NASA CR-120830

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(NASA-CR-120830) POWER CONVERSION IN ELECTRICAL NETWORKS (Harvard Univ.) 132 p HC $2.25 CSCL 09C

N74-34644

Unclas

G3/09 51040

POWER CONVERSION IN ELECTRICAL NETWORKS

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prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

NASA Lewis Research Center

Contract NGR 22-007-172

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-120830		

4. Title and Subtitle	5. Report Date June 1974 6. Performing Organization Code
POWER CONVERSION IN ELECTRICAL NETWORKS	
7. Author(s) Jonathan Rotherham Wood	8. Performing Organization Report No. 10. Work Unit No.
9. Performing Organization Name and Address Harvard University Division of Engineering and Applied Physics Cambridge, Massachusetts 02130	11. Contract or Grant No. 13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Project Manager, Vincent R. Lalli, Spacecraft Technology Division, NASA Lewis Research Center, Cleveland, Ohio

16. Abstract

This report is concerned with DC-DC conversion. In Chapter 1 we study a class of switching voltage regulators from a stability viewpoint. In Chapter 2 we discuss background concepts of Nonlinear System Theory which are used in Chapter 1, including the problem of obtaining suitable realizations for a class of positive operators. In Chapter 3 we show that the state evolution equations for a power conversion network are in general of bilinear form, and that the theory of Lie Groups and Lie Algebras is useful in analyzing such systems. In Chapter 4 we discuss the feedback stabilization of a class of bilinear systems whose state space is a manifold.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Power Conditioning Electronics Frequency Domain Stability Criteria Lie Algebras	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	viii + 171	

* For sale by the National Technical Information Service, Springfield, Virginia 22151

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TOPICAL REPORT

POWER CONVERSION IN ELECTRICAL NETWORKS

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NOTATION

$x \in A$ means that x is a member of the set A.

$A \Rightarrow B$ means that A implies B.

$\{x | x \text{ has property } A\}$ denotes the set of all x such that x has property A.

$G: A \to B$ means that the operator (or function) G maps the set A into the set B.

$\dot{x}$ denotes $\frac{dx}{dt}$.

$\mathrm{Re}\, z$ denotes the real part of the complex number z.

$\mathrm{Im}\, z$ denotes the imaginary part of z.

$x < \infty$ means that x is finite.

$x(t) \equiv a$ means that $x(t) = a$ for all t.

$\lim_{t\to\infty} x(t) = a$, or $x(t) \to a$ as $t \to \infty$, means that for all $\eta > 0$ there is a T such that $|x(t)-a| < \eta$ for all $t \geqslant T$.

$\sup_{x \in A} x$ denotes the supremum (or least upper bound) of the set of numbers A, i.e. the least number y such that $x \leqslant y$ for all $x \in A$.

$\inf_{x \in A} x$ denotes the infimum (or greatest lower bound) of the set of numbers A, i.e. the greatest number z such that $x \geqslant z$ for all $x \in A$.

$\underset{(-a,b)}{\mathrm{stp}}\ \sigma$ is the function on the real line defined by

$$\underset{(-a,b)}{\mathrm{stp}}\ \sigma = \begin{cases} -a, & \sigma < 0 \\ 0, & \sigma = 0 \\ b, & \sigma > 0 \end{cases}.$$

$\underset{(-a,b)}{\mathrm{Sod}}\ \sigma$ is the function defined by

$$\underset{(-a,b)}{\mathrm{Sod}}\ \sigma = \sigma \underset{(-a,b)}{\mathrm{stp}}\ \sigma .$$

$\mathbb{R}$ denotes the set of all real numbers.

$\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices.

$\mathbb{R}^n$ denotes the set of all n-dimensional real vectors.

A matrix is denoted $\underline{A}$, a column- or row-vector is denoted $\underline{b}$.

A scalar is a one-dimensional vector, i.e. a real number.

$\underline{0}$ denotes the zero matrix.

$\underline{I}$ denotes the unit matrix.

$\underline{A}'$ denotes the transpose of the matrix $\underline{A}$.

tr $\underline{A}$ denotes the trace of $\underline{A}$.

det $\underline{A}$ denotes the determinant of $\underline{A}$.

S^{n-1} denotes the set $\{\underline{x} \in \mathbb{R}^n | \underline{x}'\underline{x} = 1\}$.

$\underline{A} > \underline{B}$ means that the matrix $(\underline{A}-\underline{B})$ is positive definite, i.e. $\underline{x}'(\underline{A}-\underline{B})\underline{x} > 0$ for all $\underline{x} \in \mathbb{R}^n$, $\underline{x} \neq 0$.

QED denotes the end of a proof.

CHAPTER 1

SWITCHING VOLTAGE REGULATORS

§1.1 Introduction: Design Objectives

This chapter addresses itself to the question of stability in DC to DC convertors. In order to better understand the role of the stability question, we shall first review the convertor design task as a whole.

The designer of a DC to DC convertor usually has in mind four primary design objectives, which are (i) Efficiency (ii) Stability (iii) Regulation (iv) Smoothing. There are secondary considerations such as size, weight, and cost, but these can be considered for any one design only when the primary objectives have been met.

In order to meet high efficiency requirements any conversion process considered should be inherently lossless, i.e. given ideal components the process considered would yield a 100% efficient convertor. We shall therefore restrict ourselves at first to the study of lossless circuits, and concentrate our efforts on the theoretical problems involved in meeting simultaneously the requirements of stability, good regulation, and good smoothing. Our components available are inductors, capacitors, transformers, and switches (i.e. ideal transistors, thyristors, and diodes). The only resistors involved are the load resistance and possibly the source resistance. The intentional introduction of resistors in the conversion process is disallowed. In practice, of course, the non-ideal nature of the components used will involve small losses. Our justification for ignoring these in our initial analyses is that such losses tend to aid in stabilization rather than hinder, so that the most difficult design case occurs when these losses vanish. In fact the idea that dissipation aids in stabilization is expressed formally in the Positive Operator Theorem, which we use in Chapter 1 and which we shall discuss further in Chapter 2. We therefore phrase our theoretical problem as "Given an input DC voltage source, produce a stable power convertor to deliver power at some other DC voltage to a possibly time-varying load, using only inductors, capacitors, transformers, and switches, and meeting some prespecified requirements on regulation and smoothing."

There are two types of regulation to consider:

(i) "Line regulation": to make the output voltage insensitive to input voltage variation,

(ii) "Load regulation": to make the output voltage insensitive to load resistance variation.

As well as being concerned with the steady-state regulating capabilities of the convertor, the designer may be interested in ensuring that the convertor also has a good transient response, e.g. a quick recovery from a temporary short circuit. For a periodically varying load, the output impedance of the convertor as a function of frequency may be important.

Since all DC to DC convertors necessarily involve some AC process, ([23], [38], [39]), it follows from the smoothing requirement that in general a low pass filter must be utilized on the output side of the convertor. However a low pass filter has an inherent lag associated with it, so that by lowering the cut-off frequency of the output filter the designed may very well degrade the transient response performance. For a given low pass filter the transient response can be improved by increasing the closed loop feedback gain, but stability considerations impose limits on this gain.

If the smoothing requirement is dropped completely, then the problem may be easily solvable in the ideal case. For instance, one may have a pulse-width-modulated system with the design requirement that the integral of the output voltage over each modulation cycle be constant. Designing a modulator to meet this requirement is then easy, in principle.

To summarize, we see that the basic theoretical problem confronting the designer of a DC to DC convertor is that of meeting the three simultaneous requirements of stability, regulation, and smoothing, while using only lossless components. Further, these three requirements cannot be considered independently, and in order to obtain an optimum compromise between the demands of good regulation and good smoothing, a designer must thoroughly understand the stability problem.

§1.2 A Classification of Convertor Types

The stability question can be tackled only after an extensive classification of the different types of convertor, since different mathematical methods will be applicable in each case. The first characteristic of a convertor is whether its output voltage is to be larger or smaller than the input voltage, and the second characteristic is whether or not transformers are involved. In this chapter we consider transformerless downconvertors, or switching voltage regulators. Even in this class there are many different analysis situations, depending for example on whether or not a load is present, whether or not the load is purely resistive, whether or not the smoothing filter is lossless, whether or not the voltage source has an internal impedance, whether or not the source impedance is purely resistive, and whether or not the source voltage is time-varying.

§1.3 Switching Voltage Regulators without Source Impedance

Given a fixed DC source voltage E, we wish to obtain a near-constant voltage of αE with $0 < \alpha < 1$. The scheme we consider is that shown in Fig. 1.1. The low-pass filter is of the form shown in Fig. 1.2. The switch represents a suitable interconnection of diodes and transistors or thyristors. The scalar u is the control variable, which takes on the values 0 or 1 depending on the position of the control switch. A load resistance may or may not be present at the output of the low pass filter.

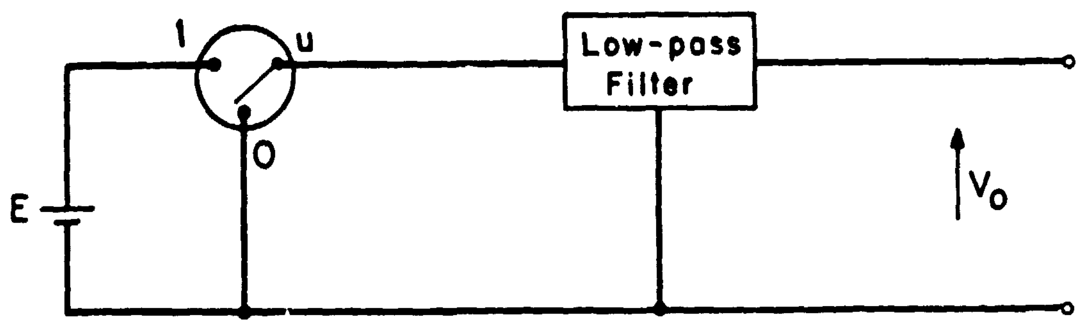


Fig. 1.1

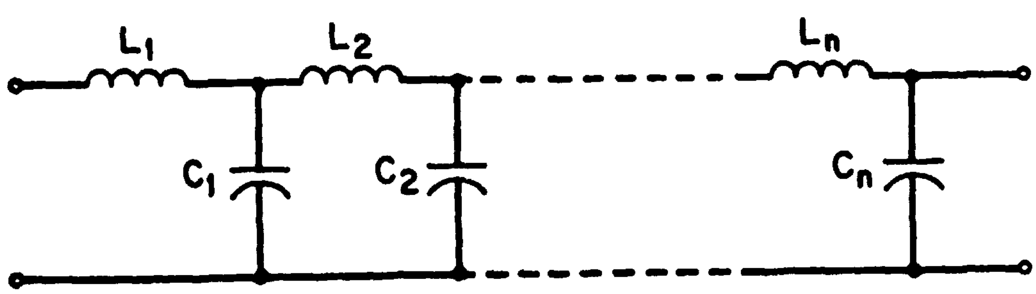


Fig. 1.2

The approach we follow here is a state-feedback approach, that is, we assume knowledge of the capacitor voltages and inductor currents, and use this to determine the desired position for the switch at any one time. We must choose a feedback control law which brings the output voltage V_0 as close as possible to the desired value αE, and gives the overall system good regulatory and smoothing characteristics. The assumption that we have full knowledge of the state is not unreasonable, since a fully satisfactory performance will not be obtained without it, and in practice it should not be too difficult to obtain a good estimate of the state from measurements (of the capacitor voltages, for instance). The state-feedback approach is the most reasonable in a situation such as this. To adopt some other scheme such as pulse-width-modulation is to arbitrarily impose constraints which can only hinder the attempt to obtain an op[illegible]m overall performance.

