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ATTITUDE CONTROL CONCEPTS FOR PRECISION-POINTING NONRIGID SPACECRAFT

PETER W. LIKINS

UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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ATTITUDE CONTROL CONCEPTS FOR PRECISION-POINTING NONRIGID SPACECRAFT

Final Report

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Principal Investigator: Peter W. Likins
Postgraduate Research Engineer: Yoshiaki Ohkami

Mechanics and Structures Department
School of Engineering and Applied Science
University of California
Los Angeles, California

ABSTRACT

This report covers the scope of Mod. 4 of Contract NAS8-28358, covering a period of less than one year which is interior to the total time span of the contract. Thus what follows is a "final report" only in the sense of providing final documentation of those tasks which have been completed during this period. Specifically, this report includes the following subjects: "The Influence of Spacecraft Flexibility on System Controllability and Observability;" "Commutativity of Coordinate Truncation and Transformation Matrix Inversion for Flexible Spacecraft Dynamic Analysis;" and "Matched Asymptotic Expansion Modal Analysis of Rotating Beams."

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OVERVIEW

Contract NAS8-28358 began on 15 February, 1972, and is currently scheduled through 14 December, 1974. The present report covers only "Mod. 4" of the contract, which was awarded in late Spring of 1973 and covers the period from 15 February 1973 to 15 February 1974. In this report there appears a final documentation of those tasks presently deemed completed; other work still in progress will be documented at the conclusion of "Mod. 5" of this contract, in December of 1974.

The three topics treated here are as follows:

Chapter 1. "The Influence of Spacecraft Flexibility on System Controllability and Observability;"

Chapter 2. "Commutativity of Coordinate Truncation and Transformation Matrix Inversion for Flexible Spacecraft Dynamic Analysis;"

Chapter 3. "Matched Asymptotic Expansion Modal Analysis of Rotating Beams."

These chapters have been written by the Principal Investigator in such a way that each can be submitted (perhaps in abridged form) for conference presentation and/or journal publication.

Other topics under study during the Mod. 4 period of the contract include:

(1) Development of general-purpose simulation equations for arbitrary spacecraft;

(2) Evaluation of the influence of sensor and actuator location on flexible spacecraft performance; and

(3) Preliminary evaluation of the concept of the disturbance-insensitive control system.

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During the contract period, support funds were extended to Dr. Yoshiaki Ohkami and (to a minimal degree) Mr. Oluyemisi Olusola, a student who is no longer with the project; substantial contributions to total progress have also been made by Mr. Robert Skelton and Mr. Stewart Hopkins (both current students) and by Dr. André Colin (a former student).

CHAPTER I

The Influence of Spacecraft Flexibility on System Controllability and Observability*

Y. Ohkami††

and

P. Likins†

ABSTRACT. Literal criteria are developed for the controllability and observability of general models of flexible spacecraft. Results are interpreted in special cases and in physical terms, permitting in some cases the identification of uncontrollable and unobservable states simply by examination of scalars composed of modal parameters and location matrices for sensors and actuators. A procedure is established for isolation of uncontrollable states, whereby sensor and actuator configurations assure that uncontrollable flexible mode states are also unobservable; in many applications such states can then be removed by coordinate truncation.

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*Based on research supported by NASA Marshall Space Flight Center under Contract NAS8-28358.

†Professor, School of Engineering and Applied Science, University of California, Los Angeles, California, USA.

††NASA International Fellow on leave at the University of California, Los Angeles, from the National Aerospace Laboratory, Tokyo, Japan.

INTRODUCTION

Scientific spacecraft now in the planning stages (as typified by the NASA Large Space Telescope) are vehicles of dynamically significant flexibility which must be capable of maintaining inertial orientations to within thousandths of an arcsecond over periods of hours. Sensors and actuators are typically distributed over the spacecraft, and influenced in their performance by vehicle flexibility. The attitude control problems posed by these spacecraft are of unprecedented difficulty, and their solution will require unparalleled preflight investigation and perhaps conceptual innovation, such as system optimization or adaptive control. Although problems of spacecraft attitude control have in the past been resolved primarily by a combination of digital computer simulation and the traditional design procedures of linear control theory, in the future the techniques of modern control theory must increasingly be employed. The purpose of this paper is to examine the fundamental concepts of controllability and observability as they apply to flexible spacecraft, in order to establish a foundation on which to build future applications of modern control theory. As our results emerge, it will become apparent that they are useful not only in projected control system synthesis but also in the modal coordinate truncation process which is essential to system simulation.

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EQUATIONS OF STATE

Although a variety of mathematical models have been adopted for flexible spacecraft, the most common idealization consists of a rigid primary body with attached elastic appendages, which are themselves modeled sometimes as continua [1] but most often as discretized assemblages of particles or rigid nodal bodies interconnected by massless elastic elements called finite elements. [2] In some cases, the appendage mass is distributed throughout the finite elements. [3] Whether the original appendage model is continuous or discretized, its displacements relative to the primary body are finally characterized by distributed or modal coordinates, whereas the inertial orientation of the primary body is characterized by discrete coordinates, such as a set of three attitude angles. In many applications additional discrete coordinates are employed to describe the motion of system components, such as rotors and scientific instruments. A representation of vehicle kinematics in terms of a combination of discrete and distributed coordinates is called a hybrid coordinate formulation.

In what follows, it is assumed at the outset that the vehicle model consists of a rigid primary body with discretized elastic appendages having n rigid nodal bodies. After coordinate transformation, the resulting dynamic equations have a structure that permits their interpretation as representative also of appendage models consisting of elastic continua or distributed-mass finite element systems. An example in Appendix A illustrates the applicability of our results to continuous appendages.

It is further assumed in this paper that the vehicle experiences only small deviations from a nominal state in which the central body is of inertially fixed orientation and the appendages have no deformations. Moreover, the vehicle has no inherently nonlinear elements, so that all analysis can be based on linearized variational equations. Implications for nonlinear systems can sometimes be drawn from the physical interpretations of the conclusions presented here.

From Eq. (278) of [2], the vehicle rotational equations take the form

$$I^*\ddot{\theta} + \Delta^T\ddot{q} = T \tag{1}$$

where T is the 3x1 matrix representing the external torque vector about the vehicle mass center c, for an orthogonal vector basis in the primary body b, I^* is the 3x3 inertia matrix of the total vehicle for c, θ is the 3x1 matrix of 1-2-3 inertial attitude angles of b, q is a 6nx1 matrix consisting of a sequential ordering of 6x1 matrices typified for nodal body j by $[u_1^j\ u_2^j\ u_3^j\ \beta_1^j\ \beta_2^j\ \beta_3^j]^T$, where for $\alpha = 1,2,3$, u_α^j is the translation of the nodal body mass center relative to b in the direction defined by axis α fixed in b, and β_α^j is the small rotation of nodal body j about axis α. The 6nx3 matrix Δ is established by the geometry and mass distribution characteristics of the appendages, as in Eq. (278) of [2].

The appendage deformation equations in Eq. (277) of [2] may be written

$$M'\ddot{q} + Kq + \Delta\ddot{\theta} = \lambda \tag{2}$$

under the assumption that the resultant external force applied to the vehicle is zero. Here K is the appendage stiffness matrix and M' is the inertia matrix appropriate for the appendage attached to a translationally free but rotationally constrained rigid body. Both M' and K are symmetric and positive definite, and all quantities in Eqs. (1) and (2) are real. The 6nx1 matrix λ consists of n 6x1 partitions for the n nodal bodies, as typified for the j^{th} body by $[f_1^j\ f_2^j\ f_3^j\ \ell_1^j\ \ell_2^j\ \ell_3^j]$, where for $\alpha = 1,2,3$, f_α^j and ℓ_α^j are the scalar components in direction α of the external force and torque applied to nodal body j. In the present application, λ contains only forces and/or torques associated with attitude control actuators. Although in practice these actuators might themselves be defined by their own dynamical equations, it serves our present objectives to treat them as pure <u>thrusters</u> or <u>torquers</u>.

The relationship between λ and T can be expressed in general terms involving the torque T_o applied directly to b and the forces and torques in λ as

$$T = \kappa\lambda + T_o \tag{3}$$

for some 3x6n matrix κ which sums the torques on the nodal bodies and cross-multiplies the forces by the vector $\underline{R}^j$ locating nodal body j relative to c; thus

$$\kappa = [\tilde{R}^1 \mid U_3 \mid \tilde{R}^2 \mid U_3 \mid --- \mid \tilde{R}^n \mid U_3] \tag{4}$$

where U_3 is the 3x3 unit matrix and in terms of the scalar components R_1^j, R_2^j, R_3^j of $\underline{R}^j$ in the vector basis fixed in b,

$$[\tilde{R}^j] \triangleq \begin{bmatrix} 0 & -R_3^j & R_2^j \\ R_3^j & 0 & -R_1^j \\ -R_2^j & R_1^j & 0 \end{bmatrix} \tag{5}$$

In Eq. (3) there are potentially 6n+3 independent controls embodied in T_o and λ.

In the simplest case the 3 elements of T are each identified as independent controllers (each perhaps associated with a single actuator, such as an idealized control moment gyro); then it is most convenient to replace λ in Eq. (2) by

$$\lambda = \mathscr{L}_c T \tag{6}$$

in which the 6nx3 matrix $\mathscr{L}_c$ establishes the location and type of the attitude control actuators. For example, if the actuator is conceived as a pure torquer located on the k^{th} nodal body, then

$$\mathscr{L}_c^T = \left[0 \mid 0 \mid --- \mid 0 \mid U_3 \mid --- \mid 0 \mid 0 \right] \tag{7}$$

where U_3 is a 3x3 unit matrix appearing in the $2k^{th}$ position of $\mathscr{L}_c$. If instead the actuation system consists of a system of synchronized attitude control jets responding to only three independent commands, then Eq. (6) still provides an appropriate actuation model, although $\mathscr{L}_c$ becomes more complex than the simple case illustrated by Eq. (7). (See the first example, case (c), in Appendix A.)

Since the nature of the three-axis control problem clearly demands at least three independent controllers, Eq. (6) represents a minimal actuator model.

In the next section we address the question: Is the system controllable with the minimal actuator model? An affirmative answer implies controllability with the more general actuator model of Eq. (3), but the system could theoretically be uncontrollable with the minimal model of Eq. (6) and yet controllable with the model of Eq. (3).

The dynamical equations (1) and (2) must be augmented by an observation equation. Again we are concerned only with the minimal sensor model, which provides only three attitude readings. Thus we adopt the observation equation

$$y = \theta + \mathscr{L}_o^T q \tag{8}$$

where the 6nx3 matrix $\mathscr{L}_o$ locates the sensors. For example, if a sensor on the k^{th} nodal body provides the three inertial attitude angles of that body, then $\mathscr{L}_o^T$ is identical to the $\mathscr{L}_c^T$ matrix illustrated in Eq (7), and the sensor output is

$$y = \theta + \beta^k \tag{9}$$

Physical arguments suggest that observability is unchanged by the acquisition of attitude rate information at the same locations for which attitude information is established by Eq. (8). Generalization of this minimal sensor model to permit more than three independent attitude measurements may, however, extend observability to additional states.

In the hybrid coordinate formulation, Eqs. (2) provide the basis for the transformation

$$q = \phi\eta \tag{10}$$

in which ϕ is a 6nx6n matrix whose columns are the eigenvectors of that portion of Eq. (2) involving only q. The noted properties of M' and K assure that these eigenvectors are independent, and that ϕ is nonsingular and has orthogonality properties such that $\phi^T M' \phi$ and $\phi^T K \phi$ are diagonal. (See [2], pp. 43 and 53.)

Eqs. (1), (2), (7), (8), and (10) become, after pre-multiplication of Eq. (2) by ϕ^T and of Eq. (1) by I^{*-1},

$$\ddot{\theta} - J\ddot{\eta} = u \tag{11}$$

$$\ddot{\eta} + \sigma^2\eta - \delta\ddot{\theta} = (\phi^T M'\phi)^{-1}\phi^T \mathscr{L}_c I^* u \tag{12}$$

$$y = \theta + \mathscr{L}_o^T \phi\eta \tag{13}$$

where

$$u \triangleq I^{*-1} T$$

$$\delta \triangleq -(\phi^T M'\phi)^{-1}\phi^T \Delta$$

$$J \triangleq -I^{*-1}\Delta^T\phi = I^{*-1}\delta^T(\phi^T M'\phi)$$

$$\sigma^2 = (\phi^T M'\phi)^{-1}\phi^T K\phi$$

Eqs. (11) - (13), although obtained here for an appendage model [2] in which all mass is concentrated in nodal bodies, apply also to vehicles with continuous appendages [1] or with distributed-mass finite element appendage models [3]. In any case the dimension of η represents the number of appendage modes, which henceforth is called N (rather than the symbol 6n which is appropriate only for the fully discretized model without coordinate truncation.)

In what follows the eigenvectors in ϕ are normalized so that $\phi^T M'\phi = U_N$; then the diagonal elements σ_i^2 of the NxN matrix σ^2 represent squares of non-zero natural frequencies of vibrations which could occur independently of each other if the primary body were translationally free but constrained against rotation, as noted in [2], page 55.

In terms of the (2N + 6)x1 state variable

$$x \triangleq \begin{bmatrix} \theta \\ \hline \dot{\theta} \\ \hline \eta \\ \hline \dot{\eta} \end{bmatrix}$$

Eqs. (11) - (13) may be written as

$$\dot{x} = Ax + Bu \tag{14}$$

$$y = Cx \tag{15}$$

where

$$A = \left[\begin{array}{c|c|c|c} U_3 & 0 & 0 & 0 \\ \hline 0 & U_3 & 0 & -J \\ \hline 0 & 0 & U_N & 0 \\ \hline 0 & -\delta & 0 & U_N \end{array}\right]^{-1} \left[\begin{array}{c|c|c|c} 0 & U_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & U_N \\ \hline 0 & 0 & -\sigma^2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c} 0 & U_3 & 0 & 0 \\ \hline 0 & 0 & -JM_1\sigma^2 & 0 \\ \hline 0 & 0 & 0 & U_N \\ \hline 0 & 0 & -M_1\sigma^2 & 0 \end{array}\right] \tag{16}$$

$$B = \left[\begin{array}{c|c|c|c} U_3 & 0 & 0 & 0 \\ \hline 0 & U_3 & 0 & -J \\ \hline 0 & 0 & U_N & 0 \\ \hline 0 & -\delta & 0 & U_N \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ \hline U_3 \\ \hline 0 \\ \hline \phi^T \mathscr{L}_c \underline{T}^* \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline M_2 + JM_1\phi^T \mathscr{L}_c I^* \\ \hline 0 \\ \hline \delta M_2 + M_1 \phi^T \mathscr{L}_c I^* \end{array}\right] \tag{17}$$

and

$$C = [U_3 \mid 0 \mid \mathscr{L}_o^T \phi \mid 0] \tag{18}$$

with the NxN matrix M_1 defined by

$$M_1 \triangleq (U_N - \delta J)^{-1} \tag{19}$$

and the 3x3 matrix M_2 defined by

$$M_2 = (U_3 - J\delta)^{-1} \tag{20}$$

Note that for $\phi^T M' \phi = U_N$ the matrices δJ and M_1 are symmetric, although $J\delta$ and M_2 are not. The inversion in Eqs. (16) and (17) has been accomplished by means of the partitioning formula in Eq. (191) of [2].

As proven in Appendix B, one can interpret J and δ in terms of primitive definitions and obtain the physical interpretation

$$M_2 = I^{o^{-1}} I^* \tag{21}$$

where I^o is the inertia matrix of the primary body referred to its own mass center. This interpretation guarantees that M_2 is nonsingular, which as shown in Appendix B implies that M_1 is nonsingular.

CONTROLLABILITY

The system characterized by Eq. (14) is said to be <u>completely controllable</u> [6] if any initial state $x(t_o)$ can be brought to any finite state $x(T)$ in a finite time interval $T-t_o$ by some control function $u(t)$.

A basic controllability theorem [6] indicates that the system of Eq. (14) is completely controllable if and only if the <u>controllability matrix</u>, Q_c, as defined by

$$Q_c = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots \dots \vdots\; A^{2N+5}B \right] \tag{22}$$

has full rank, which means rank equal to the dimension of x, namely 2N+6. (Note that Q_c has dimensions (2N+6) x (2N+6)3). With the notation

$$B_2 \triangleq M_2 + JM_1\phi^T \; {}_cI^*$$

$$B_4 \triangleq \delta M_2 + M_1\phi^T \; {}_cI^*$$

we have

$$Q = \left[\begin{array}{c|c|c|c|c|c} 0 & B_2 & 0 & J(-M_1\sigma^2)B_4 & \dots & J(-M_1\sigma^2)^{N+2}B_4 \\ \hline B_2 & 0 & J(-M_1\sigma^2)B_4 & 0 & \dots & 0 \\ \hline 0 & B_4 & 0 & (-M_1\sigma^2)B_4 & \dots & (-M_1\sigma^2)^{N+2}{}_4 \\ \hline B_4 & 0 & (-M_1\sigma^2)B_4 & 0 & \dots & 0 \end{array}\right] \tag{23}$$

The rank of Q_c is twice the rank of the (N+3)x3(N+3) matrix Q_c', as given by

$$Q_c' = \left[\begin{array}{c|c|c|c|c} B_2 & J(-M_1\sigma^2)B_4 & J(-M_1\sigma^2)^2 B_4 & \ldots & J(-M_1\sigma^2)^{N+2}B_4 \\ \hline B_4 & (-M_1\sigma^2)B_4 & (-M_1\sigma^2)^2 B_4 & \ldots & (-M_1\sigma^2)^{N+2}B_4 \end{array}\right] \tag{24}$$

In evaluating the rank of Q_c' it is convenient to premultiply by a matrix V which has $|V|$ =1 and which results in a block triangular form for $Q_c'V$. By proceeding in three stages one can determine that the desired matrix is

$$V = \left[\begin{array}{c|c} U_3 & 0 \\ \hline -\phi^T \quad {}_c M_2^{-1} I^* & U_N \end{array}\right] \left[\begin{array}{c|c} U_3 & -JM_1 \\ \hline 0 & U_N \end{array}\right] \left[\begin{array}{c|c} U_3 & 0 \\ \hline -\delta & U_N \end{array}\right] \tag{25}$$

Since the determinant of a product is the product of determinants, the requirement $|V|$ = 1 is satisfied, and premultiplication by V won't change the rank of Q_c'. Moreover, the product VQ_c' becomes (with the definitions of B_2, B_4, and M_1)

$$VQ_c' = \left[\begin{array}{c|c} U_3 & 0 \\ \hline -\phi^T \quad {}_c M_2^{-1} I^* & U_N \end{array}\right] \left[\begin{array}{c|c} U_3 & -JM_1 \\ \hline 0 & U_N \end{array}\right] \left[\begin{array}{c|c|c|c} B_2 & J(-M_1\sigma^2)B_4 & \ldots & J\ (-M_1\sigma^2\ ^{N+2}B_1 \\ \hline \phi^T \quad {}_c I^* & M_1^{-1}\ -M_1\sigma^2)B_4 & \ldots & M_1^{-1}(-M_1\sigma^2)^{N+2}B_4 \end{array}\right]$$

$$= \left[\begin{array}{c|c} U_3 & 0 \\ \hline -\phi^T \mathscr{L}_c M_2^{-1} I^* & U_N \end{array}\right] \left[\begin{array}{c|c|c|c} M_2 & 0 & \ldots & 0 \\ \hline \phi^T \mathscr{L}_c I^* & M_1^{-1}(-M_1\sigma^2)B_4 & \ldots & M_1^{-1}(-M_1\sigma^2)^{N+2}B_4 \end{array}\right]$$

$$= \left[\begin{array}{c|c} M_2 & 0 \\ \hline \phi^T \mathscr{L}_c (I^* - M_2^{-1} I^* M_2) & -\sigma^2 Q_{CN}' \end{array}\right] \tag{26}$$

where

$$Q_{CN}' \triangleq \left[\begin{array}{c|c|c|c|c} B_4 & (-M_1\sigma^2)B_4 & (-M_1\sigma^2)^2 B_4 & \ldots & (-M_1\sigma^2)^{N+1}B_4 \end{array}\right] \tag{27}$$

Eq. (21) guarantees that M_2 is nonsingular, so that the first three columns of VQ_c' in Eq. (26) are independent, and by virtue of the null partition these three columns are independent of all others in VQ_c'. Moreover, the possibility of normalizing $\phi^T M' \phi$ so as to make σ^2 the diagonal matrix of squares of nonzero natural frequencies of the appendage indicates that σ^2 is nonsingular. Thus the controllability condition becomes

$$\text{Rank}(Q'_{CN}) = N \tag{28}$$

In the interpretation of this result, it becomes convenient to rewrite Q'_{CN} from Eq. (27) in terms of

$$\delta_c \triangleq \delta + \phi^T \mathscr{L}_c I^* \tag{29}$$

The identities [4]

$$M_2 = (U_3 - J\delta)^{-1} = U_3 + J(U - \delta J)^{-1}\delta = U_3 + JM_1\delta \tag{30}$$

and

$$M_1 = (U_N - \delta J)^{-1} = U_N + \delta(U_3 - J\delta)^{-1}J = U_N + \delta M_2 J \tag{31}$$

permit recognition of the further identities

$$JM_1 = J(U_N + \delta M_2 J) = (U_3 + J\delta M_2)J = (M_2^{-1} + J\delta)M_2 J = M_2 J \tag{32}$$

and

$$\delta M_2 = \delta + \delta JM_1\delta = \delta + \delta M_2 J\delta = M_1\delta \tag{33}$$

and

$$M_1 = U_N + \delta M_2 J = U_N + M_1\delta J \tag{34a}$$

and

$$M_1 = U_N + \delta M_2 J = U_N + \delta JM_1 \tag{34b}$$

and these permit B_4 to be written as

$$B_4 = M_1\delta + M_1\phi^T \mathscr{L}_c I^* = M_1\,\delta_c \tag{35}$$

and thus permit Q'_{CN} to be written in terms of δ_c. A further simplification is afforded by multiplying Q'_{CN} by the block diagonal matrix with blocks

U_N, $(-U_N)$, $(-U_N)^2$, ... $(-U_N)^{N+1}$, which removes the explicit minus signs from Eq. (27) without changing its rank. If this product is called Q_{CN}, then the necessary and sufficient condition for complete controllability becomes

$$\mathrm{Rank}(Q_{CN}) = N \tag{36}$$

where

$$Q_{CN} = \left[M_1\delta_c \,\middle|\, (M_1\sigma^2)M_1\delta_c \,\middle|\, \ldots \,\middle|\, (M_1\sigma^2)^{N+1}M_1\delta_c \right] \tag{37}$$

Before attempting to interpret this criterion in useful physical terms, we establish a parallel criterion for observability.

