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Troy, New York 12181

(NASA-CR-140057) PARAMETER ESTIMATION FOR TERRAIN MODELING FROM GRADIENT DATA (Rensselaer Polytechnic Inst.) 96 p HC $8.00 CSCL 22A

N74-33292

Unclas

G3/30 48768

R.P.I. Technical Report MP-46

PARAMETER ESTIMATION FOR TERRAIN MODELING FROM GRADIENT DATA

by

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NASA Grant NGL 33-018-091

Analysis and Design of a Capsule Landing System
an Surface Vehicle Control System for Mars Exploration

Rensselaer Polytechnic Institute
Troy, New York
May, 1974

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ACKNOWLEDGEMENT

The author would like to thank Dr. C. N. Shen for his assistance and supervision throughout the year, and especially in the formulation of this report.

ABSTRACT

This paper develops a method for modeling terrain surfaces for use on Rensselaer Polytechnic Institute's unmanned Martian roving vehicle. The modeling procedure employs a two-step process which uses gradient as well as height data in order to improve the accuracy of the model's gradient. Least square approximation is used in order to stochastically determine the parameters which describe the modeled surface. A complete error analysis of the modeling procedure is included which determines the effect of instrumental measurement errors on the model's accuracy. Computer simulation is used as a means of testing the entire modeling process which includes the acquisition of data points, the two-step modeling process and the error analysis. Finally, to illustrate the procedure, a numerical example is included.

PART 1

INTRODUCTION

An autonomous navigation system is necessary to allow the Martian rover to safely traverse the unknown Martian surface. A forty-minute time lag in communication between Earth and Mars makes remote control impractical.

One of the tasks involved is the development of a mathematical model to represent the surface terrain in front of the vehicle. This model is for use in the vehicle's path selection system which will decide whether the terrain is passable or impassable. It will then choose the appropriate course of action.

The vehicle has a laser rangefinder which gives all of the data used in modeling. In the proposed system, the laser would scan a specified area in front of the vehicle. This area is then divided into a number of sections and the terrain modeled independently in each one of those sections.

PART 2

DATA ACQUISITION

Surface information used in modeling is obtained by a laser rangefinder attached to a mast extending from the vehicle. The mast was assumed to be 3.0 meters high. The vehicle and its coordinate system are shown in Figure 1. Here the h", a", b"-coordinate system is attached to the vehicle with the h" axis along the mast and the b" axis in the forward direction of motion. The h,a,b-coordinate system is formed by the local vertical and an axis in a plane containing the heading and the local vertical. The two systems are coincident at the origin, and are related by the pitch angle ξ and roll angle ϕ.

The laser beam is transmitted at a specified elevation angle, β, and azimuth angle θ. It returns a range value, R, of the surface data point. The vehicle also measures the corresponding value of ξ and ϕ.

The specific scanning pattern used is illustrated in Figure 2. This shows two "W" shaped scan rows separated by an elevation increment, β_{inc}. $\Delta\beta$ and $\Delta\theta$ are constant for the scan. Consecutive points in each row are taken within a millisecond of each other, a technique referred to as rapid scan.[1] With this method, the roll and pitch angles of the vehicle essentially do not change for neighboring data points. However, because of the large number of data points in each row, the vehicle does change its angular position between corresponding points on different scan rows.

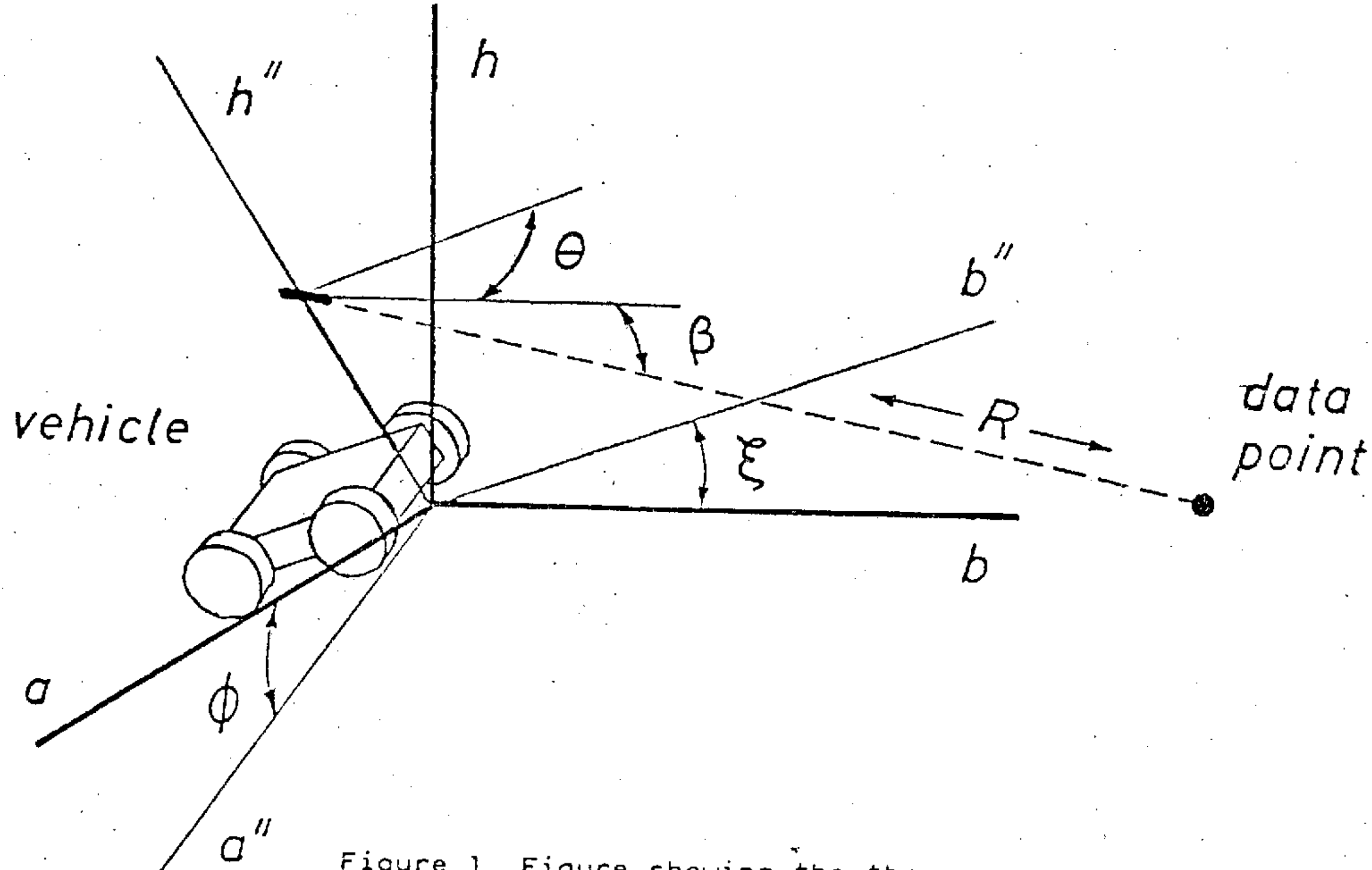

Figure 1 Figure showing the three measured quantities β, R and θ, the two coordinate systems and the transformation angles ϕ and ξ

PART 3

MODELING PROCEDURE STEP I

A. Formation of planes

The first step in the modeling procedure stochastically models planes from sets of four data points, using a previously developed procedure.[1]

The four points used for each plane are chosen from the same scan row such as points (1,1),(1,2), (1,3) and (1,4) in Figure 2. By choosing the points in this manner and utilizing the rapid scan technique, the plane can be modeled in the h",a",b"-coordinate system. This is valid since the four neighboring points are taken with essentially the same ξ and ϕ angles. This would not be true if points were taken from different scan rows, such as points (1,1), (2,1), (1,2), (2,2) in Figure 2 since the points are no longer taken with the same ξ and ϕ angles.

In Figure 3, the notation for a modeled section is introduced which will be used in the remainder of this report. Here the superscript 'n' refers to the plane number. In the procedure described below, four planes are modeled for each section; therefore, n=1,2,3,4. The double primed superscript " refers to whether the quantity named is in the h",a",b"-coordinate system, or if it is missing, the h,a,b-coordinate system.

If the subscript is a number, then the quantity refers to a measured data point. If the subscript is a 'p', then this quantity refers to a modeled center point which is defined later.

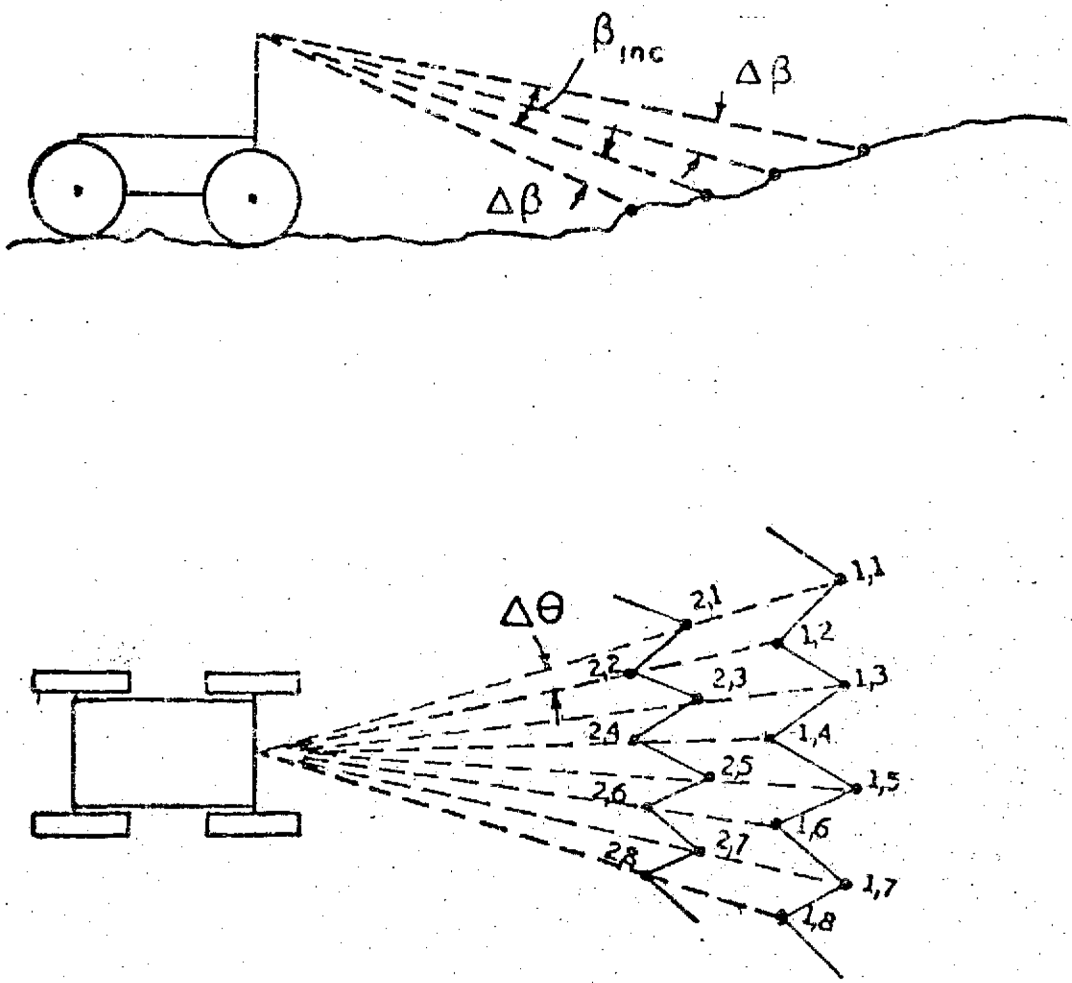

Figure 2 Scanning scheme used to obtain data points

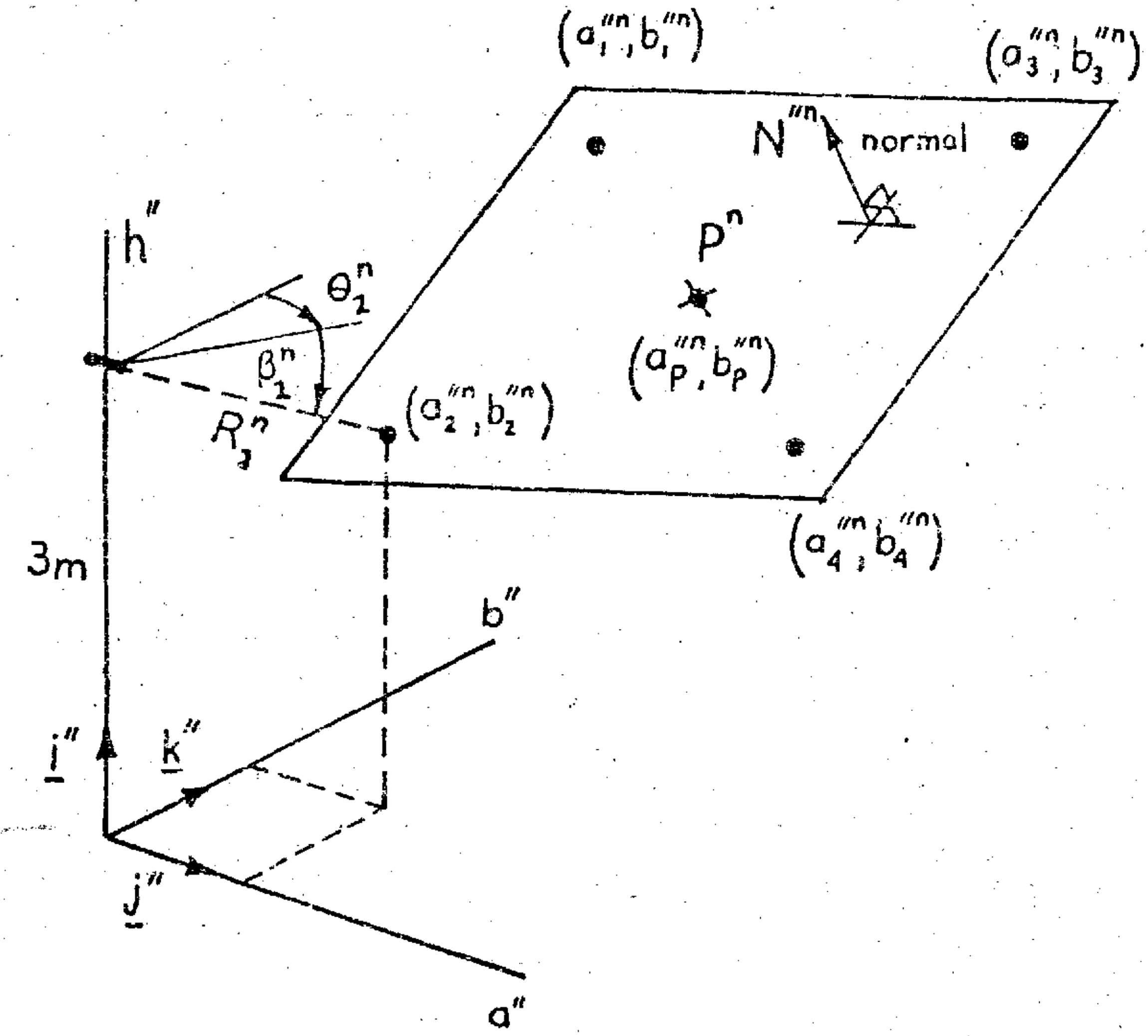

Figure 3 Notation used in the modeling process

The procedure is outlined below for finding the equation of the plane 'n'. An equation of the form

$$h'' = a'' x_1''^{n} + b'' x_2''^{n} + x_3''^{n} \tag{1}$$

is used to describe the plane in the h",a",b",-coordinate system.

where

$$x_1''^{n} = \frac{\partial h''}{\partial a''}$$

$$x_2''^{n} = \frac{\partial h''}{\partial b''}$$

and $x_3''^{n}$ is the height of the plane at a"=o, b"=o. These x_i'''s are constant parameters which must be determined.

In order to determine the parameters, first the spherical coordinates of the data points, θ_i^n, β_i^n, R_i^n are converted into the h",a",b"-coordinates by

$$h_i''^{n} = 3 - R_i^n \sin\beta_i^n \tag{2}$$

$$a_i''^{n} = R_i^n \cos\beta_i^n \sin\theta_i^n \tag{3}$$

$$b_i''^{n} = R_i^n \cos\beta_i^n \cos\theta_i^n \tag{4}$$

$$i = 1,2,3,4$$

A matrix equation is then written utilizing the location of the four data points, and a least square estimation of the parameters is formed by

$$\underline{x}''^{n} = \left(A''^{T} A''\right)^{-1} A''^{T} \underline{h}''^{n} \tag{5}$$

where
$$\underline{x}''^{n} = \left(x_1''^{n},\ x_2''^{n},\ x_3''^{n}\right)^T$$

$$\underline{h}''^{n} = \left(h_1''^{n},\ h_2''^{n},\ h_3''^{n},\ h_4''^{n}\right)^T$$

and

$$A'' = \begin{bmatrix} a_1''^{n} & b_1''^{n} & 1 \\ a_2''^{n} & b_2''^{n} & 1 \\ a_3''^{n} & b_3''^{n} & 1 \\ a_4''^{n} & b_4''^{n} & 1 \end{bmatrix}$$

The equation of the plane is now determined.

The center point, P^n, (Figure 3) is a point on the modeled plane centrally located between the four data points used in determining the plane. Its h",a",b"-coordinates can be found from

$$a_p''^{n} = \left(a_1''^{n} + a_2''^{n} + a_3''^{n} + a_4''^{n}\right)\frac{1}{4} \tag{6}$$

$$b_p''^{n} = \left(b_1''^{n} + b_2''^{n} + b_3''^{n} + b_4''^{n}\right)\frac{1}{4} \tag{7}$$

$$h_p''^{n} = x_1''^{n} a_p''^{n} + x_2''^{n} b_p''^{n} + x_3''^{n} \tag{8}$$

B. Transformation of center point information

The location, height, cross-path and in-path slopes of the center point, P^n, have been found above for the h",a",b"-coordinate system. However, this information must be transformed into the h,a,b-coordinate system to be used for the terrain model.

The height and location can be transformed by

$$\begin{bmatrix} h_p^n \\ a_p^n \\ b_p^n \end{bmatrix} = C(\phi^n)\,B(\xi^n) \begin{bmatrix} h_p''^n \\ a_p''^n \\ b_p''^n \end{bmatrix} \tag{9}$$

where

$$C^n = \begin{bmatrix} \cos\phi^n & -\sin\phi^n & 0 \\ \sin\phi^n & \cos\phi^n & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B^n = \begin{bmatrix} \cos\xi^n & 0 & \sin\xi^n \\ 0 & 1 & 0 \\ -\sin\xi^n & 0 & \cos\xi^n \end{bmatrix}$$

In order to find the cross-path and in-path slope of the modeled plane in the unprimed system, the normal to the plane, N''^n, is used. The equation for the plane is rewritten as:

$$0 = x_4''^n h'' + x_1''^n a'' + x_2''^n b'' + x_3''^n \tag{10}$$

where $\quad x_4''^n = -1$

If the unit vectors $\underline{i}''$, $\underline{j}''$, $\underline{k}''$ are defined as in Figure 3, then N''^n can be written[2]

$$N''^n = x_4''^n \underline{i}'' + x_1''^n \underline{j}'' + x_2''^n \underline{k}'' \tag{11}$$

Since the normal is a vector, the normal in the primed system (N''^n), must be the same vector as the normal in the unprimed system (N^n). With the unit vectors shown in Figure 4 the vector N^n is written as

$$N^n = N_h^n \underline{i} + N_a^n \underline{j} + N_b^n \underline{k} \tag{12}$$

Using the transformation from the primed to the unprimed system

$$N^n = C(\phi^n)B(\xi^n) \tag{13}$$

Or rewriting with $x_4''^n = -1$

$$\begin{bmatrix} N_h^n \\ N_a^n \\ N_b^n \end{bmatrix} = C(\phi^n)B(\xi^n)\begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} \tag{14}$$

Using the normal N^n the modeled plane is written in the h,a,b-coordinate system as

$$0 = N_h^n h + N_a^n a + N_b^n b + K_n \tag{15}$$

where K_n is a constant.

Now using Eqn. 15 as the modeled plane equation in the unprimed system, the cross-path and in-path slopes are found and labeled as:

$$\text{cross-path slope} = x_1^n = \frac{\partial h}{\partial a} = -\frac{N_a^n}{N_h^n} \tag{16}$$

$$\text{in-path slope} = x_2^n = \frac{\partial h}{\partial b} = -\frac{N_b^n}{N_h^n} \tag{17}$$

Therefore, the location, height, cross-path and in-path slopes are known for the center point P^n.

Repeating this process for a total of four planes results in four center points. This situation is shown in Figure 4 where the location, height and derivatives are found at each center point. Thus, there are twelve known quantities which are used in the terrain modeling process for each section modeled.

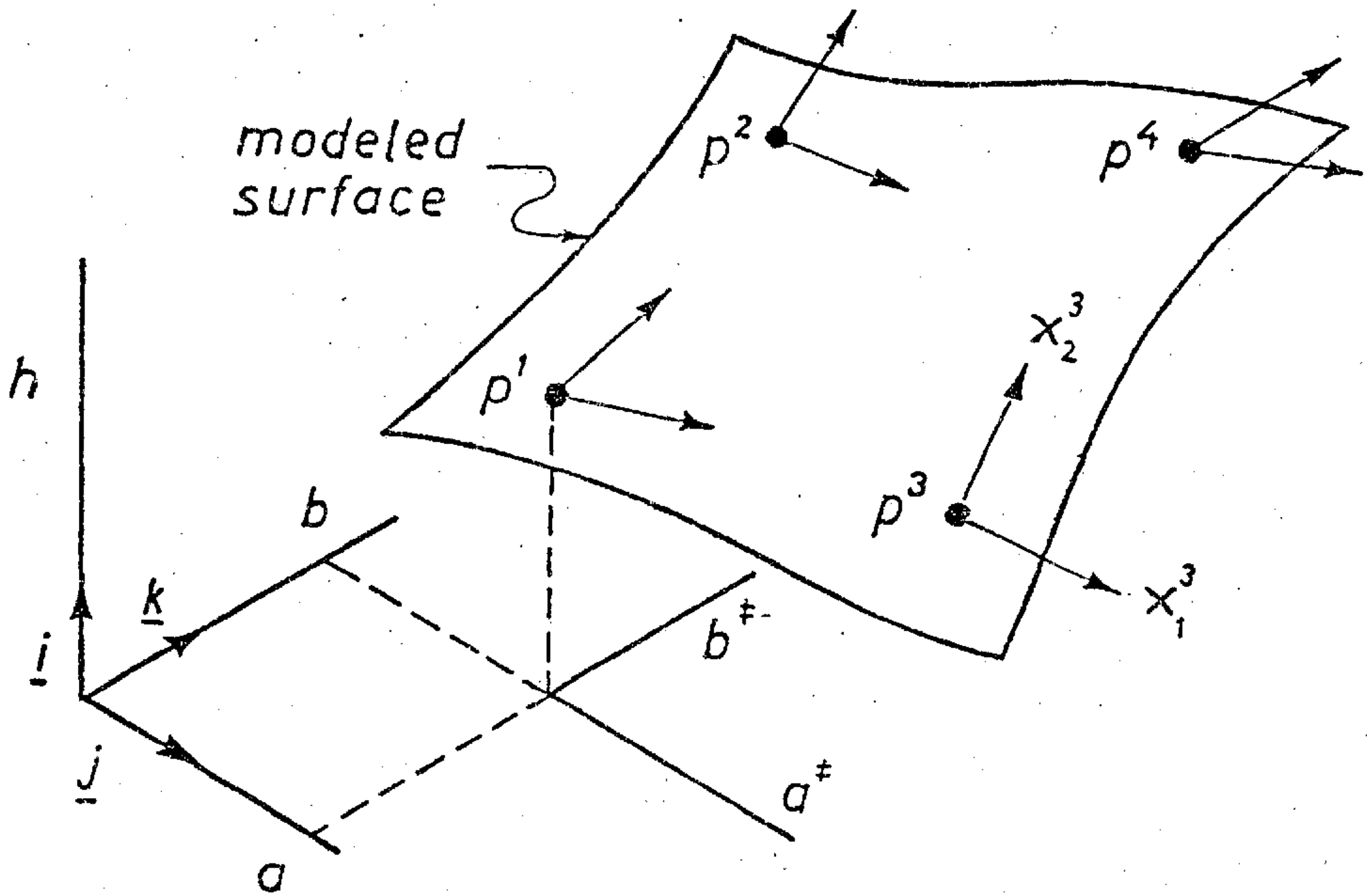

Figure 4 Illustration of a modeled surface with the four center points indicated

PART 4

MODELING PROCEDURE STEP II

A. Surface equation

To represent the terrain, a two-dimensional third order polynomial

$$h = C_{00} + C_{10} a + C_{01} b + C_{20} \frac{a^2}{2} + C_{11} ab + C_{02} \frac{b^2}{2} + C_{30} \frac{a^3}{6} + C_{21} \frac{a^2}{2} b + C_{12} a \frac{b^2}{2} + C_{03} \frac{b^3}{6} \tag{18}$$

was chosen where the C_{ij} 's are unknown parameters which must be determined in order to represent the surface.