In steady-state operation it is fairly clear that [illegible]e switch will be working in a periodic way, such that the average value of voltage on the output side of the switch is αE. The amount of ripple present at the filter output will be determined by the frequency of the switch operation, and by the size and number of filter components. Since the filter is a low-pass one, a higher switching frequency will yield a smaller output ripple. However because the particular thyristor or transistor switch used will have a finite switching time during which it dissipates some power, the operating efficiency decreases as the switching frequency increases. We assume that a lower limit on the allowable efficiency is prescribed, and thus that an upper limit on the switching frequency is obtained. The feedback controller should insure that the switching frequency does not exceed this allowable limit. Once this limit is prescribed, the ripple will be determined by the number and size of the filter inductors and capacitors. Because of physical bulk it is important to reduce the total inductance and capacitance involved. For a given total inductance and total capacitance, better high-frequency rejection for a filter of the type shown in Fig. 1.2 is obtained as the number of filter components n increases. Consider for example the filters shown in Figs. 1.3 and 1.4, which have the same total inductance and total capacitance.

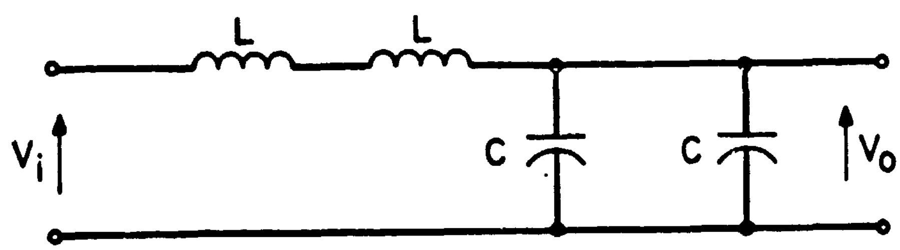


Fig. 1.3

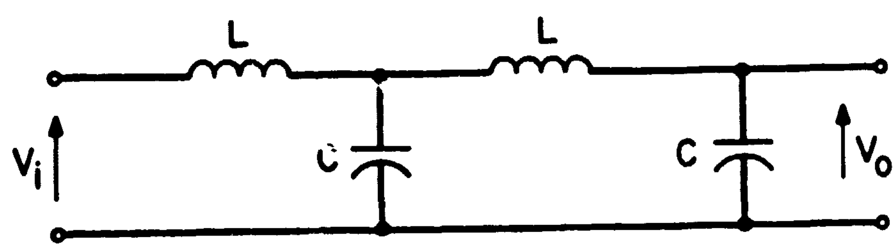


Fig. 1.4

For Fig. 1.3

$$H_2(s) = \frac{V_i(s)}{V_0(s)} = 1 + 4LCs^2$$

and for Fig. 1.4

$$H_4(s) = \frac{V_i(s)}{V_o(s)} = 1 + 3LCs^2 + L^2C^2s^4$$

So for a sinusoidal input signal of angular frequency ω

$$\frac{H_4(j\omega)}{H_2(j\omega)} = \frac{1 - 3LC\omega^2 + L^2C^2\omega^4}{1 - 4LC\omega^2}$$

$$= \frac{1 - 3\left(\frac{\omega}{\omega_o}\right)^2 + \left(\frac{\omega}{\omega_o}\right)^4}{1 - 4\left(\frac{\omega}{\omega_o}\right)^2} \qquad \text{where } \omega_o^{\ 2} = \frac{1}{LC} \quad .$$

If $\frac{\omega}{\omega_o} >> 1$,

$$\left|\frac{H_4(j\omega)}{H_2(j\omega)}\right| \doteq \frac{1}{4}\left(\frac{\omega}{\omega_o}\right)^2 \quad ,$$

i.e. the higher-order filter is much more efficient. Thus we are particularly interested in high-order filters. Unfortunately, the stability question becomes increasingly difficult to analyze as n increases. In the limit, one might conclude that a transmission line would be the best type of filter to use, but difficulties with the stability analysis and with the state estimation probably rule out this possibility.

§1.4 Second-Order Lossless Regulator

(a) Choice of a Feedback Law

We now consider the first member of the series of regulators of Figs. 1.1 and 1.2. For the particular control law which we choose the stability analysis is fairly simple. We devote some time to it, however, since it turns out that this is the only member of the series for which the analysis is simple, yet it illustrates most of the features of the higher-order regulators. Fig. 1.5 shows the system under consideration. The load resistance R is assumed constant; $R = \infty$ denotes the no-load condition.

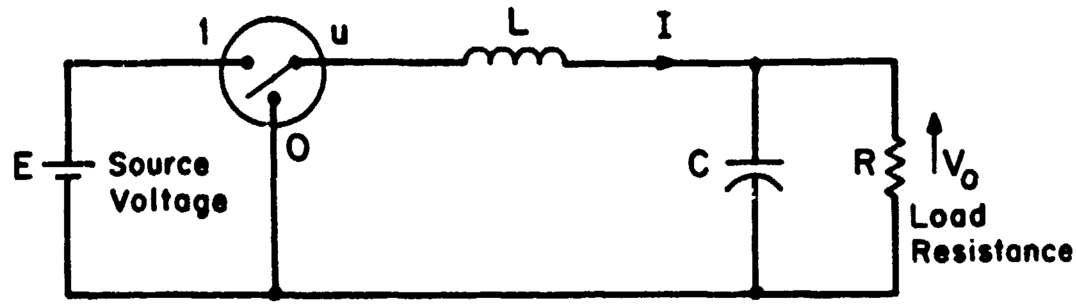


Fig. 1.5

The governing equations are

$$\begin{cases} L\dot{I} = uE - V_o \\ C\dot{V}_o = I - \left(\frac{1}{R}\right)V_o \end{cases} \qquad (\dot{I} \text{ denotes } \frac{dI}{dt}) \ .$$

Letting $y_1 = I\sqrt{L}$ and $y_2 = V_o\sqrt{C}$ we obtain

$$\begin{cases} \dot{y}_1 = u\left(\frac{E}{\sqrt{L}}\right) - \left(\frac{1}{\sqrt{LC}}\right) y_2 \\ \dot{y}_2 = \left(\frac{1}{\sqrt{LC}}\right) y_1 - \left(\frac{1}{RC}\right) y_2 \end{cases}$$

i.e.

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 \\ \omega_o & -\omega_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

where $\omega_0 = \left(\frac{1}{\sqrt{LC}}\right)$, $\omega_1 = \left(\frac{1}{RC}\right)$, $b = \left(\frac{E}{\sqrt{L}}\right)$.

We now introduce time- and amplitude-scaling which allows us to assume without loss of generality that $E = L = C = 1$. For, letting $\tau = \omega_0 t$ and $z_i = \frac{\omega_0 y_i}{b}$ we obtain

$$\begin{bmatrix} \frac{dz_1}{d\tau} \\ \\ \frac{dz_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \\ 1 & -\left(\frac{\omega_1}{\omega_0}\right) \end{bmatrix} \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} u$$

i.e.

$$\frac{d}{d\tau} \underline{z} = \underline{A}\,\underline{z} + \underline{b}\,u \quad .$$

In what follows we assume that $\omega_0 = 1$, i.e., $t = \tau$. z_1 and z_2 are called the <u>state variables</u> and the vector $\underline{z}$ is called the <u>state vector</u>.

We want to make u a function of $\underline{z}$, (i.e. the control u a function of the measured variables z_1 and z_2) so that z_2 comes as close as possible to its desired value α (where $0 < \alpha < 1$). That is, we want

$$\lim_{t \to \infty} z_2(t) = \alpha \ ,$$

for any initial condition $(z_1(0), z_2(0))$. We call this <u>global asymptotic stability about $z_2 = \alpha$</u>. Now if $z_2 < \alpha$ one would expect that the switch should be in the $u = 1$ position, and if $z_2 > \alpha$ that the switch should be in the $u = 0$ position. This leads us to try the feedback control

$$u = \begin{cases} 1 \ , & z_2 < \alpha \\ 0 \ , & z_2 > \alpha \end{cases} \quad .$$

(We postpone till part (b) a discussion of what happens at $z_2 = 0$.) Using the methods of §1.4 (b), (c), (d), or (e) we can show that this feedback control does give the desired asymptotic stability about $z_2 = \alpha$ provided that R has a finite value. When the load is removed ($R = \infty$) asymptotic stability is not obtained. Indeed in §1.9 we give consideration to the effect of an inevitable small lag in the feedback controller, and find that in this situation a small lag yields instability about $z_2 = \alpha$; in fact an oscillation whose frequency is of the order of $\frac{1}{2\pi\sqrt{LC}}$ is obtained. Note that the no-load case is the most difficult to stabilize, and for this reason we shall henceforth analyze only the no-load case.

The reason that this initial choice of control law is unsatisfactory is that it takes no account of the inductor current z_1. In conventional servomechanism theory terms, we need to provide some "rate feedback" as well as "output feedback". If z_2 is slightly less than α while z_1 is large and positive then it should be fairly clear that we want $u = 0$ rather than $u = 1$. Since $z_1 = \dot{z}_2$ when $R = \infty$ let us consider the control law

$$u = \begin{cases} 1 \ , & z_2 + \beta\dot{z}_2 < \alpha \\ 0 \ , & z_2 + \beta\dot{z}_2 > \alpha \end{cases} \qquad \text{for some } \beta > 0 \ .$$

In §1.4 (b), (d) and (e) we shall show that this control law gives a regulator with good stability about $z_2 = \alpha$ for all values of R. We discuss implementation of this control in §1.9. The parameter β can be chosen by the designer to give a good transient response to changes in operating conditions, such as changes in α, in E, or in R. One value of β cannot give an optimum response for every condition, so a compromise value will have to be chosen. An analog computer simulation shows that setting β between 0.8 and 1.0 gives a reasonable overall performance.

(b) Stability by Phase Plane Analysis

First we make a change of variable so that the desired stability is about 0. Let

$$\begin{cases} x_1 = 2z_1 \\ x_2 = 2z_2 - 2\alpha \end{cases}$$

so that our system is

$$\begin{cases} \dot{x}_1 = -x_2 + 2(u - \alpha) \\ \dot{x}_2 = x_1 \end{cases} ,$$

the desired equilibrium point being the origin in the (x_1, x_2) plane.

Next we introduce the "step" function

$$\underset{(-a,b)}{\text{stp}}\ x = \begin{cases} -a , & x < 0 \\ 0 , & x = 0 \\ b , & x > 0 \end{cases}$$

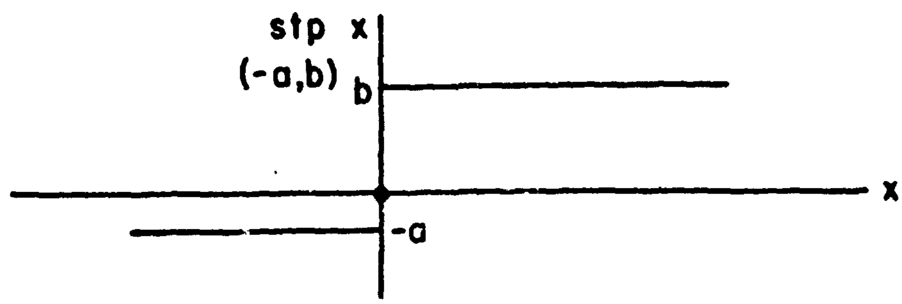


Fig. 1.6

We now let

$$u = \begin{cases} 1 \quad , & \beta\dot{x}_2 + x_2 < 0 \\ \alpha \quad , & \beta\dot{x}_2 + x_2 = 0 \\ 0 \quad , & \beta\dot{x}_2 + x_2 > 0 \end{cases}$$

i.e.

$$u = \frac{1}{2}\left[2\alpha - \underset{(-2+2\alpha,2\alpha)}{\text{stp}} (\beta\dot{x}_2+x_2)\right]$$

so that our system is

$$\begin{cases} \dot{x}_1 = -x_2 - \underset{(-2+2\alpha,2\alpha)}{\text{stp}} (\beta\dot{x}_2+x_2) \\ \dot{x}_2 = x_1 \end{cases} .$$

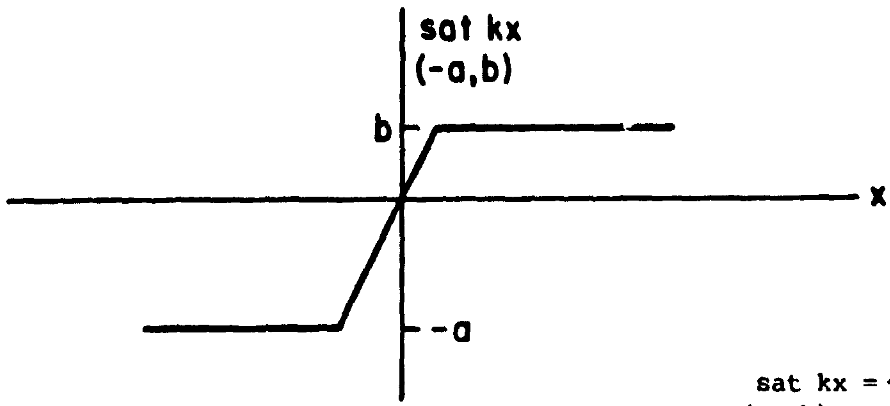
We assume henceforth that the value of α is known, and abbreviate

$\underset{(-2+2\alpha,2\alpha)}{\text{stp}}$ to stp.