OBSERVABILITY

A state $x_j(t)$ characterized by Eqs. (14) and (15) is said to be observable [10] if knowledge of u(t) and y(t) over a finite time segment $t_o<\tau\leq t$ completely determines $x_j(t)$; if all states in a system are observable the system is said to be completely observable.

A basic observability theorem [8] indicates that the system of Eqs. (14) and (15) is completely observable if and only if the observability matrix Q_o, as defined by

$$Q_o = \left[C^T \,\middle|\, A^TC^T \,\middle|\, (A^T)^2C^T \,\middle|\, \text{----} \,\middle|\, (A^T)^{2N+5}C^T \right] \tag{38}$$

has full rank, which means rank equal to the dimension of x, namely 2N+6.

Substitution from Eqs. (16) and (18), with the notation

$$\delta_o \triangleq \phi^T M'\phi\delta + \phi^T \mathscr{L}_o I^* = \delta + \phi^T \mathscr{L}_o I^* \tag{39}$$

(to be compared with Eq. (30), noting the eigenvector normalization $\phi^T M'\phi = U_N$), so that, with the definition of J,

$$J^T+(\mathscr{L}_o^T\phi)^T = (\phi^T M'\phi)^T\delta(I^{*-1})^T+\phi^T\mathscr{L}_o = (\phi^T M'\phi\delta+\phi^T\mathscr{L}_o I^*)I^{*-1} = \delta_o I^{*-1}$$

produces for the observability matrix (in which $M_1 = M_1^T$)

$$Q_o = \left[\begin{array}{c|c|c|c|c|c|c} U_3 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & U_3 & 0 & 0 & \cdots & 0 & 0 \\ \hline \phi^T\mathscr{L}_o & 0 & (-\sigma^2 M_1)(\delta_o I^{*-1}) & 0 & \cdots & (-\sigma^2 M_1)^{N+2}(\delta_o I^{*-1}) & 0 \\ \hline 0 & \phi^T\mathscr{L}_o & 0 & (-\sigma^2 M_1)(\delta_o I^{*-1}) & \cdots & 0 & (-\sigma^2 M_1)^{N+2}(\delta_o I^{*-1}) \end{array}\right] \tag{40}$$

The rank of Q_o is twice the rank of Q_o', where

$$Q_o' = \left[\begin{array}{c|c|c|c|c} U_3 & 0 & 0 & \cdots & 0 \\ \hline \phi^T\mathscr{L}_o & (-\sigma^2 M_1)(\delta_o I^{*-1}) & (-\sigma^2 M_1)^2(\delta_o I^{*-1}) & \cdots & (-\sigma^2 M_1)^{N+2}(\delta_o I^{*-1}) \end{array}\right] \tag{41}$$

and since the first three columns of Q_o' are independent of each other and of all other columns the rank of Q_o' is three plus the rank of the residual matrix obtained by deleting these three columns and the three top empty rows from Q_o'. Because $|I^{*-1}| \neq 0$, and $|\sigma^2| \neq 0$, this residual matrix has the same rank as the matrix

$$Q_{ON} \triangleq [M_1\delta_o \mid (M_1\sigma^2)M_1\delta_o \mid \ldots \mid (M_1\sigma^2)^{N+1}M_1\delta_o] \tag{42}$$

(Note that, as in the controllability case, minus signs have been removed, without influencing rank.)

Thus the necessary and sufficient condition for complete observability becomes

$$\text{Rank}(Q_{ON}) = N \tag{43}$$

The observability conditions in Eqs. (42) and (43) are now very similar to the controllability conditions in Eqs. (36) and (37), differing only in the exchange of δ_o for δ_c.

INTERPRETATION OF RESULTS

The reduction of the controllability matrix to Q_{CN} (Eq. (37)) and the observability matrix to Q_{ON} (Eq. (42)) implies that controllability and observability of the system of dimension 2N+6 in Eqs. (14) and (15) can be inferred from that of the system of dimension N given by

$$\dot{v} = M_1\sigma^2 v + M_1\delta_c u \tag{44}$$

$$w = \delta_o^T M_1 v \tag{45}$$

With this interpretation comes the realization that simpler necessary and sufficient conditions for controllability and observability are given respectively by the requirements for full rank (N) of the matrices

$$\bar{Q}_{CN} \triangleq [M_1\delta_c \mid (M_1\sigma^2)M_1\delta_c \mid \cdots \mid (M_1\sigma^2)^{N-1}M_1\delta_c] \tag{46}$$

and

$$\bar{Q}_{ON} \triangleq [M_1\delta_o \mid (M_1\sigma^2)M_1\delta_o \mid \cdots \mid (M_1\sigma^2)^{N-1}M_1\delta_o] \tag{47}$$

These conditions and the relationships between them are most interpretable in physical terms when special cases are considered.

When all of the sensors and actuators are attached to a single rigid body in the model, we can call that body the primary body, so that the location matrices $\mathscr{L}_c$ and $\mathscr{L}_o$ are both zero, $\delta_c = \delta_o = \delta$, and the controllability and observability conditions become identical, and simplify greatly. In this case Q_{CN} in Eq. (46) and Q_{ON} in Eq. (47) can be written with Eq. (33) in the form

$$Q_{CN} = Q_{ON} = [\delta M_2 \mid (M_1\sigma^2)\delta M_2 \mid \cdots \mid (M_1\sigma^2)^{N-1}\delta M_2] \tag{48}$$

Since by Eq. (21) the common factor M_2 is nonsingular, the rank of the matrix in Eq. (48) is unchanged by removing M_2 throughout; thus the controllability and observability criterion becomes

$$\mathrm{Rank}[\delta \mid (M_1\sigma^2)\delta \mid \ldots \mid (M_1\sigma^2)^{N-1}\delta] = N \tag{49}$$

Eq. (49) simplifies dramatically when subjected to the following corollary of Kalman's canonical structure theorem [7]: "If a system characterized by the pair (A,B) is completely controllable and F is any matrix of appropriate dimension, then the system characterized by (A+BF, B) is completely controllable." (This corollary is not proven in reference [7], so a proof is included here as Appendix C.)

Eq. (34b) permits Eq. (49) to be written as

$$\mathrm{Rank}[\delta \mid (\sigma^2+\delta JM_1\sigma^2)\delta \mid \ldots \mid (\sigma^2+\delta JM_1\sigma^2)^{N-1}\delta] = N$$

or, with the noted corollary,

$$\mathrm{Rank}[\delta \mid \sigma^2\delta \mid \cdots \mid (\sigma^2)^{N-1}\delta] = N \tag{50}$$

A typical Nx3 partition in the preceding matrix is

$$\sigma^{2i}\delta = \begin{bmatrix} \sigma_1^{2i} & & & \\ & \sigma_2^{2i} & & \\ & & \ddots & \\ & & & \sigma_N^{2i} \end{bmatrix} \begin{bmatrix} \delta^1 \\ \delta^2 \\ \vdots \\ \delta^N \end{bmatrix} = \left[\begin{array}{c|c|c} \sigma_1^{2i}\delta_1^1 & \sigma_1^{2i}\delta_2^1 & \sigma_1^{2i}\delta_3^1 \\ \sigma_2^{2i}\delta_1^2 & \sigma_2^{2i}\delta_2^2 & \sigma_2^{2i}\delta_3^2 \\ \vdots & \vdots & \vdots \\ \sigma_N^{2i}\delta_1^N & \sigma_N^{2i}\delta_2^N & \sigma_N^{2i}\delta_3^N \end{array}\right]$$

The Nx1 columns in Eq. (50) can be interchanged without changing the rank, so that Eq. (50) becomes

$$\mathrm{Rank}\left[\begin{array}{cccc|ccc|ccc} \delta_1^1 & \sigma_1^2\delta_1^1 & \cdots & \sigma_1^{2(N-1)}\delta_1^1 & \delta_2^1 & \cdots & \sigma_1^{2(N-1)}\delta_2^1 & \delta_3^1 & \cdots & \sigma_1^{2(N-1)}\delta_3^1 \\ \delta_1^2 & \sigma_2^2\delta_1^2 & \cdots & \sigma_2^{2(N-1)}\delta_1^2 & \delta_2^2 & \cdots & \sigma_2^{2(N-1)}\delta_2^2 & \delta_3^2 & \cdots & \sigma_2^{2(N-1)}\delta_3^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \delta_1^N & \sigma_N^2\delta_1^N & \cdots & \sigma_N^{2(N-1)}\delta_1^N & \delta_2^N & \cdots & \sigma_N^{2(N-1)}\delta_2^N & \delta_3^N & \cdots & \sigma_N^{2(N-1)}\delta_3^N \end{array}\right] = N \tag{51}$$

or

$$\mathrm{Rank}[A_1\hat{\sigma} \mid A_2\hat{\sigma} \mid A_3\hat{\sigma}] = \mathrm{Rank}[A_1 \mid A_2 \mid A_3] \begin{bmatrix} \hat{\sigma} & 0 & 0 \\ 0 & \hat{\sigma} & 0 \\ 0 & 0 & \hat{\sigma} \end{bmatrix} = N \qquad (52)$$

where

$$A_i \triangleq \begin{bmatrix} \delta_i^1 & & & 0 \\ & \delta_i^2 & & \\ & & \ddots & \\ 0 & & & \delta_i^N \end{bmatrix}$$

and

$$\hat{\sigma} = \begin{bmatrix} 1 & \sigma_1^2 & \sigma_1^4 & \sigma_1^{2(N-1)} \\ 1 & \sigma_2^2 & \sigma_2^4 & \sigma_2^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \sigma_N^2 & \sigma_N^4 & \sigma_N^{2(N-1)} \end{bmatrix}$$

Since the determinant of the Vandermonde matrix $\hat{\sigma}$ is given by [8]

$$|\hat{\sigma}| = \Pi(\sigma_i^2 - \sigma_j^2) \qquad 1 \le i < j \le N$$

then $\hat{\sigma}$ is nonsingular if for all $i \neq j$, $\sigma_i \neq \sigma_j$. Under this supposition Eq. (50) is equivalent to

$$\mathrm{Rank}[A_1 \mid A_2 \mid A_3] = N$$

or

$$\mathrm{Rank}[A \mid A \mid A] \begin{bmatrix} A_1^T \\ \hline A_2^T \\ \hline A_3^T \end{bmatrix} = \mathrm{Rank} \begin{bmatrix} \delta^1\delta^{1^T} & & & \\ & \delta^2\delta^{2^T} & & 0 \\ & & \ddots & \\ & 0 & & \delta^N\delta^{N^T} \end{bmatrix} = N$$

thus for $\sigma_i \neq \sigma_j$ for all $i \neq j$ the condition

$$\delta^i\delta^{i^T} \neq 0 \qquad i = 1, \ldots, N \qquad (53)$$

is necessary and sufficient for both controllability and observability.

In the case of repeated roots, we may assume that the first p roots are repeated and the remainder all distinct, and then re-examine Eq. (51). Now the first p rows are independent if and only if $\delta^1, \ldots, \delta^p$ are independent. (Note that this is impossible for $p>3$, or in general for p exceeding the dimension of θ, which establishes the number of control axes in the problem.) If the top p rows are deleted from Eq. (51), then what remains has a rank at least as large as that portion of it given by a new version of Eq. (52) with N replaced by N-p and A_1 and $\hat{\sigma}$ beginning with σ_{p+1} and δ_i^{p+1} rather than σ_1 and δ_i^1. Thus in the case of p repeated roots $\sigma_1, \ldots, \sigma_p$ it is sufficient for controllability and observability that $\delta^1, \ldots, \delta^p$ be independent and $\delta^i\delta^{i^T} \neq 0$ for $i = p+1, \ldots, N$. (The authors suspect that this is also a necessary condition, but the proof has eluded them.)

Eq. (53) admits an appealing physical interpretation when written as

$$\mathrm{tr}(\delta^{i^T} \delta^i) \neq 0 \tag{54}$$

since $\delta^T\delta = \Sigma\, \delta^{i^T} \delta^i$ has for the normalization $\phi^T M' \phi = U_N$ previously been interpreted ([2], pp. 69-70) as the inertia matrix of the appendage referred to the system mass center c plus the difference in the inertia matrices of the primary body referred to c and to its own mass center. The quantity $\delta^{i^T} \delta^i$ has been called the "effective inertia matrix" of the i^{th} mode, and has been recognized ([2], pp. 69-70) as a measure of the coupling between i^{th} mode vibration and primary body rotation. If $\delta^i\delta^{i^T} = 0$, the i^{th} mode would for output θ be deleted from the model by coordinate truncation; thus the truncated system is completely controllable and observable.

This success in dealing with the special case in which sensors and actuators are all attached to one rigid body (here called the primary body) has not been matched for the general case of arbitrarily distributed sensors and actuators. Since the observability condition (Eq. (38)) depends only upon C and A while the controllability condition (Eq. (22)) depends only upon B and A, one could cope with the special case in which all sensors are on one rigid body and all actuators on another by applying Eq. (53) twice, with two different selections for the primary body and two different interpretations of all of the symbols, but this case is less interesting than the common primary body case solved here or the general case in which sensors and actuators are both distributed over the body (as in the case of the telescopic sensors on the NASA Large Space Telescope). For this general case the controllability and observability conditions are full rank requirements for the matrices in Eqs. (46) and (47) respectively. These requirements can be placed in slightly simpler form by observing that $|M_1| \neq 0$ (Appendix B) and factoring M_1 to the left; necessary and sufficient conditions for controllability and observability then become respectively

$$\text{Rank}\left[\delta_c \;\middle|\; (\sigma^2 M_1)\delta_c \;\middle|\; \ldots \;\middle|\; (\sigma^2 M_1)^{N-1}\delta_c\right] = N \tag{55}$$

and

$$\text{Rank}\left[\delta_o \;\middle|\; (\sigma^2 M_1)\delta_o \;\middle|\; \ldots \;\middle|\; (\sigma^2 M_1)^{N-1}\delta_o\right] = N \tag{56}$$

These criteria correspond to the state and observation equations

$$\dot{v} = \sigma^2 M_1 v + \delta_c U \tag{57}$$

$$w = \delta_o^T v \tag{58}$$

For purposes of preliminary analysis, it may be useful to consider a flexible appendage model characterized by a single mode. Eqs. (55) and (56) then require simply that the 1x3 matrices δ_c and δ_o be of rank one, or the scalar equivalents

$$\delta_c \delta_c^T \neq 0 \tag{59}$$

and

$$\delta_o \delta_o^T \neq 0 \tag{60}$$

Experience with specific examples (see Appendix A) and with the special case represented by Eq. (53) suggests that when many appendage modes are included the satisfaction of Eqs. (59) and (60) for each mode is necessary but not sufficient for complete controllability and observability respectively, with an additional frequency criterion stipulating further criteria in the event that eigenvalues of $\sigma^2 M_1$ are repeated. This proposition has however not been proven; the corollary which produced Eq. (50) from Eq. (49) and simplified the special case governed by these equations is not helpful in the general case.

CONCLUSIONS

Necessary and sufficient conditions for controllability and observability have been established in the form of full rank requirements for Nx3N matrices for systems characterized by state equations of dimension 2N+6. For the special case in which all sensors and actuators are attached to the same "primary" rigid body, the controllability and observability conditions are identical and reduce to the requirements that N scalars $\delta^i \delta^{i^T}$ (i=1, ... , N) be nonzero and either appendage frequencies σ_1, ... , σ_N be distinct or effective inertia matrices $\delta^{i^T} \delta^i$ be independent.

With these conclusions come a new reason for the study of controllability and observability. This investigation was initially stimulated by the realization that much of the progress in optimal control theory depends upon the assumptions of complete controllability and observability [9]; if we are to address the problem of flexible spacecraft control with the techniques of modern control theory we must first determine whether or not our system is completely controllable and observable. It has been apparent from the outset that this issue is clouded by the realization that in the flexible spacecraft modeling process there is an ambiguity stemming from the foreknowledge that not all possible modes of vibration are dynamically significant or even computationally available, and many must be truncated from the system description prior to simulation. The selection of which modes to truncate and which to preserve is a major decision in flexible spacecraft simulation. Now we can see that the preceding conditions for controllability and observability not only provide answers required for the application of the techniques of optimal control, they also provide a new rationale for modal coordinate truncation, as required for any system simulation.

It has been recognized previously ([2], pp. 69-70) that the "effective inertia matrix" of each mode affords a rationale for truncation, since this matrix provides a quantitative measure of the dynamic coupling between vibration in that mode and primary body rotation. Now we see from Eq. (53) that the trace of this matrix is a formal measure of controllability and observability when all sensors and actuators are on the primary body. Moreover, we find that in the more general case in which controllers and observers are distributed over the body, we have a new truncation criterion in the controllability and observability conditions available from Eqs. (55) and (56). As we entertain various truncations, our objective is to obtain a system of minimal dimension consistent with the preservation of the fidelity of the output matrix

$$z = Dx \tag{61}$$

which describes the salient aspects of the spacecraft mission performance. Since the control law for u in Eq. (11) depends upon the sensor observations y in Eq. (13), it is necessary that all significantly observable states be retained in the coordinate truncation process. Although in general it is possible for an uncontrollable state to influence z and hence be among the states which must be retained in truncation, in the special case of states describing flexible body modes experience suggests that only controllable states will influence z. Under this assumption, it becomes desirable to truncate all those uncontrollable states which are also unobservable. This leads to the adoption of the design objective of configuring sensors so that any uncontrollable states which do not influence z are also unobservable; this strategy of uncontrollable mode isolation permits in many flexible spacecraft control applications the truncation of all uncontrollable states, which are by design also unobservable. The result is often a completely controllable and observable system model, which is more amenable to formal optimal control, simulation, and conventional control system design.

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APPENDIX A. EXAMPLES OF CONTROLLABILITY AND OBSERVABILITY CALCULATIONS

Example 1. Consider single axis rotation of a spacecraft with symmetric appendages of continuous thin beams as shown in Fig. A1.