B. Coordinate shifting

The polynomial used for the surface equation is centered around the origin and it models well near the origin and worsens as the distance from the origin increases. Therefore, the polynomial should be centered as close as possible to the section in which it is used. This can be accomplished by shifting the coordinate axis. A new coordinate system $a^{\ddagger}, b^{\ddagger}$ is formed by shifting the axis so that the point $(a_p^{\prime}, b_p^{\prime})$ is located at $(0,0)$ in the $a^{\ddagger}, b^{\ddagger}$ system (Figure 4). This is accomplished by the transformation

$$a^{\ddagger} = a - a_p^{\prime} \tag{19a}$$

$$b^{\ddagger} = b - b_p^{\prime} \tag{19b}$$

A new set of parameters is used to write the surface equation

$$h = C_{00}^{\ddagger} + C_{10}^{\ddagger} a^{\ddagger} + C_{01}^{\ddagger} b^{\ddagger} + C_{20}^{\ddagger} \frac{(a^{\ddagger})^2}{2} + C_{11}^{\ddagger} a^{\ddagger} b^{\ddagger} + C_{02}^{\ddagger} \frac{(b^{\ddagger})^2}{2} + C_{30}^{\ddagger} \frac{(a^{\ddagger})^3}{6} + C_{21}^{\ddagger} \frac{(a^{\ddagger})^2}{2} b^{\ddagger} + C_{12}^{\ddagger} a^{\ddagger} \frac{(b^{\ddagger})^2}{2} + C_{03}^{\ddagger} \frac{(b^{\ddagger})^3}{6} \tag{20}$$

Or this can be written in matrix notation as:

$$h = HM\,\underline{C}^{\ddagger} \tag{21}$$

where
$$HM = HM(a^{\ddagger}, b^{\ddagger}) = \left[1,\ a^{\ddagger},\ b^{\ddagger},\ \frac{(a^{\ddagger})^2}{2},\ a^{\ddagger}b^{\ddagger},\ \frac{(b^{\ddagger})^2}{2},\ \frac{(a^{\ddagger})^3}{6},\ \frac{(a^{\ddagger})^2}{2}b^{\ddagger},\ a^{\ddagger}\frac{(b^{\ddagger})^2}{2},\ \frac{(b^{\ddagger})^3}{6}\right] \tag{21a}$$

$$\underline{C}^{\ddagger} = \left[C_{00}, C_{10}, C_{01}, C_{20}, C_{11}, C_{02}, C_{30}, C_{21}, C_{12}, C_{03}\right]^T \tag{21b}$$

Expressions can easily be found for $\partial h/\partial a^{\ddagger}$ and $\partial h/\partial b^{\ddagger}$ from Eqn. 20. These are written in matrix notation as:

$$\partial h/\partial a^{\ddagger} = V\,\underline{C}^{\ddagger} \tag{22a}$$

$$\partial h/\partial b^{\ddagger} = Y\,\underline{C}^{\ddagger} \tag{22b}$$

where
$$V = \left[0, 1, 0, a^{\ddagger}, b^{\ddagger}, 0, (a^{\ddagger})^2/2, a^{\ddagger}b^{\ddagger}, (b^{\ddagger})^2/2, 0\right]$$

$$Y = \left[0, 0, 1, 0, a^{\ddagger}, b^{\ddagger}, 0, (a^{\ddagger})^2/2, a^{\ddagger}b^{\ddagger}, (b^{\ddagger})^2/2\right]$$

Since at each center point the location, height, $\partial h/\partial a^{\ddagger}$, and $\partial h/\partial b^{\ddagger}$, are known, 12 eqns. can be written relating the known quantities to the unknown parameters.

$$h_p^n = HM\left(a_p^{\ddagger n}, b_p^{\ddagger n}\right)\underline{C}^{\ddagger} \tag{23a}$$

$$\left(\partial h/\partial a^{\ddagger}\right)_p^n = X_1^n = V\left(a_p^{\ddagger n}, b_p^{\ddagger n}\right)\underline{C}^{\ddagger} \tag{23b}$$

$$\left(\partial h/\partial b^{\ddagger}\right)_p^n = X_2^n = Y\left(a_p^{\ddagger n}, b_p^{\ddagger n}\right)\underline{C}^{\ddagger} \qquad n = 1,2,3,4 \tag{23c}$$

These 12 eqns. are written in a single matrix equation by

$$\hat{W} = \hat{T}\,\underline{C}^{\ddagger} \tag{24}$$

where $$\hat{W} = \left[h_p^1, x_1^1, x_2^1, h_p^2, x_1^2, x_2^2, h_p^3, x_1^3, x_2^3, h_p^4, x_1^4, x_2^4\right]^T$$

$$\hat{T} = \begin{bmatrix} HM\left(a_p^{\ddagger 1}, b_p^{\ddagger 1}\right) \\ V\left(a_p^{\ddagger 1}, b_p^{\ddagger 1}\right) \\ Y\left(a_p^{\ddagger 1}, b_p^{\ddagger 1}\right) \\ HM\left(a_p^{\ddagger 2}, b_p^{\ddagger 2}\right) \\ V\left(a_p^{\ddagger 2}, b_p^{\ddagger 2}\right) \\ Y\left(a_p^{\ddagger 2}, b_p^{\ddagger 2}\right) \\ \vdots \\ HM\left(a_p^{\ddagger 4}, b_p^{\ddagger 4}\right) \\ V\left(a_p^{\ddagger 4}, b_p^{\ddagger 4}\right) \\ Y\left(a_p^{\ddagger 4}, b_p^{\ddagger 4}\right) \end{bmatrix}$$

C. Matrix order reduction

Because of the coordinate shift, by definition the $a^{\ddagger}, b^{\ddagger}$-coordinates of center point number one are $(0,0)$. This allows three parameters to be written immediately by utilizing Eqns. 23. Thus

$$\begin{aligned} h_p^1 &= C_{00}^{\ddagger} \\ x_1^1 &= C_{10}^{\ddagger} \\ x_2^1 &= C_{01}^{\ddagger} \end{aligned} \tag{25}$$

By utilizing the above values for the first three parameters and eliminating the first three rows of Eqn. 24, a new matrix equation is written as

$$W = T\underline{C1}^{*} \tag{26}$$

where

$$W = \begin{bmatrix} h_p^2 - h_p^1 - a_p^{*2}x_1^1 - b_p^{*2}x_2^1 \\ x_1^2 - x_1^1 \\ x_2^2 - x_2^1 \\ h_p^3 - h_p^1 - a_p^{*3}x_1^1 - b_p^{*3}x_2^1 \\ x_1^3 - x_1^1 \\ x_2^3 - x_2^1 \\ h_p^4 - h_p^1 - a_p^{*4}x_1^1 - b_p^{*4}x_2^1 \\ x_1^4 - x_1^1 \\ x_2^4 - x_2^1 \end{bmatrix} \tag{26a}$$

and

$$T = \begin{bmatrix} \frac{(a_p^{*2})^2}{2} & a_p^{*2}b_p^{*2} & \frac{(b_p^{*2})^2}{2} & \frac{(a_p^{*2})^3}{6} & \frac{(a_p^{*2})^2}{2}b_p^{*2} & a_p^{*2}\frac{(b_p^{*2})^2}{2} & \frac{(b_p^{*2})^3}{6} \\ a_p^{*2} & b_p^{*2} & 0 & \frac{(a_p^{*2})^2}{2} & a_p^{*2}b_p^{*2} & \frac{(b_p^{*2})^2}{2} & 0 \\ 0 & a_p^{*2} & b_p^{*2} & 0 & \frac{(a_p^{*2})^2}{2} & a_p^{*2}b_p^{*2} & \frac{(b_p^{*2})^2}{2} \\ & & & \vdots & & & \\ 0 & a_p^{*4} & b_p^{*4} & 0 & \frac{(a_p^{*4})^2}{2} & a_p^{*4}b_p^{*4} & \frac{(b_p^{*4})^2}{2} \end{bmatrix} \tag{26b}$$

and $$\underline{C1}^{\ddagger} = \left[C_{20}^{\ddagger}, C_{11}^{\ddagger}, C_{02}^{\ddagger}, C_{30}^{\ddagger}, C_{21}^{\ddagger}, C_{12}^{\ddagger}, C_{03}^{\ddagger} \right]^T \qquad (26c)$$

This manipulation reduces $\hat{W}$ (12 x 1) to W (9 x 1), $\hat{T}$ (12 x 10) to T (9 x 7), and $\underline{C}^{\ddagger}$ (10 x 1) to $\underline{C1}^{\ddagger}$ (7 x 1).

D. Stochastic fitting

Both W and T in Eqn. 26 are known quantities and $\underline{C1}^{\ddagger}$ is the unknown vector to be determined. Since the order of W is higher than that of $\underline{C1}^{\ddagger}$ (9 to 7) this system of equations is overdetermined. Therefore, a stochastic fit must be used. The method of least squares estimation was chosen to perform the stochastic fit. This is accomplished by the matrix equation

$$\underline{C1}^{\ddagger} = \left(T^T T\right)^{-1} T^T W \qquad (27)$$

Eqn. 27 requires the inversion of $\left(T^T T\right)$ a 7 x 7 matrix.

Once the $\underline{C1}^{\ddagger}$ vector is determined, the $\underline{C}^{\ddagger}$ vector is also determined. Thus, the surface polynomial, Eqn. 20, can be written. This allows the calculation of the modeled height, cross-path and in-path slopes for any location (a,b). With this information, the gradient at location (a,b) can be calculated by

$$\text{Gradient} = SG = \left[\left(\partial h/\partial a\right)^2 + \left(\partial h/\partial b\right)^2\right]^{1/2} \qquad (28)$$

PART 5

ERROR ANALYSIS

A. Covariance matrix for the data points

Because of instrumental inaccuracies in measuring θ, β and R, there is error involved in the determination of a", b" and h" for each measured data point. This error can be expressed by an error covariance matrix for each data point.[1]

$$M_\ell''^{n} = E\left\{ \begin{bmatrix} \delta h_\ell''^{n} \\ \delta a_\ell''^{n} \\ \delta b_\ell''^{n} \end{bmatrix} \begin{bmatrix} \delta h_\ell''^{n} & \delta a_\ell''^{n} & \delta b_\ell''^{n} \end{bmatrix} \right\} \tag{29}$$

$$= G_\ell^{n} \begin{bmatrix} E(\delta R)^2 & 0 & 0 \\ 0 & E(\delta\beta)^2 & 0 \\ 0 & 0 & E(\delta\theta)^2 \end{bmatrix} G_\ell^{nT} \qquad \begin{matrix} \ell = 1,2,3,4 \\ n = 1,2,3,4 \end{matrix}$$

where

$$G_\ell^{n} = \begin{bmatrix} -\sin\beta_\ell^{n} & -R_\ell^{n}\cos\beta_\ell^{n} & 0 \\ \cos\beta_\ell^{n}\sin\theta_\ell^{n} & -R_\ell^{n}\sin\beta_\ell^{n}\sin\theta_\ell^{n} & R_\ell^{n}\cos\beta_\ell^{n}\cos\theta_\ell^{n} \\ \cos\beta_\ell^{n}\cos\theta_\ell^{n} & -R_\ell^{n}\sin\beta_\ell^{n}\cos\theta_\ell^{n} & -R_\ell^{n}\cos\beta_\ell^{n}\sin\theta_\ell^{n} \end{bmatrix}$$

where E denotes expected value and δR, $\delta\beta$, & $\delta\theta$ are assumed uncorrelated. This equation relates the standard deviations of h''^{n}, a''^{n}, b''^{n} for each data point to the standard deviation of R, β and θ which are known quantities. There are 16 of these matricies.

B. Covariance matrix of the slopes in the primed system

Because of inaccuracies in measuring the data points, the modeled plane is also subject to inaccuracies. The error covariance matrix for the slopes can be expressed as a function of the covariances of the four data points which are used in modeling the plane.[1]

$$E\left\{\begin{bmatrix}\delta x_1''^n\\ \delta x_2''^n\\ \delta x_3''^n\end{bmatrix}\begin{bmatrix}\delta x_1''^n & \delta x_2''^n & \delta x_3''^n\end{bmatrix}\right\} = F\left\{E\left[\delta\underline{h}''^n\,\delta\underline{h}''^{nT}\right]\right. \tag{30}$$

$$\left.-E\left[\left(\delta A''\underline{x}''^n\right)\delta\underline{h}''^{nT}\right]-E\left[\delta\underline{h}''^n\left(\delta A''\underline{x}''^n\right)^T\right]+E\left[\left(\delta A''^n\underline{x}''^n\right)\left(\delta A''^n\underline{x}''^n\right)^T\right]\right\}F^T$$

where

$$F=\left(A''^TA''\right)^{-1}A''^T$$

These values can easily be evaluated and they relate the covariance matrix of the slopes to the standard deviation of the measured quantities. There are four of these matricies for each section modeled.

C. Covariance matrix for the center points in the primed system

Using perturbation technique, the error covariance matrix for the center points in the primed system can be evaluated. Since there are six variables of interest, the covariance matrix for the center points is a 6 x 6 matrix. Because some of the quantities in this matrix are a function of other quantities in the same matrix, the covariance matrix may best be evaluated by partitions, as shown in Eqn. 31.

$$M_p^{\prime\prime n} = E\left\{ \begin{bmatrix} \delta h_p^{\prime\prime n} \\ \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \\ \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta h_p^{\prime\prime n} & \delta a_p^{\prime\prime n} & \delta b_p^{\prime\prime n} & \delta x_1^{\prime\prime n} & \delta x_2^{\prime\prime n} & \delta x_3^{\prime\prime n} \end{bmatrix} \right\}$$

$$= E\left\{ \left[\begin{array}{c:c:c} \begin{bmatrix} \delta h_p^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta h_p^{\prime\prime n} \end{bmatrix} & \multicolumn{2}{c}{\begin{bmatrix} \delta h_p^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta a_p^{\prime\prime n} & \delta b_p^{\prime\prime n} & \delta x_1^{\prime\prime n} & \delta x_2^{\prime\prime n} & \delta x_3^{\prime\prime n} \end{bmatrix}} \\ \hdashline \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \\ \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta h_p^{\prime\prime n} \end{bmatrix} & \begin{array}{c} \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta a_p^{\prime\prime n} & \delta b_p^{\prime\prime n} \end{bmatrix} \\ \hdashline \begin{bmatrix} \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta a_p^{\prime\prime n} & \delta b_p^{\prime\prime n} \end{bmatrix} \end{array} & \begin{array}{c} \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta x_1^{\prime\prime n} & \delta x_2^{\prime\prime n} & \delta x_3^{\prime\prime n} \end{bmatrix} \\ \hdashline \begin{bmatrix} \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta x_1^{\prime\prime n} & \delta x_2^{\prime\prime n} & \delta x_3^{\prime\prime n} \end{bmatrix} \end{array} \end{array} \right] \right\}$$

$$M_p''^{n} = \left\{ \left[\begin{array}{c:cc} \mathrm{I} & \multicolumn{2}{c}{\mathrm{III}} \\ \hdashline & \mathrm{IV} & \mathrm{VI} \\ \cdashline{2-3} \mathrm{II} & \mathrm{V} & \mathrm{VII} \end{array} \right] \right\} \tag{31}$$

Block IV can be evaluated as shown in Appendix A and yields

$$E\left\{ \begin{bmatrix} \delta a_p''^{n} \\ \delta b_p''^{n} \end{bmatrix} \begin{bmatrix} \delta a_p''^{n} \\ \delta b_p''^{n} \end{bmatrix}^T \right\} = Q\, E\left\{ \begin{bmatrix} \delta a_1''^{n} \\ \delta a_2''^{n} \\ \vdots \\ \delta b_4''^{n} \end{bmatrix} \begin{bmatrix} \delta a_1''^{n} \\ \delta a_2''^{n} \\ \vdots \\ \delta b_4''^{n} \end{bmatrix}^T \right\} Q^T \tag{32}$$

where

$$Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

The separate terms in Block V can be evaluated as

$$E\left(\delta x_j''^{n}\, \delta a_p''^{n}\right) =$$

$$\frac{1}{4}\left[1, -x_1''^{n}, -x_2''^{n}\right] \left\{ f_{j1} \begin{bmatrix} E\left(\delta h_1''^{n}\, \delta a_1''^{n}\right) \\ E\left(\delta a_1''^{n}\, \delta a_1''^{n}\right) \\ E\left(\delta b_1''^{n}\, \delta a_1''^{n}\right) \end{bmatrix} + f_{j2} \begin{bmatrix} E\left(\delta h_2''^{n}\, \delta a_2''^{n}\right) \\ E\left(\delta a_2''^{n}\, \delta a_2''^{n}\right) \\ E\left(\delta b_2''^{n}\, \delta a_2''^{n}\right) \end{bmatrix} + \cdots \right\} \tag{33}$$

$$j = 1, 2, 3$$

and

$$E\left(\delta x_j''^{n}\, \delta b_p''^{n}\right) =$$

$$\frac{1}{4}\left[1, -x_1''^n, -x_2''^n\right]\left\{ f_{j1}\begin{bmatrix} E(\delta h_1''^n \delta b_1''^n) \\ E(\delta a_1''^n \delta b_1''^n) \\ E(\delta b_1''^n \delta b_1''^n) \end{bmatrix} + f_{j2}\begin{bmatrix} E(\delta h_2''^n \delta b_2''^n) \\ E(\delta a_2''^n \delta b_2''^n) \\ E(\delta b_2''^n \delta b_2''^n) \end{bmatrix} + \cdots \right\} \quad (34)$$

Where f_{ji} are elements of $F = (A^T A)^{-1} A^T$. $j = 1,2,3$

These terms are derived in Appendix B and can be calculated directly from Eqn. 29.

Because the covariance matrix (Eqn. 31) is symmetric, Block VI is just the transpose of Block V.

Block VII is equal to the covariance matrix found in Eqn. 30.

Block II can be expressed as

$$E\left\{ \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} \begin{bmatrix} \delta h_p''^n \end{bmatrix} \right\} = E\left\{ \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix}^T \right\} S^{nT} \quad (35)$$

where

$$S^n = \left[x_1''^n \;\; x_2''^n \;\; a_p''^n \;\; b_p''^n \;\; 1 \right]$$

This equation is derived in Appendix C. Note that the expected value matrix used in Eqn. 35 is a subset of the covariance matrix in Eqn. 31. This subset $(M1''^n)$ is composed of blocks IV, VI, V, and VII. These blocks have been calculated previously; therefore, Eqn. 35 can be calculated. Similarly, Block III is the transpose of Block II.

Finally, the last block of Eqn. 31 can be calculated by the equation

$$E\left\{\left[\delta h_p^{''n}\right]\left[\delta h_p^{''n}\right]\right\} = S^n M1^{''n} S^{nT} \tag{36}$$

where the quantities S^n and $M1^{''n}$ are defined in Eqn. 35.

Thus, the covariance matricies in the primed system can be calculated for the four center points. This matrix relates the standard deviations of the quantities at the center points to the standard deviations of R, θ, and β.

D. <u>Covariance matrix of the center points in the unprimed system</u>

In the modeling process, the center points were transformed into the unprimed system before the polynomial parameters were evaluated. Therefore, the error covariance matrix of the center points must also be transformed into the unprimed system. In this stage, the error in roll and pitch measurement is introduced into the model.

The covariance matrix of the center points in the unprimed system is defined as:

$$M_p^n = E\left\{\begin{bmatrix}\delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \\ \delta X_1^n \\ \delta X_2^n\end{bmatrix}\begin{bmatrix}\delta h_p^n & \delta a_p^n & \delta b_p^n & \delta X_1^n & \delta X_2^n\end{bmatrix}\right\} \tag{37}$$

Note the absence of any δX_3^n terms. These are not included since they are not used in the modeling process. Again breaking this matrix into blocks

$$M_p^n = E\left\{\left[\begin{array}{c|c} \begin{bmatrix}\delta h_p^n\\ \delta a_p^n\\ \delta b_p^n\end{bmatrix}\begin{bmatrix}\delta h_p^n & \delta a_p^n & \delta b_p^n\end{bmatrix} & \begin{bmatrix}\delta h_p^n\\ \delta a_p^n\\ \delta b_p^n\end{bmatrix}\begin{bmatrix}\delta x_1^n & \delta x_2^n\end{bmatrix} \\ \hline \begin{bmatrix}\delta x_1^n\\ \delta x_2^n\end{bmatrix}\begin{bmatrix}\delta h_p^n & \delta a_p^n & \delta b_p^n\end{bmatrix} & \begin{bmatrix}\delta x_1^n\\ \delta x_2^n\end{bmatrix}\begin{bmatrix}\delta x_1^n & \delta x_2^n\end{bmatrix}\end{array}\right]\right\} \tag{38}$$

$$M_p^n = E\left\{\left[\begin{array}{c|c} A & C \\ \hline B & D\end{array}\right]\right\}$$

Block A may be evaluated by using the expression

$$E\left\{\begin{bmatrix}\delta h_p^n\\ \delta a_p^n\\ \delta b_p^n\end{bmatrix}\begin{bmatrix}\delta h_p^n & \delta a_p^n & \delta b_p^n\end{bmatrix}\right\} = \tag{39}$$

$$D^n\begin{bmatrix}E(\delta\phi)^2 & 0\\ 0 & E(\delta\xi)^2\end{bmatrix}D^{nT} + C^n B^n E\left\{\begin{bmatrix}\delta h_p''^n\\ \delta a_p''^n\\ \delta b_p''^n\end{bmatrix}\begin{bmatrix}\delta h_p''^n & \delta a_p''^n & \delta b_p''^n\end{bmatrix}\right\}B^{nT}C^{nT}$$

where[1] D^n =

$$\begin{bmatrix} -h_p''^n \sin\phi^n \cos\xi^n - a_p''^n \cos\phi^n - b_p''^n \sin\phi^n \sin\xi^n & -h_p''^n \cos\phi^n \sin\xi^n + b_p''^n \cos\phi^n \cos\xi^n \\ h_p''^n \cos\phi^n \cos\xi^n - a_p''^n \sin\phi^n + b_p''^n \cos\phi^n \sin\xi^n & -h_p''^n \sin\phi^n \sin\xi^n + b_p''^n \sin\phi^n \cos\xi^n \\ 0 & -h_p''^n \cos\xi^n - b_p''^n \sin\xi^n \end{bmatrix}$$

and B^n and C^n are defined in Eqn. 9.

The derivation for the equation is shown in Appendix D. Notice that this expression is a function of the standard deviation of pitch and roll and also of the covariance matrix of the center points in the primed system, Eqn. 31. This will be the case for all of the blocks in Eqn. 38.

For evaluating Block B, the expression

$$E\left\{\begin{bmatrix}\delta x_1^n \\ \delta x_2^n\end{bmatrix}\begin{bmatrix}\delta h_p^n & \delta a_p^n & \delta b_p^n\end{bmatrix}\right\} = \tag{40}$$

$$U^n D_x^n \begin{bmatrix} E(\delta\phi)^2 & 0 \\ 0 & E(\delta\xi)^2 \end{bmatrix} D^{nT} + U^n C^n B^n E\left\{\begin{bmatrix} 0 \\ \delta x_1''^n \\ \delta x_2''^n \end{bmatrix}\begin{bmatrix}\delta h_p''^n & \delta a_p''^n & \delta b_p''^n\end{bmatrix}\right\} B^{nT} C^{nT}$$

where

$$U^n = \begin{bmatrix} \dfrac{N_a^n}{(N_h^n)^2} & -\dfrac{1}{N_h^n} & 0 \\ \dfrac{N_b^n}{(N_h^n)^2} & 0 & -\dfrac{1}{N_h^n} \end{bmatrix}$$

and

$$D_x^n = \begin{bmatrix} \sin\phi^n \cos\xi^n - x_1''^n \cos\phi^n - x_2''^n \sin\phi^n \sin\xi^n & \cos\phi^n \sin\xi^n + x_2''^n \cos\phi^n \cos\xi^n \\ -\cos\phi^n \cos\xi^n - x_1''^n \sin\phi^n + x_2''^n \cos\phi^n \sin\xi^n & \sin\phi^n \sin\xi^n + x_2''^n \sin\phi^n \cos\xi^n \\ 0 & \cos\xi^n - x_2''^n \sin\xi^n \end{bmatrix}$$

is found. The derivation is shown in Appendix E. Since the covariance matrix Eqn. 38 is symmetric, Block C is just the transpose of Block B.