Some remarks are necessary concerning the use which we shall be making of the stp function. This is a discontinuous function, and a statement such as

$$\frac{d}{dx}|x| = \underset{(-1,1)}{\text{stp}} \; x = \text{sgn } x$$

requires rigorous mathematical treatment in order to make strict sense for all values of x. However, we shall use the stp function merely for its notational and conceptual convenience, since it is the limit of various classes of continuous functions such as sat kx (Fig. 1.7) or tanh kx (Fig. 1.8) where k is an arbitrarily large positive real number; such functions are more accurate descriptions of a "real-world" switching function than stp x is. In order to put our treatise on a firm mathematical footing we need only replace stp x by sat kx (or some other suitably smooth function) each time it is used, and show that the relevant



$$\underset{(-a,b)}{\text{sat } kx} = \begin{cases} -a, & x < \frac{-a}{k} \\ kx, & -\frac{a}{k} \leq x \leq \frac{b}{k} \\ b, & x > \frac{b}{k} \end{cases}$$

Fig. 1.7

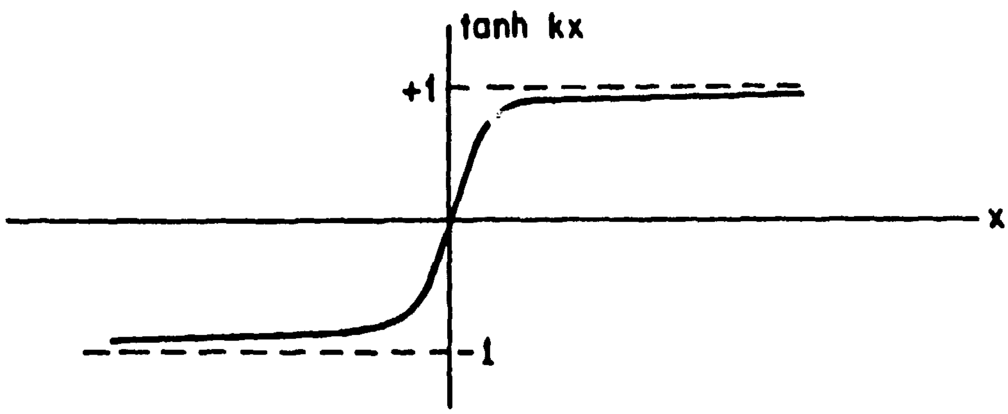


Fig. 1.8

conclusions still hold true. In fact, many of our results hold true if stp x is replaced by f(x) where f is any monotone nonlinearity with f(0) = 0. (A monotone function f(0) is one for which $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$.)

Now we continue with the phase-plane analysis of

$$\ddot{x} + x = -\text{stp}(\beta\dot{x} + x) \quad .$$

The trajectories in the $(x,\dot{x})$ plane are arcs of circles, representing piecewise-simple-harmonic motion. If $x + \beta\dot{x} > 0$ these arcs are centered on $(0,-2\alpha)$, and if $x + \beta\dot{x} < 0$ the arcs are centered on $(0,2-2\alpha)$. Fig. 1.9 shows the trajectories obtained when $\alpha = \frac{1}{2}$ and $\beta = 1$. The line $x + \beta\dot{x} = 0$ is called the switching line, and we need to consider carefully what happens at this line. We have said that the feedback nonlinearity is in practice a continuous function, and one way to determine what happens along the switching line would be to replace stp σ by a function like sat $k\sigma$, with k very large. However because of the chattering behavior along switching lines which is observed in practice, it is better to consider the feedback nonlinearity either as a stp function preceded by a small time delay, or as a stp function modified to include a small amount of hysteresis. A small delay in the feedback path will always be unavoidable in practice, representing for instance the switching-time of the power transistor used as the control switch. In addition, the designer will want to include either a small fixed delay or a small amount of hysteresis, in order to limit the switching frequency of the power transistor because of efficiency considerations, as discussed in §1.1 and §1.3. In this thesis we adopt the policy that hysteresis is undesirable, and that a predetermined small fixed delay is used. The disadvantage with the hysteresis method of frequency limiting is that the switching frequency depends on the load resistance, and furthermore this dependence is difficult to analyze. By using the fixed delay method the designer can easily set the switching frequency to some desired value which will be independent of the load resistance. However, whether a fixed delay or hysteresis or a combination of both are employed, the conclusions which we shall reach concerning behavior along the switching line are essentially the same.

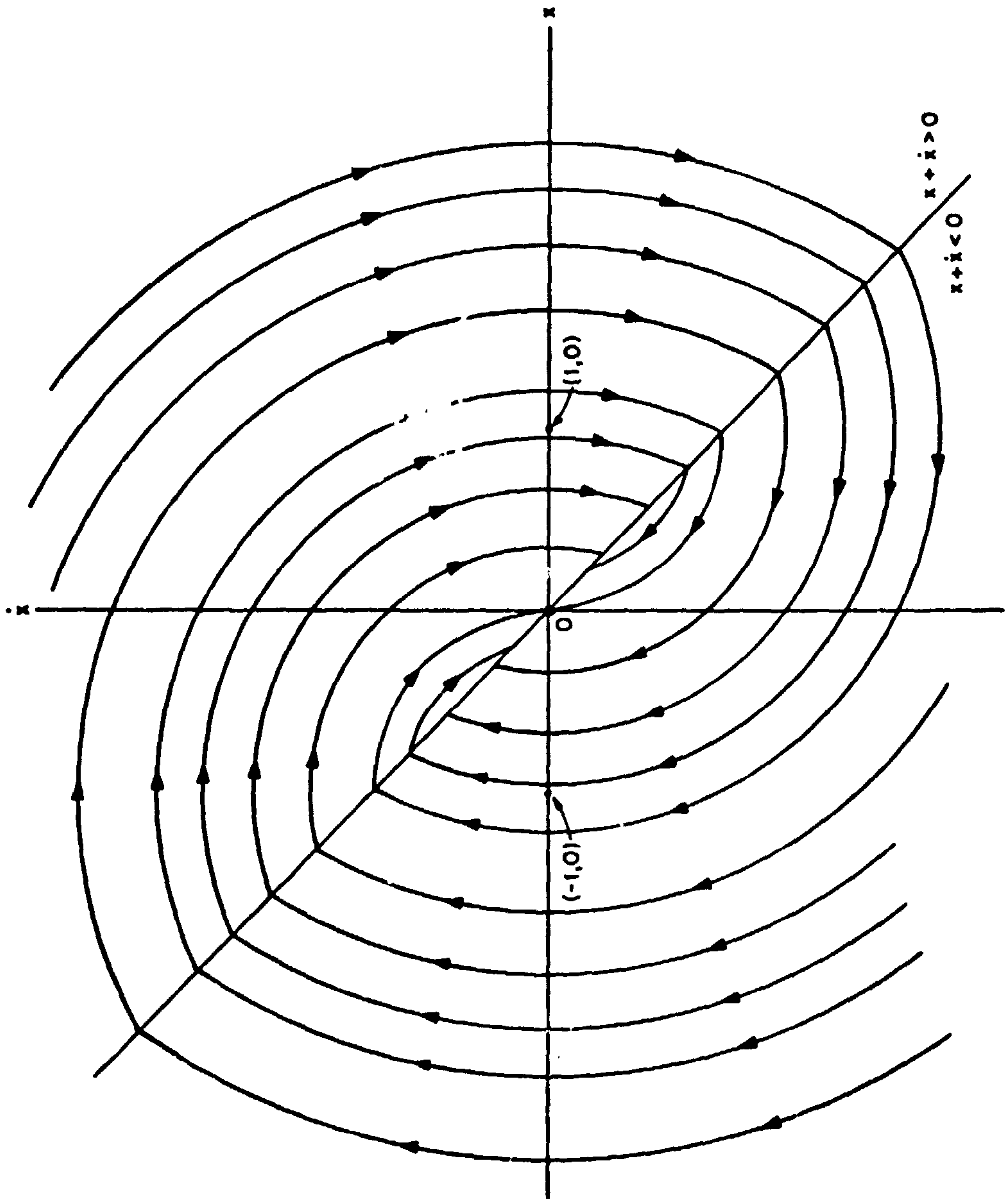


Fig. 1.9

Trajectories of $\ddot{x}+x = -\mathrm{sgn}(\dot{x}+x)$

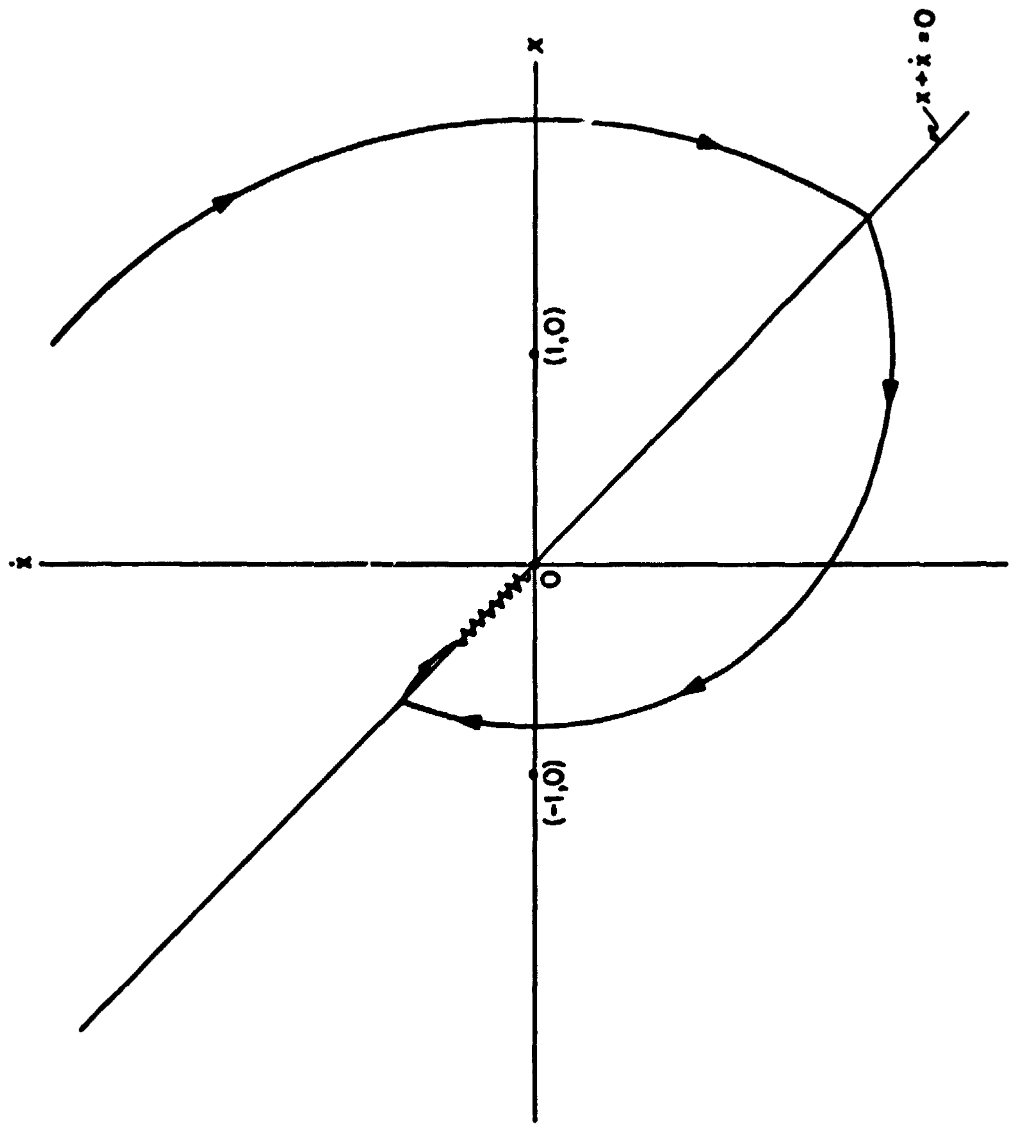


Fig. 1.10

Typical path for $\ddot{x}+x = -\mathrm{sgn}(\dot{x}+x)$

We now consider in some detail the behavior of feedback systems which have a stp function in the feedback path. For a rigorous treatment the reader is referred to the work of Filippov [13]. Other useful references are [1] (Chapters 6 and 12), [18] (Chapter 6), and [22].

