)
      VD=(VPX3-(VDDD*UPX2+UDDD*VPX2))+(VQ3*UPX1+UQ3*VPX1)
      UAAA=2.0*UQ3
      VAAA=2.0*VQ3
      UAAA=UAAA+1.0
      UBBB=UDDD*UDDD-VDDD*VDDD
      VBBB=VDDD*UDDD+UDDD*VDDD
      UCCC=UQ3*UQ3-VQ3*VQ3
      VCCC=VQ3*UQ3+UQ3*VQ3
      UB=((UAAA*UPX3-VAAA*VPX3)-(UBBB*UPX2-VBBB*VPX2))+(UCCC*UPX1-VCCC*V
     1PX1)
      VB=((VAAA*UPX3+UAAA*VPX3)-(VBBB*UPX2+UBBB*VPX2))+(VCCC*UPX1+UCCC*V
     1PX1)
      UC=UDDD*UPX3-VDDD*VPX3
      VC=VDDD*UPX3+UDDD*VPX3
      UDISC=(UB*UB-VB*VB)-(4.0*(UD*UC-VD*VC))
      VDISC=(2.0*(VB*UB))-(4.0*(VD*UC+UD*VC))
      AAA=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(AAA.EQ.0.0) GO TO 5
      GO TO 7
    5 THETA=0.0
      GO TO 9
    7 THETA=DATAN2(VDISC,UDISC)
    9 RAD=DSQRT(AAA)
      ANGLE=THETA/2.0
      UTEST=RAD*DCOS(ANGLE)
      VTEST=RAD*DSIN(ANGLE)
      UDEN1=UB+UTEST
      VDEN1=VB+VTEST
      UDEN2=UB-UTEST
      VDEN2=VB-VTEST
      ARG1=UDEN1*UDEN1+VDEN1*VDEN1
```

TABLE C.V (Continued)

```
      ARG2=UDEN2*UDEN2+VDEN2*VDEN2
      AAA=DSQRT(ARG1)
      BBB=DSQRT(ARG2)
      IF(AAA.LT.BBB) GO TO 10
      IF(AAA.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN1+VAAA*VDEN1)/ARG1
      VQ4=(VAAA*UDEN1-UAAA*VDEN1)/ARG1
      GO TO 50
   10 IF(BBB.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN2+VAAA*VDEN2)/ARG2
      VQ4=(VAAA*UDEN2-UAAA*VDEN2)/ARG2
      GO TO 50
   50 UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      RETURN
   60 UQ4=1.0
      VQ4=0.0
      GO TO 50
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE TEST(UX3,VX3,UX4,VX4,CONV)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE TEST CHECKS FOR CONVERGENCE OF THE SEQUENCE OF APPROX-    *
C     * IMATIONS BY TESTING THE EXPRESSION                                   *
C     * ABSOLUTE VALUE OF (X(N+1)-X(N))/ABSOLUTE VALUE OF X(N+1).            *
C     * WHEN IT IS AS SMALL AS DESIRED, CONVERGENCE IS OBTAINED.             *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX3,VX3,UX4,VX4,EPSRT,EPSO,EPS,AAA,UDUMMY,VDUMMY,
     1DENOM
      LOGICAL CONV
      COMMON EPSRT,EPSO,EPS,IO2,MAX
      UDUMMY=UX4-UX3
      VDUMMY=VX4-VX3
      AAA=DSQRT(UDUMMY*UDUMMY+VDUMMY*VDUMMY)
      DENOM=DSQRT(UX4*UX4+VX4*VX4)
      IF(DENOM.LT.EPSO) GO TO 20
      IF(AAA/DENOM.LT.EPS) GO TO 10
    5 CONV=.FALSE.
      GO TO 100
   10 CONV=.TRUE.
      GO TO 100
   20 IF(AAA.LT.EPSO) GO TO 10
      GO TO 5
  100 RETURN
      END
```

```
      SUBROUTINE HORNER(NA,UA,VA,UX,VX,UB,VB,UPX,VPX)
C     ************************************************************************
C     *                                                                      *
C     * HORNER'S METHOD COMPUTES THE VALUE OF THE POLYNOMIAL P(X) AT A POINT D.  *
C     * SYNTHETIC DIVISION IS USED TO DEFLATE THE POLYNOMIAL BY DIVIDING OUT THE *
C     * FACTOR (X-D).                                                        *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX,VX,UPX,VPX,UB,VB,UA,VA
      DIMENSION UA(26),VA(26),UB(26),VB(26)
      UB(1)=UA(1)
      VB(1)=VA(1)
      NUM=NA+1
      DO 10 I=2,NUM
      UB(I)=UA(I)+(UB(I-1)*UX-VB(I-1)*VX)
   10 VB(I)=VA(I)+(VB(I-1)*UX+UB(I-1)*VX)
      UPX=UB(NUM)
      VPX=VB(NUM)
      RETURN
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE QUAD(UA,VA,NA,UROOT,VROOT,NROOT,MULTI,EPST)
C     ***************************************************************************
C     *                                                                         *
C     * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES  *
C     * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE   *
C     * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                          *
C     *                                                                         *
C     ***************************************************************************
      DOUBLE PRECISION UA,VA,UROOT,VROOT,BBB,UAAA,VAAA,UDISC,VDISC,UDUMM
     1Y,VDUMMY,RDUMMY,SDUMMY,EPST,UBBB,VBBB
      DIMENSION UA(26),VA(26),UROOT(25),VROOT(25),MULTI(25)
      IF(NA.EQ.2) GO TO 7
      IF(NA.EQ.1) GO TO 5
      UROOT(NROOT+1)=0.0
      VROOT(NROOT+1)=0.0
      MULTI(NROOT+1)=1
      NROOT=NROOT+1
      GO TO 50
    5 BBB=UA(1)*UA(1)+VA(1)*VA(1)
      UROOT(NROOT+1)=(-UA(2)*UA(1)-VA(2)*VA(1))/BBB
      VROOT(NROOT+1)=(-VA(2)*UA(1)+UA(2)*VA(1))/BBB
      MULTI(NROOT+1)=1
      NROOT=NROOT+1
      GO TO 50
    7 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(1)*UA(3)-VA(1)*VA(3)))
      VDISC=(VA(2)*UA(2)+UA(2)*VA(2))-(4.0*(VA(1)*UA(3)+UA(1)*VA(3)))
      BBB=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(BBB.LT.EPST) GO TO 10
      CALL COMSQT(UDISC,VDISC,UDUMMY,VDUMMY)
      UBBB=-UA(2)+UDUMMY
      VBBB=-VA(2)+VDUMMY
      RDUMMY=-UA(2)-UDUMMY
      SDUMMY=-VA(2)-VDUMMY
      UAAA=2.0*UA(1)
      VAAA=2.0*VA(1)
      BBB=UAAA*UAAA+VAAA*VAAA
      UROOT(NROOT+1)=(UBBB*UAAA+VBBB*VAAA)/BBB
      VROOT(NROOT+1)=(VBBB*UAAA-UBBB*VAAA)/BBB
      UROOT(NROOT+2)=(RDUMMY*UAAA+SDUMMY*VAAA)/BBB
      VROOT(NROOT+2)=(SDUMMY*UAAA-RDUMMY*VAAA)/BBB
      MULTI(NROOT+1)=1
      MULTI(NROOT+2)=1
      NROOT=NROOT+2
      GO TO 50
   10 UAAA=2.0*UA(1)
      VAAA=2.0*VA(1)
      BBB=UAAA*UAAA+VAAA*VAAA
      UROOT(NROOT+1)=(-UA(2)*UAAA-VA(2)*VAAA)/BBB
      VROOT(NROOT+1)=(-VA(2)*UAAA+UA(2)*VAAA)/BBB
      MULTI(NROOT+1)=2
      NROOT=NROOT+1
   50 RETURN
      END
```

TABLE C.V (Continued)

```
      SUBROUTINE CONSQT(UX,VX,UY,VY)
C   ***********************************************************************
C   *                                                                     *
C   * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.      *
C   *                                                                     *
C   ***********************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

APPENDIX D

SPECIAL FEATURES OF THE G.C.D. AND THE REPEATED G.C.D. PROGRAMS

Several special features have been provided in each program as an aid to the user and to improve accuracy of the results. These are explained and illustrated below.*

1. Generating Approximations

If the user does not have initial approximations available, subroutine GENAPP can systematically generate, for an N^{th} degree polynomial, N initial approximations of increasing magnitude, beginning with the magnitude specified by XSTART. If XSTART is 0., XSTART is automatically initialized to 0.5 to avoid the approximation 0. + 0.i. The approximations are generated according to the formula:

$$X_K = (\text{XSTART} + 0.5K)\ (\text{Cos}\ \beta + i\ \text{Sin}\ \beta)$$

where

$$\beta = \frac{\Pi}{12} + K\frac{\Pi}{6}\ ,\ K = 0,1,2,\ldots$$

To accomplish this, the user defines the number of initial approximations to be read (NAPP) on the control card to be zero (0) or these columns

*These illustrations are representative of G.C.D.-Newton's method in double precision. Control cards for other methods should be prepared accordingly.

(7-8) may be left blank. If XSTART is left blank, it is interpreted as 0.

For example, a portion of a control card which generates initial approximations beginning at the origin for a seventh degree polynomial is shown in Example D.1.

Card Columns	1 2		4 5		7 8		64 70		72 78		80
Variable Name	NOPOLY		NP		NAPP		XSTART				
Example	1		7								

Example D.1

The approximations are generated in a spiral configuration as illustrated in Figure A.1.

Example D.2 shows a portion of a control card which generates initial approximations beginning at a magnitude of 25.0 for a sixth degree polynomial.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		NP		NAPP		XSTART				
2		6				2.5D+01				

Example D.2

Note that if the approximations are generated beginning at the origin, the order in which the roots are found will probably be of increasing magnitude. Roots obtained in this way are usually more accurate.

2. Altering Approximations

If an initial approximation, X_0, does not produce convergence to a root within the maximum number of iterations, it is systematically altered a maximum of five times until convergence is possibly obtained according to the following formulas:

If the number of the alteration is odd: $(j = 1,3)$

$$X_{j+1} = |X_0| \, (\text{Cos } \beta + i \text{ Sin } \beta) \text{ where}$$

$$\beta = \text{Tan}^{-1} \frac{\text{Im } X_0}{\text{Re } X_0} + K \frac{\Pi}{3} \; ; \; \begin{array}{l} K = 1 \text{ if } j = 1 \\ K = 2 \text{ if } j = 3. \end{array}$$

If the number of the alteration is even: $(j = 0,2,4)$

$$X_{j+1} = -X_j .$$

Each altered approximation is then taken as a starting approximation. If none of the six starting approximations produce convergence, the next initial approximation is taken, and the process repeated. The six approximations are spaced 60 degrees apart on a circle of radius $|X_0|$ centered at the origin as illustrated in Figure A.2.

3. Searching the Complex Plane

By use of initial approximations and the altering technique, any region of the complex plane in the form of an annulus centered at the origin can be searched for roots. This procedure can be accomplished in two ways.

The first way is more versatile but requires more effort on the part of the user. Specifically selected initial approximation can be used to define particular regions to be searched. For example, if the roots of a particular polynomial are known to have magnitudes between 20 and 40 an annulus of inner radius 20 and outer radius 40 could be searched by using the initial approximations 20. + i, 23. + i, 26. + i, 29. + i, 32. + i, 35. + i, 38. + i, 40. + i.

By generating initial approximations internally, the program can search an annulus centered at the origin of inner radius XSTART and outer radius XEND. Values for XSTART and XEND are supplied on the control card by the user. Example D.3 shows a portion of a control card to search the above annulus of inner radius 20.0 and outer radius 40.0.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		NP		NAPP		XSTART		XEND		
1		7				2.0D+01		4.0D+01		

Example D.3

Note that since not less than N initial approximations can be generated at one time, the outer radius of the annulus actually searched may be greater than XEND but not greater than XEND + .5N.

Example D.4 shows a control card to search a circle of radius 15.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		NP		NAPP		XSTART		XEND		
1		7						1.5D+01		

Example D.4

Figure A.3 shows the distribution of initial and altered approximations for an annulus of width 2 and inner radius a.

4. Improving Zeros Found

After the zeros of a polynomial are found, they are printed under the heading "Roots of Q(X)." They are then used as initial approximations with Newton's (Muller's) method applied each time to the full (undeflated) polynomial Q(X), which contains only distinct roots. In most cases, zeros that have lost accuracy due to roundoff error in the deflation process are improved. The improved zeros are then printed under the heading "Roots of P(X)." Since each root is used as an approximation to the original (undeflated) polynomial Q(X), it is possible that the root may converge to an entirely different root. This is especially true where several zeros are close together. Therefore, the user should check both lists of zeros to determine whether or not this has occurred.

5. Solving Quadratic Polynomial

After N-2 roots of an N^{th} degree polynomial have been extracted, the remaining quadratic, $aX^2 + bX + c$, is solved using the quadratic formula

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the two remaining roots. These are indicated by the words "Results of Subroutine QUAD" in the initial approximation column. If only a polynomial of degree 1 is to be solved, the solution is found directly as $(X - C) = 0$ implies $X = C$.

6. Missing Roots

If not all N roots of an N^{th} degree polynomial are found, the coefficients of the remaining deflated polynomial are printed under the heading "Coefficients of Deflated Polynomial For Which No Zeros Were Found." The user may then work with this polynomial in an attempt to find the remaining roots. The leading coefficient (coefficient of the highest degree term) will be printed first (Exhibit 6.11)

7. Miscellaneous

By using various combinations of values for NAPP, XSTART, and XEND, the user has several options available as illustrated below.

Example D.5 shows the control card for a seventh degree polynomial. Three initial approximations are supplied by the user. At most three roots will be found and the coefficients of the remaining deflated polynomial will be printed.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		NP		NAPP		XSTART		XEND		
1		7		3						

Example D.5

Note that if several roots are known to the user, they may be "divided out" of the original polynomial by using this procedure.

Example D.6 indicates that 2 initial approximations are supplied by the user to a 7^{th} degree polynomial. After these approximations are used the circle of radius 15 will be searched for the remaining roots.

1 2		4 5		7 8		64 70		72 78		80
NOPOLY		NP		NAPP		XSTART		XEND		
1		7		2				1.5D+01		

Example D.6

By defining XSTART between 0. and 15. an annulus instead of the circle will be searched.

APPENDIX E

G.C.D. - NEWTON'S METHOD

1. Use of the Program

A double precision FORTRAN IV program using the G.C.D. method with Newton's method as a supporting method is presented here. Flow charts for this program are given in Figure E.6 while Table E.VII gives a FORTRAN IV listing of this program. Single precision variables are listed in some of the tables. The simple precision variables are used in the flow charts and the corresponding double precision variables can be obtained from the appropriate tables.

This program is designed to solve polynomials having degree less than or equal to 25. In order to solve polynomials of degree N where N > 25, the data statement and array dimensions given in Talbe E.I must be changed.

In this program both the leading coefficient and the constant coefficient are assumed to be non-zero.

TABLE E.I

PROGRAM CHANGES NECESSARY TO SOLVE POLYNOMIALS OF DEGREE GREATER THAN 25 BY G.C.D. - NEWTON'S METHOD

Main Program

Data Entry/1H1,1H2,...,1H9,2H10,2H11,...,2HXX/where XX = N+1

UP(N+1), VP(N+1)
UAPP(N), VAPP(N)
UROOT(N), VROOT(N)
MULT(N)
UDP(N+1), VDP(N+1)
UD(N+1), VD(N+1)
UQ(N+1), VQ(N+1)
UQQ(N+1), VQQ(N+1)
UAP(N), VAP(N)
UQD(N+1), VQD(N+1)
ENTRY(N+1)
UROOTS(N), VROOTS(N)

Subroutine GENAPP

APPR(N), APPI(N)

Subroutine GCD

UR(N+1), VR(N+1)
US(N+1), VS(N+1)
USS(N+1), VSS(N+1)
URR(N+1), VRR(N+1)
UT(N+1), VT(N+1)

Subroutine QUAD

UA(N+1), VA(N+1)
UROOT(N), VROOT(N)
MULT(N)

Subroutine NEWTON

UP(N+1), VP(N+1)
UB(N+1), VB(N+1)

Subroutine DIVIDE

UP(N+1), VP(N+1)
UD(N+1), VD(N+1)
UQ(N+1), VQ(N+1)

TABLE E.I (Continued)

Subroutine HORNER

UP(N+1), VP(N+1)
UB(N+1), VB(N+1)

Subroutine DERIV

UP(N+1), VP(N+1)
UA(N+1), VA(N+1)

Subroutine MULTI

UP(N+1), VP(N+1)
UROOT(N), VROOT(N)
UA(N+1), VA(N+1)
UB(N+1), VB(N+1)
MULT(N)

2. Input Data for G.C.D. - Newton's Method

The input data for G.C.D. - Newton's method is grouped into polynomial data sets. Each polynomial data set consists of the data for one and only one polynomial. As many polynomials as the user desires may be solved by placing the polynomial data sets one behind the other. Each polynomial data set consists of three kinds of information placed in the following order:

1. Control information.
2. Coefficients of the polynomial.
3. Initial approximations. These may be omitted as described in Appendix D, § 1.

An end card follows the entire collection of data sets. It indicates that there is no more data to follow and terminates execution of the program. This information is displayed in Figure E.1 and described below. All data should be right justified and the D-type specification should be used. The recommendations given in Table E.II are those found to give best results on the IBM 360/50 computer which has a 32 bit word.

Control Information

The control card is the first card of the polynomial data set and contains the information given in Table E.II. See Figure E.2.

TABLE E.II

CONTROL DATA FOR G.C.D. - NEWTON'S METHOD

Variable Name	Card Columns	Description
NOPOLY	c.c. 1-2	Number of the polynomial. Integer. Right justified.
NP	c.c. 4-5	Degree of the polynomial. Integer. Right justified.
NAPP	c.c. 7-8	Number of initial approximations to be read. Integer. Right justified. If no initial approximations are given, leave blank.
MAX	c.c. 19-21	Maximum number of iterations. Integer. Right justified. 200 is recommended.
EPS1	c.c. 23-28	Test for zero in subroutine GCD. Double precision. Right justify. 1.D-03 is recommended.

TABLE E.II (Continued)

Variable Name	Card Columns	Description
EPS2	c.c. 30-35	Convergence requirement. Double precision. Right justify. 1.D-10 is recommended.
EPS3	c.c. 37-42	Test for zero in subroutine QUAD. Double precision. Right justify. 1.D-20 is recommended.
EPS4	c.c. 44-49	Multiplicity requirement. Double precision. Right justify. 1.D-02 is recommended.
XSTART	c.c. 64-70	Magnitude at which to begin generating initial approximations. Double precision. Right justify. This is a special feature of the program and may be omitted.
XEND	c.c. 72-78	Magnitude at which to end the generating of initial approximations. Double precision. Right justify. This is a special feature of the program and may be omitted.
KCHECK	c.c. 80	This should be left blank.

Coefficients of the Polynomial

The coefficient cards follow the control card. For an N^{th} degree polynomial, N+1 coefficients must be entered one per card. The coefficient of the highest degree term is entered first; that is, the leading coefficient is entered first. For example, if the polynomial $X^5 + 3X^4 + 2X + 5$ were to be solved for its zeros, the order in which the coefficients would be entered is: 1, 3, 0, 0, 2, 5. Each real or complex coefficient is entered, one per card, as described in Table E.III and illustrated in Figure E.3.

TABLE E.III

COEFFICIENT DATA FOR G.C.D. - NEWTON'S METHOD

Variable Name	Card Columns	Description
UP (P in single precision)	c.c. 1-30	Real part of complex coefficient. Double precision. Right justify. If none, leave blank or enter 0.0D00.
VP (P in single precision)	c.c. 31-60	Imaginary part of complex coefficient. Double precision. Right justify. If none, leave blank or enter 0.0D00.

Initial Approximations

The initial approximation cards follow the set of coefficient cards. The number of initial approximations read must be the number specified on the control card and are entered, one per card, as given in Table E.IV and illustrated in Figure E.4.

TABLE E.IV

INITIAL APPROXIMATION DATA FOR G.C.D. - NEWTON'S METHOD

Variable Name	Card Columns	Description
UAPP (APP in single precision)	c.c. 1-30	Real part of complex number. Double precision. Right justify. If none, leave blank or enter 0.0D00.
VAPP (APP in single precision)	c.c. 31-60	Imaginary part of complex number. Double precision. Right justify. If none, leave blank or enter 0.0D00.

End Card

The end card is the last card of the input data to the program. It indicates that there is no more data to be read. When this card is read, program execution is terminated. This card is described in Table E.V and illustrated in Figure E.5.

TABLE E.V

DATA TO END EXECUTION OF G.C.D. - NEWTON'S METHOD

Variable Name	Card Column	Description
KCHECK	c.c. 80	Must contain the number 1. Integer.

3. Variables Used in G.C.D. - Newton's Method

The definitions of the major variables used in G.C.D. - Newton's method are given in Table E.VI. The symbols used to indicate type are:

R - real variable

I - integer variable

D - double precision

C - complex variable

L - logical variable

A - alphanumeric variable

When two variables are listed, the one on the left is the real part of the corresponding single precision complex variable; the one on the right is the imaginary part. The symbols used to indicate disposition are:

E - entered

R - returned

ECR - entered, changed, and returned

C - variable in common

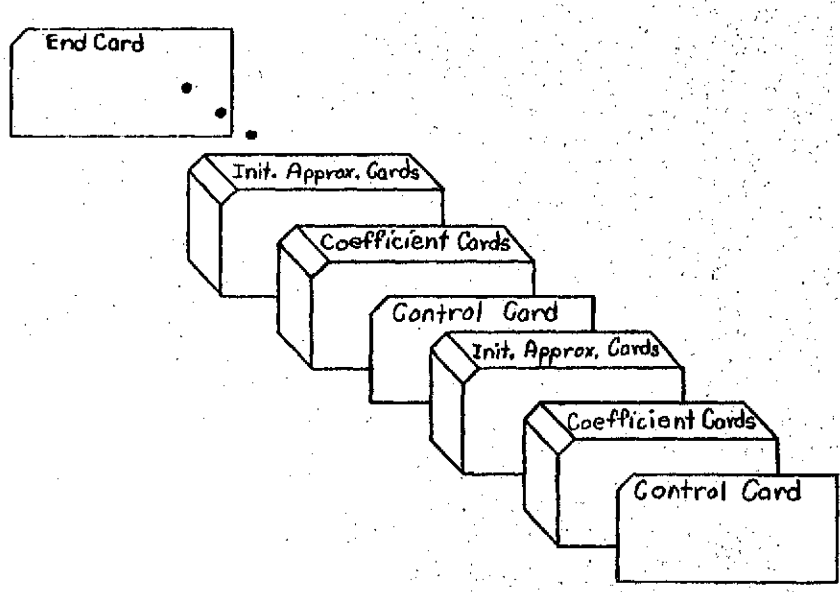


Figure E.1. Sequence of Input Data for G.C.D.-Newton's Method

Variable Name	Card Columns	Example
NOPOLY	01–02	1
	03	
NP	04–05	7
	06	
NAPP	07–08	7
	09–18	
MAX	19–21	200
	22	
EPS1	23–28	1.D-03
	29	
EPS2	30–35	1.D-10
	36	
EPS3	37–42	1.D-20
	43	
EPS4	44–49	1.D-02
	50–63	
XSTART	64–70	1.0D+01
	71	
XEND	72–78	5.0D+02
	79	
KCHECK	80	

Figure E.2. Control Card for G.C.D. - Newton's Method

Variable Name	Card Columns	Example
UP	01–30	+0.125768D+01
VP	31–60	-0.37225D+02
	61–80	

Figure E.3. Coefficient Card for G.C.D. - Newton's Method

Variable Name

Card Columns

000000000111111111122222222223 123456789012345678901234567890	333333333444444444455555555556 123456789012345678901234567890	66666666677777777778 12345678901234567890
UAPP	VAPP	
+0.15D+01	-0.35D+01	

Example

Figure E.4. Initial Approximation Card for G.C.D. - Newton's Method

Variable Name

Card Columns

0000000001111111111222222222233333333334444444444555555555566666666667777777777 1234567890123456789012345678901234567890123456789012345678901234567890123456789	8 0
	K C H E C K
	1

Example

Figure E.5. End Card for G.C.D. - Newton's Method

TABLE E.VI

VARIABLES USED IN G.C.D. - NEWTON'S METHOD

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Main Program
J	I	J	I		Number of distinct roots found
ITIME	I	ITIME	I		Program control
NOPOLY	I	NOPOLY	I		Number of the polynomial
NP	I	NP	I		Degree of the original polynomial
P	C	UP,VP	D		Array of coefficients of original polynomial, P(X)
NAPP	I	NAPP	I		Number of initial approximation to be read
EPS1	R	EPS1	D		Tolerance check for zero (0) in Subroutine GCD
EPS2	R	EPS2	D		Tolerance check for convergence
EPS3	R	EPS3	D		Tolerance check for zero (0) in Subroutine QUAD
MAX	I	MAX	I		Maximum number of iterations permitted
IO1	I	IO1	I		Unit number of input device
IO2	I	IO2	I		Unit number of output device
KCHECK	I	KCHECK	I		Program control, KCHECK = 1 implies stop execution
APP	C	UAPP,VAPP	D		Array of initial approximations
XSTART	R	XSTART	D		Magnitude at which to start search for roots
XEND	R	XEND	D		Magnitude at which to end search for roots
ANAME	A	ANAME	A		Contains name of method used "NEWTONS"
ROOT	C	UROOT,VROOT	D		Array of roots found
MULT	I	MULT	I		Array of multiplicities
DP	C	UDP,VDP	D		Array containing coefficients of the derivative, (P'(X)), of P(X)
NDP	I	NDP	I		Degree of the derivative of original polynomial
D	C	UD,VD	D		Array of coefficients of the greatest common divisor of P(X) and P'(X)
ND	I	ND	I		Degree of g.c.d. of P(X) and P'(X)
Q	C	UQ,VQ	D		Array of coefficients of quotient polynomial P(X)/g.c.d.

TABLE E.VI (Continued)

Single Precision Variable	Single Precision Type	Double Precision Variable	Double Precision Type	Disposition of Argument	Description
NQ	I	NQ	I		Degree of quotient polynomial Q(X)
ZRO	C	UZRO,VZRO	D		Value at which to evaluate or deflate polynomial
DUMMY	C	UDUMMY,VDUMMY	D		Dummy variable
QQ	C	UQQ,VQQ	D		Working array of coefficients of current polynomial
NQQ	I	NQQ	I		Degree of current polynomial, QQ(X)
IALTER	I	IALTER	I		Number of alterations of an initial approximation
CONV	L	CONV	L		CONV = TRUE implies convergence to a root
EPS4	R	EPS4	D		Tolerance for checking multiplicities
AP	C	UAP,VAP	D		Array of approximations (initial or altered) producing convergence
QD	C	UQD,VQD	D		Array of coefficients of newly deflated polynomial
JAP	I	JAP	I		Number of distinct roots found by iterative process i.e. not as a result of Subroutine QUAD
J1	I	J1	I		Number of distinct roots found in the attempt to improve roots
ROOTS	C	UROOTS,VROOTS	D		Array of improved roots
NEWT	L	NEWT	L		Program control. NEWT = TRUE implies that Newton's method was used instead of Subroutine QUAD

Subroutine NEWTON

Single Precision Variable	Single Precision Type	Double Precision Variable	Double Precision Type	Disposition of Argument	Description
X	C	UX,VX	D	E	Starting approximation (initial or altered)
N	I	N	I	E	Degree of current polynomial
P	C	UP,VP	D	E	Array of coefficients of current polynomial
MAX	I	MAX	I	C	Maximum number of iterations
EPSLON	R	EPSLON	D	C	Tolerance for checking convergence
X0	C	UX0,VX0	D	R	Current approximation to root
B	C	UB,VB	D		Array of coefficients of newly deflated polynomial
DPX0	C	UDPX0,VDPX0	D		Derivative of the polynomial at X0

TABLE E.VI (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
DIFF	C	UDIFF,VDIFF	D		PXO/DPXO
PXO	C	UPXO,VPXO	D		Value of polynomial at XO
CONV	L	CONV	L	R	CONV = TRUE implies convergence to root
				Subroutine HORNER	
X	C	UX,VX	D	E	Value at which to evaluate or deflate polynomial
N	I	N	I	E	Degree of polynomial
P	C	UP,VP	D		Array of coefficients of polynomial
C	C	UC,VC	D	R	Updated at each iteration to yield derivative of polynomial at X
B	C	UB,VB	D		Array of coefficients of newly deflated polynomial
				Subroutine QUAD	
N	I	N	I	E	Degree of polynomial to be solved
A	C	UA,VA	D	E	Array of coefficients of polynomial to be solved
J	I	J	I	ECR	Number of distinct roots found of original polynomial (J = -1 implies original polynomial is of degree 2 or 1)
ROOT	C	UROOT,VROOT	D	ECR	Array of roots found
MULT	I	MULT	I	ECR	Array of multiplicities
DISC	C	UDISC,VDISC	D		Discriminate of quadratic
TEMP	C	UTEMP,VTEMP	D		$\sqrt{DISC}$
EPSLON	R	EPSLON	D	C	Tolerance for zero (0)
D	C	UD,VD	D		Twice leading coefficient of quadratic

TABLE E.VI (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine GCD
R	C	UR,VR	D	E	Array of coefficients of original polynomial, P(X)
S	C	US,VS	D	E	Array of coefficients of derivative polynomial, P'(X)
N	I	N	I	E	Degree of original polynomial, P(X)
M	I	M	I	E	Degree of derivative polynomial, P'(X)
RR	C	URR,VRR	D		Array of coefficients of dividend polynomial
SS	C	USS,VSS	D	R	Array of coefficients of divisor polynomial also array of coefficients of g.c.d. of P(X) and P'(X) when returned
N1	I	N1	I		Degree of dividend polynomial, RR(X)
M1	I	M1	I	R	Degree of divisor polynomial, SS(X), also degree of g.c.d. of P(X) and P'(X) when returned
D	C	UD,VD	D		Quotient RR_{N1+1}/SS_{M1+1}
T	C	UT,VT	D		Array of coefficients of difference polynomial(RR - D(SS))
K	I	K	I		Degree of difference polynomial T(X)
EPSLON	R	EPSLON	D	C	Tolerance check for zero (0)
					Subroutine MULTI
N	I	N	I	E	Degree of original polynomial, P(X)
P	C	UP,VP	D	E	Array of coefficients of original polynomial, P(X)
J	I	J	I	E	Number of distinct roots of P(X)
ROOT	C	UROOT,VROOT	D	E	Array of distinct roots of P(X)
A	C	UA,VA	D		Working array of coefficients of current polynomial
M	I	M	I		Degree of current polynomial, A(X)
MULT	I	MULT	I	R	Array of multiplicities of the roots
IO2	I	IO2	I	C	Unit number of output device
B	C	UB,VB	D		Array of coefficients of newly deflated polynomial
C	C	UC,VC	D		Derivative of polynomial at $ROOT_i$
EPSLON	R	EPSLON	D	C	Tolerance for checking multiplicities

TABLE E.VI (Continued)

Single Precision Variable	Type	Doubel Precision Variable	Type	Disposition of Argument	Description
					Subroutine DERIV
N	I	N	I	E	Degree of polynomial, P(X)
P	C	UP,VP	D	E	Array of coefficients of polynomial, P(X)
A	C	UA,VA	D	R	Array of coefficients of derivative, P'(X)
M	I	M	I	R	Degree of derivative polynomial, P'(X)
					Subroutine DIVIDE
P	C	UP,VP	D	E	Array of coefficients of dividend polynomial
N	I	N	I	E	Degree of dividend polynomial
D	C	UD,VD	D	E	Array of coefficients of divisor polynomial
M	I	M	I	E	Degree of divisor polynomial
Q	C	UQ,VQ	D	R	Array of coefficients of quotient polynomial P(X)/D(X)
K	I	K	I	R	Degree of quotient polynomial, Q(X)
J	I	J	I		Counter
TERM	C	UTERM,VTERM	D		Dummy variable used for temporary storage of products
KK	I	KK	I		Number of coefficients of quotient polynomial, Q(X)
					Subroutine GENAPP
APP	C	APPR,APPI	D	R	Array containing initial approximations
NAPP	I	NAPP	I	E	Number of initial approximations to be generated
XSTART	R	XSTART	D	ECR	Magnitude at which to begin generating approximations; also magnitude of the approximation being generated
BETA	R	BETA	D		Argument of complex approximation being generated
U	R	APPR(I)	D		Real part of complex approximation
V	R	APPI(I)	D		Imaginary part of complex approximation

TABLE E.VI (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine ALTER
XOLD	C	XOLDR,XOLDI	D	ECR	Old approximation to be altered to new approximation
NALTER	I	NALTER	I	ECR	Number of alterations performed on an initial approximation
ITIME	I	ITIME	I	E	Program control
MAX	I	MAX	I	C	Maximum number of iterations permitted
Y	R	XOLDI	D		Imaginary part of original initial approximation (unaltered)
X	R	XOLDR	D		Real part of original, unaltered initial approximation
R	R	ABXOLD	D		Magnitude of original unaltered initial approximation
BETA	R	BETA	D		Argument of new approximation
XOLDR	R	XOLDR	D		Real part of new approximation
XOLDI	R	XOLDI	D		Imaginary part of new approximation
IO2	I	IO2	I	C	Unit number of output device
					Subroutine COMSQT
		UX,VX	D	E	Complex number for which the square root is desired
		UY,VY	D	R	Square root of the complex number

4. Description of Program Output

The output from G.C.D. - Newton's method consists of the following information.

The heading is "GREATEST COMMON DIVISOR METHOD USED WITH NEWTON'S METHOD TO FIND ZEROS OF POLYNOMIALS NUMBER XX." XX represents the number of the polynomial.

As an aid to ensure that the control information is correct, the number of initial approximations given, maximum number of iterations, test for zero in subroutine GCD, test for convergence, test for zero in subroutine QUAD, test for multiplicities, radius to start search, and radius to end search are printed as read from the control card.

The coefficients of the polynomial are printed under the heading "THE DEGREE OF P(X) IS XX THE COEFFICIENTS ARE." XX represents the degree of the polynomial. The coefficient of the highest degree term is printed first.

The polynomial obtained after dividing the original polynomial, P(X), by the greatest common divisor of P(X) and its derivative, P'(X), is printed under the heading "Q(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE DISTINCT ROOTS OF P(X). THE DEGREE OF Q(X) IS XX THE COEFFICIENTS ARE." XX represents the degree of this polynomial. This polynomial contains all distinct roots and is solved by Newton's method. The coefficient of the highest degree term is printed first; that is, the leading coefficient is printed first.

The zeros found before the attempt to improve accuracy are printed under the heading "ROOTS OF Q(X)."

The initial approximation producing convergence to a root is printed to the right of the corresponding root and headed by "INITIAL APPROXIMATION." The initial approximations may be those supplied by the user, or generated by the program or a combination of both. The message "RESULTS OF SUBROUTINE QUAD" indicates that the corresponding root was obtained by subroutine QUAD. See Appendix D, § 5.

The zeros found after the attempt to improve accuracy are printed under the heading "ROOTS OF P(X)." The corresponding initial approximation producing convergence is printed as described above.

The multiplicity of each zero is given under the title "MULTIPLICITIES."

5. Informative Messages and Error Messages

The output may contain informative or error messages. These are intended as an aid to the user and are described as follows.

If not all roots of a polynomial were found before the attempt to improve accuracy, the remaining unsolved polynomial will be printed, with the leading coefficient first, under the heading "COEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO ZEROS WERE FOUND." See Appendix D, § 6.

"NO ROOTS FOR INITIAL APPROXIMATION ROOT XX = YYY." This message is printed if a root fails to produce convergence when trying to improve accuracy. XX represents the number of the root and YYY represents the value of the root before the attempt to improve accuracy.

"NO ROOTS FOR THE POLYNOMIAL Q(X) OF DEGREE XX WITH GENERATED INITIAL APPROXIMATIONS." XX represents the degree of the polynomial Q(X). This message is printed if none of the roots produce convergence in the attempt to improve accuracy.

"THE EPSILON (XXX) CHECK IN SUBROUTINE MULTI INDICATES THAT ROOT YY = ZZZ IS NOT CLOSE ENOUGH TO BE A TRUE ROOT. IT IS PRINTED BELOW WITH MULTIPLICITY 0." XXX represents the multiplicity requirement (EPS4 on the control card), YY represents the number of the root, and ZZZ represents the value of the root after the attempt to improve accuracy. The message indicates that this root does not meet the requirement for multiplicities. It is, however, usually a good approximation to the true root since convergence was obtained both before and after the attempt to improve accuracy.