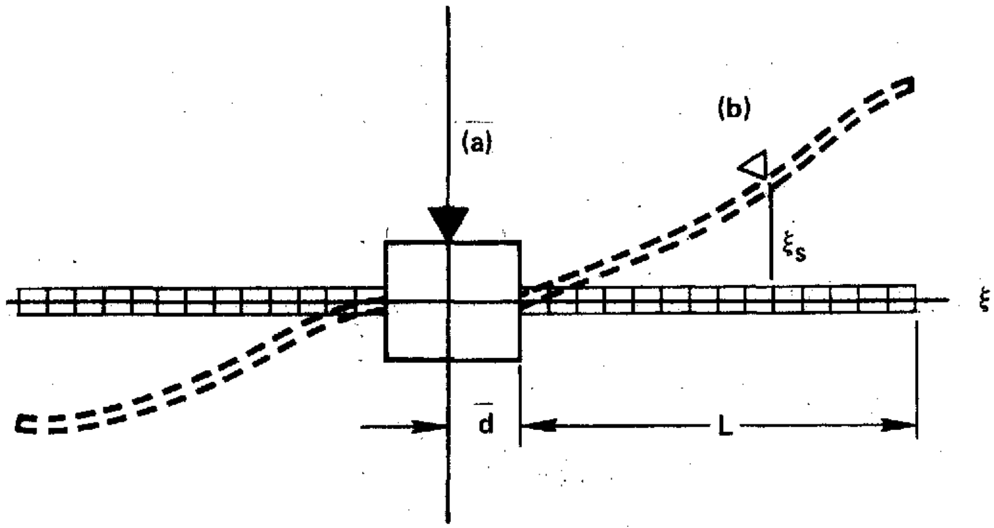

Figure A1. Model of Example 1.

The rotational motion of the model is characterized by Eqs. (11) and (12) with the observation equation (13). However, the assumptions that the rotational motion is limited to the single axis and that the mass distribution is continuous (so that each mass element has infinitesimally small moment of inertia) necessitate a slight modification of definitions of δ, J, ϕ, $\mathscr{L}_c$ and $\mathscr{L}_o$. Furthermore, mode shapes are assumed to be given by a function of ξ, i.e., $\phi^j(\xi)$ which is at least once differentiable with respect to ξ over the intervals $d<|\xi|<L+d$, or more specifically,

$$\phi^j(\xi) = \begin{cases} 0 & ,0<|\xi|<d \\ \phi_+^j(\xi) & ,d\leq\xi\leq L+d \\ \phi_-^j(\xi) & ,-(L+d)\leq\xi\leq -d \end{cases} \tag{A-1}$$

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Under these assumptions, we may express δ^j (scalar) in terms of mode shape and inertial parameters as follows.

$$\delta^j = -\frac{m}{L}\int_{-(L+d)}^{L+d} \phi^j(\xi)\xi d\xi = -\frac{m}{L}\int_{d}^{L+d} \{\phi_+^j(\xi)-\phi_-^j(-\xi)\}\xi d\xi \qquad \text{(A-2)}$$

For the purpose of illustration, we further assume that only two modes are retained (N=2). The first modal coordinate, η_1, is associated with an asymmetric mode which satisfies

$$\phi_-^1(-\xi) = -\phi_+^1(\xi)$$

and the second is with a symmetric mode which satisfies

$$\phi_-^2(-\xi) = \phi_+^2(\xi)$$

The modal deformation coordinates thus defined are illustrated in Fig. A.2.

(1) $\theta = \eta_2 = 0; \eta_1 \neq 0$ (2) $\theta = \eta_1 = 0; \eta_2 \neq 0$

Figure A2. Asymmetric and Symmetric Modes.

Corresponding to these modes, δ^j's (j=1,2) are given by

$$\delta^1 = -\frac{2m}{L}\int_{d}^{L+d} \phi_+^1(\xi)\xi d\xi \qquad \text{(A-3a)}$$

$$\delta^2 = 0 \qquad \text{(A-3b)}$$

We examine the following three cases.

a) If both sensor and actuator are located on the primary body, the rotational equation is characterized by

$$\begin{cases} \ddot{\theta} - \dfrac{\delta^1}{I^*}\ddot{\eta} = u & \text{(A-4)} \\ \ddot{\eta}_1 + \sigma_1^2 \eta_1 = -\delta^1 \ddot{\theta} & \text{(A-5)} \\ \ddot{\eta}_2 + \sigma_2^2 \eta_2 = 0 & \text{(A-6)} \end{cases}$$

with observation equation

$$y = \theta \qquad \text{(A-7)}$$

Since Eq. (A-6) is independent of u, θ and η_1 and does not affect Eq. (A-7), we can readily identify η_2 as uncontrollable and unobservable. Thus, we may limit our analysis to the truncated system of equations in which Eq. (A-6) is deleted.

<u>Direct approach:</u> For comparison, we examine the controllability and observability matrices directly. In terms of the state variable $x \triangleq [\theta, \dot{\theta}, \eta_1, \dot{\eta}_1]^T$ the state equation is $\dot{x} = Ax+Bu$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-\delta^1\sigma_1^2}{I^*-(\delta^1)^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-I^*\sigma_1^2}{I^*-(\delta^1)^2} & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0 \\ \dfrac{1}{I^*-(\delta^1)^2} \\ 0 \\ \dfrac{\delta^1}{I^*-(\delta^1)^2} \end{bmatrix}$$

and with the observation equation $y = Cx$ with

$$C = [1 \quad 0 \quad 0 \quad 0]$$

Now, the controllability matrix is

$$Q_c = [B \mid AB \mid A^2B \mid A^3B]$$

and the observability matrix is

$$Q_o = [C^T \mid A^TC^T \mid (A^T)^2C^T \mid (A^T)^3C^T].$$

Matrix algebra produces, with $\alpha \triangleq (I^* - (\delta^1)^2)$ (>0),

$$\alpha^2 Q_c = \begin{bmatrix} 0 & \alpha & 0 & -(\delta^1)^2\sigma_1^2 \\ \alpha & 0 & -(\delta^1)^2\sigma_1^2 & 0 \\ 0 & \delta^1\alpha & 0 & -I^*\delta^1\sigma_1^2 \\ \delta^1\alpha & 0 & -I\,\delta^1\sigma_1^2 & 0 \end{bmatrix}$$

and

$$\alpha Q_o = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & -\delta^1\sigma_1^2 & 0 \\ 0 & 0 & 0 & -\delta^1\sigma_1^2 \end{bmatrix}$$

Consequently,

$$|Q_c| = (-\delta^1)^3\sigma_1^6/\alpha^2$$

$$|Q_o| = (\delta^1)^2\sigma_1^4/\alpha^2$$

Since $\sigma_1 \neq 0$, $\alpha>0$, the full rank conditions reduce to

$$\delta^1 \neq 0$$

for controllability and observability.

<u>The result</u> (Eq. (53)) on the controllability and the observability produces the same conclusion as follows.

The fact that $\delta^2 = 0$ implies that the state including η_2 is not completely controllable, and is not completely observable either because the sensor is attached to the same body as the actuator.

If η_2 is deleted from the state, then Eq. (53) simply requires $\delta^1 \neq 0$, noting that $N = 1$,

b) If the actuator is attached to the primary body and the sensor is attached at $\xi = \xi_s$ as shown in Fig. A1, then it reads

$$y = \theta + [\phi_s^{1'} \quad \phi_s^{2'}] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{A-8}$$

where for simplicity of notation we write

$$\frac{\partial}{\partial \xi} \phi_+^j(\xi) \Big|_{\xi = \xi_s} \triangleq \phi_s^{j'}, \quad j = 1, 2$$

In this case, Eqs.(A-4)-(A-6) remain unchanged so that the controllability condition is the same as the previous result.

The observability is to be discussed with respect to the state including η_2 in view of the difference between Eqs. (A-7) and (A-8). By definitions of Eqs. (19) and (39),

$$M_1 = \left\{ U_2 - \begin{bmatrix} \delta^1 \\ 0 \end{bmatrix} \left(\frac{1}{I^*}\right) \begin{bmatrix} \delta^1 \\ 0 \end{bmatrix}^T \right\}^{-1} = \begin{bmatrix} \frac{I^*}{\alpha} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\delta_o = \begin{bmatrix} \delta_o^1 \\ \delta_o^2 \end{bmatrix} = \begin{bmatrix} \delta^1 \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_s^{1'} \\ \phi_s^{2'} \end{bmatrix} I^* = \begin{bmatrix} \delta^1 + I^* \phi_s^{1'} \\ I^* \phi_s^{2'} \end{bmatrix}$$

and consequently,

$$M_1 \delta_o = \begin{bmatrix} \frac{I^*}{\alpha} \delta_o^1 \\ I^* \delta_o^2 \end{bmatrix}, \quad M_1 \sigma^2 = \begin{bmatrix} \frac{I^*}{\alpha} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Thus, we can construct $\bar{Q}_{ON}$ defined by Eq. (47)

$$\bar{Q}_{ON} = \begin{bmatrix} \frac{I^*}{\alpha}\delta_o^1 & (\frac{I^*}{\alpha})^2 \sigma_1^2\delta_o^1 \\ I^*\delta_o^2 & I^*\sigma_2^2\delta_o^2 \end{bmatrix}$$

Since $\bar{Q}_{ON}$ is a square matrix, the full rank property is simply examined by its determinant, i.e.,

$$|\bar{Q}| = \frac{I^{*2}}{\alpha}\delta_o^1\delta_o^2(\sigma_2^2 - \frac{I^*}{\alpha}\sigma_1^2)$$

Thus, observability requires three conditions:

(i) $\delta_o^1 = \delta^1 + I^*\phi_s^{1'} \neq 0$

(ii) $\delta_o^2 = I^*\phi_s^{2'} \neq 0$

(iii) $\sigma_2^2 - \frac{I^*}{\alpha}\sigma_1^2 \neq 0$

We see that the conditions (i) and (ii) are exactly what Eq. (60) requires. Especially, the second requirement which reduces to $\phi_s^{2'} \neq 0$ is physically clear because if $\phi_s^{2'} = 0$ then Eq. (A-8) is not affected by η_2 as in the previous case.

The condition (iii) deserves particular attention. This prohibits the diagonal elements of $M_1\sigma^2$ to be equal or the eigenvalues of $M_1\sigma^2$ to be repeated. Eq. (44) should be referred to with consideration that in case of diagonal $M_1\sigma^2$, repeated eigenvalues are not accepted for the single input system to be either completely controllable or observable. (For similar discussion, see Ref. [9]).

c) Assume that the sensor is attached to the primary body and a pair of gas jets are attached on the beams at $\xi = \ell$ and $\xi = -\ell$. Further, assume that two actuators synchronize but their forces are not necessarily of equal magnitude, i.e., the one at $\xi = -\ell$ produces a force f_2 which is given by $f_2 = c'f_1$ with $-1 \leq c' < 1$ where f_1 is a force produced by the jet at $\xi = \ell$. Such dependency comes from the minimal actuator model. Then Eqs. (A-4) - (A-6) are rewritten as

$$\ddot{\theta} - \frac{\delta^1}{I^*}\ddot{\xi} = \frac{\ell}{I^*}(1-c')f = u$$

$$\eta + \sigma^2\eta - \delta\theta = \begin{bmatrix} \phi_+^1(\ell)(1-c') \\ \\ \phi_+^2(\ell)(1+c') \end{bmatrix} f_1 = \begin{bmatrix} \phi_+^1(\ell)I^*/\ell \\ \\ \phi_+^2(\ell)I^*c/\ell \end{bmatrix} u$$

where $c \triangleq \frac{1+c'}{1-c'}$

By the definition of Eq. (29),

$$\delta_c = \begin{bmatrix} \delta_c^1 \\ \\ \delta_c^2 \end{bmatrix} = \begin{bmatrix} \delta^1 + \phi_+^1(\ell)I^*\ell \\ \\ \phi_+^2 \ \ell I^*/\ell \end{bmatrix}$$

Similarly as previously, we can construct Q defined by Eq. (47)

$$\bar{Q}_{CN} = \begin{bmatrix} \frac{I^*}{\alpha}\delta_c^1 & \left(\frac{I^*}{\alpha}\right)^2\sigma^2\delta_c^1 \\ I^*\delta_c^2 & I^*\sigma_1^2\sigma_c^2 \end{bmatrix}$$

and

$$|\bar{Q}_{CN}| = \frac{I^{*2}}{\alpha}\delta_c^1\delta_c^2(\delta_2^2 - \frac{I^*}{\alpha}\sigma_-^2)$$

Thus, complete controllability requires three conditions:

(i) $\delta_c^1 = \delta^1 + \phi_+^1(\ell) I^* / \ell \neq 0$

(ii) $\delta_c^2 = \phi_+^2(\ell) I^* c / \ell \neq 0$

(iii) $\sigma_2^2 - \frac{I^*}{\alpha} \sigma_1^2 \neq 0$

The condition (iii) is identical to that of (b). (i) and (ii) are what Eq. (59) requires. The condition (i) together with (A3-a) implies that if the mode $\phi_+^1(\xi)$ satisfies the integral equation

$$-\frac{2m}{L} \int_d^{L+d} \phi_+^1(\xi) \xi d\xi + \frac{I^*}{\ell} \phi_+^1(\ell) = 0$$

then η_1 is uncontrollable. (Equivalently, if the actuator location parameter, ℓ, satisfies this, then η_1 is uncontrollable.)

The condition (ii) requires

$$\phi_+^2(\ell) \neq 0 \quad \text{and} \quad c \neq 0$$

$c = 0$ implies $c' = -1$ and $f_2 = -f_1$, which is realized by a pair of equal magnitude jets. From the practical point of view, it is rather desirable to locate the jets at ℓ satisfying $\phi_+^2(\ell) = 0$, because it can eliminate the effect of η_2 which causes an unnecessary vibration due to some imperfection of the jets ($c \neq -1$ or $f_2 \neq -f_1$). If this is possible, then we can truncate η_2 for this particular location, although otherwise we must retain it.

APPENDIX B. INTERPRETATION OF $U_3 - J\delta$

By the definitions following Eq. (13),

$$U_3 - J\delta = U_3 - I^{*^{-1}}\delta^T(\phi^T M'\phi)\delta = I^{*^{-1}}[I^* - \delta^T(\phi^T M'\phi)\delta] \qquad \text{(B-1)}$$

Moreover, in view of the symmetry of M' and the definition of δ,

$$\begin{aligned} \delta^T\phi^T M'\phi\delta &= \Delta^T\phi(\phi^T M'\phi)^{-1}(\phi^T M'\phi)(\phi^T M'\phi)^{-1}\phi^T\Delta \\ &= \Delta^T\phi\phi^{-1}M'^{-1}\phi^{T^{-1}}\phi^T\Delta = \Delta^T M'^{-1}\Delta \end{aligned} \qquad \text{(B-2)}$$

From [2], Eq. (277), with minor notational revision,

$$M' \triangleq M(U_{6n} - \Sigma_{UO}\Sigma_{UO}^T M/\mathscr{M}) \qquad \text{(B-3)}$$

and

$$\Delta \triangleq M(\Sigma_{OU} - \tilde{R}\Sigma_{UO}) \qquad \text{(B-4)}$$

where M is the block diagonal 6nx6n matrix containing in alternating sequence along the main diagonal the mass and inertia matrices of the nodal bodies, as designated by the 3x3 matrices m^1, I^1, m^2, I^2, --- , m^n, I^n. The symbol $\mathscr{M}$ denotes the total vehicle mass, and the 6nx3 matrix binary operators Σ_{UO} and Σ_{OU} are defined in terms of 3x3 zeros and unit matrices by

$$\Sigma_{UO} = [U_3 \mid 0 \mid U_3 \mid 0 \mid \ldots \mid U_3 \mid 0]^T \qquad \text{(B-5)}$$

and

$$\Sigma^{OU} = [0 \mid U_3 \mid 0 \mid U_3 \mid \ldots \mid 0 \mid U_3]^T \qquad \text{(B-6)}$$

Finally, the $\tilde{R}$ in Eq. (B-4) is defined in terms of 3x3 zeros and the matrices in Eq. (5) by

$$\tilde{R} \triangleq \begin{bmatrix} \tilde{R}^1 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & \tilde{R}^2 & & & \\ & & & \ddots & & \\ & & & & \tilde{R}^n & 0 \\ & & & & 0 & 0 \end{bmatrix} \tag{B-7}$$

Thus Eq. (B-2) becomes, in view of the symmetry of M,

$$\Delta^T M'^{-1} \Delta = (\Sigma_{OU} - \tilde{R}\Sigma_{UO})^T M(U_{6n} - \Sigma_{UO}\Sigma_{UO}^T M/\mathcal{M})^{-1}(\Sigma_{OU} - \tilde{R}\Sigma_{UO})$$

with the identity [4]

$$(U_{6n} - \Sigma_{UO}\Sigma_{UO}{}^T M/\mathcal{M})^{-1} = U_{6n} + \Sigma_{UO}[U_{6n} - \Sigma_{UO}^T (M/\mathcal{M})\,\Sigma_{UO}]^{-1}\Sigma_{UO}^T M/\mathcal{M}$$

and the recognition that

$$\Sigma_{UO}^T M \Sigma_{UO} = \sum_{i=1}^{n} m^i = \sum_{i=1}^{n} m_i U_3$$

where m_i is the mass of nodal body i, we can identify the scalar

$$(1 - \sum_{i=1}^{n} m_i/\mathcal{M})^{-1} = \mathcal{M}/m$$

where m_0 is the mass of the primary body, and write (noting the skew-symmetry of $\tilde{R}$)

$$\Delta^T M'^{-1} \Delta = (\Sigma_{OU}^T + \Sigma_{UO}^T \tilde{R}) M (U_{6n} + \Sigma_{UO}\Sigma_{UO}^T M/m_0)(\Sigma_{OU} - \tilde{R}\Sigma_{UO}) \tag{B-8}$$

With the identities

$$\Sigma_{OU}^T M \Sigma_{UO} = 0$$

$$\Sigma_{UO}^T M \Sigma_{OU} = 0$$

$$\Sigma_{UO}^{T}\tilde{R}M\Sigma_{OU} = 0$$

$$\Sigma_{OU}^{T}M\tilde{R}\Sigma_{UO} = 0$$

$$\Sigma_{OU}^{T}M\Sigma_{OU} = \sum_{j=1}^{n} I^{j}$$

$$\Sigma_{UO}^{T}\tilde{R}M\tilde{R}\Sigma_{UO} \quad \sum_{j=1}^{n} m_j\tilde{R}^{j}\tilde{R}^{j}$$

$$\Sigma_{UO}^{T}\tilde{R}M\Sigma_{UO} = \Sigma_{UO}^{T}M\tilde{R}\Sigma_{UO} = \sum_{j=1}^{n} m_j\tilde{R}^{j}$$

Eq. (B-8) becomes

$$\Delta^{T}M'^{-1}\Delta = \sum_{j=1}^{n} I^{j} - \sum_{j=1}^{n} m_j\tilde{R}^{j}\tilde{R}^{j} - [\sum_{j=1}^{n} m_j\tilde{R}^{j}]^{2}/m_o \tag{B-9}$$

The first two summations can be identified by the inertia matrix reference point transfer theorem [5] as the inertia matrix of the set of n appendage nodal bodies referred to the system mass center c. Thus, if I' is the inertia matrix of the primary body referred to c, then, with Eqs. (B-9) and B-2), Eq. (B-1) becomes

$$U_3 - J\delta = I^{*^{-1}}[I' + (\sum_{j=1}^{n} m_j\tilde{R}^{j})^{2}/m_o] \tag{B-10}$$

The sum $\Sigma m_j R^j$ is by system mass center definition equal to m_o times the vector from c to the mass center of the primary body; thus by the noted reference point transfer theorem the expression in square brackets in Eq. (10) is the inertia matrix I^o for the primary body referred to its own mass center. This matrix is symmetric and positive definite, which guarantees that the matrix

$$U_3 - J\delta = I^{*^{-1}} I^{o} \tag{B-11}$$

is nonsingular, and the matrix

$$M_2 = (U_3 - J\delta)^{-1} = I^{o^{-1}} I^{*} \tag{B-12}$$

exists, so that

$$| \; U_3 - J\delta \; | \neq 0 \tag{B-13}$$

From the relationship [4]

$$| \; P \; | = \det \left[\begin{array}{c:c} P_{11} & P_{12} \\ \hdashline P_{21} & P_{22} \end{array} \right] =$$

$$|P_{11}| \quad |P_{22} - P_{21} P_{11}^{-1} P_{12}| \;=\; |P_{22}| \quad |P_{11} - P_{12} P_{22}^{-1} P_{21}| \tag{B-14}$$

with $P_{11} = U_3$, $P_{22} = U_N$, $P_{12} = J$ and $P_{21} = \delta$, we see that

$$|U_N - J\delta| = |U_3 - \delta J| \tag{B-15}$$

so that the nonsingularity of M_1 is assured by the nonsingularity of M_2.

APPENDIX C. PROOF OF CONTROLLABILITY COROLLARY

Corollary: If a system characterized by the pair (A,B) is completely state controllable and F is any matrix of appropriate dimension, then the system characterized by (A+BF,B) is completely state controllable.

Note: This corollary is applicable whenever there exists a constant linear feedback law, such that the control variable u(t) can be written in the form

$$u(t) = u_c(t)+Fx(t)$$

where F is a constant matrix. Then the corollary states that the system controllability is unaffected by the linear feedback term Fx(t).