Consider the feedback system shown in Fig. 1.11, described by the equations

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x}(t),\ u(t),\ t) \\ u(t) = -\underset{(-a,b)}{\mathrm{stp}}\ \underline{c}\ \underline{x}(t) \end{cases}$$

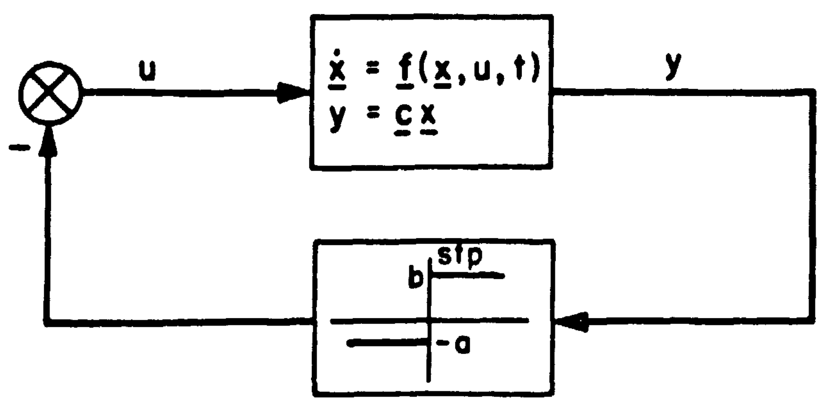


Fig. 1.11

where $\underline{x}(t) \in \mathbb{R}^n$ and $u(t)$, $y(t)$ are scalars. The switching surface $\underline{c}\ \underline{x} = 0$ (which is a line when $n = 2$, a plane when $n = 3$, etc.) divides the state space into two regions, in one of which $y > 0$, $u = -b$, and in the other $y < 0$, $u = a$. We shall use the symbol S to denote the switching surface. The scalar quantity $\rho = \underline{c}\ \underline{x}$ is the distance of the point $\underline{x}$ from S, with due attention to algebraic sign. The scalar $\dot{\rho} = \underline{c}\ \dot{\underline{x}}$ gives the rate at which the point $\underline{x}$ on a trajectory is approaching S. If we consider an arbitrary point $\underline{x}_0$ on S together with values of ρ and $\dot{\rho}$ nearby we can identify several situations with respect to the possible signs of ρ and $\dot{\rho}$. Fig. 1.12 shows the four main cases of interest (we are not considering cases where $\dot{\rho} = 0$).

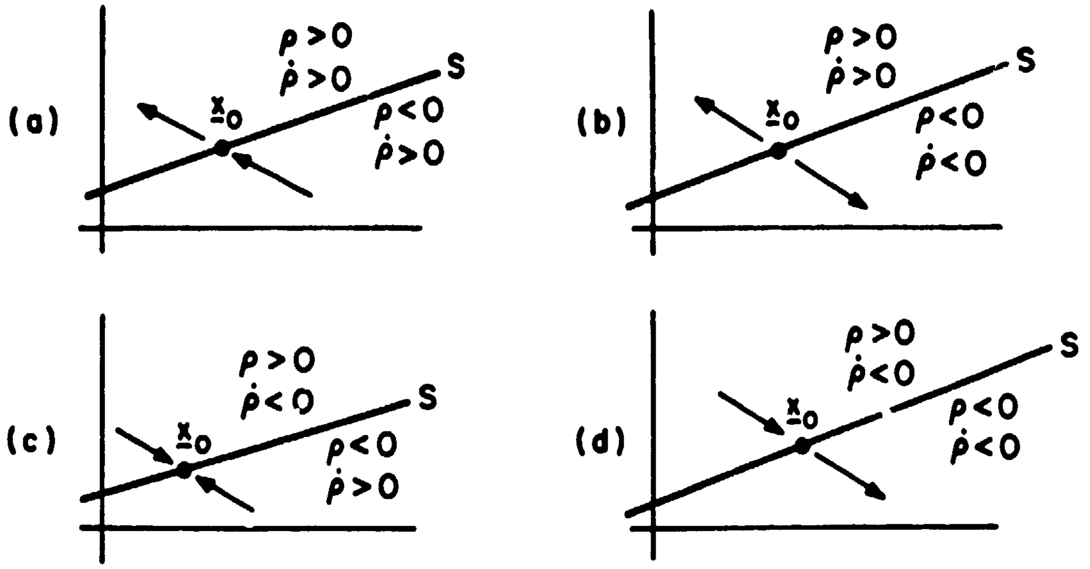


Fig. 1.12

The figures in Fig. 1.12 show typical trajectories on each side of S. In cases (a) and (d) switching is instantaneous, since trajectories approach S, cross it, and move away on the other side. In case (b) no trajectories approach S, thus no switching occurs; if a system is started in a state $\underline{x}_0$ on S, the subsequent trajectory could leave S on either side. In case (c) trajectories on both sides head toward S: no trajectories leave $\underline{x}_0$, and the differential equation $\dot{\underline{x}} = \underline{f}(\underline{x}, u, t)$ apparently cannot be solved beyond this point. Such a point is called an "endpoint". To obtain a solution we can redefine the stp function in one of two ways: either we can precede it by a short time delay τ, or we can include a small amount of hysteresis. We follow the former approach here, for the reasons given earlier. Thus, if ρ changes sign instantaneously at time t_1 from a negative to a positive value, u changes from a to -b at time $t_1 + \tau$. This will give rise to a zigzag type of trajectory along S, the frequency of the crossings increasing as τ decreases. This chattering behavior is observed in practice, even though the function labelled stp is actually a continuous function with bounded slope at the origin: if this slope at the origin were small enough the chattering would cease. The system is now described by

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), u(t - \tau), t) \\ u(t) = - \operatorname{stp} \underline{c}\, \underline{x}(t) \end{cases}$$

We adopt the notation $\underline{f}^+$ to denote $\underline{f}(\underline{x}, -b, t)$, i.e. $\underline{f}^+$ is $\underline{f}$ when $\underline{c}\,\underline{x} > 0$. Similarly $\underline{f}^-$ denotes $\underline{f}(\underline{x}, a, t)$, i.e. $\underline{f}^-$ is $\underline{f}$ when $\underline{c}\,\underline{x} < 0$. We obtain first a condition for a state $\underline{x}_0$ on the switching surface $\underline{c}\,\underline{x} = 0$ to be an end-point. From Fig. 1.12(c) we see that $\rho > 0$ implies $\dot{\rho} < 0$ in the vicinity of $\underline{x}_0$, i.e. $\underline{c}\,\underline{f}^+ < 0$. Similarly $\rho < 0$ implies $\dot{\rho} > 0$, i.e. $\underline{c}\,\underline{f}^- > 0$. Taken together these two conditions are sufficient for $\underline{x}_0$ to be an endpoint, i.e. $\underline{c}\,\underline{f}^+ < 0$ and $\underline{c}\,\underline{f}^- > 0$. Now we determine the rate at which the state chatters along S. Suppose that for $t < t_1$ the state lies in the region $\underline{c}\,\underline{x} < 0$, and that $\underline{x}(t_1) = \underline{x}_0$ (where $\underline{c}\,\underline{x}_0 = 0$), i.e. the trajectory meets S at time t_1. Let $\underline{x}_2 = \underline{x}(t_1 + \tau)$ which is the state when u switches from a to -b, and let $\underline{x}_3 = \underline{x}(t_1 + \Delta t)$ be the state when the trajectory next intersects S. In the interval $[t_1, t_1 + \tau]$ we have $u = a$, so that

$$\underline{x}_2 = \underline{x}_0 + \tau\underline{f}^- + O[\tau^2] \quad .$$

For $t > t_1 + \tau$ we have $u = -b$, so

$$\begin{aligned} \underline{x}_3 &= \underline{x}_2 + (\Delta t - \tau)\ \underline{f}^+ + O[(\Delta t - \tau)^2] \\ &= \underline{x}_0 + \Delta t\ \underline{f}^+ + \tau(\underline{f}^- - \underline{f}^+) + O[\tau^2] + O[(\Delta t - \tau)^2]. \end{aligned}$$

Multiplying this from the left by $\underline{c}$ and remembering that $\underline{c}\,\underline{x}_3 = \underline{c}\,\underline{x}_0 = 0$ we obtain

$$\Delta t = -\tau \frac{\underline{c}(\underline{f}^- - \underline{f}^+)}{\underline{c}\ \underline{f}^+}$$

ignoring the higher order terms in τ and Δt. Thus

$$\frac{\underline{x}_3 - \underline{x}_0}{\Delta t} = \underline{f}^+ - (\underline{f}^- - \underline{f}^+) \frac{\underline{c}\ \underline{f}^+}{\underline{c}(\underline{f}^- - \underline{f}^+)}$$

$$= \frac{\underline{c}\ \underline{f}^- \underline{f}^+ - \underline{c}\ \underline{f}^+ \underline{f}^-}{\underline{c}(\underline{f}^- - \underline{f}^+)} \quad .$$

Now if we were to consider a switching transition going in the other direction we would obtain this same result, even if the switching delay was different from τ: this can be seen easily by observing that the result is unchanged when $\underline{f}^+$ is exchanged with $\underline{f}^-$ and that the result is independent of τ. Thus, letting $\tau \to 0$ we see that the system trajectory approaches arbitrarily close to the trajectory defined by

$$\dot{\underline{x}} = \underline{f}^+ - (\underline{f}^- - \underline{f}^+) \frac{\underline{c}\ \underline{f}^+}{\underline{c}(\underline{f}^- - \underline{f}^+)}$$

$$= \frac{\underline{c}\ \underline{f}^- \underline{f}^+ - \underline{c}\ \underline{f}^+ \underline{f}^-}{c(\underline{f}^- - \underline{f}^+)} \quad .$$

That this trajectory remains on $\underline{c}\ \underline{x} = 0$ can easily be checked by evaluating $\underline{c}\ \dot{\underline{x}}$, which is 0. This formula has a simple graphical interpretation. In Fig. 1.13 the vectors $\overrightarrow{AB} = \underline{f}^+$ and $\overrightarrow{AD} = \underline{f}^-$ are the velocities on each side of S, at the point A on S. Note that the arrows denoting f^+ and f^- in Fig. 1.13 are drawn on the opposite side of S to those of Fig. 1.12. The resulting velocity vector is $\dot{\underline{x}} = \overrightarrow{AC}$ where C is the intersection of BD with S. When the situation of Fig. 1.13 is as in Fig. 1.14 the resulting velocity along S will be to the left or to the right according to whether AG is larger or smaller than AH.