MAIN PROGRAM

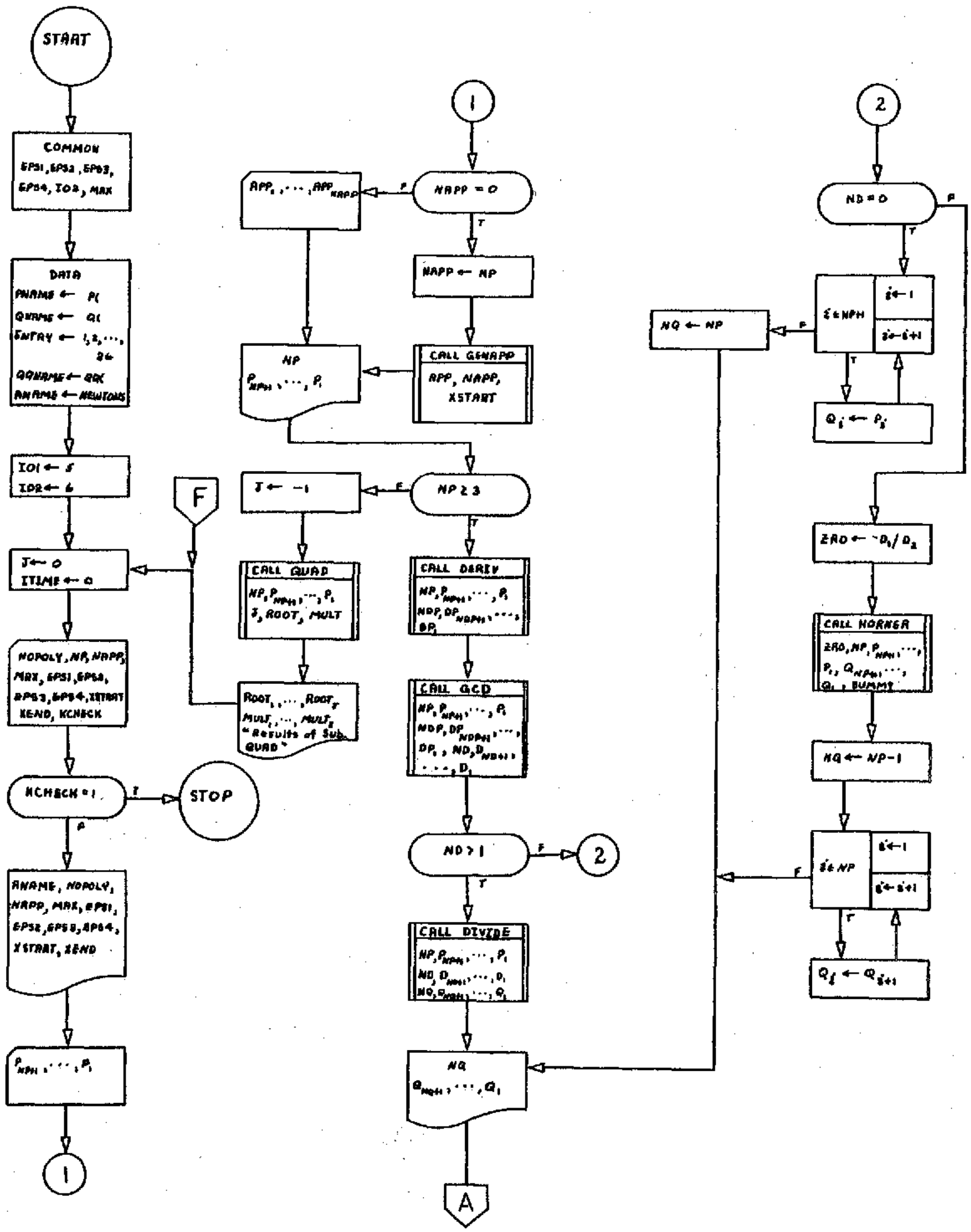


Figure E.6. Flow Charts for G.C.D.-Newton's Method

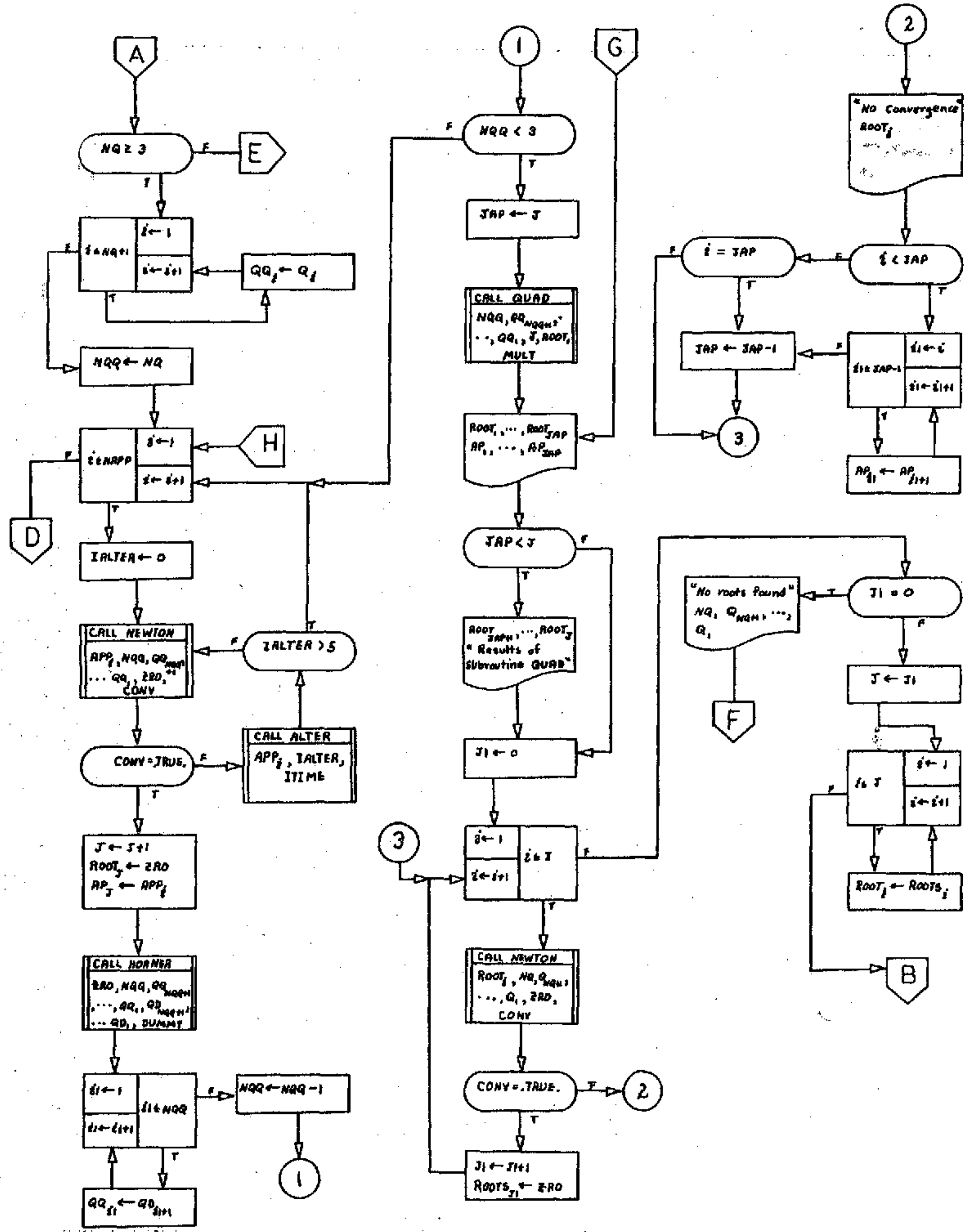


Figure E.6. (Continued)

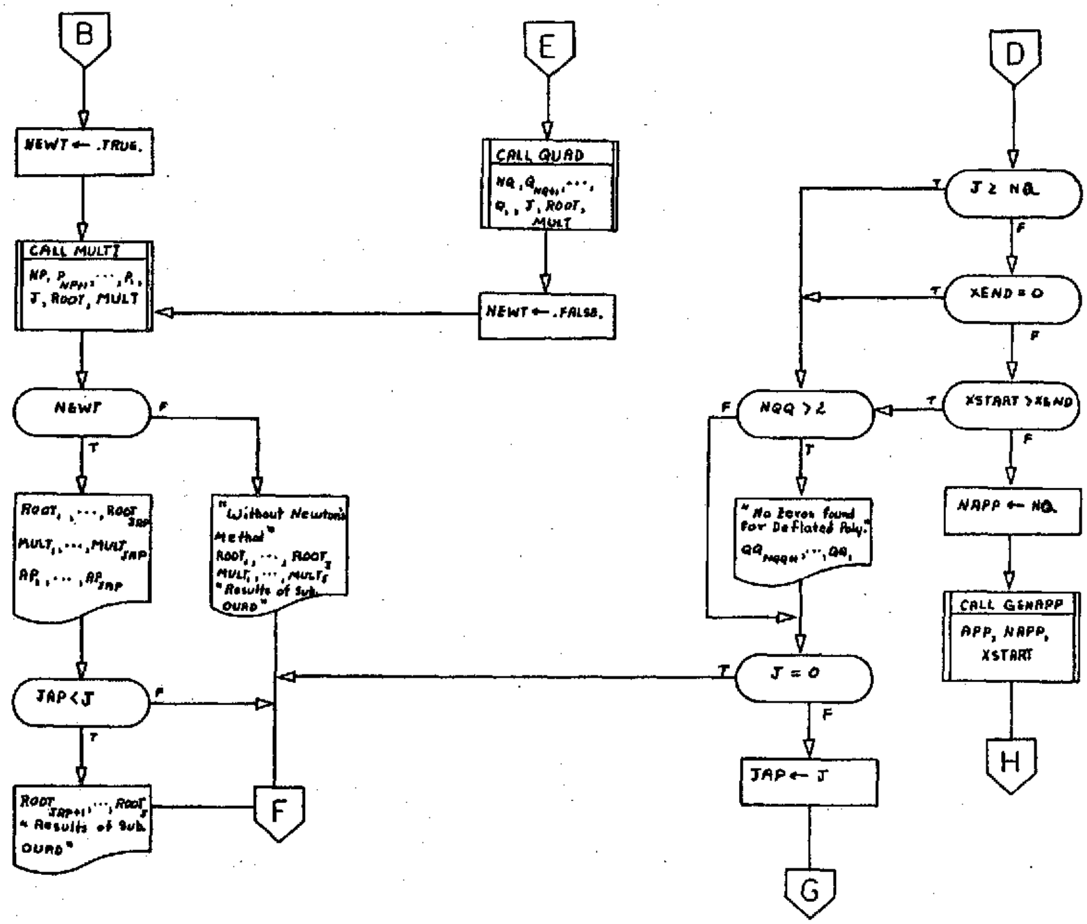
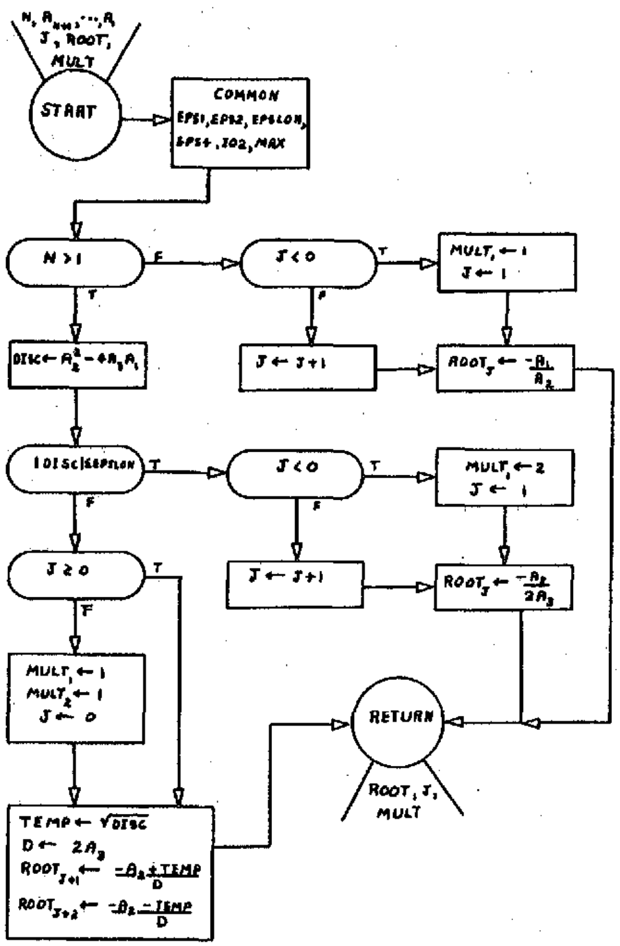


Figure E.6. (Continued)



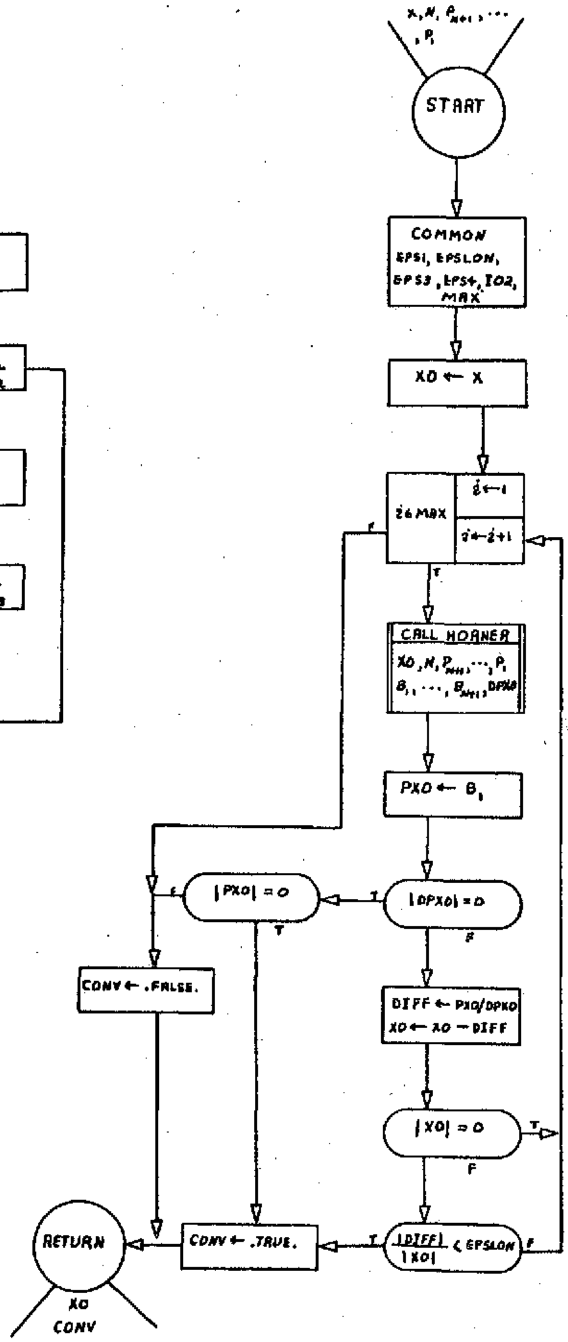
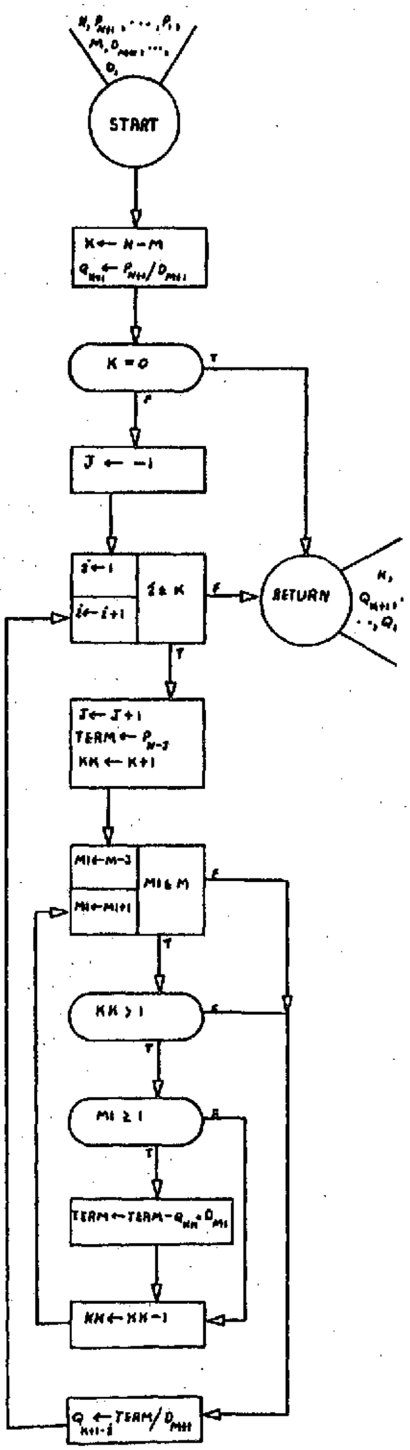
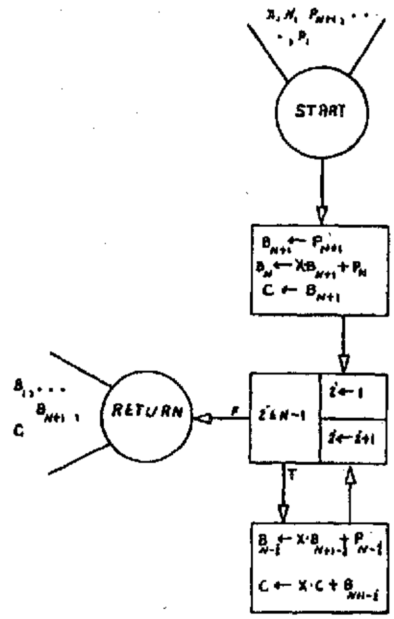


Figure E.6. (Continued)

DIVIDE



HORNER



DERIV

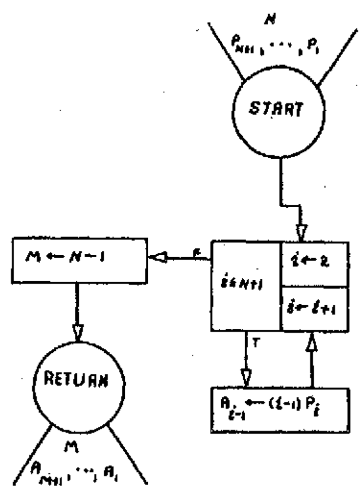


Figure E.6. (Continued)

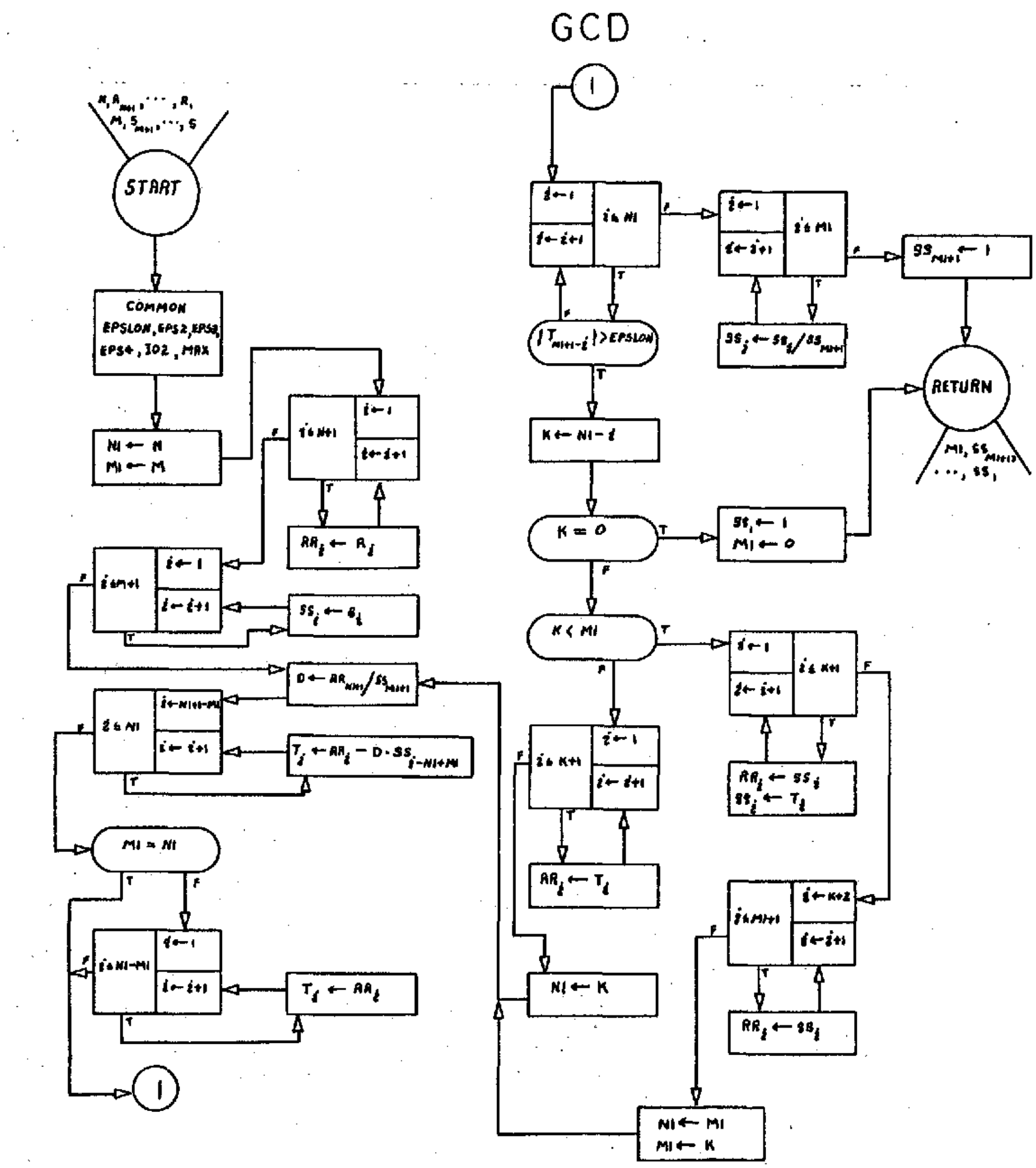


Figure E.6. (Continued)

MULTI

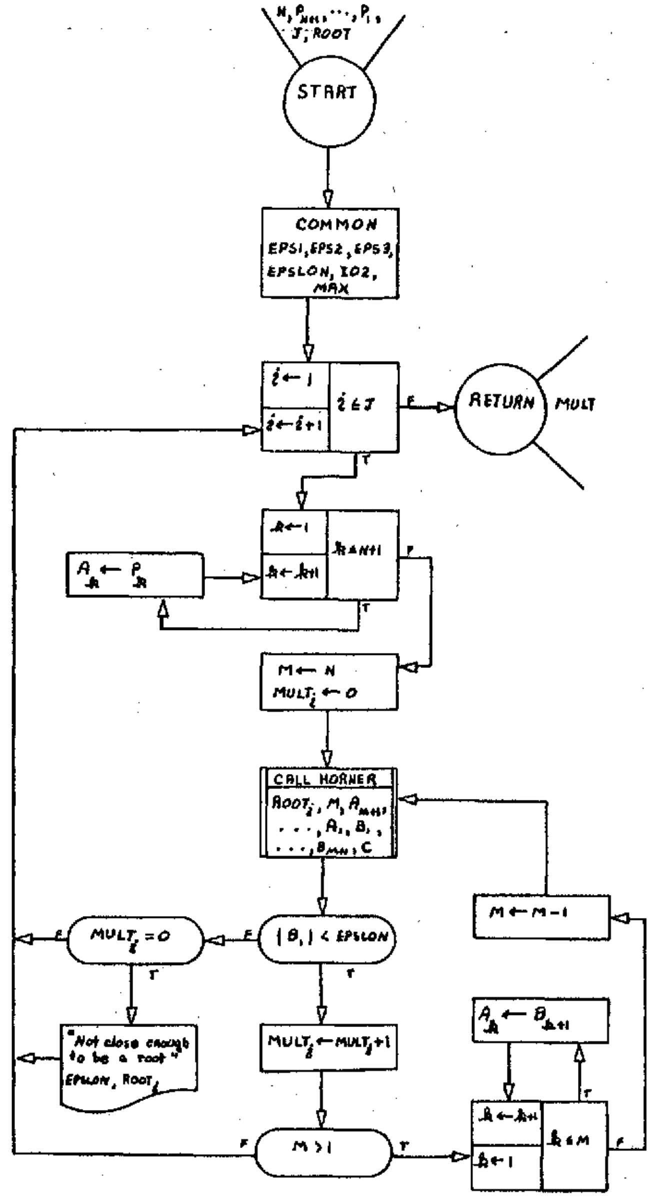


Figure E.6. (Continued)

COMSQT

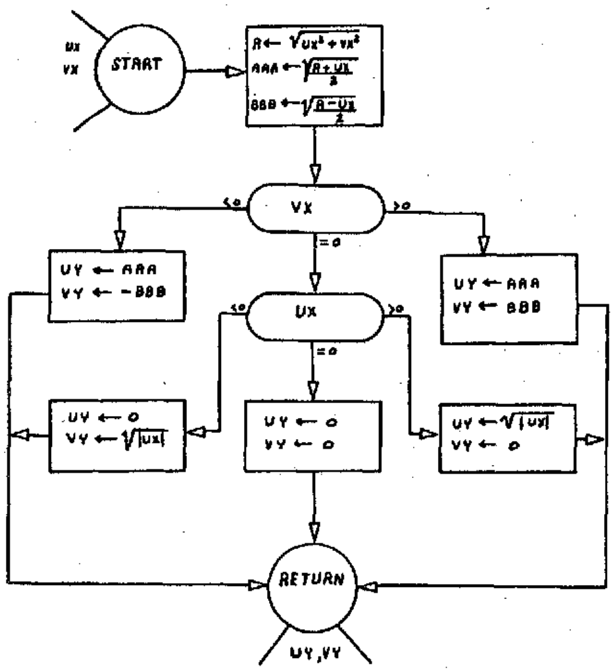
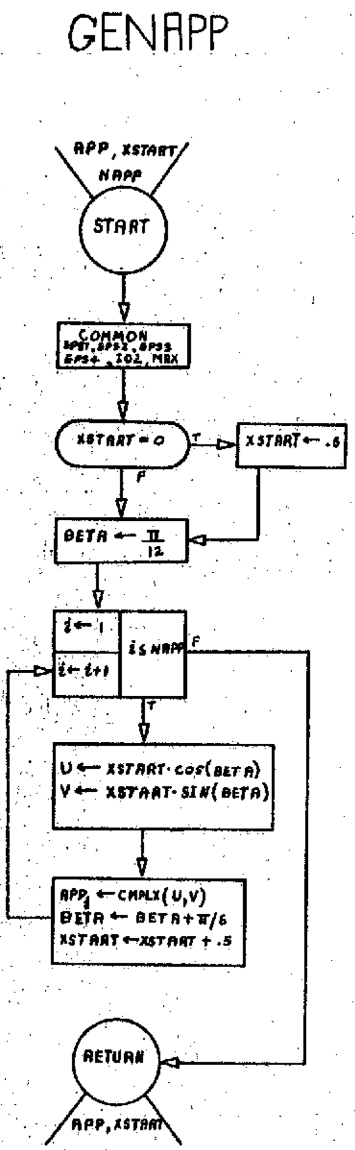


Figure E.6. (Continued)



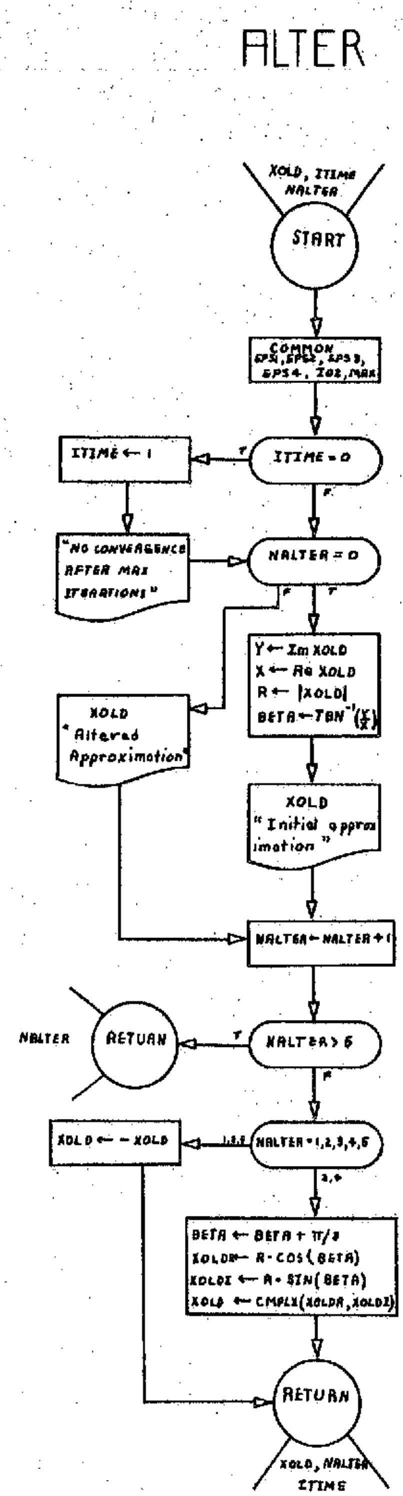


Figure E.6. (Continued)

TABLE E.VII

PROGRAM FOR G.C.D.-NEWTON'S METHOD

```
C     ***********************************************************************
C     *                                                                     *
C     * DOUBLE PRECISION PROGRAM FOR G.C.D. - NEWTON'S METHOD              *
C     *                                                                     *
C     *                                                                     *
C     * THE G.C.D. METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A  *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25.  ALL MULTIPLE ROOTS ARE REMOVED BY *
C     * DIVIDING THE POLYNOMIAL BY THE GREATEST COMMON DIVISOR OF THE POLYNOMIAL *
C     * AND ITS DERIVATIVE.  THE ZEROS OF THE RESULTING POLYNOMIAL ARE EXTRACTED *
C     * AND THEIR MULTIPLICITIES DETERMINED.                                *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UP,VP,UAPP,VAPP,UROOT,VROOT,UDP,VDP,UD,VD,UZRO,VZ
     1RO,UQ,VQ,UDUMMY,VDUMMY,UQQ,VQQ,UAP,VAP,UQD,VQD,UROOTS,VROOTS,EPS1,
     2EPS2,EPS3,EPS4
      DOUBLE PRECISION XSTART
      DOUBLE PRECISION XEND
      DIMENSION UP(26),VP(26),UAPP(25),VAPP(25),UROOT(25),VROOT(25),MULT
     1(25),UDP(26),VDP(26),UD(26),VD(26),UQ(26),VQ(26),UQQ(26),VQQ(26),U
     2AP(25),VAP(25),UQD(26),VQD(26),ANAME(2),ENTRY(26),UROOTS(25),VROOT
     3S(25)
      COMMON EPS1,EPS2,EPS3,EPS4,IO2,MAX
      LOGICAL NEWT,CONV
      DATA PNAME,QNAME,QQNAME/2HP(,2HQ(,3HQQ(/
      DATA ENTRY/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,2H10,2H11,2H12,2H13
     1,2H14,2H15,2H16,2H17,2H18,2H19,2H20,2H21,2H22,2H23,2H24,2H25,2H26/
      DATA ANAME(1),ANAME(2)/4HNEWT,4HONS /
      IO1=5
      IO2=6
   10 J=0
      ITIME=0
      READ(IO1,1000) NOPOLY,NP,NAPP,MAX,EPS1,EPS2,EPS3,EPS4,XSTART,XEND,
     1KCHECK
      IF(KCHECK.EQ.1) STOP
      WRITE(IO2,1020) ANAME(1),ANAME(2),NOPOLY
      WRITE(IO2,2000) NAPP
      WRITE(IO2,2010) MAX
      WRITE(IO2,2070) EPS1
      WRITE(IO2,2020) EPS2
      WRITE(IO2,2080) EPS3
      WRITE(IO2,2030) EPS4
      WRITE(IO2,2040) XSTART
      WRITE(IO2,2050) XEND
      WRITE(IO2,2060)
      KKK=NP+1
      NNN=KKK+1
      DO 20 I=1,KKK
      JJJ=NNN-I
   20 READ(IO1,1010) UP(JJJ),VP(JJJ)
      IF(NAPP.NE.0) GO TO 22
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 23
   22 READ(IO1,1015) (UAPP(I),VAPP(I),I=1,NAPP)
   23 WRITE(IO2,1030) NP
      KKK=NP+1
      NNN=KKK+1
      DO 25 I=1,KKK
```

TABLE E.VII (Continued)

```
      JJJ=NNN-1
   25 WRITE(IO2,1040) PNAME,ENTRY(JJJ),UP(JJJ),VP(JJJ)
      IF(NP.GE.3) GO TO 30
      J=-1
      CALL QUAD(NP,UP,VP,J,UROOT,VROOT,MULT)
      WRITE(IO2,1070)
      WRITE(IO2,1165) (I,UROOT(I),VROOT(I),MULT(I),I=1,J)
      GO TO 10
   30 CALL DERIV(NP,UP,VP,NDP,UDP,VDP)
      CALL GCD(NP,UP,VP,NDP,UDP,VDP,ND,UD,VD)
      IF(ND.GT.1) GO TO 70
      IF(ND.EQ.0) GO TO 65
      UDUMMY=(UD(2)*UD(2))+(VD(2)*VD(2))
      UZRO=(-(UD(1)*UD(2))-(VD(1)*VD(2)))/UDUMMY
      VZRO=(-(UD(2)*VD(1))+(UD(1)*VD(2)))/UDUMMY
      CALL HORNER(UZRO,VZRO,NP,UP,VP,UQ,VQ,UDUMMY,VDUMMY)
      NQ=NP-1
      DO 60 I=1,NP
      UQ(I)=UQ(I+1)
   60 VQ(I)=VQ(I+1)
      GO TO 80
   65 KKK=NP+1
      DO 66 I=1,KKK
      UQ(I)=UP(I)
   66 VQ(I)=VP(I)
      NQ=NP
      GO TO 80
   70 CALL DIVIDE(NP,UP,VP,ND,UD,VD,NQ,UQ,VQ)
   80 WRITE(IO2,1120) NQ
      KKK=NQ+1
      NNN=KKK+1
      DO 83 I=1,KKK
      JJJ=NNN-I
   83 WRITE(IO2,1040) QNAME,ENTRY(JJJ),UQ(JJJ),VQ(JJJ)
      IF(NQ.GE.3) GO TO 85
      GO TO 110
   85 KKK=NQ+1
      DO 90 I=1,KKK
      UQQ(I)=UQ(I)
   90 VQQ(I)=VQ(I)
      NQQ=NQ
      GO TO 120
  110 CALL QUAD(NQ,UQ,VQ,J,UROOT,VROOT,MULT)
      NEWT=.FALSE.
      GO TO 310
  120 DO 200 I=1,NAPP
      IALTER=0
  130 CALL NEWTON(UAPP(I),VAPP(I),NQQ,UQQ,VQQ,UZRO,VZRO,CONV)
      IF(CONV) GO TO 160
      CALL ALTER(UAPP(I),VAPP(I),IALTER,ITIME)
      IF(IALTER.GT.5) GO TO 200
      GO TO 130
  160 J=J+1
      UROOT(J)=UZRO
      VROOT(J)=VZRO
      UAP(J)=UAPP(I)
      VAP(J)=VAPP(I)
      CALL HORNER(UZRO,VZRO,NQQ,UQQ,VQQ,UQD,VQD,UDUMMY,VDUMMY)
```

TABLE E.VII (Continued)

```
      DO 180 I1=1,NQQ
      UQQ(I1)=UQD(I1+1)
  180 VQQ(I1)=VQD(I1+1)
      NQQ=NQQ-1
      IF(NQQ.LT.3) GO TO 220
  200 CONTINUE
      IF(J.GE.NQ) GO TO 205
      IF(XEND.EQ.0.0) GO TO 205
      IF(XSTART.GT.XEND) GO TO 205
      NAPP=NQ
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 120
  205 IF(NQQ.LE.2) GO TO 210
      WRITE(IO2,1200)
      KKK=NQQ+1
      NNN=KKK+1
      DO 157 L=1,KKK
      JJJ=NNN-L
  157 WRITE(IO2,1100) QQNAME,ENTRY(JJJ),UQQ(JJJ),VQQ(JJJ)
  210 IF(J.EQ.0) GO TO 10
      JAP=J
      GO TO 230
  220 JAP=J
      CALL QUAD(NQQ,UQQ,VQQ,J,UROOT,VROOT,MULT)
  230 WRITE(IO2,1132)
      WRITE(IO2,1133) (I,UROOT(I),VROOT(I),UAP(I),VAP(I),I=1,JAP)
      IF(JAP.LT.J) GO TO 235
      GO TO 240
  235 KKK=JAP+1
      WRITE(IO2,1134) (I,UROOT(I),VROOT(I),I=KKK,J)
  240 J1=0
      DO 300 I=1,J
      CALL NEWTON(UROOT(I),VROOT(I),NQ,UQ,VQ,UZRO,VZRO,CONV)
      IF(CONV) GO TO 280
      WRITE(IO2,1140) I,UROOT(I),VROOT(I)
      IF(I.LT.JAP) GO TO 241
      IF(I.EQ.JAP) GO TO 250
      GO TO 300
  241 KKK=JAP-1
      DO 245 I1=I,KKK
      UAP(I1)=UAP(I1+1)
  245 VAP(I1)=VAP(I1+1)
  250 JAP=JAP-1
      GO TO 300
  280 J1=J1+1
      UROOTS(J1)=UZRO
      VROOTS(J1)=VZRO
  300 CONTINUE
      IF(J1.EQ.0) GO TO 305
      J=J1
      DO 303 I=1,J
      UROOT(I)=UROOTS(I)
  303 VROOT(I)=VROOTS(I)
      GO TO 307
  305 WRITE(IO2,1150) NQ
      KKK=NQ+1
      NNN=KKK+1
      DO 306 L=1,KKK
```

TABLE E.VII (Continued)

```
      JJJ=NNN-L
  306 WRITE(IO2,1040) QNAME,ENTRY(JJJ),UQ(JJJ),VQ(JJJ)
      GO TO 10
  307 NEWT=.TRUE.
  310 CALL MULTI(NP,UP,VP,J,UROOT,VROOT,MULT)
      IF(NEWT) GO TO 330
      WRITE(IO2,1070)
      WRITE(IO2,1165) (L,UROOT(L),VROOT(L),MULT(L),L=1,J)
      GO TO 10
  330 WRITE(IO2,1180)
      WRITE(IO2,1190) (L,UROOT(L),VROOT(L),MULT(L),UAP(L),VAP(L),L=1,JAP
     1)
      KKK=JAP+1
      IF(JAP.LT.J) WRITE(IO2,1165) (L,UROOT(L),VROOT(L),MULT(L),L=KKK,J)
      GO TO 10
 1000 FORMAT(3(I2,1X),9X,I3,1X,4(D6.0,1X),13X,2(D7.0,1X),I1)
 1010 FORMAT(2D30.0)
 1015 FORMAT(2D30.0)
 1020 FORMAT(1H1,10X,41HGREATEST COMMON DIVISOR METHOD USED WITH ,2(A4),
     135HMETHOD TO FIND ZEROS OF POLYNOMIALS/11X,18HPOLYNOMIAL NUMBER ,I
     22///)
 1030 FORMAT(1X,22HTHE DEGREE OF P(X) IS ,I2,22H  THE COEFFICIENTS ARE//
     1)
 1040 FORMAT(2X,A2,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1070 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES//)
 1080 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,10X,I2)
 1100 FORMAT(2X,A3,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1120 FORMAT(///1X,73HQ(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE
     1DISTINCT ROOTS OF P(X)./1X,22HTHE DEGREE OF Q(X) IS ,I2,22H  THE C
     2OEFFICIENTS ARE//)
 1200 FORMAT(///1X,70HCOEFFICIENTS OF THE DEFLATED POLYNOMIAL FOR WHICH
     1NO ZEROS WERE FOUND.//)
 1132 FORMAT(///1X,13HROOTS OF Q(X),84X,21HINITIAL APPROXIMATION//)
 1133 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,17X,D23.16,3H
     1 + ,D23.16,2H I)
 1134 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,20X,26HRESULT
     1S OF SUBROUTINE QUAD)
 1140 FORMAT(///,1X,40HNO ROOTS FOR INITIAL APPROXIMATION ROOT(,I2,4H) =
     1 ,D23.16,3H + ,D23.16,2H I)
 1150 FORMAT(///,1X,45HNO ROOTS FOR THE POLYNOMIAL Q(X) OF DEGREE = ,I2,
     138H WITH GENERATED INITIAL APPROXIMATIONS//)
 1165 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,10X,26H
     1RESULTS OF SUBROUTINE QUAD)
 1180 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
     1 APPROXIMATION//)
 1190 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,7X,D23.
     116,3H + ,D23.16,2H I)
 2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.   ,I2)
 2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
 2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
 2030 FORMAT(1X,24HTEST FOR MULTIPLICITIES.,10X,D9.2)
 2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
 2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
 2060 FORMAT(//1X)
 2070 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE GCD.  ,D9.2)
 2080 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE QUAD. ,D9.2)
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C    ***************************************************************************
C    *                                                                         *
C    * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE   *
C    * DEGREE OF THE ORIGINAL POLYNOMIAL.                                      *
C    *                                                                         *
C    ***************************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,BETA,    EPS1,EPS2,EPS3,EPS4
      DIMENSION APPR(25),APPI(25)
      COMMON EPS1,EPS2,EPS3,EPS4,IO2,MAX
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I)=XSTART*DCOS(BETA)
      APPI(I)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE ALTER(XOLDR,XOLDI,NALTER,ITIME)
C     ***********************************************************************
C     *                                                                     *
C     * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO  *
C     * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION XOLDR,XOLDI,ABXOLD,BETA,EPS1,EPS2,EPS3,EPS4
      COMMON EPS1,EPS2,EPS3,EPS4,IO2,MAX
      IF(ITIME.NE.0) GO TO 5
      ITIME =1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) XOLDR,XOLDI
      GO TO 20
   10 ABXOLD=DSQRT((XOLDR*XOLDR)+(XOLDI*XOLDI))
      BETA=DATAN2(XOLDI,XOLDR)
      WRITE(IO2,1020) XOLDR,XOLDI
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 XOLDR=-XOLDR
      XOLDI=-XOLDI
      GO TO 50
   40 BETA=BETA+1.0471976
      XOLDR=ABXOLD*DCOS(BETA)
      XOLDI=ABXOLD*DSIN(BETA)
   50 RETURN
 1000 FORMAT(1X,D23.16,3H + ,D23.16,2H I,10X,21HALTERED APPROXIMATION)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     ITER ,I3,12H ITERATIONS.//)
 1020 FORMAT(/1X,D23.16,3H + ,D23.16,2H I,10X,21HINITIAL APPROXIMATION)
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE GCD(N,UR,VR,M,US,VS,M1,USS,VSS)
C     ***********************************************************************
C     *                                                                     *
C     * GIVEN POLYNOMIALS P(X) AND DP(X) WHERE DEG. DP(X) IS LESS THAN DEG. *
C     * P(X), SUBROUTINE GCD COMPUTES THE GREATEST COMMON DIVISOR OF P(X) AND *
C     * DP(X).                                                              *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION USSSSS,VSSSSS
      DOUBLE PRECISION UR,VR,US,VS,USS,VSS,URR,VRR,UD,VD,UT,VT,EPSLON,EP
     1S2,EPS3,EPS4,BBB
      DIMENSION UR(26),VR(26),US(26),VS(26),USS(26),VSS(26),URR(26),VRR(
     126),UT(26),VT(26)
      COMMON EPSLON,EPS2,EPS3,EPS4,IO2,MAX
      N1=N
      M1=M
      KKK=N+1
      DO 20 I=1,KKK
      URR(I)=UR(I)
   20 VRR(I)=VR(I)
      KKK=M+1
      DO 25 I=1,KKK
      USS(I)=US(I)
   25 VSS(I)=VS(I)
   30 BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      UD=(URR(N1+1)*USS(M1+1)+VRR(N1+1)*VSS(M1+1))/BBB
      VD=(USS(M1+1)*VRR(N1+1)-URR(N1+1)*VSS(M1+1))/BBB
      KKK=N1+1-M1
      DO 40 I=KKK,N1
      UT(I)=URR(I)-(UD*USS(I-N1+M1)-VD*VSS(I-N1+M1))
   40 VT(I)=VRR(I)-(UD*VSS(I-N1+M1)+VD*USS(I-N1+M1))
      IF(M1.EQ.N1) GO TO 70
      KKK=N1-M1
      DO 60 I=1,KKK
      UT(I)=URR(I)
   60 VT(I)=VRR(I)
   70 DO 90 I=1,N1
      BBB=DSQRT(UT(N1+1-I)*UT(N1+1-I)+VT(N1+1-I)*VT(N1+1-I))
      IF(BBB.GT.EPSLON) GO TO 100
   90 CONTINUE
      DO 95 I=1,M1
      BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      USSSSS=(USS(I)*USS(M1+1)+VSS(I)*VSS(M1+1))/BBB
      VSSSSS=(VSS(I)*USS(M1+1)-USS(I)*VSS(M1+1))/BBB
      USS(I)=USSSSS
   95 VSS(I)=VSSSSS
      USS(M1+1)=1.0
      VSS(M1+1)=0.0
      GO TO 200
  100 K=N1-I
      IF(K.EQ.0) GO TO 170
      IF(K.LT.M1) GO TO 140
      KKK=K+1
      DO 130 J=1,KKK
      URR(J)=UT(J)
  130 VRR(J)=VT(J)
      N1=K
      GO TO 30
```