Proof

Define a tridiagonal matrix V by

$$V \triangleq \begin{bmatrix} U & -FB & -FAB & -FA^2B & \cdots & -FA^{n-2}B \\ 0 & U & -FB & -FAB & \cdots & -FA^{n-1}B \\ 0 & 0 & U & -FB & \cdots & \vdots \\ 0 & 0 & 0 & U & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & -FA^2B \\ 0 & 0 & 0 & 0 & \ddots & -FAB \\ 0 & 0 & 0 & 0 & & -FB \\ 0 & 0 & 0 & 0 & \cdots & U \end{bmatrix}$$

Let

$$Q_c \triangleq [B \mid (A+BF)B \mid (A+BF)^2B \mid --- \mid (A+BF)^{n-1}B]$$

so that

$$VQ_c = [B \mid AB \mid A^2B \mid --- \mid A^{n-1}B]$$

Since $|V| = 1$, the rank of Q_c is the rank of VQ_c, and the corollary is proven.

CHAPTER II

Commutativity of Coordinate Truncation and Transformation Matrix Inversion for Flexible Spacecraft Dynamic Analysis†

P. Likins*

and

Y. Ohkami**

ABSTRACT. Necessary and sufficient conditions are established for the commutativity of matrix inversion involved in certain coordinate transformations and the coordinate truncations that are essential to practical structural dynamics. Computational constraints demand that truncations precede the generation of explicit transformed equations, while mathematical arguments imply the opposite sequence. Results jeopardize the utility of two of the three transformation procedures considered here for rotating flexible bodies.

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*Professor

**Postgraduate Research Engineer; on leave from National Aerospace Laboratory, Tokyo, Japan

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INTRODUCTION

It is now commonplace to characterize spacecraft with flexible appendages in terms of hybrid coordinates, consisting of discrete coordinates for rigid bodies in the system and distributed or modal coordinates for the flexible appendages. The selection of mode shapes and frequencies associated with the modal coordinates is usually accomplished by adopting a finite element model of the appendage and writing equations of motion for the small vibrations of the appendage with respect to a base which has a nominally constant (perhaps zero) inertial angular velocity; the eigenvectors of these equations then provide the mode shapes and the eigenvalues provide the natural frequencies of appendage vibration. This information is used in various ways to construct a transformation matrix which transforms the discrete nodal coordinates of the finite element model into distributed coordinates; these then are reduced in number by a process of coordinate truncation, which sacrifices mathematical rigor for computational feasibility (hopefully without doing violence to the salient features of the mathematical model). This paper addresses an aspect of the truncation problem which arises in the case of modal analysis of an elastic appendage on a rotating base. Several alternative transformations have been proposed for this case, but comparison among them is difficult without detailed numerical studies. In this paper we consider three alternative transformations, two of which involve the inversion of the transformation matrix. Whereas mathematical arguments demand that this matrix be inverted prior to truncation, practical considerations demand that truncation precede inversion (which then requires a pseudo-inverse). In this paper we establish necessary and sufficient conditions for the commutativity of truncation and inversion, in order to define the formal limits of these two procedures. We conclude with recommendations for the selection of coordinate transformations for elastic appendages on rotating bodies.

EQUATIONS OF APPENDAGE VIBRATION

As shown in Eq. (64) of [1], the equations of vibration of a finite element model of a flexible appendage on a rotating base have the structure

$$M'\ddot{q} + D'\dot{q} + G'\dot{q} + K'q + A'q = L' \qquad (1)$$

where for an appendage with n nodes q is the 6nx1 matrix of small deformations relative to a nominal (deformed) state, the matrices M', D', and K' are symmetric, and the matrices G' and A' are skew-symmetric, with M' positive definite and hence nonsingular. These equations are applicable whether the appendage mass is concentrated into rigid nodal bodies (as in [2], Eq. (140)), or also distributed over the finite elements of the model (as in [1]). The matrix L' in Eq. (1) holds the vibration forcing functions, which include base motions, but the coefficient matrices in Eq. (1) also depend on the inertial angular velocity of the base. If and only if this velocity experiences only small deviations from a nominal constant can these coefficient matrices be treated (after linearization) as constants.

COORDINATE TRANSFORMATIONS AND TRUNCATION FOR NONROTATING ELASTIC APPENDAGES

It is well known [3] that when the base is nonrotating and the appendage is undamped the homogeneous counterpart to Eq. (1) becomes

$$M'\ddot{q} + K'q = 0 \qquad (2)$$

and that the transformation to modal coordinates

$$q = \phi\eta \qquad (3)$$

followed by premultiplication by ϕ^T produces an uncoupled system of scalar equations

$$\ddot{\eta}_i + \sigma_i^2\eta_i = 0 \qquad i = 1,\ldots,6n \qquad (4)$$

where the columns of ϕ are the eigenvectors of Eq. (2) and σ_i^2 represents the

i^{th} eigenvalue. The inhomogeneous counterpart to Eq. (4) may be written as the scalar equation

$$\ddot{\eta}_i + \sigma_i^2 \eta_i = (\phi^T L')_i \qquad i = 1,\ldots,6n \tag{5}$$

where $(\phi^T L')_i$ is the i^{th} element of the 6nx1 matrix in parentheses. The vibration equations in Eq. (5) could theoretically be used in conjunction with the vehicle rotation equations to fully characterize the spacecraft attitude dynamics, but in practice this is not done because most of the modal coordinates η_i ($i = 1,\ldots,6n$) have no significant influence on the motion of the primary rigid body to which the flexible appendage is attached. Thus the modal vibration equations are ignored except for a small number $N < 6n$ (typically N ranges from one to twenty and 6n is measured in hundreds or thousands) and Eqs. (3) and (5) are replaced by

$$q = \bar{\phi}\bar{\eta} \tag{6}$$

and

$$\ddot{\eta}_i + \sigma_i^2 \eta_i = (\bar{\phi}^T L')_i \qquad i = 1,\ldots,N \tag{7a}$$

or

$$\ddot{\bar{\eta}} + \bar{\sigma}^2 \bar{\eta} = \bar{\phi}^T L' \tag{7b}$$

where $\bar{\eta}$ is the Nx1 matrix of truncated modal coordinates, $\bar{\sigma}^2$ is the NXN diagonal matrix of corresponding squared natural frequencies, and $\bar{\phi}$ is a Nx6n matrix whose columns are the eigenvectors corresponding to those modal coordinates which are retained. In practice one computes only the N eigenvectors in $\bar{\phi}$ and the corresponding eigenvalues, which normally include the lower eigenvalues; calculation of all 6n eigenvalues and eigenvectors would be not only prohibitively expensive but also computationally infeasible for typical values of n.

Thus in the special case of Eq. (1) represented by Eq. (2) the transformation matrix ϕ (in Eq. (3)) need not be inverted; since the required transposition of ϕ is commutative with truncation, there is no mathematical obstacle to the pragmatically motivated practice of truncating ϕ (as in Eq. (6)) before transposing it (as in Eq. (7)).

GENERAL PROBLEM OF TRANSFORMATION AND TRUNCATION OF COORDINATES

Although other special cases of Eq. (1) permit the use of special transformation matrices which circumvent all matrix inversion (see [1], pp. 726-729 and [2], pp. 46-56), we are here concerned with the more general case, for which there exists no transformation of the structure of Eq. (3) which transforms the homogeneous counterpart to Eq. (1) into uncoupled scalar second order equations such as those in Eq. (4). In the general case one must rewrite Eq. (1) as a first order matrix (state) equation before attempting a transformation to uncoupled scalar equations.

In terms of the state variables in the 12nx1 matrix

$$Q \triangleq \left[\begin{array}{c} q \\ \hline \dot{q} \end{array}\right] \tag{8}$$

Eq. (1) can be written as

$$\mathscr{A}\dot{Q} + \mathscr{B}Q = \mathscr{L} \tag{9}$$

or alternatively as

$$\dot{Q} = BQ + L \tag{10}$$

where

$$\mathscr{A} \triangleq \left[\begin{array}{c|c} K'+A' & 0 \\ \hline 0 & M' \end{array}\right] \quad ; \quad \mathscr{B} \triangleq \left[\begin{array}{c|c} 0 & -K'-A' \\ \hline K'+A' & D'+G' \end{array}\right]$$

$$B \triangleq \left[\begin{array}{c|c} 0 & U \\ \hline -(M')^{-1}(K'+A') & -(M')^{-1}(G'+D') \end{array}\right]$$

$$L \triangleq \left[\frac{0}{(M')^{-1}L'}\right] \qquad ; \qquad \mathscr{L} \triangleq \left[\frac{0}{L'}\right]$$

The eigenvalues $\lambda_1, \ldots, \lambda_{12n}$ and the eigenvectors $\Phi^1, \ldots, \Phi^{12n}$ associated with the differential operators on Q in Eqs. (9) and (10) are the same, and since all quantities in these equations are real we are assured that any complex eigenvalues or eigenvectors appear in conjugate pairs. In what follows we assume that the eigenvectors are linearly independent, so that the matrix Φ whose columns are these eigenvectors is nonsingular. The validity of this assumption is assured if the eigenvalues are distinct [4], and assured even in the presence of repeated eigenvalues if $A'=D'=0$ and the null solution of the homogeneous counterpart to Eq. (9) is Liapunov stable ([2], page 43).

The coordinate transformation

$$Q = \Phi Y \tag{11}$$

can then be used to good purpose in either Eq. (9) or Eq. (10). The first of these alternatives is developed in [1], with results only summarized here. If Φ' is the matrix whose columns are the eigenvectors of the equation adjoint to the homogeneous part of Eq. (9), namely,

$$\mathscr{A}^T \dot{Q}' + \mathscr{B}^T Q = 0$$

then by substituting Eq. (11) into Eq. (9) and premultiplying by Φ' one can obtain

$$\dot{Y} = \Lambda Y + (\Phi'^T \mathscr{A} \Phi)^{-1} \Phi'^T \mathscr{L} \tag{12}$$

where $(\Phi'^T \mathscr{A} \Phi)$ is diagonal and

$$\Lambda = (\Phi'^T \mathscr{A} \Phi)^{-1} (\Phi'^T \mathscr{B} \Phi)$$

is the diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_{12n}$.

As noted in the context of Eqs. (5) - (7), practical considerations mandate the truncation of the transformed coordinates, so that the 12nx1 matrix Y must be replaced by the 2Nx1 matrix $\overline{Y}$. It is irrelevant whether the truncation is imposed on Eq. (12) or on Φ in Eq. (11) and its counterpart Φ', as may be established formally by partitioning

$$\Phi \triangleq \left[\overline{\Phi} \;\middle|\; \overline{\overline{\Phi}}\right] \tag{13}$$

and

$$\Phi' \triangleq \left[\overline{\Phi}' \;\middle|\; \overline{\overline{\Phi}}'\right]$$

into the 12nx2N partitions with one overbar to be preserved in truncation and the remaining 12nx(12n-2N) matrices of eigenvectors to be removed in truncation, and writing

$$\left\{\begin{bmatrix} \overline{\Phi}'^T \\ \hline \overline{\overline{\Phi}}'^T \end{bmatrix}\begin{bmatrix} \mathscr{A} \end{bmatrix}\begin{bmatrix} \overline{\Phi} & \vline & \overline{\overline{\Phi}} \end{bmatrix}\right\}^{-1} = \left[\begin{array}{c|c} \overline{\Phi}'^T\mathscr{A}\overline{\Phi} & 0 \\ \hline 0 & \overline{\overline{\Phi}}'^T\mathscr{A}\overline{\overline{\Phi}} \end{array}\right]^{-1}$$

$$= \left[\begin{array}{c|c} (\overline{\Phi}'^T\mathscr{A}\overline{\Phi})^{-1} & 0 \\ \hline 0 & (\overline{\overline{\Phi}}'^T\mathscr{A}\overline{\overline{\Phi}})^{-1} \end{array}\right] \tag{14}$$

The indicated commutativity of truncation of the transformation matrix and inversion of $\Phi'^T\mathscr{A}\Phi$ is here established as a consequence of the diagonal structure of the latter. Obviously it is advantageous to truncate before inversion, because then there is no need even to calculate $\overline{\overline{\Phi}}$ and $\overline{\overline{\Phi}}'$, which are generally of much larger dimension than $\overline{\Phi}$ and $\overline{\Phi}'$. Thus Eq. (12) can for practical purposes be replaced by its truncated counterpart

$$\dot{\overline{Y}} = \overline{\Lambda}\overline{Y} + (\overline{\Phi}'^T\mathscr{A}\overline{\Phi})^{-1}\overline{\Phi}'^T\mathscr{L} \tag{15}$$

If the transformation in Eq. (11) is introduced into Eq. (10) rather than into Eq. (9), then the calculation of the adjoint eigenvectors is not necessary, since premultiplication of the result by Φ^{-1} produces

$$\dot{Y} = \Lambda Y + \Phi^{-1}L \tag{16}$$

If truncation is imposed on Eq. (16), the result is

$$\dot{\overline{Y}} = \overline{\Lambda Y} + (\overline{\Phi^{-1}})L \tag{17}$$

where $(\overline{\Phi^{-1}})$ consists of the first 2N rows of the matrix Φ^{-1}. As a practical matter, however, Φ^{-1} is not generally available, since for typical values of n it is not feasible even to calculate all of the eigenvectors comprising the columns of Φ. In order to circumvent the problem, it was proposed in [2] that Eq. (17) be replaced by

$$\dot{\overline{Y}} = \overline{\Lambda Y} + (\overline{\Phi})^{\dagger}L \tag{18}$$

where

$$\overline{\Phi}^{\dagger} \triangleq (\overline{\Phi}^T\overline{\Phi})^{-1}\overline{\Phi}^T \tag{19}$$

is a pseudoinverse of the 12nx2N matrix $\overline{\Phi}$. This substitution amounts to the assumption that inversion and truncation of Φ are commutative operations.

COUNTER-EXAMPLE

Dr. William Hooker of the Lockheed Palo Alto Research Laboratories has noted (in personal correspondence) that the above assumption is not always valid, as demonstrated by a counter-example such as the following:

If

$$\Phi = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

then

$$\Phi^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

and

$$(\overline{\Phi^{-1}}) = \frac{1}{2} [\ -4 \quad 3\ ]$$

whereas

$$\overline{\Phi}^{\dagger} = (\overline{\Phi}^T\overline{\Phi})^{-1}\overline{\Phi}^T = \frac{1}{5} [\ 1 \quad 2\ ] \neq (\overline{\Phi^{-1}})$$

Dr. Hooker pointed out that Eq. (18) is the formal consequence of the substitution of

$$Q = \bar{\Phi}\bar{Y} \tag{20}$$

instead of Eq. (11) into Eq. (10), noting that while this substitution may appear to be plausible its consequences are not equivalent to Eq. (17).

PURPOSE

Since in practical terms Eq. (17) is not directly available and Eq. (18) is available upon inversion of the 2Nx2N (generally complex) matrix $\bar{\Phi}^T\bar{\Phi}$, it is important that we discover those conditions under which these equations are equivalent. This is the primary purpose of this paper.

CONDITIONS FOR COMMUTATIVITY OF INVERSION AND TRUNCATION

Proposition: For any nonsingular square matrix M partitioned as

$$M = \left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]$$

and its truncated counterpart

$$\bar{M} \triangleq \left[\begin{array}{c} a \\ \hline c \end{array}\right]$$

the condition

$$c^T d + a^T b = 0 \tag{21}$$

is necessary and sufficient for the equivalence

$$\bar{M}^{\dagger} = \overline{(M^{-1})} \tag{22}$$

where

$$\bar{M}^{\dagger} = (\bar{M}^T\bar{M})^{-1}\bar{M}^T \tag{23}$$

and $\overline{(M^{-1})}$ is the uppermost row-partition of M^{-1} having the dimensions of $\bar{M}^T$.

Proof: Since [5]

$$M^{-1} = \left[\begin{array}{c|c} (a-bd^{-1}c)^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ \hline -d^{-1}c(a-bd^{-1}c)^{-1} & (d-ca^{-1}b)^{-1} \end{array}\right] \tag{24}$$

then

$$\overline{(M^{-1})} = \left[\begin{array}{c|c} (a-bd^{-1}c)^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \end{array}\right]$$

while

$$\overline{M}^{\dagger} = \left(\left[\begin{array}{c|c} a^T & c^T \end{array}\right]\left[\begin{array}{c} a \\ \hline c \end{array}\right]\right)^{-1}\left[\begin{array}{c|c} a^T & c^T \end{array}\right] = \left[a^Ta+c^Tc\right]^{-1}\left[\begin{array}{c|c} a^T & c^T \end{array}\right]$$

$$= \left[\begin{array}{c|c} (a^Ta+c^Tc)^{-1}a^T & (a^Ta+c^Tc)^{-1}c^T \end{array}\right]$$

As necessary and sufficient conditions for $\overline{(M^{-1})} = \overline{M}^{\dagger}$ we have

$$(a^Ta+c^Tc)^{-1}a^T = (a-bd^{-1}c)^{-1} \tag{25}$$

and

$$(a^Ta+c^Tc)^{-1}c^T = -a^{-1}b(d-ca^{-1}b)^{-1} \tag{26}$$

The identities

$$(a^Ta+c^Tc)^{-1}a^T = (a^Ta+c^Tc)^{-1}[(a^T)^{-1}]^{-1} = [(a^T)^{-1}(a^Ta+c^Tc)]^{-1}$$

and similarly

$$(a^Ta+c^Tc)^{-1}c^T = [(c^T)^{-1}(a^Ta+c^Tc)]^{-1}$$

and

$$-a^{-1}b(d-ca^{-1}b)^{-1} = -[(d-ca^{-1}b)(a^{-1}b)^{-1}]^{-1}$$

permit the inverses of Eqs. (25) and (26) to be written respectively as

$$(a^T)^{-1}(a^Ta+c^Tc) = a-bd^{-1}c$$

and

$$(c^T)^{-1}(a^Ta+c^Tc) = -(d-ca^{-1}b)(a^{-1}b)^{-1}$$

or as

$$a + (a^T)^{-1}c^Tc = a-bd^{-1}c$$

and

$$(a^Ta+c^Tc)(a^{-1}b) = -c^T(d-ca^{-1}b)$$

or as

$$[(a^T)^{-1}c^T+bd^{-1}]c = 0$$

and

$$a^Tb + c^Tca^{-1}b + c^Td - c^Tca^{-1}b = 0$$

or as

$$(c^Td+a^Tb)d^{-1}c = 0 \tag{27}$$

and

$$(c^Td+a^Tb) = 0 \tag{28}$$

Satisfaction of Eq. (21) is sufficient for Eq. (27) and both necessary and sufficient for Eq. (28), so the Proposition is proven.

INTERPRETATIONS FOR STRUCTURAL DYNAMICS

Eq. (22) is obviously valid when M is block diagonal, since then b=c=0 and Eq. (21) is trivially satisfied. This result is consistent with Eq. (14).

Eq. (22) is also valid when

$$c = -a \quad \text{and} \quad d = b \tag{29}$$

and this result has an important physical interpretation for structural dynamics.

In assessing conditions for the equivalence of Eqs. (17) and (18), we must recall that the columns of Φ are the eigenvectors of Eqs. (9) and (10), and hence the "mode shapes" of independent structural vibrations for certain boundary conditions. The columns of the partitions $\overline{\Phi}$ and $\overline{\overline{\Phi}}$ (see Eq. (14)) of Φ represent respectively those modes which are preserved and those which are abandoned in truncation. Although several truncation criteria are useful, when the elastic structure is an appendage on a rigid body over which one must maintain precise attitude control but not precise position control (as in the spacecraft problem) the essential criterion is the truncation of modes which have no influence on the attitude of the primary rigid body to which the appendage is attached. If the mode shape is symmetric with respect to the primary body, as in Fig. 1, then clearly it must be removed in truncation, while antisymmetric modes, as in Fig. 2, must be preserved. In the special case in which all modes are either symmetric or antisymmetric, it becomes possible to formally justify removal of the former by the proposition of the previous section, as will be shown.