Finally, Block D is found by squaring Eqn. E-10 and taking the expected value

$$E\left\{\begin{bmatrix}\delta x_1^n \\ \delta x_2^n\end{bmatrix}\begin{bmatrix}\delta x_1^n & \delta x_2^n\end{bmatrix}\right\} =$$

$$E\left\{\left(U^n D_x^n\begin{bmatrix}\delta\phi \\ \delta\xi\end{bmatrix} + U^n C^n B^n\begin{bmatrix}0 \\ \delta x_1''^n \\ \delta x_2''^n\end{bmatrix}\right)\left(U^n D_x^n\begin{bmatrix}\delta\phi \\ \delta\xi\end{bmatrix} + U^n C^n B^n\begin{bmatrix}0 \\ \delta x_1''^n \\ \delta x_2''^n\end{bmatrix}\right)^T\right\} \tag{41}$$

Now eliminating non-correllated terms

$$E\left\{\begin{bmatrix}\delta x_1^n \\ \delta x_2^n\end{bmatrix}\begin{bmatrix}\delta x_1^n & \delta x_2^n\end{bmatrix}\right\} = U^n D_x^n\begin{bmatrix}E(\delta\phi)^2 & 0 \\ 0 & E(\delta\xi)^2\end{bmatrix}D_x^{nT}U^{nT}$$

$$+ U^n C^n B^n E\left\{\begin{bmatrix}0 \\ \delta x_1''^n \\ \delta x_2''^n\end{bmatrix}\begin{bmatrix}0 & \delta x_1''^n & \delta x_2''^n\end{bmatrix}\right\}B^{nT}C^{nT}U^{nT} \tag{42}$$

Therefore, the error covariance matricies for the four center points in the unprimed coordinate system can be determined.

E. Covariance matrix of the modeling parameters

The covariance matrix of the $C_{ij}^{\ddagger}$ parameters used in the surface polynomial must be found. This 10 x 10 matrix is defined as

$$M_c = E\left\{ \begin{bmatrix} \delta C_{00}^{\ddagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger} \\ \vdots \\ \delta C_{03}^{\ddagger} \end{bmatrix} \begin{bmatrix} \delta C_{00}^{\ddagger} & \delta C_{10}^{\ddagger} & \delta C_{01}^{\ddagger} & \cdots & \delta C_{03}^{\ddagger} \end{bmatrix} \right\} \tag{43}$$

Matrix M_c may be broken down into blocks as

$$M_c = E\left\{ \left[\begin{array}{c:c} \begin{bmatrix} \delta C_{00}^{\ddagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger} \end{bmatrix} \begin{bmatrix} \delta C_{00}^{\ddagger} & \delta C_{10}^{\ddagger} & \delta C_{01}^{\ddagger} \end{bmatrix} & \begin{bmatrix} \delta C_{00}^{\ddagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger} \end{bmatrix} \delta \underline{C1}^{\ddagger T} \\ \hdashline \delta \underline{C1}^{\ddagger} \begin{bmatrix} \delta C_{00}^{\ddagger} & \delta C_{10}^{\ddagger} & \delta C_{01}^{\ddagger} \end{bmatrix} & \delta \underline{C1}^{\ddagger} \; \delta \underline{C1}^{\ddagger T} \end{array} \right] \right\}$$

$$M_c = E\left\{ \left[\begin{array}{c:c} M_c\mathrm{I} & M_c\mathrm{III} \\ \hdashline M_c\mathrm{II} & M_c\mathrm{IV} \end{array} \right] \right\} \tag{44}$$

Covariance Block $M_c I$ may be easily determined as (See Appendix F)

$$M_c I = R M_p^1 R^T \tag{45}$$

where

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Block $M_c II$ can be evaluated by the expression derived in Appendix G

$$M_c II = -Z \left\{ \begin{bmatrix} \Omega_{22} \\ \Omega_{33} \\ \Omega_{44} \end{bmatrix} \right\} M_p^1 R^T \tag{46}$$

where $Z = (T^T T)^{-1} T^T$

and Ω_{ij} is defined by Eqns. G-16a, G-19, and G-11a.

M_p^n is the covariance matrix for the center point found in Part III d.

Block $M_c III$ is the transpose of Block $M_c II$ since M_c is a symmetric matrix.

The expression for Block III is derived in Appendix H and is given by

$$M_c III = Z \left\{ \begin{bmatrix} \Omega_{12} M_p^2 \Omega_{12}^T & 0 & 0 \\ 0 & \Omega_{13} M_p^3 \Omega_{13}^T & 0 \\ 0 & 0 & \Omega_{14} M_p^4 \Omega_{14}^T \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} \Omega_{22} M_p^1 \Omega_{22}^T & \cdots & \Omega_{22} M_p^1 \Omega_{44}^T \\ \vdots & & \vdots \\ \Omega_{44} M_p^1 \Omega_{22}^T & \cdots & \Omega_{44} M_p^1 \Omega_{44}^T \end{bmatrix} \right\} Z^T \tag{47}$$

Thus the covariance matrix of the C_{ij} parameters can be determined.

F. Standard deviation of height

Once the covariance matrix of the parameters have been determined, the standard deviation of height can be found. This value is a function of location. From Eqn. 21 the height of any point $(a^\ddagger, b^\ddagger)$ can be determined. Perturbing Eqn. 21 yields

$$\delta h = HM\, \delta \underline{C}^\ddagger \tag{48}$$

Finding the expected value of Eqn. 48

$$E(\delta h\, \delta h^T) = HM\; \delta \underline{C}^\ddagger \delta \underline{C}^{\ddagger T}\, HM^T \tag{49}$$

However, from Eqn. 43, which defines $\delta \underline{C}^{\ddagger} \delta \underline{C}^{\ddagger T}$ as M_c

$$E(\delta h)^2 = HM(M_c)HM^T \tag{50}$$

From the definition of standard deviation,

$$\sigma_H = \left\{ HM(M_c)HM^T \right\}^{1/2} \tag{51}$$

where $\sigma_H = \left\{ E(\delta h)^2 \right\}^{1/2}$ standard deviation of height.

G. Standard deviation of gradient

The values of $\delta\left(\partial h/\partial a^{\ddagger}\right)$ and $\delta\left(\partial h/\partial b^{\ddagger}\right)$ can be found by perturbing Eqns. 22a and 22b.

$$\begin{bmatrix} \delta\left(\dfrac{\partial h}{\partial a^{\ddagger}}\right) \\ \delta\left(\dfrac{\partial h}{\partial b^{\ddagger}}\right) \end{bmatrix} = \begin{bmatrix} V\,\delta\underline{C}^{\ddagger} \\ Y\,\delta\underline{C}^{\ddagger} \end{bmatrix} \tag{52}$$

The covariance matrix of the slopes is defined as

$$M_{SLOPE} = E\left\{ \begin{bmatrix} \delta\left(\dfrac{\partial h}{\partial a^{\ddagger}}\right) \\ \delta\left(\dfrac{\partial h}{\partial b^{\ddagger}}\right) \end{bmatrix} \begin{bmatrix} \delta\left(\dfrac{\partial h}{\partial a^{\ddagger}}\right) & \delta\left(\dfrac{\partial h}{\partial b^{\ddagger}}\right) \end{bmatrix} \right\} \tag{53}$$

By using Eqn. 52, this becomes

$$M_{SLOPE} = E\left\{\begin{bmatrix} V\,\delta\underline{C}^{\ddagger} \\ Y\,\delta\underline{C}^{\ddagger} \end{bmatrix}\begin{bmatrix} \delta\underline{C}^{\ddagger T}V^T & \delta\underline{C}^{\ddagger T}Y^T \end{bmatrix}\right\}$$

$$= E\begin{Bmatrix} V\,\delta\underline{C}^{\ddagger}\,\delta\underline{C}^{\ddagger T}V^T & V\,\delta\underline{C}^{\ddagger}\,\delta\underline{C}^{\ddagger T}Y^T \\ Y\,\delta\underline{C}^{\ddagger}\,\delta\underline{C}^{\ddagger T}V^T & Y\,\delta\underline{C}^{\ddagger}\,\delta\underline{C}^{\ddagger T}Y^T \end{Bmatrix} \tag{54}$$

However, from Eqn. 43

$$M_{SLOPE} = \begin{bmatrix} V M_c V^T & V M_c Y^T \\ Y M_c V^T & Y M_c Y^T \end{bmatrix} \tag{55}$$

From Eqn. 28, the value for gradient, S_g, is given as

$$\text{Gradient} = SG = \left[\left(\partial h/\partial a\right)^2 + \left(\partial h/\partial b\right)^2\right]^{1/2} \tag{56}$$

Perturbing Eqn. 56 yields

$$\delta SG = \left[\left(\partial h/\partial a\right)^2 + \left(\partial h/\partial b\right)^2\right]^{-1/2}\left(\partial h/\partial a\right)\,\delta\left(\partial h/\partial a\right)$$

$$+ \left[\left(\partial h/\partial a\right)^2 + \left(\partial h/\partial b\right)^2\right]^{-1/2}\left(\partial h/\partial b\right)\,\delta\left(\partial h/\partial b\right) \tag{57}$$

Writing in matrix notation

$$\delta SG = \frac{1}{SG} L \begin{bmatrix} \delta\left(\frac{\partial h}{\partial a}\right) \\ \delta\left(\frac{\partial h}{\partial b}\right) \end{bmatrix} \tag{58}$$

Where $$\frac{1}{SG} = \left[\left(\partial h/\partial a\right)^2 + \left(\partial h/\partial b\right)^2\right]^{-1/2}$$

$$L = \left[\left(\partial h/\partial a\right) \quad \left(\partial h/\partial b\right)\right]$$

Finding the expected value of gradient from Eqn. 58

$$\sigma_{SG}^{2} = E\left\{(\delta SG)(\delta SG)^T\right\} = \frac{1}{SG} L\left[M_{SLOPE}\right] L^T \frac{1}{SG} \tag{59}$$

or finally

$$\sigma_{SG} = \frac{1}{SG}\left\{L\left[M_{SLOPE}\right]L^T\right\}^{1/2} \tag{60}$$

Thus, the standard deviation of any point on the modeled surface can be calculated.

PART 6

NUMERICAL RESULTS

The modeling procedure explained in Parts 2-5 was simulated, using a computer program. This program simulates the scanning process, models the polynomial and performs the error analysis. These results vary with every terrain configuration and scanning parameters. Therefore, only one detailed example will be presented here as an illustration of the developed modeling procedure.

The example terrain surface is shown in Figure 5. Here the vehicle is located on level ground, traveling towards the center of a mound located 23 meters away from the front of the vehicle. The mound is a gaussian hill larger in width than in depth, with a maximum height of two meters. The equation of this hill is

$$h = 2e^{-0.08\ (b-23)^2 - 0.05a^2}$$

where h,a,b refer to the inertial coordinate system.

Enough data points were taken to allow ten polynomial sections to be modeled. Only one section will be looked at in detail. The data points for this section are shown graphically in Figure 5. Shown here, also, are the two "W" shaped scan rows. Although there is roll and pitch of the vehicle between each row, the points in a row are assumed to be taken so fast that the vehicle essentially does not roll or pitch between data points.

The actual scan was carried out by using constant

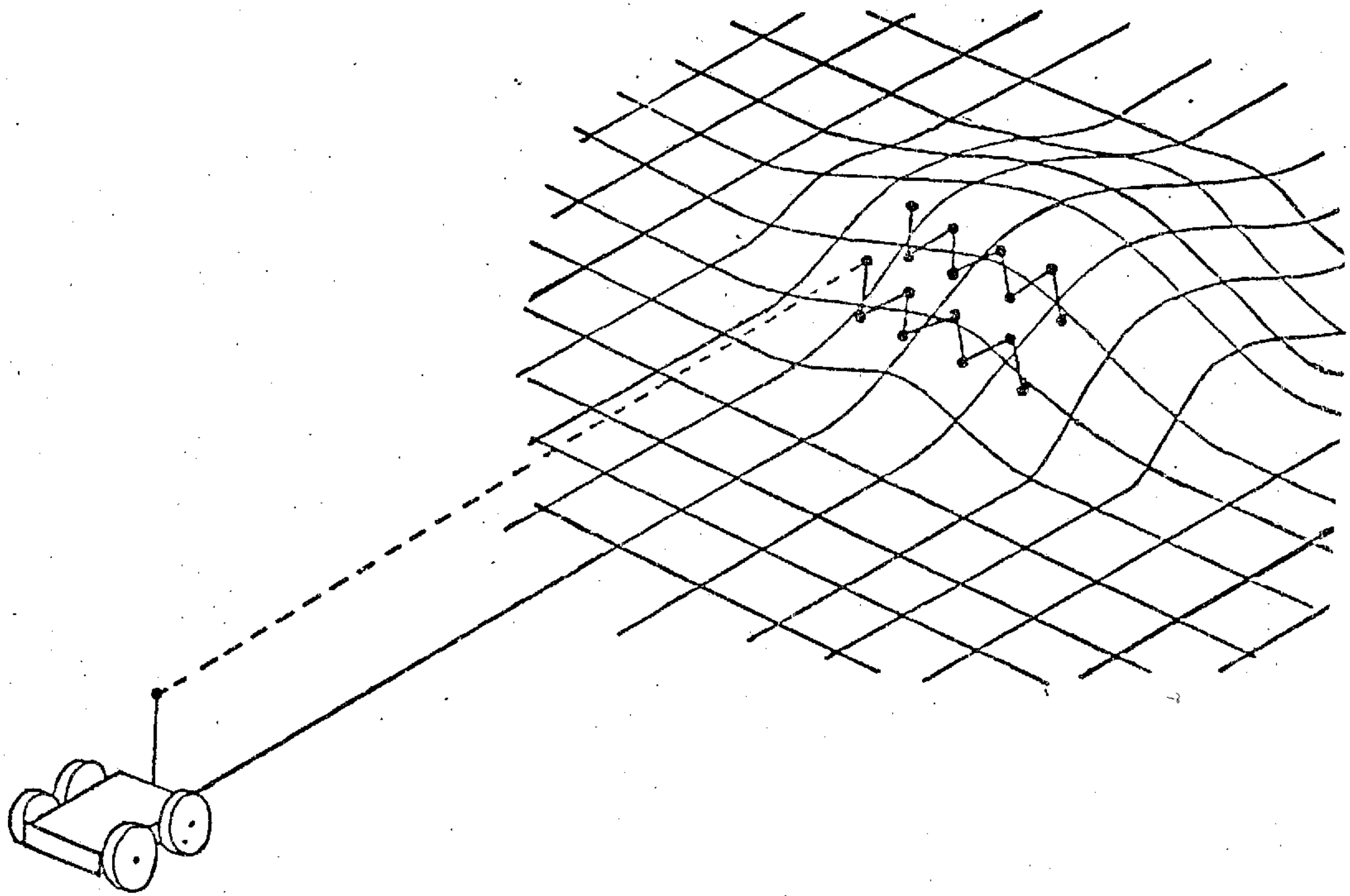

Figure 5 Illustration of gaussian hill used for the example

$\Delta\beta$ and $\Delta\theta$ increments. The $\Delta\beta$ angle between successive points in a scan row was .03261 rad., while the β_{INC} angle between corresponding points in the two scan rows was .06523 rad. The $\Delta\theta$ spacing between successive points in both rows was .016305 rad. The actual data for the 16 data points is shown in Table 1. Here the data point number refers to the row number (first column) and the number of the data point left to right (second column). For each measured point, the vehicle transmits the laser beam at a certain elevation and azimuth angle and receives the range measurement. It also measures the roll and pitch angle corresponding to that point.

The first step in the modeling process is to model planes in the vehicle's coordinate system from sets of four data points. Here no data point overlap was used. The data points which were used in each plane are listed below:

Plane 1 - (1,1), (1,2), (1,3), (1,4)
Plane 2 - (1,5), (1,6), (1,7), (1,8)
Plane 3 - (2,1), (2,2), (2,3), (2,4)
Plane 4 - (2,5), (2,6), (2,7), (2,8)

The height, location, cross-path and in-path slopes of the center points were found in the h",a",b"-coordinate system. These quantities were then transformed into the h,a,b-coordinate system.

The numerical results are shown in Table 2, where HP is the height of the center point, AP and BP are the a and b coordinates of the center point and XP1 and XP2 are the cross-path and in-path slopes respectively.

The next step is forming the surface equation

DATA POINT	ELEVATION	AZIMUTH	RANGE	ROLL	PITCH	HEIGHT	A	B
1, 1	0.05268	-0.05000	22.42480	0.0	0.0	1.81926	-1.11917	22.36472
1, 2	0.08529	-0.03369	20.69824	0.0	0.0	1.23689	-0.69473	20.61031
1, 3	0.05268	-0.01739	22.05371	0.0	0.0	1.83882	-0.38295	22.01880
1, 4	0.08529	-0.00108	20.63574	0.0	0.0	1.24222	-0.02231	20.55974
1, 5	0.05268	0.01522	22.04395	0.0	0.0	1.83923	0.33505	22.01176
1, 6	0.08529	0.03153	20.69043	0.0	0.0	1.23756	0.64976	20.60399
1, 7	0.05268	0.04783	22.38184	0.0	0.0	1.82155	1.06858	22.32423
1, 8	0.08529	0.06413	20.87207	0.0	0.0	1.22192	1.33291	20.75439
2, 1	0.11791	-0.05000	19.60449	0.0	0.0	0.69393	-0.97296	19.44305
2, 2	0.15052	-0.03369	18.22363	0.0	0.0	0.26720	-0.60702	18.00830
2, 3	0.11791	-0.01739	19.53223	0.0	0.0	0.70220	-0.33731	19.39462
2, 4	0.15052	-0.00108	18.20410	0.0	0.0	0.27013	-0.01953	17.99922
2, 5	0.11791	0.01522	19.53027	0.0	0.0	0.70266	0.29516	19.39143
2, 6	0.15052	0.03153	18.22168	0.0	0.0	0.26779	0.56782	18.00572
2, 7	0.11791	0.04783	19.59863	0.0	0.0	0.69438	0.93058	19.44125
2, 8	0.15052	0.06413	18.27637	0.0	0.0	0.25958	1.15804	18.03160

Table 1 Data points used for terrain model

CENTER POINT NUMBER 1		
HP	=	1.53408
AP	=	-0.55481
BP	=	21.38936
XP1	=	0.12824
XP2	=	0.39474

CENTER POINT NUMBER 2		
HP	=	1.53014
AP	=	0.84657
BP	=	21.42360
XP1	=	-0.14778
XP2	=	0.37392

CENTER POINT NUMBER 3		
HP	=	0.48339
AP	=	-0.48420
BP	=	18.71082
XP1	=	0.02486
XP2	=	0.30909

CENTER POINT NUMBER 4		
HP	=	0.48101
AP	=	0.73792
BP	=	18.71799
XP1	=	-0.03171
XP2	=	0.30584

Table 2 Center point information

C00	=	1.534870
C10	=	0.128256
C01	=	0.394753
C20	=	0.212926
C11	=	0.029409
C02	=	-0.066491
C30	=	0.023077
C21	=	-0.056413
C12	=	-0.005039
C03	=	-0.075679

Table 3 Modeled polynomial parameters

polynomial. Here the C_{ij} parameters were computed using the information at the four center points. The values which were found are shown in Table 3.

Since the model parameters are determined, the height and gradient for any point can be found. In order to find the modeled surface shape, 100 test points were taken in an area bounded by the four center points. The location of these points is shown in Table 4.

The height of the modeled surface corresponding to the 100 test points is shown in Table 5. Also shown is the actual height and the error of the modeled height compared to the actual height. The maximum error for this section was 6.6 cm. Graphs of four different cross sections of the hill are shown in Figure 6. Here the modeled surface can be seen to approximate the actual surface shape very well.

Tables of the modeled gradient, actual gradient, and gradient error are given in Table 6, again for the 100 test points in Table 4. Figure 7 shows the gradient graphically for four different cross sections of the hill. The modeled gradient can be seen to approximate the actual gradient very well, deviating from the true gradient by only 2.4°, which is fairly small. However, since the maximum slope which the vehicle can climb is 25°, the actual surface indicates an impassable object while the model indicates it passable. Therefore, the maximum threshold must be lowered to compensate for modeling error.

For the error analysis, the standard deviation for

Table 4 a and b coordinate location of test points

A LOCATION OF TEST POINTS

-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658
-0.55481	-0.39910	-0.24339	-0.08768	0.06803	0.22374	0.37945	0.53516	0.69087	0.84658

B LOCATION OF TEST POINTS

21.42358	21.42358	21.42358	21.42358	21.42358	21.42358	21.42358	21.42358	21.42358	21.42358
21.12216	21.12216	21.12216	21.12216	21.12216	21.12216	21.12216	21.12216	21.12216	21.12216
20.82074	20.82074	20.82074	20.82074	20.82074	20.82074	20.82074	20.82074	20.82074	20.82074
20.51932	20.51932	20.51932	20.51932	20.51932	20.51932	20.51932	20.51932	20.51932	20.51932
20.21790	20.21790	20.21790	20.21790	20.21790	20.21790	20.21790	20.21790	20.21790	20.21790
19.91649	19.91649	19.91649	19.91649	19.91649	19.91649	19.91649	19.91649	19.91649	19.91649
19.61507	19.61507	19.61507	19.61507	19.61507	19.61507	19.61507	19.61507	19.61507	19.61507
19.31364	19.31364	19.31364	19.31364	19.31364	19.31364	19.31364	19.31364	19.31364	19.31364
19.01222	19.01222	19.01222	19.01222	19.01222	19.01222	19.01222	19.01222	19.01222	19.01222
18.71082	18.71082	18.71082	18.71082	18.71082	18.71082	18.71082	18.71082	18.71082	18.71082

Table 5 Values of height modeled, height actual and height error for the test points

HEIGHT MODELED

1.54824	1.56587	1.57829	1.58567	1.58810	1.58567	1.57847	1.56658	1.55009	1.52909
1.42726	1.44359	1.45522	1.46222	1.46469	1.46271	1.45636	1.44575	1.43094	1.41204
1.30198	1.31704	1.32780	1.33436	1.33679	1.33518	1.32963	1.32021	1.30702	1.29014
1.17456	1.18828	1.19611	1.20415	1.20647	1.20517	1.20033	1.19204	1.18039	1.16547
1.04709	1.05939	1.06822	1.07366	1.07581	1.07474	1.07055	1.06332	1.05314	1.04010
0.92163	0.93245	0.94020	0.94498	0.94688	0.94597	0.94235	0.93611	0.92733	0.91610
0.81026	0.80951	0.81612	0.82017	0.82174	0.82093	0.81752	0.81249	0.80504	0.79554
0.68504	0.69267	0.69806	0.70130	0.70248	0.70169	0.69901	0.69453	0.68833	0.68051
0.57806	0.58399	0.58809	0.59046	0.59117	0.59033	0.58800	0.58429	0.57928	0.57306
0.48228	0.48552	0.48826	0.48970	0.48988	0.48891	0.48688	0.48387	0.47997	0.47527

HEIGHT ACTUAL

1.61430	1.62641	1.63457	1.63879	1.63904	1.63532	1.62766	1.61611	1.60075	1.58171
1.48536	1.49643	1.50393	1.50781	1.50804	1.50462	1.49757	1.48695	1.47282	1.45530
1.34692	1.35696	1.36377	1.36729	1.36750	1.36439	1.35800	1.34837	1.33556	1.31967
1.20376	1.21274	1.21682	1.22196	1.22215	1.21938	1.21367	1.20505	1.19361	1.17940
1.06030	1.06820	1.07356	1.07633	1.07649	1.07405	1.06902	1.06143	1.05135	1.03884
0.92046	0.92732	0.93197	0.93437	0.93452	0.93239	0.92803	0.92144	0.91269	0.90183
0.79752	0.79339	0.79737	0.79943	0.79955	0.79774	0.79400	0.78836	0.78088	0.77158
0.66406	0.66901	0.67237	0.67410	0.67421	0.67268	0.66952	0.66477	0.65846	0.65062
0.55188	0.55599	0.55878	0.56022	0.56031	0.55903	0.55642	0.55247	0.54722	0.54071
0.45293	0.45540	0.45768	0.45886	0.45893	0.45789	0.45575	0.45251	0.44821	0.44288

HEIGHT ERROR

-0.06606	-0.06054	-0.05628	-0.05312	-0.05094	-0.04965	-0.04919	-0.04953	-0.05066	-0.05262
-0.05809	-0.05283	-0.04871	-0.04559	-0.04336	-0.04191	-0.04121	-0.04120	-0.04188	-0.04326
-0.04494	-0.03993	-0.03596	-0.03293	-0.03071	-0.02921	-0.02837	-0.02816	-0.02854	-0.02953
-0.02920	-0.02446	-0.02071	-0.01782	-0.01568	-0.01421	-0.01334	-0.01301	-0.01321	-0.01394
-0.01321	-0.00881	-0.00534	-0.00266	-0.00069	0.00069	0.00153	0.00189	0.00179	0.00126
0.00115	0.00513	0.00823	0.01061	0.01236	0.01358	0.01433	0.01467	0.01464	0.01427
0.01274	0.01612	0.01875	0.02074	0.02219	0.02319	0.02352	0.02412	0.02416	0.02396
0.02098	0.02366	0.02569	0.02720	0.02828	0.02901	0.02948	0.02975	0.02987	0.02988
0.02618	0.02800	0.02931	0.03024	0.03087	0.03129	0.03159	0.03183	0.03206	0.03235
0.02935	0.03013	0.03059	0.03083	0.03094	0.03102	0.03113	0.03135	0.03176	0.03239