TABLE E.VII (Continued)

```
  140 KKK=K+1
      DO 150 J=1,KKK
      URR(J)=USS(J)
      VRR(J)=VSS(J)
      USS(J)=UT(J)
  150 VSS(J)=VT(J)
      KKK=K+2
      NNN=M1+1
      DO 160 J=KKK,NNN
      URR(J)=USS(J)
  160 VRR(J)=VSS(J)
      N1=M1
      M1=K
      GO TO 30
  170 USS(1)=1.0
      VSS(1)=0.0
      M1=0
  200 RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE QUAD(N,UA,VA,J,UROOT,VROOT,MULT)
C     ***********************************************************************
C     *                                                                     *
C     * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES *
C     * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE *
C     * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                      *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UA,VA,UROOT,VROOT,UDISC,VDISC,UTEMP,VTEMP,UD,VD,E
     1PS1,EPS2,EPS4,EPSLON,BBB
      DIMENSION UA(26),VA(26),UROOT(25),VROOT(25),MULT(25)
      COMMON EPS1,EPS2,EPSLON,EPS4,IO2,MAX
      IF(N.GT.1) GO TO 60
      IF(J.LT.0) GO TO 40
      J=J+1
      GO TO 50
   40 MULT(1)=1
      J=1
   50 BBB=UA(2)*UA(2)+VA(2)*VA(2)
      UROOT(J)=-(UA(1)*UA(2)+VA(1)*VA(2))/BBB
      VROOT(J)=-(VA(1)*UA(2)-UA(1)*VA(2))/BBB
      GO TO 200
   60 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(3)*UA(1)-VA(3)*VA(1)))
      VDISC=(2.0*UA(2)*VA(2))-(4.0*(UA(3)*VA(1)+VA(3)*UA(1)))
      BBB=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(BBB.LE.EPSLON) GO TO 100
      IF(J.GE.0) GO TO 80
      MULT(1)=1
      MULT(2)=1
      J=0
   80 CALL COMSQT(UDISC,VDISC,UTEMP,VTEMP)
      UD=2.0*UA(3)
      VD=2.0*VA(3)
      BBB=UD*UD+VD*VD
      UROOT(J+1)=((-UA(2)+UTEMP)*UD+(-VA(2)+VTEMP)*VD)/BBB
      VROOT(J+1)=((-VA(2)+VTEMP)*UD-(-UA(2)+UTEMP)*VD)/BBB
      UROOT(J+2)=((-UA(2)-UTEMP)*UD+(-VA(2)-VTEMP)*VD)/BBB
      VROOT(J+2)=((-VA(2)-VTEMP)*UD-(-UA(2)-UTEMP)*VD)/BBB
      J=J+2
      GO TO 200
  100 IF(J.LT.0) GO TO 110
      J=J+1
      GO TO 130
  110 MULT(1)=2
      J=1
  130 UD=2.0*UA(3)
      VD=2.0*VA(3)
      BBB=UD*UD+VD*VD
      UROOT(J)=(-UA(2)*UD-VA(2)*VD)/BBB
      VROOT(J)=(-VA(2)*UD+UA(2)*VD)/BBB
  200 RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE NEWTON(UX,VX,N,UP,VP,UXO,VXO,CONV)
C     ***********************************************************************
C     *                                                                     *
C     * THIS SUBROUTINE CALCULATES A NEW APPROXIMATION FROM THE OLD APPROX- *
C     * IMATION BY USING THE ITERATION FORMULA                              *
C     *              X(N+1) = X(N)-P(X(N))/P'(X(N)).                        *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UX,VX,UP,VP,UXO,VXO,UB,VB,UDPXO,VDPXO,UPXO,VPXO,U
     1DIFF,VDIFF,EPS1,EPSLON,EPS3,EPS4,AAA,BBB
      DOUBLE PRECISION DDD
      DOUBLE PRECISION ABPXO
      DIMENSION UP(26),VP(26),UB(26),VB(26)
      COMMON EPS1,EPSLON,EPS3,EPS4,IO2,MAX
      LOGICAL CONV
      UXO=UX
      VXO=VX
      DO 10 I=1,MAX
      CALL HORNER(UXO,VXO,N,UP,VP,UB,VB,UDPXO,VDPXO)
      UPXO=UB(1)
      VPXO=VB(1)
      DDD=DSQRT(UDPXO*UDPXO+VDPXO*VDPXO)
      IF(DDD.NE.0.0) GO TO 5
      ABPXO=DSQRT(UPXO*UPXO+VPXO*VPXO)
      IF(ABPXO.EQ.0.0) GO TO 20
      GO TO 15
    5 BBB=UDPXO*UDPXO+VDPXO*VDPXO
      UDIFF=(UPXO*UDPXO+VPXO*VDPXO)/BBB
      VDIFF=(VPXO*UDPXO-UPXO*VDPXO)/BBB
      UXO=UXO-UDIFF
      VXO=VXO-VDIFF
      AAA=DSQRT(UDIFF*UDIFF+VDIFF*VDIFF)
      BBB=DSQRT(UXO*UXO+VXO*VXO)
      IF(BBB.EQ.0.0) GO TO 10
      IF(AAA/BBB.LT.EPSLON) GO TO 20
   10 CONTINUE
   15 CONV=.FALSE.
      RETURN
   20 CONV=.TRUE.
      RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE DIVIDE(N,UP,VP,M,UD,VD,K,UQ,VQ)
C     ***********************************************************************
C     *                                                                     *
C     * GIVEN TWO POLYNOMIALS F(X) AND G(X), SUBROUTINE DIVIDE COMPUTES THE *
C     * QUOTIENT POLYNOMIAL  H(X) = F(X)/G(X).                              *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UP,VP,UD,VD,UQ,VQ,UTERM,VTERM,UDUMMY
      DIMENSION UP(26),VP(26),UD(26),VD(26),UQ(26),VQ(26)
      K=N-M
      UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1)=(UP(N+1)*UD(M+1)+VP(N+1)*VD(M+1))/UDUMMY
      VQ(K+1)=(VP(N+1)*UD(M+1)-UP(N+1)*VD(M+1))/UDUMMY
      IF(K.EQ.0) GO TO 100
      J=-1
      DO 50 I=1,K
      J=J+1
      UTERM=UP(N-J)
      VTERM=VP(N-J)
      KK=K+1
      NNN=M-J
      DO 40 M1=NNN,M
      IF(KK.GT.1) GO TO 10
      GO TO 45
   10 IF(M1.GE.1) GO TO 20
      GO TO 40
   20 UTERM=UTERM-(UQ(KK)*UD(M1)-VQ(KK)*VD(M1))
      VTERM=VTERM-(UQ(KK)*VD(M1)+VQ(KK)*UD(M1))
   40 KK=KK-1
   45 UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1-I)=(UTERM*UD(M+1)+VTERM*VD(M+1))/UDUMMY
   50 VQ(K+1-I)=(VTERM*UD(M+1)-UTERM*VD(M+1))/UDUMMY
  100 RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE HORNER(UX,VX,N,UP,VP,UB,VB,UC,VC)
C    ***********************************************************************
C    *                                                                     *
C    * HORNER'S METHOD COMPUTES THE VALUE OF THE POLYNOMIAL P(X) AT A      *
C    * POINT D AND ITS DERIVATIVE AT D.  SYNTHETIC DIVISION IS USED TO     *
C    * DEFLATE THE POLYNOMIAL BY DIVIDING OUT THE FACTOR (X - D).          *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION UX,VX,UP,VP,UB,VB,UC,VC
      DOUBLE PRECISION UDUMMY,VDUMMY
      DIMENSION UP(26),VP(26),UB(26),VB(26)
      UB(N+1)=UP(N+1)
      VB(N+1)=VP(N+1)
      UB(N)=(UX*UB(N+1)-VX*VB(N+1))+UP(N)
      VB(N)=(UX*VB(N+1)+VX*UB(N+1))+VP(N)
      UC=UB(N+1)
      VC=VB(N+1)
      KKK=N-1
      DO 10 I=1,KKK
      UB(KKK+1-I)=(UX*UB(KKK+2-I)-VX*VB(KKK+2-I))+UP(KKK+1-I)
      VB(KKK+1-I)=(UX*VB(KKK+2-I)+VX*UB(KKK+2-I))+VP(KKK+1-I)
      UDUMMY=UX*UC-VX*VC
      VDUMMY=UX*VC+VX*UC
      UC=UDUMMY+UB(KKK+2-I)
   10 VC=VDUMMY+VB(KKK+2-I)
      RETURN
      END
```

```
      SUBROUTINE DERIV(N,UP,VP,M,UA,VA)
C    ***********************************************************************
C    *                                                                     *
C    * GIVEN A POLYNOMIAL P(X), SUBROUTINE DERIV COMPUTES THE COEFFICIENTS OF *
C    * ITS DERIVATIVE P'(X).                                               *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION UP,VP,UA,VA,AAA
      DIMENSION UP(26),VP(26),UA(26),VA(26)
      KKK=N+1
      DO 10 I=2,KKK
      AAA=I-1
      UA(I-1)=AAA*UP(I)
   10 VA(I-1)=AAA*VP(I)
      M=N-1
      RETURN
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE MULTI(N,UP,VP,J,UROOT,VROOT,MULT)
C   ************************************************************************
C   *                                                                      *
C   * GIVEN N ZEROS OF A POLYNOMIAL, SUBROUTINE MULTI COMPUTES THEIR       *
C   * MULTIPLICITIES.                                                      *
C   *                                                                      *
C   ************************************************************************
      DOUBLE PRECISION UP,VP,UROOT,VROOT,UA,VA,UB,VB,UC,VC,EPS1,EPS2,EPS
     1LON,EPS3,BBB
      DIMENSION UP(26),VP(26),UROOT(25),VROOT(25),UA(26),VA(26),UB(26),V
     1B(26),MULT(25)
      COMMON EPS1,EPS2,EPS3,EPSLON,IO2,MAX
      DO 100 I=1,J
      KKK=N+1
      DO 10 K=1,KKK
      UA(K)=UP(K)
   10 VA(K)=VP(K)
      M=N
      MULT(I)=0
   20 CALL HORNER(UROOT(I),VROOT(I),M,UA,VA,UB,VB,UC,VC)
      BBB=DSQRT(UB(1)*UB(1)+VB(1)*VB(1))
      IF(BBB.LT.EPSLON) GO TO 50
      IF(MULT(I).EQ.0) GO TO 40
      GO TO 100
   40 WRITE(IO2,1000) EPSLON,I,UROOT(I),VROOT(I)
      GO TO 100
   50 MULT(I)=MULT(I)+1
      IF(M.GT.1) GO TO 60
      GO TO 100
   60 DO 70 K=1,M
      UA(K)=UB(K+1)
   70 VA(K)=VB(K+1)
      M=M-1
      GO TO 20
  100 CONTINUE
      RETURN
 1000 FORMAT(///15H THE EPSILON  (,D10.3,48H) CHECK IN SUBROUTINE MULTI
     1INDICATES THAT ROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,/80H IS NO
     2T CLOSE ENOUGH TO BE A TRUE ROOT.  IT IS PRINTED BELOW WITH MULTIP
     3LICITY 0//)
      END
```

TABLE E.VII (Continued)

```
      SUBROUTINE CONSQT(UX,VX,UY,VY)
C    ************************************************************************
C    *                                                                      *
C    * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.        *
C    *                                                                      *
C    ************************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

APPENDIX F

G.C.D. - MULLER'S METHOD

1. Use of the Program

A double precision FORTRAN IV program using the G.C.D. method with Muller's method as a supporting method is presented here. Flow charts for this program are given in Figure F.1 while Table F.III gives a FORTRAN IV listing of this program. Single precision variables are listed in Table F.II. The single precision variables are used in the flow charts and the corresponding double precision variables can be obtained from Table F.II.

This program is designed to solve polynomials having degree less than or equal to 25. In order to solve polynomials of degree N where N > 25, the data statement and array dimensions given in Table F.I must be changed.

In this program both the leading coefficient and the constant coefficient are assumed to be non-zero.

TABLE F.I

PROGRAM CHANGES NECESSARY TO SOLVE POLYNOMIALS OF DEGREE GREATER THAN 25 BY G.C.D. - MULLER'S METHOD

Main Program

Data Entry/1H1,1H2,...,1H9,2H10,2H11,...,2HXX/where XX = N+1

URAPP(N,3), VRAPP(N,3)
UAPP(N,3), VAPP(N,3)
UP(N+1), VP(N+1)
UROOT(N), VROOT(N)
MULT(N)
UDP(N+1), VDP(N+1)
UD(N+1), VD(N+1)
UQ(N+1), VQ(N+1)
UQQ(N+1), VQQ(N+1)
UB(N+1), VB(N+1)
ENTRY(N+1)

Subroutines MULTI, DIVIDE, DERIV, GCD, and QUAD

See corresponding subroutines in Table E.I.

Subroutine MULLER

UROOT(N), VROOT(N)
MULT(N)
UAPP(N,3), VAPP(n,3)
UWORK(N+1), VWORK(N+1)
UB(N+1), VB(N+1)
UA(N+1), VA(N+1)
URAPP(N,3), VRAPP(N,3)

Subroutine BETTER

UROOT(N), VROOT(N)
UA(N+1), VA(N+1)
UBAPP(N,3), VBAPP(N,3)
UB(N+1), VB(N+1)
UROOTS(N), VROOTS(N)
URAPP(N,3), VRAPP(N,3)
MULT(N)

Subroutine GENAPP

APPR(N,3) APPI(N,3)

Subroutine HORNER

UA(N+1), VA(N+1)
UB(N+1), VB(N+1)

2. Input Data for G.C.D. - Muller's Method

The input data for G.C.D. - Muller's method is prepared exactly as described in Appendix E, § 2 for G.C.D. - Newton's method.

3. Variables Used in G.C.D. - Muller's Method

The main variables used in G.C.D. - Muller's method are given in Table F.II. The symbols used to indicate type and disposition are described in Appendix E, § 3. For variables not listed in Table F.II, see the main program or corresponding subprogram of Table E.VI.

4. Description of Program Output

The output from G.C.D. - Muller's method is identical to that for G.C.D. - Newton's method as described in Apptendix E, § 4, keeping in mind that Muller's instead of Newton's method is used. The expression "SOLVED BY DIRECT METHOD" is equivalent to "RESULTS OF SUBROUTINE QUAD." Only one initial approximation, X_0, (not three) is printed. The other two required by Muller's method were $.9X_0$ and $1.1X_0$.

5. Informative Messages and Error Messages

The informative messages and error messages in this program are described as follows. For other messages not listed here, see Appendix E, § 5.

"THE EPSILON (XXX) CHECK IN SUBROUTINE MULTI INDICATES THAT ROOT YY = ZZZ IS NOT CLOSE ENOUGHT TO BE A TRUE ROOT. IT IS PRINTED BELOW WITH MULTIPLICITY 0." This message is described in Appendix E, § 5.

"COEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO ZEROS WERE FOUND." This message is described in Apptendix E, § 5.

"NO ZEROS WERE FOUND FOR POLYNOMIAL NUMBER XX." XX represents the number of the polynomial for which no zeros were extracted.

"IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT XX = YYY DID NOT CONVERGE AFTER ZZZ ITERATIONS." This message indicates that a root did not produce convergence during the attempt to improve accuracy. XX represents the number of the root before the attempt to improve accuracy, YYY represents its value, and ZZZ represents the maximum number of iterations. The following message then follows. "THE PRESENT APPROXIMATION IS AAA." AAA represents the present approximation to the root after the maximum number of iterations.

TABLE F.II

VARIABLES USED IN G.C.D. - MULLER'S METHOD

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine MULLER
NP	I	NP	I	E	Degree of polynomial P(X)
NROOT	I	NROOT	I	R	Number of distinst roots found
NOMULT	I	NOMULT	I		Number of roots (counting multiplicities)
ROOT	C	UROOT,VROOT	D	R	Array containing the roots
NAPP	I	NAPP	I	E	Number of initial approximations to be read in
APP	C	UAPP,VAPP	D	E	Array of initial approximations
WORK	C	UWORK,VWORK	D		Working array containing coefficients of current polynomial
B	C	UB,VB	D		Array containing coefficients of deflated polynomial
A	C	UA,VA	D	E	Array containing coefficients of original polynomial, P(X)
RAPP	C	URAPP,VRAPP	D	R	Array of initial or altered approximation for which convergence was obtained
X1	C	UX1,VX1	D		One of three current approximations to a root
X2	C	UX2,VX2	D		One of three current approximations to a root
X3	C	UX3,VX3	D		One of three current approximations to a root
PX1	C	UPX1,VPX1	D		Value of polynomial P(X) at X1
PX2	C	UPX2,VPX2	D		Value of polynomial P(X) at X2
PX3	C	UPX3,VPX3	D		Value of polynomial P(X) at X3
X4	C	UX4,VX4	D		Newest approximation (X_{n+1}) to root
PX4	C	UPX4,VPX4	D		Value of polynomial P(X) at X4
MULT	I	MULT	I		Array containing the multiplicities of each root found
ITER	I	ITER	I		Counter for iterations
IO1	I	IO1	I		Unit number of input device
IO2	I	IO2	I	C	Unit number of output device
EPSRT	R	EPSRT	D	C	Number used in subroutine BETTER to generate two approximations from the one given
NOPOLY	I	NOPOLY	I	E	Number of the polynomial

TABLE F.II (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
MAX	I	MAX	I	C	Maximum number of iterations
EPS	R	EPS	D	C	Tolerance check for convergence
EPSO	R	EPSO	D	C	Tolerance check for zero (0)
EPSM	R	EPSM	D	C	Tolerance check for multiplicities
KCHECK	I	KCHECK	I		Program control, KCHECK = 1 stops execution of program
XSTART	R	XSTART	D	E	Magnitude at which to start generating initial approximations
XEND	R	XEND	D	E	Magnitude at which to end generating initial approximations
NWORK	I	NWORK	I		Degree of current deflated polynomial whose coefficients are in WORK
ITIME	I	ITIME	I		Program control
NALTER	I	NALTER	I		Number of alterations which have been performed on an initial approximation
IAPP	I	IAPP	I		Counter for number of initial approximations used
CONV	L	CONV	L		When CONV is true, convergence has been obtained
IROOT	I	IROOT	I	R	Number of distinct roots solved by Muller's method, i.e. not solved directly by subroutine *QUAD*
				Subroutine HORNER	
A	C	UA,VA	D	E	Array of current polynomial coefficients (to be deflated or evaluated)
NA	I	NA	I	E	Degree of polynomial to be deflated or evaluated
X	C	UX,VX	D	E	Approximation at which to evaluate or deflate the polynomial
B	C	UB,VB	D	R	Array containing the coefficients of the deflated polynomial
PX	C	UPX,VPX	D	R	Value of the polynomial at X
NUM	I	NUM	I		Number of coefficients of polynomial to be deflated

TABLE F.II (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Subroutine TEST
X3	C	UX3,VX3	D	E	Approximation to root (old) (X_n)
X4	C	UX4,VX4	D	E	New approximation to root (X_{n+1})
CONV	L	CONV	L	R	CONV = True implies convergence has been obtained
EPS	R	EPS	D	C	Tolerance for convergence test
EPSO	R	EPSO	D	C	Tolerance check for zero (0)
DENOM	R	DENOM	D		Magnitude of new approximation, (X_{n+1})
					Subroutine BETTER
MULT	I	MULT	I	ECR	Array of multiplicities of each root
A	C	UA,VA	D	E	Array of coefficients of original undeflated polynomial
NP	I	NP	I	E	Degree of original polynomial
ROOT	C	UROOT,VROOT	D	ECR	Array of ROOTS
NROOT	I	NROOT	I	ECR	Number of roots stored in ROOT
BAPP	C	UBAPP,VBAPP	D	E	Array of initial approximations (old roots)
IROOT	I	IROOT	I	ECR	Number of roots solved by the iterative process (Not QUAD)
ROOTS	C	UROOTS,VROOTS	D		Temporary storage for new (better) roots
L	I	L	I		Number of roots found by BETTER
EPSRT	R	EPSRT	D	C	A small number used to generate two of the three approximations when given one
ITER	I	ITER	I	C	Counter for number of iterations
B	C	UB,VB	D		Array containing coefficients of deflated polynomial
X1	C	UX1,VX1	D		One of three approximations to the root
X2	C	UX2,VX2	D		One of three approximations to the root
X3	C	UX3,VX3	D		One of three approximations to the root
PX1	C	UPX1,VPX1	D		Value of polynomial (P(X)) at X1
PX2	C	UPX2,VPX2	D		Value of polynomial (P(X)) at X2
PX3	C	UPX3,VPX3	D		Value of polynomial (P(X)) at X3

TABLE F.II (Continued)

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
CONV	L	CONV	L		CONV = true implies convergence has been obtained
X4	C	UX4,VX4	D		Newest approximation to root
J	I	J	I		Program control - counts the number of roots used as initial approximations
MAX	I	MAX	I	C	Maximum number of iterations permitted
IO2	I	IO2	I	C	Unit number of output device

Subroutine ALTER

X1	C	X1R,X1I	D	ECR	One of the three approximations to be altered
X2	C	X2R,X2I	D	ECR	One of the three approximations to be altered
X3	C	X3R,X3I	D	ECR	One of the three approximations to be altered
X2R	R	X2R	D		Real part of complex approximation
X2I	R	X2I	D		Imaginary part of complex approximation

Subroutine CALC

These variables are dummy variables used for temporary storage and thus, are not defined.

MAIN PROGRAM

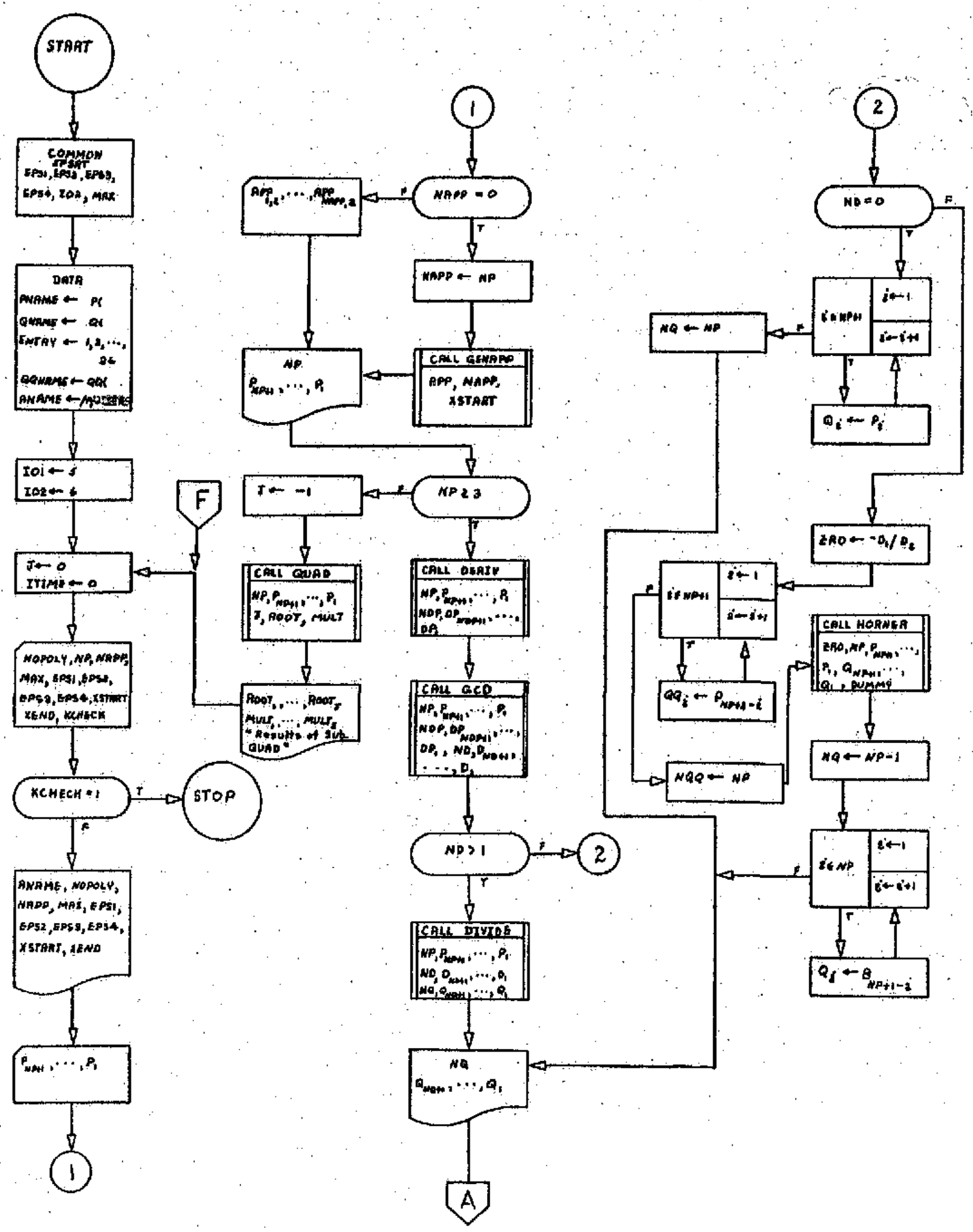


Figure F.1. Flow Charts for G.C.D.-Muller's Method

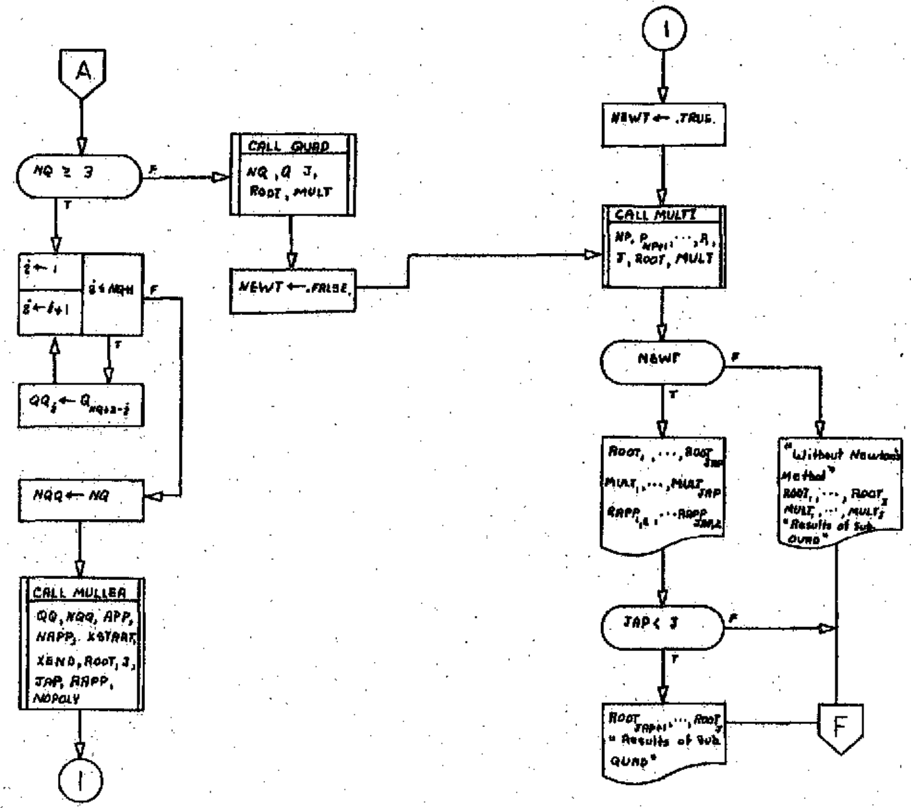


Figure F.1. (Continued)

MULLER

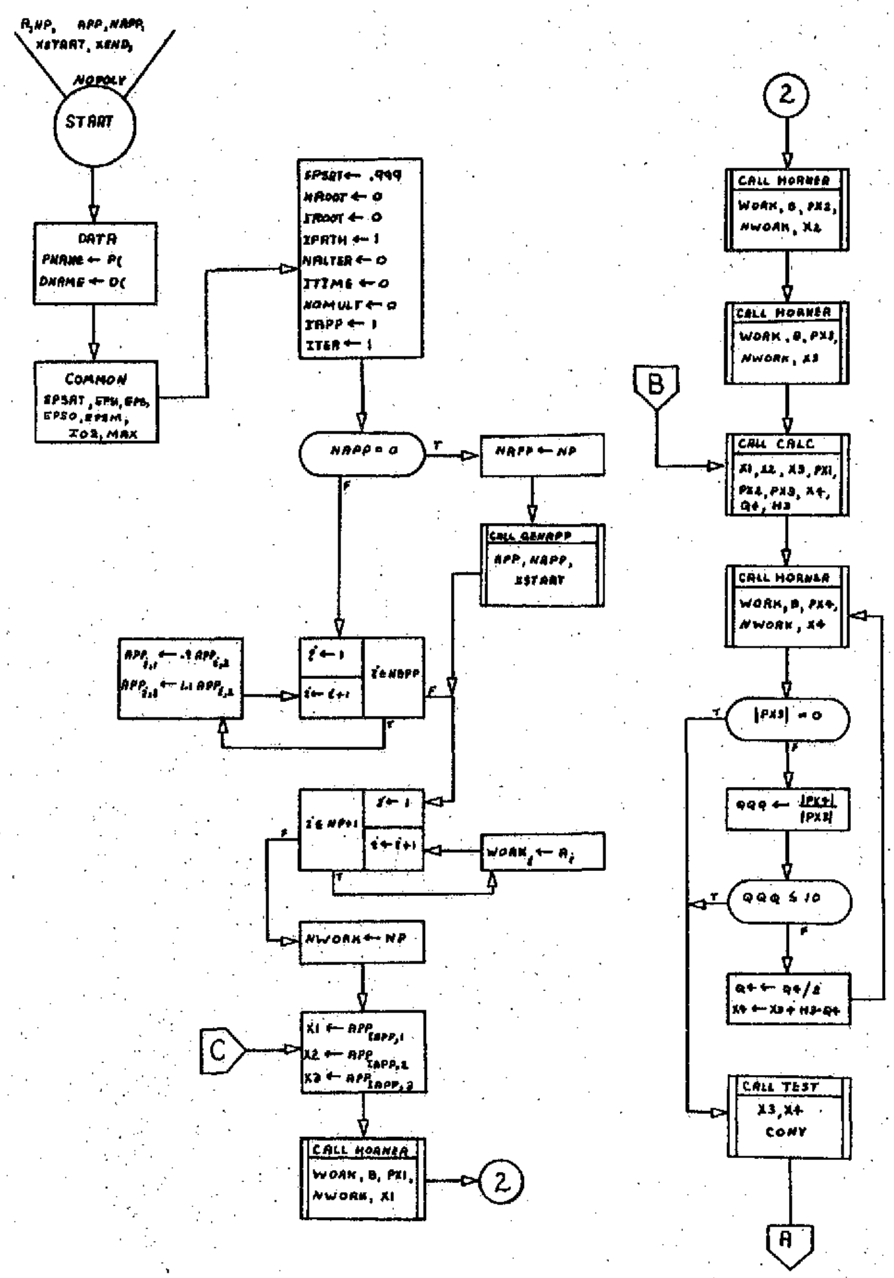


Figure F.1. (Continued)

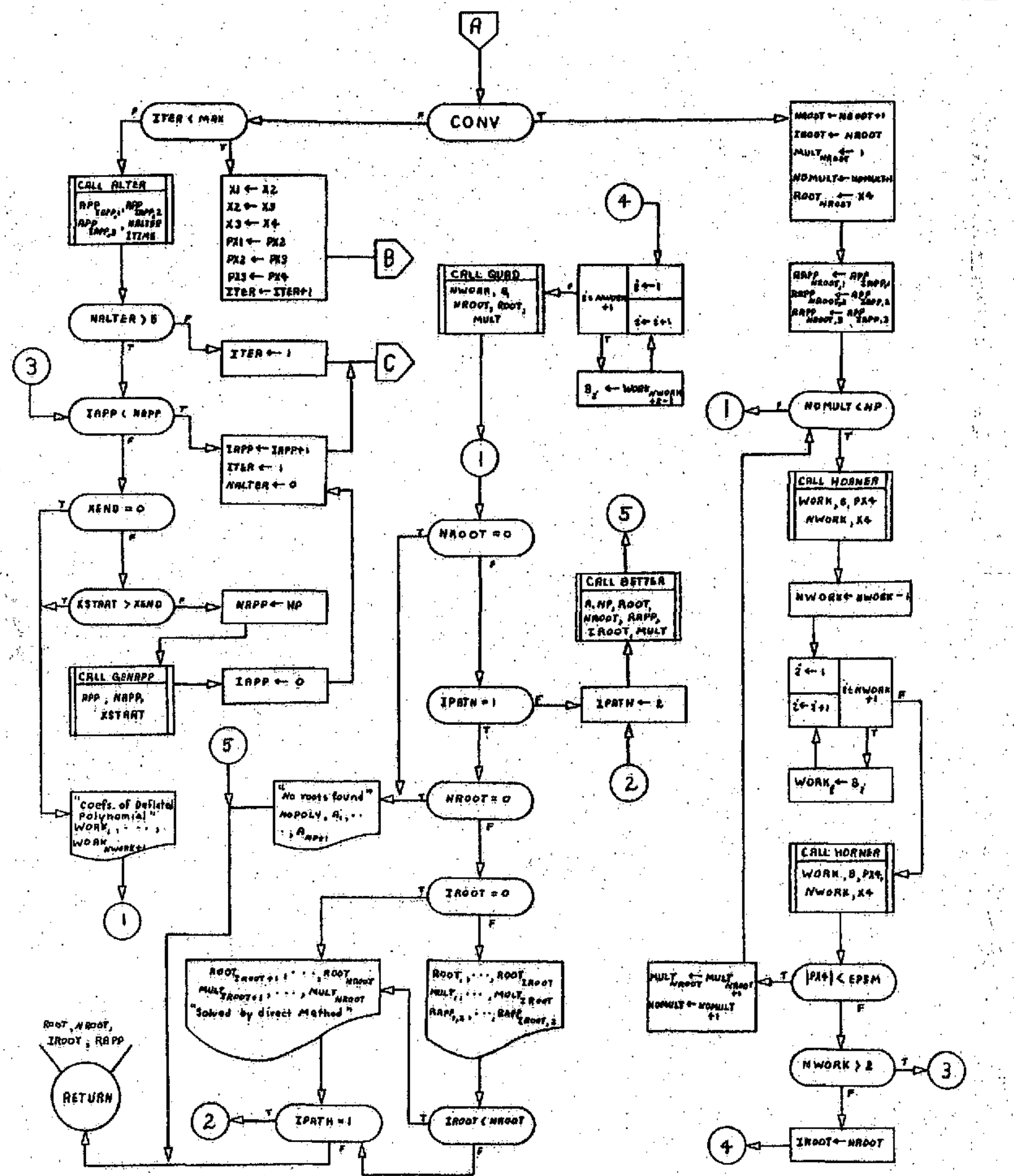


Figure F.1. (Continued)

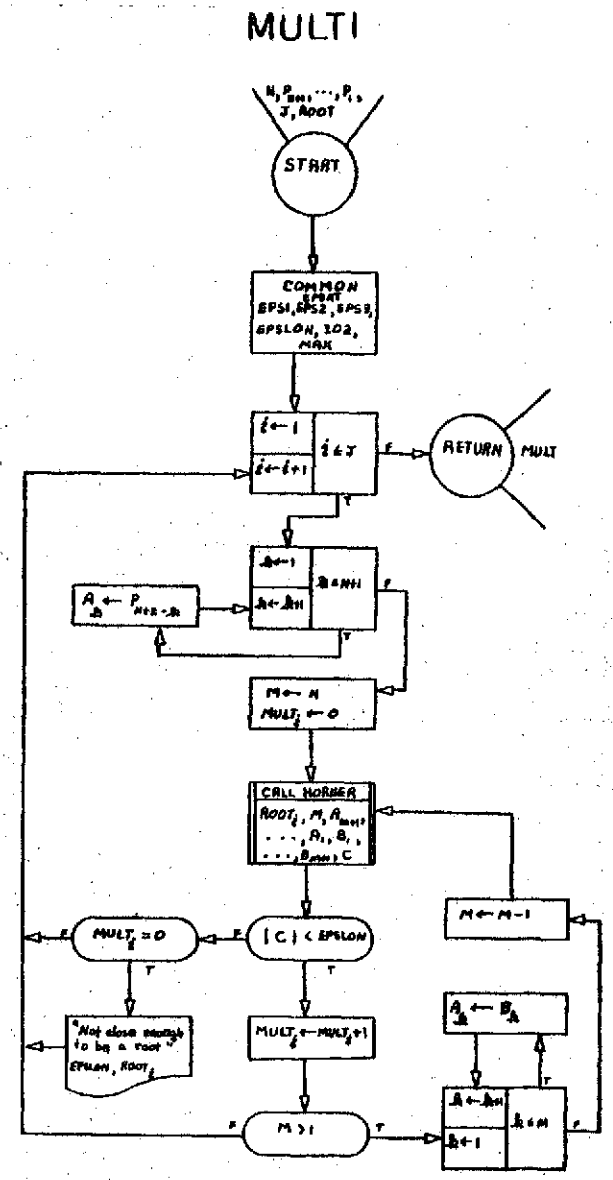
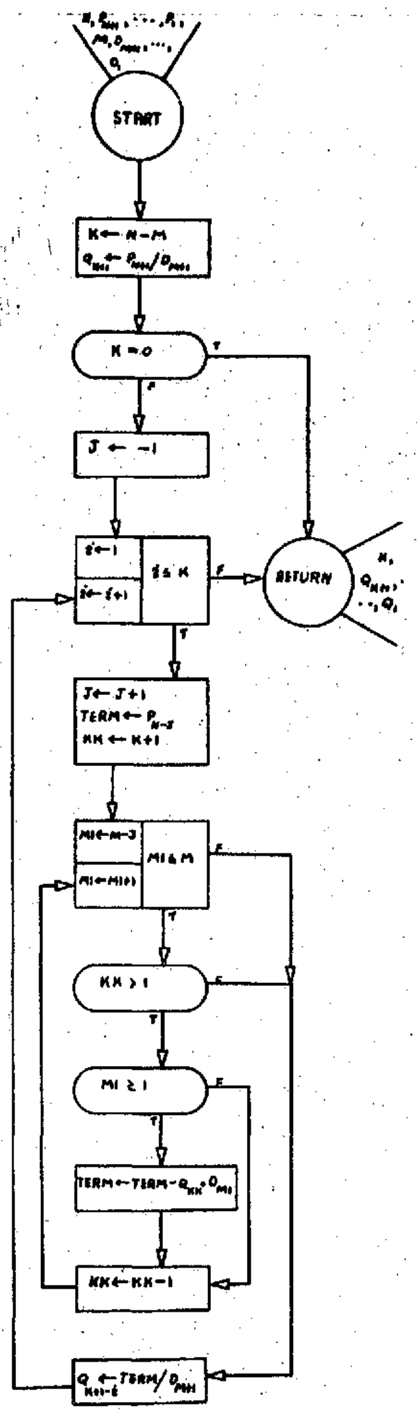


Figure F.1. (Continued)

DIVIDE



DERIV

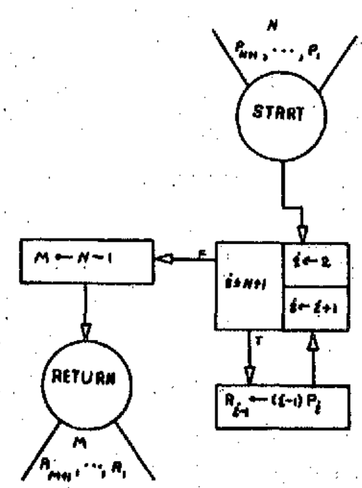


Figure F.1. (Continued)

GCD

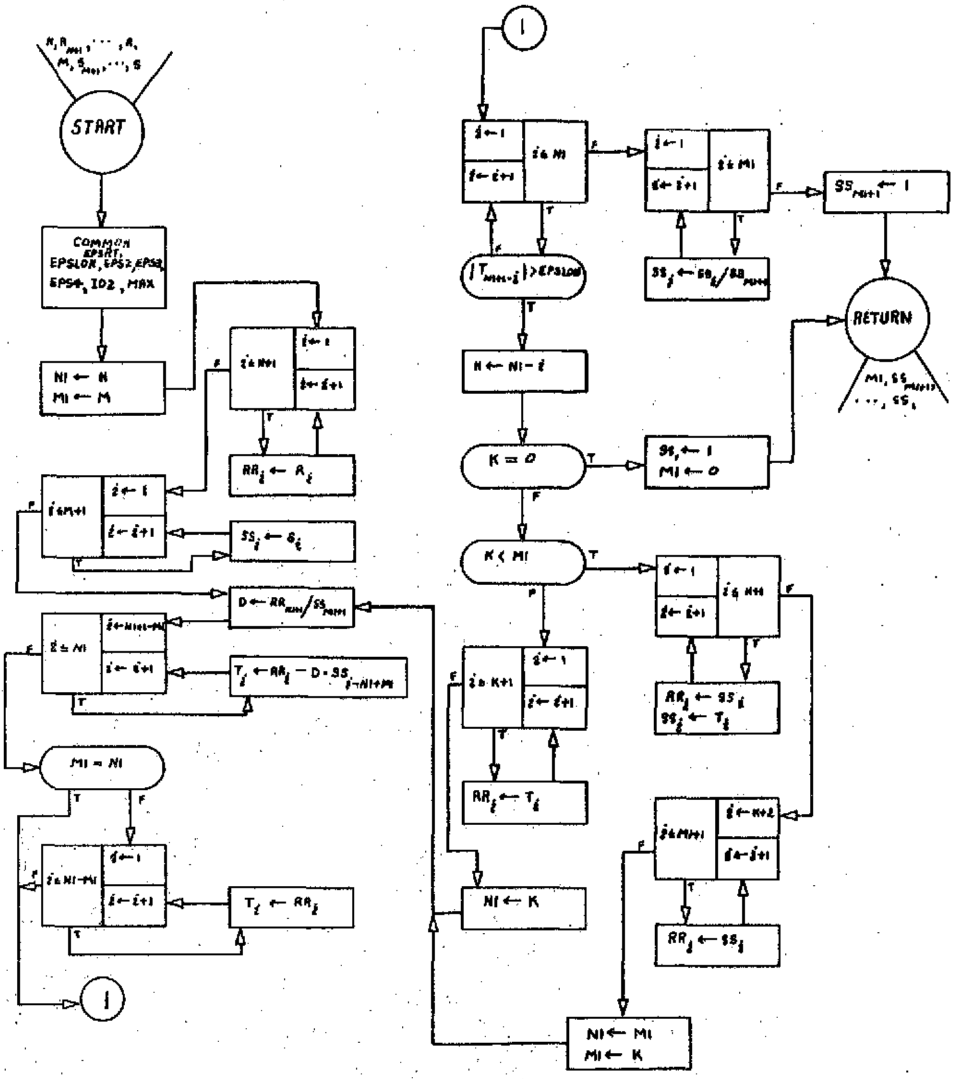


Figure F.1. (Continued)

QUAD

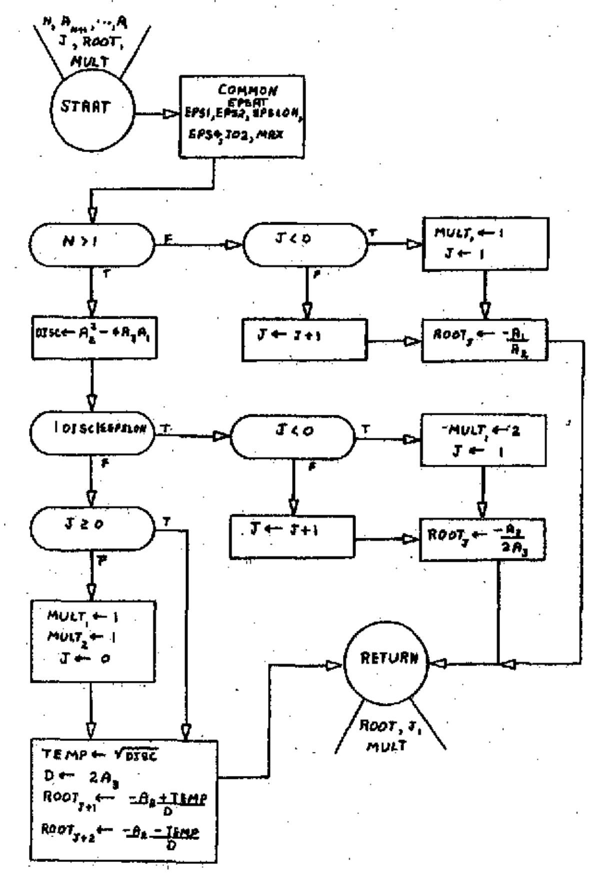


Figure F.1. (Continued)

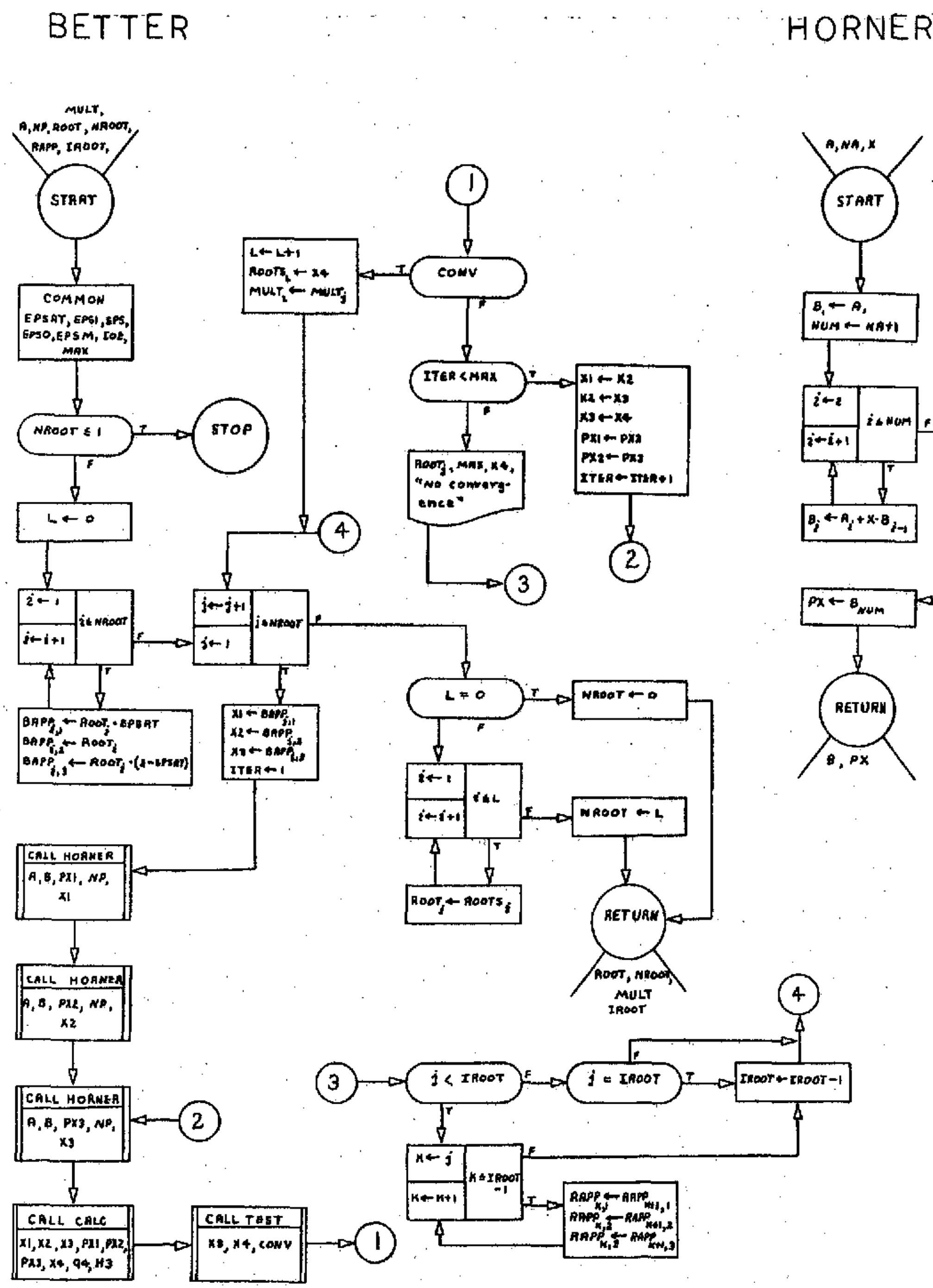
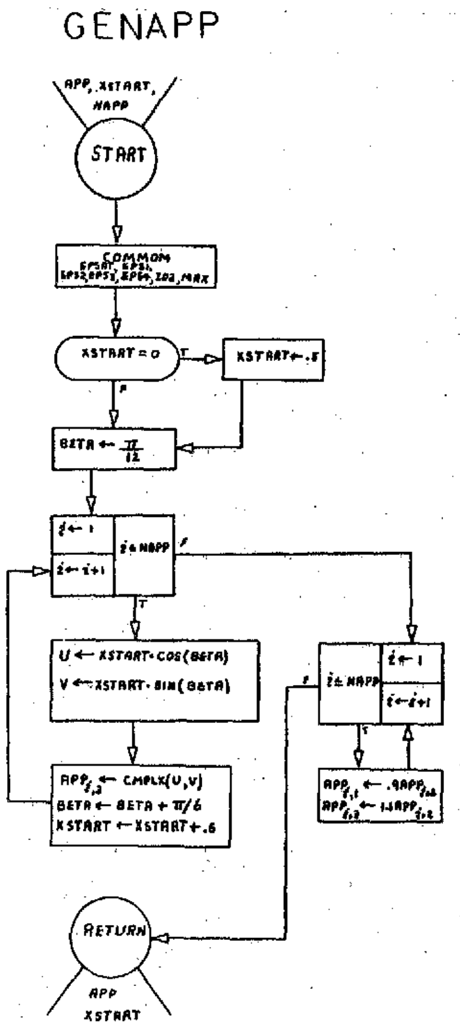


Figure F.1. (Continued)



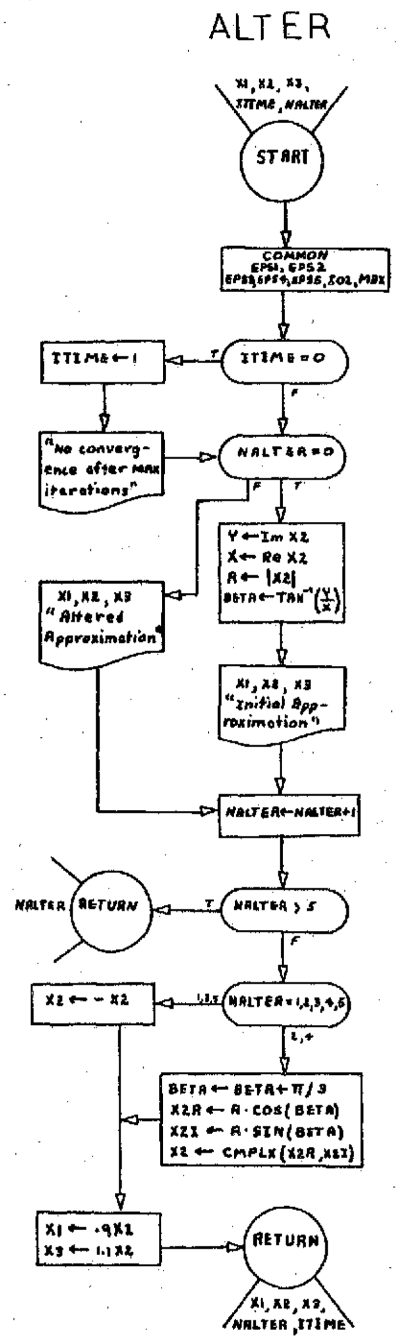
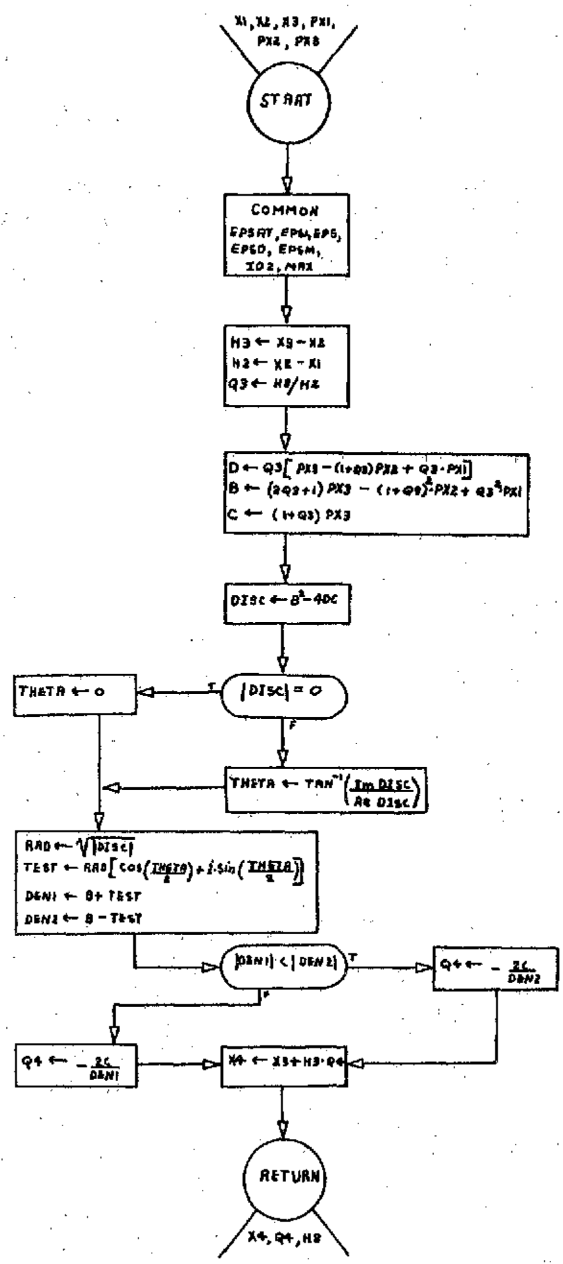


Figure F.1. (Continued)

CALC



TEST

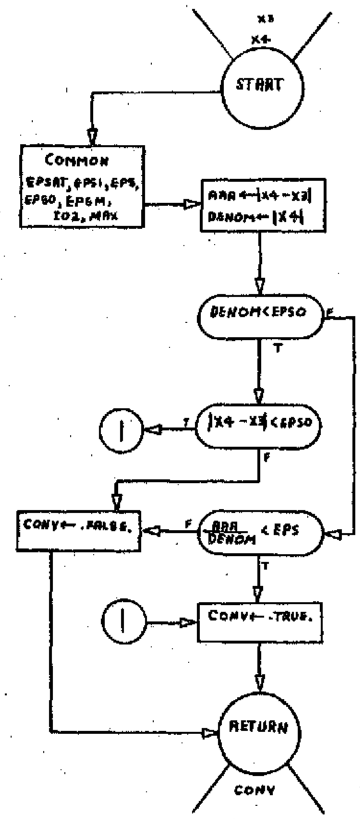


Figure F.1. (Continued)

COMSQT

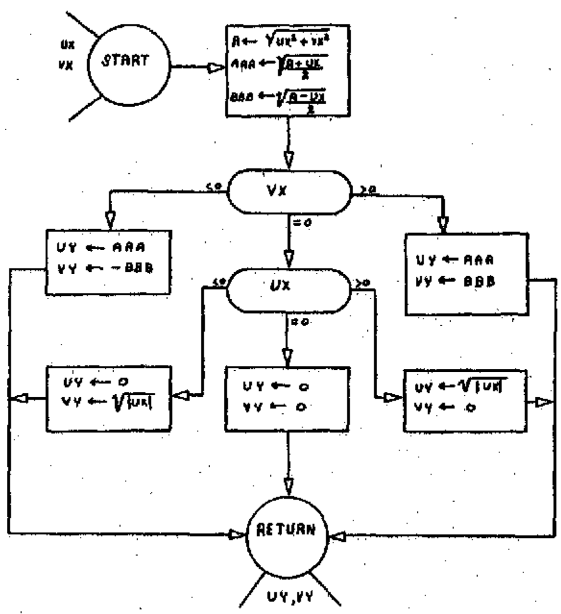


Figure F.1. (Continued)

TABLE F.III

PROGRAM FOR G.C.D.-MULLER'S METHOD