Eq. (8) indicates that Eq. (13) must in greater detail be representable as

$$\Phi = \left[\begin{array}{c|c} \bar{\Phi} & \bar{\bar{\Phi}} \\ \hline \bar{\Phi}\bar{\Lambda} & \bar{\bar{\Phi}}\bar{\bar{\Lambda}} \end{array}\right] \tag{30}$$

where $\bar{\lambda}$ and $\bar{\bar{\lambda}}$ are diagonal matrices of eigenvalues in the preserved and deleted categories, respectively, while $\bar{\phi}$ and $\bar{\bar{\phi}}$ establish corresponding mode shapes for the position variables in q. Application of the criterion in Eq. (21) to Eq. (30) produces the requirement

$$\overline{\lambda\phi}^T\bar{\bar{\phi}}\bar{\bar{\lambda}} + \bar{\phi}^T\bar{\bar{\phi}} = 0 \tag{31}$$

for the equivalence of Eqs. (17) and (18), and this is for all practical purposes never satisfied, since it reduces to the requirement

$$\bar{\lambda}_\alpha\bar{\bar{\lambda}}_\beta = 1 \qquad (\alpha = 1,\ldots,N; \quad \beta = 1,\ldots,6n-N) \tag{32}$$

However, if the problem is restructured so that Eq. (8) is replaced by

$$\tilde{Q} \triangleq [q_1\dot{q}_1q_2\dot{q}_2 \cdots q_{6n}\dot{q}_{6n}]^T \tag{33}$$

in which $q_1,\ldots,q_{3n}$ and $q_{3n+1},\ldots,q_{6n}$ are coordinates defining parallel motions of appendage nodes located symmetrically with respect to a central rigid body (as would be possible for the system in Figs. 1 and 2, but not generally so), then instead of Eq. (30) we can write

$$\tilde{\Phi} = \left[\begin{array}{c|c} A & S \\ \hline -A & S \end{array}\right] \tag{34}$$

where the left partitions (comprising $\tilde{\Phi}$) contain the antisymmetric modes and the right partitions contain the symmetric modes. Now Eq. (29) is satisfied, and (with equations restructured as required by Eq. (33)) Eqs. (17) and (18) are identical, since inversion and truncation are commutative.

SPECIAL CASE WITH $D' = A' = 0$

When energy dissipation is ignored, and any rigid nodal bodies in the appendage model are in the nominal state spinning about principal axes of inertia, then $D' = A' = 0$ [1] in Eq. (1), which becomes

$$M'\ddot{q} + G'\dot{q} + K'q = L' \tag{35}$$

In the undamped stable case of interest here all eigenvalues from Eqs. (9) and (10) are imaginary, and may be designated in conjugate pairs as $\pm i\sigma_\alpha$ for $\alpha = 1,\ldots,6n$. The corresponding eigenvectors are however complex, suggesting the designation

$$\Phi^\alpha = \Psi^\alpha + i\Gamma^\alpha \tag{36}$$

where the form of Eq. (8) requires

$$\Psi^\alpha = \left[\begin{array}{c} \psi^\alpha \\ \hline -\gamma^\alpha\sigma_\alpha \end{array}\right] \quad \text{and} \quad \Gamma^\alpha = \left[\begin{array}{c} \gamma^\alpha \\ \hline \psi^\alpha\sigma_\alpha \end{array}\right] \tag{37}$$

If in Eq. (35) we have $L'=0$, we can without contradiction substitute

$$q = \psi z + \gamma\sigma^{-1}\dot{z} \tag{38a}$$

$$\dot{q} = -\gamma\sigma z + \psi\dot{z} \tag{38b}$$

$$\ddot{q} = -\gamma\sigma\dot{z} + \psi\ddot{z} \tag{38c}$$

into the homogeneous part of Eq. (35) to obtain

$$M'(-\gamma\sigma\dot{z} + \psi\ddot{z}) + G'(-\gamma\sigma z + \psi\dot{z}) + K'(\psi z + \gamma\sigma^{-1}\dot{z}) = 0$$

or

$$M'\psi\ddot{z} + (-M'\gamma\sigma + G'\psi + K'\gamma\sigma^{-1})\dot{z} + (-G\gamma\sigma + K'\psi)z = 0 \tag{39}$$

A homogeneous solution of Eq. (35) is

$$q_h = (\psi^\alpha + i\gamma^\alpha)e^{i\sigma_\alpha t} \tag{40}$$

which when substituted yields

$$M'(\psi^\alpha + i\gamma^\alpha)(-\sigma_\alpha^2) + G'(\psi^\alpha + i\gamma^\alpha)(i\sigma_\alpha) + K'(\psi^\alpha + i\gamma^\alpha) = 0$$

or

$$-M'\psi^\alpha\sigma_\alpha{}^2 - G'\gamma^\alpha\sigma_\alpha + K'\psi^\alpha = 0 \tag{41a}$$

and

$$-M'\gamma^\alpha\sigma_\alpha{}^2 + G'\psi^\alpha\sigma_\alpha + K'\gamma^\alpha = 0 \tag{41b}$$

and Eqs. (41a) and (41b) considered for $\alpha = 1,\ldots,6n$ imply

$$-M'\psi\sigma^2 - G'\gamma\sigma + K'\psi = 0 \tag{42a}$$

$$-M'\gamma\sigma + G'\psi + K'\gamma\sigma^{-1} = 0 \tag{42b}$$

Thus with Eq. (42b) Eq. (39) reduces to

$$M'\psi\ddot{z} + (-G\gamma\sigma + K'\psi)z = 0$$

which after premultiplication by ψ^T becomes

$$\psi^T M'\psi\ddot{z} + (\psi^T K'\psi - \psi^T G\gamma\sigma)z = 0$$

and with ψ^T times Eq. (42a) this is

$$\psi^T M'\psi(\ddot{z} + \sigma^2 z) = 0$$

or with nonsingular $\psi^T M'\psi$,

$$\ddot{z} + \sigma^2 z = 0 \tag{43}$$

This same result arises if Eq. (38b) is equated to the derivative of Eq. (38a), so that contradiction is avoided. The transformations in Eq. (38) are <u>not</u> acceptable in the inhomogeneous case of Eq. (35) of primary interest here, however, because these transformations directly imply Eq. (43). Instead

Eq. (35) must in the inhomogeneous case be written in first order form, as in Eqs. (9) and (10). In the latter case, one can accomplish the desired transformation to uncoupled (pairs of) scalar equations with the real transformation

$$Q = PZ \tag{44}$$

where

$$P \triangleq \left[\begin{array}{c|c} \psi & \gamma \\ \hline -\gamma\sigma & \psi\sigma \end{array}\right] \tag{45}$$

and P^{-1} exists if the eigenvectors of the system are independent, as required for stability in this undamped case. Substituting Eq. (45) into Eq. (10) with $D' = A' = 0$ and premultiplying by P^{-1} produces

$$\dot{Z} = \left[\begin{array}{c|c} 0 & \sigma \\ \hline -\sigma & 0 \end{array}\right] Z + P^{-1}L \tag{46}$$

as may be confirmed by considering the homogeneous case and defining

$$Z = \left[\begin{array}{c} z \\ \hline \sigma^{-1}\dot{z} \end{array}\right]$$

Eq. (46) offers a substantial advantage over Eqs. (12) and (16), which involve complex numbers in Y, Λ, Φ, and Φ', although Eq. (12) possesses an advantage in requiring the inversion of only a diagonal matrix. The inversion of P in Eq. (46) becomes particularly critical when (as required for practical structural dynamics) this inversion is to be followed by, or precede, coordinate truncation. We must once again face the critical question raised in this paper, "Are truncation and inversion commutative?" More explicitly, we must determine whether or not the truncation of Eq. (46) to

$$\dot{\overline{Z}} = \left[\begin{array}{c|c} 0 & \overline{\sigma} \\ \hline -\overline{\sigma} & 0 \end{array}\right] \overline{Z} + \overline{(P^{-1})}L \tag{47}$$

provides the same result as would be obtained by substituting

$$Q = \bar{P}\bar{Z} \triangleq \left[\begin{array}{c|c} \bar{\psi} & \bar{\gamma} \\ \hline -\bar{\gamma}\bar{\sigma} & \bar{\psi}\bar{\sigma} \end{array}\right] \bar{Z} \tag{48}$$

into Eq. (10) with $D' = A' = 0$ and premultiplying by

$$\bar{P}^{t} \triangleq (\bar{P}^{T}\bar{P})^{-1}\bar{P}^{T} \tag{49}$$

This issue was not confronted in [2], where an affirmative answer was presumed. This presumption requires for its validity that

$$\bar{P}^{\dagger}B\bar{P} = \left[\begin{array}{c|c} 0 & \bar{\sigma} \\ \hline -\bar{\sigma} & 0 \end{array}\right] \tag{50}$$

and

$$\bar{P}^{\dagger} = \overline{(P^{-1})} \tag{51}$$

Eq. (50) is proven in an appendix of the referenced report which underlies [1], but the validity of Eq. (51) has not been examined previously. With the proposition presented as the central result of this paper (see Eqs. (21) and (22)), it becomes evident that Eq. (51) is correct if any only if

$$-\sigma^{T}\gamma^{T}\psi\sigma + \psi^{T}\gamma = 0 \tag{52}$$

and this is an untenable hypothesis. Eq. (51) must be rejected, and the very appealing real transformation represented by Eq. (48) cannot be accepted as the formal equivalent of the truncation of the rigorously valid Eq. (46).

CONCLUSIONS

The necessary and sufficient conditions presented in Eq. (21) for the commutativity of coordinate truncation and transformation matrix inversion (see Eq. (22)) are quite severe. Two of the three transformation procedures considered here for flexible appendages on constantly rotating rigid bodies are essentially destroyed by the noncommutativity of these operations, since for systems of realistic dimension it is not feasible to perform the necessary matrix inversions prior to truncation, as an honest interpretation of the mathematics of the problem seems to require. This discouraging result is not yet a definitive conclusion, because truncation is at best a process of approximation. It may yet be true that transformations such as Eqs. (20) and (48) give satisfactory results in many cases; at this point however there appears to be no acceptable evidence supporting this possibility.

The only one of the three transformation procedures considered here which survives the commutativity test is that culminating in Eq. (15). This procedure involves no inversion except for a diagonal matrix, but it is a transformation to complex coordinates, and it generally requires not only the selected system eigenvectors in $\overline{\Phi}$ but also the corresponding adjoint eigenvectors in $\overline{\Phi}'$. In the important special case for which $D' = A' = 0$ this procedure is much simplified by the relationship [1]

$$\overline{\Phi}' = \overline{\Phi}^* \tag{53}$$

but unless the appendage is nonrotating (so that $G' = 0$) the transformations involve complex numbers.

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Figure 1. Symmetric Mode.

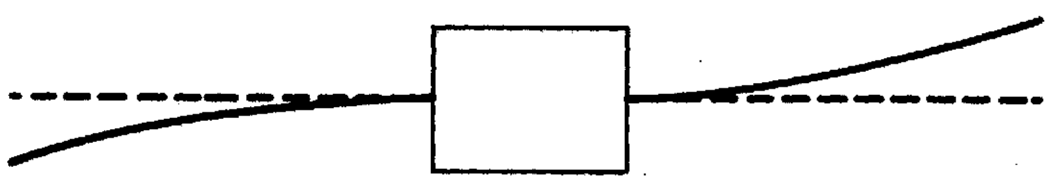

Figure 2. Antisymmetric Mode.

CHAPTER III

MATCHED ASYMPTOTIC EXPANSION MODAL ANALYSIS OF ROTATING BEAMS*

André D. Colin†

and

Peter W. Likins††

University of California, Los Angeles

ABSTRACT

Modal analyses are presented for the transverse vibrations of a uniform Euler-Bernoulli beam, rotating about an axis orthogonal to the beam. Results are obtained for a cantilever beam emanating radially from a base which is rotating but not translating in inertial space, and for a beam with built-in ends spanning the diameter of a ring which is rotating about its inertially fixed symmetry axis. The equations are formulated in terms of a small parameter ε which is proportional to beam stiffness and inversely related to mass density, length, rotation rate, and the pretension of the diametral beam. The resulting singular perturbation problem is analyzed by the method of matched asymptotic expansions, employing asymptotic expansions for the central region and for boundary layers at either end. Results are expressed literally, with sample numerical examples.

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†-------------------, Belgian Army; formerly UCLA graduate student.

††Professor, Mechanics and Structures Department; Associate Fellow, AIAA.

MATCHED ASYMPTOTIC EXPANSION

MODAL ANALYSIS OF ROTATING BEAMS

Introduction

The rotating elastic beam has technological significance in the development of helicopters and spin-stabilized space vehicles, and it has been analyzed extensively. Because the linearized partial differential equations of vibration have coefficients depending on the independent spatial variable, closed form exact solutions are not available, and recourse to approximate methods is necessary. Most investigators have adopted numerical analysis procedures, obtaining results that are limited to the particular parameter values considered. Yntema[1] provides a good illustration of the extensive application of Galerkin's method to many specific cases, expanding the modes sought in terms of the nonrotating beam modes; Renard and Rakowski[2] and Hughes and Fung[3] use similar numerical methods to address specific problems of interest in their work.

In this paper we seek literal expressions for approximate representations of natural frequencies and mode shapes, using the method of matched asymptotic expansions.[4] Modal analyses of rotating structures have been developed previously by Boyce and Handelman[5] and by Abel and Kerr.[6] In the former paper the authors generate zeroth order solutions for a rotating beam with a tip mass, applying a method developed by Moser[7] in which the linearly independent solutions are taken in the form of $B(x,\eta) \exp[\eta^{-1}h(x)]$, where η is of the order of magnitude of the small perturbation parameter of the problem and $B(x,\eta)$ and $h(x)$ are functions to be determined. In the paper by Abel and Kerr[6] the method of asymptotic expansions is applied to a rotating cable-counterweighted space station for cables with small flexural rigidity. Matching of central and boundary layer expansions is employed to establish those constants of integration not established by boundary conditions.

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In what follows the rotating Euler-Bernoulli beam vibration equations are presented and a central solution expansion developed, and then boundary layer expansions are developed and matched for the boundary conditions appropriate for rotating beams of both "radial" and "diametral" configuration.

Rotating Beam Equations

The small transverse vibration $W(\xi,t)$ of an Euler-Bernoulli beam with constant flexural stiffness EI and constant mass per unit length μ, under the external axial load per unit length $P(\xi)$, is characterized by[8]

$$EI \frac{\partial^4 W}{\partial \xi^4} - \frac{\partial}{\partial \xi}\left[P(\xi) \frac{\partial W}{\partial \xi}\right] + \mu \frac{\partial^2 W}{\partial t^2} = 0 \tag{1}$$

In what follows we address two distinct problems of beam vibration: the radial rotating beam (Fig. 1), and the diametral rotating beam (Fig. 2). In each case the beam is attached to a body B which has a prescribed constant inertial angular velocity $\Omega \underline{\hat{e}}_\zeta$ with the ξ,η,ζ cartesian coordinate system axes fixed in B and its origin 0 fixed both in B and in inertial space. In each case the beam supports are built into B, constrained against translation and rotation relative to B. Thus the boundary conditions are as follows:

$$\text{Radial Beam: } W(0,t) = \frac{\partial W}{\partial \xi}(0,t) = \frac{\partial^2 W}{\partial \xi^2}(L,t) = \frac{\partial^3 W}{\partial \xi^3}(L,t) = 0 \tag{2}$$

$$\text{Diametral Beam: } W(L,t) = W(-L,t) = \frac{\partial W}{\partial \xi}(L,t) = \frac{\partial W}{\partial \xi}(-L,t) = 0 \tag{3}$$

In each case the rotation induces axial strains in the beam, with accompanying change in transverse stiffness characteristics. As has been formally established by the use of nonlinear strain-displacement equations,[9] one can for the small strain case continue to represent small transverse vibrations by Equation (1), incorporating into $P(\xi)$ the "centrifugal force." Thus for the radial beam

$$P(\xi) = \int_{\xi}^{L} \mu\Omega^2\bar{\xi}d\bar{\xi} = \mu\Omega^2(L^2-\xi^2)/2 \tag{4}$$

For the diametral beam, a pretensile load T is required to avoid instability of the solution $W(\xi,t) \equiv 0$; thus the effective axial "force" is

$$P(\xi) = T - \int_{0}^{\xi} \mu\Omega^2\bar{\xi}d\bar{\xi} = T - \mu\Omega^2\xi^2/2 \tag{5}$$

with

$$T > \mu\Omega^2L^2/2 \tag{6}$$

By defining k^2 such that

$$T \triangleq k^2\mu\Omega^2L^2/2$$

one can write Equations (4) and (5) together as

$$P(\xi) = \mu\Omega^2(k^2L^2-\xi^2)/2 \tag{8}$$

where for the radial beam $k^2 = 1$, and for the diametral beam $k^2 > 1$.

Equation (1) can now be written in a form applicable to both radial and diametral beams as

$$EI\,\frac{\partial^4 W}{\partial\xi^4} - \frac{\mu\Omega^2}{2}\,\frac{\partial}{\partial\xi}\left[(k^2L^2-\xi^2)\,\frac{\partial W}{\partial\xi}\right] + \mu\,\frac{\partial^2 W}{\partial t^2} = 0 \tag{9}$$

Representation in terms of dimensionless quantities is accomplished by introducing

$$w \triangleq W/L$$

$$x \triangleq \xi/(Lk)$$

$$\tau \triangleq \Omega t$$

and dividing Equation (1) by $\mu\Omega^2L/2$, to obtain

$$\varepsilon\,\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left[(1-x^2)\,\frac{\partial w}{\partial x}\right] + 2\,\frac{\partial^2 w}{\partial\tau^2} = 0 \tag{10}$$

where

$$\varepsilon \triangleq \frac{2EI}{\mu\Omega^2 L^4 k^4} \tag{11}$$

Equation (10) admits the separable product solution

$$w(x,\tau) = \phi(x)\ \sigma(\tau) \tag{12}$$

with

$$\varepsilon\phi^{IV} - [(1-x^2)\phi']' - \lambda^2\phi = 0 \tag{13}$$

and

$$\ddot{\sigma} + \frac{1}{2}\lambda^2\sigma = 0 \tag{14}$$

for some constant λ^2, with prime denoting d/dx and dot denoting d/dτ. Those discrete constant values of λ^2 for which Equation (13) has nonzero solutions are the system <u>eigenvalues</u>.

In terms of the separated dimensionless quantities in Equation (13), the boundary conditions become for the radial beam (for which k = 1)

$$\phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0 \tag{15}$$

from Equation (2), and for the diametral beam

$$\phi(k^{-1}) = \phi(-k^{-1}) = \phi'(k^{-1}) = \phi'(-k^{-1}) = 0 \tag{16}$$

from Equation (3).

The objective of this paper is to obtain explicit solutions for the modal functions $\phi(x)$ in Equation (13) as asymptotic expansions in ε, for boundary conditions as in Equations (15) and (16). This process must also yield asymptotic expansions for the system eigenvalues, which produce natural frequencies of vibration from Equation (14) and the time normalization $\tau = \Omega t$ as

$$\omega = \lambda\Omega/\sqrt{2} \tag{17}$$

The solution technique employed is the method of matched asymptotic expansions, as presented by Cole.[4] Because in the limit as $\varepsilon \to 0$ Equation (13) becomes only a second order differential equation, the solution of which cannot satisfy the four boundary conditions in Equation (15) or Equation (16), this problem is classified as singular. The solution will be obtained as the combination of an asymptotic expansion valid in the central region of the beam, obtained from the reduced order differential equation arising in the limit $\varepsilon \to 0$, and asymptotic expansions valid near the boundaries of the beam.

Central Solution Expansion

For the central region of the beam, we represent the dependence of the solution of Equation (13) upon the small parameter ε by the asymptotic expansion

$$\phi(x,\varepsilon) = h_0(x) + \nu_1(\varepsilon)h_1(x) + \nu_2(\varepsilon)h_2(x) + \ldots \tag{18}$$

where $\nu_i(\varepsilon)$ is for $i = 1,2,\ldots$ an asymptotic sequence to be established. Similarly the eigenvalues represented by λ^2 in Equation (13) are expanded as

$$\lambda^2 = \Lambda_0^2 + \kappa_1(\varepsilon)\Lambda_1^2 + \kappa_2(\varepsilon)\Lambda_2^2 + \ldots \tag{19}$$

for some asymptotic sequence $\kappa_i(\varepsilon)$, $i = 1,2,\ldots$

Substitution of Equations (18) and (19) into Equation (13) and consideration of the limit $\varepsilon \to 0$ requires

$$[(1-x^2)h_0']' + \Lambda_0^2 h_0 = 0 \tag{20}$$

Equation (20) is a Legendre equation of order n, where $n(n+1) = \Lambda_0^2$. The general solution of Equation (20) is given by[10]

$$h_0 = AU_n(x) + BV_n(x) \tag{21}$$

where

$$U_n(x) \triangleq 1 - \frac{\Lambda_0^2}{2!}x^2 + \frac{\Lambda_0^2(\Lambda_0^2-6)}{4!}x^4 - \ldots \tag{22}$$

and

$$V_n(x) \triangleq x - \frac{(\Lambda_0^2-2)}{3!}x^3 + \frac{(\Lambda_0^2-2)(\Lambda_0^2-12)}{5!}x^5 - \ldots \tag{23}$$

The Legendre functions $U_n(x)$ and $V_n(x)$ converge for arbitrary Λ_0^2 over the range $-1 < x < 1$, and either $U_n(x)$ or $V_n(x)$ terminates for those discrete values of Λ_0^2 for which

$$\Lambda_0^2 = n(n+1) \qquad n = 0,1,2,\ldots \tag{24}$$

with $U_n(x)$ terminating for n even and $V_n(x)$ terminating for n odd. The terminated series can then be expressed in terms of a Legendre polynomial $P_n(x)$ by

$$U_n(x) = P_n(x)U_n(1) \qquad n = 0,2,4,\ldots \tag{25a}$$

or

$$V_n(x) = P_n(x)V_n(1) \qquad n = 1,3,5,\ldots \tag{25b}$$

and the nonterminating series can be written in terms of Legendre functions of the second kind, $Q_n(x)$, which are unbounded in the limit $x = 1$. These observations are particularly significant for the radial beam, as indicated by the boundary conditions in Equations (15) and (16).