Table 6 Values of gradient modeled, gradient actual and height error for the test points

GRADIENT MODELED

22.44908	22.18481	21.98991	21.85904	21.78664	21.76709	21.79501	21.86536	21.97363	22.11592
23.12668	22.95464	22.82712	22.73968	22.68657	22.66190	22.66109	22.68601	22.71527	22.76418
23.49298	23.39677	23.32718	23.27785	23.24321	23.21928	23.19875	23.18106	23.16234	23.14047
23.54607	23.51809	23.49615	23.48090	23.46454	23.44273	23.41146	23.36754	23.30836	23.23207
23.28262	23.31087	23.33582	23.35178	23.35379	23.33760	23.29961	23.23686	23.14700	23.02820
22.69562	22.77715	22.84334	22.88889	22.90927	22.90060	22.85960	22.78352	22.67014	22.51762
21.77473	21.90608	22.01126	22.08542	22.12444	22.12485	22.08266	21.99838	21.86685	21.68729
20.59697	20.68506	20.92782	20.93035	20.98811	20.99892	20.95982	20.86853	20.72305	20.52164
18.87298	19.04805	19.27754	19.40759	19.48508	19.50737	19.47717	19.37749	19.22151	19.00255
16.85687	17.12703	17.34241	17.49991	17.59700	17.62156	17.60173	17.50567	17.34242	17.10991

GRADIENT ACTUAL

22.62240	22.59446	22.49916	22.46976	22.46797	22.49297	22.54617	22.62146	22.71541	22.82239
24.40979	24.39702	24.38683	24.38107	24.38072	24.38583	24.39554	24.40613	24.42105	24.43115
25.43369	25.46555	25.48611	25.49643	25.49702	25.48795	25.46872	25.43837	25.39543	25.33810
25.75409	25.81650	25.85805	25.87930	25.88057	25.86185	25.82288	25.76314	25.68182	25.57794
25.42710	25.51993	25.57545	25.60467	25.60570	25.58065	25.52841	25.44907	25.34210	25.20705
24.55944	24.65500	24.71928	24.75238	24.75433	24.72517	24.66484	24.57326	24.45030	24.29588
23.27750	23.35916	23.37801	23.41350	23.41562	23.38432	23.31969	23.22173	23.09055	22.92674
21.47425	21.57866	21.64868	21.68480	21.68695	21.65511	21.56937	21.41982	21.25675	21.19041
19.41596	19.56723	19.63562	19.67090	19.67300	19.64189	19.57768	19.48055	19.35080	19.18887
17.23748	17.35292	17.44745	17.48074	17.48274	17.45337	17.39279	17.30122	17.17902	17.02667

GRADIENT ERROR

-0.18332	-0.36964	-0.50925	-0.61072	-0.68134	-0.72688	-0.75116	-0.75610	-0.74178	-0.70647
-1.28311	-1.44258	-1.55971	-1.64119	-1.69415	-1.72094	-1.73445	-1.72812	-1.70578	-1.66698
-1.94073	-2.06877	-2.15894	-2.21856	-2.25381	-2.26967	-2.26597	-2.25731	-2.23309	-2.19763
-2.20802	-2.34060	-2.36159	-2.39850	-2.41603	-2.41911	-2.41139	-2.39560	-2.37346	-2.34587
-2.11449	-2.21506	-2.22964	-2.25224	-2.25191	-2.24795	-2.22881	-2.21220	-2.19510	-2.17885
-1.86795	-1.87785	-1.87595	-1.86349	-1.84506	-1.82457	-1.80124	-1.78973	-1.78017	-1.77826
-1.43727	-1.40308	-1.36674	-1.32808	-1.29118	-1.25948	-1.23607	-1.22334	-1.22369	-1.23895
-0.96881	-0.89360	-0.82086	-0.75475	-0.69854	-0.65619	-0.62955	-0.62129	-0.63370	-0.66878
-0.55308	-0.46918	-0.35809	-0.26331	-0.18793	-0.13452	-0.10551	-0.10306	-0.12929	-0.18633
-0.43000	-0.25591	-0.10504	0.01917	0.11426	0.17819	0.20894	0.20465	0.16341	0.08324

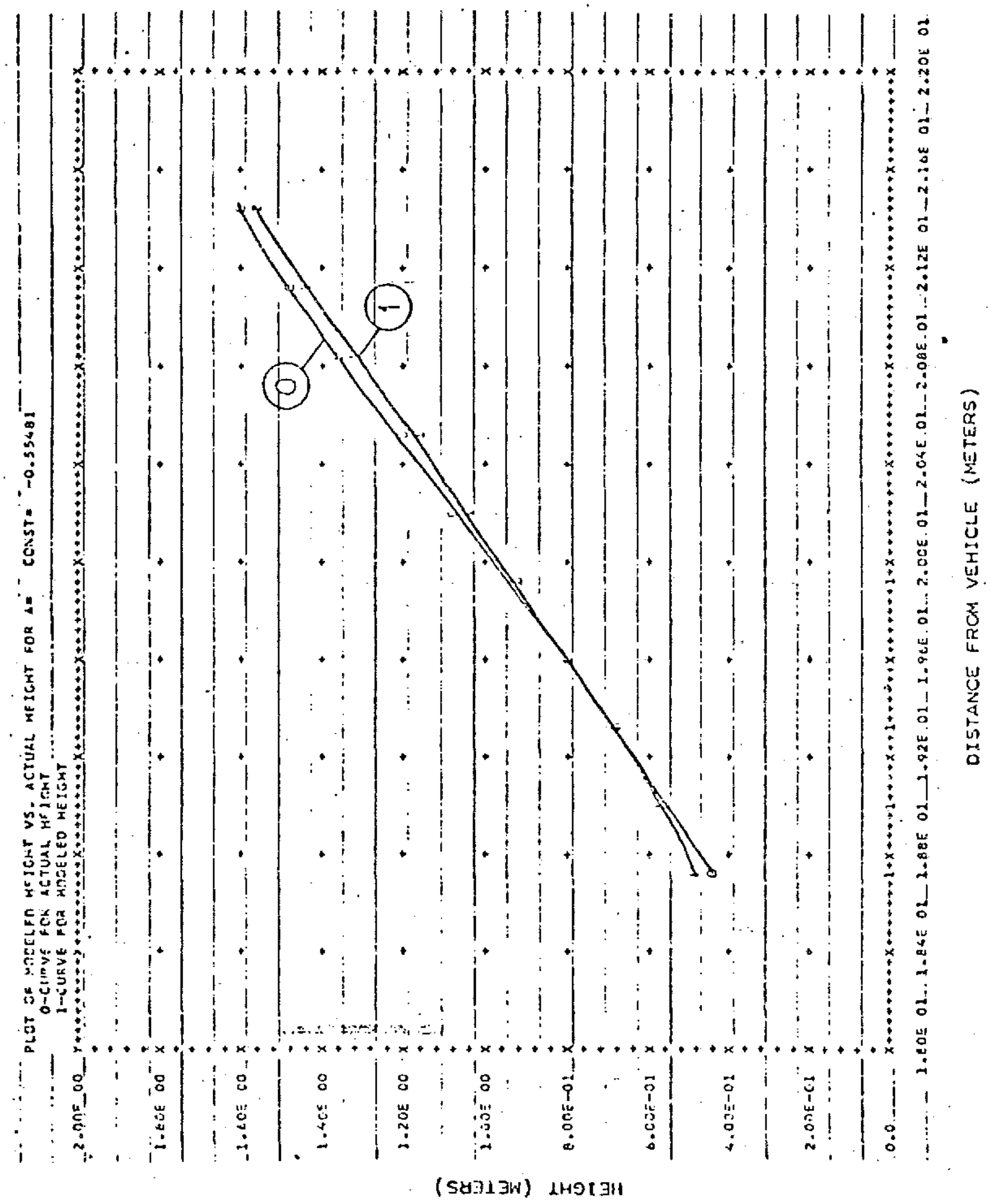

Figure 6 Plot of height modeled and height actual vs. distance for four hill cross sections

6a Cross section at A = -0.55481 m.

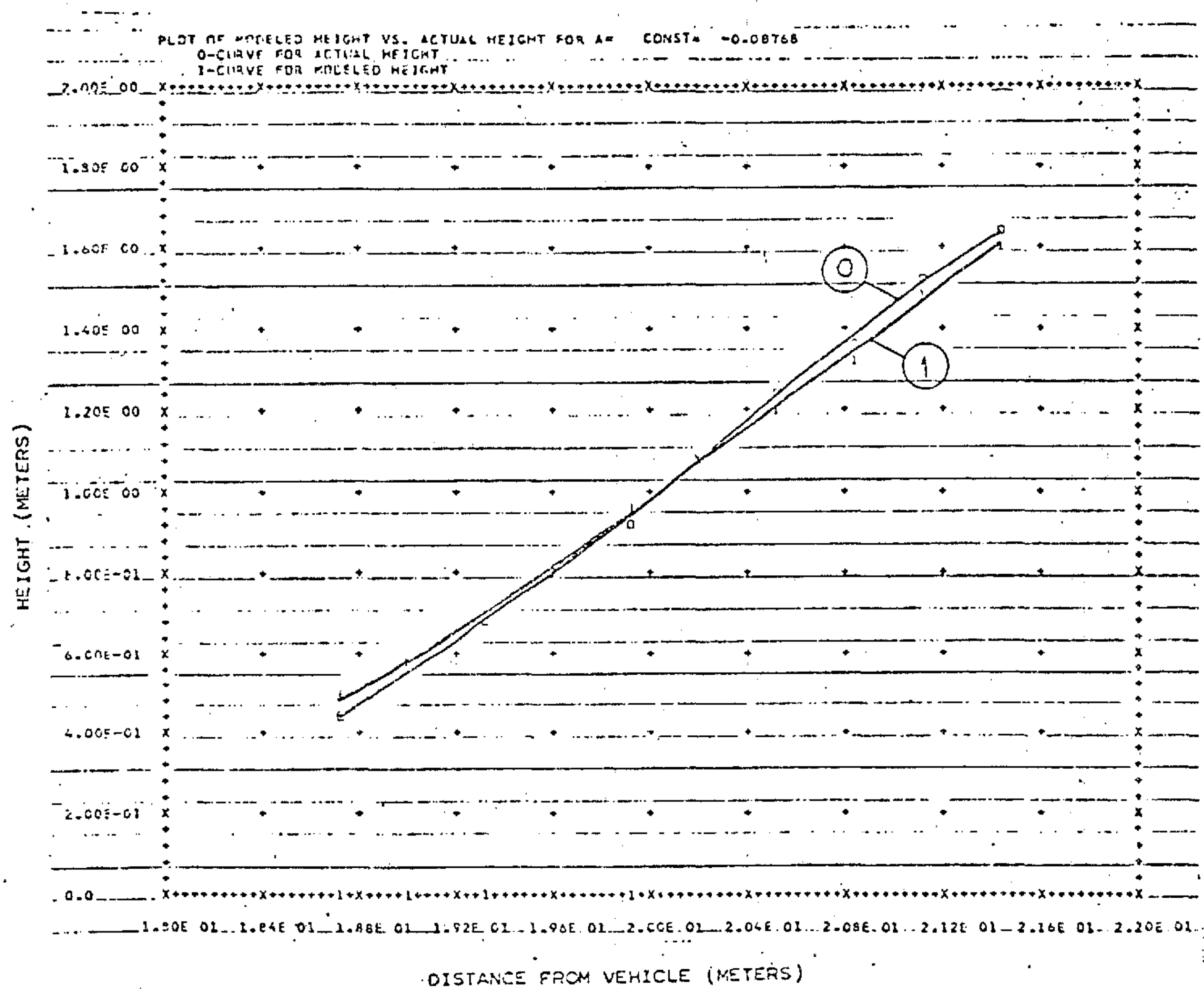

Figure 6b Cross section at A = -0.08768 m.

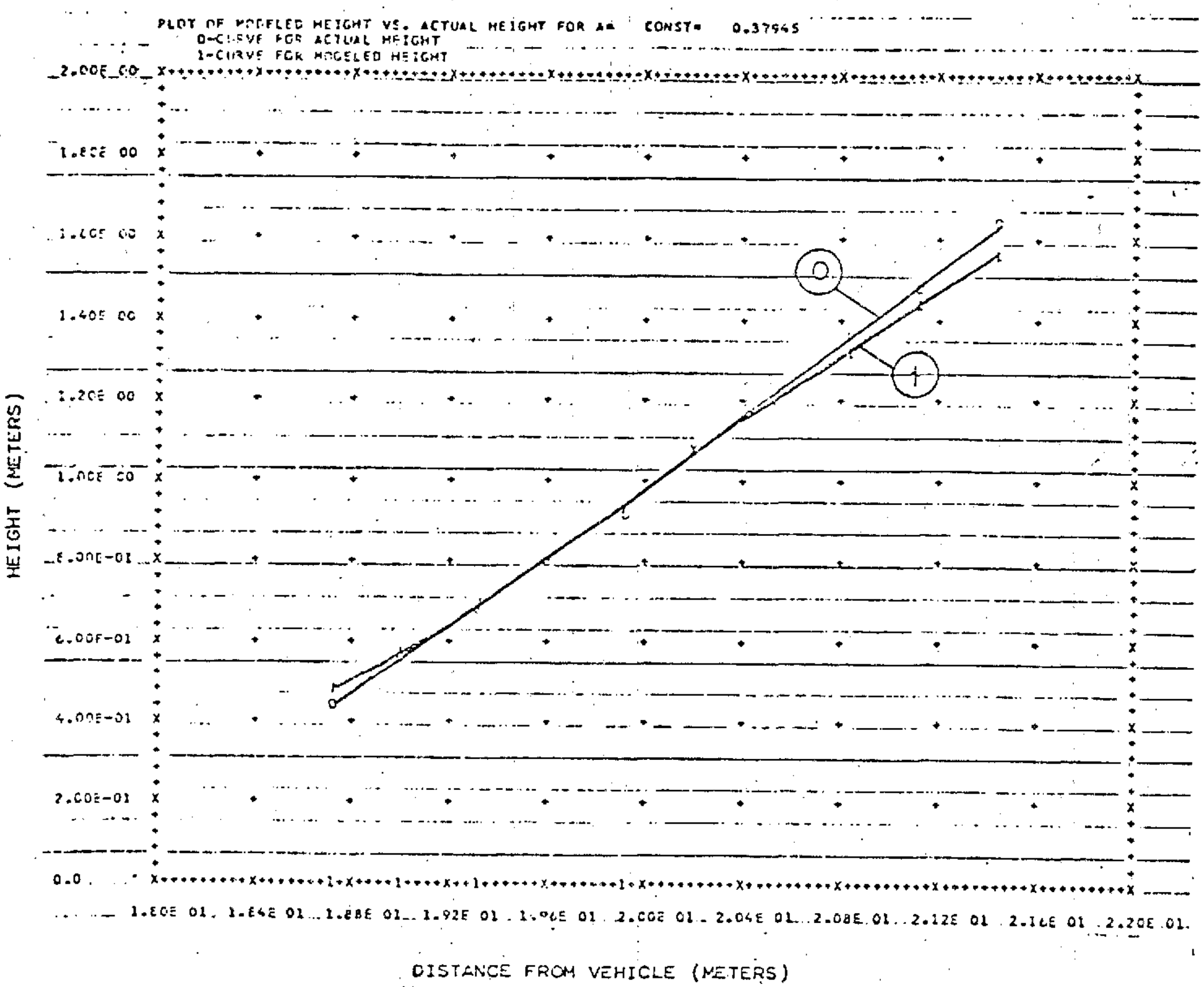

Figure 6c Cross section at A = 0.37945 m.

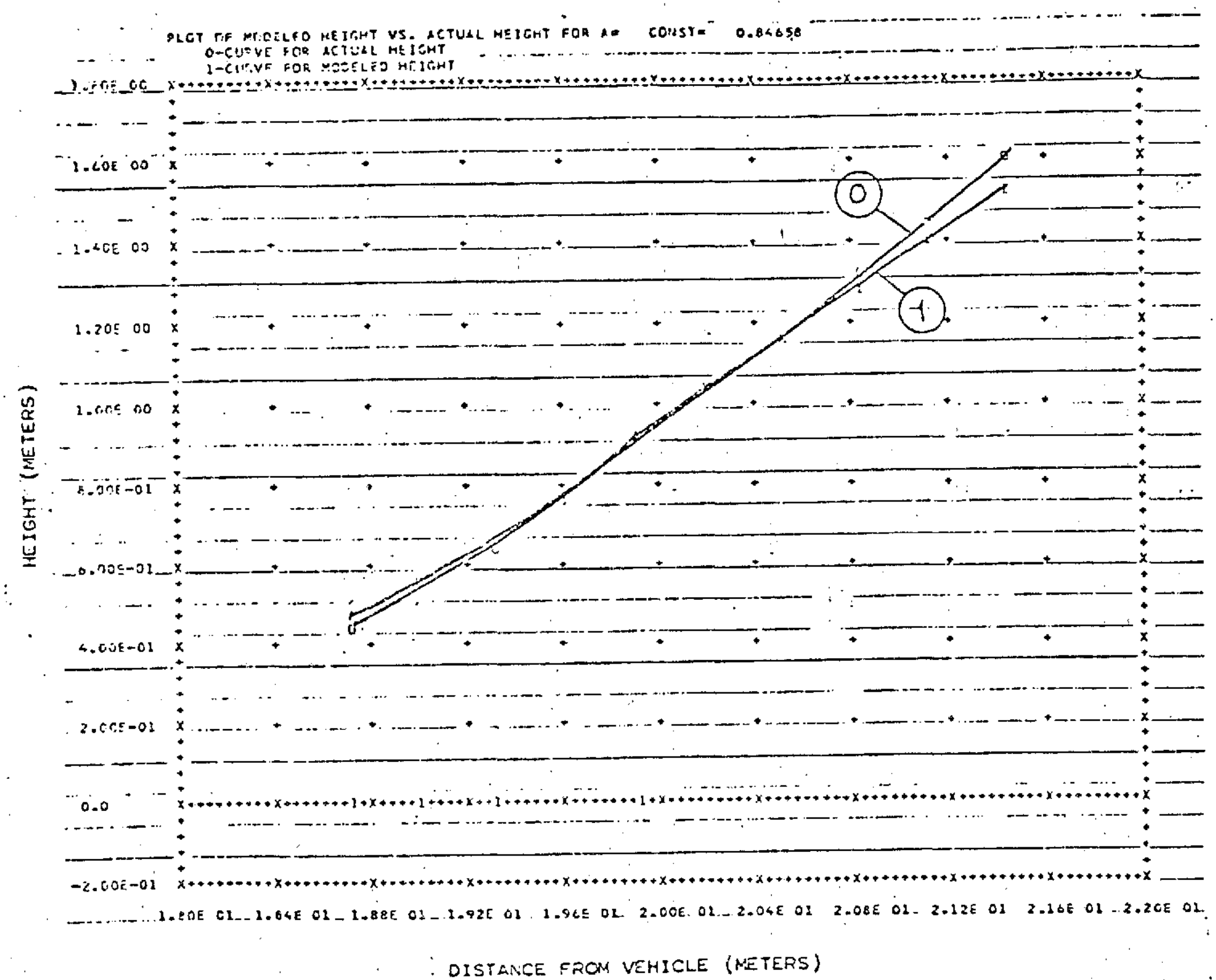

Figure 6d Cross section at A = 0.84658 m.

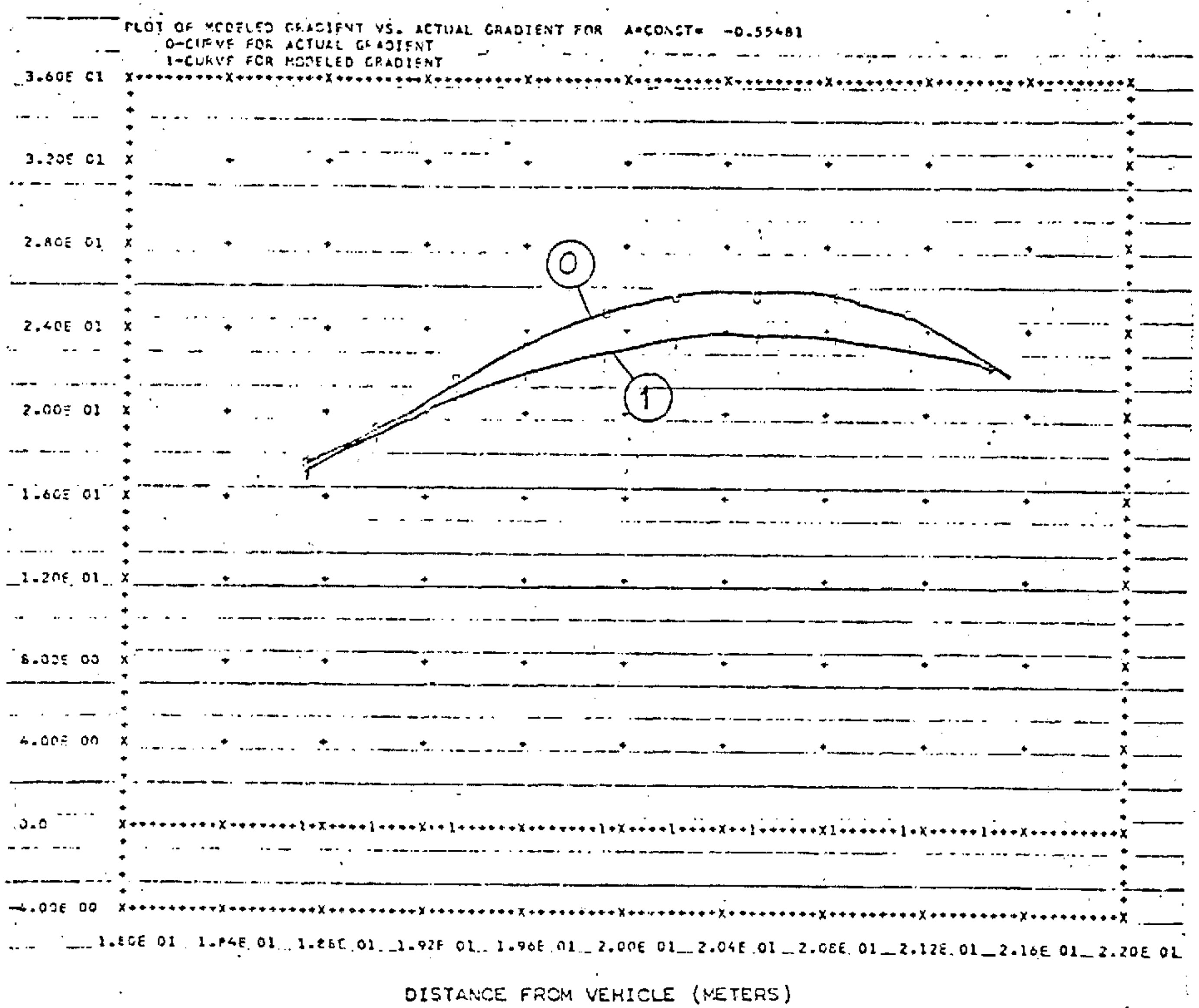

Figure 7 Plot of gradient modeled and gradient actual vs. distance for four hill cross sections

7a Cross section at A = -0.55481 m.