```
C     ***********************************************************************
C     *                                                                     *
C     * DOUBLE PRECISION PROGRAM FOR G.C.D. - MULLER'S METHOD               *
C     *                                                                     *
C     *                                                                     *
C     * THE G.C.D. METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A  *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25.  ALL MULTIPLE ROOTS ARE REMOVED BY *
C     * DIVIDING THE POLYNOMIAL BY THE GREATEST COMMON DIVISOR OF THE POLYNOMIAL *
C     * AND ITS DERIVATIVE.  THE ZEROS OF THE RESULTING POLYNOMIAL ARE EXTRACTED *
C     * AND THEIR MULTIPLICITIES DETERMINED.                                *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION URAPP,VRAPP
      DOUBLE PRECISION UP,VP,UAPP,VAPP,UROOT,VROOT,UDP,VDP,UD,VD,UZRO,VZ
     1RO,UQ,VQ,UDUMMY,VDUMMY,UQQ,VQQ,UB,VB,EPS1,EPS2,EPS3,EPS4
      DIMENSION URAPP(25,3),VRAPP(25,3),UAPP(25,3),VAPP(25,3)
      DIMENSION UP(26),VP(26),UROOT(25),VROOT(25),MULT(25),UDP(26),VDP(2
     16),UD(26),VD(26),UQ(26),VQ(26),UQQ(26),VQQ(26),UB(26),VB(26),ANAME
     2(2),ENTRY(26)
      DOUBLE PRECISION XSTART
      DOUBLE PRECISION XEND
      DOUBLE PRECISION EPSRT
      COMMON EPSRT,EPS1,EPS2,EPS3,EPS4,IO2,MAX
      DATA PNAME,QNAME,QQNAME/2HP(,2HQ(,3HQQ(/
      DATA ENTRY/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,2H10,2H11,2H12,2H13
     1,2H14,2H15,2H16,2H17,2H18,2H19,2H20,2H21,2H22,2H23,2H24,2H25,2H26/
      DATA ANAME(1),ANAME(2)/4HMULL,4HERS /
      LOGICAL NEWT
      IO1=5
      IO2=6
   10 J=0
      ITIME=0
      READ(IO1,1000) NOPOLY,NP,NAPP,MAX,EPS1,EPS2,EPS3,EPS4,XSTART,XEND,
     1KCHECK
      IF(KCHECK.EQ.1) STOP
      WRITE(IO2,1020) ANAME(1),ANAME(2),NOPOLY
      WRITE(IO2,2000) NAPP
      WRITE(IO2,2010) MAX
      WRITE(IO2,2070) EPS1
      WRITE(IO2,2020) EPS2
      WRITE(IO2,2080) EPS3
      WRITE(IO2,2030) EPS4
      WRITE(IO2,2040) XSTART
      WRITE(IO2,2050) XEND
      WRITE(IO2,2060)
      KKK=NP+1
      NNN=KKK+1
      DO 20 I=1,KKK
      JJJ=NNN-I
   20 READ(IO1,1010) UP(JJJ),VP(JJJ)
      IF(NAPP.NE.0) GO TO 22
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 23
   22 READ(IO1,1015) (UAPP(I,2),VAPP(I,2),I=1,NAPP)
   23 WRITE(IO2,1030) NP
      KKK=NP+1
      NNN=KKK+1
```

TABLE F.III (Continued)

```
      DO 25 I=1,KKK
      JJJ=NNN-I
   25 WRITE(IO2,1040) PNAME,ENTRY(JJJ),UP(JJJ),VP(JJJ)
      IF(NP.GE.3) GO TO 30
      J=-1
      CALL QUAD(NP,UP,VP,J,UROOT,VROOT,MULT)
      WRITE(IO2,1070)
      WRITE(IO2,1165) (I,UROOT(I),VROOT(I),MULT(I),I=1,J)
      GO TO 10
   30 CALL DERIV(NP,UP,VP,NDP,UDP,VDP)
      CALL GCD(NP,UP,VP,NDP,UDP,VDP,ND,UD,VD)
      IF(ND.GT.1) GO TO 70
      IF(ND.EQ.0) GO TO 65
      UDUMMY=UD(2)*UD(2)+VD(2)*VD(2)
      UZRO=-(UD(1)*UD(2)+VD(1)*VD(2))/UDUMMY
      VZRO=-(UD(2)*VD(1)-UD(1)*VD(2))/UDUMMY
      KKK=NP+1
      DO 55 I=1,KKK
      UQQ(I)=UP(KKK+1-I)
   55 VQQ(I)=VP(KKK+1-I)
      NQQ=NP
      CALL HORNER(NQQ,UQQ,VQQ,UZRO,VZRO,UB,VB,UDUMMY,VDUMMY)
      NQ=NP-1
      DO 60 I=1,NP
      UQ(I)=UB(NP+1-I)
   60 VQ(I)=VB(NP+1-I)
      GO TO 80
   65 KKK=NP+1
      DO 66 I=1,KKK
      UQ(I)=UP(I)
   66 VQ(I)=VP(I)
      NQ=NP
      GO TO 80
   70 CALL DIVIDE(NP,UP,VP,ND,UD,VD,NQ,UQ,VQ)
   80 WRITE(IO2,1120) NQ
      KKK=NQ+1
      NNN=KKK+1
      DO 83 I=1,KKK
      JJJ=NNN-I
   83 WRITE(IO2,1040) QNAME,ENTRY(JJJ),UQ(JJJ),VQ(JJJ)
      IF(NQ.GE.3) GO TO 85
      GO TO 110
   85 KKK=NQ+1
      DO 90 I=1,KKK
      UQQ(I)=UQ(KKK+1-I)
   90 VQQ(I)=VQ(KKK+1-I)
      NQQ=NQ
      GO TO 120
  110 CALL QUAD(NQ,UQ,VQ,J,UROOT,VROOT,MULT)
      NEWT=.FALSE.
      GO TO 310
  120 CALL MULLER(UQQ,VQQ,NQQ,UAPP,VAPP,NAPP,XSTART,XEND,UROOT,VROOT,J,J
     1AP,URAPP,VRAPP,NOPOLY)
      NEWT=.TRUE.
  310 CALL MULTI(NP,UP,VP,J,UROOT,VROOT,MULT)
      IF(NEWT) GO TO 330
      WRITE(IO2,1070)
      WRITE(IO2,1165) (L,UROOT(L),VROOT(L),MULT(L),L=1,J)
```

TABLE F.III (Continued)

```
      GO TO 10
  330 WRITE(IO2,1180)
      DO 350 L=1,JAP
  350 WRITE(IO2,1190) L,UROOT(L),VROOT(L),MULT(L),URAPP(L,2),VRAPP(L,2)
      KKK=JAP+1
      IF(JAP.LT.J) WRITE(IO2,1165) (L,UROOT(L),VROOT(L),MULT(L),L=KKK,J)
      GO TO 10
 1000 FORMAT(3(I2,1X),9X,I3,1X,4(D6.0,1X),13X,2(D7.0,1X),I1)
 1010 FORMAT(2D30.0)
 1015 FORMAT(2D30.0)
 1020 FORMAT(1H1,10X,41HGREATEST COMMON DIVISOR METHOD USED WITH ,2(A4),
     135HMETHOD TO FIND ZEROS OF POLYNOMIALS/11X,18HPOLYNOMIAL NUMBER ,I
     22///)
 1030 FORMAT(1X,22HTHE DEGREE OF P(X) IS ,I2,22H  THE COEFFICIENTS ARE//
     1)
 1040 FORMAT(2X,A2,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1070 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES//)
 1080 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,10X,I2)
 1100 FORMAT(2X,A3,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1120 FORMAT(///1X,73HQ(X) IS THE POLYNOMIAL WHICH HAS AS ITS ROOTS THE
     1DISTINCT ROOTS OF P(X)./1X,22HTHE DEGREE OF Q(X) IS ,I2,22H  THE C
     2OEFFICIENTS ARE//)
 1165 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,10X,26H
     1RESULTS OF SUBROUTINE QUAD)
 1180 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
     1 APPROXIMATION//)
 1190 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,9X,D23.
     116,3H + ,D23.16,2H I)
 2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.  ,I2)
 2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
 2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
 2030 FORMAT(1X,24HTEST FOR MULTIPLICITIES.,10X,D9.2)
 2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
 2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
 2060 FORMAT(//1X)
 2070 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE GCD.  ,D9.2)
 2080 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE QUAD. ,D9.2)
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE MULTI(N,UP,VP,J,UROOT,VROOT,MULT)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN N ZEROS OF A POLYNOMIAL, SUBROUTINE MULTI COMPUTES THEIR       *
C     * MULTIPLICITIES.                                                      *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UP,VP,UROOT,VROOT,UA,VA,UB,VB,UC,VC,EPS1,EPS2,EPS
     1LON,EPS3,BBB
      DIMENSION UP(26),VP(26),UROOT(25),VROOT(25),UA(26),VA(26),UB(26),V
     1B(26),MULT(25)
      DOUBLE PRECISION EPSRT
      COMMON EPSRT,EPS1,EPS2,EPS3,EPSLON,IO2,MAX
      DO 100 I=1,J
      KKK=N+1
      DO 10 K=1,KKK
      UA(K)=UP(KKK+1-K)
   10 VA(K)=VP(KKK+1-K)
      M=N
      MULT(I)=0
   20 CALL HORNER(M,UA,VA,UROOT(I),VROOT(I),UB,VB,UC,VC)
      BBB=DSQRT(UC*UC+VC*VC)
      IF(BBB.LT.EPSLON) GO TO 50
      IF(MULT(I).EQ.0) GO TO 40
      GO TO 100
   40 WRITE(IO2,1000) EPSLON,I,UROOT(I),VROOT(I)
      GO TO 100
   50 MULT(I)=MULT(I)+1
      IF(M.GT.1) GO TO 60
      GO TO 100
   60 DO 70 K=1,M
      UA(K)=UB(K)
   70 VA(K)=VB(K)
      M=M-1
      GO TO 20
  100 CONTINUE
      RETURN
 1000 FORMAT(///15H THE EPSILON  (,D10.3,48H) CHECK IN SUBROUTINE MULTI
     1INDICATES THAT ROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,/80H IS NO
     2T CLOSE ENOUGH TO BE A TRUE ROOT.  IT IS PRINTED BELOW WITH MULTIP
     3LICITY 0//)
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE DIVIDE(N,UP,VP,M,UD,VD,K,UQ,VQ)
C   ***********************************************************************
C   *                                                                     *
C   * GIVEN TWO POLYNOMIALS F(X) AND G(X), SUBROUTINE DIVIDE COMPUTES THE *
C   * QUOTIENT POLYNOMIAL  H(X) = F(X)/G(X).                              *
C   *                                                                     *
C   ***********************************************************************
      DOUBLE PRECISION UP,VP,UD,VD,UQ,VQ,UTERM,VTERM,UDUMMY
      DIMENSION UP(26),VP(26),UD(26),VD(26),UQ(26),VQ(26)
      K=N-M
      UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1)=(UP(N+1)*UD(M+1)+VP(N+1)*VD(M+1))/UDUMMY
      VQ(K+1)=(VP(N+1)*UD(M+1)-UP(N+1)*VD(M+1))/UDUMMY
      IF(K.EQ.0) GO TO 100
      J=-1
      DO 50 I=1,K
      J=J+1
      UTERM=UP(N-J)
      VTERM=VP(N-J)
      KK=K+1
      NNN=M-J
      DO 40 M1=NNN,M
      IF(KK.GT.1) GO TO 10
      GO TO 45
   10 IF(M1.GE.1) GO TO 20
      GO TO 40
   20 UTERM=UTERM-(UQ(KK)*UD(M1)-VQ(KK)*VD(M1))
      VTERM=VTERM-(UQ(KK)*VD(M1)+VQ(KK)*UD(M1))
   40 KK=KK-1
   45 UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1-I)=(UTERM*UD(M+1)+VTERM*VD(M+1))/UDUMMY
   50 VQ(K+1-I)=(VTERM*UD(M+1)-UTERM*VD(M+1))/UDUMMY
  100 RETURN
      END
```

```
      SUBROUTINE DERIV(N,UP,VP,M,UA,VA)
C   ***********************************************************************
C   *                                                                     *
C   * GIVEN A POLYNOMIAL P(X), SUBROUTINE DERIV COMPUTES THE COEFFICIENTS OF *
C   * ITS DERIVATIVE P'(X).                                               *
C   *                                                                     *
C   ***********************************************************************
      DOUBLE PRECISION UP,VP,UA,VA,AAA
      DIMENSION UP(26),VP(26),UA(26),VA(26)
      KKK=N+1
      DO 10 I=2,KKK
      AAA=I-1
      UA(I-1)=AAA*UP(I)
   10 VA(I-1)=AAA*VP(I)
      M=N-1
      RETURN
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE GCD(N,UR,VR,M,US,VS,M1,USS,VSS)
C     ***********************************************************************
C     *                                                                     *
C     * GIVEN POLYNOMIALS P(X) AND DP(X) WHERE DEG. DP(X) IS LESS THAN DEG. *
C     * P(X), SUBROUTINE GCD COMPUTES THE GREATEST COMMON DIVISOR OF P(X) AND *
C     * DP(X).                                                              *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION EPSRT
      DOUBLE PRECISION USSSSS,VSSSSS
      DOUBLE PRECISION UR,VR,US,VS,USS,VSS,URR,VRR,UD,VD,UT,VT,EPSLON,EP
     1S2,EPS3,EPS4,BBB
      DIMENSION UR(26),VR(26),US(26),VS(26),USS(26),VSS(26),URR(26),VRR(
     126),UT(26),VT(26)
      COMMON EPSRT,EPSLON,EPS2,EPS3,EPS4,IO2,MAX
      N1=N
      M1=M
      KKK=N+1
      DO 20 I=1,KKK
      URR(I)=UR(I)
   20 VRR(I)=VR(I)
      KKK=M+1
      DO 25 I=1,KKK
      USS(I)=US(I)
   25 VSS(I)=VS(I)
   30 BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      UD=(URR(N1+1)*USS(M1+1)+VRR(N1+1)*VSS(M1+1))/BBB
      VD=(USS(M1+1)*VRR(N1+1)-URR(N1+1)*VSS(M1+1))/BBB
      KKK=N1+1-M1
      DO 40 I=KKK,N1
      UT(I)=URR(I)-(UD*USS(I-N1+M1)-VD*VSS(I-N1+M1))
   40 VT(I)=VRR(I)-(UD*VSS(I-N1+M1)+VD*USS(I-N1+M1))
      IF(M1.EQ.N1) GO TO 70
      KKK=N1-M1
      DO 60 I=1,KKK
      UT(I)=URR(I)
   60 VT(I)=VRR(I)
   70 DO 90 I=1,N1
      BBB=DSQRT(UT(N1+1-I)*UT(N1+1-I)+VT(N1+1-I)*VT(N1+1-I))
      IF(BBB.GT.EPSLON) GO TO 100
   90 CONTINUE
      DO 95 I=1,M1
      BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      USSSSS=(USS(I)*USS(M1+1)+VSS(I)*VSS(M1+1))/BBB
      VSSSSS=(VSS(I)*USS(M1+1)-USS(I)*VSS(M1+1))/BBB
      USS(I)=USSSSS
   95 VSS(I)=VSSSSS
      USS(M1+1)=1.0
      VSS(M1+1)=0.0
      GO TO 200
  100 K=N1-I
      IF(K.EQ.0) GO TO 170
      IF(K.LT.M1) GO TO 140
      KKK=K+1
      DO 130 J=1,KKK
      URR(J)=UT(J)
  130 VRR(J)=VT(J)
      N1=K
```

TABLE F.III (Continued)

```
      GO TO 30
  140 KKK=K+1
      DO 150 J=1,KKK
      URR(J)=USS(J)
      VRR(J)=VSS(J)
      USS(J)=UT(J)
  150 VSS(J)=VT(J)
      KKK=K+2
      NNN=M1+1
      DO 160 J=KKK,NNN
      URR(J)=USS(J)
  160 VRR(J)=VSS(J)
      N1=M1
      M1=K
      GO TO 30
  170 USS(1)=1.0
      VSS(1)=0.0
      M1=0
  200 RETURN
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE QUAD(N,UA,VA,J,UROOT,VROOT,MULT)
C   ***********************************************************************
C   *                                                                     *
C   * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES *
C   * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE  *
C   * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                       *
C   *                                                                     *
C   ***********************************************************************
      DOUBLE PRECISION EPSRT
      DOUBLE PRECISION UA,VA,UROOT,VROOT,UDISC,VDISC,UTEMP,VTEMP,UD,VD,E
     1PS1,EPS2,EPS4,EPSLON,BBB
      DIMENSION UA(26),VA(26),UROOT(25),VROOT(25),MULT(25)
      COMMON EPSRT,EPS1,EPS2,EPSLON,EPS4,IO2,MAX
      IF(N.GT.1) GO TO 60
      IF(J.LT.0) GO TO 40
      J=J+1
      GO TO 50
   40 MULT(1)=1
      J=1
   50 BBB=UA(2)*UA(2)+VA(2)*VA(2)
      UROOT(J)=-(UA(1)*UA(2)+VA(1)*VA(2))/BBB
      VROOT(J)=-(VA(1)*UA(2)-UA(1)*VA(2))/BBB
      GO TO 200
   60 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(3)*UA(1)-VA(3)*VA(1)))
      VDISC=(2.0*UA(2)*VA(2))-(4.0*(UA(3)*VA(1)+VA(3)*UA(1)))
      BBB=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(BBB.LE.EPSLON) GO TO 100
      IF(J.GE.0) GO TO 80
      MULT(1)=1
      MULT(2)=1
      J=0
   80 CALL COMSQT(UDISC,VDISC,UTEMP,VTEMP)
      UD=2.0*UA(3)
      VD=2.0*VA(3)
      BBB=UD*UD+VD*VD
      UROOT(J+1)=((-UA(2)+UTEMP)*UD+(-VA(2)+VTEMP)*VD)/BBB
      VROOT(J+1)=((-VA(2)+VTEMP)*UD-(-UA(2)+UTEMP)*VD)/BBB
      UROOT(J+2)=((-UA(2)-UTEMP)*UD+(-VA(2)-VTEMP)*VD)/BBB
      VROOT(J+2)=((-VA(2)-VTEMP)*UD-(-UA(2)-UTEMP)*VD)/BBB
      J=J+2
      GO TO 200
  100 IF(J.LT.0) GO TO 110
      J=J+1
      GO TO 130
  110 MULT(1)=2
      J=1
  130 UD=2.0*UA(3)
      VD=2.0*VA(3)
      BBB=UD*UD+VD*VD
      UROOT(J)=(-UA(2)*UD-VA(2)*VD)/BBB
      VROOT(J)=(-VA(2)*UD+UA(2)*VD)/BBB
  200 RETURN
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE MULLER(UA,VA,NP,UAPP,VAPP,NAPP,XSTART,XEND,UROOT,VROOT,
     1NROOT,IROOT,URAPP,VRAPP,NOPOLY)
C     ************************************************************************
C     *                                                                      *
C     * MULLER'S METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A     *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25.  THROUGH THREE GIVEN POINTS THE     *
C     * POLYNOMIAL IS APPROXIMATED BY A QUADRATIC.  THE ZERO OF THE QUADRATIC *
C     * CLOSEST TO THE OLD APPROXIMATION IS TAKEN AS THE NEW APPROXIMATION.  *
C     * IN THIS MANNER A SEQUENCE IS OBTAINED CONVERGING TO A ZERO.          *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UROOT,VROOT,UX1,VX1,UAPP,VAPP
     1,UX2,VX2,UWORK,VWORK,UX3,VX3,UB,VB,UX4,VX4,UA,VA,UPX1,VPX1,URAPP,V
     2RAPP,UPX4,VPX4,EPSRT,EPSO,EPS,CCC,EPSM,UH3,VH3,UQ4,VQ4,ABPX4,ABPX3
     3,QQQ,XSTART,XEND
      DIMENSION UROOT(25),VROOT(25),MULT(25),UAPP(25,3),VAPP(25,3),UWORK
     1(26),VWORK(26),UB(26),VB(26),UA(26),VA(26),URAPP(25,3),VRAPP(25,3)
      LOGICAL CONV
      DOUBLE PRECISION EPS1
      COMMON EPSRT,EPS1,EPS,EPSO,EPSM,IO2,MAX
      DATA PNAME,DNAME/2HP(,2HD(/
      EPSRT=0.999
      NROOT=0
      IROOT=0
      IPATH=1
      NOMULT=0
      NALTER=0
      ITIME=0
      IAPP=1
      ITER=1
      IF(NAPP.NE.0) GO TO 18
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 27
   18 DO 25 I=1,NAPP
      UAPP(I,1)=0.9*UAPP(I,2)
      VAPP(I,1)=0.9*VAPP(I,2)
      UAPP(I,3)=1.1*UAPP(I,2)
   25 VAPP(I,3)=1.1*VAPP(I,2)
   27 KKK=NP+1
      DO 30 I=1,KKK
      UWORK(I)=UA(I)
   30 VWORK(I)=VA(I)
      NWORK=NP
   40 UX1=UAPP(IAPP,1)
      VX1=VAPP(IAPP,1)
      UX2=UAPP(IAPP,2)
      VX2=VAPP(IAPP,2)
      UX3=UAPP(IAPP,3)
      VX3=VAPP(IAPP,3)
      CALL HORNER(NWORK,UWORK,VWORK,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NWORK,UWORK,VWORK,UX2,VX2,UB,VB,UPX2,VPX2)
      CALL HORNER(NWORK,UWORK,VWORK,UX3,VX3,UB,VB,UPX3,VPX3)
   50 CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   60 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      ABPX4=DSQRT(UPX4*UPX4+VPX4*VPX4)
      ABPX3=DSQRT(UPX3*UPX3+VPX3*VPX3)
```

TABLE F.III (Continued)

```
      IF(ABPX3.EQ.0.0) GO TO 70
      QQQ=ABPX4/ABPX3
      IF(QQQ.LE.10.) GO TO 70
      UQ4=0.5*UQ4
      VQ4=0.5*VQ4
      UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      GO TO 60
   70 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 120
      IF(ITER.LT.MAX) GO TO 110
      CALL ALTER(UAPP(IAPP,1),VAPP(IAPP,1),UAPP(IAPP,2),VAPP(IAPP,2),UAP
     1P(IAPP,3),VAPP(IAPP,3),NALTER,ITIME)
      IF(NALTER.GT.5) GO TO 75
      ITER=1
      GO TO 40
   75 IF(IAPP.LT.NAPP) GO TO 100
      IF(XEND.EQ.0.0) GO TO 77
      IF(XSTART.GT.XEND) GO TO 77
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      IAPP=0
      GO TO 100
   77 WRITE(102,1090)
      KKK=NWORK+1
      WRITE(102,1035) (DNAME,J,UWORK(J),VWORK(J),J=1,KKK)
   80 IF(NROOT.EQ.0) GO TO 90
      IF(IPATH.EQ.1) GO TO 82
   81 IPATH=2
      CALL BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MULT)
      RETURN
   82 IF(NROOT.EQ.0)GO TO 90
      IF(IROOT.EQ.0) GO TO 85
      WRITE(102,1080)
      DO 55 I=1,IROOT
   55 WRITE(102,1085) I,UROOT(I),VROOT(I),URAPP(I,2),VRAPP(I,2)
      IF(IROOT.LT.NROOT) GO TO 85
      GO TO 87
   85 KKK=IROOT+1
      WRITE(102,1086) (I,UROOT(I),VROOT(I),I=KKK,NROOT)
   87 IF(IPATH.EQ.1) GO TO 81
      RETURN
   90 WRITE(102,1070) NOPOLY
      RETURN
  100 IAPP=IAPP+1
      ITER=1
      NALTER=0
      GO TO 40
  120 NROOT=NROOT+1
      IROOT=NROOT
      MULT(NROOT)=1
      NOMULT=NOMULT+1
      UROOT(NROOT)=UX4
      VROOT(NROOT)=VX4
      URAPP(NROOT,1)=UAPP(IAPP,1)
      VRAPP(NROOT,1)=VAPP(IAPP,1)
      URAPP(NROOT,2)=UAPP(IAPP,2)
      VRAPP(NROOT,2)=VAPP(IAPP,2)
```

TABLE F.III (Continued)

```
      URAPP(NROOT,3)=UAPP(IAPP,3)
      VRAPP(NROOT,3)=VAPP(IAPP,3)
  125 IF(NOMULT.LT.NP) GO TO 130
      GO TO 80
  130 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      NWORK=NWORK-1
      KKK=NWORK+1
      DO 140 I=1,KKK
      UWORK(I)=UB(I)
  140 VWORK(I)=VB(I)
      CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      CCC=DSQRT(UPX4*UPX4+VPX4*VPX4)
      IF(CCC.LT.EPSM) GO TO 150
      IF(NWORK.GT.2) GO TO 75
      IROOT=NROOT
      KKK=NWORK+1
      DO 145 I=1,KKK
      UB(I)=UWORK(KKK+1-I)
  145 VB(I)=VWORK(KKK+1-I)
      CALL QUAD(NWORK,UB,VB,NROOT,UROOT,VROOT,MULT)
      GO TO 80
  150 MULT(NROOT)=MULT(NROOT)+1
      NOMULT=NOMULT+1
      GO TO 125
  110 UX1=UX2
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      UPX3=UPX4
      VPX3=VPX4
      ITER=ITER+1
      GO TO 50
 1090 FORMAT(///,1X,65HCOEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO
     1ZEROS WERE FOUND//)
 1080 FORMAT(///1X,13HROOTS OF Q(X),83X,21HINITIAL APPROXIMATION//)
 1070 FORMAT(//,43H NO ZEROS WERE FOUND FOR POLYNOMIAL NUMBER ,I2)
 1086 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,19X,23HSOLVED
     1 BY DIRECT METHOD)
 1035 FORMAT(3X,A2,I2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1050 FORMAT(82X,D23.16,3H + ,D23.16,2H I/82X,D23.16,3H + ,D23.16,2H I/)
 1085 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,18X,D23.16,3H
     1 + ,D23.16,2H I)
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MUL
     1T)
C     ***********************************************************************
C     *                                                                     *
C     * SUBROUTINE BETTER ATTEMPTS TO IMPROVE THE ACCURACY OF THE ZEROS FOUND  *
C     * BY USING THEM AS INITIAL APPROXIMATIONS WITH MULLER'S METHOD APPLIED TO *
C     * THE FULL, UNDEFLATED POLYNOMIAL.                                     *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UROOT,VROOT,UA,VA,UBAPP,VBAPP,UX1,VX1,UX2,VX2,UX3
     1,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UB,VB,UROOTS,VROOTS,EPSRT,UX4,V
     2X4,URAPP,VRAPP,EPSO,EPS,UQ4,VQ4,UH3,VH3
      LOGICAL CONV
      DIMENSION UROOT(25),VROOT(25),UA(26),VA(26),UBAPP(25,3),VBAPP(25,3
     1),UB(26),VB(26),UROOTS(25),VROOTS(25),URAPP(25,3),VRAPP(25,3),MULT
     3(25)
      DOUBLE PRECISION EPS1,EPSM
      COMMON EPSRT,EPS1,EPS,EPSO,EPSM,IO2,MAX
      IF(NROOT.LE.1) RETURN
      L=0
      DO 10 I=1,NROOT
      UBAPP(I,1)=UROOT(I)*EPSRT
      VBAPP(I,1)=VROOT(I)*EPSRT
      UBAPP(I,2)=UROOT(I)
      VBAPP(I,2)=VROOT(I)
      UBAPP(I,3)=UROOT(I)*(2.0-EPSRT)
   10 VBAPP(I,3)=VROOT(I)*(2.0-EPSRT)
      DO 100 J=1,NROOT
      UX1=UBAPP(J,1)
      VX1=VBAPP(J,1)
      UX2=UBAPP(J,2)
      VX2=VBAPP(J,2)
      UX3=UBAPP(J,3)
      VX3=VBAPP(J,3)
      ITER=1
      CALL HORNER(NP,UA,VA,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NP,UA,VA,UX2,VX2,UB,VB,UPX2,VPX2)
   20 CALL HORNER(NP,UA,VA,UX3,VX3,UB,VB,UPX3,VPX3)
      CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   30 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 50
      IF(ITER.LT.MAX) GO TO 40
      WRITE(IO2,1000) J,UROOT(J),VROOT(J),MAX
      WRITE(IO2,1010) UX4,VX4
      IF(J.LT.IROOT) GO TO 33
      IF(J.EQ.IROOT) GO TO 35
      GO TO 100
   33 KKK=IROOT-1
      DO 34 K=J,KKK
      URAPP(K,1)=URAPP(K+1,1)
      VRAPP(K,1)=VRAPP(K+1,1)
      URAPP(K,2)=URAPP(K+1,2)
      VRAPP(K,2)=VRAPP(K+1,2)
      URAPP(K,3)=URAPP(K+1,3)
   34 VRAPP(K,3)=VRAPP(K+1,3)
   35 IROOT=IROOT-1
      GO TO 100
```

TABLE F.III (Continued)