In addition to the zeroth order approximation given by Equation (20), one obtains higher order term equations from the substitution of Equations (18) and (19) into Equation (13). If no restrictions on $\nu_1(\varepsilon)$ or $\kappa_1(\varepsilon)$ were stipulated, the first equation involving terms above the zeroth order would be

$$\varepsilon h_0{}^{IV} - \nu_1(\varepsilon)[(1-x^2)h_1']' - \nu_1(\varepsilon)\Lambda_0^2 h_1(\varepsilon) = \kappa_1(\varepsilon)\Lambda_1^2 h_0 \tag{26}$$

Further progress requires knowledge of the order relationship among $\kappa_1(\varepsilon)$, $\nu_1(\varepsilon)$ and ε. The possibility that* $\kappa_1(\varepsilon) = o(\nu_1(\varepsilon))$ and $\kappa_1(\varepsilon) = o(\varepsilon)$ is rejected, because Equation (26) then indicates that $\Lambda_1^2 = 0$. Similarly the possibility that $\varepsilon = o(\nu_1(\varepsilon))$ and $\varepsilon = o(\kappa_1(\varepsilon))$ is rejected because of the implication that $h_0{}^{IV} = 0$. Subject to later confirmation, we assume that $\nu_1(\varepsilon) = o(\varepsilon)$ and $\kappa_1(\varepsilon) = o(\varepsilon)$. Thus Equation (26) becomes either

*By notational convention $a(\ ) = o(b(\ ))$ implies $\lim_{\varepsilon \to 0} (\frac{a}{b}) = 0$.

$$[(1-x^2)h_1']' + \Lambda_0^2 h_1 = -\Lambda_1^2 h_0 \qquad [\nu_1(\varepsilon) = \kappa_1(\varepsilon)] \tag{27a}$$

or

$$[(1-x^2)h_1']' + \Lambda_0^2 h_1 = 0 \qquad [\nu_1(\varepsilon) = o(\kappa_1(\varepsilon))] \tag{27b}$$

In order to proceed further with this calculation we require $\nu_1(\varepsilon)$, which must be established by the requirements of matching the expansions which are valid at the boundaries. These expansions are different for the radial beam and the diametral beam, and these cases must be treated separately.

Radial Beam Inner Expansion

In the neighborhood of the boundary x = 0, the term $\varepsilon\phi^{IV}$ in Equation (13) becomes important, and the previous solution is invalid. In order to develop a solution which is valid in the "boundary layer" near x = 0, we introduce a stretched coordinate

$$\tilde{x} \triangleq x\,\sigma^{-1}(\varepsilon) \tag{28}$$

assuming $\sigma(\varepsilon) \to 0$ when $\varepsilon \to 0$. Into the original Equation (13) we now substitute the asymptotic expansion

$$\phi(x,\varepsilon) = \rho_0(\varepsilon)g_0(\tilde{x}) + \rho_1(\varepsilon)g_1(\tilde{x}) + \rho_2(\varepsilon)g_2(\tilde{x}) + \ldots \tag{29}$$

where $\rho_i(\varepsilon)$ constitute an asymptotic sequence for i = 0,1,2,..., to obtain

$$\varepsilon\left[\frac{\rho_0(\varepsilon)}{\sigma^4(\varepsilon)}\frac{d^4g_0(\tilde{x})}{d\tilde{x}^4} + \frac{\rho_1(\varepsilon)}{\sigma^4(\varepsilon)}\frac{d^4g_1(\tilde{x})}{d\tilde{x}^4} + \ldots\right]$$

$$- (1-\sigma^2(\varepsilon)\tilde{x}^2)\left[\frac{\rho_0(\varepsilon)}{\sigma^2(\varepsilon)}\frac{d^2g_0(\tilde{x})}{d\tilde{x}^2} + \frac{\rho_1(\varepsilon)}{\sigma^2(\varepsilon)}\frac{d^2g_1(\tilde{x})}{d\tilde{x}^2} + \ldots\right]$$

$$+ 2\sigma(\varepsilon)\tilde{x}\left[\frac{\rho_0(\varepsilon)}{\sigma(\varepsilon)}\frac{dg_0}{d\tilde{x}} + \frac{\rho_1(\varepsilon)}{\sigma(\varepsilon)}\frac{dg_1(\tilde{x})}{d\tilde{x}} + \ldots\right]$$

$$- \Lambda_0^2\,[\rho_0(\varepsilon)\,g_0(\tilde{x}) + \rho_1(\varepsilon)\,g_1(\tilde{x}) + \ldots]$$

$$- \kappa_1(\varepsilon)\Lambda_1^2[\rho_0(\varepsilon)\,g_0(\tilde{x}) + \rho_1(\varepsilon)\,g_1(\tilde{x}) + \ldots] = 0$$

The coordinate scaling factor $\sigma(\varepsilon)$ is chosen so as to maintain the highest order derivative and an additional term or terms in the lowest order approximation of this expansion. The choice

$$\sigma = \varepsilon^{1/2} \tag{30}$$

then produces the dominant boundary layer equation

$$\frac{d^4 g_0(\tilde{x})}{d\tilde{x}^4} - \frac{d^2 g_0(\tilde{x})}{d\tilde{x}^2} = 0 \tag{31}$$

which after two integrations provides

$$\frac{d^2 g_0(\tilde{x})}{d\tilde{x}^2} = C_0 e^{-\tilde{x}} + C_1 e^{\tilde{x}} \tag{32}$$

The inner expansion in Equation (29) in the limit as $\tilde{x} \to \infty$ must match the central expansion in Equation (18) in the limit as $x \to 0$, so that C_1 in Equation (32) must be zero. Two more integrations then provide $g_0(\tilde{x})$, which with the boundary conditions

$$g_0(\tilde{x})\Big|_{\tilde{x}=0} = \frac{dg_0}{d\tilde{x}}\Big|_{\tilde{x}=0} = 0 \tag{33}$$

must be

$$g_0(\tilde{x}) = C_0(\tilde{x} - 1 + e^{-\tilde{x}}) \tag{34}$$

In order to match the approximation of the inner expansion (Equation (29)) provided by Equation (34) with the approximation of the central expansion (Equation (18)) provided by Equation (21) we can introduce an intermediate limit defined by the coordinate measure

$$x_\eta \triangleq x\eta^{-1}(\varepsilon)$$

with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\varepsilon^{1/2} = o(\eta(\varepsilon))$, so that in the limit as $\varepsilon \to 0$ with x_η fixed, $x = \eta(\varepsilon)x_\eta \to 0$ and $\tilde{x} = \eta(\varepsilon)\varepsilon^{-1/2}x_\eta \to \infty$. Matching then requires

$$\lim_{\substack{\varepsilon\to 0 \\ x_\eta \text{ fixed}}} \{h_0(\eta x_\eta) + \nu_1(\varepsilon)h_1(\eta x_\eta) + \ldots - \rho_0(\varepsilon)g_0(\eta\varepsilon^{-1/2}x_\eta) + \ldots\} = 0 \tag{35}$$

Expanding $h_0(x)$ and $h_1(x)$ as Taylor series about the origin $x = 0$ and substituting Equation (34) for $g_0(\tilde{x})$, we find

$$\lim_{\substack{\varepsilon\to 0 \\ x_\eta \text{ fixed}}} \{h_0(0) + \eta x_\eta h'_0(0) + \ldots + \nu_1(\varepsilon)h_1(0) + \nu_1(\varepsilon)\eta x_\eta h_1'(0) + \ldots$$
$$- \rho_0(\varepsilon)C_0(\eta\varepsilon^{-1/2}x_\eta - 1 + e^{-\eta\varepsilon^{-1/2}x_\eta}) + \ldots\} = 0 \tag{36}$$

Equation (36) is satisfied by

$$\rho_0(\varepsilon) = \varepsilon^{1/2} \tag{37}$$

$$\nu_1(\varepsilon) = \varepsilon^{1/2} \tag{38}$$

$$h_0(0) = 0 \tag{39}$$

$$h_0'(0) = C_0 \tag{40}$$

$$h_1(0) = -C_0 \tag{41}$$

which conforms to the assumption $\nu_1(\varepsilon) = o(\varepsilon)$ preceding Equation (27a).

With Equations (39) and (40) we can return to Equations (21) - (23) to conclude that

$$A = 0 \qquad \text{and} \qquad B = C_0 \tag{42}$$

Thus first approximations are fully established for both inner expansion (Equation (29)) and central expansion (Equation (18)), in terms of the constant C_0. Moreover, we can now recognize that the first approximation of the eigenvalue expansion in Equation (19) must belong to the set of discrete values for which the Legendre function $V_n(x)$ in Equation (23) is a finite series; otherwise $V_n(1)$ and $\phi(1)$ become infinite. Thus Equation (24) must be satisfied for n odd, so that

$$\Lambda_0^2 = 2,\ 12,\ 30,\ \ldots\ n(n+1) \text{ for } n = 1,\ 3,\ \ldots \tag{43}$$

and the natural frequencies identified in Equation (17) are approximately

$$\omega_{10} = \Omega,\ \Omega\sqrt{6},\ \Omega\sqrt{15},\ \ldots,\ \Omega[n(n+1)/2]^{1/2} \text{ for } n = 1, 3, \ldots \tag{44}$$

The next step is the determination of $h_1(x)$ in Equation (18). We make use here of the orthonormality of the two different solutions $\phi_\alpha(x,\varepsilon)$ and $\phi_\beta(x,\varepsilon)$ corresponding to the distinct eigenvalues λ_α and λ_β, which requires

$$\int_0^1 \phi_\alpha \phi_\beta d_x = \delta_{\alpha\beta} \tag{45}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Equation (45) can be proven for $\alpha \neq \beta$ by substituting ϕ_α for ϕ in Equation (13), multiplying the equation by ϕ_β and integrating twice by parts, using Equation (15) to obtain

$$-\varepsilon\phi_\alpha''\phi_\beta''\Big|_0^1 + \varepsilon\int_0^1 \phi_\alpha''\phi_\beta'' dx + \int_0^1 (1-x^2)\phi_\alpha'\phi_\beta' dx - \lambda_\alpha^2 \int_0^1 \phi_\alpha\phi_\beta = 0$$

and then repeating the process with α and β exchanged, finally subtracting these results from one another and noting that $\lambda_\alpha^2 - \lambda_\beta^2 \neq 0$. The validity of Equation (45) for $\beta = \alpha$ is freely prescribed, since there is a free constant factor to be prescribed in any solution to Equation (13).

If ϕ_α and ϕ_β are asymptotic expansions in ε, then Equation (45) equates an asymptotic expansion to $\delta_{\alpha\beta}$. For this equation to be valid for any ε, the zeroth order term must be $\delta_{\alpha\beta}$, and all other terms must be zero. We have established that in the inner boundary layer the first term is of order $\varepsilon^{1/2}$ (see Equations (29) and (37)), and moreover the boundary layer "thickness" is of order $\varepsilon^{1/2}$ (see Equation (30)). Thus the contribution of the inner expansion to Equation (45) is limited to terms of order ε and above. If we now <u>assume</u> (subject to later confirmation) that the outer expansion contribution to Equation (45) is also of lower order (higher power) than $\varepsilon^{1/2}$, then we can substitute the central expansion (Equation (18)) alone into Equation (45) to obtain, for a single mode ($\beta = \alpha$)

$$\int_0^1 h_0^2(x)dx = 1 \tag{46}$$

$$\int_0^1 h_0(x)h_1(x)dx = 0 \tag{47}$$

The combination of Equations (21), (42), and (46) yields

$$C_0^2 = \left[\int_0^1 V_n^2(x)dx\right]^{-1} \tag{48}$$

Equation (47) is useful in resolving the choice between Equations (27a) and (27b); in the latter case $h_1(x)$ is simply a multiple of $h_0(x)$, and Equation (47) is impossible. Thus we now conclude (noting Equation (38)) that

$$\kappa_1(\varepsilon) = \nu_1(\varepsilon) = \varepsilon^{1/2} \tag{49}$$

and $h_1(x)$ must satisfy Equation (27a) and Equation (41).

Equation (27a) involves Λ_1^2, which can be determined by multiplying Equation (27a) by $h_0(x)$ and integrating by parts from 0 to 1, to obtain

$$(1-x^2)h_1'h_0 \Big|_0^1 - \int_0^1 (1-x^2)h_1'h_0'dx$$

$$+ \Lambda_0^2 \int_0^1 h_1h_0dx = - \Lambda_1^2 \int_0^1 h_0^2dx$$

or, with Equations (39), (46), and (47) and another integration by parts,

$$- (1-x^2)h_0'h_1 \Big|_0^1 + \int_0^1 [(1-x^2)h_0']'h_1dx = - \Lambda_1^2$$

Substituting $- \Lambda_0^2h_0$ from Equation (20) into the final integrand and utilizing Equation (47) again, we find

$$\Lambda_1^2 = - h_0'(0)h_1(0)$$

With Equations (40), (41), and (48), this becomes

$$\Lambda_1^2 = \left[\int_0^1 V_n^2(x)dx\right]^{-1} \tag{50}$$

More specifically, corresponding to the first two modes defined by Equation (43),

$$\Lambda_{11}^2 = \left[\int_0^1 V_1^2(x)dx\right]^{-1} = \left[\int_0^1 x^2dx\right]^{-1} = 3 \tag{51a}$$

and

$$\Lambda_{12}^2 = \left[\int_0^1 V_2^2(x)dx\right]^{-1}$$

$$= \left[\int_0^1 (x - \frac{10}{6} x^3)^2dx\right]^{-1} = \frac{63}{4} = 15.75 \tag{51b}$$

(It may be noted that Equations (43) and (49) for Λ_0^2 and Λ_1^2 could alternatively have been obtained from Equation (45).)

Finally we can return to Equation (27a) with Λ_1^2 from Equation (49) and seek $h_1(x)$ from that combination. The homogeneous solution is of course merely some factor k_1 times h_0, and since $h_0(0) = 0$ this leaves the particular solution $\overline{h}_1(x)$ to satisfy the boundary condition on $h_1(0)$ in Equation (41). Thus

$$h_1(x) = \overline{h}_1(x) + k_1h_0 \tag{52}$$

where from Equation (47)

$$0 = \int_0^1 h_1(x)h_0(x)dx = \int_0^1 \overline{h}_1(x)h_0(x)dx + k_1 \int_0^1 h_0^2dx$$

so that

$$k_1 = - \int_0^1 \overline{h}_1(x)h_0(x)dx \tag{53}$$

We can proceed without a more explicit expression for $\overline{h}_1(x)$ to accomplish the necessary matching with an outer asymptotic expansion to be obtained for a boundary layer near the free end of the beam.

<u>Radial Beam Outer Expansion</u>

At the free end of the radial beam in Figure 1 we must again satisfy boundary conditions which cannot be reconciled with the central expansion. We therefore introduce another boundary layer coordinate

$$x^* \triangleq (1-x)\alpha^{-1}(\varepsilon) \tag{54}$$

with $\alpha(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Into the original Equation (13) we now substitute the asymptotic expansion

$$\phi(x,\varepsilon) = \delta_0(\varepsilon)f_0(x^*) + \delta_1(\varepsilon)f_1(x^*) + \ldots \tag{55}$$

where $\delta_i(\varepsilon)$ constitute an asymptotic sequence for $i = 0, 1, 2, \ldots$, to obtain

$$\begin{aligned}
&\varepsilon\left[\frac{\delta_0(\varepsilon)}{\alpha^4(\varepsilon)}\frac{d^4f_0(x^*)}{dx^{*4}} + \frac{\delta_1(\varepsilon)}{\alpha^4(\varepsilon)}\frac{d^4f_1(x^*)}{dx^{*4}} + \ldots\right] \\
&- \left[1 - (1-\alpha(\varepsilon)x^*)^2\right]\left[\frac{\delta_0(\varepsilon)}{\alpha^2(\varepsilon)}\frac{d^2f_0(x^*)}{dx^{*2}} + \frac{\delta_1(\varepsilon)}{\alpha^2(\varepsilon)}\frac{d^2f_1(x^*)}{dx^{*2}} + \ldots\right] \\
&- 2\,(1-\alpha(\varepsilon)x^*)\left[\frac{\delta_0(\varepsilon)}{\alpha(\varepsilon)}\frac{df_0(x^*)}{dx^*} + \frac{\delta_1(\varepsilon)}{\alpha(\varepsilon)}\frac{df_1(x^*)}{dx^*} + \ldots\right] \\
&- \Lambda_0^2\,[\delta_0(\varepsilon)f_0(x^*) + \delta_1(\varepsilon)f_1(x^*) + \ldots] \\
&- \kappa_1(\varepsilon)\,\Lambda_1^2\,[\delta_0(\varepsilon)f_0(x^*) + \delta_1(\varepsilon)f_1(x^*) + \ldots] = 0
\end{aligned} \tag{56}$$

The choice

$$\alpha(\varepsilon) = \varepsilon^{1/3} \tag{57}$$

produces from Equation (56) the dominant boundary layer equation

$$\frac{d^4f_0(x^*)}{dx^{*4}} - 2x^*\frac{d^2f_0(x^*)}{dx^{*2}} - 2\frac{df_0(x^*)}{dx^*} = 0 \tag{58}$$

Boundary conditions at x = 1 must be satisfied; from Equations (15) and (54)

$$\left.\frac{d^3 f_0(x^*)}{dx^{*3}}\right|_{x^*=0} = 0 \tag{59}$$

and

$$\left.\frac{d^2 f_0(x^*)}{dx^{*2}}\right|_{x^*=0} = 0 \tag{60}$$

Integrating Equation (58) and enforcing Equation (59) produces

$$\frac{d^3 f_0(x^*)}{dx^{*3}} = x^* \frac{df_0(x^*)}{dx^*} \tag{61}$$

which can be recognized as an Airy equation in the variable $df_0(x^*)/dx^*$, with a boundary condition established by Equation (60), and solutions available[11] as a linear combination of the Airy integrals $Ai(x^*)$ and $Bi(x^*)$. However, $Bi(x^*)$ cannot appear in the solution for $df_0(x^*)/dx^*$ because it grows exponentially with some power of x^*, and matching with the central solution expansion in the limit as $x^* \to \infty$ would be impossible. Moreover, $Ai(x^*)$ cannot appear in the solution because it cannot meet the boundary condition at $x^* = 0$ imposed by Equation (60). Thus the required solution to Equation (61) must be zero, implying

$$f_0(x^*) = D_0 \tag{62}$$

a constant.