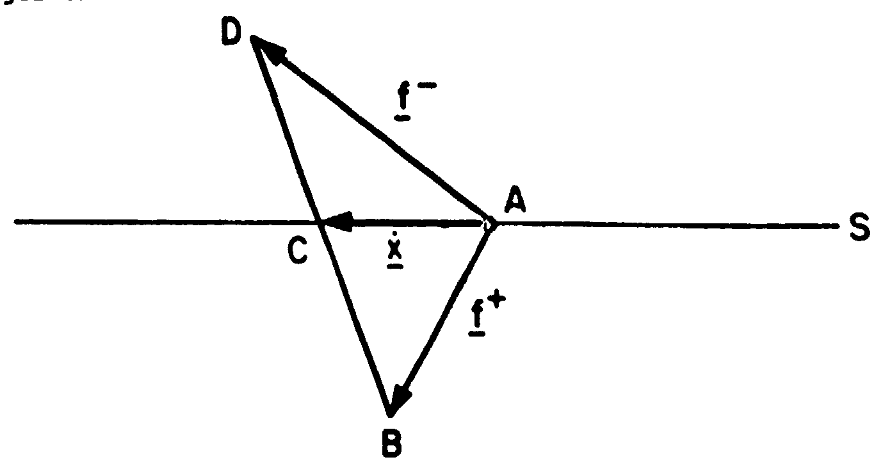


Fig. 1.13

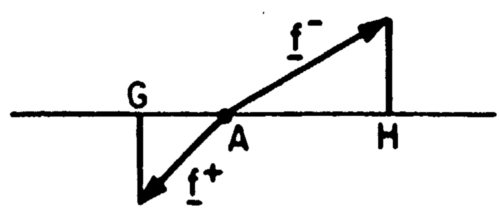


Fig. 1.14

If the operator in the forward path of Fig. 1.11 is linear and time-invariant we have the situation depicted in Fig. 1.15.

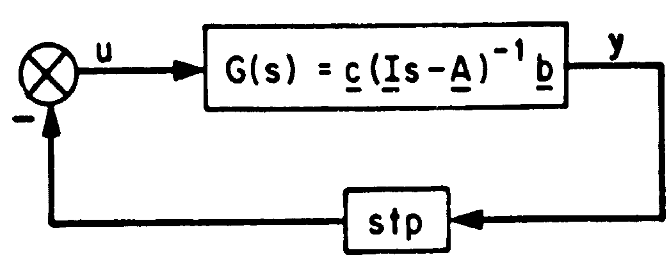


Fig. 1.15

Here

$$\begin{cases} \dot{\underline{x}} = f(\underline{x},u,t) = \underline{A}\,\underline{x} + \underline{b}\,u \\ y = \underline{c}\,\underline{x} \\ u = -\underset{(-a,b)}{\mathrm{stp}}\, y \end{cases}$$

$$\underline{f}^{+} = \underline{A}\,\underline{x} - b\,\underline{b}$$

$$\underline{f}^{-} = \underline{A}\,\underline{x} + a\,\underline{b} \quad .$$

The condition for a point $\underline{x}$ on $\underline{c}\,\underline{x} = 0$ to be an endpoint is

$$\underline{c}\,\underline{A}\,\underline{x} - b\,\underline{c}\,\underline{b} < 0 \quad \text{and} \quad \underline{c}\,\underline{A}\,\underline{x} + a\,\underline{c}\,\underline{b} > 0$$

i.e.,

$$-a\,\underline{c}\,\underline{b} < \underline{c}\,\underline{A}\,\underline{x} < b\,\underline{c}\,\underline{b}\,,$$

and the chattering motion is described by

$$\dot{\underline{x}} = \frac{(\underline{c}\,\underline{A}\,\underline{x} + a\,\underline{c}\,\underline{b})(\underline{A}\,\underline{x} - b\,\underline{b}) - (\underline{c}\,\underline{A}\,\underline{x} - b\,\underline{c}\,\underline{b})(\underline{A}\,\underline{x} + a\,\underline{b})}{(a+b)\underline{c}\,\underline{b}}$$

$$= \underline{A}\,\underline{x} - \frac{(\underline{c}\,\underline{A}\,\underline{x})\underline{b}}{\underline{c}\,\underline{b}}$$

i.e.

$$\dot{\underline{x}} = \left(\underline{I} - \frac{\underline{b}\,\underline{c}}{\underline{c}\,\underline{b}}\right)\underline{A}\,\underline{x} \quad .$$

A plausible but nonrigorous argument suggesting this result is as follows. A trajectory on $\underline{c}\,\underline{x} = 0$ satisfies $\underline{c}\,\dot{\underline{x}} = 0$, i.e. $\underline{c}\,\underline{A}\,\underline{x} - \underline{c}\,\underline{b}$ stp $\underline{c}\,\underline{x} = 0$, so stp $\underline{c}\,\underline{x}$ can be replaced by $\dfrac{\underline{c}\,\underline{A}\,\underline{x}}{\underline{c}\,\underline{b}}$. Then the equation $\dot{\underline{x}} = \underline{A}\,\underline{x} - \underline{b}$ stp $\underline{c}\,\underline{x}$ becomes

$$\dot{\underline{x}} = \underline{A}\,\underline{x} - \left(\frac{\underline{c}\,\underline{A}\,\underline{x}}{\underline{c}\,\underline{b}}\right)\underline{b}$$

i.e.

$$\dot{\underline{x}} = \left(\underline{I} - \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\right)\underline{A}\ \underline{x} \quad .$$

This will define an asymptotically stable motion along S if and only if the matrix $\underline{F} = \left(\underline{I} - \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\right)\underline{A}$ has n-1 eigenvalues with negative real parts; the remaining eigenvalue must be zero since $\underline{c}\ \underline{F} = \underline{c}\left(\underline{I} - \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\right)\underline{A} = 0$. It is an interesting and useful fact that the nonzero eigenvalues of $\underline{F}$ are the zeros of the numerator polynomial of $G(s) = \underline{c}(\underline{I}\ s - \underline{A})^{-1}\underline{b}$, as we now prove:

Theorem 1.1

Let

$$\begin{aligned} G(s) &= \underline{c}(\underline{I}\ s - \underline{A})^{-1}\underline{b} \\ &= \frac{q(s)}{p(s)} \\ &= \frac{q_{n-1}\ s^{n-1} + q_{n-2}\ s^{n-2} + \ldots + q_0}{s^n + p_{n-1}\ s^{n-1} + p_{n-2}\ s^{n-2} + \ldots + p_0} \end{aligned}$$

where $(\underline{A}, \underline{b}, \underline{c})$ is a minimal realization of G(s) and assume that $q_{n-1} \neq 0$. Let $\underline{F} = \left(\underline{I} - \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\right)\underline{A}$. Then $\det(\underline{I}\ s - \underline{F}) = \frac{s q(s)}{q_{n-1}}$.

Proof

Let

$$\underline{A}_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 1 \\ -p_0 & -p_1 & \cdot & \ldots & -p_{n-1} \end{bmatrix}, \quad \underline{b}_1 = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad \underline{c}_1 = [q_0\ q_1 \ldots q_{n-1}]$$

and let

$$\underline{H} = \left(\underline{I} - \frac{\underline{b}_1\ \underline{c}_1}{\underline{c}_1\ \underline{b}_1}\right) \underline{A}_1 \ .$$

Then

$$\underline{H} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & & & & 0 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & 1 \\ 0 & \frac{-q_0}{q_{n-1}} & \frac{-q_1}{q_{n-1}} & & & & \frac{-q_{n-2}}{q_{n-1}} \end{bmatrix}$$

so that

$$\det(\underline{I}\ s - \underline{H}) = \frac{s}{q_{n-1}}\left(q_{n-1}\ s^{n-1} + q_{n-2}\ s^{n-2} + \ldots + q_0\right)$$

$$= \frac{s\ q(s)}{q_{n-1}} \ .$$

Now $(\underline{A},\underline{b},\underline{c})$ is minimal, and $(\underline{A}_1,\underline{b}_1,\underline{c}_1)$ is also minimal, being the standard controllable realization of G(s), ([7], section 17), so there exists a a matrix $\underline{P}$ such that

$$\underline{A} = \underline{P}\ \underline{A}_1\underline{P}^{-1}\ ,\ \underline{b} = \underline{P}\ \underline{b}_1\ ,\ \underline{c} = \underline{c}_1\underline{P}^{-1}\ .$$

Therefore

$$\det(\underline{I}\ s - \underline{F}) = \det\left(\underline{I}\ s - \underline{A} + \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\ \underline{A}\right)$$

$$= \det\left(\underline{I}\ s - \underline{P}\ \underline{A}_1\underline{P}^{-1} + \frac{\underline{P}\ \underline{b}_1\underline{c}_1\underline{P}^{-1}}{\underline{c}_1\underline{b}_1}\ \underline{P}\ \underline{A}_1\underline{P}^{-1}\right)$$

$$= \det\ \underline{P}\left(\underline{I}\ s - \underline{A}_1 + \frac{\underline{b}_1\underline{c}_1}{\underline{c}_1\underline{b}_1}\ \underline{A}_1\right)\ \underline{P}^{-1}$$

$$= \det\ \underline{P}(\underline{I}\ s - \underline{H})\ \underline{P}^{-1}$$

$$= \det \ (\underline{I}\, s - \underline{H})$$

$$= \frac{s\, q(s)}{q_{n-1}} \qquad \text{QED.}$$

Theorem 1.1 also follows from equation (13) of reference [5].

<u>Corollary</u>: The feedback system of Fig. 1.16 is asymptotically stable in the chattering mode if and only if the numerator polynomial of $G(s) = \underline{c}(Is-\underline{A})^{-1}\,\underline{b}$ is strictly Hurwitz, (i.e. has all its zeros in Re $s < 0$).

A plausible argument suggesting this corollary is as follows: If the stp function feedback operator of Fig. 1.16 is replaced by the sat kx function of Fig. 1.7, asymptotic stability in a neighborhood of the origin is obtained if and only if $q(s) + \frac{1}{k}\,p(s)$ is strictly Hurwitz. Letting k tend to infinity makes sat kx approach stp x, and the zeros of $q(s) + \frac{1}{k}\,p(s)$ approach the zeros of q(s), and the result follows.

Now we return to the lossless second order regulator described by

$$\ddot{x} + x = -\underset{(-2+2\alpha,\,2\alpha)}{\mathrm{stp}} \ (\beta\dot{x}+x)$$

i.e.

$$\dot{\underline{x}} = \underline{A}\,\underline{x} - \underline{b}\ \underset{(-a,b)}{\mathrm{stp}}\ \underline{c}\,\underline{x}$$

where

$$\underline{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \underline{c} = [\beta \quad 1], \ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix},$$

$$a = 2 - 2\alpha, \quad b = 2\alpha \ .$$

The switching line S is $x_2 + \beta x_1 = 0$, and $\underline{x}$ is an endpoint if

$$-a\,\underline{c}\,\underline{b} < \underline{c}\,\underline{A}\,\underline{x} < b\,\underline{c}\,\underline{b}$$

which becomes

$$-\beta(2-2\alpha) < -\beta x_2 + x_1 < 2\alpha\beta$$

or

$$-2 + 2\alpha < -\left(1 + \frac{1}{\beta^2}\right) x_2 < 2\alpha$$

since $x_1 = \frac{-x_2}{\beta}$, i.e.

$$-2\alpha\left(\frac{\beta^2}{\beta^2+1}\right) < x_2 < (2-2\alpha)\left(\frac{\beta^2}{\beta^2+1}\right).$$

Motion along S is governed by $\underline{\dot{x}} = \underline{F}\ \underline{x}$ where

$$\underline{F} = \left(\underline{I} - \frac{\underline{b}\ \underline{c}}{\underline{c}\ \underline{b}}\right)\underline{A}$$

$$= \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \frac{1}{\beta}\begin{pmatrix}\beta & 1\\ 0 & 0\end{pmatrix}\right]\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

i.e.

$$\underline{F} = \begin{bmatrix}-\frac{1}{\beta} & 0\\ 1 & 0\end{bmatrix}$$

We note that $\det(\underline{I}s-\underline{F}) = \frac{s}{\beta}(\beta s+1)$, in accord with the theorem. Thus chattering occurs along the switching line $x_2 + \beta x_1 = 0$ in the region $-2\alpha\left(\frac{\beta^2}{\beta^2+1}\right) < x_2 < (2-2\alpha)\left(\frac{\beta^2}{\beta^2+1}\right)$, with motion in this region being determined by $\dot{x}_1 = -\left(\frac{1}{\beta}\right) x_1$.