```
   40 UX1=UX2
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      ITER=ITER+1
      GO TO 20
   50 L=L+1
      UROOTS(L)=UX4
      VROOTS(L)=VX4
  100 CONTINUE
      IF(L.EQ.0) GO TO 120
      DO 110 I=1,L
      UROOT(I)=UROOTS(I)
  110 VROOT(I)=VROOTS(I)
      NROOT=L
      RETURN
  120 NROOT=0
      RETURN
 1000 FORMAT(///42H IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(,I2,4H) = ,
     1D23.16,3H + ,D23.16,2H I/24H DID NOT CONVERGE AFTER ,I3,11H ITERAT
     2IONS)
 1010 FORMAT(30H THE PRESENT APPROXIMATION IS ,D23.16,3H + ,D23.16,2H I/
     1/)
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE ALTER(X1R,X1I,X2R,X2I,X3R,X3I,NALTER,ITIME)
C    ***********************************************************************
C    *                                                                     *
C    * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO *
C    * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION X1R,X1I,X2R,X2I,X3R,X3I,EPS1,EPS2,EPS3,R,BETA
      DOUBLE PRECISION EPS4,EPS5
      COMMON EPS1,EPS2,EPS3,EPS4,EPS5,IO2,MAX
      IF(ITIME.NE.0) GO TO 5
      ITIME=1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) X1R,X1I,X2R,X2I,X3R,X3I
      GO TO 20
   10 R=DSQRT(X2R*X2R+X2I*X2I)
      BETA=DATAN2(X2I,X2R)
      WRITE(IO2,1020) X1R,X1I,X2R,X2I,X3R,X3I
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 X2R=-X2R
      X2I=-X2I
      GO TO 50
   40 BETA=BETA+1.0471976
      X2R=R*DCOS(BETA)
      X2I=R*DSIN(BETA)
   50 X1R=0.9*X2R
      X1I=0.9*X2I
      X3R=1.1*X2R
      X3I=1.1*X2I
      RETURN
 1000 FORMAT(1X,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HALTERED APPROXIM
     1ATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2,D23.16,2H I/)
 1020 FORMAT(1H0,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HINITIAL APPROXI
     1MATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2 ,D23.16,2H I/)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     1TER ,I3,12H ITERATIONS.//)
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C     ***********************************************************************
C     *                                                                     *
C     * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE *
C     * DEGREE OF THE ORIGINAL POLYNOMIAL.                                  *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,EPS1,EPS2,EPS3,BETA
      DOUBLE PRECISION EPSRT,EPS4
      DIMENSION APPR(25,3),APPI(25,3)
      COMMON EPSRT,EPS1,EPS2,EPS3,EPS4,IO2,MAX
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I,2)=XSTART*DCOS(BETA)
      APPI(I,2)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      DO 20 I=1,NAPP
      APPR(I,1)=0.9*APPR(I,2)
      APPI(I,1)=0.9*APPI(I,2)
      APPR(I,3)=1.1*APPR(I,2)
   20 APPI(I,3)=1.1*APPI(I,2)
      RETURN
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE TEST(UX3,VX3,UX4,VX4,CONV)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE TEST CHECKS FOR CONVERGENCE OF THE SEQUENCE OF APPROX-    *
C     * IMATIONS BY TESTING THE EXPRESSION                                   *
C     * ABSOLUTE VALUE OF (X(N+1)-X(N))/ABSOLUTE VALUE OF X(N+1).            *
C     * WHEN IT IS AS SMALL AS DESIRED, CONVERGENCE IS OBTAINED.             *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX3,VX3,UX4,VX4,EPSRT,EPSO,EPS,AAA,UDUMMY,VDUMMY,
     1DENOM
      LOGICAL CONV
      DOUBLE PRECISION EPS1,EPSM
      COMMON EPSRT,EPS1,EPS,EPSO,EPSM,IO2,MAX
      UDUMMY=UX4-UX3
      VDUMMY=VX4-VX3
      AAA=DSQRT(UDUMMY*UDUMMY+VDUMMY*VDUMMY)
      DENOM=DSQRT(UX4*UX4+VX4*VX4)
      IF(DENOM.LT.EPSO) GO TO 20
      IF(AAA/DENOM.LT.EPS) GO TO 10
    5 CONV=.FALSE.
      GO TO 100
   10 CONV=.TRUE.
      GO TO 100
   20 IF(AAA.LT.EPSO) GO TO 10
      GO TO 5
  100 RETURN
      END
```

```
      SUBROUTINE HORNER(NA,UA,VA,UX,VX,UB,VB,UPX,VPX)
C     ************************************************************************
C     *                                                                      *
C     * HORNER'S METHOD COMPUTES THE VALUE OF THE POLYNOMIAL P(X) AT A POINT D. *
C     * SYNTHETIC DIVISION IS USED TO DEFLATE THE POLYNOMIAL BY DIVIDING OUT THE *
C     * FACTOR (X-D).                                                        *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX,VX,UPX,VPX,UB,VB,UA,VA
      DIMENSION UA(26),VA(26),UB(26),VB(26)
      UB(1)=UA(1)
      VB(1)=VA(1)
      NUM=NA+1
      DO 10 I=2,NUM
      UB(I)=UA(I)+(UB(I-1)*UX-VB(I-1)*VX)
   10 VB(I)=VA(I)+(VB(I-1)*UX+UB(I-1)*VX)
      UPX=UB(NUM)
      VPX=VB(NUM)
      RETURN
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,V
C    ************************************************************************
C    *                                                                      *
C    * GIVEN THREE APPROXIMATIONS X(N-2), X(N-1), AND X(N), SUBROUTINE CALC *
C    * APPROXIMATES THE POLYNOMIAL BY A QUADRATIC AND SOLVES FOR THE ZERO OF *
C    * THE QUADRATIC CLOSEST TO X(N). THIS ZERO IS THE NEW APPROXIMATION    *
C    * X(N+1) TO THE ZERO OF THE POLYNOMIAL.                                *
C    *                                                                      *
C    ************************************************************************
     1PX3,UX4,VX4,UQ4,VQ4,UH3,VH3)
      DOUBLE PRECISION ARG1,ARG2
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UX1,VX1,UX2,VX2,UX3,VX3,UPX1,
     1VPX1,UH3,VH3,UH2,VH2,UQ3,VQ3,UD,VD,UB,VB,UC,VC,UDISC,VDISC,UCCC,VC
     2CC,UDEN1,VDEN1,UDEN2,VDEN2,UQ4,VQ4,UX4,VX4,EPSRT,EPSO,EPS,UDDD,VDD
     3D,AAA,BBB,RAD,UAAA,VAAA,UBBB,VBBB
      DOUBLE PRECISION THETA,ANGLE,UTEST,VTEST
      DOUBLE PRECISION EPS1,EPSM
      COMMON EPSRT,EPS1,EPS,EPSO,EPSM,IO2,MAX
      UH3=UX3-UX2
      VH3=VX3-VX2
      UH2=UX2-UX1
      VH2=VX2-VX1
      BBB=UH2*UH2+VH2*VH2
      UQ3=(UH3*UH2+VH3*VH2)/BBB
      VQ3=(VH3*UH2-UH3*VH2)/BBB
      UDDD=1.0+UQ3
      VDDD=VQ3
      UD=(UPX3-(UDDD*UPX2-VDDD*VPX2))+(UQ3*UPX1-VQ3*VPX1)
      VD=(VPX3-(VDDD*UPX2+UDDD*VPX2))+(VQ3*UPX1+UQ3*VPX1)
      UAAA=2.0*UQ3
      VAAA=2.0*VQ3
      UAAA=UAAA+1.0
      UBBB=UDDD*UDDD-VDDD*VDDD
      VBBB=VDDD*UDDD+UDDD*VDDD
      UCCC=UQ3*UQ3-VQ3*VQ3
      VCCC=VQ3*UQ3+UQ3*VQ3
      UB=((UAAA*UPX3-VAAA*VPX3)-(UBBB*UPX2-VBBB*VPX2))+(UCCC*UPX1-VCCC*V
     1PX1)
      VB=((VAAA*UPX3+UAAA*VPX3)-(VBBB*UPX2+UBBB*VPX2))+(VCCC*UPX1+UCCC*V
     1PX1)
      UC=UDDD*UPX3-VDDD*VPX3
      VC=VDDD*UPX3+UDDD*VPX3
      UDISC=(UB*UB-VB*VB)-(4.0*(UD*UC-VD*VC))
      VDISC=(2.0*(VB*UB))-(4.0*(VD*UC+UD*VC))
      AAA=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(AAA.EQ.0.0) GO TO 5
      GO TO 7
    5 THETA=0.0
      GO TO 9
    7 THETA=DATAN2(VDISC,UDISC)
    9 RAD=DSQRT(AAA)
      ANGLE=THETA/2.0
      UTEST=RAD*DCOS(ANGLE)
      VTEST=RAD*DSIN(ANGLE)
      UDEN1=UB+UTEST
      VDEN1=VB+VTEST
      UDEN2=UB-UTEST
      VDEN2=VB-VTEST
```

TABLE F.III (Continued)

```
      ARG1=UDEN1*UDEN1+VDEN1*VDEN1
      ARG2=UDEN2*UDEN2+VDEN2*VDEN2
      AAA=DSQRT(ARG1)
      BBB=DSQRT(ARG2)
      IF(AAA.LT.BBB) GO TO 10
      IF(AAA.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN1+VAAA*VDEN1)/ARG1
      VQ4=(VAAA*UDEN1-UAAA*VDEN1)/ARG1
      GO TO 50
   10 IF(BBB.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN2+VAAA*VDEN2)/ARG2
      VQ4=(VAAA*UDEN2-UAAA*VDEN2)/ARG2
      GO TO 50
   50 UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      RETURN
   60 UQ4=1.0
      VQ4=0.0
      GO TO 50
      END
```

TABLE F.III (Continued)

```
      SUBROUTINE CONSQT(UX,VX,UY,VY)
C     ******************************************************************
C     *                                                                *
C     * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.  *
C     *                                                                *
C     ******************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

APPENDIX G

REPEATED G.C.D. - NEWTON'S METHOD

1. Use of the Program

A double precision FORTRAN IV program using the repeated G.C.D. method with Newton's method as a supporting method is presented here. Flow charts for this program are given in Figure G.2 while Table G.III gives a FORTRAN IV listing of this program. Single precision variables are listed in Table G.II. The single precision variables are used in the flow charts and the corresponding double precision variables can be obtained from Table G.II.

This program is designed to solve polynomials having degree less than or equal to 25. In order to solve polynomials of degree N where N > 25, the data statement and array dimensions given in Table G.I must be changed.

In this program both the leading coefficient and the constant coefficient are assumed to be non-zero.

TABLE G.I

PROGRAM CHANGES NECESSARY TO SOLVE POLYNOMIALS OF DEGREE GREATER THAN 25 BY THE REPEATED G.C.D. - NEWTON'S METHOD

Main Program

Data Entry/1H1,1H2,...,1H9,2H10,2H11,...,2HXX/where XX = N+1
UP(N+1), VP(N+1)
UAPP(N), VAPP(N)
UD0(N+1), VD0(N+1)
UDD0(N+1), VDD0(N+1)
UD1(N+1), VD1(N+1)
UD2(N+1), VD2(N+1)
UDD1(N+1), VDD1(N+1)
UG(N+1), VG(N+1)
UD3(2N+1), VD3(2N+1)
UD4(2N+1), VD4(2N+1)
UZROS(N), VZROS(N)
UAP(N), VAP(N)
UROOT(N), VROOT(N)
NULT(N)
ENTRY(N+1)

Subroutine PROD

UH(2N+1), VH(2N+1)
UF(N+1), VF(N+1)
UG(N+1), VG(N+1)

Subroutine ZROS

UAPP(N), VAPP(N)
UROOT(N), VROOT(N)
UQ(N+1), VQ(N+1)
UQQ(N+1), VQQ(N+1)
UAP(N), VAP(N)
UQD(N+1), VQD(N+1)
ENTRY(N+1)
UROOTS(N), VROOTS(N)

Subroutines GENAPP, GCD, NEWTON, DIVIDE, HORNER, and DERIV

See corresponding subroutine in Table E.I.

Subroutine QUAD

UROOT(N), VROOT(N)
UA(N+1), VA(N+1)

2. Input Data for Repeated G.C.D. - Newton's Method

The input data for repeated G.C.D. - Newton's method is prepared as described for G.C.D. - Newton's method in Appendix E, § 2 except that the item EPS4 on the control card (Figure E.2) is omitted. An example control card for the repeated G.C.D. - Newton's method is given in Figure G.1.

3. Variables Used in Repeated G.C.D. - Newton's Method

The definitions of variables used in repeated G.C.D. - Newton's method are given in Table G.II. For definitions of variables not listed in this table, see the main program or corresponding subprogram of Table E.VI. The notation and symbols used are defined in Appendix E, § 3.

4. Description of Program Output

The number of the polynomial, control data, degree and coefficients of the polynomial are printed as described in Appendix E, § 4.

All roots of multiplicity one are extracted first. Following the first row of asterixes, the message "THE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE ROOTS OF P(X) WHICH HAVE MULTIPLICITY 1." This is followed by the coefficients of G(X) with the leading coefficient listed first. If there are no roots of multiplicity one, then the message "NO ROOTS OF MULTIPLICITY ONE" is printed.

The roots of G(X) are printed under the heading "ROOTS OF G(X)." These are the roots obtained before the attempt to improve accuracy. The initial approximations producing convergence to the corresponding root are printed under the heading "INITIAL APPROXIMATION." The message "RESULTS OF SUBROUTINE QUAD" means that the corresponding root was obtained from subroutine QUAD.

The roots found as a result of attempting to improve accuracy are printed under the heading "ROOTS OF P(X)." Their multiplicity is given under the heading "MULTIPLICITIES." The initial approximation is printed above where "NO INITIAL APPROXIMATION" means the same as "RESULTS OF SUBROUTINE QUAD."

A line of asterixes is then printed. This procedure is then repeated for the roots of multiplicity 2,3,4, etc. until all roots have been found.

5. Informative Messages and Error Messages

The informative messages and error messages for repeated G.C.D. - Newton's method are given below. For those not listed, see Appendix E, § 5.

"NOT ALL ROOTS OF THE ABOVE POLYNOMIAL, G, WERE FOUND." This message indicates that some of the roots of the polynomial G(X) were not extracted.

"QUAD FOUND XXX TO BE A MULTIPLE ROOT." XXX represents the value of the root found as a multiple root by Subroutine QUAD.

NOPOLY	NP	NAPP	MAX	EPS1	EPS2	EPS3	XSTART	XEND	KCHECK
1	7	7	200	1.D-03	1.D-10	1.D-20	1.0D+01	2.0D+01	

Figure G.1 Control Card for Repeated G.C.D. - Newton's Method

TABLE G.II

REPEATED GCD - NEWTON'S METHOD

Single Precision Variable	Type	Double Precision Variable	Type	Disposition of Argument	Description
					Main Program
KD	I	KD	I		Number of distinct roots found
K	I	K	I		Number of roots found
J1	I	J1	I		Multiplicity of given root
D0	C	UD0,VD0	D		Array of coefficients of original polynomial
ND0	I	ND0	I		Degree of original polynomial
DD0	C	UDD0,VDD0	D		Array of coefficients of derivative of D0(X) i.e. D0'(X)
NDD0	I	NDD0	I		Degree of DD0(X)
D1	C	UD1,VD1	D		Array of coefficients of g.c.d. of D0(X) and DD0(X)
ND1	I	ND1	I		Degree of D1(X)
DD1	C	UDD1,VDD1	D		Array of coefficients of derivative of D1(X) i.e. D1'(X)
NDD1	I	NDD1	I		Degree of DD1(X)
D2	C	UD2,VD2	D		Array of coefficients of g.c.d. of D1(X) and DD1(X)
ND2	I	ND2	I		Degree of D2(X)
D3	C	UD3,VD3	D		Array of coefficients of the product of D0(X) and D2(X)
ND3	I	ND3	I		Degree of D3(X)
D4	C	UD4,VD4	D		Array of coefficients of the square of D1(X)
ND4	I	ND4	I		Degree of D4(X)
G	C	UG,VG	D		Array of coefficients of the quotient D3(X)/D4(X)
NG	I	NG	I		Degree of G(X)
ZROS	C	UZROS,VZROS	D		Array of roots of G(X)
					Subroutine ZROS
APROX	C	UAPROX,VAPROX	D	R	Starting approximation (initial or altered)

TABLE G.II (Continued)

Single Precision Variable	Single Precision Type	Double Precision Variable	Double Precision Type	Disposition of Argument	Description
					Subroutine PROD
M	I	M	I	E	Degree of polynomial to be multiplied
F	C	UF,VF	D	E	Array of coefficients of polynomial to be multiplied
N	I	N	I	E	Degree of polynomial to be multiplied
G	C	UG,VG	D	E	Array of coefficients of polynomial to be multiplied
MN	I	MN	I	R	Degree of product polynomial H(X)
H	C	UH,VH	D	R	Array of coefficients of product polynomial
LIMIT	I	LIMIT	I		Number of coefficients of polynomial F(X)
K	I	K	I		Counter

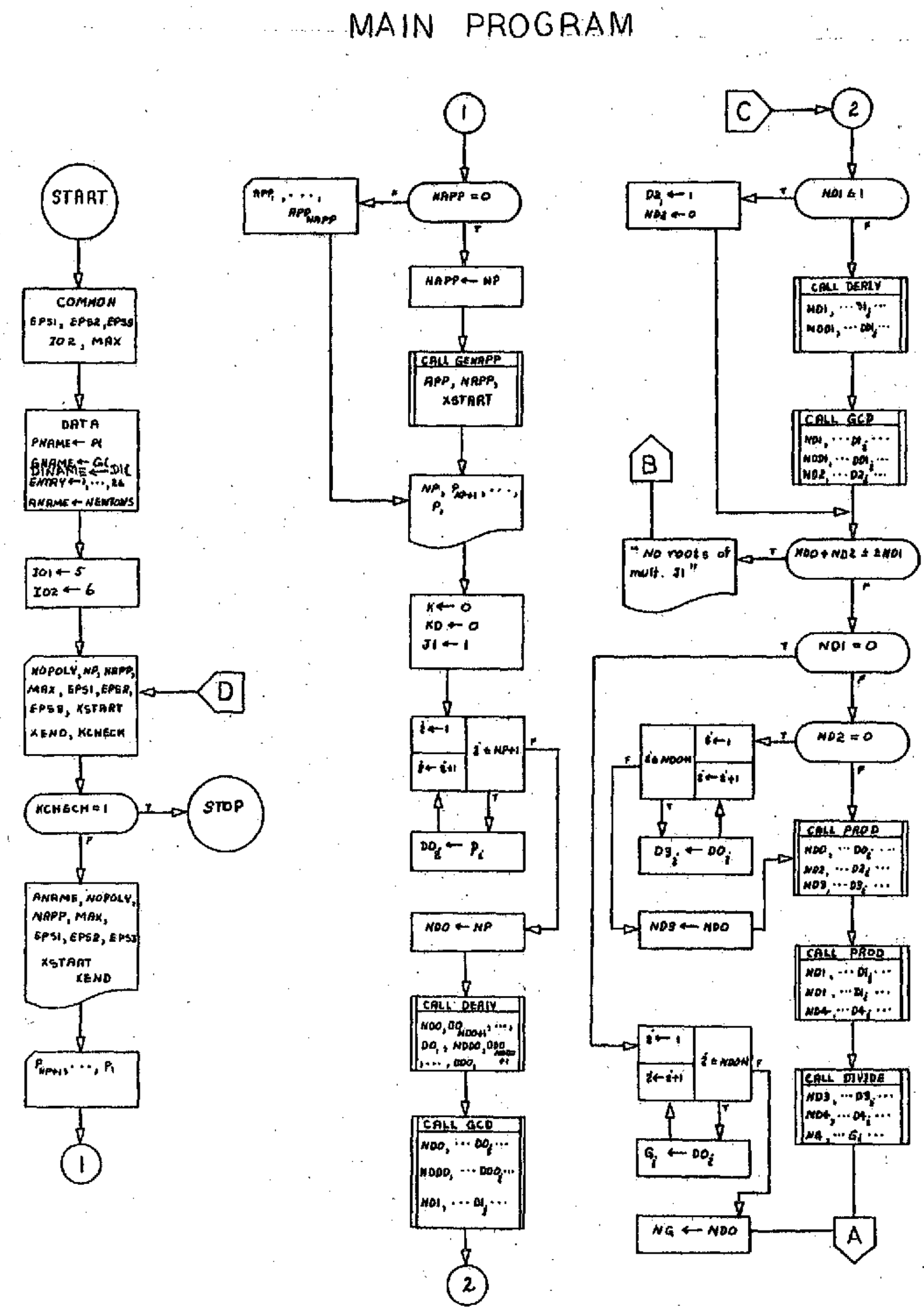


Figure G.2. Flow Charts for Repeated G.C.D.-Newton's Method

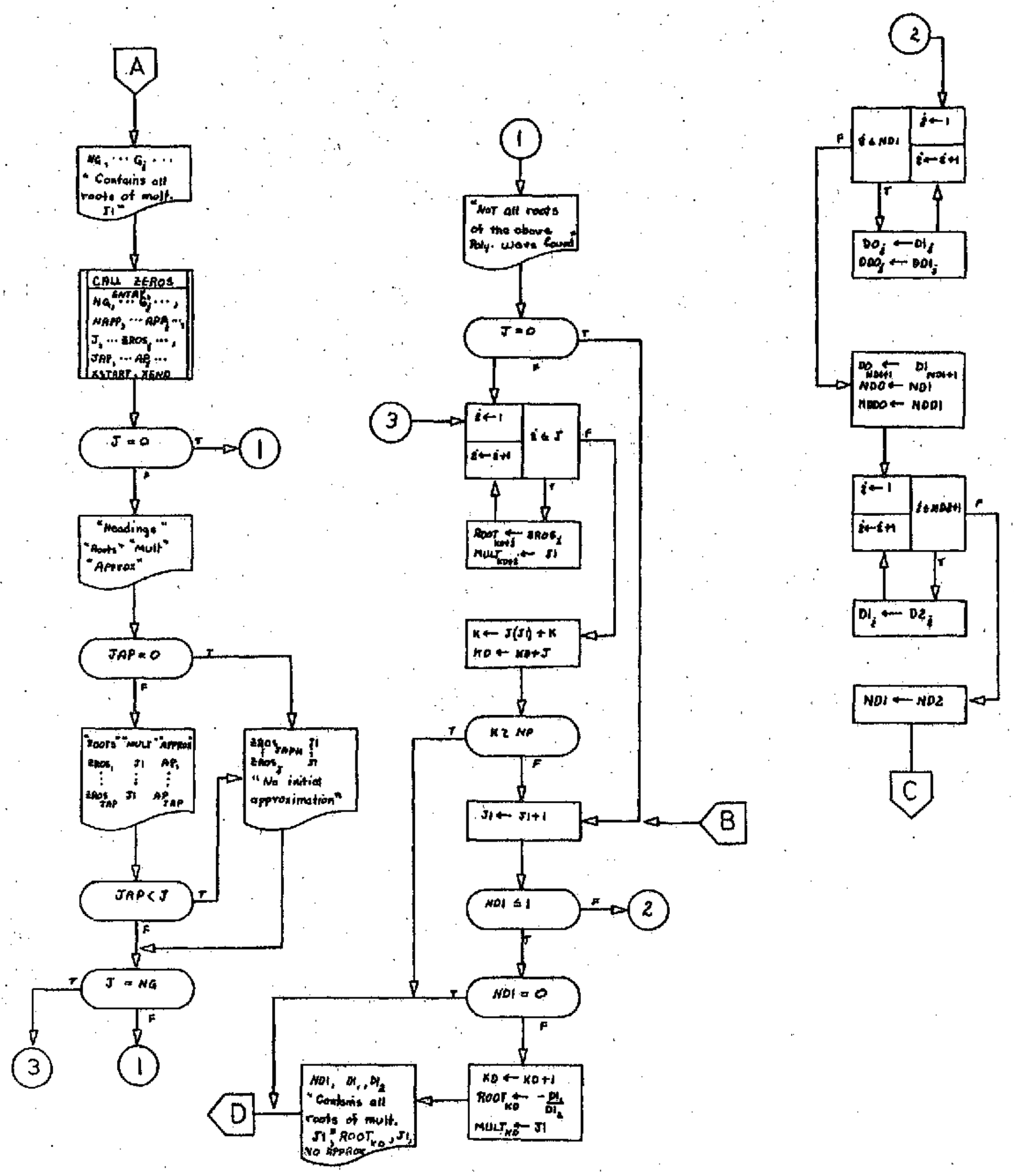


Figure G.2. (Continued)

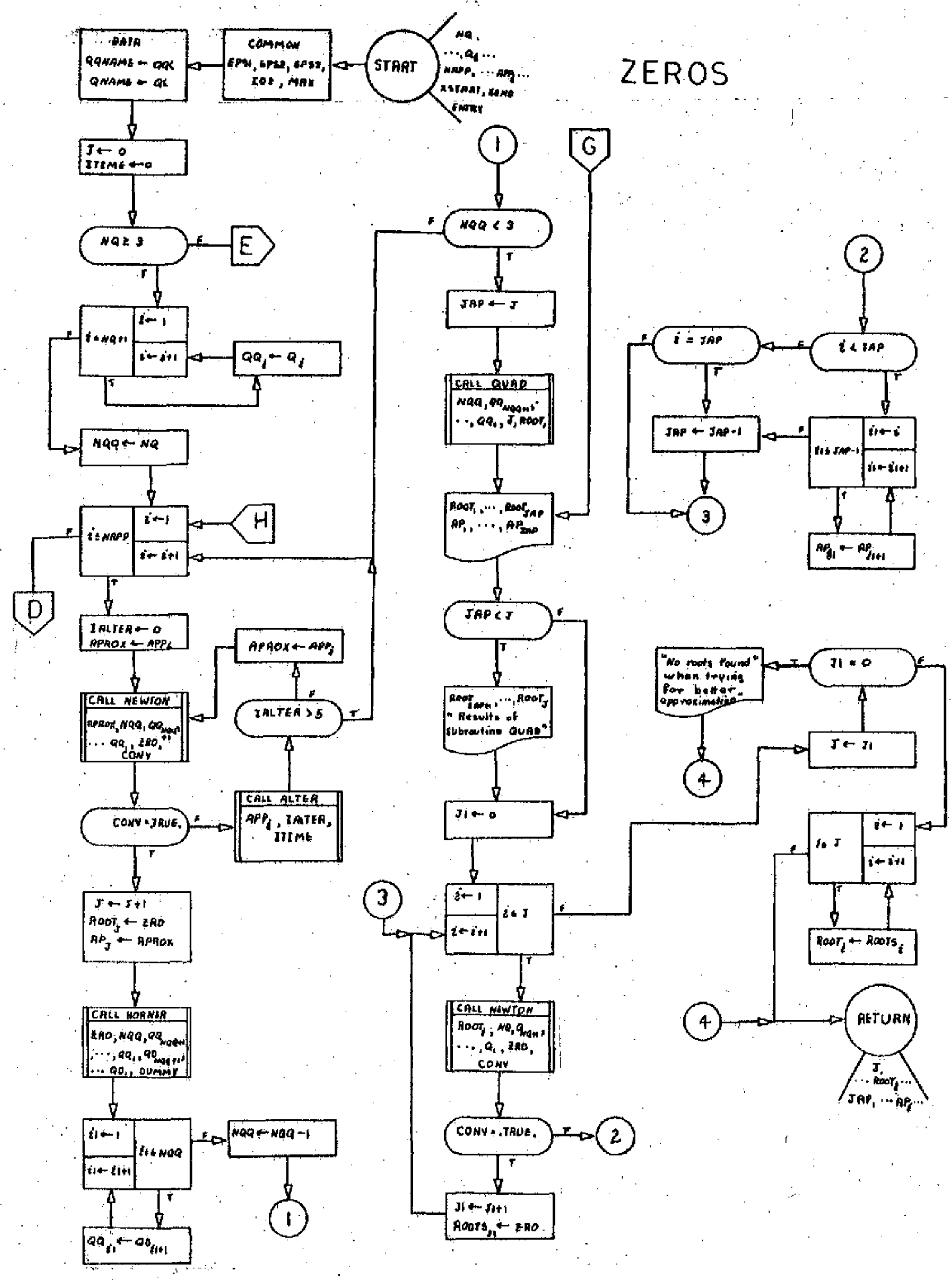


Figure G.2. (Continued)

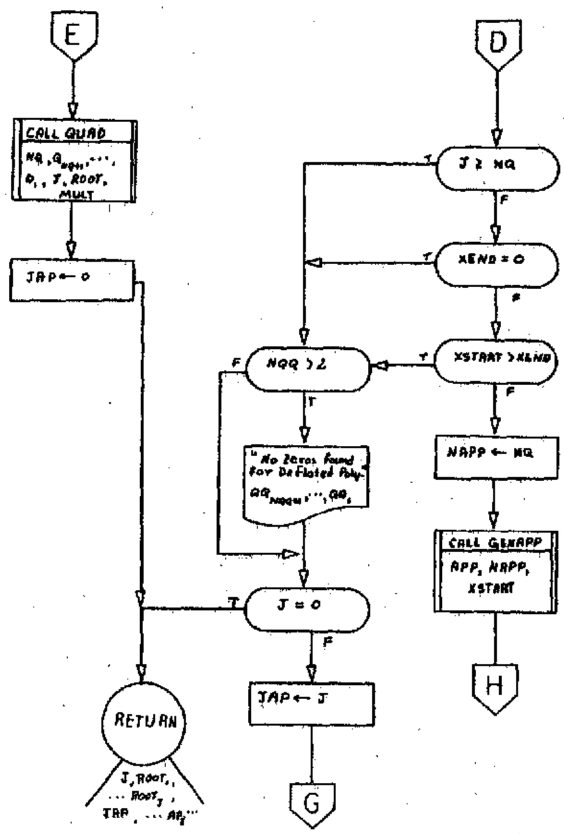
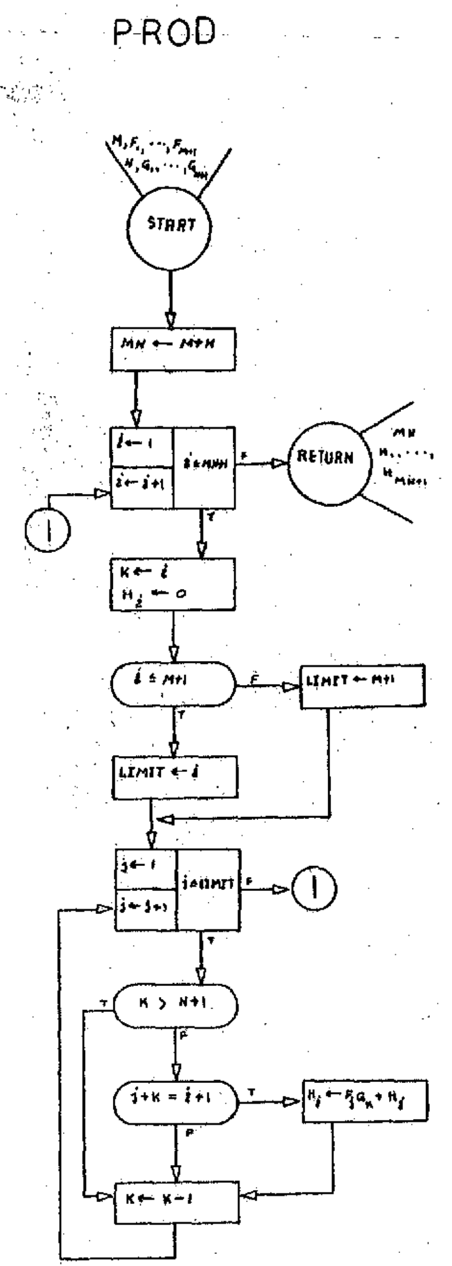


Figure G.2. (Continued)



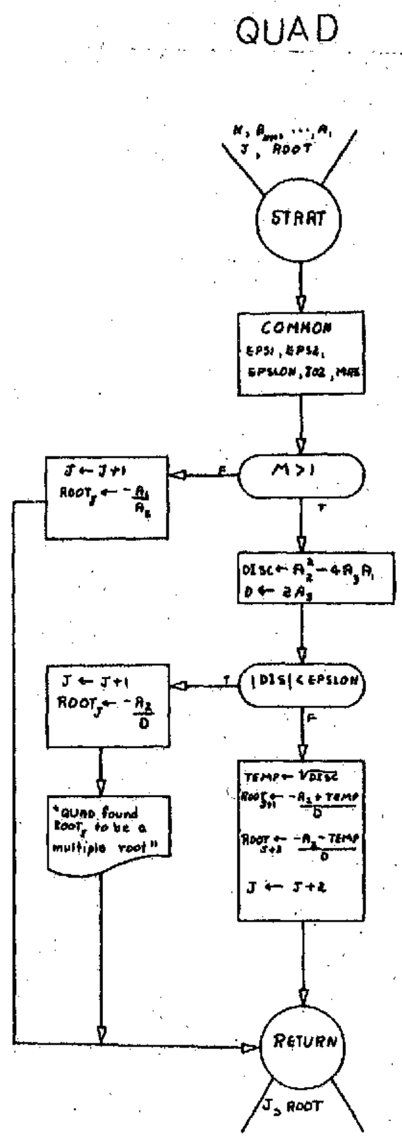


Figure G.2. (Continued)

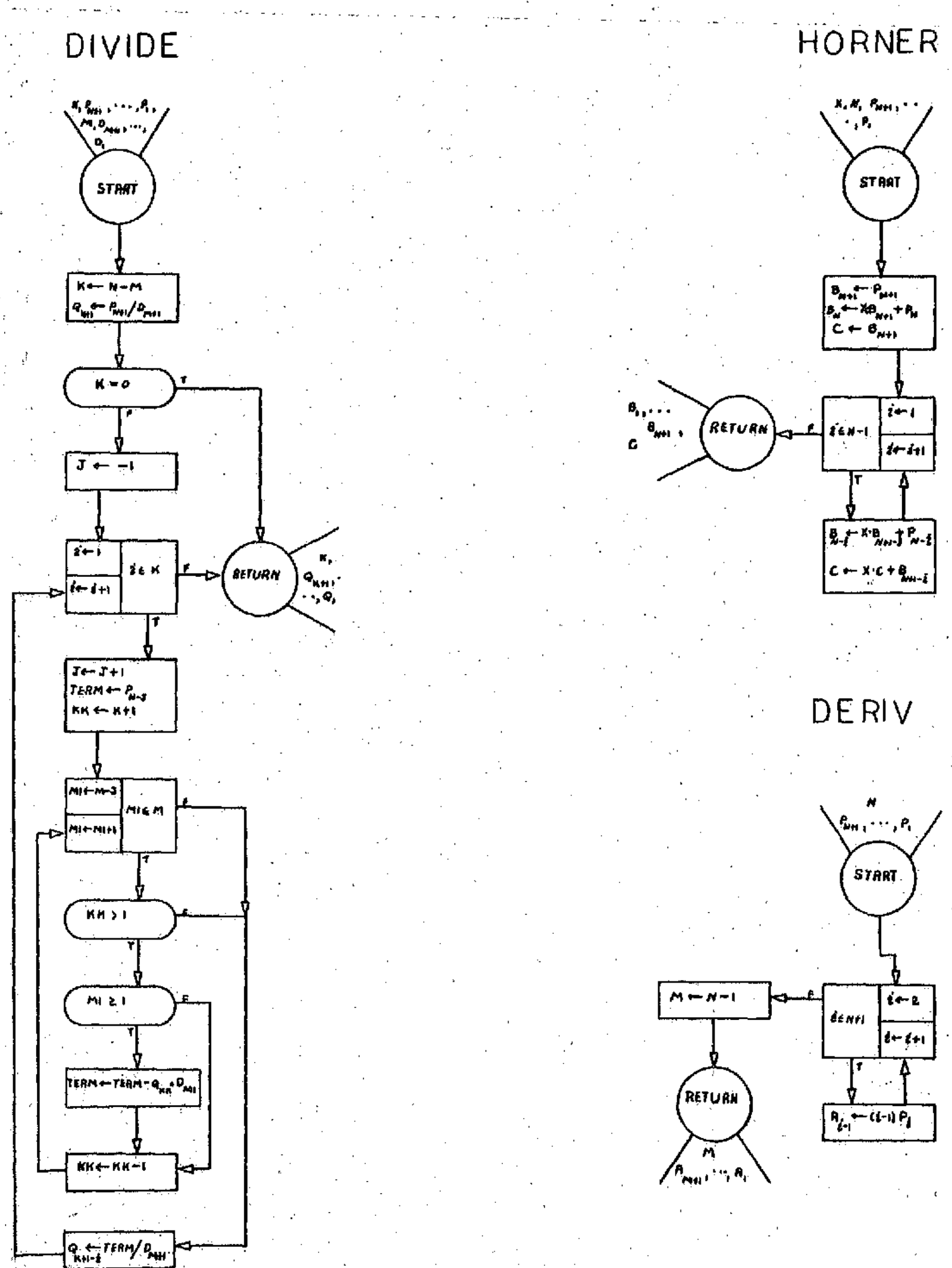


Figure G.2. (Continued)

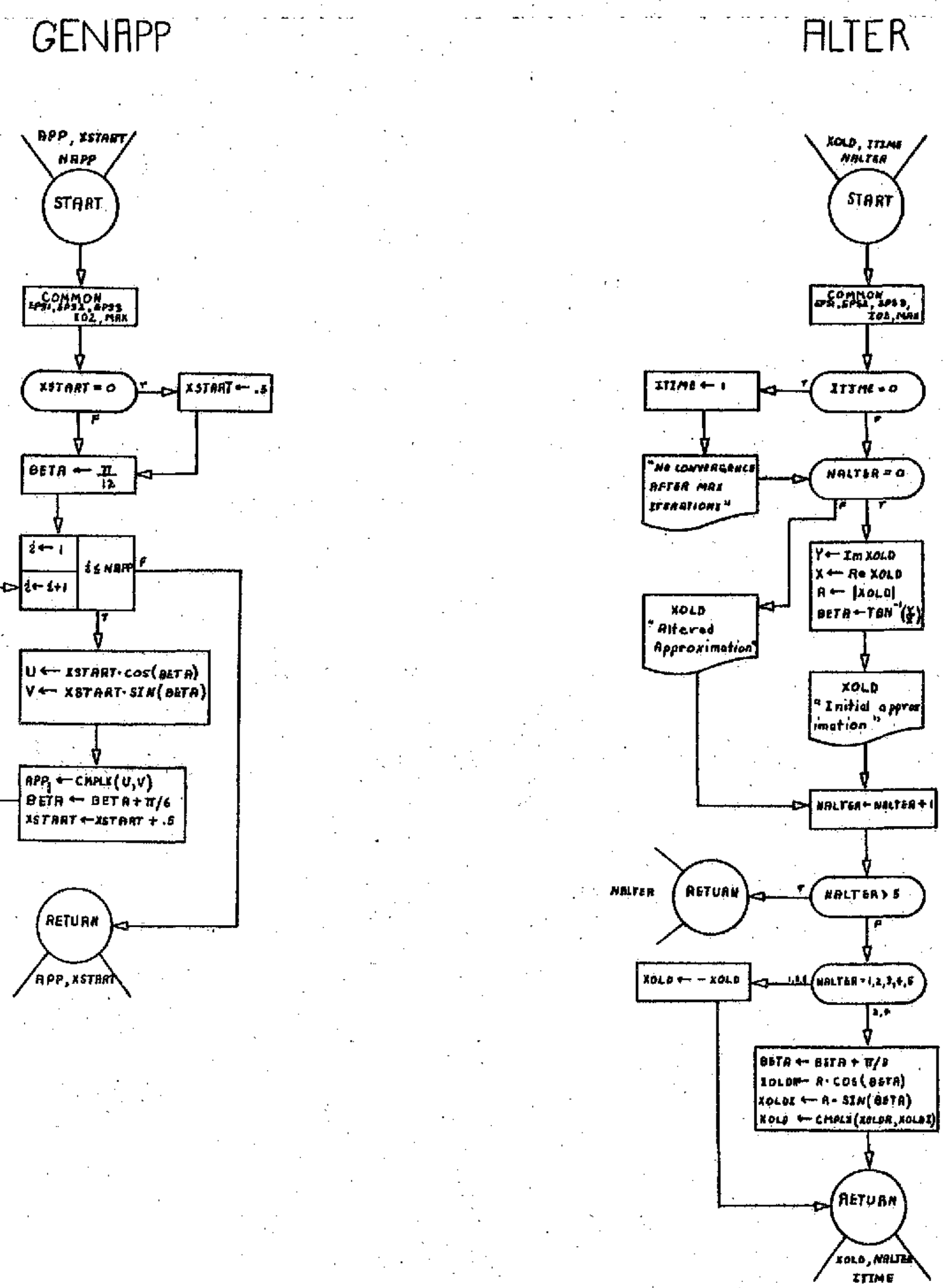


Figure G.2. (Continued)