In order to match the outer expansion in Equation (55) with the central expansion in Equation (18), we again introduce an intermediate limit defined by the coordinate measure

$$x_\eta = (1 - x)\eta(\varepsilon)^{-1}$$

with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\varepsilon^{1/3} = o(\eta(\varepsilon))$, so that in the limit as $\varepsilon \to 0$ with x_η fixed, $(1-x) = x_\eta \eta(\varepsilon) \to 0$ and $x^* = (1-x)\varepsilon^{-1/3} = x_\eta \eta(\varepsilon)\varepsilon^{-1/3} \to \infty$. Matching in this limit then requires (in view of Equation (38))

$$\lim_{\substack{\varepsilon\to 0\\ x_\eta \text{ fixed}}} \{h_0(1-\eta x_\eta) + \varepsilon^{1/2}h_1(1-\eta x_\eta) + \ldots$$

$$- \delta_0(\varepsilon)f_0(\varepsilon^{-1/3}\eta x_\eta) + \delta_1(\varepsilon)f_1(\varepsilon^{-1/3}\eta x_\eta) + \ldots\} = 0$$

Expanding h_0, h_1 and h_2 as Taylor series about $x_\eta = 0$ yields, with Equation (62),

$$\lim_{\substack{\varepsilon\to 0\\ x_\eta \text{ fixed}}} \{h_0(1) - h_0'(1)\eta x_\eta + \frac{1}{2!}h_0''(1)\eta^2 x_\eta{}^2 + \frac{1}{3!}h_0'''(1)\eta^3 x_\eta{}^3 + \ldots$$

$$+ \varepsilon^{1/2}h_1(1) - \varepsilon^{1/2}h_1'(1)\eta x_\eta + \ldots + \nu_2(\varepsilon)h_2(1) + \ldots$$

$$- \delta_0(\varepsilon)D_0 - \delta_1(\varepsilon)f_1(\varepsilon^{-1/3}\eta x_\eta) - \delta_2(\varepsilon)f_2(\varepsilon^{-1/3}\eta x_\eta) + \ldots\} = 0 \quad (63)$$

Matching requirements suggest the tentative adoption of the following equations:

$$\nu_2(\varepsilon) = \varepsilon \quad (64)$$

$$h_0(1) = D_0 \quad (65)$$

$$\delta_0(\varepsilon) = \varepsilon^0 = 1 \quad (66)$$

$$\delta_1(\varepsilon) = \varepsilon^{1/3} \quad (67)$$

$$\delta_2(\varepsilon) = \varepsilon^{1/2} \quad (68)$$

$$\delta_3(\varepsilon) = \varepsilon^{2/3} \quad (69)$$

$$\delta_4(\varepsilon) = \varepsilon^{5/6} \quad (70)$$

$$\delta_5(\varepsilon) = \varepsilon \quad (71)$$

and so forth. These values are compatible with the assumption made prior to Equation (46), and provide the means to return to Equation (56) to establish differential equations to be solved for $f_1(x^*)$, $f_2(x^*)$, ... With Equations (49), (57), and (66) - (71), Equation (56) can be rewritten as follows:

$$\varepsilon^{-1/3}\left[\frac{d^4f_0}{dx^{*4}} - 2x^*\frac{d^2f_0}{dx^{*2}} - 2\frac{df_0}{dx^*}\right]$$

$$+ \varepsilon^{0}\left[\frac{d^4f_1}{dx^{*4}} - 2x^*\frac{d^2f_1}{dx^{*2}} + x^{*2}\frac{d^2f_0}{dx^{*2}} - 2\frac{df_1}{dx^*} + 2x^*\frac{df_0}{dx^*} - \Lambda_0^2 f_0\right]$$

$$+ \varepsilon^{1/6}\left[\frac{d^4f_2}{dx^{*4}} - 2x^*\frac{d^2f_2}{dx^{*2}} - 2\frac{df_2}{dx^*}\right]$$

$$+ \varepsilon^{1/3}\left[\frac{d^4f_3}{dx^{*4}} - 2x^*\frac{d^2f_3}{dx^{*2}} + x^{*2}\frac{d^2f_1}{dx^{*2}} - 2\frac{df_3}{dx^*} + 2x^*\frac{df_1}{dx^*} - \Lambda_0^2 f_1\right]$$

$$+ \varepsilon^{1/2}\left[\frac{d^4f_4}{dx^{*4}} - 2x^*\frac{d^2f_4}{dx^{*2}} + x^{*2}\frac{d^2f_2}{dx^{*2}} - 2\frac{df_4}{dx^*} + 2x^*\frac{df_2}{dx^*} - \Lambda_0^2 f_2 - \Lambda_1^2 f_0\right]$$

$$+ \varepsilon^{2/3}\left[\frac{d^4f_5}{dx^{*4}} - 2x^*\frac{d^2f_5}{dx^{*2}} + x^{2}\frac{d^2f_3}{dx^{*2}} - 2\frac{df_5}{dx^*} + 2x^*\frac{df_3}{dx^*} - \Lambda_0^2 f_3\right]$$

$$+ \ldots = 0 \tag{72}$$

assuming that $\kappa_2(\varepsilon) = o(\varepsilon^{2/3})$ in Equation (19).

Tha validity of Equation (72) for any ε requires that each of the expressions in square brackets be zero. The first such equation has already been recorded as Equation (58), and its solution for the required boundary conditions has been given as Equation (62). The second bracketed expression in Equation (72) then simplifies to

$$\frac{d^4f_1}{dx^{*4}} - 2x^*\frac{d^2f_1}{dx^{*2}} - 2\frac{df_1}{dx^*} = \Lambda_0^2 D_0 \tag{73}$$

For $f_1(x^*)$ and terms of higher index the boundary conditions on the free end are

$$\left.\frac{d^2f_j}{dx^{*2}}\right|_{x^*=0} = 0 \qquad , \qquad j = 1, 2, 3, \ldots \tag{74}$$

$$\left.\frac{d^3f_j}{dx^{*3}}\right|_{x^*=0} = 0 \qquad j = 1, 2, 3, \ldots \tag{75}$$

Equation (73) has the same homogeneous solution as Equation (58), and in addition the particular solution

$$\overline{f}_1(x^*) = -\frac{1}{2}\Lambda_0^2 D_0 x^* \tag{76}$$

Satisfaction of the boundary conditions again eliminates the Airy integrals in the general solution, so that Equation (76) with Equation (65) provides the total solution.

Now we can evaluate the validity of Equation (67) by returning to Equation (63) to examine the matching of the f_1 term and the $h_0'(1)$ term, which together become

$$-h_0'(1)\eta x_\eta - \delta_1(\varepsilon)f_1(\varepsilon^{-1/3}\eta x_\eta) =$$

$$[-h_0'(1) + \frac{1}{2}\varepsilon^{1/3}\Lambda_0^2 h_0(1)\varepsilon^{-1/3}]\eta x_\eta = [-h_0'(1) + \frac{1}{2}\Lambda_0^2 h_0(1)]\eta x_\eta$$

The expression in brackets is indeed zero, as required by Equation (20) evaluated at x = 1, and these terms are properly matched.

The third bracketed expression in Equation (72) has the same structure as the first, and admits a solution reduced by boundary conditions to a constant, which for matching in Equation (63) must be $h_1(1)$; thus

$$f_2(x^*) = h_1(1) \tag{78}$$

The bracketed term multiplied by $\varepsilon^{1/3}$ in Equation (72) combines with Equations (76) and (65) to provide

$$\frac{d^4 f_3}{dx^{*4}} - 2x^* \frac{d^2 f_3}{dx^{*2}} - 2\frac{df_3}{dx^*} =$$

$$(2 - \Lambda_0^2)\,\Lambda_0^2 h_0(1)x^*/2 \tag{79}$$

with the particular solution

$$\overline{f}_3(x^*) = \Lambda_0^2(\Lambda_0^2 - 2)\,h_0(1)x^{*2}/16 \tag{80}$$

and the general solution involving Airy integrals Ai and Bi as previously. Again Bi must be rejected for its exponential growth, which precludes matching for $x^* \to \infty$, but here Ai cannot be rejected, and the general solution to Equation (79) becomes

$$f_3(x^*) = \Lambda_0^2(\Lambda_0^2 - 2)h_0(1)x^{*2}/16 + D_1 \int_0^{x^*} Ai(\xi)d\xi + D_2 \tag{81}$$

Equation (74) imposes on solution (81) a boundary condition providing

$$D_1 = \frac{\Lambda_0^2(\Lambda_0^2 - 2)h_0(1)}{8Ai'(0)} \tag{82}$$

with $Ai'(0)$ from tables.[11]

Matching in Equation (63) requires that the term involving $h_0''(1)$ sum to zero in the limit with the unwritten term $-\delta_3(\varepsilon)f_3(\varepsilon^{-1/3}\eta x_\eta)$. That portion of $f_3(x^*)$ involving D_1 and D_2 decays exponentially, so that matching requires only

$$\frac{1}{2}h_0''(1)\eta^2 x_\eta{}^2 - \varepsilon^{2/3}\Lambda_0^2(\Lambda_0^2 - 2)h_0(1)\varepsilon^{-2/3}\eta^2 x_\eta{}^2/16$$

$$= [8h_0''(1) - (\Lambda_0^2 - 2)\Lambda_0^2 h_0(1)]\eta^2 x_\eta{}^2/16 = 0$$

This equality holds since the expression in brackets is zero by virtue of Equation (20), which may be written in terms of Taylor series about x = 1 as

$$[-2h_0'(1) + \Lambda_0^2 h_0(1)] + [4h_0''(1) + 2h_0'(1) - \Lambda_0^2 h_0'(1)](1 - x)$$

$$+ [-3h_0'''(1) - 3h_0''(1) + \frac{1}{2}\Lambda_0^2 h_0''(1)](1 - x)^2 + \ldots = 0 \tag{83}$$

so that each of the bracketed expressions is zero.

Solving the three equations implied by Equation (83) simultaneously produces

$$h_0'(1) = \frac{1}{2}\Lambda_0^2 h_0(1) \tag{84}$$

$$h_0''(1) = \frac{1}{4}(\Lambda_0^2-2)\Lambda_0^2 h_0(1) \tag{85}$$

$$h_0'''(1) = \frac{1}{48}(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1) \tag{86}$$

The next step is the solution of the equation involving the coefficient of $\varepsilon^{1/2}$ in Equation (72). With Equations (78), (62) and (65), this equation becomes

$$\frac{d^4f_4}{dx^{*4}} - 2x^* \frac{d^2f_4}{dx^{*4}} - 2\frac{df_4}{dx^*} = \Lambda_0^2 h_1(1) + \Lambda_0^2 h_0(1) \tag{87}$$

The boundary conditions in Equations (74) and (75) limit the solution of Equation (87) to

$$f_4(x^*) = -\frac{1}{2}(\Lambda_0^2 h_1(1) + \Lambda_1^2 h_0(1))x^* \tag{88}$$

Matching of the $f_4(x^*)$ term with the $h_1'(1)$ term in Equation (63) requires

$$-\varepsilon^{1/2}h_1'(1)\eta x_\eta + \frac{1}{2}\varepsilon^{5/6}(\Lambda_0^2 h_1(1) + \Lambda_1^2 h_0(1))\varepsilon^{-1/3}\eta x_\eta$$

$$= [-h_1'(1) + \frac{1}{2}(\Lambda_0^2 h_1(1) + \Lambda_1^2 h_0(1))]\varepsilon^{1/2}\eta x_\eta = 0$$

which is assured by Equation (27a) with $x = 1$.

To complete the projected treatment of the outer boundary layer, we set the last bracketed expression in Equation (72) to zero, noting Equation (81). Since our concern here is not with an exact expression for $f_5(x^*)$ but only with enough information about $f_5(x^*)$ as $x^* \to \infty$ to establish the matching and verify Equation (71), we write the equation for $f_5(x^*)$ as

$$\frac{d^4f_5}{dx^{*4}} - 2x^* \frac{d^2f_5}{dx^{*2}} - 2\frac{df_5}{dx^*} = (\Lambda_0^2-6)(\Lambda_0^2-2)\frac{\Lambda_0^2}{16}h_0(1)x^{*2} + \text{TST} \tag{89}$$

where TST represents transcendentally small terms which have no influence on the matching process. Equation (89) is easily integrated, to obtain a result presented here as

$$\frac{df_5}{dx^*} = \frac{-(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)x^{*2}}{96} + \frac{1}{2x^*}\frac{d^3 f_5}{dx^{*3}} - \frac{D_3}{2x^*} + \text{TST} \tag{90}$$

Two differentiations yield the approximation*

$$\frac{d^3 f_5}{dx^{*3}} = -\frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)}{48} + O(x^{*-3})$$

which in turn permits Equation (90) to be written as

$$\frac{df_5}{dx^*} = -\frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)x^{*2}}{96}$$

$$-\left[D_3 + \frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)}{48}\right]\frac{1}{2x^*}$$

$$+ O(x^{*-4}) + \text{TST}$$

Integration now yields the approximation

$$f_5(x^*) \cong -\frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)x^{*3}}{288}$$

$$-\left[D_3 + \frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)}{48}\right]\frac{\ell n x^*}{2} + D_4$$

$$+ O(x^{*-3}) + \text{TST} \tag{91}$$

The constants D_3 and D_4 must for matching be given by

$$D_3 = -\frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)}{48}$$

*The notation $X = O(x^{*-3})$ implies that $\lim_{x^* \to \infty} X x^{*3}$ is bounded and in view of Equation (64),

$$D_4 = h_2(1)$$

Now matching in Equation (63) is assured by the identity

$$\frac{1}{3!} h_0'''(1)\eta^3 x^3_\eta - \varepsilon f_5(\varepsilon^{-1/3}\eta x_\eta) =$$

$$\left[\frac{h_0'''(1)}{3!} + \frac{(\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)}{288}\right]\eta^3 x^3_\eta = 0$$

which follows from Equation (86).

Thus the outer expansion in Equation (55) has been completed to order ε.

Radial Beam Solution Summary

In brief summary, the rotating radial beam in Figure 1, as characterized by transverse vibration equations (Equation (10)) with cantilever boundary conditions (Equation (15)) has natural frequencies given by Equation (17), with λ^2 from Equation (19) and (49) available as

$$\lambda^2 = \Lambda_0^2 + \varepsilon^{1/2}\Lambda_1^2 + \ldots$$

where Λ_0^2 is given for the sequence of modes by Equation (43) and Λ_1^2 is given for all modes by Equation (50) and for the first two modes explicitly by Equation (51). Although these results could be compared for specific spin rates to the computer-generated frequencies presented by Yntema and others, it is more revealing to compare to the formula for the first mode used by Vigneron:

$$\omega_1^2 = \omega_{NR}^2 + 1.193\Omega^2$$

where ω_{NR}^2 is the lowest natural frequency of the nonrotating beam, which is, in view of Equation (11), given by

$$\omega_{NR}^2 = (3.515)^2\ EI/(\mu L^4) = (3.515)^2\Omega^2\varepsilon/2$$

Thus, in terms of our ε, Vigneron uses

$$\omega_1^2 = \Omega^2(1.193 + 3.515^2\varepsilon/2) \tag{92}$$

For comparison, our approximation is

$$\omega_1^2 = \Omega^2(1 + 2.12\sqrt{\varepsilon/2}) \tag{93}$$

These two approximations are protrayed graphically in Figure 3. More precise estimates indicate that Equation (93) gives better results for sufficiently small ε (and yields the correct limiting case of the cable frequency when $\varepsilon \to 0$), but Equation (92) is more accurate for sufficiently large ε.

The mode shapes have been developed in this paper in terms of asymptotic expansions as follows:

Inner boundary layer expansion

$$\phi(x,\varepsilon) = C_0\varepsilon^{1/2}(\varepsilon^{-12}x - 1 + e^{-x\varepsilon^{-12}}) + \ldots$$

Central expansion

$$\phi(x,\varepsilon) = h_0(x) + \varepsilon^{1/2}h_1(x) + \varepsilon h_2(x) + \ldots$$

Outer boundary layer expansion

$$\begin{aligned}\phi(x,\varepsilon) &= h_0(1) - \tfrac{1}{2}\Lambda_0^2 h_0(1)(1-x) + \varepsilon^{1/2}h_1(1)\\ &+ \Lambda_0^2(\Lambda_0^2-2)h_0(1)(1-x)^2/16\\ &- \tfrac{1}{2}\varepsilon^{1/2}(\Lambda_0^2 h_1(1) + \Lambda_1^2 h_0(1))(1-x)\\ &- (\Lambda_0^2-6)(\Lambda_0^2-2)\Lambda_0^2 h_0(1)(1-x)^3/288\\ &+ h_2(1) + \ldots\end{aligned}$$

In these expressions C_0, $h_0(x)$, $h_1(x)$, Λ_0^2 and Λ_1^2 come respectively from Equations (48), (21) and (42), (52), (43), and (50). Note that despite the large number of terms evaluated in the outer boundary layer the final expansion contains no terms involving powers of ε above 1/2.

Diametral Beam Outer Expansions

The differential equation and central solution expansion are the same for radial and diametral beams, so that with the exception of the boundary conditions in Equations (2) - (3) and (15) - (16) the development is the same for both beams through Equation (27), with $k = 1$ for the radial beam and $k > 1$ for the diametral beam. Different boundary conditions imply different boundary layer solutions, and in the case of the diametral beam symmetry indicates that the same boundary layer expansions apply at $x = k^{-1}$ and $x = -k^{-1}$. In what follows we develop the solution near $x = k^{-1}$.

As previously, we define a stretched coordinate

$$\bar{x} = (k^{-1} - x)\beta^{-1}(\varepsilon)$$

where $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and introduce the expansion

$$\phi(\bar{x},\varepsilon) = \gamma_0(\varepsilon)F_0(\bar{x}) + \gamma_1(\varepsilon)F_1(\bar{x}) + \ldots \tag{94}$$

where $\gamma_i(\varepsilon)$ constitute an asymptotic sequence, $i = 0, 1, 2, \ldots.$ Equation (13) then becomes

$$\begin{aligned}
&\varepsilon\left[\frac{\gamma_0(\varepsilon)}{\beta^4(\varepsilon)}\frac{d^4F_0}{d\bar{x}^4} + \frac{\gamma_1(\varepsilon)}{\beta^4(\varepsilon)}\frac{d^4F_1}{d\bar{x}^4} + \ldots\right] \\
&- \left[1 - (k^{-1} - \beta(\varepsilon)\bar{x})^2\right]\left[\frac{\gamma_0(\varepsilon)}{\beta^2(\varepsilon)}\frac{d^2F_0}{d\bar{x}^2} + \frac{\gamma_1(\varepsilon)}{\beta^2(\varepsilon)}\frac{d^2F_1}{d\bar{x}^2} + \ldots\right] \\
&- 2(k^{-1} - \beta(\varepsilon)\bar{x})\left[\frac{\gamma_0(\varepsilon)}{\beta(\varepsilon)}\frac{dF_0}{d\bar{x}} + \frac{\gamma_1(\varepsilon)}{\beta(\varepsilon)}\frac{dF_1}{d\bar{x}} + \ldots\right] \\
&- \Lambda_0^2\,[\gamma_0(\varepsilon)F_0 + \gamma_1(\varepsilon)F_1 + \ldots] \\
&- \kappa_1(\varepsilon)\Lambda_1^2[\gamma_0(\varepsilon)F_0 + \gamma_1(\varepsilon)F_1 + \ldots] = 0
\end{aligned} \tag{95}$$

Under the assumption that pretension T is sufficient to maintain $1 - k^{-2} = O(1)$, we can choose

$$\beta(\varepsilon) = \varepsilon^{1/2} \tag{96}$$

and obtain from Equation (95) the dominant boundary layer equation

$$\frac{d^4F_0(\bar{x})}{d\bar{x}^4} - k^{*2}\frac{d^2F_0(\bar{x})}{d\bar{x}^2} = 0 \tag{97}$$

where $k^{*2} \triangleq (1 - k^{-2})$. Equation (97) can be solved as was Equation (31), since F_0 and g_0 have the same boundary conditions (see Equation (33)) with the result

$$F_0(\bar{x}) = A_0(k^*\bar{x} - 1 + e^{-k^*\bar{x}}) \tag{98}$$

where A_0 is to be determined by matching with the central solution expansion.

Matching requires the introduction of the intermediate limit defined by

$$x_\eta = (k^{-1} - x)\eta^{-1}(\varepsilon) \tag{99}$$

with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\varepsilon^{1/2} = o(\eta(\varepsilon))$, so that in the limit as $\varepsilon \to 0$ with x_η fixed, $(k^{-1} - x) = x_\eta\eta \to 0$ and $\bar{x} = (k^{-1} - x)\varepsilon^{-1/2} = x_\eta\eta\varepsilon^{-1/2} \to \infty$. Matching in this limit requires, from Equations (18) and (95)

$$\lim_{\substack{\varepsilon\to 0\\ x_\eta \text{ fixed}}} \left\{ h_0(x) + \nu_1(\varepsilon)h_1(x) + \ldots - \gamma_0(\varepsilon)F_0(\bar{x}) - \gamma_1(\varepsilon)F_1(\bar{x}) - \ldots \right\} = 0$$

Expanding in Taylor series about $x = k^{-1}$ and substituting Equation (98) produces

$$\begin{aligned}\lim_{\substack{\varepsilon\to 0\\ x_\eta \text{ fixed}}} \Big\{ & h_0(k^{-1}) - h_0'(k^{-1})\eta x_\eta + \tfrac{1}{2}h_0''(k^{-1})\eta^2 x_\eta^2 + \ldots + \nu_1(\varepsilon)h_1(k^{-1}) \\ & - \nu_1(\varepsilon)h_1'(k^{-1})\eta x_\eta + \ldots - \gamma_0(\varepsilon)A_0(k^*\varepsilon^{-1/2}\eta x_\eta - 1 + e^{-k^*\varepsilon^{-1/2}\eta x_\eta}) \\ & - \gamma_1(\varepsilon)F_1(\varepsilon^{-1/2}\eta x_\eta) - \ldots \Big\} = 0 \end{aligned} \tag{100}$$

Matching requirements suggest the following:

$$h_0(k^{-1}) = 0 \tag{101}$$

$$\gamma_0(\varepsilon) = \varepsilon^{1/2} \tag{102}$$

$$A_0k^* = -h_0'(k^{-1}) \tag{103}$$

$$\nu_1(\varepsilon) = \varepsilon^{1/2} \tag{104}$$

$$A_0 = -h_1(k^{-1}) \tag{105}$$

With this identification, we can proceed to establish $h_1(x)$ and $F_1(\bar{x})$, and then return to Equation (100) to confirm the matching. We can also use Equation (101) in conjunction with the Legendre function solution for h_0 in Equation (21) to obtain values for Λ_0^2.