Three conclusions of practical significance arising from these results are:

(i) That on the switching surface a small value of β is desirable for quick settling (i.e. a short transient response), though this is not necessarily true for the overall transient response,

(ii) The state vector $\underline{x}$ will not reach the origin in finite time,

(iii) The turn-on and turn-off delays of the switch need not be equal for the foregoing analysis to apply.

An examination of the trajectories of Fig. 1.9 shows that indeed the desired stability about (0,0) is obtained, for any initial condition. Fig. 1.10 shows a typical path.

We can see why asymptotic stability is not obtained when $\beta = 0$ by considering Fig. 1.16, in which we see that the switching line $x + \beta\dot{x} = 0$ is now the $\dot{x}$ axis. The paths form closed trajectories representing sustained oscillations.

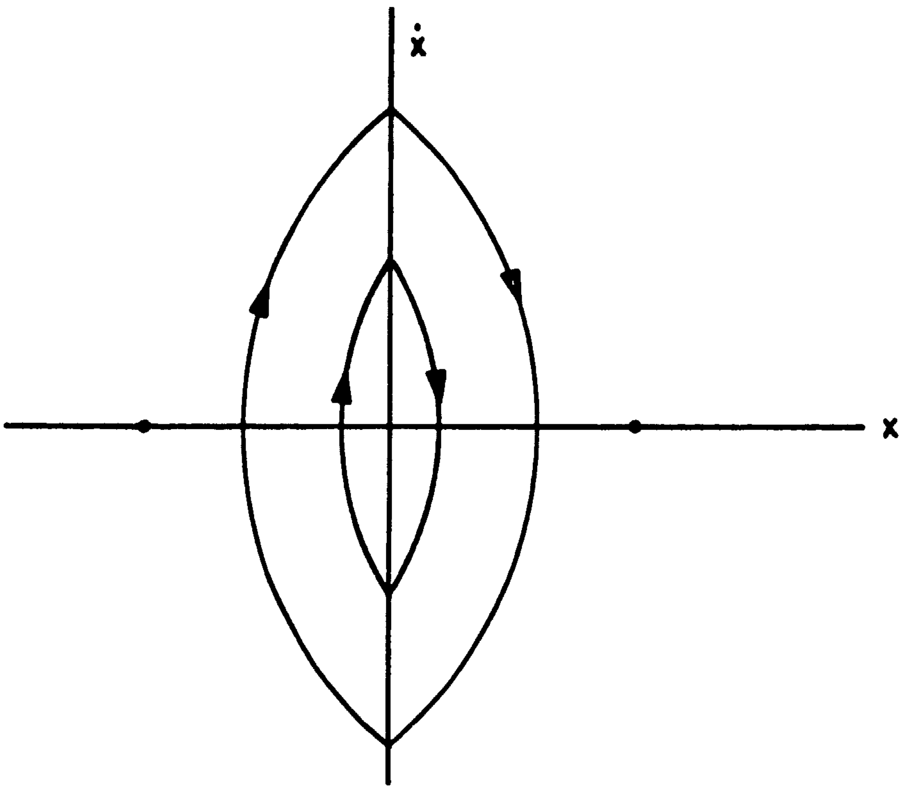


Fig. 1.16

(c) Stability by Total Gain Linearization

The system we are considering is

$$\ddot{x} + x = - \operatorname{stp}(\beta\dot{x}+x)$$

By taking Laplace transforms we can represent this in the usual feedback system form of Fig. 1.17. Consider now a feedback system with transfer function G(s) in the forward path, and nonlinearity f(σ) in the feedback

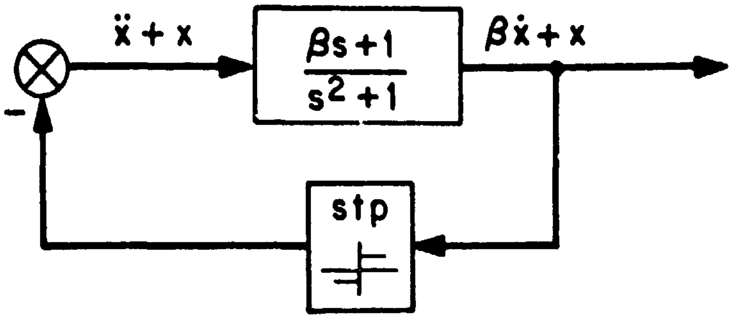


Fig. 1.17

path, as in Fig. 1.18. Assume that f(0) = 0. We can associate with this the linear system of Fig. 1.19, with feedback gain k.

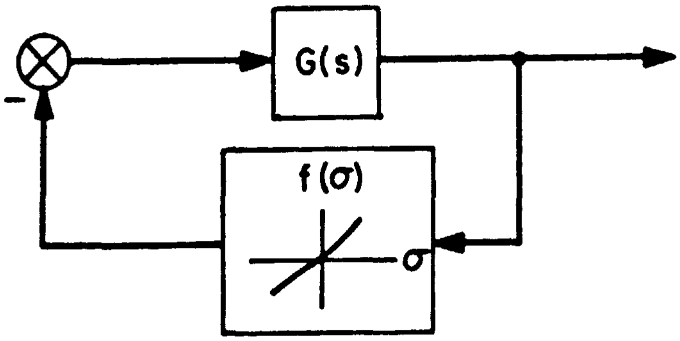


Fig. 1.18

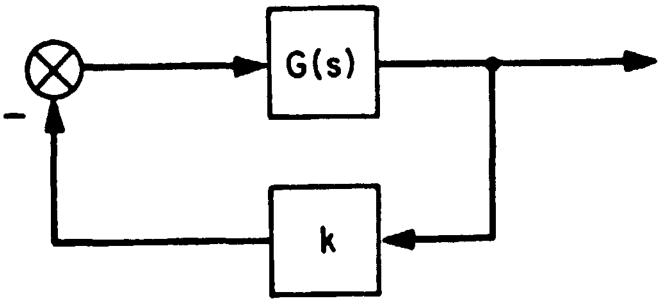


Fig. 1.19

Aizerman's Conjecture ([34], Chapter 7) states that for the system of Fig. 1.18 we would expect to have global asymptotic stability if the associated system of Fig. 1.19 is globally asymptotically stable for all values of k lying in the range of values taken on by $\frac{f(\sigma)}{\sigma}$ as σ varies along the real line. This method of investigating the stability of a feedback system is called the method of <u>total gain linearization</u>. Aizerman's Conjecture is not true in general, though it appears to be true for the type of systems considered in this thesis, and it can be shown to be true for second-order systems ([34] Chapter 7). The nonlinearity stp σ lies in the first and third quadrants, with $\frac{\text{stp}\,\sigma}{\sigma}$ taking on all values between 0 and $+\infty$. The total gain linearization stability conditions are thus fulfilled if the Nyquist locus of G(s) does not cross the negative real axis. This is so for the system of Fig. 1.17 for which the Nyquist locus is as shown in Fig. 1.20.

The Circle Criterion is not useful here, because the interior of the disc is the left half plane, and the Nyquist locus enters this region. The Popov Criterion does prove stability for this second order system, however we shall not consider its application because it is a special case of the Positive Operator Theorem method we consider next: the Popov Criterion makes use of a first-order multiplier, which is useful for our purposes only for the second-order regulator.

(d) <u>Stability by the Positive Operator Theorem</u>

We now make use of some concepts from the theory of Positive (or Dissipative) Operators. The most important result which we use from this theory is called the Positive Operator Theorem. It is felt that one of the main contributions of this thesis is in showing the utility of the

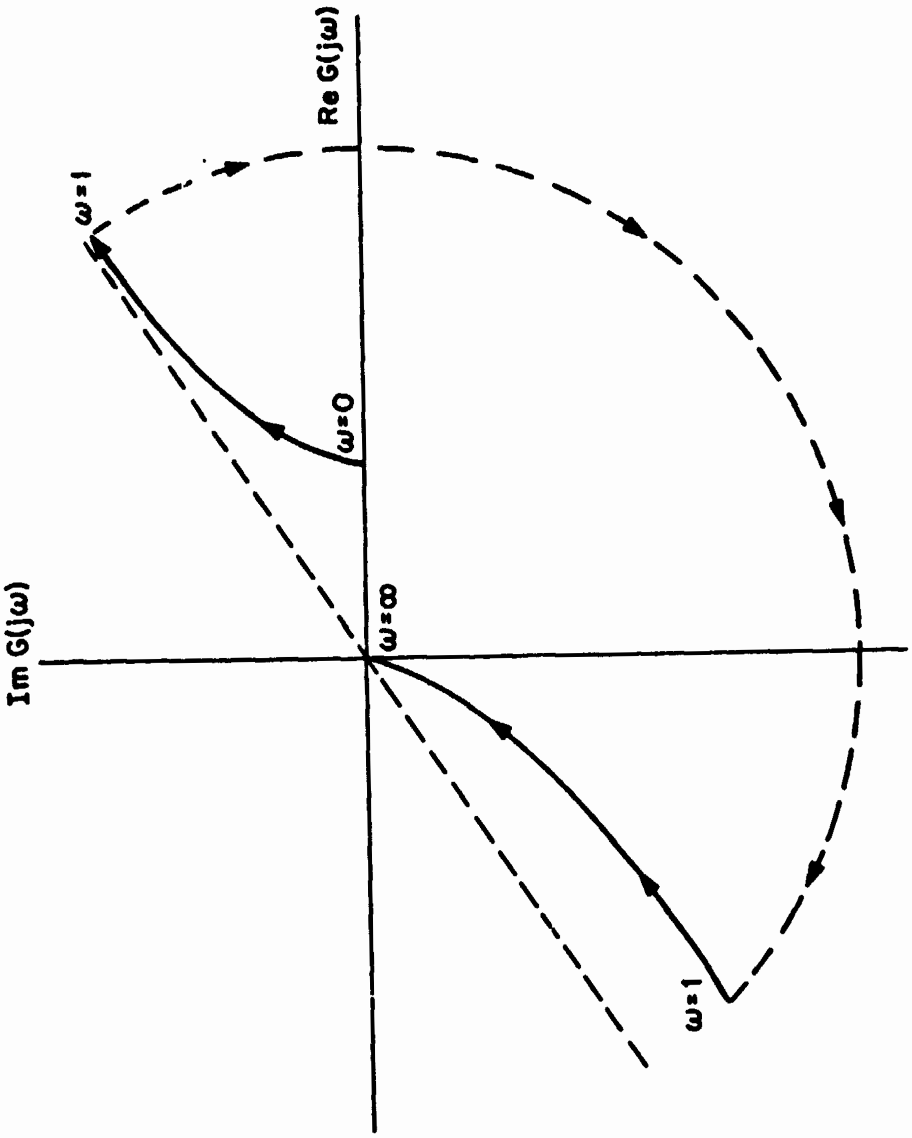


Fig. 1.20

Nyquist locus of $\frac{s+1}{s^2+1}$

ideas associated with this theorem. We devote more attention to positive operators in Chapter 2. For our purposes here, a <u>positive operator</u> is an operator with input u(t) and output y(t), where $0 \leq t < \infty$, for which $\int_0^T uy\,dt \geq 0$ for all $T \geq 0$, and for which $y(t) = 0$ for all t whenever $u(t) = 0$ for all t. This last requirement can be written $G0 = 0$. If the operator is a convolution operator, and if it can be represented by a rational transfer function G(s), then it can be shown ([36] Theorem 1,[41], [21]) that positivity is equivalent to the requirement that G(s) have no poles in the right half plane, that any poles on the imaginary axis be simple with real positive residues, and that $\mathrm{Re}\,G(j\omega) \geq 0$ for all ω, i.e. the Nyquist locus of G(s) must lie entirely in the right half plane. An equivalent requirement ([2], [16]) is that $\mathrm{Re}\,G(j\omega) \geq 0$ and if $G(s) = \frac{q(s)}{p(s)}$ then $p(s) + q(s)$ must be strictly Hurwitz, (i.e. all its zeros must lie in $\mathrm{Re}\,s < 0$). Such functions are called <u>positive real</u> and play an important role in electrical network synthesis, since the driving point impedance of a linear passive network is positive real, and any positive real function is the driving point impedance of an RLC network.