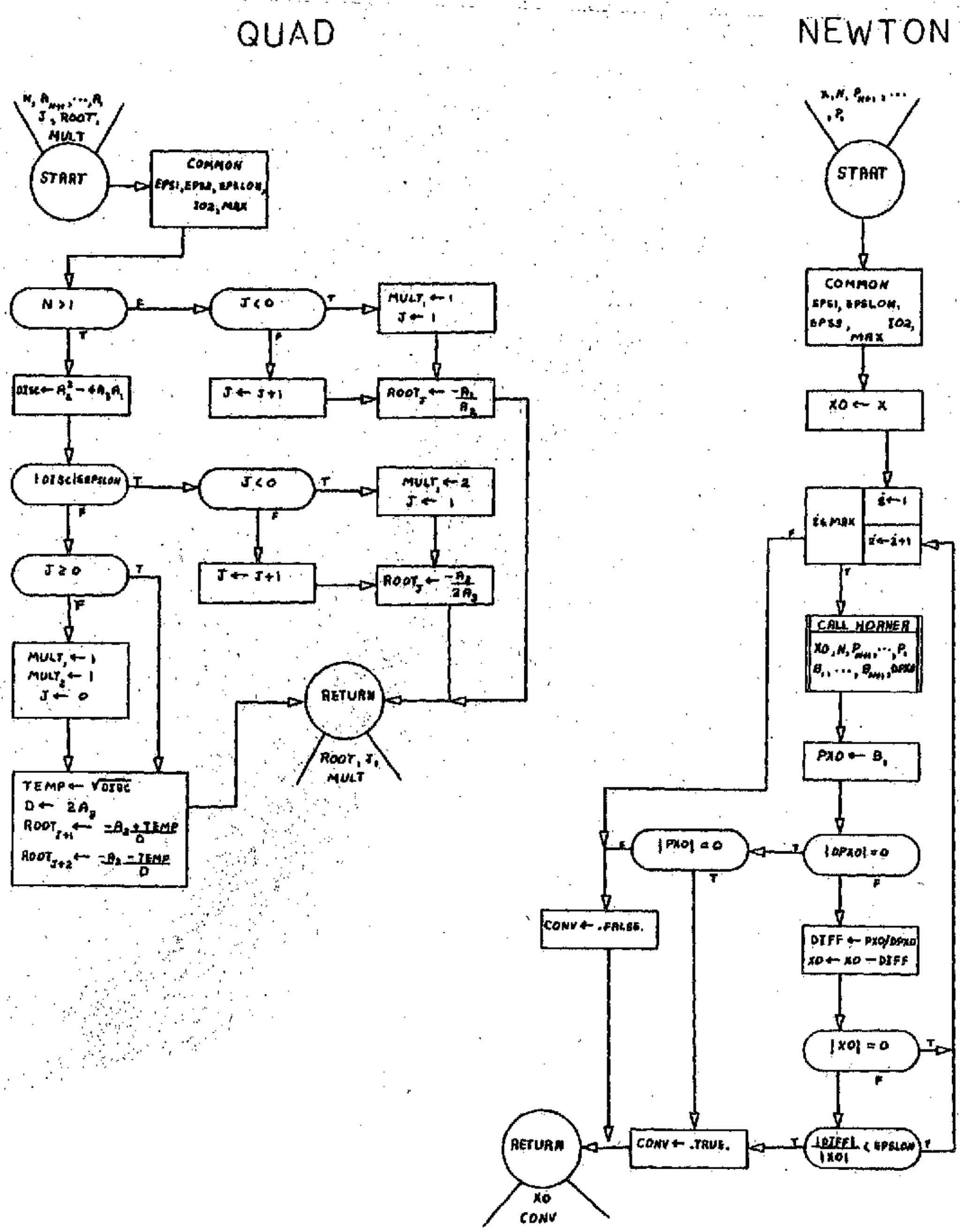


Figure G.2. (Continued)

GCD

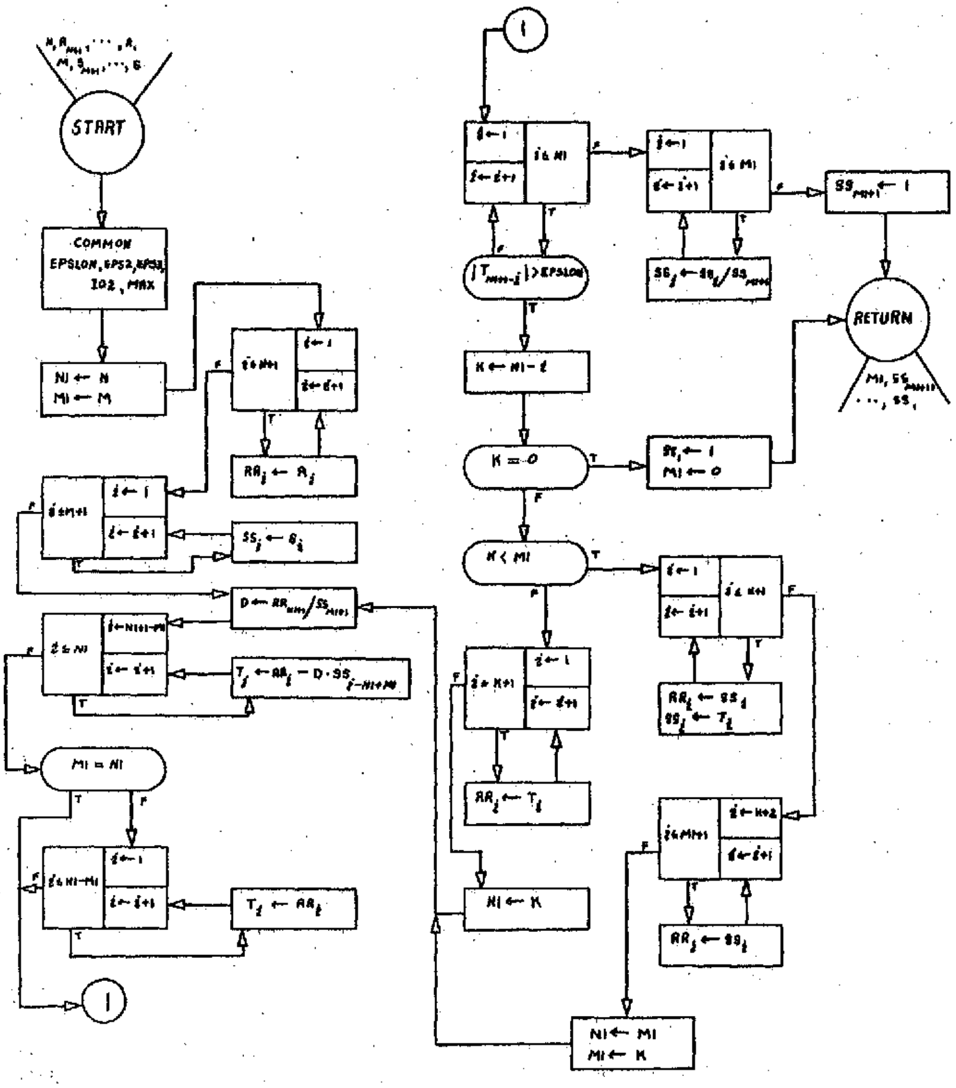


Figure G.2. (Continued)

COMSQT

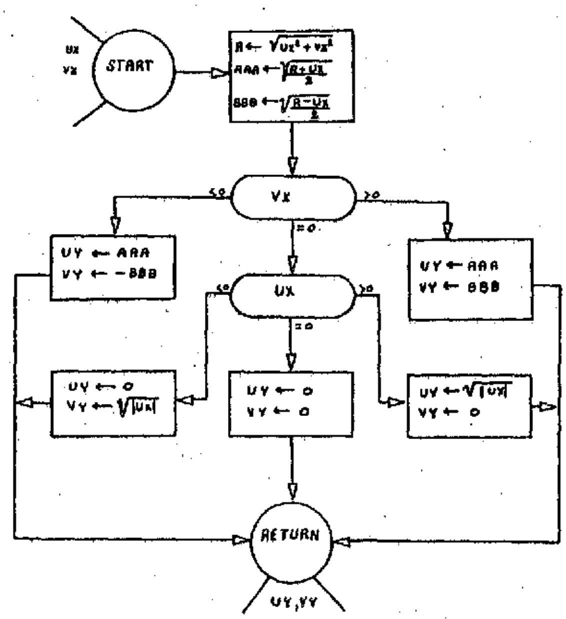


Figure G.2. (Continued)

TABLE G.III

PROGRAM FOR REPEATED G.C.D.-NEWTON'S METHOD

```
C     ***********************************************************************
C     *                                                                     *
C     * DOUBLE PRECISION PROGRAM FOR THE REPEATED G.C.D. - NEWTON'S METHOD  *
C     *                                                                     *
C     *                                                                     *
C     * THIS METHOD REPEATEDLY FINDS THE GREATEST COMMON DIVISOR OF TWO     *
C     * POLYNOMIALS IN ORDER TO EXTRACT THE ZEROS IN GROUPS ACCORDING TO    *
C     * MULTIPLICITY USING NEWTON'S METHOD.  ALL ZEROS OF MULTIPLICITY 1    *
C     * ARE EXTRACTED FOLLOWED BY THOSE OF MULTIPLICITY 2, ETC.             *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION EPS1,EPS2,EPS3,UP,VP,UAPP,VAPP,UD0,VD0,UDD0,VDD0,
     1UD1,VD1,UD2,VD2,UDD1,VDD1,UG,VG,UD3,VD3,UD4,VD4,UZROS,VZROS,UAP,VA
     2P,UROOT,VROOT,DENOM
      DOUBLE PRECISION XSTART
      DOUBLE PRECISION XEND
      DIMENSION ANAME(2),UP(26),VP(26),UAPP(25),VAPP(25),UD0(26),VD0(26)
     1,UDD0(26),VDD0(26),UD1(26),VD1(26),UD2(26),VD2(26),UDD1(26),VDD1(2
     26),UG(26),VG(26),UD3(51),VD3(51),UD4(51),VD4(51),UZROS(25),VZROS(2
     35),UAP(25),VAP(25),UROOT(25),VROOT(25),MULT(25),ENTRY(26)
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      DATA ASTER/4H****/
      DATA PNAME,GNAME/2HP(,2HG(/, DINAME/3HD)(/
      DATA ENTRY/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,2H10,2H11,2H12,2H13
     1,2H14,2H15,2H16,2H17,2H18,2H19,2H20,2H21,2H22,2H23,2H24,2H25,2H26/
      DATA ANAME(1),ANAME(2)/4HNEWT,4HONS /
      IO1=5
      IO2=6
    1 READ(IO1,1000) NOPOLY,NP,NAPP,MAX,EPS1,EPS2,EPS3,XSTART,XEND,KCHEC
     1K
      IF(KCHECK.EQ.1) STOP
      WRITE(IO2,1020) ANAME(1),ANAME(2),NOPOLY
      WRITE(IO2,2000) NAPP
      WRITE(IO2,2010) MAX
      WRITE(IO2,2070) EPS1
      WRITE(IO2,2020) EPS2
      WRITE(IO2,2080) EPS3
      WRITE(IO2,2040) XSTART
      WRITE(IO2,2050) XEND
      WRITE(IO2,2060)
      KKK=NP+1
      NNN=KKK+1
      DO 5 I=1,KKK
      JJJ=NNN-I
    5 READ(IO1,1010) UP(JJJ),VP(JJJ)
      IF(NAPP.NE.0) GO TO 22
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 23
   22 READ(IO1,1015) (UAPP(I),VAPP(I),I=1,NAPP)
   23 WRITE(IO2,1030) NP
      KKK=NP+1
      NNN=KKK+1
      DO 8 I=1,KKK
      JJJ=NNN-I
    8 WRITE(IO2,1040) PNAME,ENTRY(JJJ),UP(JJJ),VP(JJJ)
      K=0
      KD=0
```

TABLE G.III (Continued)

```
      J1=1
      KKK=NP+1
      DO 10 I=1,KKK
      UD0(I)=UP(I)
   10 VD0(I)=VP(I)
      ND0=NP
      CALL DERIV(ND0,UD0,VD0,NDD0,UDD0,VDD0)
      CALL GCD(ND0,UD0,VD0,NDD0,UDD0,VDD0,ND1,UD1,VD1)
   20 WRITE(IO2,3000) (ASTER,I=1,33)
      IF(ND1.LE.1) GO TO 30
      GO TO 40
   30 UD2(1)=1.0
      VD2(1)=0.0
      ND2=0
      GO TO 50
   40 CALL DERIV(ND1,UD1,VD1,NDD1,UDD1,VDD1)
      CALL GCD(ND1,UD1,VD1,NDD1,UDD1,VDD1,ND2,UD2,VD2)
   50 IF(ND0+ND2.LE.2*ND1) GO TO 60
      GO TO 70
   60 WRITE(IO2,1025) J1
      GO TO 170
   70 IF(ND1.EQ.0) GO TO 80
      GO TO 90
   80 KKK=ND0+1
      DO 85 I=1,KKK
      UG(I)=UD0(I)
   85 VG(I)=VD0(I)
      NG=ND0
      GO TO 110
   90 IF(ND2.EQ.0) GO TO 115
      CALL PROD(ND0,UD0,VD0,ND2,UD2,VD2,ND3,UD3,VD3)
  100 CALL PROD(ND1,UD1,VD1,ND1,UD1,VD1,ND4,UD4,VD4)
      CALL DIVIDE(ND3,UD3,VD3,ND4,UD4,VD4,NG,UG,VG)
  110 WRITE(IO2,1035) J1
      KKK=NG+1
      NNN=KKK+1
      DO 112 I=1,KKK
      JJJ=NNN-I
  112 WRITE(IO2,1040) GNAME,ENTRY(JJJ),UG(JJJ),VG(JJJ)
      CALL ZEROS(NG,UG,VG,NAPP,UAPP,VAPP,J,UZROS,VZROS,JAP,UAP,VAP,ENTRY
     1,XSTART,XEND)
      IF(J.EQ.0) GO TO 150
      WRITE(IO2,1180)
      IF(JAP.EQ.0) GO TO 120
      GO TO 130
  115 KKK=ND0+1
      DO 116 I=1,KKK
      UD3(I)=UD0(I)
  116 VD3(I)=VD0(I)
      ND3=ND0
      GO TO 100
  120 KKK=JAP+1
      WRITE(IO2,1085) (I,UZROS(I),VZROS(I),J1,I=KKK,J)
      GO TO 140
  130 WRITE(IO2,1190) (I,UZROS(I),VZROS(I),J1,UAP(I),VAP(I),I=1,JAP)
      IF(JAP.LT.J) GO TO 120
  140 IF(J.EQ.NG) GO TO 155
  150 WRITE(IO2,1095)
```

TABLE G.III (Continued)

```
      IF(J.EQ.0) GO TO 170
  155 DO 160 I=1,J
      UROOT(KD+I)=UZROS(I)
      VROOT(KD+I)=VZROS(I)
  160 MULT(KD+I)=J1
      K=(J*J1)+K
      KD=KD+J
      IF(K.GE.NP) GO TO 1
  170 J1=J1+1
      IF(ND1.LE.1) GO TO 200
      DO 180 I=1,ND1
      UD0(I)=UD1(I)
      VD0(I)=VD1(I)
      UDD0(I)=UDD1(I)
  180 VDD0(I)=VDD1(I)
      UD0(ND1+1)=UD1(ND1+1)
      VD0(ND1+1)=VD1(ND1+1)
      ND0=ND1
      NDD0=NDD1
      KKK=ND2+1
      DO 190 I=1,KKK
      UD1(I)=UD2(I)
  190 VD1(I)=VD2(I)
      ND1=ND2
      GO TO 20
  200 IF(ND1.EQ.0) GO TO 1
      KD=KD+1
      DENOM=UD1(2)*UD1(2)+VD1(2)*VD1(2)
      UROOT(KD)=(-UD1(1)*UD1(2)-VD1(1)*VD1(2))/DENOM
      VROOT(KD)=(-VD1(1)*UD1(2)+UD1(1)*VD1(2))/DENOM
      MULT(KD)=J1
      WRITE(IO2,3000) (ASTER,I=1,33)
      WRITE(IO2,1035) J1
      KKK=ND1+1
      NNN=KKK+1
      DO 210 I=1,KKK
      JJJ=NNN-I
  210 WRITE(IO2,1100) DINAME,ENTRY(JJJ),UD1(JJJ),VD1(JJJ)
      WRITE(IO2,1180)
      WRITE(IO2,1085) KD,UROOT(KD),VROOT(KD),J1
      GO TO 1
 1020 FORMAT(1H1,10X,48HREPEATED USE OF THE GREATEST COMMON DIVISOR AND
     1,A4,A4,58H METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIA
     2LS/11X,18HPOLYNOMIAL NUMBER ,I2///)
 1025 FORMAT(///1X,25HNO ROOTS OF MULTIPLICITY ,I2//)
 1035 FORMAT(///1X,87HTHE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE R
     1OOTS OF P(X) WHICH HAVE MULTIPLICITY ,I2//)
 1085 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,18X,25H
     1NO INITIAL APPROXIMATIONS)
 1095 FORMAT(///1X,51HNOT ALL ROOTS OF THE ABOVE POLYNOMIAL,G, WERE FOUN
     1D//)
 1000 FORMAT(3(I2,1X),9X,I3,1X,3(D6.0,1X),20X,2(D7.0,1X),I1)
 1010 FORMAT(2D30.0)
 1015 FORMAT(2D30.0)
 1030 FORMAT(1X,22HTHE DEGREE OF P(X) IS ,I2,22H  THE COEFFICIENTS ARE//
     1)
 1040 FORMAT(2X,A2,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1100 FORMAT(2X,A3,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
```

TABLE G.III (Continued)

```
0150          1180 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
                  1 APPROXIMATION//)
0151          1190 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,7X,I2,7X,D23.
                  116,3H + ,D23.16,2H I)
0152          2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.  ,I2)
0153          2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
0154          2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
0155          2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
0156          2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
0157          2060 FORMAT(//1X)
0158          2070 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE GCD.  ,D9.2)
0159          2080 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE QUAD. ,D9.2)
0160          3000 FORMAT(/////1X,A3,32A4)
0161               END
```

TABLE G.III (Continued)

```
      SUBROUTINE PROD(M,UF,VF,N,UG,VG,MN,UH,VH)
C   ***************************************************************************
C   *                                                                         *
C   * GIVEN POLYNOMIALS R(X) AND S(X), THIS SUBROUTINE COMPUTES THE           *
C   * COEFFICIENTS OF THE PRODUCT POLYNOMIAL T(X) = R(X).S(X).                *
C   *                                                                         *
C   ***************************************************************************
      DOUBLE PRECISION UH,VH,UF,VF,UG,VG
      DIMENSION UH(51),VH(51),UF(26),VF(26),UG(26),VG(26)
      MN=M+N
      KKK=MN+1
      DO 100 I=1,KKK
      K=I
      UH(I)=0.0
      VH(I)=0.0
      IF(I.LE.M+1) GO TO 10
      LIMIT=M+1
      GO TO 20
   10 LIMIT=I
   20 DO 50 J=1,LIMIT
      IF(K.GT.N+1) GO TO 50
      IF(J+K.EQ.I+1) GO TO 40
      GO TO 50
   40 UH(I)=UH(I)+(UF(J)*UG(K)-VF(J)*VG(K))
      VH(I)=VH(I)+(VF(J)*UG(K)+UF(J)*VG(K))
   50 K=K-1
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C   ***************************************************************************
C   *                                                                         *
C   * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE    *
C   * DEGREE OF THE ORIGINAL POLYNOMIAL.                                      *
C   *                                                                         *
C   ***************************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,BETA,     EPS1,EPS2,EPS3
      DIMENSION APPR(25),APPI(25)
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I)=XSTART*DCOS(BETA)
      APPI(I)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE ALTER(XOLDR,XOLDI,NALTER,ITIME)
C    ***********************************************************************
C    *                                                                     *
C    * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO *
C    * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION XOLDR,XOLDI,ABXOLD,BETA,EPS1,EPS2,EPS3
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      IF(ITIME.NE.0) GO TO 5
      ITIME =1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) XOLDR,XOLDI
      GO TO 20
   10 ABXOLD=DSQRT((XOLDR*XOLDR)+(XOLDI*XOLDI))
      BETA=DATAN2(XOLDI,XOLDR)
      WRITE(IO2,1020) XOLDR,XOLDI
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 XOLDR=-XOLDR
      XOLDI=-XOLDI
      GO TO 50
   40 BETA=BETA+1.0471976
      XOLDR=ABXOLD*DCOS(BETA)
      XOLDI=ABXOLD*DSIN(BETA)
   50 RETURN
 1000 FORMAT(1X,D23.16,3H + ,D23.16,2H I,10X,21HALTERED APPROXIMATION)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     1TER ,I3,12H ITERATIONS.//)
 1020 FORMAT(/1X,D23.16,3H + ,D23.16,2H I,10X,21HINITIAL APPROXIMATION)
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE ZEROS(NQ,UQ,VQ,NAPP,UAPP,VAPP,J,UROOT,VROOT,JAP,UAP,VAP
     1,ENTRY,XSTART,XEND)
C     ******************************************************************
C     *                                                                *
C     * NEWTONS METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25 BY COMPUTING A SEQUENCE OF APPROX- *
C     * IMATIONS CONVERGING TO A ZERO OF THE POLYNOMIAL USING THE ITERATION *
C     * FORMULA                                                        *
C     *              X(N+1) = X(N)-P(X(N))/P'(X(N)).                   *
C     *                                                                *
C     ******************************************************************
      DOUBLE PRECISION UAPP,VAPP,UROOT,VROOT,UZRO,VZRO,UQ,VQ,UDUMMY,VDUM
     1MY,UQQ,VQQ,UAP,VAP,UQD,VQD,UROOTS,VROOTS,EPS1,EPS2,EPS3,UAPROX,VAP
     2ROX
      DOUBLE PRECISION XEND,XSTART
      DIMENSION UAPP(25),VAPP(25),UROOT(25),VROOT(25),UQ(26),VQ(26),UQQ(
     126),VQQ(26),UAP(25),VAP(25),UQD(26),VQD(26),ENTRY(26),UROOTS(25),V
     2ROOTS(25)
      COMMON EPS1,EPS2,EPS3,IO2,MAX
      DATA QQNAME,QNAME/3HQQ(,2HQ(/
      LOGICAL CONV
      J=0
      ITIME=0
      IF(NQ.GE.3) GO TO 85
      GO TO 110
   85 KKK=NQ+1
      DO 90 I=1,KKK
      UQQ(I)=UQ(I)
   90 VQQ(I)=VQ(I)
      NQQ=NQ
      GO TO 120
  110 CALL QUAD(NQ,UQ,VQ,J,UROOT,VROOT)
      JAP=0
      GO TO 310
  120 DO 200 I=1,NAPP
      IALTER=0
      UAPROX=UAPP(I)
      VAPROX=VAPP(I)
  130 CALL NEWTON(UAPROX,VAPROX,NQQ,UQQ,VQQ,UZRO,VZRO,CONV)
      IF(CONV) GO TO 160
      CALL ALTER(UAPP(I),VAPP(I),IALTER,ITIME)
      IF(IALTER.GT.5) GO TO 200
      UAPROX=UAPP(I)
      VAPROX=VAPP(I)
      GO TO 130
  160 J=J+1
      UROOT(J)=UZRO
      VROOT(J)=VZRO
      UAP(J)=UAPROX
      VAP(J)=VAPROX
      CALL HORNER(UZRO,VZRO,NQQ,UQQ,VQQ,UQD,VQD,UDUMMY,VDUMMY)
      DO 180 I1=1,NQQ
      UQQ(I1)=UQD(I1+1)
  180 VQQ(I1)=VQD(I1+1)
      NQQ=NQQ-1
      IF(NQQ.GE.3) GO TO 200
      JAP=J
      GO TO 220
```

TABLE G.III (Continued)

```
  200 CONTINUE
      IF(J.GE.NQ) GO TO 205
      IF(XEND.EQ.0.0) GO TO 205
      IF(XSTART.GT.XEND) GO TO 205
      NAPP=NQ
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 120
  205 IF(NQQ.LE.2) GO TO 210
      WRITE(IO2,1200)
      KKK=NQQ+1
      NNN=KKK+1
      DO 157 L=1,KKK
      JJJ=NNN-L
  157 WRITE(IO2,1100) QQNAME,ENTRY(JJJ),UQQ(JJJ),VQQ(JJJ)
  210 IF(J.EQ.0) GO TO 310
      JAP=J
      GO TO 230
  220 CALL QUAD(NQQ,UQQ,VQQ,J,UROOT,VROOT)
  230 WRITE(IO2,1132)
      WRITE(IO2,1133) (I,UROOT(I),VROOT(I),UAP(I),VAP(I),I=1,JAP)
      IF(JAP.LT.J) GO TO 235
      GO TO 240
  235 KKK=JAP+1
      WRITE(IO2,1134) (I,UROOT(I),VROOT(I),I=KKK,J)
  240 J1=0
      DO 300 I=1,J
      CALL NEWTON(UROOT(I),VROOT(I),NQ,UQ,VQ,UZRO,VZRO,CONV)
      IF(CONV) GO TO 280
      WRITE(IO2,1140) I,UROOT(I),VROOT(I),MAX,NQ
      KKK=NQ+1
      NNN=KKK+1
      DO 242 L=1,KKK
      JJJ=NNN-L
  242 WRITE(IO2,1040) QNAME,ENTRY(JJJ),UQ(JJJ),VQ(JJJ)
      IF(I.LT.JAP) GO TO 241
      IF(I.EQ.JAP) GO TO 250
      GO TO 300
  241 KKK=JAP-1
      DO 245 I1=1,KKK
      UAP(I1)=UAP(I1+1)
  245 VAP(I1)=VAP(I1+1)
  250 JAP=JAP-1
      GO TO 300
  280 J1=J1+1
      UROOTS(J1)=UZRO
      VROOTS(J1)=VZRO
  300 CONTINUE
      J=J1
      IF(J.EQ.0) GO TO 305
      DO 303 I=1,J
      UROOT(I)=UROOTS(I)
  303 VROOT(I)=VROOTS(I)
      GO TO 310
  305 WRITE(IO2,1150) NQ
      KKK=NQ+1
      NNN=KKK+1
      DO 306 L=1,KKK
      JJJ=NNN-L
```

TABLE G.III (Continued)

```
0103        306 WRITE(IO2,1040) QNAME,ENTRY(JJJ),UQ(JJJ),VQ(JJJ)
0104        310 RETURN
0105       1200 FORMAT(///1X,70HCOEFFICIENTS OF THE DEFLATED POLYNOMIAL FOR WHICH
               1NO ZEROS WERE FOUND.//)
0106       1132 FORMAT(///1X,13HROOTS OF G(X),84X,21HINITIAL APPROXIMATION//)
0107       1133 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,17X,D23.16,3H
               1 + ,D23.16,2H I)
0108       1134 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,22X,26HRESULT
               1S OF SUBROUTINE QUAD)
0109       1140 FORMAT(///,1X,40HNO ROOTS FOR INITIAL APPROXIMATION ROOT(,I2,4H) =
               1 ,D23.16,3H + ,D23.16,2H I/6H  AND ,I3,40H ITERATIONS ON THE POLYN
               2OMIAL OF DEGREE ,I2,18H WITH COEFFICIENTS//)
0110       1150 FORMAT(///,1X,45HNO ROOTS FOR THE POLYNOMIAL Q(X) OF DEGREE = ,I2,
               138H WITH GENERATED INITIAL APPROXIMATIONS//)
0111       1040 FORMAT(2X,A2,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
0112       1100 FORMAT(2X,A3,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
0113            END
```

TABLE G.III (Continued)

```
      SUBROUTINE GCD(N,UR,VR,M,US,VS,M1,USS,VSS)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN POLYNOMIALS P(X) AND DP(X) WHERE DEG. DP(X) IS LESS THAN DEG.  *
C     * P(X), SUBROUTINE GCD COMPUTES THE GREATEST COMMON DIVISOR OF P(X) AND *
C     * DP(X).                                                               *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION USSSSS,VSSSSS
      DOUBLE PRECISION UR,VR,US,VS,USS,VSS,URR,VRR,UD,VD,UT,VT,EPSLON,EP
     1S2,EPS3,BBB
      DIMENSION UR(26),VR(26),US(26),VS(26),USS(26),VSS(26),URR(26),VRR(
     126),UT(26),VT(26)
      COMMON EPSLON,EPS2,EPS3,IO2,MAX
      N1=N
      M1=M
      KKK=N+1
      DO 20 I=1,KKK
      URR(I)=UR(I)
   20 VRR(I)=VR(I)
      KKK=M+1
      DO 25 I=1,KKK
      USS(I)=US(I)
   25 VSS(I)=VS(I)
   30 BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      UD=(URR(N1+1)*USS(M1+1)+VRR(N1+1)*VSS(M1+1))/BBB
      VD=(USS(M1+1)*VRR(N1+1)-URR(N1+1)*VSS(M1+1))/BBB
      KKK=N1+1-M1
      DO 40 I=KKK,N1
      UT(I)=URR(I)-(UD*USS(I-N1+M1)-VD*VSS(I-N1+M1))
   40 VT(I)=VRR(I)-(UD*VSS(I-N1+M1)+VD*USS(I-N1+M1))
      IF(M1.EQ.N1) GO TO 70
      KKK=N1-M1
      DO 60 I=1,KKK
      UT(I)=URR(I)
   60 VT(I)=VRR(I)
   70 DO 90 I=1,N1
      BBB=DSQRT(UT(N1+1-I)*UT(N1+1-I)+VT(N1+1-I)*VT(N1+1-I))
      IF(BBB.GT.EPSLON) GO TO 100
   90 CONTINUE
      DO 95 I=1,M1
      BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      USSSSS=(USS(I)*USS(M1+1)+VSS(I)*VSS(N1+1))/BBB
      VSSSSS=(VSS(I)*USS(M1+1)-USS(I)*VSS(M1+1))/BBB
      USS(I)=USSSSS
   95 VSS(I)=VSSSSS
      USS(M1+1)=1.0
      VSS(M1+1)=0.0
      GO TO 200
  100 K=N1-I
      IF(K.EQ.0) GO TO 170
      IF(K.LT.M1) GO TO 140
      KKK=K+1
      DO 130 J=1,KKK
      URR(J)=UT(J)
  130 VRR(J)=VT(J)
      N1=K
      GO TO 30
```

TABLE G.III (Continued)

```
  140 KKK=K+1
      DO 150 J=1,KKK
      URR(J)=USS(J)
      VRR(J)=VSS(J)
      USS(J)=UT(J)
  150 VSS(J)=VT(J)
      KKK=K+2
      NNN=M1+1
      DO 160 J=KKK,NNN
      URR(J)=USS(J)
  160 VRR(J)=VSS(J)
      N1=M1
      M1=K
      GO TO 30
  170 USS(1)=1.0
      VSS(1)=0.0
      M1=0
  200 RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE NEWTON(UX,VX,N,UP,VP,UXO,VXO,CONV)
C     ************************************************************************
C     *                                                                      *
C     * THIS SUBROUTINE CALCULATES A NEW APPROXIMATION FROM THE OLD APPROX-  *
C     * IMATION BY USING THE ITERATION FORMULA                               *
C     *              X(N+1) = X(N)-P(X(N))/P'(X(N)).                         *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX,VX,UP,VP,UXO,VXO,UB,VB,UDPXO,VDPXO,UPXO,VPXO,U
     1DIFF,VDIFF,EPS1,EPSLON,EPS3,AAA,BBB
      DOUBLE PRECISION DDD
      DOUBLE PRECISION ABPXO
      DIMENSION UP(26),VP(26),UB(26),VB(26)
      COMMON EPS1,EPSLON,EPS3,IO2,MAX
      LOGICAL CONV
      UXO=UX
      VXO=VX
      DO 10 I=1,MAX
      CALL HORNER(UXO,VXO,N,UP,VP,UB,VB,UDPXO,VDPXO)
      UPXO=UB(1)
      VPXO=VB(1)
      DDD=DSQRT(UDPXO*UDPXO+VDPXO*VDPXO)
      IF(DDD.NE.0.0) GO TO 5
      ABPXO=DSQRT(UPXO*UPXO+VPXO*VPXO)
      IF(ABPXO.EQ.0.0) GO TO 20
      GO TO 15
    5 BBB=UDPXO*UDPXO+VDPXO*VDPXO
      UDIFF=(UPXO*UDPXO+VPXO*VDPXO)/BBB
      VDIFF=(VPXO*UDPXO-UPXO*VDPXO)/BBB
      UXO=UXO-UDIFF
      VXO=VXO-VDIFF
      AAA=DSQRT(UDIFF*UDIFF+VDIFF*VDIFF)
      BBB=DSQRT(UXO*UXO+VXO*VXO)
      IF(BBB.EQ.0.0) GO TO 10
      IF(AAA/BBB.LT.EPSLON) GO TO 20
   10 CONTINUE
   15 CONV=.FALSE.
      RETURN
   20 CONV=.TRUE.
      RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE DIVIDE(N,UP,VP,M,UD,VD,K,UQ,VQ)
C    ************************************************************************
C    *                                                                      *
C    * GIVEN TWO POLYNOMIALS F(X) AND G(X), SUBROUTINE DIVIDE COMPUTES THE  *
C    * QUOTIENT POLYNOMIAL  H(X) = F(X)/G(X).                               *
C    *                                                                      *
C    ************************************************************************
      DOUBLE PRECISION UP,VP,UD,VD,UQ,VQ,UTERM,VTERM,UDUMMY
      DIMENSION UP(26),VP(26),UD(26),VD(26),UQ(26),VQ(26)
      K=N-M
      UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1)=(UP(N+1)*UD(M+1)+VP(N+1)*VD(M+1))/UDUMMY
      VQ(K+1)=(VP(N+1)*UD(M+1)-UP(N+1)*VD(M+1))/UDUMMY
      IF(K.EQ.0) GO TO 100
      J=-1
      DO 50 I=1,K
      J=J+1
      UTERM=UP(N-J)
      VTERM=VP(N-J)
      KK=K+1
      NNN=M-J
      DO 40 M1=NNN,M
      IF(KK.GT.1) GO TO 10
      GO TO 45
   10 IF(M1.GE.1) GO TO 20
      GO TO 40
   20 UTERM=UTERM-(UQ(KK)*UD(M1)-VQ(KK)*VD(M1))
      VTERM=VTERM-(UQ(KK)*VD(M1)+VQ(KK)*UD(M1))
   40 KK=KK-1
   45 UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1-I)=(UTERM*UD(M+1)+VTERM*VD(M+1))/UDUMMY
   50 VQ(K+1-I)=(VTERM*UD(M+1)-UTERM*VD(M+1))/UDUMMY
  100 RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE HORNER(UX,VX,N,UP,VP,UB,VB,UC,VC)
C     ************************************************************************
C     *                                                                      *
C     * HORNER'S METHOD COMPUTES THE VALUE OF THE POLYNOMIAL P(X) AT A       *
C     * POINT D AND ITS DERIVATIVE AT D.  SYNTHETIC DIVISION IS USED TO      *
C     * DEFLATE THE POLYNOMIAL BY DIVIDING OUT THE FACTOR (X - D).           *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX,VX,UP,VP,UB,VB,UC,VC
      DOUBLE PRECISION UDUMMY,VDUMMY
      DIMENSION UP(26),VP(26),UB(26),VB(26)
      UB(N+1)=UP(N+1)
      VB(N+1)=VP(N+1)
      UB(N)=(UX*UB(N+1)-VX*VB(N+1))+UP(N)
      VB(N)=(UX*VB(N+1)+VX*UB(N+1))+VP(N)
      UC=UB(N+1)
      VC=VB(N+1)
      KKK=N-1
      DO 10 I=1,KKK
      UB(KKK+1-I)=(UX*UB(KKK+2-I)-VX*VB(KKK+2-I))+UP(KKK+1-I)
      VB(KKK+1-I)=(UX*VB(KKK+2-I)+VX*UB(KKK+2-I))+VP(KKK+1-I)
      UDUMMY=UX*UC-VX*VC
      VDUMMY=UX*VC+VX*UC
      UC=UDUMMY+UB(KKK+2-I)
   10 VC=VDUMMY+VB(KKK+2-I)
      RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE QUAD(N,UA,VA,J,UROOT,VROOT)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES *
C     * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE *
C     * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                       *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION EPS1,EPS2,EPSLON,UROOT,VROOT,UA,VA,UDISC,VDISC,UD
     1,VD,DDD,UTEMP,VTEMP,BBB
      DIMENSION UROOT(25),VROOT(25),UA(26),VA(26)
      COMMON EPS1,EPS2,EPSLON,IO2,MAX
      IF(N.GT.1) GO TO 10
      J=J+1
      BBB=UA(2)*UA(2)+VA(2)*VA(2)
      UROOT(J)=-(UA(1)*UA(2)+VA(1)*VA(2))/BBB
      VROOT(J)=-(VA(1)*UA(2)-UA(1)*VA(2))/BBB
      GO TO 100
   10 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(3)*UA(1)-VA(3)*VA(1)))
      VDISC=(2.0*UA(2)*VA(2))-(4.0*(UA(3)*VA(1)+VA(3)*UA(1)))
      UD=2.0*UA(3)
      VD=2.0*VA(3)
      DDD=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(DDD.LT.EPSLON) GO TO 20
      CALL COMSQT(UDISC,VDISC,UTEMP,VTEMP)
      BBB=UD*UD+VD*VD
      UROOT(J+1)=((-UA(2)+UTEMP)*UD+(-VA(2)+VTEMP)*VD)/BBB
      VROOT(J+1)=((-VA(2)+VTEMP)*UD-(-UA(2)+UTEMP)*VD)/BBB
      UROOT(J+2)=((-UA(2)-UTEMP)*UD+(-VA(2)-VTEMP)*VD)/BBB
      VROOT(J+2)=((-VA(2)-VTEMP)*UD-(-UA(2)-UTEMP)*VD)/BBB
      J=J+2
      GO TO 100
   20 J=J+1
      BBB=UD*UD+VD*VD
      UROOT(J)=(-UA(2)*UD-VA(2)*VD)/BBB
      VROOT(J)=(-VA(2)*UD+UA(2)*VD)/BBB
      WRITE(IO2,1000) UROOT(J),VROOT(J)
 1000 FORMAT(///1X,11HQUAD FOUND ,D23.16,3H + ,D23.16,2H I,22H TO BE A M
     1ULTIPLE ROOT//)
  100 RETURN
      END
```

TABLE G.III (Continued)

```
      SUBROUTINE DERIV(N,UP,VP,M,UA,VA)
C     ***********************************************************************
C     *                                                                     *
C     * GIVEN A POLYNOMIAL P(X), SUBROUTINE DERIV COMPUTES THE COEFFICIENTS OF *
C     * ITS DERIVATIVE P'(X).                                               *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UP,VP,UA,VA,AAA
      DIMENSION UP(26),VP(26),UA(26),VA(26)
      KKK=N+1
      DO 10 I=2,KKK
      AAA=I-1
      UA(I-1)=AAA*UP(I)
   10 VA(I-1)=AAA*VP(I)
      M=N-1
      RETURN
      END
```

```
      SUBROUTINE CONSQT(UX,VX,UY,VY)
C     ***********************************************************************
C     *                                                                     *
C     * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.       *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

APPENDIX H

REPEATED G.C.D. - MULLER'S METHOD

1. Use of the Program

A double precision FORTRAN IV program using the repeated G.C.D. method with Muller's method as a supporting method is presented here. Flow charts for this program are given in Figure H.1 while Table H.III gives a FORTRAN IV listing of this program.

This program is designed to solve polynomials having degree less than or equal to 25. In order to solve polynomials of degree N where N > 25, the data statement and array dimensions given in Table H.I must be changed.

In this program both the leading coefficient and the constant coefficient are assumed to be non-zero.

TABLE H.I

PROGRAM CHANGES NECESSARY TO SOLVE POLYNOMIALS OF DEGREE GREATER THAN 25 BY THE REPEATED G.C.D. - MULLER'S METHOD

Main Program