The singularity of the Legendre functions for unity argument does not present a problem for the diametral beam, since $k^{-1} < 1$. Thus both $U_n(x)$ and $V_n(x)$ are permissible solutions, subject to Equation (101). The symmetric or "even" modes are represented by $U_n(x)$, and the antisymmetric or "odd" modes are represented by $V_n(x)$, as indicated by Equations (22) and (23), which together with Equation (101) provide solutions for Λ_0^2. These results are illustrated numerically in the section which follows the present section on the boundary layer expansion.

Orthogonality of the modal functions $\phi(x)$ corresponding to different discrete values of λ_α^2, say λ_α^2 and λ_β^2, can be established as for the radial beam, to obtain the orthogonality relationship (as in Equation (45))

$$\int_{-k^{-1}}^{k^{-1}} \phi_\alpha \phi_\beta dx = \delta_{\alpha\beta} \tag{106}$$

For the terms in the asymptotic expansion of a single modal function, we now have (in parallel with Equations (46) and (47))

$$\int_{-k^{-1}}^{k^{-1}} h_0^2(x) dx = 1 \tag{107}$$

$$\int_{-k^{-1}}^{k^{-1}} h_0(x) h_1(x) dx = 0 \tag{108}$$

under the same conditions affecting the boundary layer solution.

Thus we have for the even modes, from Equations (107) and (21), with B = 0,

$$\int_{-k^{-1}}^{k^{-1}} A^2 U_n^2(x)dx = 1$$

so that

$$A^2 = \left[\int_{-k^{-1}}^{k^{-1}} U_n^2(x)dx \right]^{-1} \tag{109}$$

and similarly for the odd modes A = 0 and

$$B^2 = \left[\int_{-k^{-1}}^{k^{-1}} V_n^2(x)dx \right]^{-1} \tag{110}$$

Equations (107) and (108) indicate that Equations (49) and (27a) apply to the diametral beam also. We can now obtain a new expression for Λ_1^2 by multiplying Equation (27a) by h_0 and then integrating from $-k^{-1}$ to k^{-1}, to obtain, after noting Equations (107) and (108),

$$\int_{-k^{-1}}^{k^{-1}} [(1-x^2)h_1']'h_0dx = -\Lambda_1^2$$

Two integrations by parts, utilizing boundary conditions established by Equation (101) and its counterpart at $x = -k^{-1}$, and involving Equation (20) and Equation (108), produce

$$\Lambda_1^2 = (1 - x^2)h_0'h_1 \Big|_{-k^{-1}}^{k^{-1}}$$

For the odd modes the slope is the same at both ends but the displacement is opposite in sign; for the even modes this relationship is reversed. Thus in any case

$$\Lambda_1^2 = 2(1 - k^{-2})h_0'(k^{-1})h_1(k^{-1}) \tag{111}$$

By using Equation (105) and then Equation (103), we can rewrite Equation (111) as

$$\Lambda_1^2 = 2k^{-1}(k^2 - 1)^{1/2}[h_0'(k^{-1})]^2 \tag{112}$$

To obtain numerical answers from Equation (112) one must use the derivative of $h_0(x)$ as found in Equation (21) together with Equation (109) for even modes and Equation (110) for odd modes.

Now we could return with Λ_1^2 to Equation (27a) to solve for $h_1(x)$, using the boundary conditions established by matching in Equation (100). Satisfied that $h_1(x)$ is fully determined, we now seek to establish the second term in the asymptotic expansion in Equation (95). The matching requirement in Equation (100) suggests that

$$\gamma_1(\varepsilon) = \varepsilon \tag{113}$$

Under this assumption, and with Equations (102) and (96), we can obtain from Equation (95) the following equation to be solved for $F_1(x)$:

$$\frac{d^4F_1}{d\bar{x}^4} - (1-k^{-2})\frac{d^2F_1}{d\bar{x}^2} = 2k^{-1}\frac{dF_0}{d\bar{x}} + 2k^{-1}\bar{x}\frac{d^2F_0}{d\bar{x}^2}$$

or, in view of Equations (98) and the definition of k^*,

$$\frac{d^4F_1}{d\bar{x}^4} - k^{*2}\frac{d^2F_1}{d\bar{x}^2} = 2k^{-1}A_0k^*[1 - e^{-k^*\bar{x}} + \bar{x}k^*e^{-k^*\bar{x}}] \tag{114}$$

Recall that A_0k^* is fully determined by Equation (103).

Equation (114) has the same homogeneous solution as Equation (97), and in addition the particular solution $\overline{F}_1(\bar{x})$ such that

$$\frac{d^2\overline{F}_1}{d\bar{x}^2} = 2k^{-1}A_0k^*\left[-\frac{1}{k^{*2}} + \frac{1}{4k^*}\bar{x}e^{-k^*\bar{x}} - \frac{1}{4}\bar{x}^2e^{-k^*\bar{x}}\right] \tag{115}$$

Proceeding again in parallel with the solution of Equation (31), we observe that for the total solution

$$\frac{d^2F_1}{d\bar{x}^2} = A_1 e^{-k^*\bar{x}} + \frac{d^2\bar{F}_1}{d\bar{x}^2}$$

After two integrations, with integration constants A_2 and A_3 introduced, and with A_1 and A_2 removed by imposition of boundary conditions on $F_1(\bar{x})$ and $dF_1/d\bar{x}$, the final solution becomes

$$F_1(\bar{x}) = -k^{-1}A_0 k^{*2}\bar{x}^2 e^{-k^*\bar{x}} - \frac{3k^{-1}A_0}{2k^{*2}}\bar{x}e^{-k^*\bar{x}} - A_3 e^{-k^*\bar{x}}$$

$$- \frac{k^{-1}A_0}{k^*}\bar{x}^2 + \left(\frac{2k^{-1}A_0}{k^{*2}} - k^*A_3\right)\bar{x} + A_3 \tag{116}$$

Since A_0 in Equation (116) is known from Equation (103), the only free constant in this expression is A_3; this unknown will be established by the matching process. Examination of Equation (116) reveals that matching in Equation (100) requires that

$$\lim_{\substack{\varepsilon\to 0 \\ x_\eta \text{ fixed}}} \left\{ \frac{1}{2} h_0''(k^{-1})\eta^2 x^2_\eta - \nu_1(\varepsilon)h_1'(k^{-1})\eta x_\eta - \gamma_1(\varepsilon)F_1(\varepsilon^{-1/2}\eta x_\eta) \right\} = 0 \tag{117}$$

If the first three terms in $F_1(\bar{x})$ in Equation (116) are ignored as transcendentally small, and Equations (104) and (113) are substituted, Equation (117) is satisfied up to terms below ε if

$$\frac{2k^{-1}A_0}{k^{*2}} - k^*A_3 = h_1'(k^{-1}) \tag{118}$$

and

$$\frac{k^{-1}A_0}{k^*} = -\frac{1}{2}h_0''(k^{-1}) \tag{119}$$

Equation (118) provides the required value for A_3, but Equation (119) must be an identity if the proposed solution is to be verified. This equation can be confirmed by using Equation (103) to rewrite Equation (119), noting the definition of k^*, as

$$(1 - k^{-2})h_0''(k^{-1}) - 2k^{-1}h_0'(k^{-1}) = 0 \tag{120}$$

The validity of Equation (120) follows from Equation (20) with $x = k^{-1}$, since $h_0(k^{-1}) = 0$ by Equation (101). Thus the proposed solution is established.

Diametral Beam Solution Summary

The rotating pretensioned diametral beam in Figure 2, as characterized by transverse vibration equations (Equation (10)) with fixed end boundary conditions (Equation (16)) has natural frequencies given by Equation (17), with λ^2 from Equation (19) and (49) available as

$$\lambda^2 = \Lambda_0^2 + \varepsilon^{1/2}\Lambda_1^2 + \ldots$$

where Λ_1^2 is given by Equation (112) in terms of h_0 from Equation (21), and where Λ_0^2 is given for even modes by the combination of Equations (22) and (101) in the form

$$U_n(k^{-1}) = 1 - \frac{1}{2!}\mu_0^2 + \frac{1}{4!}\mu_0^2(\mu_0^2 - 6k^{-2})$$
$$- \frac{1}{6!}\mu_0^2(\mu_0^2 - 6k^{-2})(\mu_0^2 - 20k^{-2}) + \ldots = 0 \tag{121}$$

where $\mu_0 \triangleq \Lambda_0^2/k^2$, and for the odd modes from Equations (23) and (101) by

$$V_n(k^{-1}) = 1 - \frac{1}{3!}(\mu_0^2 - 2k^{-2}) + \frac{1}{5!}(\mu_0^2 - 2k^{-2})(\mu_0^2 - 12k^{-2}) - \ldots = 0 \tag{122}$$

These equations have obvious solutions for particular values of k^2; for example with $\mu_0^2 = 6k^{-2}$ Equation (121) yields $\mu_0^2 = 2$ and hence $k^2 = 3$ and $\Lambda_0^2 = 6$, and with $\mu_0^2 = 20k^{-2}$ Equation (121) yields either the "first mode" with $\mu_0^2 = 14.8$ and hence $k^2 = 1.35$ and $\Lambda_0^2 = 20$ or the "third" mode with $\mu_0^2 = 2.32$ and hence $k^2 = 8.6$ and $\Lambda_0^2 = 20$. More generally however one finds for any value of parameter k a spectrum of discrete solutions for μ_0^2. In the limiting case $k^2 \to \infty$ the Legendre function $U_n(k^{-1})$ for the even modes becomes the cosine function, implying $\mu_0^2 = \pi^2/4$, and corresponding to the vibrating taut string, and in the neighborhood of the singularity at $k^2 = 1$ we find $\mu_0^2 \to 0$. Similarly for the odd modes $V_n(k^{-1})$ becomes the sine for $k^2 \to \infty$, implying $\mu_0^2 = \pi^2$, and at $k^2 = 1$, $\mu_0^2 = 2$. Figures 4 and 5 portray the variations of μ_0^2 (and hence natural frequencies) with k^2 for the first and second modes respectively.

In addition to the natural frequency equations, we have developed expressions for mode shapes in the form of asymptotic expansions as follows:

Central expansion

$$\phi(x,\varepsilon) = h_0(x) + \varepsilon^{1/2} h_1(x) + \ldots$$

Boundary layer expansion near $x = k^{-1}$

$$\phi(x,\varepsilon) = -(k^{-1} - x)h_0'(k^{-1}) + \frac{\varepsilon^{1/2} h_0'(k^{-1})}{k^*} [1 - e^{-k(k^{-1}-x)\varepsilon^{-1/2}}]$$

$$+ k^{-1} h_0'(k^{-1}) k^* (k^{-1} - x)^2 e^{-k^*(k^{-1}-x)\varepsilon^{-1/2}}$$

$$+ \frac{3}{2} \varepsilon^{1/2} k^{-1} \frac{h_0'(k^{-1})}{k^{*3}} (k^{-1} - x) e^{-k^*(k^{-1}-x)\varepsilon^{-1/2}}$$

$$+ \varepsilon \left[\frac{2k^{-1} h_0'(k^{-1})}{k^{*4}} + h_1'(k^{-1}) \right] e^{-k^*(k^{-1}-x)\varepsilon^{-1/2}}$$

$$+ \frac{k^{-1} h_0'(k^{-1})}{k^{*2}} (k^{-1} - x)^2 + \varepsilon^{1/2} h_1'(k^{-1})(k^{-1} - x)$$

$$- \varepsilon \left[\frac{2k^{-1} h_0'(k^{-1})}{k^{*4}} + h_1'(k^{-1}) \right] + \ldots$$

and a similar boundary layer near $x = -k^{-1}$. In these expressions $k^{*2} = (1 - k^{-2})$, $h_1(x)$ is the solution of Equation (27a) with Λ_1^2 from Equation (112), and $h_0(x)$

is given by Equation (21), with the constants A and B in Equation (21) established by Equations (109) and (110). Figures 6 and 7 are plots of $h_0(s)$ for modes one and two respectively, with $s \triangleq \xi/L$.

Conclusions

The method of matched asymptotic expansions has a unique advantage in the modal analysis of rapidly rotating beams, in that the results are literal rather than numerical, and thus are applicable at once to a range of values of the system parameters, which include beam density, stiffness, pretension, and spin rate. The primary disadvantage is the approximate nature of the results, although the order of the approximation is well established.

Application of this method to modal analysis of rotating plates is also feasible in some cases,[12] as will be reported in a separate paper.

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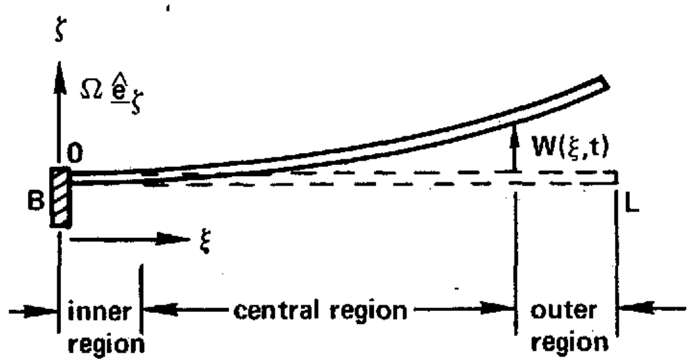

Figure 1. Radial Rotating Beam.

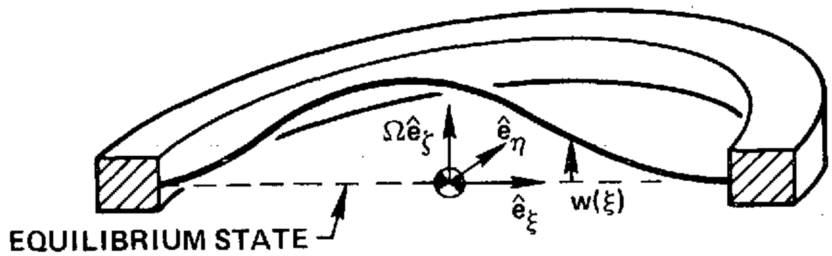

Figure 2. Diametral Rotating Beam.

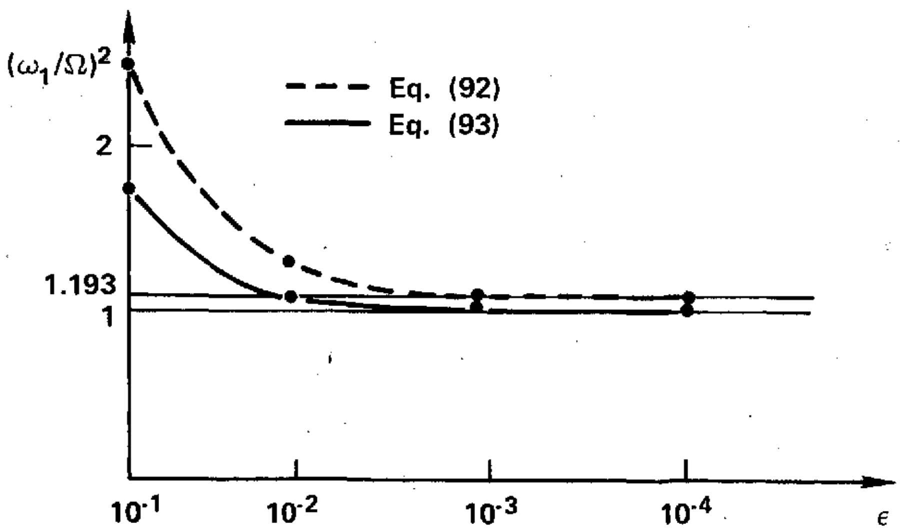

Figure 3. First Mode Frequency Estimates.

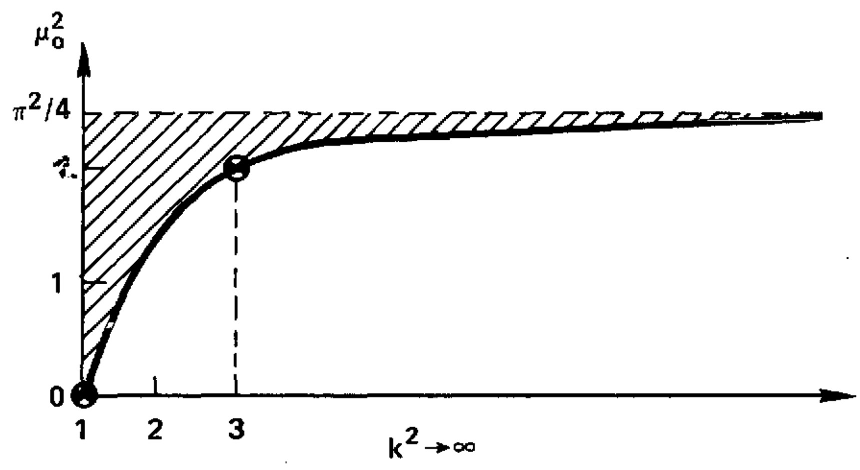

Figure 4. Natural Frequencies for First Mode.

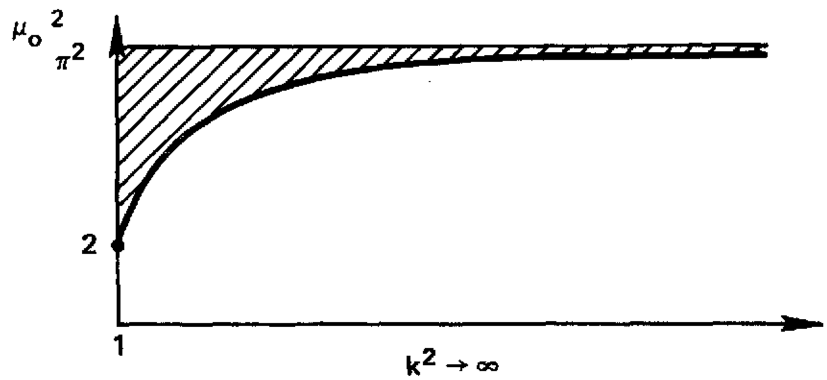

Figure 5. Natural Frequencies for Second Mode.

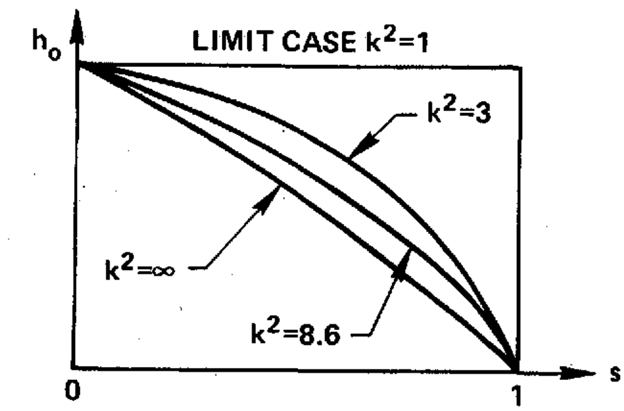

Figure 6. Mode Shape Approximation for First Mode.

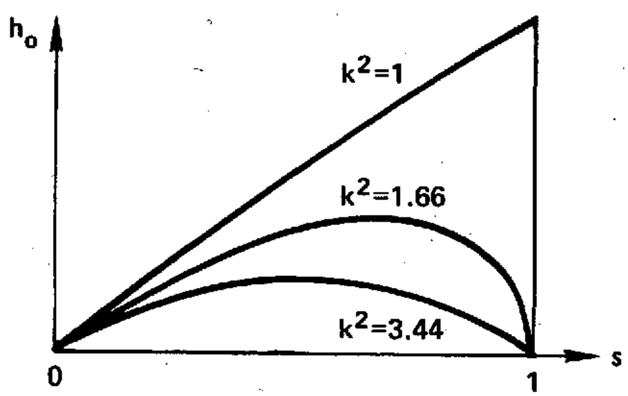

Figure 7. Mode Shape Approximation for Second Mode.