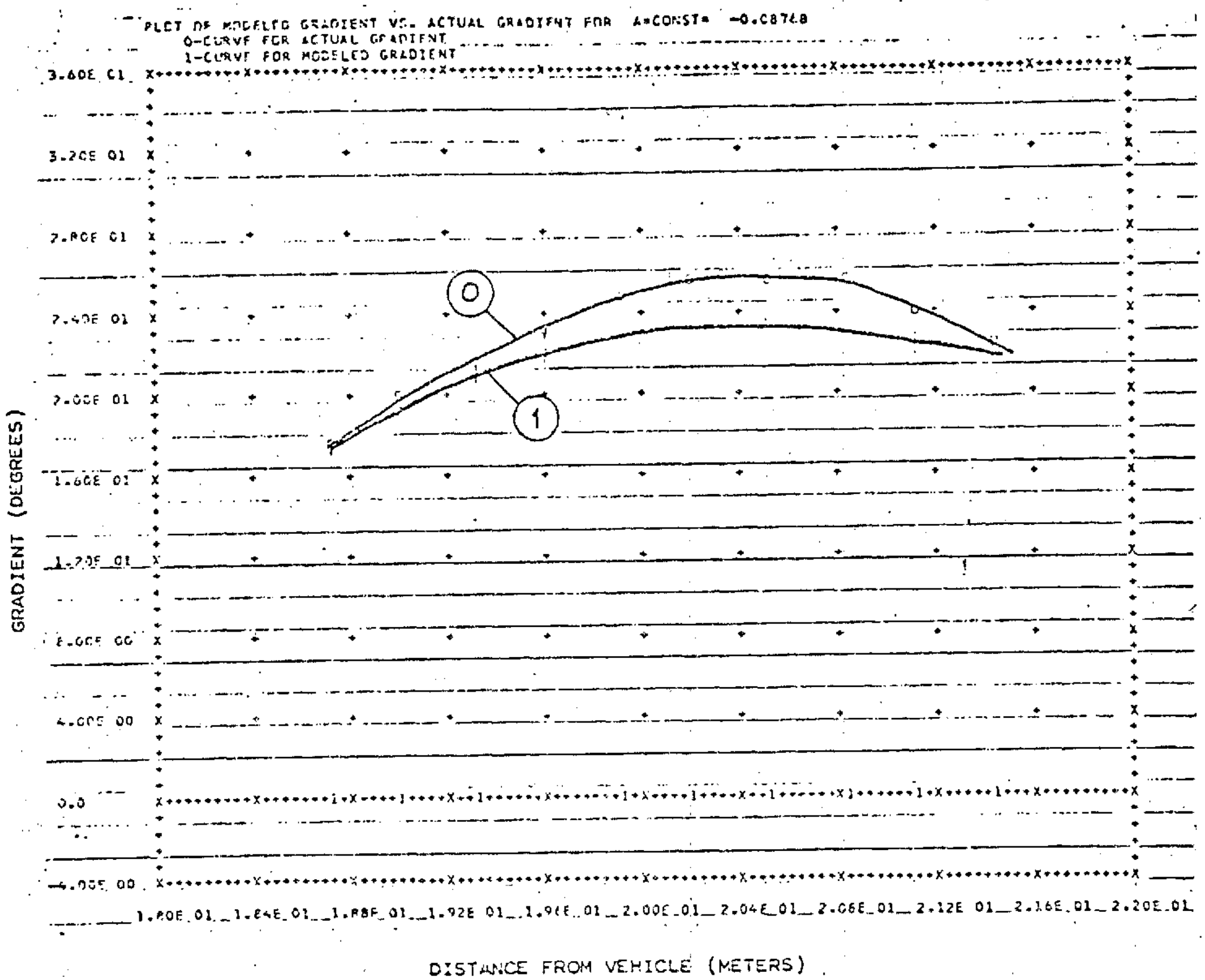

Figure 7b Cross section at A = -0.08768 m.

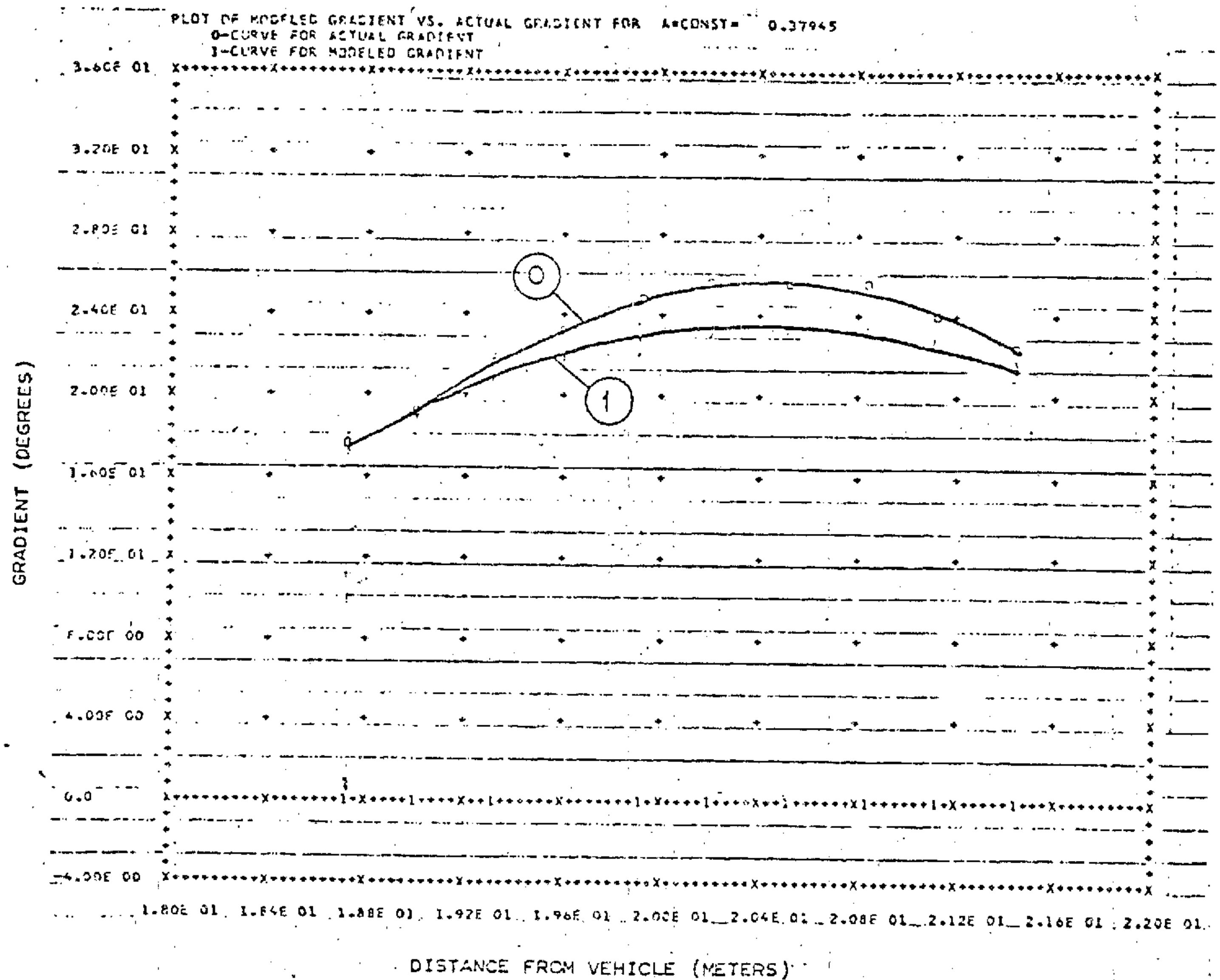

Figure 7c Cross section at A = 0.37945 m.

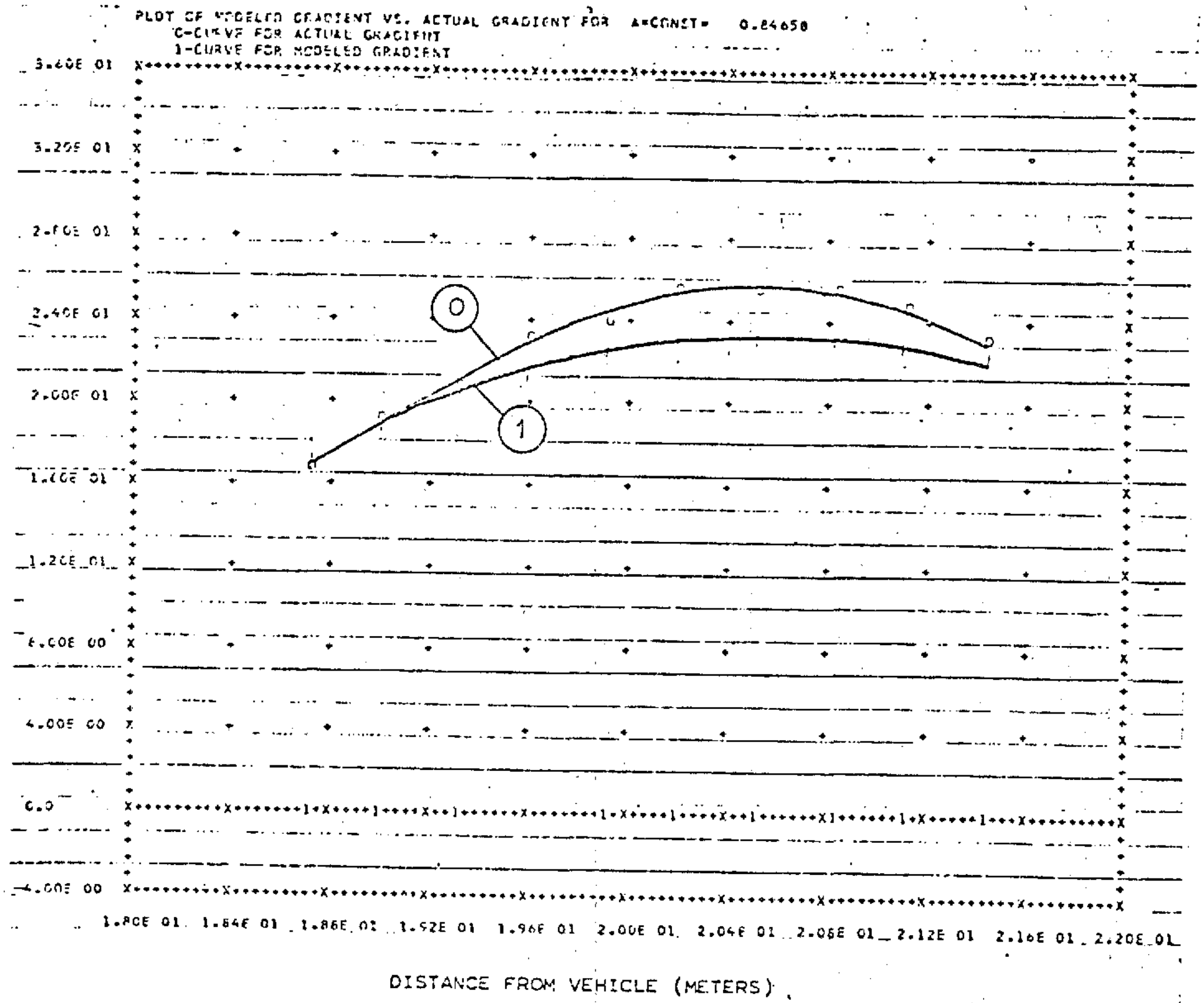

Figure 7d Cross section at A = 0.84658 m.

elevation and azimuth angles were set at one arc minute, while the standard deviation of range was set at 5 cm. The standard deviation of roll (σ_{ROLL}) and pitch (σ_{PITCH}) angles was set at 0°, 0.25°, 0.5° and 1.0°. The covariance matricies of the parameters, Eqn. 43, are shown in Table 7 for the four values of σ_{ROLL} and σ_{PITCH}. The values of this matrix increase with an increase in σ_{ROLL} and σ_{PITCH} as expected.

The standard deviation of height (σ_H) can now be found for the 100 test points in Table 4 for the four different values of σ_R (here $\sigma_R = \sigma_{ROLL} = \sigma_{PITCH}$). These values are shown in Table 8. These values are also plotted for four different cross sections of the hill. (Figure 8). In order to keep σ_H below a value such as 20 cm would require a σ_R of 0.5° or less.

The standard deviation of gradient, σ_{SG}, is likewise calculated for the 100 test points and for the four values of σ_R. These are shown in Table 9. Figure 9 shows plots of modeled σ_{SG} and actual σ_{SG} vs. distance for four different hill cross sections. For σ_{SG} to be below a value such as 6° would require a σ_R of 0.5° or less. For σ_{SG} to be below 3° requires σ_R to be 0.25° or less.

Table 7 Covariance matricies of the parameter for four values of σ_r

COVARIENCE MATRIX OF THE PARAMETERS FOR SIGMA ROLL= 0.0 DEG SIGMA PITCH= 0.0 DEG

0.00001	-0.00000	-0.00001	-0.00007	0.00000	-0.00002	0.00010	-0.00000	0.00000	-0.00002
-0.00000	0.00134	0.00039	-0.00349	0.00012	0.00052	0.00351	-0.00041	-0.00018	0.00021
-0.00001	0.00039	0.00033	-0.00106	-0.00008	0.00042	0.00106	-0.00012	-0.00014	0.00018
-0.00007	-0.00349	-0.00106	0.01263	-0.00020	-0.00142	-0.01527	0.00080	0.00033	-0.00066
0.00000	0.00012	-0.00008	-0.00020	0.00026	-0.00008	0.00010	-0.00019	0.00006	-0.00004
-0.00002	0.00052	0.00042	-0.00142	-0.00008	0.00076	0.00153	-0.00012	-0.00014	0.00045
0.00010	0.00351	0.00106	-0.01527	0.00010	0.00152	0.02032	-0.00050	-0.00024	0.00078
-0.00000	-0.00041	-0.00012	0.00080	-0.00019	-0.00012	-0.00050	0.00042	0.00009	-0.00002
0.00000	-0.00018	-0.00014	0.00033	0.00006	-0.00014	-0.00024	0.00009	0.00012	-0.00003
-0.00002	0.00021	0.00018	-0.00066	-0.00004	0.00045	0.00078	-0.00002	-0.00003	0.00032

COVARIENCE MATRIX OF THE PARAMETERS FOR SIGMA ROLL= 0.250 DEG SIGMA PITCH= 0.250 DEG

0.00073	0.00001	0.00046	-0.02218	-0.00046	-0.00463	0.02106	-0.00039	0.00026	-0.00335
0.00001	0.00136	0.00040	-0.00356	0.00013	0.00051	0.00360	-0.00042	-0.00018	0.00020
0.00046	0.00040	0.00036	-0.00225	-0.00011	0.00017	0.00274	-0.00014	-0.00012	0.00000
-0.02218	-0.00356	-0.00225	0.13457	0.00220	0.00221	-0.18771	0.00246	-0.00065	0.00203
-0.00046	0.00013	-0.00011	0.00220	0.00034	0.00001	-0.00323	-0.00015	0.00004	-0.00000
-0.00463	0.00051	0.00017	0.00221	0.00001	0.00659	-0.00469	0.00003	-0.00029	0.00483
0.02106	0.00360	0.00274	-0.18771	-0.00323	-0.00469	0.26448	-0.00278	0.00112	-0.00281
-0.00039	-0.00042	-0.00014	0.00246	-0.00015	0.00003	-0.00278	0.00046	0.00007	0.00008
0.00026	-0.00018	-0.00012	-0.00065	0.00004	-0.00029	0.00112	0.00007	0.00013	-0.00013
-0.00335	0.00020	0.00000	0.00203	-0.00000	0.00483	-0.00281	0.00008	-0.00013	0.00362

COVARIENCE MATRIX OF THE PARAMETERS FOR SIGMA ROLL= 0.500 DEG SIGMA PITCH= 0.500 DEG

0.03458	0.00004	0.00187	-0.08850	-0.00183	-0.01846	0.12395	-0.00154	0.00103	-0.01335
0.00004	0.00142	0.00040	-0.00378	0.00014	0.00050	0.00385	-0.00043	-0.00019	0.00019
0.00187	0.00040	0.00043	-0.00584	-0.00018	-0.00058	0.00776	-0.00021	-0.00008	-0.00054
-0.08850	-0.00378	-0.00584	0.50036	0.00941	0.01710	-0.70504	0.00746	-0.00358	0.01011
-0.00183	0.00014	-0.00018	0.00941	0.00057	0.00028	-0.01320	0.00000	-0.00004	0.00012
-0.01846	0.00050	-0.00058	0.01710	0.00028	0.02407	-0.02232	0.00049	-0.00073	0.01796
0.12395	0.00385	0.00776	-0.70504	-0.01320	-0.02232	0.99694	-0.00963	0.00518	-0.01357
-0.00154	-0.00043	-0.00021	0.00746	0.00000	0.00049	-0.00963	0.00059	0.00001	0.00037
0.00103	-0.00019	-0.00008	-0.00358	-0.00004	-0.00073	0.00518	0.00001	0.00017	-0.00043
-0.01335	0.00019	-0.00054	0.01011	0.00012	0.01796	-0.01357	0.00037	-0.00043	0.01353

COVARIENCE MATRIX OF THE PARAMETERS FOR SIGMA ROLL= 1.000 DEG SIGMA PITCH= 1.000 DEG

0.13947	0.00016	0.00752	-0.35381	-0.00735	-0.07376	0.49551	-0.00617	0.00411	-0.05335
0.00016	0.00165	0.00043	-0.00462	0.00017	0.00045	0.00486	-0.00049	-0.00020	0.00014
0.00752	0.00043	0.00074	-0.02020	-0.00047	-0.00356	0.02785	-0.00046	0.00008	-0.00270
-0.35381	-0.00462	-0.02020	1.96354	0.03825	0.07267	-2.77433	0.02743	-0.01532	0.04242
-0.00735	0.00017	-0.00047	0.03825	0.00152	0.00138	-0.05308	0.00060	-0.00037	0.00060
-0.07376	0.00045	-0.00356	0.07267	0.00138	0.09401	-0.09785	0.00233	-0.00248	0.07051
0.49551	0.00486	0.02785	-2.77433	-0.05308	-0.09785	3.92680	-0.03704	0.02143	-0.05663
-0.00617	-0.00049	-0.00046	0.02743	0.00060	0.00233	-0.03704	0.00112	-0.00024	0.00153
0.00411	-0.00020	0.00008	-0.01532	-0.00037	-0.00248	0.02143	-0.00024	0.00031	-0.00164
-0.05335	0.00014	-0.00270	0.04242	0.00060	0.07051	-0.05663	0.00153	-0.00164	0.05314

STANDARD DEVIATION OF HEIGHT FOR SIGMA ROLL= 0.0 DEG SIGMA PITCH= 0.0 DEG

0.00559	0.00599	0.00836	0.00957	0.01090	0.01181	0.01267	0.01337	0.01373	0.01392
0.00558	0.00511	0.00651	0.00790	0.00924	0.01067	0.01210	0.01326	0.01391	0.01413
0.00803	0.00638	0.00661	0.00752	0.00883	0.01046	0.01216	0.01357	0.01442	0.01474
0.00967	0.00777	0.00750	0.00803	0.00917	0.01070	0.01253	0.01403	0.01500	0.01547
0.01055	0.00899	0.00871	0.00910	0.01008	0.01155	0.01322	0.01469	0.01570	0.01631
0.01124	0.01011	0.01006	0.01048	0.01135	0.01266	0.01416	0.01552	0.01650	0.01722
0.01180	0.01114	0.01123	0.01184	0.01265	0.01380	0.01513	0.01634	0.01726	0.01804
0.01245	0.01200	0.01230	0.01283	0.01359	0.01462	0.01579	0.01686	0.01769	0.01850
0.01343	0.01273	0.01263	0.01272	0.01388	0.01480	0.01585	0.01681	0.01757	0.01844
0.01537	0.01375	0.01320	0.01316	0.01356	0.01434	0.01532	0.01623	0.01699	0.01797

STANDARD DEVIATION OF HEIGHT FOR SIGMA ROLL= 0.250000 DEG SIGMA PITCH= 0.250000 DEG

0.09257	0.09107	0.08475	0.07708	0.07112	0.06950	0.07273	0.07872	0.08431	0.08675
0.09056	0.08814	0.08188	0.07425	0.06828	0.06663	0.06981	0.07563	0.08092	0.08295
0.08438	0.08207	0.07581	0.06808	0.06205	0.06055	0.06403	0.07009	0.07538	0.07714
0.07677	0.07465	0.06841	0.06058	0.05453	0.05335	0.05744	0.06398	0.06939	0.07095
0.06957	0.06805	0.06201	0.05422	0.04850	0.04788	0.05264	0.05958	0.06504	0.06636
0.06567	0.06427	0.05876	0.05171	0.04664	0.04670	0.05179	0.05867	0.06385	0.06483
0.06502	0.06407	0.05938	0.05343	0.04945	0.05005	0.05492	0.06122	0.06578	0.06637
0.06644	0.06631	0.06244	0.05759	0.05459	0.05545	0.05993	0.06532	0.06915	0.06935
0.06920	0.06868	0.06539	0.06171	0.05890	0.05983	0.06371	0.06846	0.07164	0.07151
0.06955	0.06990	0.06593	0.06210	0.05992	0.06077	0.06427	0.06850	0.07121	0.07090

STANDARD DEVIATION OF HEIGHT FOR SIGMA ROLL= 0.500000 DEG SIGMA PITCH= 0.500000 DEG

0.18704	0.18185	0.16809	0.15222	0.14097	0.13748	0.14360	0.15573	0.16693	0.17182
0.18086	0.17605	0.16338	0.14796	0.13561	0.13198	0.13803	0.14950	0.16004	0.16408
0.16818	0.16377	0.15120	0.13554	0.12316	0.11973	0.12632	0.13820	0.14867	0.15215
0.15273	0.14868	0.13621	0.12037	0.10789	0.10506	0.11281	0.12562	0.13634	0.13934
0.13852	0.13520	0.12310	0.10748	0.09542	0.09364	0.10276	0.11642	0.12721	0.12967
0.12988	0.12734	0.11623	0.10182	0.09119	0.09079	0.10063	0.11423	0.12446	0.12619
0.12843	0.12667	0.11713	0.10487	0.09644	0.09720	0.10667	0.11911	0.12812	0.12900
0.13214	0.13057	0.12306	0.11302	0.10660	0.10798	0.11650	0.12733	0.13487	0.13494
0.13643	0.13557	0.12897	0.12046	0.11533	0.11686	0.12442	0.13379	0.14001	0.13942
0.13853	0.13570	0.12966	0.12206	0.11752	0.11998	0.12578	0.13408	0.13934	0.13834

STANDARD DEVIATION OF HEIGHT FOR SIGMA ROLL= 1.000000 DEG SIGMA PITCH= 1.000000 DEG

0.37407	0.36356	0.33747	0.30596	0.28131	0.27419	0.28675	0.31060	0.33302	0.34278
0.36159	0.35199	0.32657	0.29541	0.27075	0.26331	0.27926	0.29812	0.31917	0.32725
0.33608	0.32735	0.30218	0.27078	0.24584	0.23878	0.25175	0.27539	0.29629	0.30323
0.30460	0.29706	0.27210	0.24033	0.21520	0.20928	0.22457	0.25006	0.27143	0.27740
0.27644	0.26995	0.24574	0.21428	0.19003	0.18621	0.20424	0.23144	0.25297	0.25779
0.25904	0.25407	0.23181	0.20283	0.18171	0.18025	0.19976	0.22686	0.24727	0.25060
0.25604	0.25261	0.23344	0.20873	0.19163	0.19291	0.21173	0.22654	0.25449	0.25610
0.26339	0.26112	0.24520	0.22495	0.21190	0.21446	0.23139	0.25298	0.26800	0.26797
0.27156	0.27024	0.25678	0.23982	0.22940	0.23224	0.24733	0.26596	0.27835	0.27699
0.27176	0.27036	0.25831	0.24310	0.23386	0.23666	0.25016	0.26669	0.27712	0.27491

Table 8 Standard deviation of height (σ_H) for four values of σ_R

Table 9 Standard deviation of gradient (σ_{SG}) for four values of σ_R

STANDARD DEVIATION OF GRADIENT FOR SIGMA ROLL= 0.0 DEG SIGMA PITCH= 0.0 DEG

1.51499	1.27992	1.07634	0.96407	0.94307	0.93685	0.94150	0.93372	0.91511	0.95217
1.14977	0.91179	0.77801	0.70602	0.67670	0.67707	0.68651	0.69416	0.70394	0.78908
0.67094	0.60521	0.60090	0.56652	0.55943	0.56965	0.58599	0.60107	0.63243	0.74620
0.62108	0.51666	0.52723	0.53041	0.54307	0.55968	0.57658	0.59452	0.63479	0.75495
0.57995	0.50094	0.50393	0.52506	0.54557	0.56322	0.57833	0.59591	0.63846	0.75667
0.56016	0.48623	0.48447	0.50076	0.51761	0.53226	0.54533	0.56316	0.60827	0.72605
0.61651	0.50719	0.46455	0.45205	0.45284	0.46041	0.47253	0.49439	0.54713	0.67000
0.71175	0.59345	0.49187	0.42047	0.40087	0.39612	0.40572	0.44208	0.50026	0.63710
0.97119	0.77907	0.63796	0.54161	0.48935	0.47395	0.49001	0.53104	0.60213	0.72035
1.27577	1.07265	0.92322	0.62349	0.77059	0.75782	0.77589	0.81784	0.88336	0.98033

STANDARD DEVIATION OF GRADIENT FOR SIGMA ROLL= 0.250000 DEG SIGMA PITCH= 0.250000 DEG

1.53272	1.36715	1.21342	1.14053	0.98735	1.03969	1.20059	1.23632	1.04878	1.11718
1.67746	1.50465	1.47952	1.41235	1.37274	1.43322	1.52436	1.54815	1.43141	1.49585
2.25021	2.15214	2.14495	2.13213	2.13270	2.17144	2.21746	2.20767	2.12904	2.16767
2.85392	2.69141	2.68438	2.68359	2.66966	2.71198	2.73992	2.70707	2.64398	2.67298
3.13431	2.98609	2.96968	2.96500	2.96623	2.97474	2.97510	2.94447	2.89145	2.92537
3.14427	3.06914	2.97184	2.95554	2.94670	2.94240	2.92992	2.89473	2.85206	2.90175
2.97048	2.75069	2.65603	2.65154	2.62891	2.61169	2.58933	2.55288	2.52558	2.60822
2.46093	2.23301	2.12616	2.06371	2.02202	1.99149	1.96179	1.93104	1.93521	2.08685
1.68202	1.55863	1.37594	1.26127	1.18707	1.14113	1.11480	1.11877	1.21557	1.52235
1.63869	1.27006	1.03709	0.89111	0.81299	0.79726	0.84068	0.95763	1.18941	1.58846