```
Data Entry/1H1,1H2,...,1H9,2H10,2H11,...,2HXX/where XX = N+1
                    UAPP(N,3), VAPP(N,3)
                    URAPP(N,3), URAPP(N,3)
                    UP(N+1), VP(N+1)
                    MULT(N)
                    UDDO(N+1), VDDO(N+1)
                    UD1(N+1), VD1(N+1)
                    UDD1(N+1), VDD1(N+1)
                    UD2(N+1), VD2(N+1)
                    UG(N+1), VG(N+1)
                    UD3(2N+1), VD3(2N+1)
                    UD4(2N+1), VD4(2N+1)
                    UAP(N+1), VAP(N+1)
                    UZROS(N), VZROS(N)
                    UROOT(N), VROOT(N)
                    UDO(N+1), VDO(N+1)
                    ENTRY(N+1)
```

Subroutines PROD, QUAD

See corresponding subroutine in Table G.I.

Subroutines DERIV, GCD, and DIVIDE

See corresponding subroutine in Table E.I.

Subroutines MULLER, GENAPP, BETTER and HORNER

See corresponding subroutine in Table F.I.

2. Input Data for Repeated G.C.D. - Muller's Method

The input data to the repeated G.C.D. - Muller's method is the same as for the repeated G.C.D. - Newton's method as described in Appendix G, § 2.

3. Variables Used in Repeated G.C.D. - Muller's Method

The variables used in this program are referenced in Table H.II. The notation and symbols used in the referenced tables are described in Appendix E, § 3.

TABLE H.II

VARIABLES USED IN REPEATED G.C.D. - MULLER'S METHOD

Main Program and Subroutine PROD

See Table G.II.

Subroutines QUAD, DERIV, GCD, DIVIDE, and COMSQT

See corresponding subroutine in Table E.VI.

Subroutines CALC, MULLER, GENAPP, ALTER, BETTER, TEST, and HORNER

See corresponding subroutine in Table F.II.

4. Description of Program Output

The output for this program is the same as that for repeated G.C.D. - Newton's method as described in Appendix G, § 4. Only one initial approximation, X_0, (not three) is printed. The other two required by Muller's method are $.9X_0$ and $1.1X_0$. The message "SOLVED BY DIRECT METHOD" means that the corresponding root was obtained by Subroutine QUAD.

5. Informative Messages and Error Messages

Descriptions of the informative messages and error messages printed by this program can be found either in Appendix E, § 5, Appendix F, § 5, or Appendix G, § 5.

MAIN PROGRAM

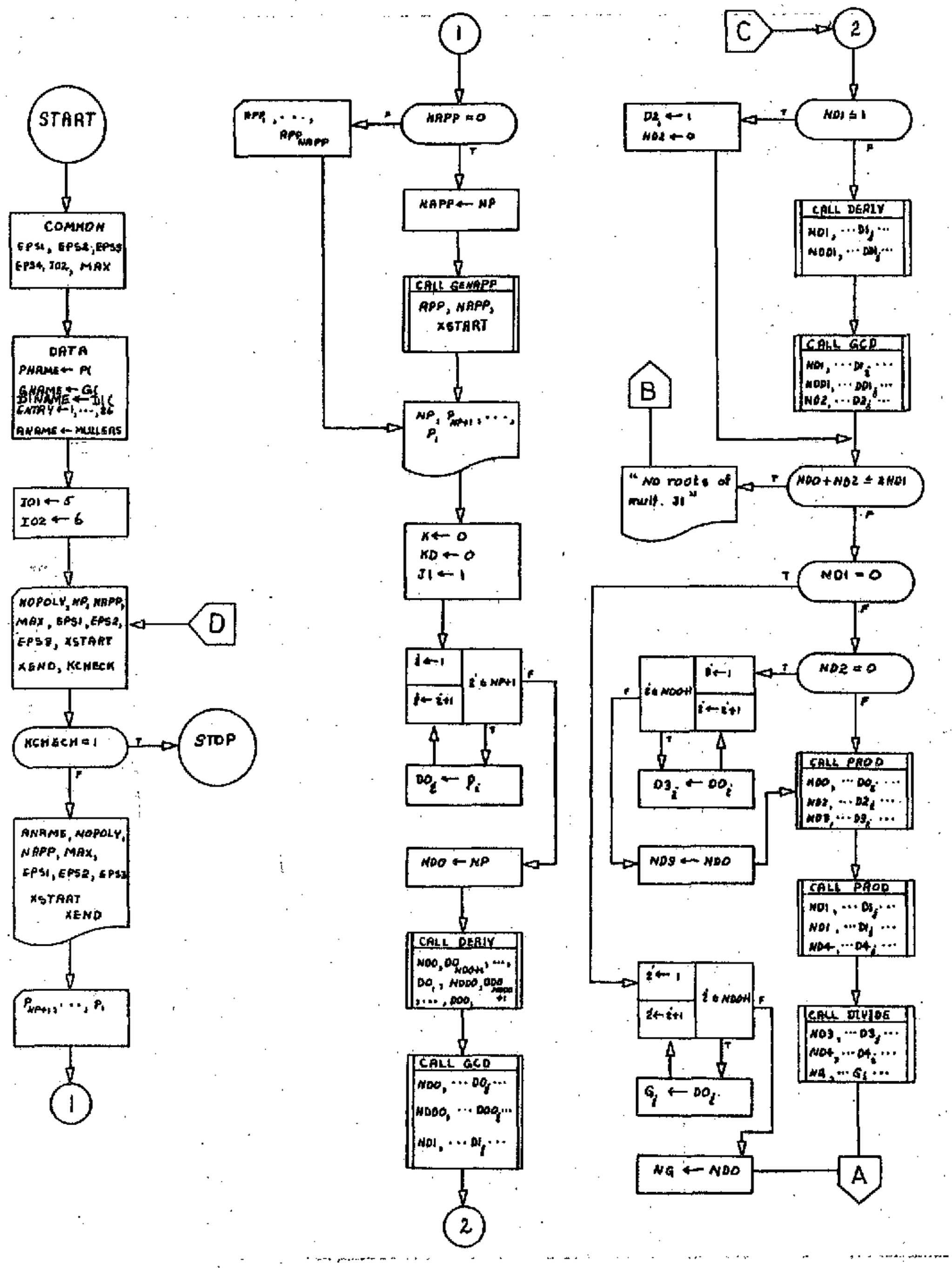


Figure H.1. Flow Charts for Repeated G.C.D.-Muller's Method

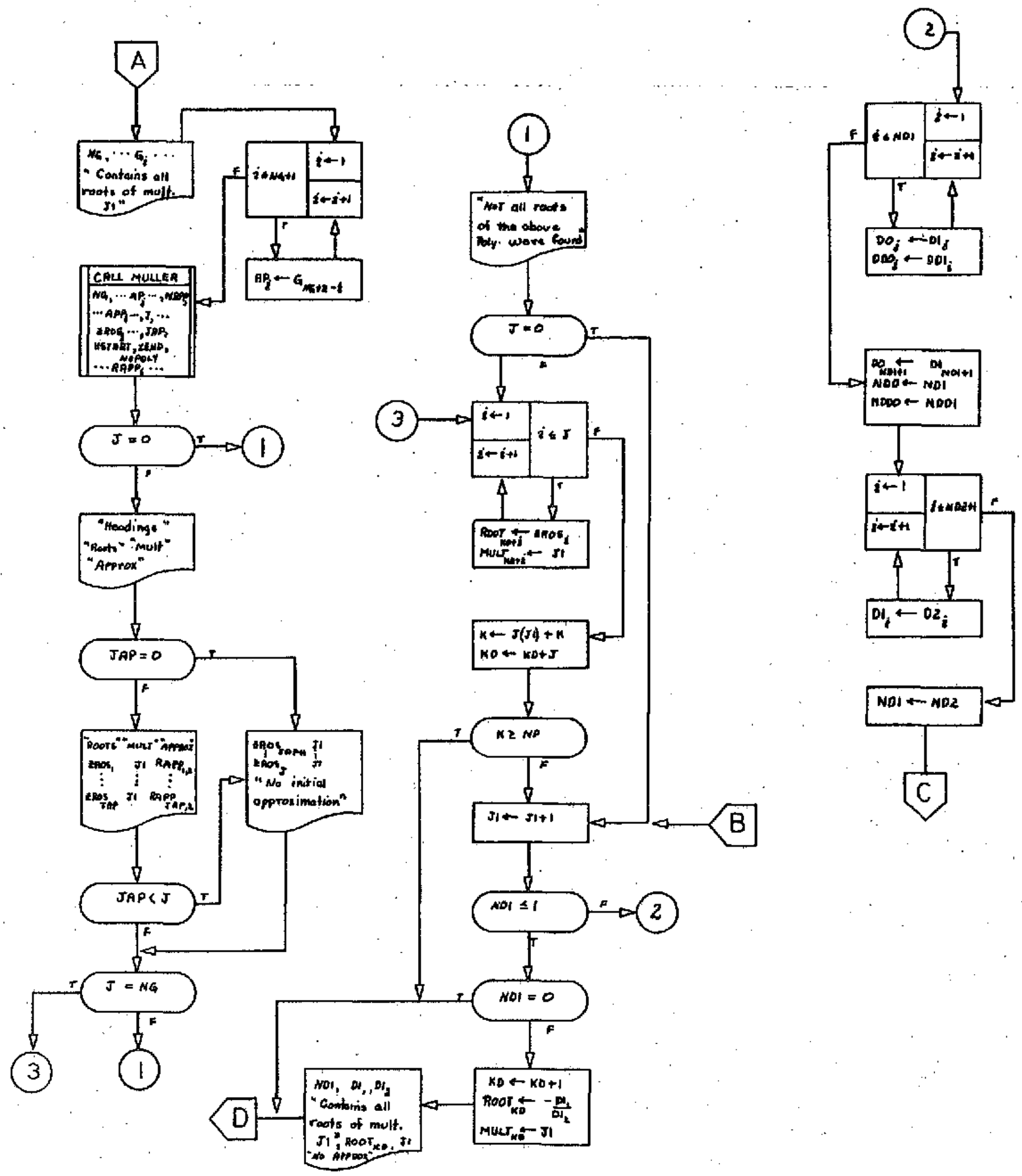


Figure H.1. (Continued)

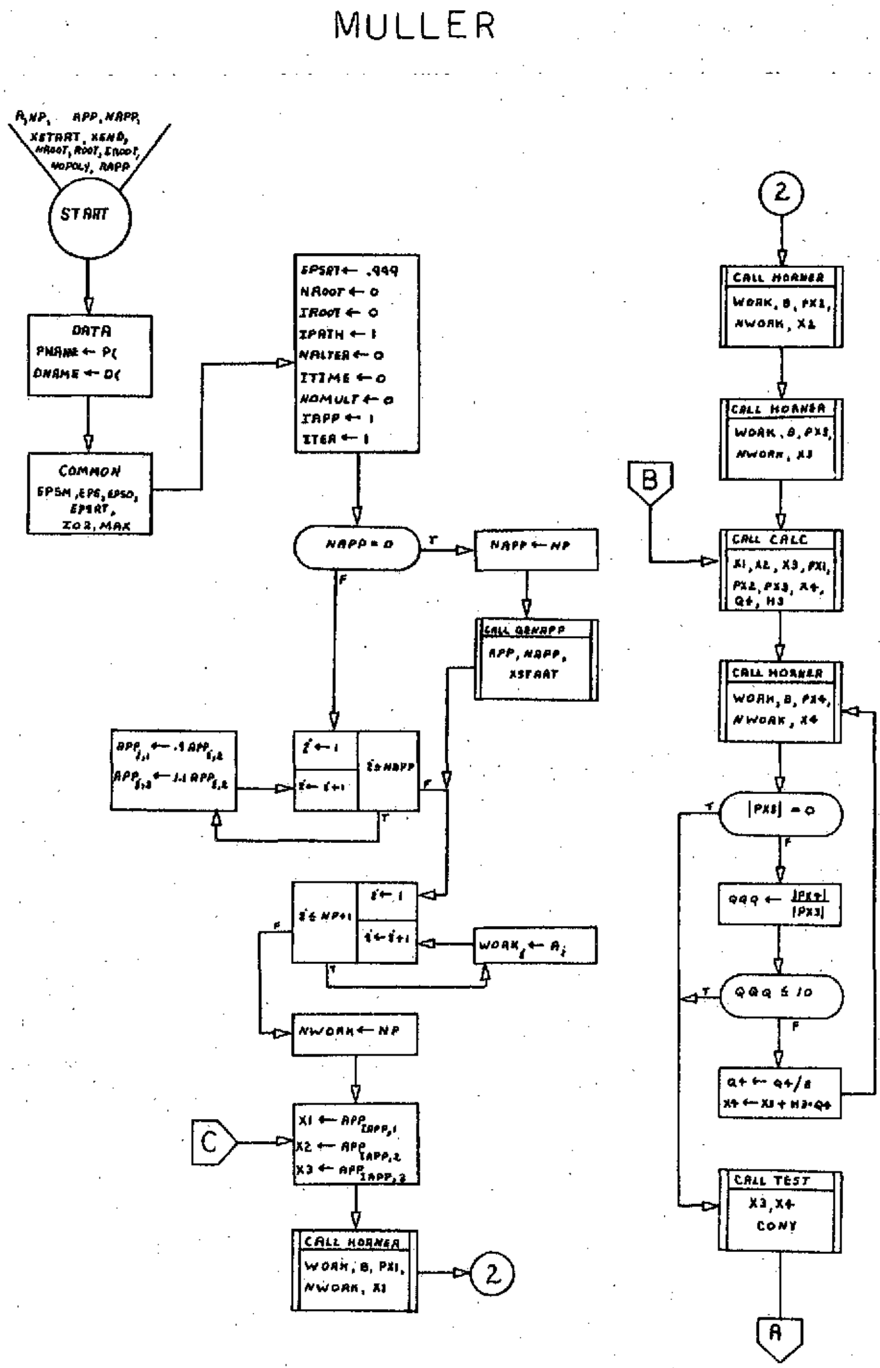


Figure H.1. (Continued)

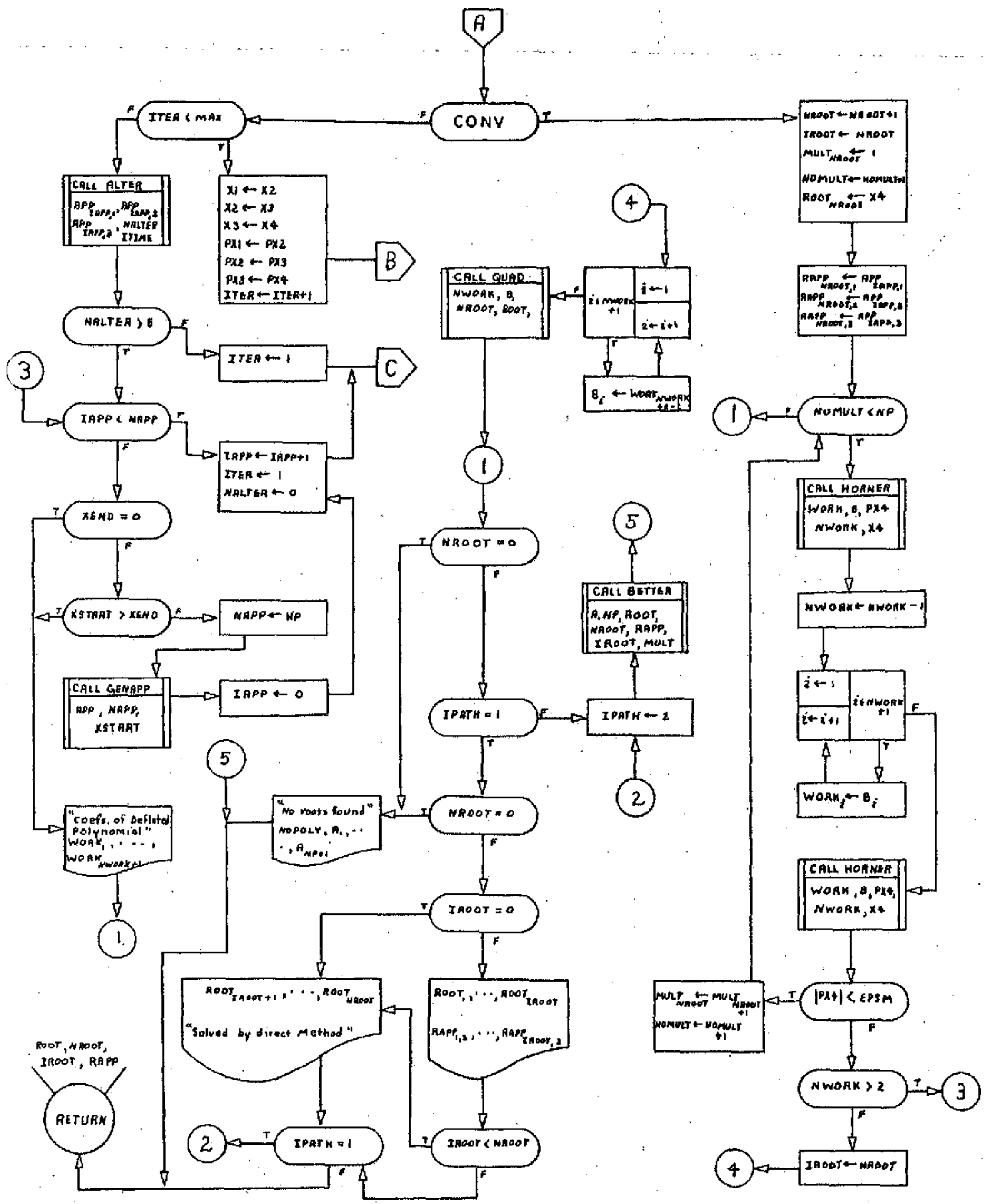
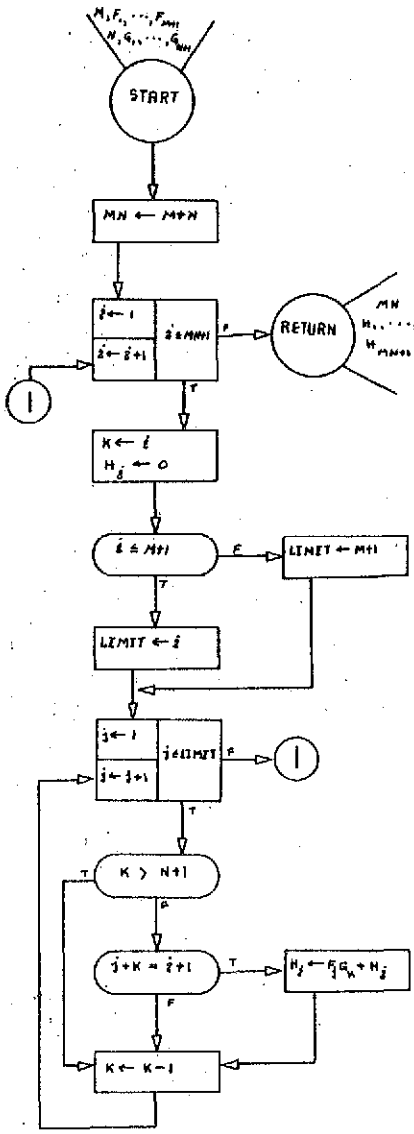


Figure H.1. (Continued)

PROD



QUAD

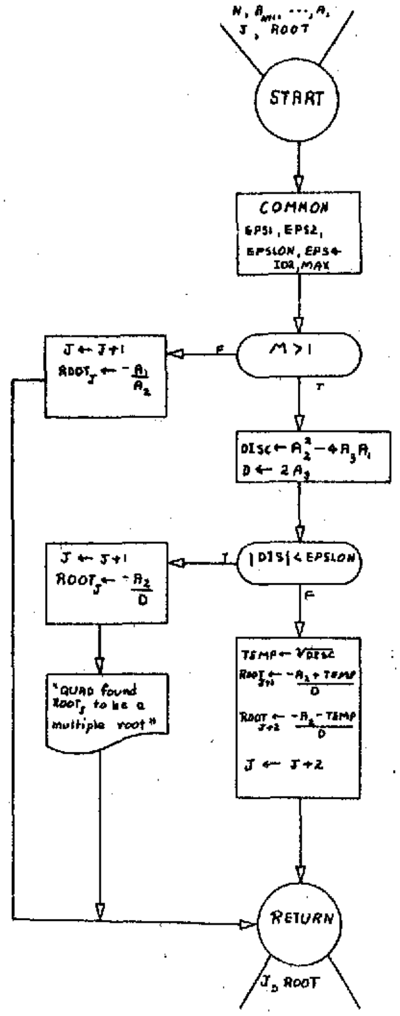
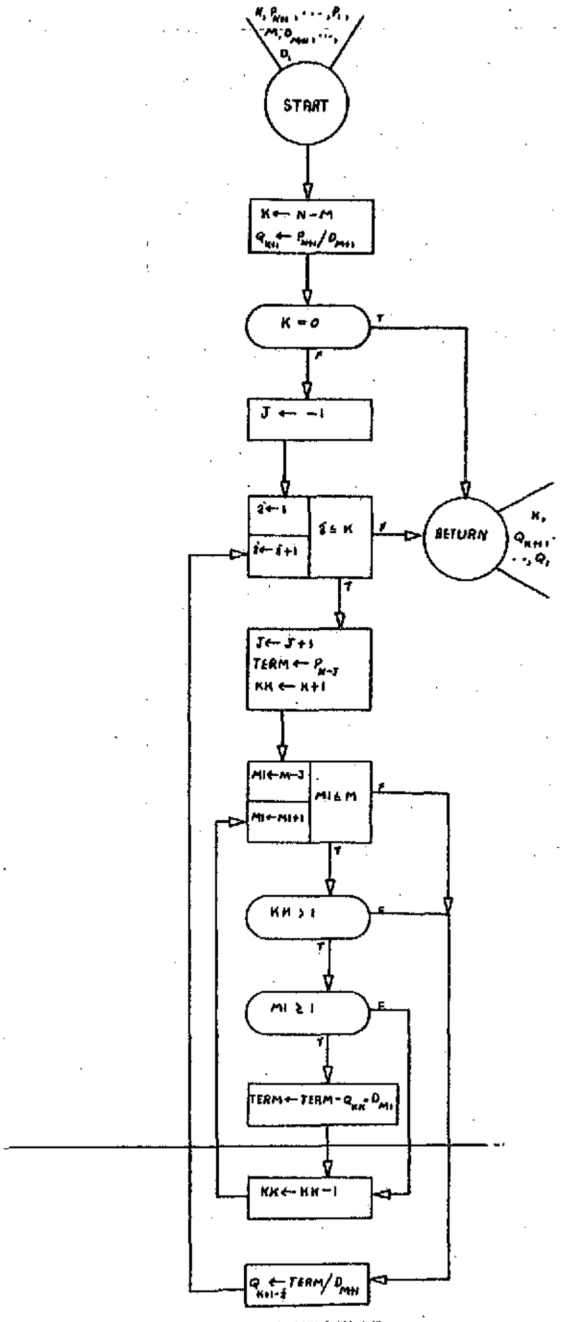


Figure H.1. (Continued)

DIVIDE



DERIV

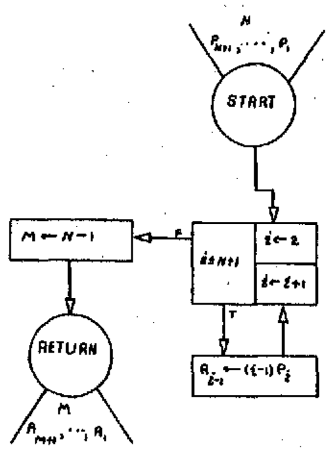


Figure H.1. (Continued)

GCD

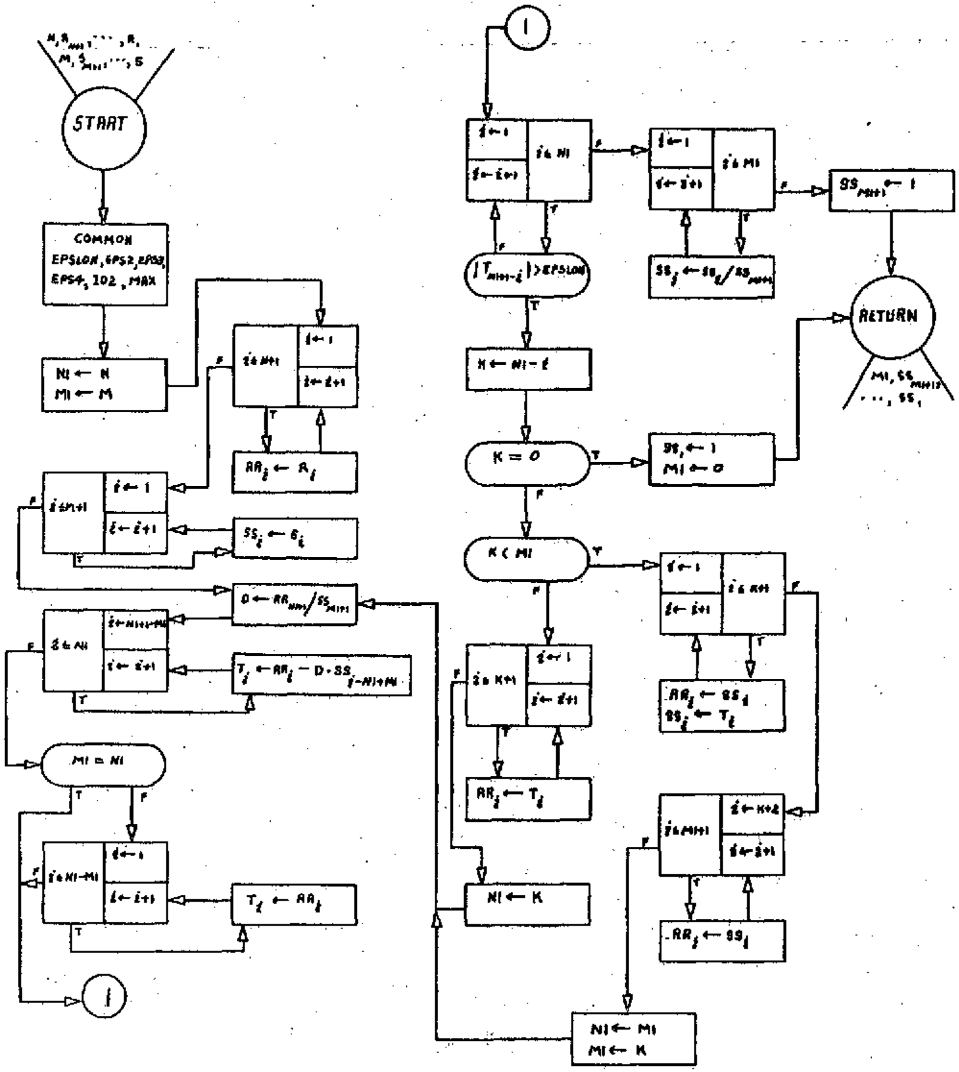
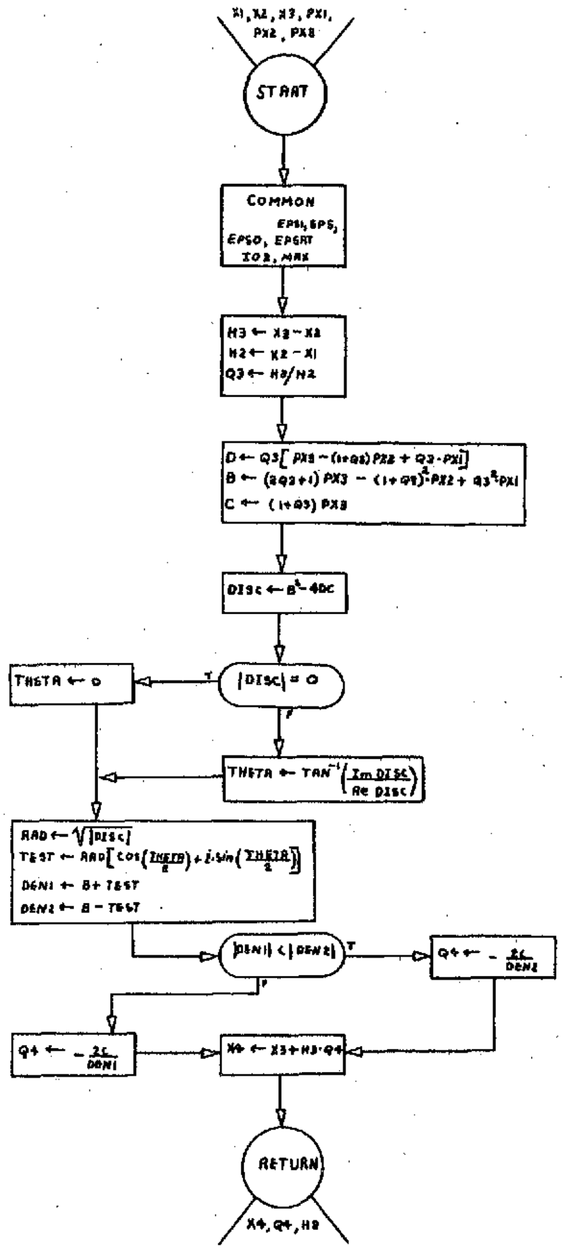


Figure H.1. (Continued)

CALC



TEST

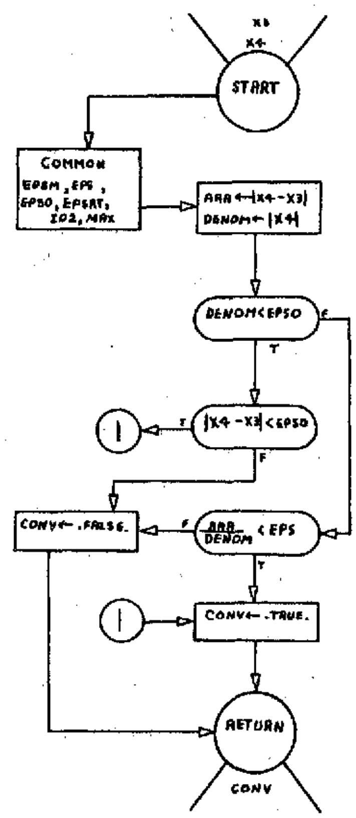
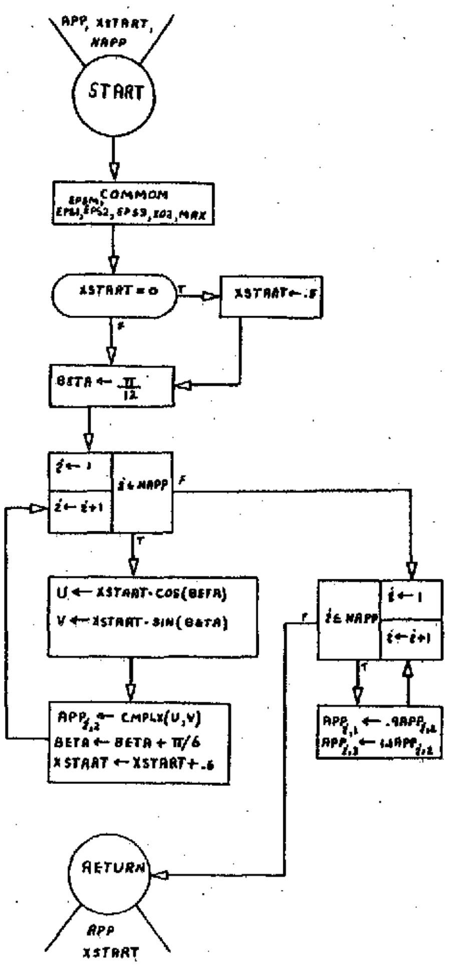


Figure H.1. (Continued)

GENAPP



ALTER

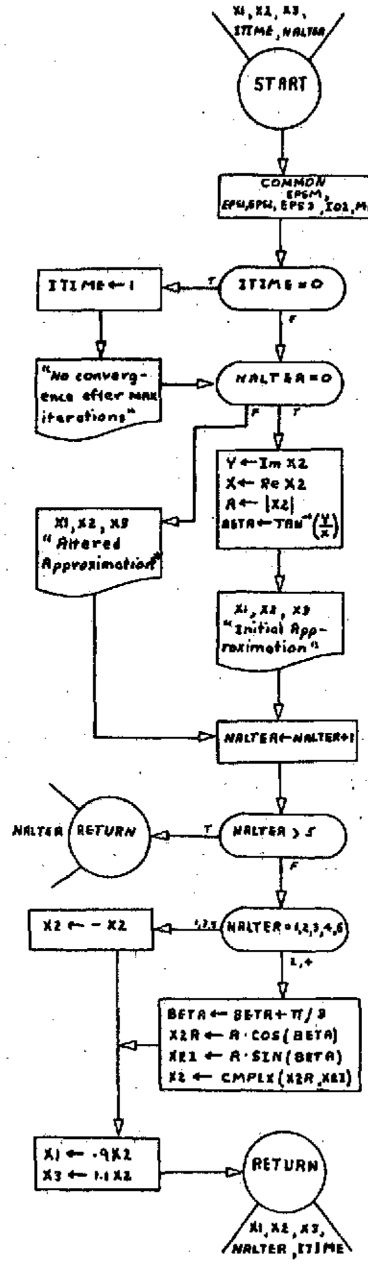


Figure H.1. (Continued)

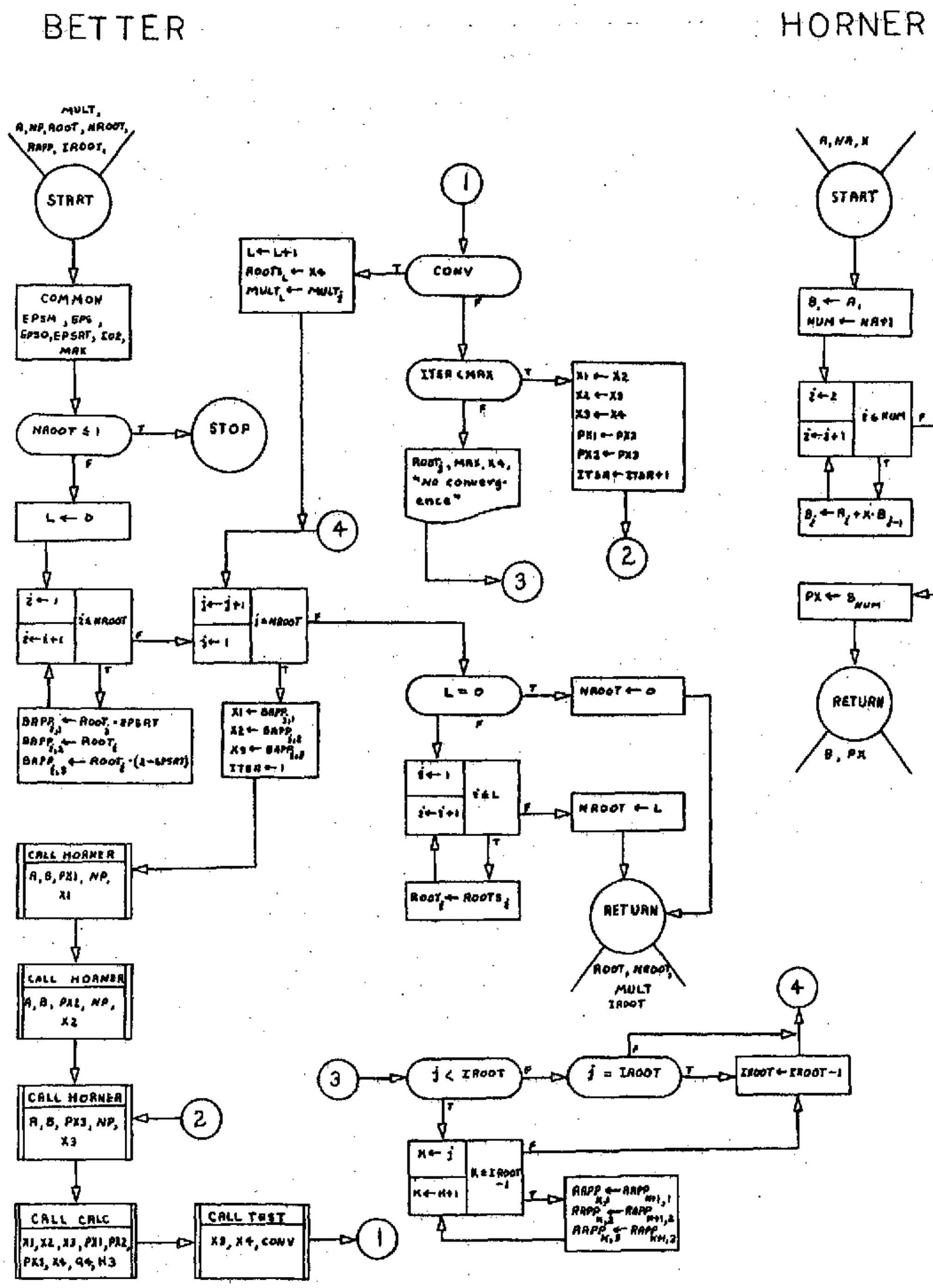
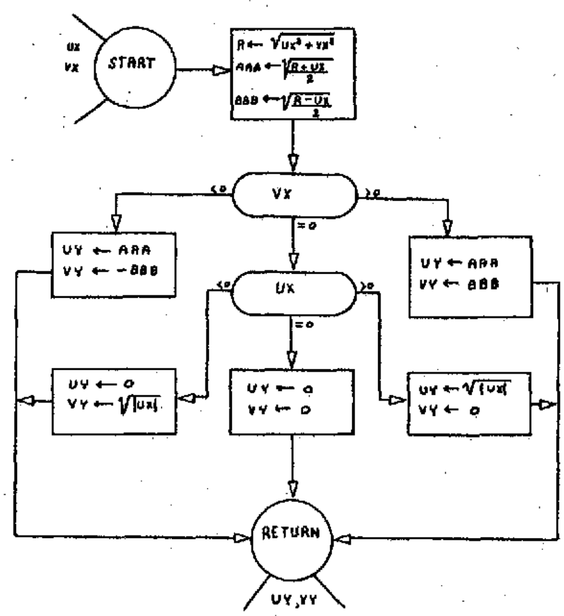


Figure H.1. (Continued)

COMSQT



Figue H.1. (Continued)

TABLE H.III

PROGRAM FOR REPEATED G.C.D.-MULLER'S METHOD

```
C     *********************************************************************
C     *                                                                   *
C     * DOUBLE PRECISION PROGRAM FOR THE REPEATED G.C.D. - MULLER'S METHOD *
C     *                                                                   *
C     *                                                                   *
C     * THIS METHOD REPEATEDLY FINDS THE GREATEST COMMON DIVISOR OF TWO    *
C     * POLYNOMIALS IN ORDER TO EXTRACT THE ZEROS IN GROUPS ACCORDING TO   *
C     * MULTIPLICITY USING NEWTON'S METHOD.  ALL ZEROS OF MULTIPLICITY 1   *
C     * ARE EXTRACTED FOLLOWED BY THOSE OF MULTIPLICITY 2, ETC.            *
C     *                                                                   *
C     *********************************************************************
      DOUBLE PRECISION EPS1,EPS2,EPS3,UP,VP,UAPP,VAPP,UD0,VD0,UDD0,VDD0,
     1UD1,VD1,UD2,VD2,UDD1,VDD1,UG,VG,UD3,VD3,UD4,VD4,UZROS,VZROS,UAP,VA
     2P,UROOT,VROOT,DENOM
      DOUBLE PRECISION XSTART
      DOUBLE PRECISION XEND
      DOUBLE PRECISION URAPP,VRAPP
      DOUBLE PRECISION EPS4
      DIMENSION UAPP(25,3),VAPP(25,3),URAPP(25,3),VRAPP(25,3)
      DIMENSION UP(26),VP(26),MULT(25),UDD0(26),VDD0(26),UD1(26),VD1(26)
     1,UDD1(26),VDD1(26),UD2(26),VD2(26),UG(26),VG(26),UD3(51),VD3(51),U
     2D4(51),VD4(51),UAP(26),VAP(26),UZROS(25),VZROS(25),UROOT(25),VROOT
     3(25),ANAME(2),UD0(26),VD0(26),ENTRY(26)
      COMMON EPS1,EPS2,EPS3,EPS4,IO2,MAX
      DATA PNAME,GNAME/2HP(,2HG(/, D1NAME/3HD1(/
      DATA ASTER/4H****/
      DATA ENTRY/1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,2H10,2H11,2H12,2H13
     1,2H14,2H15,2H16,2H17,2H18,2H19,2H20,2H21,2H22,2H23,2H24,2H25,2H26/
      DATA ANAME(1),ANAME(2)/4HMULL,4HERS /
      IO1=5
      IO2=6
    1 READ(IO1,1000) NOPOLY,NP,NAPP,MAX,EPS1,EPS2,EPS3,XSTART,XEND,KCHEC
     1K
      IF(KCHECK.EQ.1) STOP
      WRITE(IO2,1020) ANAME(1),ANAME(2),NOPOLY
      WRITE(IO2,2000) NAPP
      WRITE(IO2,2010) MAX
      WRITE(IO2,2070) EPS1
      WRITE(IO2,2020) EPS2
      WRITE(IO2,2080) EPS3
      WRITE(IO2,2040) XSTART
      WRITE(IO2,2050) XEND
      WRITE(IO2,2060)
      KKK=NP+1
      NNN=KKK+1
      DO 5 I=1,KKK
      JJJ=NNN-I
    5 READ(IO1,1010) UP(JJJ),VP(JJJ)
      IF(NAPP.NE.0) GO TO 22
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 23
   22 READ(IO1,1015) (UAPP(I,2),VAPP(I,2),I=1,NAPP)
   23 WRITE(IO2,1030) NP
      KKK=NP+1
      NNN=KKK+1
      DO 8 I=1,KKK
      JJJ=NNN-I
```

TABLE H.III (Continued)