The Positive Operator Theorem gives a sufficient condition for input-output stability of the feedback system of Fig. 1.21 in which $u_1, y_1, u_2, y_2, v_1, v_2$ are all functions of t for $0 \leq t < \infty$.

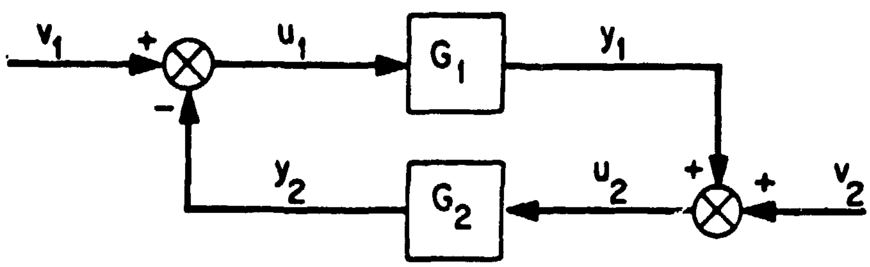


Fig. 1.21

The functions v_1 and v_2 are the inputs, and can be used to represent driving functions, driving noise, or initial condition responses. By choosing v_1 and v_2 appropriately we can use the Positive Operator Theorem to obtain conclusions about the behavior of u_1, y_1, u_2, y_2 as $t \to \infty$. To do this we need a measure of the "size" of a function of time $x(t)$; we use the L_2-norm

$$||x(t)|| = \left(\int_0^\infty x^2\, dt\right)^{\frac{1}{2}} .$$

Using this norm we can only handle functions $x(t)$ for which $\int_0^\infty x^2\, dt$ is finite; the set of all such functions is denoted L_2. There are many functions of interest which are not in L_2, for example the constant functions. This difficulty is overcome by using truncated functions, that is, functions which are zero after some time T. Now it can be shown that for any function $x(t) \in L_2$ for which $\dot{x}(t)$ is bounded or square-integrable (i.e. $\int_0^\infty \dot{x}^2\, dt < \infty$), we must have $\lim_{t \to \infty} x(t) = 0$. An operator G is said to be <u>bounded</u> if there exists a real number M such that $\frac{||Gx||}{||x||} < M$ for all x. An operator G with input $u(t)$ and output $y(t)$ is said to be <u>strictly positive</u> if $\int_0^T uy\, dt \geq \eta \int_0^T u^2\, dt$ for some $\eta > 0$, and $G0 = 0$. (Alternatively we can say that G is strictly positive if $G - \eta I$ is positive for some $\eta > 0$, I being the identity operator).

We can now state the Positive Operator Theorem ([42], [34] Chapter 4):

<u>Theorem 1.2</u>

If G_1 and G_2 are positive with one of them being strictly positive and bounded, then u_1, y_1, u_2, y_2 are all in L_2 whenever v_1 and v_2 are in L_2, and there exist positive constants ρ_1 and ρ_2 such that

$$||u_1||, ||y_1||, ||u_2||, ||y_2|| \leq \rho_1 ||v_1|| + \rho_2 ||v_2||.$$

Before applying the theorem we need to consider the requirement $G0 = 0$ when G is a convolution operator. A convolution operator $G(s)$ mapping u into y as depicted in Fig. 1.22 is usually taken to mean the operator with input $u(t)$ and output $y(t)$, $0 \leq t < \infty$, defined by the differential equation

$$p(D)\ y(t) = q(D)\ u(t) \qquad \text{where } D = \frac{d}{dt}$$

i.e.

$$p_n\ y^{(n)}(t) + p_{n-1} y^{(n-1)}(t) + \ldots + p_0\ y(t) = q_m\ u^{(m)}(t) + \ldots + q_0\ u(t),$$

together with a given set of initial conditions

$$y(0)\ ,\ y^{(1)}(0)\ ,\ldots,\ y^{(n-1)}(0)\ .$$

$$u(t) \longrightarrow \boxed{G(s) = \frac{q(s)}{p(s)} = \frac{q_m s^m + q_{m-1} s^{m-1} + \cdots + q_0}{p_n s^n + \cdots + p_0}} \longrightarrow y(t)$$

Fig. 1.22

If $m \leq n$ this means that we can describe G in state space form by

$$\begin{cases} \dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{b}\ u(t) \\ y \quad = \underline{c}\ \underline{x}(t) + d\ u(t) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

where $\underline{x}(t) \in \mathbb{R}^n$, $\underline{c}(\underline{I}s-\underline{A})^{-1}\ \underline{b} = G(s)$, and $(\underline{A},\underline{b},\underline{c},d)$ is assumed to be a minimal realization. Now the operator G defined thus has the property $G0 = 0$ if and only if $\underline{x}_0 = \underline{0}$. For, $G(u(t))$ is given [7] by

$$y(t) = \underline{c}\, e^{\underline{A}t}\, \underline{x}_0 + \int_0^t \underline{c}\, e^{\underline{A}(t-\tau)}\, \underline{b}\, u(\tau)\, d\tau + d\, u(t) \;,$$

thus

$$u(t) \equiv 0 \Longrightarrow y(t) = \underline{c}\, e^{\underline{A}c}\, \underline{x}_0$$

$$\Longrightarrow \underline{x}_0 = \underline{0} \quad \text{if } y(t) \equiv 0 \qquad \text{since } (\underline{A},\underline{c}) \text{ is observable}$$

$$\text{since } (\underline{A},\underline{b},\underline{c},d) \text{ is minimal.}$$

Further,

$$\underline{x}_0 = \underline{0} \iff \left.\frac{d^k y}{dt^k}\right|_{t=0} = 0$$

for $0 \leq k \leq n-1$, since $(\underline{A},\underline{c})$ is observable.

In what follows we shall depict the initial conditions associated with G explicitly by means of an arrow, as in Fig. 1.23. If no such

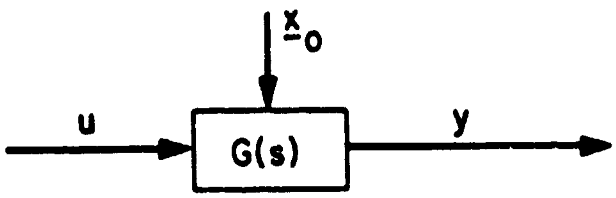


Fig. 1.23

arrow is shown we shall mean that the initial conditions are unspecified.

Now an operator G(s) with initial conditions $\underline{x}_0$ as in Fig. 1.23 can be represented as an operator with zero initial conditions followed by the addition of an external signal which is the <u>initial condition response</u> of G to $\underline{x}_0$, as depicted in Fig. 1.24. We shall make use of this equivalence in applying the Positive Operator Theorem.

Before applying the Theorem to our regulator problem we shall first consider a simple example. Let $G_1(s)$ be $\frac{(s+2)}{(s+1)(s+3)}$, and let the

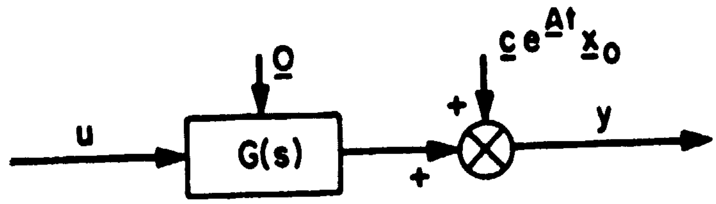


Fig. 1.24

feedback operator G_2 be sat kx, as in Fig. 1.25. We have $G_1(s) = \underline{c}(\underline{I}s-\underline{A})^{-1}\underline{b}$

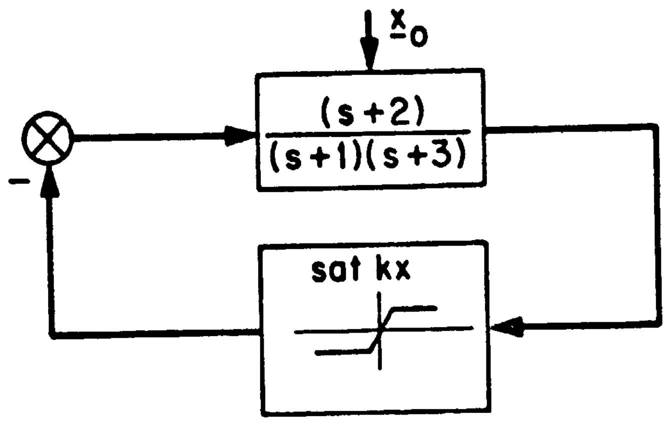


Fig. 1.25

where

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{c} = [2 \quad 1] .$$

This gives us the equivalent formulation of Fig. 1.26, in which the initial condition response is considered as an external input. The operators in Fig. 1.26 now both satisfy $G0 = 0$. Since it is a first- and third-quadrant function, sat kx is a positive operator, and $G_1(s)$ is also a positive operator, since Re $G_1(j\omega) = \dfrac{2(\omega^2+3)}{(\omega^2+1)(\omega^2+9)} > 0$ for all ω, and $p(s) + q(s) = s^2 + 5s + 5$ which is strictly Hurwitz. It is easy to show that G_2 is strictly positive and bounded, (the finite slope at the origin being necessary for this boundedness). In fact G_1 is also bounded,

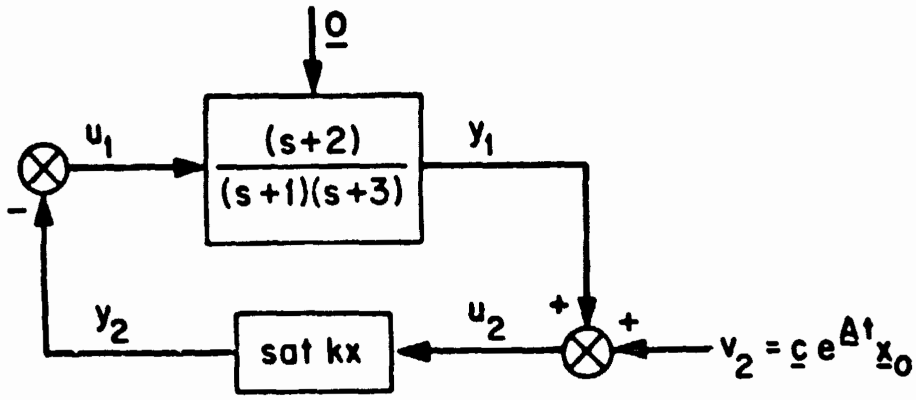


Fig. 1.26

since $|G_1(j\omega)|$ is bounded, since $G_1(s)$ has no poles on the imaginary axis. Now in Fig. 1.26 the initial condition response $v_2 = \underline{c}\ e^{\underline{A}t}\ \underline{x}_0$ is in L_2 because the eigenvalues of $\underline{A}$, which are the poles of $G_1(s)$, lie strictly in the left half plane. We can now apply the Positive Operator Theorem as stated above to deduce that the functions $u_1(t)$, $y_1(t)$, $u_2(t)$, $y_2(t)$ of Fig. 1.26 all belong to L_2. This implies that they all approach zero as $t \to \infty$, which is the desired asymptotic stability, provided that we can show that all of their derivatives are bounded or square-integrable. Now it can be shown for a system of the form

$$\begin{cases} \dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{b}\ u \\ y = \underline{c}\ \underline{x} \end{cases}$$

that if $u \in L_2$ then $\dot{y} \in L_2$, provided that $\underline{A}$ is an asymptotically stable matrix, i.e. all its eigenvalues lie in the half-plane Re $s < 0$. Thus, by describing G_1 in this state space form we see that in Fig. 1.26 $\dot{y}_1 \in L_2$.