STANDARD DEVIATION OF GRADIENT FOR SIGMA ROLL= 0.500000 DEG SIGMA PITCH= 0.500000 DEG

1.55465	1.69230	1.55040	1.51585	1.10962	1.29703	1.76228	1.87013	1.37363	1.50748
2.77016	2.56062	2.63224	2.54915	2.48172	2.61440	2.82601	2.85179	2.58910	2.66014
4.29837	4.13027	4.15648	4.14460	4.14863	4.22278	4.31134	4.28509	4.16975	4.13321
5.47026	5.25490	5.27953	5.27659	5.28541	5.37536	5.35627	5.20427	5.16223	5.17554
6.10974	5.89744	5.85978	5.84488	5.84177	5.85393	5.85030	5.78335	5.66769	5.68853
6.10537	5.94324	5.86195	5.83218	5.81009	5.79752	5.76883	5.69289	5.59296	5.65229
5.72850	5.41986	5.30027	5.23419	5.18850	5.15229	5.10261	5.02395	4.95244	5.07610
4.77979	4.24009	4.16074	4.05455	3.97925	3.91593	3.85459	3.78135	3.76504	4.02045
3.36756	2.80555	2.51914	2.34064	2.21748	2.12974	2.06103	2.02903	2.19528	2.77580
2.41973	1.75122	1.37302	1.06806	0.92790	0.90471	1.00473	1.28850	1.82085	2.68376

STANDARD DEVIATION OF GRADIENT FOR SIGMA ROLL= 1.000000 DEG SIGMA PITCH= 1.000000 DEG

1.77730	2.61514	3.19490	2.50634	1.50220	2.07349	3.12268	3.37003	2.24366	2.52285
5.01129	4.86249	5.08161	4.93243	4.61469	5.08589	5.51237	5.56264	5.02223	5.13177
8.41875	8.13423	8.20571	8.18979	8.15669	8.34564	8.51587	8.46077	8.10585	8.12433
10.75120	10.43866	10.43273	10.42649	10.44192	10.51764	10.57555	10.47070	10.18521	10.19779
12.04417	11.64193	11.56033	11.53664	11.52789	11.64917	11.53596	11.40695	11.16488	11.19436
12.21294	11.73271	11.58651	11.51502	11.46991	11.44353	11.38589	11.23538	11.03350	11.12926
11.29690	10.70812	10.48184	10.35413	10.26461	10.19272	10.09572	9.93639	9.78790	10.00931
9.32630	8.57132	8.23529	8.03525	7.89077	7.77203	7.64248	7.49186	7.44709	7.92713
6.53194	5.44082	4.90696	4.57928	4.34760	4.17237	4.02903	3.96841	4.25932	5.39804
4.30071	2.93105	2.10883	1.59114	1.29140	1.24715	1.50689	2.15293	3.30302	5.08288

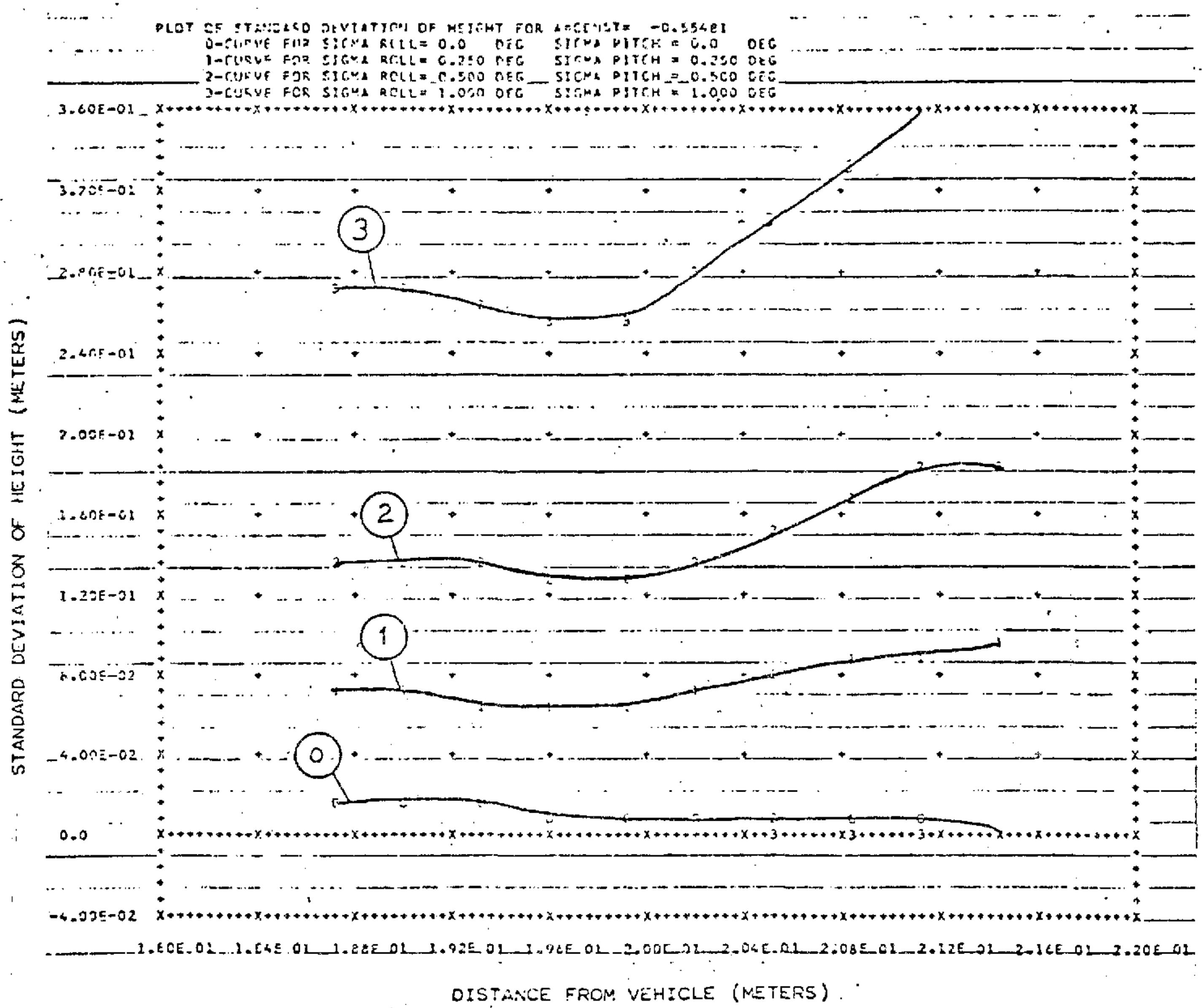

Figure 8 Plot of standard deviation of height (σ_H) for four values of σ_R for four hill cross sections

8a Cross section at A = -0.55481 m.

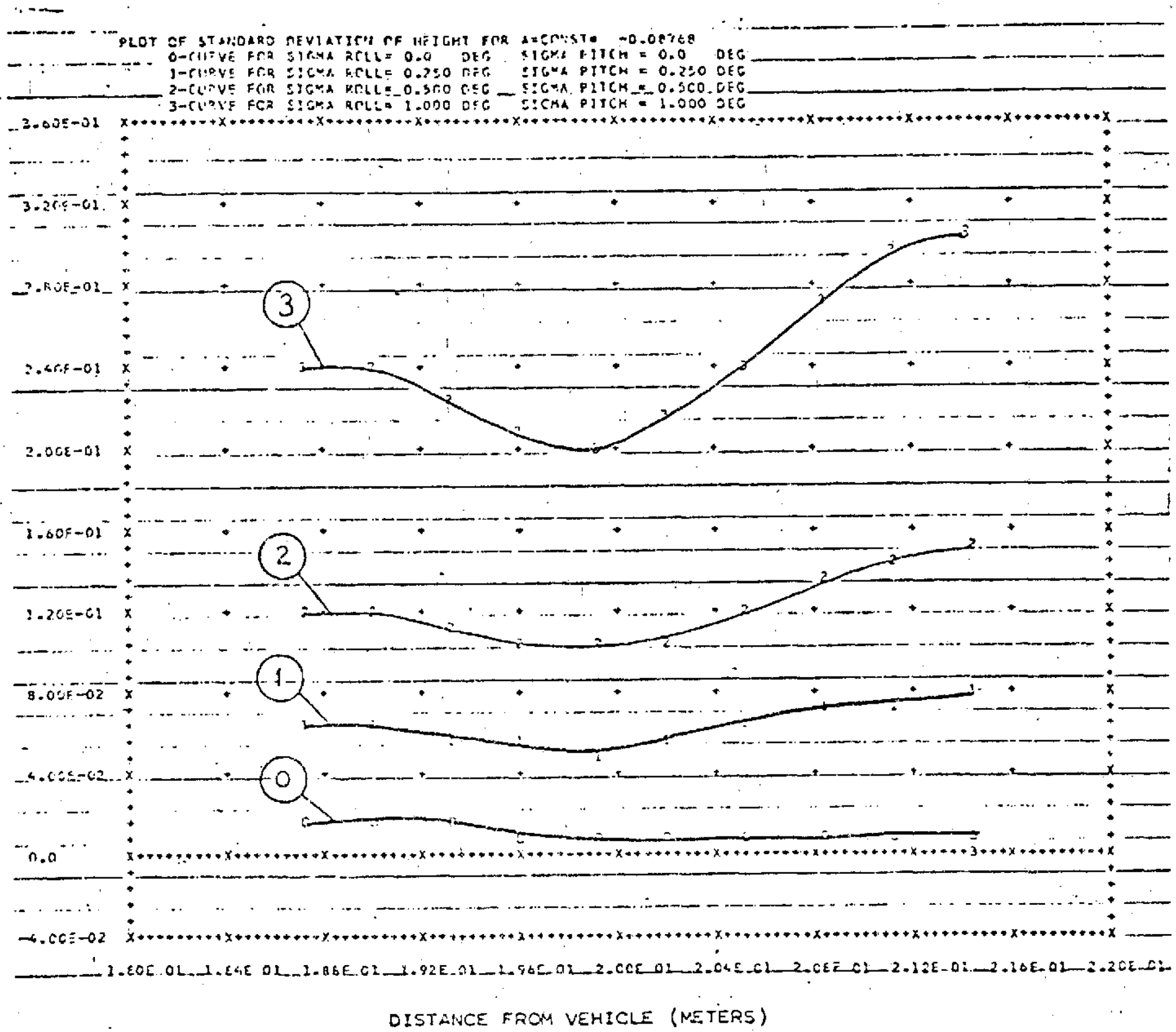

Figure 8b Cross section at A = -0.08768 m.

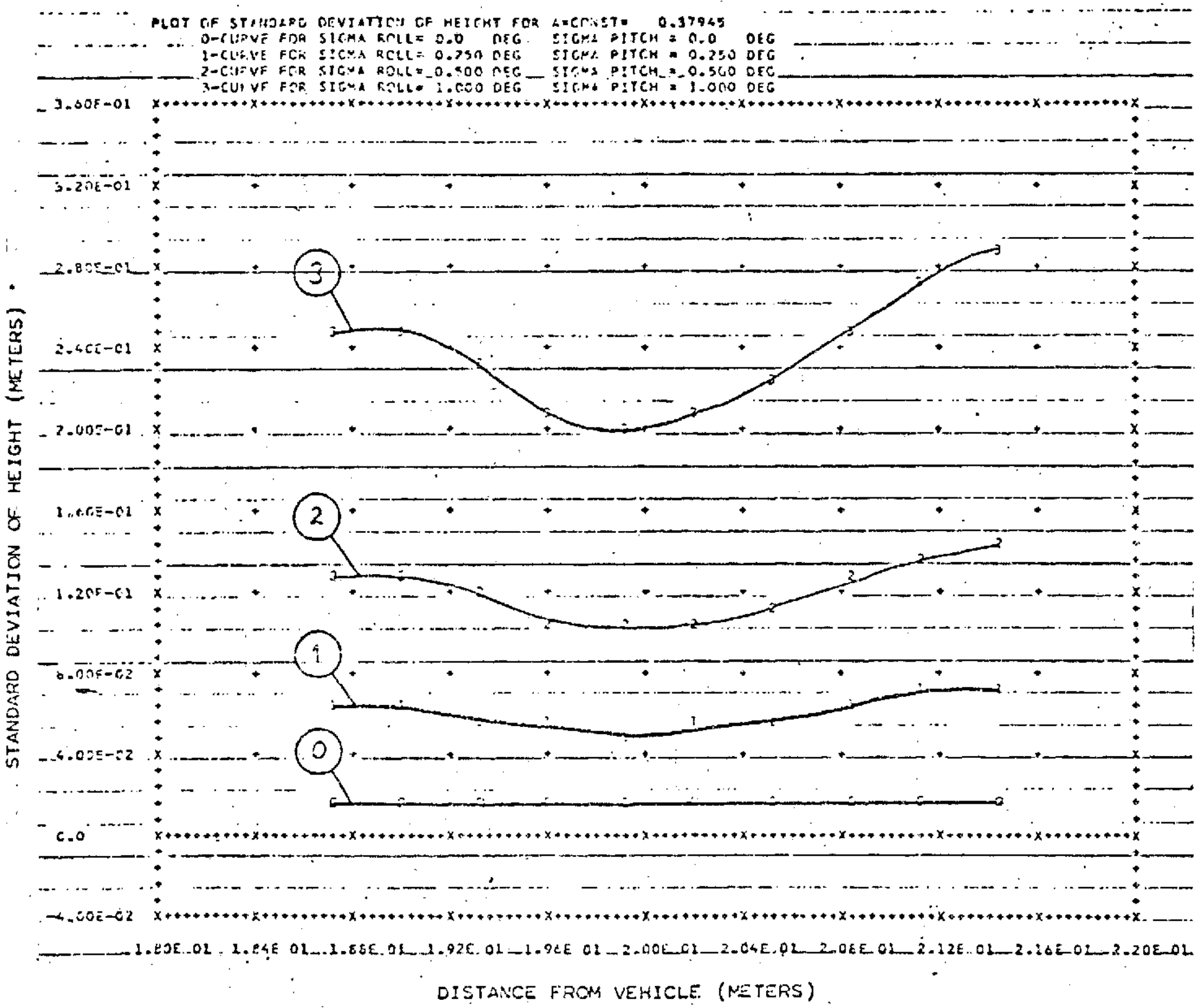

Figure 8c Cross section at A = 0.37945 m.

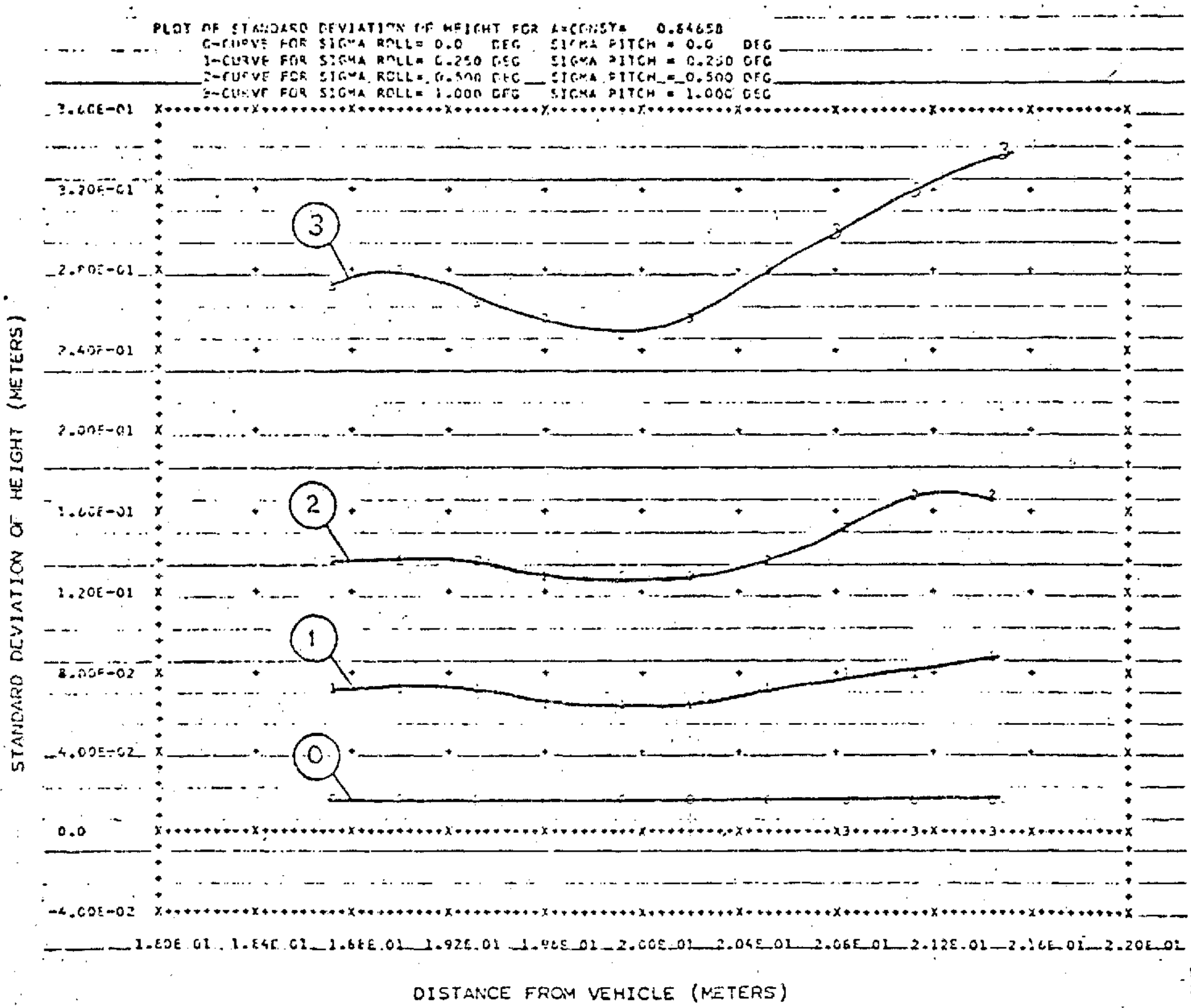

Figure 8d Cross section at A = 0.84658 m.

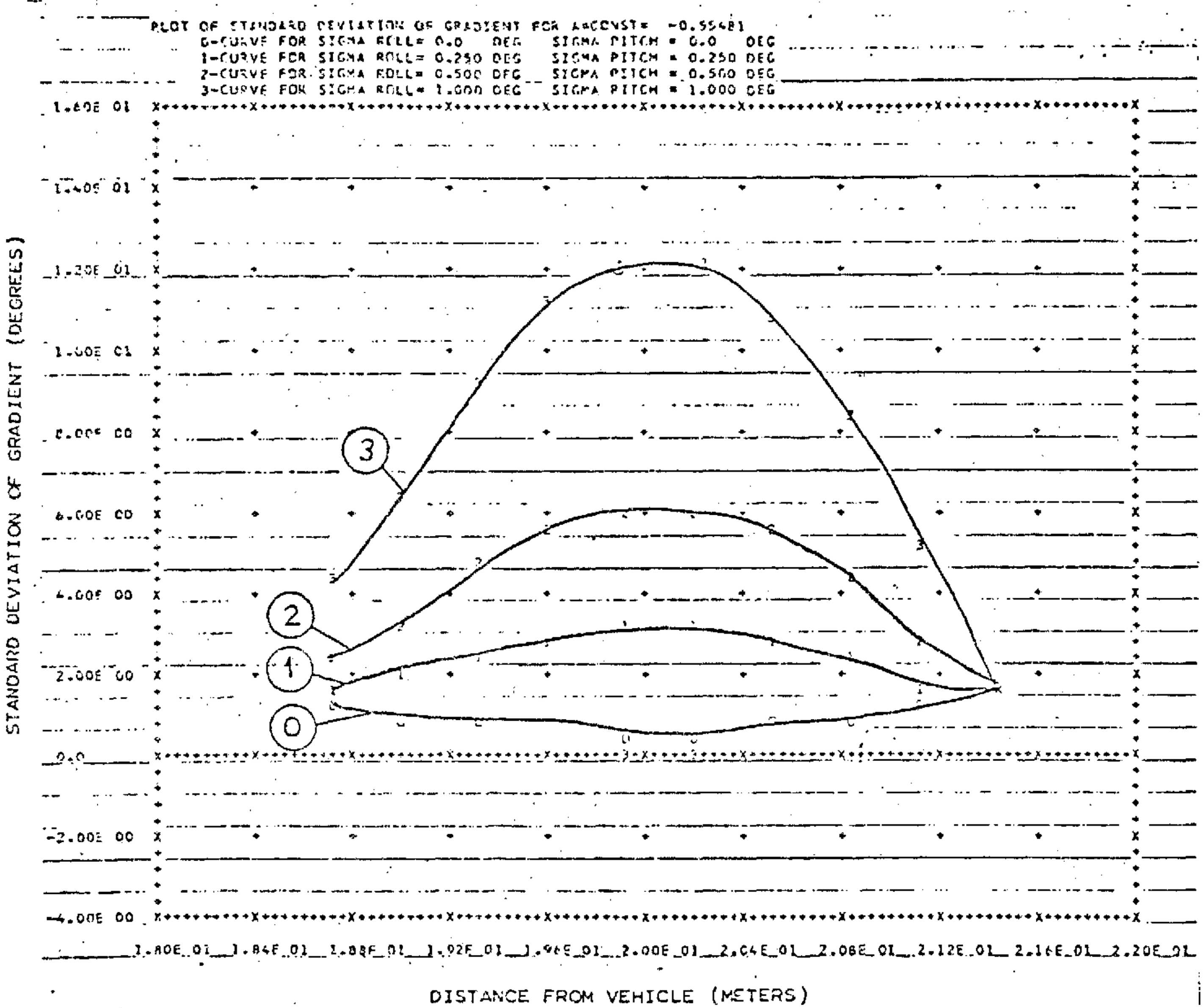

Figure 9 Plot of standard deviation of gradient (σ_{Sc}) for four values of σ_R for four hill cross sections

9a Cross section at A = -0.55481 m.

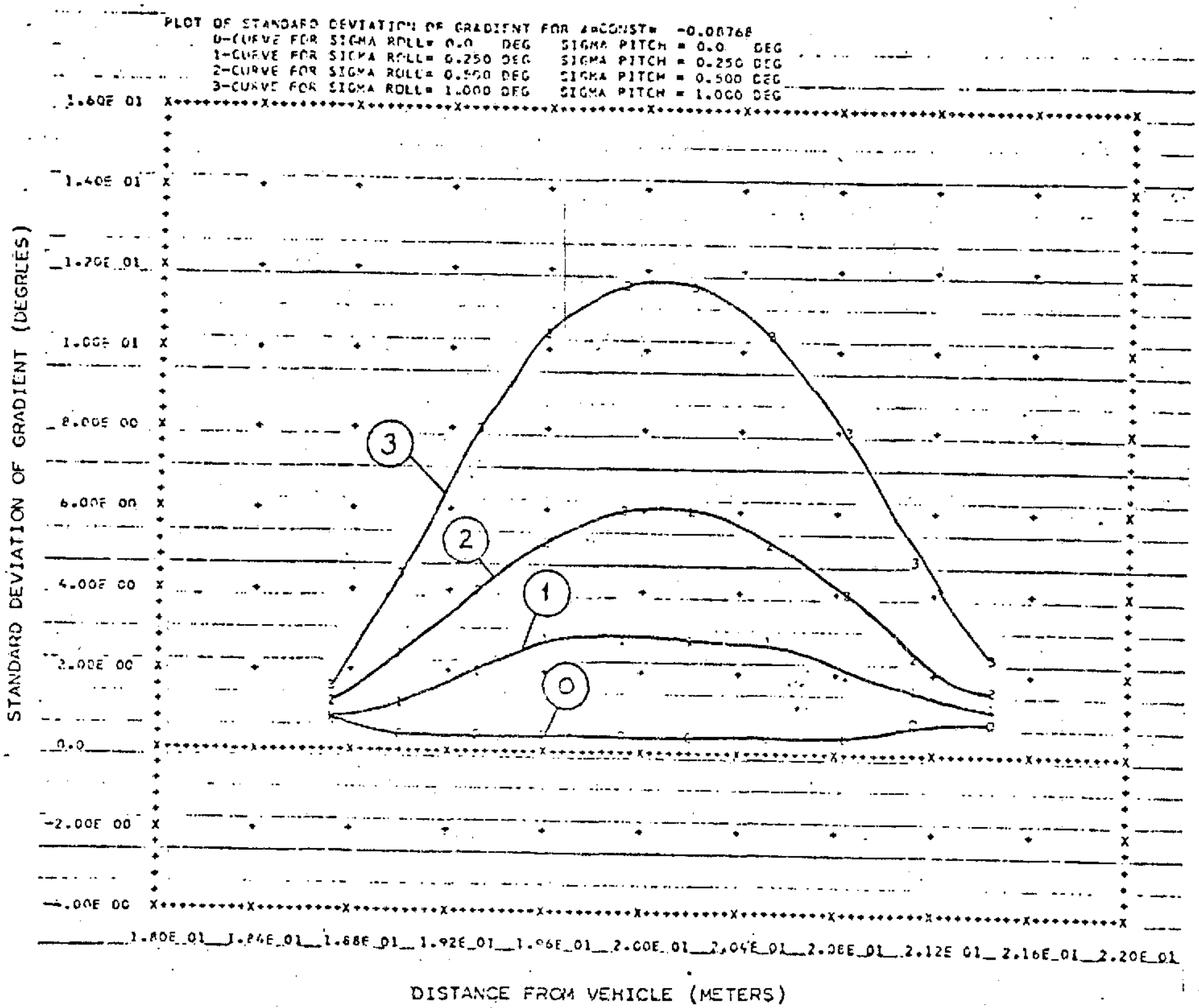

Figure 9b Cross section at A = -0.08768 m.