```
    8 WRITE(IO2,1040) PNAME,ENTRY(JJJ),UP(JJJ),VP(JJJ)
      K=0
      KD=0
      J1=1
      KKK=NP+1
      DO 10 I=1,KKK
      UDO(I)=UP(I)
   10 VDO(I)=VP(I)
      NDO=NP
      CALL DERIV(NDO,UDO,VDO,NDDO,UDDO,VDDO)
      CALL GCD(NDO,UDO,VDO,NDDO,UDDO,VDDO,ND1,UD1,VD1)
   20 WRITE(IO2,3000) (ASTER,I=1,33)
      IF(ND1.LE.1) GO TO 30
      GO TO 40
   30 UD2(1)=1.0
      VD2(1)=0.0
      ND2=0
      GO TO 50
   40 CALL DERIV(ND1,UD1,VD1,NDD1,UDD1,VDD1)
      CALL GCD(ND1,UD1,VD1,NDD1,UDD1,VDD1,ND2,UD2,VD2)
   50 IF(NDO+ND2.LE.2*ND1) GO TO 60
      GO TO 70
   60 WRITE(IO2,1025) J1
      GO TO 170
   70 IF(ND1.EQ.0) GO TO 80
      GO TO 90
   80 KKK=NDO+1
      DO 85 I=1,KKK
      UG(I)=UDO(I)
   85 VG(I)=VDO(I)
      NG=NDO
      GO TO 110
   90 IF(ND2.EQ.0) GO TO 115
      CALL PROD(NDO,UDO,VDO,ND2,UD2,VD2,ND3,UD3,VD3)
  100 CALL PROD(ND1,UD1,VD1,ND1,UD1,VD1,ND4,UD4,VD4)
      CALL DIVIDE(ND3,UD3,VD3,ND4,UD4,VD4,NG,UG,VG)
  110 WRITE(IO2,1035) J1
      KKK=NG+1
      NNN=KKK+1
      DO 112 I=1,KKK
      JJJ=NNN-I
  112 WRITE(IO2,1040) GNAME,ENTRY(JJJ),UG(JJJ),VG(JJJ)
      KKK=NG+1
      DO 113 I=1,KKK
      UAP(I)=UG(KKK+1-I)
  113 VAP(I)=VG(KKK+1-I)
      CALL MULLER(NG,UAP,VAP,NAPP,UAPP,VAPP,J,UZROS,VZROS,JAP,XSTART,XEN
     1D,NOPOLY,URAPP,VRAPP)
      IF(J.EQ.0) GO TO 150
      WRITE(IO2,1180)
      IF(JAP.EQ.0) GO TO 120
      GO TO 130
  115 KKK=NDO+1
      DO 116 I=1,KKK
      UD3(I)=UDO(I)
  116 VD3(I)=VDO(I)
      ND3=NDO
      GO TO 100
```

TABLE H.III (Continued)

```
  120 KKK=JAP+1
      WRITE(I02,1085) (I,UZROS(I),VZROS(I),J1,I=KKK,J)
      GO TO 140
  130 DO 135 I=1,JAP
  135 WRITE(I02,1190) I,UZROS(I),VZROS(I),J1,URAPP(I,2),VRAPP(I,2)
      IF(JAP.LT.J) GO TO 120
  140 IF(J.EQ.NG) GO TO 155
  150 WRITE(I02,1095)
      IF(J.EQ.0) GO TO 170
  155 DO 160 I=1,J
      UROOT(KD+I)=UZROS(I)
      VROOT(KD+I)=VZROS(I)
  160 MULT(KD+I)=J1
      K=(J*J1)+K
      KD=KD+J
      IF(K.GE.NP) GO TO 1
  170 J1=J1+1
      IF(ND1.LE.1) GO TO 200
      DO 180 I=1,ND1
      UD0(I)=UD1(I)
      VD0(I)=VD1(I)
      UDD0(I)=UDD1(I)
  180 VDD0(I)=VDD1(I)
      UD0(ND1+1)=UD1(ND1+1)
      VD0(ND1+1)=VD1(ND1+1)
      ND0=ND1
      NDD0=NDD1
      KKK=ND2+1
      DO 190 I=1,KKK
      UD1(I)=UD2(I)
  190 VD1(I)=VD2(I)
      ND1=ND2
      GO TO 20
  200 IF(ND1.EQ.0) GO TO 1
      KD=KD+1
      DENOM=UD1(2)*UD1(2)+VD1(2)*VD1(2)
      UROOT(KD)=(-UD1(1)*UD1(2)-VD1(1)*VD1(2))/DENOM
      VROOT(KD)=(-VD1(1)*UD1(2)+UD1(1)*VD1(2))/DENOM
      MULT(KD)=J1
      WRITE(I02,3000) (ASTER,I=1,33)
      WRITE(I02,1035) J1
      KKK=ND1+1
      NNN=KKK+1
      DO 210 I=1,KKK
      JJJ=NNN-I
  210 WRITE(I02,1100) DINAME,ENTRY(JJJ),UD1(JJJ),VD1(JJJ)
      WRITE(I02,1180)
      WRITE(I02,1085) KD,UROOT(KD),VROOT(KD),J1
      GO TO 1
 1020 FORMAT(1H1,10X,48HREPEATED USE OF THE GREATEST COMMON DIVISOR AND
     1,A4,A4,58H METHOD TO EXTRACT ROOTS AND MULTIPLICITIES OF POLYNOMIA
     2LS/11X,18HPOLYNOMIAL NUMBER ,I2///)
 1025 FORMAT(///1X,25HNO ROOTS OF MULTIPLICITY ,I2//)
 1035 FORMAT(///1X,87HTHE FOLLOWING POLYNOMIAL, G(X), CONTAINS ALL THE R
     1OOTS OF P(X) WHICH HAVE MULTIPLICITY ,I2//)
 1085 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,8X,I2,9X,25HN
     1O INITIAL APPROXIMATIONS)
 1095 FORMAT(///1X,51HNOT ALL ROOTS OF THE ABOVE POLYNOMIAL,G, WERE FOUN
```

TABLE H.III (Continued)

```
     10//)
1000 FORMAT(3(I2,1X),9X,I3,1X,3(D6.0,1X),20X,2(D7.0,1X),I1)
1010 FORMAT(2D30.0)
1015 FORMAT(2D30.0)
1030 FORMAT(1X,22HTHE DEGREE OF P(X) IS ,I2,22H  THE COEFFICIENTS ARE//
     1)
1040 FORMAT(2X,A2,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
1100 FORMAT(2X,A3,A2,4H) = ,D23.16,3H + ,D23.16,2H I)
1180 FORMAT(///1X,13HROOTS OF P(X),52X,14HMULTIPLICITIES,17X,21HINITIAL
     1 APPROXIMATION//)
1190 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,8X,I2,8X,D23.
     116,3H + ,D23.16,2H I)
2000 FORMAT(1X,41HNUMBER OF INITIAL APPROXIMATIONS GIVEN.  ,I2)
2010 FORMAT(1X,29HMAXIMUM NUMBER OF ITERATIONS.,11X,I3)
2020 FORMAT(1X,21HTEST FOR CONVERGENCE.,13X,D9.2)
2040 FORMAT(1X,23HRADIUS TO START SEARCH.,11X,D9.2)
2050 FORMAT(1X,21HRADIUS TO END SEARCH.,13X,D9.2)
2060 FORMAT(//1X)
2070 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE GCD.  ,D9.2)
2080 FORMAT(1X,34HTEST FOR ZERO IN SUBROUTINE QUAD. ,D9.2)
3000 FORMAT(/////1X,A3,32A4)
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE PROD(M,UF,VF,N,UG,VG,MN,UH,VH)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN POLYNOMIALS R(X) AND S(X), THIS SUBROUTINE COMPUTES THE        *
C     * COEFFICIENTS OF THE PRODUCT POLYNOMIAL T(X) = R(X).S(X).             *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UH,VH,UF,VF,UG,VG
      DIMENSION UH(51),VH(51),UF(26),VF(26),UG(26),VG(26)
      MN=M+N
      KKK=MN+1
      DO 100 I=1,KKK
      K=I
      UH(I)=0.0
      VH(I)=0.0
      IF(I.LE.M+1) GO TO 10
      LIMIT=M+1
      GO TO 20
   10 LIMIT=I
   20 DO 50 J=1,LIMIT
      IF(K.GT.N+1) GO TO 50
      IF(J+K.EQ.I+1) GO TO 40
      GO TO 50
   40 UH(I)=UH(I)+(UF(J)*UG(K)-VF(J)*VG(K))
      VH(I)=VH(I)+(VF(J)*UG(K)+UF(J)*VG(K))
   50 K=K-1
  100 CONTINUE
      RETURN
      END
```

TABLE H.III. (Continued)

```
      SUBROUTINE QUAD(N,UA,VA,J,UROOT,VROOT)
C     ***********************************************************************
C     *                                                                     *
C     * SUBROUTINE QUAD SOLVES DIRECTLY FOR THE ZEROS AND THEIR MULTIPLICITIES *
C     * OF EITHER A QUADRATIC POLYNOMIAL OR A LINEAR FACTOR.  SOLUTION OF THE  *
C     * QUADRATIC IS DONE USING THE QUADRATIC FORMULA.                      *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION EPS1,EPS2,EPSLON,UROOT,VROOT,UA,VA,UDISC,VDISC,UD
     1,VD,DDD,UTEMP,VTEMP,BBB
      DOUBLE PRECISION EPS4
      DIMENSION UROOT(25),VROOT(25),UA(26),VA(26)
      COMMON EPS1,EPS2,EPSLON,EPS4,IO2,MAX
      IF(N.GT.1) GO TO 10
      J=J+1
      BBB=UA(2)*UA(2)+VA(2)*VA(2)
      UROOT(J)=-(UA(1)*UA(2)+VA(1)*VA(2))/BBB
      VROOT(J)=-(VA(1)*UA(2)-UA(1)*VA(2))/BBB
      GO TO 100
   10 UDISC=(UA(2)*UA(2)-VA(2)*VA(2))-(4.0*(UA(3)*UA(1)-VA(3)*VA(1)))
      VDISC=(2.0*UA(2)*VA(2))-(4.0*(UA(3)*VA(1)+VA(3)*UA(1)))
      UD=2.0*UA(3)
      VD=2.0*VA(3)
      DDD=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(DDD.LT.EPSLON) GO TO 20
      CALL COMSQT(UDISC,VDISC,UTEMP,VTEMP)
      BBB=UD*UD+VD*VD
      UROOT(J+1)=((-UA(2)+UTEMP)*UD+(-VA(2)+VTEMP)*VD)/BBB
      VROOT(J+1)=((-VA(2)+VTEMP)*UD-(-UA(2)+UTEMP)*VD)/BBB
      UROOT(J+2)=((-UA(2)-UTEMP)*UD+(-VA(2)-VTEMP)*VD)/BBB
      VROOT(J+2)=((-VA(2)-VTEMP)*UD-(-UA(2)-UTEMP)*VD)/BBB
      J=J+2
      GO TO 100
   20 J=J+1
      BBB=UD*UD+VD*VD
      UROOT(J)=(-UA(2)*UD-VA(2)*VD)/BBB
      VROOT(J)=(-VA(2)*UD+UA(2)*VD)/BBB
      WRITE(IO2,1000) UROOT(J),VROOT(J)
 1000 FORMAT(///1X,11HQUAD FOUND ,D23.16,3H + ,D23.16,2H I,22H TO BE A M
     1ULTIPLE ROOT//)
  100 RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE DERIV(N,UP,VP,M,UA,VA)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN A POLYNOMIAL P(X), SUBROUTINE DERIV COMPUTES THE COEFFICIENTS OF *
C     * ITS DERIVATIVE P'(X).                                                *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UP,VP,UA,VA,AAA
      DIMENSION UP(26),VP(26),UA(26),VA(26)
      KKK=N+1
      DO 10 I=2,KKK
      AAA=I-1
      UA(I-1)=AAA*UP(I)
   10 VA(I-1)=AAA*VP(I)
      M=N-1
      RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE GCD(N,UR,VR,M,US,VS,M1,USS,VSS)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN POLYNOMIALS P(X) AND DP(X) WHERE DEG. DP(X) IS LESS THAN DEG.  *
C     * P(X), SUBROUTINE GCD COMPUTES THE GREATEST COMMON DIVISOR OF P(X) AND *
C     * DP(X).                                                               *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION USSSSS,VSSSSS
      DOUBLE PRECISION UR,VR,US,VS,USS,VSS,URR,VRR,UD,VD,UT,VT,EPSLON,EP
     1S2,EPS3,EPS4,BBB
      DIMENSION UR(26),VR(26),US(26),VS(26),USS(26),VSS(26),URR(26),VRR(
     126),UT(26),VT(26)
      COMMON EPSLON,EPS2,EPS3,EPS4,IO2,MAX
      N1=N
      M1=M
      KKK=N+1
      DO 20 I=1,KKK
      URR(I)=UR(I)
   20 VRR(I)=VR(I)
      KKK=M+1
      DO 25 I=1,KKK
      USS(I)=US(I)
   25 VSS(I)=VS(I)
   30 BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      UD=(URR(N1+1)*USS(M1+1)+VRR(N1+1)*VSS(M1+1))/BBB
      VD=(USS(M1+1)*VRR(N1+1)-URR(N1+1)*VSS(M1+1))/BBB
      KKK=N1+1-M1
      DO 40 I=KKK,N1
      UT(I)=URR(I)-(UD*USS(I-N1+M1)-VD*VSS(I-N1+M1))
   40 VT(I)=VRR(I)-(UD*VSS(I-N1+M1)+VD*USS(I-N1+M1))
      IF(M1.EQ.N1) GO TO 70
      KKK=N1-M1
      DO 60 I=1,KKK
      UT(I)=URR(I)
   60 VT(I)=VRR(I)
   70 DO 90 I=1,N1
      BBB=DSQRT(UT(N1+1-I)*UT(N1+1-I)+VT(N1+1-I)*VT(N1+1-I))
      IF(BBB.GT.EPSLON) GO TO 100
   90 CONTINUE
      DO 95 I=1,M1
      BBB=USS(M1+1)*USS(M1+1)+VSS(M1+1)*VSS(M1+1)
      USSSSS=(USS(I)*USS(M1+1)+VSS(I)*VSS(M1+1))/BBB
      VSSSSS=(VSS(I)*USS(M1+1)-USS(I)*VSS(M1+1))/BBB
      USS(I)=USSSSS
   95 VSS(I)=VSSSSS
      USS(M1+1)=1.0
      VSS(M1+1)=0.0
      GO TO 200
  100 K=N1-I
      IF(K.EQ.0) GO TO 170
      IF(K.LT.M1) GO TO 140
      KKK=K+1
      DO 130 J=1,KKK
      URR(J)=UT(J)
  130 VRR(J)=VT(J)
      N1=K
      GO TO 30
```

TABLE H.III (Continued)

```
  140 KKK=K+1
      DO 150 J=1,KKK
      URR(J)=USS(J)
      VRR(J)=VSS(J)
      USS(J)=UT(J)
  150 VSS(J)=VT(J)
      KKK=K+2
      NNN=M1+1
      DO 160 J=KKK,NNN
      URR(J)=USS(J)
  160 VRR(J)=VSS(J)
      N1=M1
      M1=K
      GO TO 30
  170 USS(1)=1.0
      VSS(1)=0.0
      M1=0
  200 RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE DIVIDE(N,UP,VP,M,UD,VD,K,UQ,VQ)
C     **********************************************************************
C     *                                                                    *
C     * GIVEN TWO POLYNOMIALS F(X) AND G(X), SUBROUTINE DIVIDE COMPUTES THE *
C     * QUOTIENT POLYNOMIAL  H(X) = F(X)/G(X).                             *
C     *                                                                    *
C     **********************************************************************
      DOUBLE PRECISION UP,VP,UD,VD,UQ,VQ,UTERM,VTERM,UDUMMY
      DIMENSION UP(26),VP(26),UD(26),VD(26),UQ(26),VQ(26)
      K=N-M
      UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1)=(UP(N+1)*UD(M+1)+VP(N+1)*VD(M+1))/UDUMMY
      VQ(K+1)=(VP(N+1)*UD(M+1)-UP(N+1)*VD(M+1))/UDUMMY
      IF(K.EQ.0) GO TO 100
      J=-1
      DO 50 I=1,K
      J=J+1
      UTERM=UP(N-J)
      VTERM=VP(N-J)
      KK=K+1
      NNN=M-J
      DO 40 M1=NNN,M
      IF(KK.GT.1) GO TO 10
      GO TO 45
   10 IF(M1.GE.1) GO TO 20
      GO TO 40
   20 UTERM=UTERM-(UQ(KK)*UD(M1)-VQ(KK)*VD(M1))
      VTERM=VTERM-(UQ(KK)*VD(M1)+VQ(KK)*UD(M1))
   40 KK=KK-1
   45 UDUMMY=UD(M+1)*UD(M+1)+VD(M+1)*VD(M+1)
      UQ(K+1-I)=(UTERM*UD(M+1)+VTERM*VD(M+1))/UDUMMY
   50 VQ(K+1-I)=(VTERM*UD(M+1)-UTERM*VD(M+1))/UDUMMY
  100 RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE COMSQT(UX,VX,UY,VY)
C   ****************************************************************************
C   *                                                                          *
C   * THIS SUBROUTINE COMPUTES THE SQUARE ROOT OF A COMPLEX NUMBER.            *
C   *                                                                          *
C   ****************************************************************************
      DOUBLE PRECISION UX,VX,UY,VY,DUMMY,R,AAA,BBB
      R=DSQRT(UX*UX+VX*VX)
      AAA=DSQRT(DABS((R+UX)/2.0))
      BBB=DSQRT(DABS((R-UX)/2.0))
      IF(VX) 10,20,30
   10 UY=AAA
      VY=-1.0*BBB
      GO TO 100
   20 IF(UX) 40,50,60
   30 UY=AAA
      VY=BBB
      GO TO 100
   40 DUMMY=DABS(UX)
      UY=0.0
      VY=DSQRT(DUMMY)
      GO TO 100
   50 UY=0.0
      VY=0.0
      GO TO 100
   60 DUMMY=DABS(UX)
      UY=DSQRT(DUMMY)
      VY=0.0
  100 RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,V
     1PX3,UX4,VX4,UQ4,VQ4,UH3,VH3)
C     ************************************************************************
C     *                                                                      *
C     * GIVEN THREE APPROXIMATIONS X(N-2), X(N-1), AND X(N), SUBROUTINE CALC *
C     * APPROXIMATES THE POLYNOMIAL BY A QUADRATIC AND SOLVES FOR THE ZERO OF *
C     * THE QUADRATIC CLOSEST TO X(N). THIS ZERO IS THE NEW APPROXIMATION    *
C     * X(N+1) TO THE ZERO OF THE POLYNOMIAL.                                *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION ARG1,ARG2
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UX1,VX1,UX2,VX2,UX3,VX3,UPX1,
     1VPX1,UH3,VH3,UH2,VH2,UQ3,VQ3,UD,VD,UB,VB,UC,VC,UDISC,VDISC,UCCC,VC
     2CC,UDEN1,VDEN1,UDEN2,VDEN2,UQ4,VQ4,UX4,VX4,EPSRT,EPSO,EPS,UDDD,VDD
     3D,AAA,BBB,RAD,UAAA,VAAA,UBBB,VBBB
      DOUBLE PRECISION THETA,ANGLE,UTEST,VTEST
      DOUBLE PRECISION EPS1
      COMMON EPS1,EPS,EPSO,EPSRT,IO2,MAX
      UH3=UX3-UX2
      VH3=VX3-VX2
      UH2=UX2-UX1
      VH2=VX2-VX1
      BBB=UH2*UH2+VH2*VH2
      UQ3=(UH3*UH2+VH3*VH2)/BBB
      VQ3=(VH3*UH2-UH3*VH2)/BBB
      UDDD=1.0+UQ3
      VDDD=VQ3
      UD=(UPX3-(UDDD*UPX2-VDDD*VPX2))+(UQ3*UPX1-VQ3*VPX1)
      VD=(VPX3-(VDDD*UPX2+UDDD*VPX2))+(VQ3*UPX1+UQ3*VPX1)
      UAAA=2.0*UQ3
      VAAA=2.0*VQ3
      UAAA=UAAA+1.0
      UBBB=UDDD*UDDD-VDDD*VDDD
      VBBB=VDDD*UDDD+UDDD*VDDD
      UCCC=UQ3*UQ3-VQ3*VQ3
      VCCC=VQ3*UQ3+UQ3*VQ3
      UB=((UAAA*UPX3-VAAA*VPX3)-(UBBB*UPX2-VBBB*VPX2))+(UCCC*UPX1-VCCC*V
     1PX1)
      VB=((VAAA*UPX3+UAAA*VPX3)-(VBBB*UPX2+UBBB*VPX2))+(VCCC*UPX1+UCCC*V
     1PX1)
      UC=UDDD*UPX3-VDDD*VPX3
      VC=VDDD*UPX3+UDDD*VPX3
      UDISC=(UB*UB-VB*VB)-(4.0*(UD*UC-VD*VC))
      VDISC=(2.0*(VB*UB))-(4.0*(VD*UC+UD*VC))
      AAA=DSQRT(UDISC*UDISC+VDISC*VDISC)
      IF(AAA.EQ.0.0) GO TO 5
      GO TO 7
    5 THETA=0.0
      GO TO 9
    7 THETA=DATAN2(VDISC,UDISC)
    9 RAD=DSQRT(AAA)
      ANGLE=THETA/2.0
      UTEST=RAD*DCOS(ANGLE)
      VTEST=RAD*DSIN(ANGLE)
      UDEN1=UB+UTEST
      VDEN1=VB+VTEST
      UDEN2=UB-UTEST
      VDEN2=VB-VTEST
```

TABLE H.III (Continued)

```
      ARG1=UDEN1*UDEN1+VDEN1*VDEN1
      ARG2=UDEN2*UDEN2+VDEN2*VDEN2
      AAA=DSQRT(ARG1)
      BBB=DSQRT(ARG2)
      IF(AAA.LT.BBB) GO TO 10
      IF(AAA.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN1+VAAA*VDEN1)/ARG1
      VQ4=(VAAA*UDEN1-UAAA*VDEN1)/ARG1
      GO TO 50
   10 IF(BBB.EQ.0.0) GO TO 60
      UAAA=-2.0*UC
      VAAA=-2.0*VC
      UQ4=(UAAA*UDEN2+VAAA*VDEN2)/ARG2
      VQ4=(VAAA*UDEN2-UAAA*VDEN2)/ARG2
      GO TO 50
   50 UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      RETURN
   60 UQ4=1.0
      VQ4=0.0
      GO TO 50
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE MULLER(NP,UA,VA,NAPP,UAPP,VAPP,NROOT,UROOT,VROOT,IROOT,
     1XSTART,XEND,NOPOLY,URAPP,VRAPP)
C     ***********************************************************************
C     *                                                                     *
C     * MULLER'S METHOD EXTRACTS THE ZEROS AND THEIR MULTIPLICITIES OF A    *
C     * POLYNOMIAL OF MAXIMUM DEGREE 25.  THROUGH THREE GIVEN POINTS THE    *
C     * POLYNOMIAL IS APPROXIMATED BY A QUADRATIC.  THE ZERO OF THE QUADRATIC *
C     * CLOSEST TO THE OLD APPROXIMATION IS TAKEN AS THE NEW APPROXIMATION. *
C     * IN THIS MANNER A SEQUENCE IS OBTAINED CONVERGING TO A ZERO.        *
C     *                                                                     *
C     ***********************************************************************
      DOUBLE PRECISION UPX3,VPX3,UPX2,VPX2,UROOT,VROOT,UX1,VX1,UAPP,VAPP
     1,UX2,VX2,UWORK,VWORK,UX3,VX3,UB,VB,UX4,VX4,UA,VA,UPX1,VPX1,URAPP,V
     2RAPP,UPX4,VPX4,EPSRT,EPSD,EPS,CCC,EPSM,UH3,VH3,UQ4,VQ4,ABPX4,ABPX3
     3,QQQ,XSTART,XEND
      DIMENSION UROOT(25),VROOT(25),MULT(25),UAPP(25,3),VAPP(25,3),UWORK
     1(26),VWORK(26),UB(26),VB(26),UA(26),VA(26),URAPP(25,3),VRAPP(25,3)
      LOGICAL CONV
      COMMON EPSM,EPS,EPSD,EPSRT,IO2,MAX
      DATA PNAME,DNAME/2HP(,2HD(/
      EPSM=0.0D00
      EPSRT=0.999
      NROOT=0
      IROOT=0
      IPATH=1
      NOMULT=0
      NALTER=0
      ITIME=0
      IAPP=1
      ITER=1
      IF(NAPP.NE.0) GO TO 18
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      GO TO 27
   18 DO 25 I=1,NAPP
      UAPP(I,1)=0.9*UAPP(I,2)
      VAPP(I,1)=0.9*VAPP(I,2)
      UAPP(I,3)=1.1*UAPP(I,2)
   25 VAPP(I,3)=1.1*VAPP(I,2)
   27 KKK=NP+1
      DO 30 I=1,KKK
      UWORK(I)=UA(I)
   30 VWORK(I)=VA(I)
      NWORK=NP
   40 UX1=UAPP(IAPP,1)
      VX1=VAPP(IAPP,1)
      UX2=UAPP(IAPP,2)
      VX2=VAPP(IAPP,2)
      UX3=UAPP(IAPP,3)
      VX3=VAPP(IAPP,3)
      CALL HORNER(NWORK,UWORK,VWORK,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NWORK,UWORK,VWORK,UX2,VX2,UB,VB,UPX2,VPX2)
      CALL HORNER(NWORK,UWORK,VWORK,UX3,VX3,UB,VB,UPX3,VPX3)
   50 CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   60 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      ABPX4=DSQRT(UPX4*UPX4+VPX4*VPX4)
      ABPX3=DSQRT(UPX3*UPX3+VPX3*VPX3)
```

TABLE H.III (Continued)

```
      IF(ABPX3.EQ.0.0) GO TO 70
      QQQ=ABPX4/ABPX3
      IF(QQQ.LE.10.) GO TO 70
      UQ4=0.5*UQ4
      VQ4=0.5*VQ4
      UX4=UX3+(UH3*UQ4-VH3*VQ4)
      VX4=VX3+(VH3*UQ4+UH3*VQ4)
      GO TO 60
   70 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 120
      IF(ITER.LT.MAX) GO TO 110
      CALL ALTER(UAPP(IAPP,1),VAPP(IAPP,1),UAPP(IAPP,2),VAPP(IAPP,2),UAP
     1P(IAPP,3),VAPP(IAPP,3),NALTER,ITIME)
      IF(NALTER.GT.5) GO TO 75
      ITER=1
      GO TO 40
   75 IF(IAPP.LT.NAPP) GO TO 100
      IF(XEND.EQ.0.0) GO TO 77
      IF(XSTART.GT.XEND) GO TO 77
      NAPP=NP
      CALL GENAPP(UAPP,VAPP,NAPP,XSTART)
      IAPP=0
      GO TO 100
   77 WRITE(IO2,1090)
      KKK=NWORK+1
      WRITE(IO2,1035) (DNAME,J,UWORK(J),VWORK(J),J=1,KKK)
   80 IF(NROOT.EQ.0) GO TO 90
      IF(IPATH.EQ.1) GO TO 82
   81 IPATH=2
      CALL BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MULT)
      RETURN
   82 IF(NROOT.EQ.0)GO TO 90
      IF(IROOT.EQ.0) GO TO 85
      WRITE(IO2,1080)
      DO 55 I=1,IROOT
   55 WRITE(IO2,1085) I,UROOT(I),VROOT(I),URAPP(I,2),VRAPP(I,2)
      IF(IROOT.LT.NROOT) GO TO 85
      GO TO 87
   85 KKK=IROOT+1
      WRITE(IO2,1086) (I,UROOT(I),VROOT(I),I=KKK,NROOT)
   87 IF(IPATH.EQ.1) GO TO 81
      RETURN
   90 WRITE(IO2,1070) NOPOLY
      RETURN
  100 IAPP=IAPP+1
      ITER=1
      NALTER=0
      GO TO 40
  120 NROOT=NROOT+1
      IROOT=NROOT
      MULT(NROOT)=1
      NOMULT=NOMULT+1
      UROOT(NROOT)=UX4
      VROOT(NROOT)=VX4
      URAPP(NROOT,1)=UAPP(IAPP,1)
      VRAPP(NROOT,1)=VAPP(IAPP,1)
      URAPP(NROOT,2)=UAPP(IAPP,2)
      VRAPP(NROOT,2)=VAPP(IAPP,2)
```

TABLE H.III (Continued)

```
      URAPP(NROOT,3)=UAPP(IAPP,3)
      VRAPP(NROOT,3)=VAPP(IAPP,3)
  125 IF(NOMULT.LT.NP) GO TO 130
      GO TO 80
  130 CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      NWORK=NWORK-1
      KKK=NWORK+1
      DO 140 I=1,KKK
      UWORK(I)=UB(I)
  140 VWORK(I)=VB(I)
      CALL HORNER(NWORK,UWORK,VWORK,UX4,VX4,UB,VB,UPX4,VPX4)
      CCC=DSQRT(UPX4*UPX4+VPX4*VPX4)
      IF(CCC.LT.EPSM) GO TO 150
      IF(NWORK.GT.2) GO TO 75
      IROOT=NROOT
      KKK=NWORK+1
      DO 145 I=1,KKK
      UB(I)=UWORK(KKK+1-I)
  145 VB(I)=VWORK(KKK+1-I)
      CALL QUAD(NWORK,UB,VB,NROOT,UROOT,VROOT)
      GO TO 80
  150 MULT(NROOT)=MULT(NROOT)+1
      NOMULT=NOMULT+1
      GO TO 125
  110 UX1=UX2
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      UPX3=UPX4
      VPX3=VPX4
      ITER=ITER+1
      GO TO 50
 1090 FORMAT(///,1X,65HCOEFFICIENTS OF DEFLATED POLYNOMIAL FOR WHICH NO
     1ZEROS WERE FOUND//)
 1080 FORMAT(///1X,13HROOTS OF G(X),83X,21HINITIAL APPROXIMATION//)
 1070 FORMAT(//,43H NO ZEROS WERE FOUND FOR POLYNOMIAL NUMBER ,I2)
 1086 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,19X,23HSOLVED
     1 BY DIRECT METHOD)
 1035 FORMAT(3X,A2,I2,4H) = ,D23.16,3H + ,D23.16,2H I)
 1085 FORMAT(2X,5HROOT(,I2,4H) = ,D23.16,3H + ,D23.16,2H I,18X,D23.16,3H
     1 + ,D23.16,2H I)
 1000 FORMAT(3(I2,1X),9X,I3,8X,3(D6.0,1X),13X,2(D7.0,1X),I1)
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE GENAPP(APPR,APPI,NAPP,XSTART)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE GENAPP GENERATES N INITIAL APPROXIMATIONS, WHERE N IS THE *
C     * DEGREE OF THE ORIGINAL POLYNOMIAL.                                   *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION APPR,APPI,XSTART,EPS1,EPS2,EPS3,BETA
      DOUBLE PRECISION EPSM
      DIMENSION APPR(25,3),APPI(25,3)
      COMMON EPSM,EPS1,EPS2,EPS3,IO2,MAX
      IF(XSTART.EQ.0.0) XSTART=0.5
      BETA=0.2617994
      DO 10 I=1,NAPP
      APPR(I,2)=XSTART*DCOS(BETA)
      APPI(I,2)=XSTART*DSIN(BETA)
      BETA=BETA+0.5235988
   10 XSTART=XSTART+0.5
      DO 20 I=1,NAPP
      APPR(I,1)=0.9*APPR(I,2)
      APPI(I,1)=0.9*APPI(I,2)
      APPR(I,3)=1.1*APPR(I,2)
   20 APPI(I,3)=1.1*APPI(I,2)
      RETURN
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE ALTER(X1R,X1I,X2R,X2I,X3R,X3I,NALTER,ITIME)
C     **********************************************************************
C     *                                                                    *
C     * SUBROUTINE ALTER ALTERS THE INITIAL APPROXIMATIONS WHICH PRODUCE NO *
C     * CONVERGENCE TO A ZERO. THIS IS DONE A MAXIMUM OF 5 TIMES FOR EACH ROOT. *
C     *                                                                    *
C     **********************************************************************
      DOUBLE PRECISION X1R,X1I,X2R,X2I,X3R,X3I,EPS1,EPS2,EPS3,R,BETA
      DOUBLE PRECISION EPSM
      COMMON EPSM,EPS1,EPS2,EPS3,IO2,MAX
      IF(ITIME.NE.0) GO TO 5
      ITIME=1
      WRITE(IO2,1010) MAX
    5 IF(NALTER.EQ.0) GO TO 10
      WRITE(IO2,1000) X1R,X1I,X2R,X2I,X3R,X3I
      GO TO 20
   10 R=DSQRT(X2R*X2R+X2I*X2I)
      BETA=DATAN2(X2I,X2R)
      WRITE(IO2,1020) X1R,X1I,X2R,X2I,X3R,X3I
   20 NALTER=NALTER+1
      IF(NALTER.GT.5) RETURN
      GO TO (30,40,30,40,30),NALTER
   30 X2R=-X2R
      X2I=-X2I
      GO TO 50
   40 BETA=BETA+1.0471976
      X2R=R*DCOS(BETA)
      X2I=R*DSIN(BETA)
   50 X1R=0.9*X2R
      X1I=0.9*X2I
      X3R=1.1*X2R
      X3I=1.1*X2I
      RETURN
 1000 FORMAT(1X,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HALTERED APPROXIM
     1ATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2,D23.16,2H I/)
 1020 FORMAT(1H0,5HX1 = ,D23.16,3H + ,D23.16,2H I,10X,22HINITIAL APPROXI
     1MATIONS/1X,5HX2 = ,D23.16,3H + ,D23.16,2H I/1X,5HX3 = ,D23.16,3H +
     2 ,D23.16,2H I/)
 1010 FORMAT(///1X,54HNO CONVERGENCE FOR THE FOLLOWING APPROXIMATIONS AF
     1TER ,I3,12H ITERATIONS.//)
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE BETTER(UA,VA,NP,UROOT,VROOT,NROOT,URAPP,VRAPP,IROOT,MUL
     1T)
C    ***********************************************************************
C    *                                                                     *
C    * SUBROUTINE BETTER ATTEMPTS TO IMPROVE THE ACCURACY OF THE ZEROS FOUND *
C    * BY USING THEM AS INITIAL APPROXIMATIONS WITH MULLER'S METHOD APPLIED TO *
C    * THE FULL, UNDEFLATED POLYNOMIAL.                                    *
C    *                                                                     *
C    ***********************************************************************
      DOUBLE PRECISION UROOT,VROOT,UA,VA,UBAPP,VBAPP,UX1,VX1,UX2,VX2,UX3
     1,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UB,VB,UROOTS,VROOTS,EPSRT,UX4,V
     2X4,URAPP,VRAPP,EPSO,EPS,UQ4,VQ4,UH3,VH3
      DOUBLE PRECISION EPSM
      LOGICAL CONV
      DIMENSION UROOT(25),VROOT(25),UA(26),VA(26),UBAPP(25,3),VBAPP(25,3
     1),UB(26),VB(26),UROOTS(25),VROOTS(25),URAPP(25,3),VRAPP(25,3),MULT
     3(25)
      COMMON EPSM,EPS,EPSO,EPSRT,IO2,MAX
      IF(NROOT.LE.1) RETURN
      L=0
      DO 10 I=1,NROOT
      UBAPP(I,1)=UROOT(I)*EPSRT
      VBAPP(I,1)=VROOT(I)*EPSRT
      UBAPP(I,2)=UROOT(I)
      VBAPP(I,2)=VROOT(I)
      UBAPP(I,3)=UROOT(I)*(2.0-EPSRT)
   10 VBAPP(I,3)=VROOT(I)*(2.0-EPSRT)
      DO 100 J=1,NROOT
      UX1=UBAPP(J,1)
      VX1=VBAPP(J,1)
      UX2=UBAPP(J,2)
      VX2=VBAPP(J,2)
      UX3=UBAPP(J,3)
      VX3=VBAPP(J,3)
      ITER=1
      CALL HORNER(NP,UA,VA,UX1,VX1,UB,VB,UPX1,VPX1)
      CALL HORNER(NP,UA,VA,UX2,VX2,UB,VB,UPX2,VPX2)
   20 CALL HORNER(NP,UA,VA,UX3,VX3,UB,VB,UPX3,VPX3)
      CALL CALC(UX1,VX1,UX2,VX2,UX3,VX3,UPX1,VPX1,UPX2,VPX2,UPX3,VPX3,UX
     14,VX4,UQ4,VQ4,UH3,VH3)
   30 CALL TEST(UX3,VX3,UX4,VX4,CONV)
      IF(CONV) GO TO 50
      IF(ITER.LT.MAX) GO TO 40
      WRITE(IO2,1000) J,UROOT(J),VROOT(J),MAX
      WRITE(IO2,1010) UX4,VX4
      IF(J.LT.IROOT) GO TO 33
      IF(J.EQ.IROOT) GO TO 35
      GO TO 100
   33 KKK=IROOT-1
      DO 34 K=J,KKK
      URAPP(K,1)=URAPP(K+1,1)
      VRAPP(K,1)=VRAPP(K+1,1)
      URAPP(K,2)=URAPP(K+1,2)
      VRAPP(K,2)=VRAPP(K+1,2)
      URAPP(K,3)=URAPP(K+1,3)
   34 VRAPP(K,3)=VRAPP(K+1,3)
   35 IROOT=IROOT-1
      GO TO 100
```

TABLE H.III. (Continued)

```
   40 UX1=UX2
      VX1=VX2
      UX2=UX3
      VX2=VX3
      UX3=UX4
      VX3=VX4
      UPX1=UPX2
      VPX1=VPX2
      UPX2=UPX3
      VPX2=VPX3
      ITER=ITER+1
      GO TO 20
   50 L=L+1
      UROOTS(L)=UX4
      VROOTS(L)=VX4
  100 CONTINUE
      IF(L.EQ.0) GO TO 120
      DO 110 I=1,L
      UROOT(I)=UROOTS(I)
  110 VROOT(I)=VROOTS(I)
      NROOT=L
      RETURN
  120 NROOT=0
      RETURN
 1000 FORMAT(///42H IN THE ATTEMPT TO IMPROVE ACCURACY, ROOT(,I2,4H) = ,
     1D23.16,3H + ,D23.16,2H I/24H DID NOT CONVERGE AFTER ,I3,11H ITERAT
     2IONS)
 1010 FORMAT(30H THE PRESENT APPROXIMATION IS ,D23.16,3H + ,D23.16,2H I/
     1/)
      END
```

TABLE H.III (Continued)

```
      SUBROUTINE TEST(UX3,VX3,UX4,VX4,CONV)
C     ************************************************************************
C     *                                                                      *
C     * SUBROUTINE TEST CHECKS FOR CONVERGENCE OF THE SEQUENCE OF APPROX-    *
C     * IMATIONS BY TESTING THE EXPRESSION                                   *
C     * ABSOLUTE VALUE OF (X(N+1)-X(N))/ABSOLUTE VALUE OF X(N+1).            *
C     * WHEN IT IS AS SMALL AS DESIRED, CONVERGENCE IS OBTAINED.             *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX3,VX3,UX4,VX4,EPSRT,EPSO,EPS,AAA,UDUMMY,VDUMMY,
     1DENOM
      DOUBLE PRECISION EPSM
      LOGICAL CONV
      COMMON EPSM,EPS,EPSO,EPSRT,IO2,MAX
      UDUMMY=UX4-UX3
      VDUMMY=VX4-VX3
      AAA=DSQRT(UDUMMY*UDUMMY+VDUMMY*VDUMMY)
      DENOM=DSQRT(UX4*UX4+VX4*VX4)
      IF(DENOM.LT.EPSO) GO TO 20
      IF(AAA/DENOM.LT.EPS) GO TO 10
    5 CONV=.FALSE.
      GO TO 100
   10 CONV=.TRUE.
      GO TO 100
   20 IF(AAA.LT.EPSO) GO TO 10
      GO TO 5
  100 RETURN
      END
```

```
      SUBROUTINE HORNER(NA,UA,VA,UX,VX,UB,VB,UPX,VPX)
C     ************************************************************************
C     *                                                                      *
C     * HORNER'S METHOD COMPUTES THE VALUE OF THE POLYNOMIAL P(X) AT A POINT D. *
C     * SYNTHETIC DIVISION IS USED TO DEFLATE THE POLYNOMIAL BY DIVIDING OUT THE *
C     * FACTOR (X-D).                                                        *
C     *                                                                      *
C     ************************************************************************
      DOUBLE PRECISION UX,VX,UPX,VPX,UB,VB,UA,VA
      DIMENSION UA(26),VA(26),UB(26),VB(26)
      UB(1)=UA(1)
      VB(1)=VA(1)
      NUM=NA+1
      DO 10 I=2,NUM
      UB(I)=UA(I)+(UB(I-1)*UX-VB(I-1)*VX)
   10 VB(I)=VA(I)+(VB(I-1)*UX+UB(I-1)*VX)
      UPX=UB(NUM)
      VPX=VB(NUM)
      RETURN
      END
```