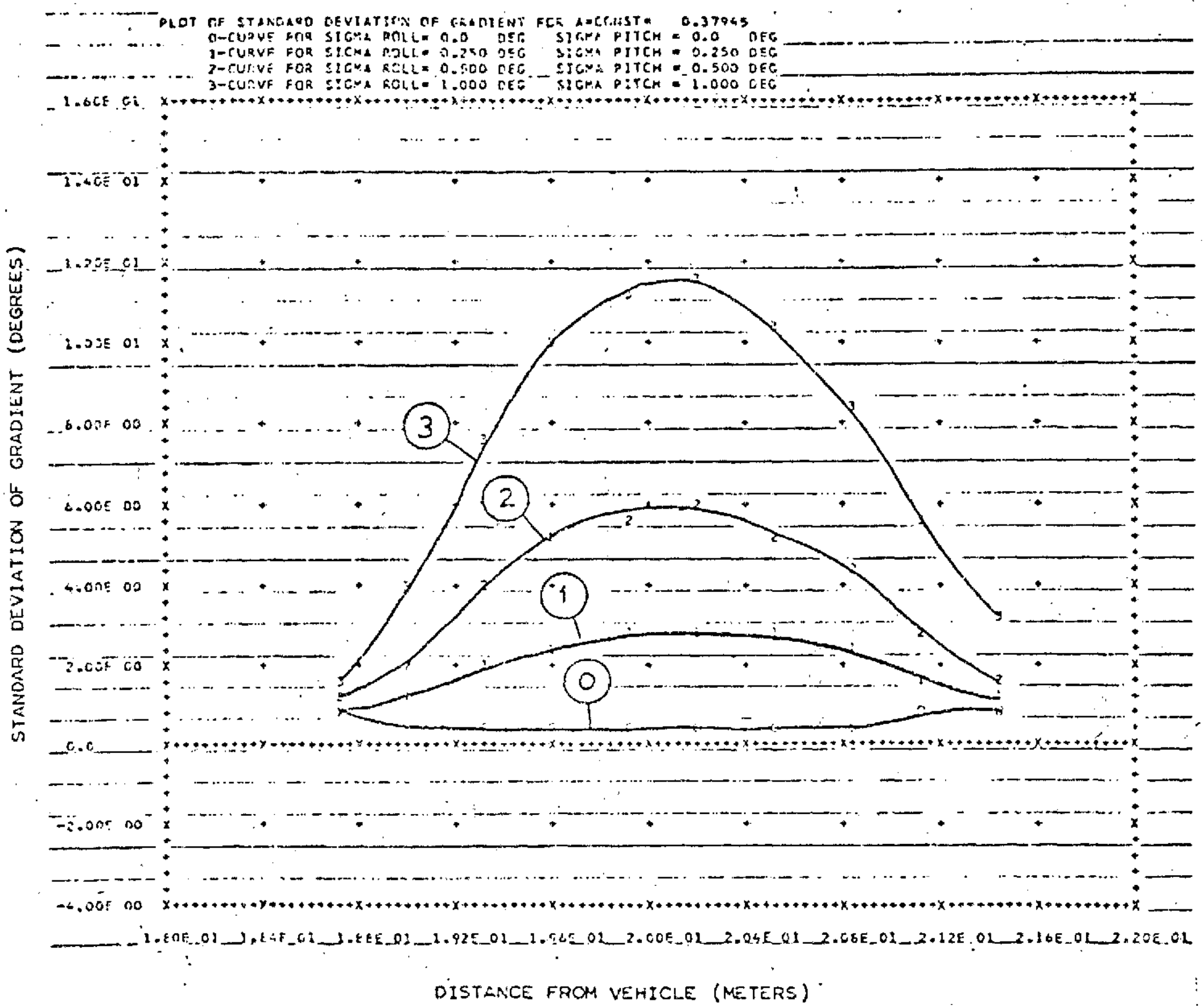

Figure 9c Cross section at A = 0.37945 m.

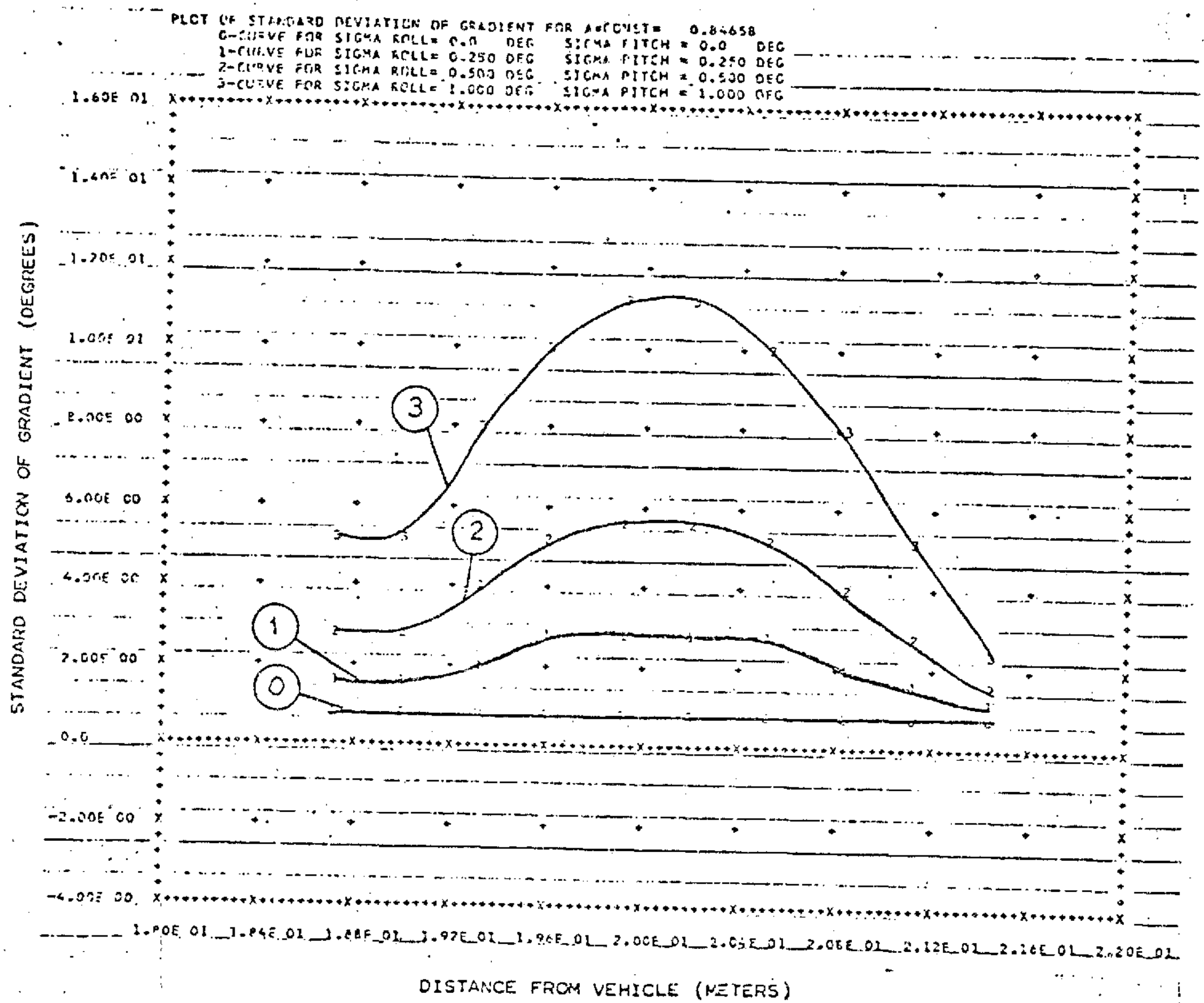

Figure 9d Cross section at A = 0.84658 m.

PART 7

CONCLUSION

This report has developed in detail a two-step terrain modeling procedure, using gradient and height information obtained via a laser rangefinder. The formulated terrain model is composed of third order polynomials which approximate the actual terrain surface. Two steps are used instead of one so that the developed models are less sensitive to instrumental error. The use of gradient information in the modeling process results in a model that follows the actual gradient closer than one using height data only. This is important since the gradient is an important factor in determining whether the terrain is passable or impassable by the vehicle. Also, this method uses few data points, thus saving time in the scanning process. This could allow the vehicle to save energy as well as increase its rate of forward travel.

From the simulation results, this method of terrain modeling seems to have the potential for usefully portraying the actual terrain contour and gradient, although much work is still needed to refine the modeling method.

The use of two scan rows for the terrain model may not prove practical unless the standard deviation of the roll and pitch measurement is reduced to about 0.25°. If this is not feasible, this modeling method may still be used if the scanning scheme can be changed so that enough points can be found rapid from one row of scan instead of the present two rows of scan.

This scheme could also be used in conjunction with an edge detection scheme. The plan here is for the edge detection scheme to locate the boundary of an obstacle. Once the boundary has been located, the terrain modeling procedure would be used to determine the frontal shape of the obstacle. By examining the terrain model, a decision could be made as to whether or not the obstacle was passable.

Another plan would be a data point saving technique, where the surface is scanned, using few data points. The terrain modeling procedure would then be used to determine a rough picture of the terrain. If any questionable areas were present, more data points could be taken of these areas to obtain a detailed model, or the area avoided completely.

PART 8

REFERENCES

1. Burger, Paul, "Stochastic Estimates of Gradient from Laser Measurements for an Autonomous Mortion Roving Vehicle," Masters Project Report, R.P.I., May, 1973.

2. Fisher, R. and Ziebur, A., Calculus and Analytic Geometry, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 2nd Edition, p. 464, 1967.

APPENDIX A

DERIVATION OF COVARIANCE BLOCK IV

The quantities $a_p''^n$ and $b_p''^n$ are related to the measured quantities of the four data points used to determine the plane by

$$a_p''^n = \left(a_1''^n + a_2''^n + a_3''^n + a_4''^n \right) \frac{1}{4} \qquad \text{(A-1)}$$

$$b_p''^n = \left(b_1''^n + b_2''^n + b_3''^n + b_4''^n \right) \frac{1}{4} \qquad \text{(A-2)}$$

Perturbing (A-1) and (A-2) yields

$$\delta a_p''^n = \left(\delta a_1''^n + \delta a_2''^n + \delta a_3''^n + \delta a_4''^n \right) \frac{1}{4} \qquad \text{(A-3)}$$

$$\delta b_p''^n = \left(\delta b_1''^n + \delta b_2''^n + \delta b_3''^n + \delta b_4''^n \right) \frac{1}{4} \qquad \text{(A-4)}$$

Rewriting this in matrix form

$$\begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \end{bmatrix} = Q \begin{bmatrix} \delta a_1''^n & \delta a_2''^n & \delta a_3''^n & \delta a_4''^n & \delta b_1''^n & \delta b_2''^n & \delta b_3''^n & \delta b_4''^n \end{bmatrix}^T \qquad \text{(A-5)}$$

where
$$Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Then multiplying (A-5) by its transpose yields

$$\begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix}^T = Q \begin{bmatrix} \delta a_1^{\prime\prime n} \\ \delta a_2^{\prime\prime n} \\ \vdots \\ \delta b_4^{\prime\prime n} \end{bmatrix} \begin{bmatrix} \delta a_1^{\prime\prime n} \\ \delta a_2^{\prime\prime n} \\ \vdots \\ \delta b_4^{\prime\prime n} \end{bmatrix}^T Q^T \qquad \text{(A-6)}$$

The expected value of (A-6) can now be taken. Since all of the data points are measured independently, all of the terms on the right of Eqn. (A-6) with non-matching subscripts are non-correlated. Therefore, their expected value is zero. Finally, this yields

$$E\left\{ \begin{vmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{vmatrix} \begin{vmatrix} \delta a_p^{\prime\prime n} & \delta b_p^{\prime\prime n} \end{vmatrix} \right\} =$$

$$Q \begin{bmatrix}
E(\delta a_1^{\prime\prime n})^2 & 0 & 0 & 0 & E(\delta a_1^{\prime\prime n}\delta b_1^{\prime\prime n}) & 0 & 0 & 0 \\
0 & E(\delta a_2^{\prime\prime n})^2 & 0 & 0 & 0 & E(\delta a_2^{\prime\prime n}\delta b_2^{\prime\prime n}) & 0 & 0 \\
0 & 0 & E(\delta a_3^{\prime\prime n})^2 & 0 & 0 & 0 & E(\delta a_3^{\prime\prime n}\delta b_3^{\prime\prime n}) & 0 \\
0 & 0 & 0 & E(\delta a_4^{\prime\prime n})^2 & 0 & 0 & 0 & E(\delta a_4^{\prime\prime n}\delta b_4^{\prime\prime n}) \\
E(\delta a_1^{\prime\prime n}\delta b_1^{\prime\prime n}) & 0 & 0 & 0 & E(\delta b_1^{\prime\prime n})^2 & 0 & 0 & 0 \\
0 & E(\delta a_2^{\prime\prime n}\delta b_2^{\prime\prime n}) & 0 & 0 & 0 & E(\delta b_2^{\prime\prime n})^2 & 0 & 0 \\
0 & 0 & E(\delta a_3^{\prime\prime n}\delta b_3^{\prime\prime n}) & 0 & 0 & 0 & E(\delta b_3^{\prime\prime n})^2 & 0 \\
0 & 0 & 0 & E(\delta a_4^{\prime\prime n}\delta b_4^{\prime\prime n}) & 0 & 0 & 0 & E(\delta b_4^{\prime\prime n})^2
\end{bmatrix} Q^T \qquad \text{(A-7)}$$

The expected value quantities in (A-7) are calculated in Eqn. 29. Therefore, Eqn. (A-7) can be evaluated directly.

APPENDIX B

DERIVATION OF EQUATIONS 33 AND 34

The perturbed values of the slopes can be written as[1]

$$\begin{bmatrix} \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} = F^n \left(\delta \underline{h}''^n - \delta A'' \underline{x}''^n \right) \qquad \text{(B-1)}$$

where $F^n = (A''^T A'')^{-1} A''^T$

$$A'' = \begin{bmatrix} a_1''^n & b_1''^n & 1 \\ a_2''^n & b_2''^n & 1 \\ a_3''^n & b_3''^n & 1 \\ a_4''^n & b_4''^n & 1 \end{bmatrix} \qquad \delta A'' \underline{x}''^n = \begin{bmatrix} \delta a_1''^n x_1''^n + \delta b_1''^n x_2''^n \\ \delta a_2''^n x_1''^n + \delta b_2''^n x_2''^n \\ \delta a_3''^n x_1''^n + \delta b_3''^n x_2''^n \\ \delta a_4''^n x_1''^n + \delta b_4''^n x_2''^n \end{bmatrix}$$

$$\delta \underline{h}''^n = \begin{bmatrix} \delta h_1''^n & \delta h_2''^n & \delta h_3''^n & \delta h_4''^n \end{bmatrix}^T$$

Then the vector ($\delta \underline{h}''^n - \delta A'' \underline{x}''^n$) can be written expressly by

$$\left(\delta \underline{h}''^n - \delta A'' \underline{x}''^n \right) = \begin{bmatrix} \delta h_1''^n - \delta a_1''^n x_1''^n - \delta b_1''^n x_2''^n \\ \delta h_2''^n - \delta a_2''^n x_1''^n - \delta b_2''^n x_2''^n \\ \delta h_3''^n - \delta a_3''^n x_1''^n - \delta b_3''^n x_2''^n \\ \delta h_4''^n - \delta a_4''^n x_1''^n - \delta b_4''^n x_2''^n \end{bmatrix} \qquad \text{(B-3)}$$

or in matrix form as

$$\left(\delta \underline{h}''^{n} - \delta A'' \underline{x}''^{n}\right) = \begin{bmatrix} \delta h_1''^{n} & \delta a_1''^{n} & \delta b_1''^{n} \\ \delta h_2''^{n} & \delta a_2''^{n} & \delta b_2''^{n} \\ \delta h_3''^{n} & \delta a_3''^{n} & \delta b_3''^{n} \\ \delta h_4''^{n} & \delta a_4''^{n} & \delta b_4''^{n} \end{bmatrix} \begin{bmatrix} 1 \\ -x_1''^{n} \\ -x_2''^{n} \end{bmatrix} \tag{B-3}$$

If f_{ij}^{n} is an element of F^{n} then the j^{th} row of Eqn. (B-1) is

$$\delta x_j''^{n} = \begin{bmatrix} f_{j1}^{n} & f_{j2}^{n} & f_{j3}^{n} & f_{j4}^{n} \end{bmatrix} \left(\delta \underline{h}''^{n} - \delta A'' \underline{x}''^{n}\right) \tag{B-4}$$

or

$$\delta x_j''^{n} = f_{j1}^{n} \begin{bmatrix} 1 & -x_1''^{n} & -x_2''^{n} \end{bmatrix} \begin{bmatrix} \delta h_1''^{n} \\ \delta a_1''^{n} \\ \delta b_1''^{n} \end{bmatrix} + f_{j2}^{n} \begin{bmatrix} 1 & -x_1''^{n} & -x_2''^{n} \end{bmatrix} \begin{bmatrix} \delta h_2''^{n} \\ \delta a_2''^{n} \\ \delta b_2''^{n} \end{bmatrix}$$

$$+ f_{j3}^{n} \begin{bmatrix} 1 & -x_1''^{n} & -x_2''^{n} \end{bmatrix} \begin{bmatrix} \delta h_3''^{n} \\ \delta a_3''^{n} \\ \delta b_3''^{n} \end{bmatrix} + f_{j4}^{n} \begin{bmatrix} 1 & -x_1''^{n} & -x_2''^{n} \end{bmatrix} \begin{bmatrix} \delta h_4''^{n} \\ \delta a_4''^{n} \\ \delta b_4''^{n} \end{bmatrix} \tag{B-5}$$

The terms in Block V can be written out as

$$E \left\{ \begin{bmatrix} \delta x_1''^{n} \\ \delta x_2''^{n} \\ \delta x_3''^{n} \end{bmatrix} \begin{bmatrix} \delta a_p''^{n} & \delta b_p''^{n} \end{bmatrix} \right\} = \begin{bmatrix} E\left(\delta x_1''^{n}\, \delta a_p''^{n}\right) & E\left(\delta x_1''^{n}\, \delta b_p''^{n}\right) \\ E\left(\delta x_2''^{n}\, \delta a_p''^{n}\right) & E\left(\delta x_2''^{n}\, \delta b_p''^{n}\right) \\ E\left(\delta x_3''^{n}\, \delta a_p''^{n}\right) & E\left(\delta x_3''^{n}\, \delta b_p''^{n}\right) \end{bmatrix} \tag{B-6}$$

For the first column the terms are now expressed using Eqn. (B-4) and Eqn. (A-1)

$$E\left(\delta x_j''^n \delta a_p''^n\right) =$$

$$E\left\{\left[f_{j1}^n \; f_{j2}^n \; f_{j3}^n \; f_{j4}^n\right]\left[\delta \underline{h}''^n - \delta A'' \underline{x}''^n\right]\left[\delta a_1''^n \; \delta a_2''^n \; \delta a_3''^n \; \delta a_4''^n\right]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}\frac{1}{4} \tag{B-7}$$

and rewriting this using Eqn. (B-5)

$$E\left(\delta x_j''^n \delta a_p''^n\right) =$$

$$E\left\{f_{j1}^n\left[1 \quad -x_1''^n \quad -x_2''^n\right]\begin{bmatrix}\delta h_1''^n\\ \delta a_1''^n\\ \delta b_1''^n\end{bmatrix}\left[\delta a_1''^n \; \delta a_2''^n \; \delta a_3''^n \; \delta a_4''^n\right]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{1}{4} + \right.$$

$$\left. f_{j2}^n\left[1 \quad -x_1''^n \quad -x_2''^n\right]\begin{bmatrix}\delta h_2''^n\\ \delta a_2''^n\\ \delta b_2''^n\end{bmatrix}\left[\delta a_1''^n \; \delta a_2''^n \; \delta a_3''^n \; \delta a_4''^n\right]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{1}{4} + \cdots\right\} \tag{B-8}$$

Because terms in the right-hand side of Eqn. (B-8) are uncorrelated if their subscripts do not match, the expected value of these terms is zero. Combining terms and simplifying Eqn. (B-8) results in

$$E\left(\delta x_j^{''n}\,\delta a_p^{''n}\right) =$$

$$\frac{1}{4}\left[1, -x_1^{''n}, -x_2^{''n}\right]\left\{ f_{j1}\begin{bmatrix} E\left(\delta h_1^{''n}\,\delta a_1^{''n}\right) \\ E\left(\delta a_1^{''n}\,\delta a_1^{''n}\right) \\ E\left(\delta b_1^{''n}\,\delta a_1^{''n}\right) \end{bmatrix} + f_{j2}\begin{bmatrix} E\left(\delta h_2^{''n}\,\delta a_2^{''n}\right) \\ E\left(\delta a_2^{''n}\,\delta a_2^{''n}\right) \\ E\left(\delta b_2^{''n}\,\delta a_2^{''n}\right) \end{bmatrix} + \cdots \right\} \qquad \text{(B-9)}$$

$$j = 1, 2, 3$$

A similar analysis can be done for the $E\left(\delta x_j^{''n}\,\delta b_p^{''n}\right)$ terms yielding

$$E\left(\delta x_j^{''n}\,\delta b_p^{''n}\right) =$$

$$\frac{1}{4}\left[1, -x_1^{''n}, -x_2^{''n}\right]\left\{ f_{j1}\begin{bmatrix} E\left(\delta h_1^{''n}\,\delta b_1^{''n}\right) \\ E\left(\delta a_1^{''n}\,\delta b_1^{''n}\right) \\ E\left(\delta b_1^{''n}\,\delta b_1^{''n}\right) \end{bmatrix} + f_{j2}\begin{bmatrix} E\left(\delta h_2^{''n}\,\delta b_2^{''n}\right) \\ E\left(\delta a_2^{''n}\,\delta b_2^{''n}\right) \\ E\left(\delta b_2^{''n}\,\delta b_2^{''n}\right) \end{bmatrix} + \cdots \right\} \qquad \text{(B-10)}$$

$$j = 1, 2, 3$$

APPENDIX C

DERIVATION OF BLOCK II

Eqn. 8, the height of $h_p''^n$, can be perturbed which yields

$$\delta h_p''^n = x_1''^n \delta a_p''^n + \delta x_1''^n a_p''^n + x_2''^n \delta b_p''^n + \delta x_2''^n b_p''^n + \delta x_3''^n \quad \text{(C-1)}$$

This can be rewritten as a matrix equation

$$\delta h_p''^n = S^n \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} \quad \text{(C-2)}$$

where $$S^n = \begin{bmatrix} x_1''^n & x_2''^n & a_p''^n & b_p''^n & 1 \end{bmatrix}$$

Therefore, Eqn. 35 can be written directly by

$$E\left\{ \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} \begin{bmatrix} \delta h_p''^n \end{bmatrix} \right\} = E\left\{ \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix} \begin{bmatrix} \delta a_p''^n \\ \delta b_p''^n \\ \delta x_1''^n \\ \delta x_2''^n \\ \delta x_3''^n \end{bmatrix}^T \right\} S^{nT} \quad \text{(C-3)}$$

$$= M1''^n S^{nT}$$

APPENDIX D

DERIVATION OF COVARIANCE BLOCK A $_{\{\delta h\}^{\prime\prime n}}$

The transformation of the center point in the primed system to the unprimed system

$$\begin{bmatrix} h_p^n \\ a_p^n \\ b_p^n \end{bmatrix} = C^n B^n \begin{bmatrix} h_p^{\prime\prime n} \\ a_p^{\prime\prime n} \\ b_p^{\prime\prime n} \end{bmatrix} \tag{D-1}$$

can be perturbed

$$\begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \end{bmatrix} = \delta C^n B^n \begin{bmatrix} h_p^{\prime\prime n} \\ a_p^{\prime\prime n} \\ b_p^{\prime\prime n} \end{bmatrix} + C^n \delta B^n \begin{bmatrix} h_p^{\prime\prime n} \\ a_p^{\prime\prime n} \\ b_p^{\prime\prime n} \end{bmatrix} + C^n B^n \begin{bmatrix} \delta h_p^{\prime\prime n} \\ \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix} \tag{D-2}$$

This equation can be reduced to yield [1]

$$\begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \end{bmatrix} = D^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} + C^n B^n \begin{bmatrix} \delta h_p^{\prime\prime n} \\ \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \end{bmatrix} \tag{D-3}$$

where D^n is written in Eqn. 39.

Multiplying Eqn. (D-3) by its transpose and taking the expected value

$$E\left\{ \begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \end{bmatrix} \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n \end{bmatrix} \right\} = E\left\{ D^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} \begin{bmatrix} \delta\phi & \delta\xi \end{bmatrix} D^{nT} + \right.$$

$$D^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} \begin{bmatrix} \delta h_p''^n & \delta a_p''^n & \delta b_p''^n \end{bmatrix} B^{nT} C^{nT} + C^n B^n \begin{bmatrix} \delta h_p''^n \\ \delta a_p''^n \\ \delta b_p''^n \end{bmatrix} \begin{bmatrix} \delta\phi & \delta\xi \end{bmatrix} D^{nT}$$

$$\left. + \; C^n B^n \begin{bmatrix} \delta h_p''^n \\ \delta a_p''^n \\ \delta b_p''^n \end{bmatrix} \begin{bmatrix} \delta h_p''^n & \delta a_p''^n & \delta b_p''^n \end{bmatrix} B^{nT} C^{nT} \right\} \qquad \text{(D-4)}$$

Since $\delta\phi$ and $\delta\xi$ are not correlated with themselves or $\delta h_p''^n$, $\delta a_p''^n$, $\delta b_p''^n$ then Eqn. (D-4) reduces to Eqn. 39.

APPENDIX E

DERIVATION OF COVARIANCE BLOCK B

From Eqns. 16 and 17, the expressions for X_1^n and X_2^n are written

$$X_1^n = -\frac{N_a^n}{N_h^n} \tag{E-1}$$

$$X_2^n = -\frac{N_b^n}{N_h^n} \tag{E-2}$$

Perturbing Eqns. (E-1) and (E-2)

$$\delta X_1^n = -\frac{\delta N_a^n N_h^n - N_a^n \delta N_h^n}{\left(N_h^n\right)^2}$$

$$= \frac{N_a^n}{\left(N_h^n\right)^2}\,\delta N_h^n - \frac{1}{N_h^n}\,\delta N_a^n \tag{E-3}$$

$$\delta X_2^n = -\frac{\delta N_b^n N_h^n - N_b^n \delta N_h^n}{\left(N_h^n\right)^2}$$

$$= \frac{N_b^n}{\left(N_h^n\right)^2}\,\delta N_h^n - \frac{1}{N_h^n}\,\delta N_b^n \tag{E-4}$$

Rewriting Eqns. (E-3) and (E-4) as a single matrix equation

$$\begin{bmatrix}\delta x_1^n \\ \delta x_2^n\end{bmatrix} = U^n \begin{bmatrix}\delta N_h^n \\ \delta N_a^n \\ \delta N_b^n\end{bmatrix} \qquad \text{(E-5)}$$

where

$$U^n = \begin{bmatrix} \dfrac{N_a^n}{(N_h^n)^2} & -\dfrac{1}{N_h^n} & 0 \\ \dfrac{N_b^n}{(N_h^n)^2} & 0 & -\dfrac{1}{N_h^n} \end{bmatrix}$$

Now from Eqn. 14

$$\begin{bmatrix} N_h^n \\ N_a^n \\ N_b^n \end{bmatrix} = C^n B^n \begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} \qquad \text{(E-6)}$$

Perturbing Eqn. (E-6)

$$\begin{bmatrix} \delta N_h^n \\ \delta N_a^n \\ \delta N_b^n \end{bmatrix} = \delta C^n B^n \begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} + C^n \delta B^n \begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} + C^n B^n \begin{bmatrix} 0 \\ \delta x_1''^n \\ \delta x_2''^n \end{bmatrix} \qquad \text{(E-7)}$$

The first two right-hand terms can be written as[1]

$$\left\{ C_A^n B^n \begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} \vdots \; C^n B_A^n \begin{bmatrix} -1 \\ x_1''^n \\ x_2''^n \end{bmatrix} \right\} \begin{vmatrix} \delta \phi \\ \delta \xi \end{vmatrix} \qquad \text{(E-8)}$$

where

$$C_A = \begin{bmatrix} -\sin\phi^n & -\cos\phi^n & 0 \\ \cos\phi^n & -\sin\phi^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_A = \begin{bmatrix} -\sin\xi^n & 0 & \cos\xi^n \\ 0 & 0 & 0 \\ -\cos\xi^n & 0 & -\sin\xi^n \end{bmatrix}$$

Substituting the values for the quantities indicated in Eqn. (E-8) and performing the operations reduces the first two terms in Eqn. (E-7) to

$$D_x^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} \tag{E-9}$$

where D_x^n is defined in Eqn. 40.

Substituting Eqns. (E-9) and (E-7) into Eqn. (E-5) yields the expression

$$\begin{bmatrix} \delta X_1^n \\ \delta X_2^n \end{bmatrix} = U^n D_x^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} + U^n C^n B^n \begin{bmatrix} 0 \\ \delta X_1''^n \\ \delta X_2''^n \end{bmatrix} \tag{E-10}$$

Multiplying Eqn. (E-10) by the transpose of Eqn. (D-3) and taking the expected value, one obtains the expression for Block B.

$$E\left\{\begin{bmatrix} \delta X_1^n \\ \delta X_2^n \end{bmatrix} \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n \end{bmatrix}\right\} = E\left\{U^n D_x^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} \begin{bmatrix} \delta\phi & \delta\xi \end{bmatrix} D^{nT} + \right.$$

$$U^n D_x^n \begin{bmatrix} \delta\phi \\ \delta\xi \end{bmatrix} \begin{bmatrix} \delta h_p''^n & \delta a_p''^n & \delta b_p''^n \end{bmatrix} B^{nT} C^{nT} + U^n C^n B^n \begin{bmatrix} 0 \\ \delta X_1''^n \\ \delta X_2''^n \end{bmatrix} \begin{bmatrix} \delta\phi & \delta\xi \end{bmatrix} D^{nT}$$

$$\left. + U^n C^n B^n \begin{bmatrix} 0 \\ \delta X_1''^n \\ \delta X_2''^n \end{bmatrix} \begin{bmatrix} \delta h_p''^n & \delta a_p''^n & \delta b_p''^n \end{bmatrix} B^{nT} C^{nT} \right\} \tag{E-11}$$

The two middle terms are eliminated since they are not correlated. Finally, Eqn. (E-11) reduces to Eqn. 40 in the text.

APPENDIX F

DERIVATION OF COVARIANCE BLOCK $M_c I$

From Eqn. 25

$$\begin{bmatrix} C_{00}^{\ddagger} \\ C_{10}^{\ddagger} \\ C_{01}^{\ddagger} \end{bmatrix} = \begin{bmatrix} h_p' \\ x_1' \\ x_2' \end{bmatrix} \tag{F-1}$$

Perturbing Eqn. (F-1) results in

$$\begin{bmatrix} \delta C_{00}^{\ddagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger} \end{bmatrix} = \begin{bmatrix} \delta h_p^{\ddagger} \\ \delta x_1' \\ \delta x_2' \end{bmatrix} \tag{F-2}$$

or in another form

$$\begin{bmatrix} \delta C_{00}^{\dagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger} \end{bmatrix} = R\,\Phi_1 \tag{F-3}$$

where

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\Phi_1 = \begin{bmatrix} \delta h_p' & \delta a_p' & \delta b_p' & \delta x_1' & \delta x_2' \end{bmatrix}^T$$

Multiplying Eqn. (F-3) by its transpose and taking the expected value results in

$$E\left\{\begin{bmatrix}\delta C_{00}^{\ddagger} \\ \delta C_{10}^{\ddagger} \\ \delta C_{01}^{\ddagger}\end{bmatrix}\begin{bmatrix}\delta C_{00}^{\ddagger} & \delta C_{10}^{\ddagger} & \delta C_{01}^{\ddagger}\end{bmatrix}\right\} = M_c I = R\,\Phi_1\,\Phi_1^T R^T \tag{F-4}$$

However, $\Phi_1 \Phi_1^T$ is just the covariance matrix of the center point number 1. Therefore,

$$M_c I = R\,M_p^1\,R^T \tag{F-5}$$

APPENDIX G

DERIVATION OF COVARIANCE BLOCK $M_c II$

The system equation is given as Eqn. 26 for the $\underline{C1}^{\ddagger}$ parameters. Perturbing this equation yields

$$\delta W = T\,\delta \underline{C1}^{\ddagger} + \delta T\,\underline{C1}^{\ddagger} \qquad \text{(G-1)}$$

Rearranging terms in Eqn. (G-1)

$$\left(\delta W - \delta T \underline{C1}^{\ddagger}\right) = T\,\delta \underline{C1}^{\ddagger} \qquad \text{(G-2)}$$

Taking the least square estimate of the perturbed parameters yields

$$\delta \underline{C1}^{\ddagger} = \left(T^T T\right)^{-1} T^T \left(\delta W - \delta T\,\underline{C1}^{\ddagger}\right) \qquad \text{(G-3)}$$

or

$$\delta \underline{C1}^{\ddagger} = Z\left(\delta W - \delta T\,\underline{C1}^{\ddagger}\right) \qquad \text{(G-4)}$$

where

$$Z = \left(T^T T\right)^{-1} T^T$$

The next step is to find an expression for the right-hand side of Eqn. (G-4). The first step in this process is to perturb the equation for W, Eqn. 26b. This yields

$$\delta W = \begin{bmatrix} \delta h_p^2 - \delta h_p^1 - \delta a_p^{\ddagger 2} x_1^1 - a_p^{\ddagger 2}\delta x_1^1 - \delta b_p^{\ddagger 2} x_2^1 - b_p^{\ddagger 2}\delta x_2^1 \\ \delta x_1^2 - \delta x_1^1 \\ \delta x_2^2 - \delta x_2^1 \\ \delta h_p^3 - \delta h_p^1 - \delta a_p^{\ddagger 3} x_1^1 - a_p^{\ddagger 3}\delta x_1^1 - \delta b_p^{\ddagger 3} x_2^1 - b_p^{\ddagger 3}\delta x_2^1 \\ \vdots \\ \delta x_2^4 - \delta x_2^1 \end{bmatrix} \qquad \text{(G-5)}$$

However, this is written in terms of the transformed coordinates. The transformation equation was given before as

$$
\begin{aligned}
a_p^{*n} &= a_p^n - a_p^1 \\
b_p^{*n} &= b_p^n - b_p^1
\end{aligned}
\qquad \text{(G-6)}
$$

Now perturbing Eqn. (G-6) yields

$$
\begin{aligned}
\delta a_p^{*n} &= \delta a_p^n - \delta a_p^1 \\
\delta b_p^{*n} &= \delta b_p^n - \delta b_p^1
\end{aligned}
\qquad \text{(G-7)}
$$

Substituting the results of Eqn. (G-7) into Eqn. (G-5) yields

$$
\delta W = \begin{bmatrix}
\left(\delta h_p^2 - \delta h_p^1 - \delta a_p^2 x_1^1 + \delta a_p^1 x_1^1 - a_p^{*2}\delta x_1^1 - \delta b_p^2 x_2^1 \right. \\
\qquad\qquad \left. + \delta b_p^1 x_2^1 - b_p^{*2}\delta x_2^1\right) \\
\left(\delta x_1^2 - \delta x_1^1\right) \\
\left(\delta x_2^2 - \delta x_2^1\right) \\
\left(\delta h_p^3 - \delta h_p^1 - \delta a_p^3 x_1^1 + \delta a_p^1 x_1^1 - a_p^{*3}\delta x_1^1 - \delta b_p^3 x_2^1 \right. \\
\qquad\qquad \left. + \delta b_p^1 x_2^1 - b_p^{*3}\delta x_2^1\right) \\
\vdots \\
\left(\delta x_2^4 - \delta x_2^1\right)
\end{bmatrix}
\qquad \text{(G-8)}
$$

This matrix can be split into two parts. One part is a function of the perturbed variables for center point 1. The other part is a function of the perturbed variables for the center points 2, 3, 4.

$$
\delta W = \begin{bmatrix} \delta h_p^2 - \delta a_p^2 x_1^1 - \delta b_p^2 x_2^1 \\ \delta x_1^2 \\ \delta x_2^2 \\ \delta h_p^3 - \delta a_p^3 x_1^1 - \delta b_p^3 x_2^1 \\ \vdots \\ \delta x_2^4 \end{bmatrix}
$$

$$
- \begin{bmatrix} \delta h_p^1 - \delta a_p^1 x_1^1 - \delta b_p^1 x_2^1 - a_p^{*2} \delta x_1^1 + b_p^{*2} \delta x_2^1 \\ \delta x_1^1 \\ \delta x_2^1 \\ \delta h_p^1 - \delta a_p^1 x_1^1 - \delta b_p^1 x_2^1 - a_p^{*3} \delta x_1^1 + b_p^{*3} \delta x_2^1 \\ \vdots \\ \delta x_2^1 \end{bmatrix} \quad \text{(G-9)}
$$

The matricies in Eqn. (G-9) can be written as partitioned matricies in the compact form..

$$
\delta W = \begin{bmatrix} \Pi_1 \, \Phi_2 \\ \Pi_1 \, \Phi_3 \\ \Pi_1 \, \Phi_4 \end{bmatrix} - \begin{bmatrix} \Pi_2 \, \Phi_1 \\ \Pi_3 \, \Phi_1 \\ \Pi_4 \, \Phi_1 \end{bmatrix} \quad \text{(G-11)}
$$

where

$$\Pi_n = \begin{bmatrix} 1 & -X_1^1 & -X_2^1 & a_p^{*n} & b_p^{*n} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(G-11a)}$$

$$\Phi_n = \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n & \delta X_1^n & \delta X_2^n \end{bmatrix}^T \qquad \text{(G-11b)}$$

In order to evaluate Eqn. (G-4), an expression for $\delta T\ \underline{C1}^*$ must also be found. Multiplying the equation for T (Eqn. 26b) by $\underline{C1}^*$ and perturbing

$$\delta T \underline{C1}^* = \begin{bmatrix} \Big(a_p^{*2}\delta a_p^{*2} C_{20}^* + \delta a_p^{*2} b_p^{*2} C_{11}^* + a_p^{*2}\delta b_p^{*2} C_{11}^* + \\ b_p^{*2}\delta b_p^{*2} C_{02}^* + \frac{(a_p^{*2})^2}{2}\delta a_p^{*2} C_{30}^* + a_p^{*2} b_p^{*2}\delta a_p^{*2} C_{21}^* + \\ \frac{(a_p^{*2})^2}{2}\delta b_p^{*2} C_{21}^* + \delta a_p^{*2}\frac{(b_p^{*2})^2}{2} C_{12}^* + a_p^{*2} b_p^{*2}\delta b_p^{*2} C_{12}^* + \\ \frac{(b_p^{*2})^2}{2}\delta b_p^{*2} C_{03}^* \Big) \\ \Big(\delta a_p^{*2} C_{20}^* + \delta b_p^{*2} C_{11}^* + 0 + a_p^{*2}\delta a_p^{*2} C_{30}^* + \\ \delta a_p^{*2} b_p^{*2} C_{21}^* + a_p^{*2}\delta b_p^{*2} C_{21}^* + b_p^{*2}\delta b_p^{*2} C_{12}^* \Big) \\ \Big(\delta a_p^{*2} C_{11}^* + \delta b_p^{*} C_{02}^* + 0 + a_p^{*2}\delta a_p^{*2} C_{21}^* + \\ a_p^{*2}\delta b_p^{*2} C_{12}^* + \delta a_p^{*2} b_p^{*2} C_{12}^* + b_p^{*2}\delta b_p^{*2} C_{03}^* \Big) \\ \vdots \end{bmatrix} \qquad \text{(G-12)}$$

$$\delta T\underline{C1}^{*} = \begin{bmatrix} TK_2 \left[\delta h_p^2 \quad \delta a_p^{*2} \quad \delta b_p^{*2} \quad \delta x_1^2 \quad \delta x_2^2\right]^T \\ TK_3 \left[\delta h_p^3 \quad \delta a_p^{*3} \quad \delta b_p^{*3} \quad \delta x_1^3 \quad \delta x_2^3\right]^T \\ TK_4 \left[\delta h_p^4 \quad \delta a_p^{*4} \quad \delta b_p^{*4} \quad \delta x_1^4 \quad \delta x_2^4\right]^T \end{bmatrix} \qquad \text{(G-13)}$$

where

$$TK_n = \begin{bmatrix} 0 & TK_n(1,2) & TK_n(1,3) & 0 & 0 \\ 0 & TK_n(2,2) & TK_n(2,3) & 0 & 0 \\ 0 & TK_n(3,2) & TK_n(3,3) & 0 & 0 \end{bmatrix}$$

$$TK_n(1,2) = a_p^{*n} C_{20}^{*} + b_p^{*n} C_{11}^{*} + \frac{(a_p^{*n})^2}{2} C_{30}^{*} + a_p^{*n} b_p^{*n} C_{21}^{*} + \frac{(b_p^{*n})^2}{2} C_{12}^{*}$$

$$TK_n(1,3) = a_p^{*n} C_{11}^{*} + b_p^{*n} C_{02}^{*} + \frac{(a_p^{*n})^2}{2} C_{21}^{*} + a_p^{*n} b_p^{*n} C_{12}^{*} + \frac{(b_p^{*n})^2}{2} C_{03}^{*}$$

$$TK_n(2,2) = C_{20}^{*} + a_p^{*n} C_{30}^{*} + b_p^{*n} C_{21}^{*}$$

$$TK_n(2,3) = C_{11}^{*} + a_p^{*n} C_{21}^{*} + b_p^{*n} C_{12}^{*}$$

$$TK_n(3,2) = C_{11}^{*} + a_p^{*n} C_{21}^{*} + b_p^{*n} C_{12}^{*}$$

$$TK_n(3,3) = C_{02}^{*} + a_p^{*n} C_{12}^{*} + b_p^{*n} C_{03}^{*} \qquad \text{(G-13a)}$$

However, since δa_p^{*n} and δb_p^{*n} are defined in Eqn. (G-7), this matrix must be broken into two matrices, one a function of the perturbed variables for the center points 2, 3, and 4, the other a function of the perturbed variables for center point 1. Using the matricies defined by Eqn. 48

$$\delta T\underline{C1}^{\ddagger} = \begin{bmatrix} TK_2\ \Phi_2 \\ TK_3\ \Phi_3 \\ TK_4\ \Phi_4 \end{bmatrix} - \begin{bmatrix} TK_2\ \Phi_1 \\ TK_3\ \Phi_1 \\ TK_4\ \Phi_1 \end{bmatrix} \tag{G-14}$$

Therefore, Eqn. (G-14) may be expressed as

$$\delta\underline{C1}^{\ddagger} = Z\left\{\begin{bmatrix} \pi_1\Phi_2 \\ \pi_1\Phi_3 \\ \pi_1\Phi_4 \end{bmatrix} - \begin{bmatrix} \pi_2\Phi_1 \\ \pi_3\Phi_1 \\ \pi_4\Phi_1 \end{bmatrix} - \begin{bmatrix} TK_2\ \Phi_2 \\ TK_3\ \Phi_3 \\ TK_4\ \Phi_4 \end{bmatrix} + \begin{bmatrix} TK_2\ \Phi_1 \\ TK_3\ \Phi_1 \\ TK_4\ \Phi_1 \end{bmatrix}\right\} \tag{G-15}$$

Combining terms yields

$$\delta\underline{C1}^{\ddagger} = Z\left\{\begin{bmatrix} \Omega_{12}\Phi_2 \\ \Omega_{13}\Phi_3 \\ \Omega_{14}\Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22}\ \Phi_1 \\ \Omega_{33}\ \Phi_1 \\ \Omega_{44}\ \Phi_1 \end{bmatrix}\right\} \tag{G-16}$$

where $$\Omega_{ij} = \left(\pi_i - TK_j\right) \tag{G-16a}$$

To find the expression for Block M_cII, Eqn. (G-16) must be multiplied by the transpose of Eqn. (F-3) and the expected value taken. This results in

$$M_c\text{II} = E\left\{\delta\underline{C1}^{\ddagger}\left[\delta C_{00}^{\ddagger}\ \delta C_{10}^{\ddagger}\ \delta C_{01}^{\ddagger}\right]\right\} \tag{G-17}$$

$$M_c II = E\left\{ Z \left\{ \begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix} \right\} \Phi_1^T R^T \right\} \tag{G-17}$$

Since the Φ_n matricies represent perturbed quantities, then the expected value of $\Phi_i \Phi_j^T$ is equal to the covariance matrix M_p^i if $i=j$, or equals 0 if $i \neq j$.

$$E\left\{\Phi_i\,\Phi_j^T\right\} = \begin{cases} 0 & \text{if } i=j \\ M_p^i & \text{if } i \neq j \end{cases} \tag{G-18}$$

Multiplying terms in Eqn. (G-17) yields

$$M_c II = E\left\{ Z \left\{ \begin{bmatrix} \Omega_{12}\,\Phi_2\,\Phi_1^T \\ \Omega_{13}\,\Phi_3\,\Phi_1^T \\ \Omega_{14}\,\Phi_4\,\Phi_1^T \end{bmatrix} - \begin{bmatrix} \Omega_{22}\,\Phi_1\,\Phi_1 \\ \Omega_{33}\,\Phi_1\,\Phi_1 \\ \Omega_{44}\,\Phi_1\,\Phi_1 \end{bmatrix} \right\} R^T \right\} \tag{G-19}$$

Using Eqn. (G-18) and the fact that the expected value of a constant is equal to itself

$$M_c II = -Z \left\{ \begin{bmatrix} \Omega_{22} \\ \Omega_{33} \\ \Omega_{44} \end{bmatrix} \right\} M_p^1 R^T \tag{G-20}$$

APPENDIX H

DERIVATION OF COVARIANCE BLOCK $M_c III$

This block is defined by the quantity

$$M_c III = E\left\{\delta \underline{C1}^{\ddagger} \quad \delta \underline{C1}^{\ddagger T}\right\} \tag{H-1}$$

Since the expression for $\delta\underline{C1}^{\ddagger}$ was derived in Appendix G (Eqn. G-16), Eqn. (H-1) may be evaluated by

$$M_c III = E\left\{ Z \left\{ \begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix} \right\} \cdot \right.$$

$$\left. \left\{ Z \left\{ \begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix} \right\} \right\}^T \right\} \tag{H-2}$$

Multiplying out the terms yields

$$M_c III = E\left\{ Z \left\{ \begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix} \begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix}^T - \right.\right.$$

$$
\begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix}
\begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix}^T
-
\begin{bmatrix} \Omega_{12}\,\Phi_2 \\ \Omega_{13}\,\Phi_3 \\ \Omega_{14}\,\Phi_4 \end{bmatrix}
\begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix}^T
$$

$$
+ \left.\left.
\begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix}
\begin{bmatrix} \Omega_{22}\,\Phi_1 \\ \Omega_{33}\,\Phi_1 \\ \Omega_{44}\,\Phi_1 \end{bmatrix}^T
\right\} Z^T \right\} \tag{H-3}
$$

Using Eqn. (G-18) to combine terms and eliminate non-correlated terms yields

$$
M_c\,\mathrm{III} = Z \left\{ \begin{bmatrix} \Omega_{12} M_p^2 \Omega_{12}^T & 0 & 0 \\ 0 & \Omega_{13} M_p^3 \Omega_{13}^T & 0 \\ 0 & 0 & \Omega_{14} M_p^4 \Omega_{14}^T \end{bmatrix} \right.
$$

$$
\left. + \begin{bmatrix} \Omega_{22} M_p^1 \Omega_{22}^T & \Omega_{22} M_p^1 \Omega_{33}^T & \Omega_{22} M_p^1 \Omega_{44} \\ \Omega_{33} M_p^1 \Omega_{22}^T & \Omega_{33} M_p^1 \Omega_{33}^T & \Omega_{33} M_p^1 \Omega_{44} \\ \Omega_{44} M_p^1 \Omega_{22}^T & \Omega_{44} M_p^1 \Omega_{33}^T & \Omega_{44} M_p^1 \Omega_{44} \end{bmatrix} \right\} Z^T \tag{H-4}
$$

The END