Then, since $\dot{v}_2 \in L_2$, $\dot{u}_2 \in L_2$. Now for the operator $G_2(x) = \text{sat}\ kx$, we have $\frac{d}{dt}G_2(x(t)) = \frac{dG_2}{dx} \cdot \frac{dx}{dt}$; thus since $\frac{dG_2}{dx} < \infty$, $\dot{y}_2 \in L_2$ and so $\dot{u}_1 \in L_2$. We therefore have the desired asymptotic stability.

Now we consider application of the Positive Operator Theorem to the feedback system of Fig. 1.27, which represents our second-order lossless voltage regulator. We have replaced the stp x function with a sat kx function, k being arbitrarily large, in order to ensure that $\frac{d}{dt}(\text{sat}\ kx(t))$ is bounded, as just discussed.

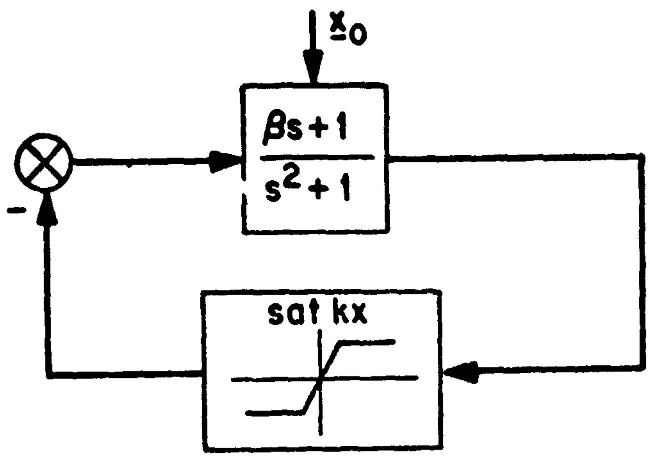


Fig. 1.27

Now $G_1(s) = \frac{\beta s+1}{s^2+1}$ is not a positive operator, since its Nyquist locus enters the left half plane, as depicted in Fig. 1.20. However, $G_2 = \text{sat}\ kx$ is "very strongly positive", and by introducing two multiplying factors into the loop we can make use of this fact, so that we end up with two positive operators. Now $G_1(s)$ has poles on the imaginary axis, and therefore its initial condition response will not be an L_2 function. To take care of this fact we can introduce a very small amount of damping, so that $G_1(s)$ becomes $\frac{\beta s+1}{s^2+rs+1}$ for some arbitrarily small $r > 0$. This represents a small amount of series resistance in the inductor of Fig. 1.5, which is a case we consider in §1.6.

Let $Z(s)$ be a transfer function, and consider the operator H with input $u(t)$ and output $y(t)$ defined by Fig. 1.28.

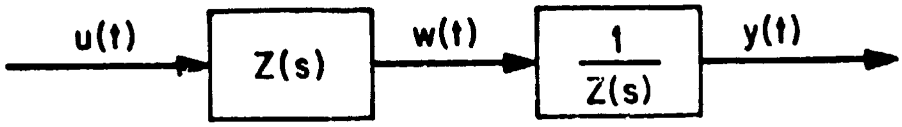


Fig. 1.28

Suppose that

$$Z(s) = \frac{p(s)}{q(s)} = \frac{p_n s^n + \ldots + p_0}{q_m s^m + \ldots + q_0} \quad .$$

Then since

$$p(D)\ u(t) = q(D)\ w(t) = p(D)\ y(t) \ ,$$

$$p(D)[u(t) - y(t)] = 0 \quad ,$$

so $u(t) \equiv y(t)$ if and only if $u^{(i)}(0) = y^{(i)}(0)$ for $0 \leq i \leq n-1$. Thus H is the identity operator if and only if the initial conditions of its second operator $Z^{-1}(s)$ match the initial conditions of the input function.

Consider now the system of Fig. 1.27 (where $G_1(s) = \frac{\beta s+1}{s^2+1}$) modified to the form of Fig. 1.29, where $\underline{M}$ is a matrix relating the initial conditions of $Z^{-1}(s)$ to those of $G_1(s)$, as just discussed. We now modify this to the form of Fig. 1.30, which still represents the same system, provided that $v_1(t)$ and $v_2(t)$ are 0 for all $0 \leq t < \infty$, and $\underline{M}$ is suitably chosen. Now let $G_1(s)\ Z(s) = \underline{c}(\underline{I}s-\underline{A})^{-1}\underline{b}$, so that we have the configuration of Fig. 1.31. The configuration of Fig. 1.31 is of the form we want, except for the initial condition $\underline{M}\ \underline{x}_0$ in the feedback operator, which we want to be zero so that the feedback operator satisfies $G0 = 0$.

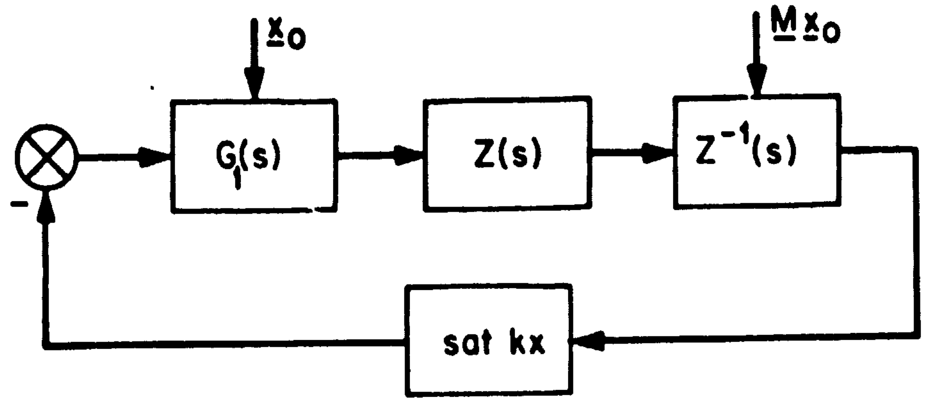


Fig. 1.29

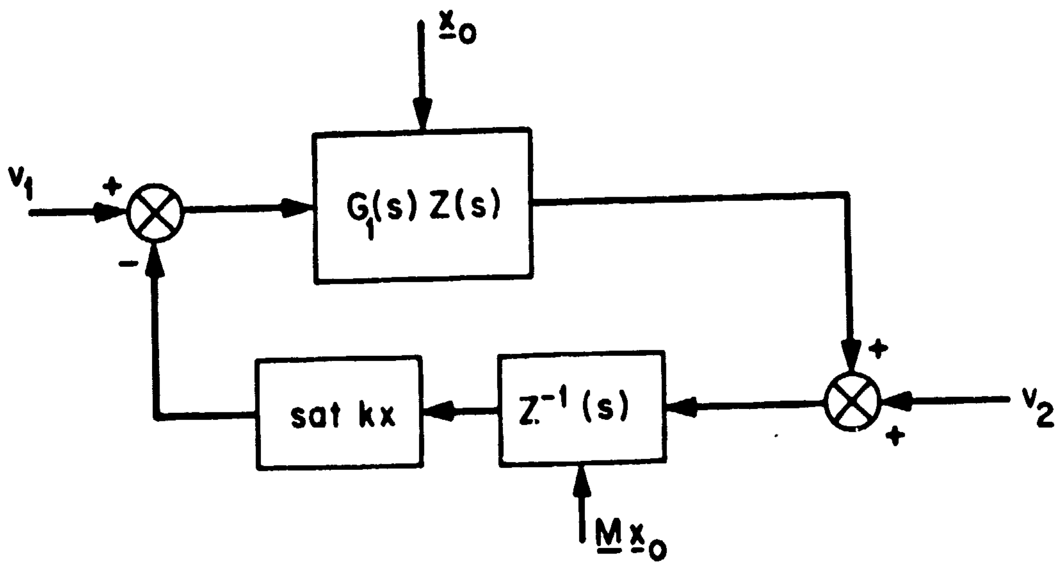


Fig. 1.30

To circumvent this difficulty we assume that the system is started at time $t = -1$ with all initial conditions zero, and that in the interval $0 \leq t \leq 1$ the inputs $v_1(t)$ and $v_2(t)$ drive the system to the state shown in Fig. 1.30 at time $t = 0$. We let v_1 and v_2 be 0 for $0 \leq t < \infty$. Now the desired stability follows from the Positive Operator Theorem as outlined above, provided that we can show that the forward and feedback operators of

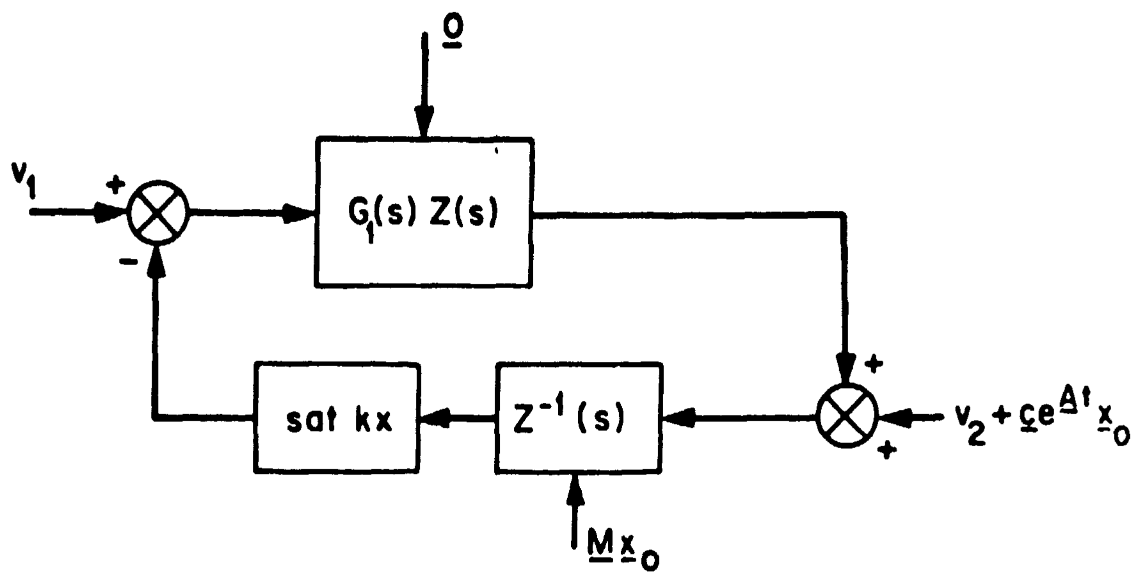


Fig. 1.31

Fig. 1.31 are positive, with one of them being strictly positive and bounded. We take $Z(s) = \gamma s+1$ where $\gamma = \frac{1}{\beta}$, so that we have

$$G_1(s)\ Z(s) = \frac{(\beta s+1)(\gamma s+1)}{s^2+1}$$

$$= 1 + \frac{(\beta+\gamma)s}{s^2+1} \qquad \text{since } \beta\gamma = 1,$$

i.e. $\mathrm{Re}\ G_1(j\omega)\ Z(j\omega) = 1 > 0$ for all ω; and $p(s) + q(s) = 2s^2 + (\beta+\gamma)s + 2$ which is strictly Hurwitz; thus $G_1(s)\ Z(s)$ is a positive real function.

We now show that the operator of Fig. 1.32 is also positive, where $f(\sigma)$ is any first- and third-quadrant nonlinearity with bounded slope at the origin, $\gamma > 0$, and $w(0) = 0$.

Let

$$\int_0^x f(\sigma)\ d\sigma = F(x)\ ,\ \geq 0 \text{ for all } x.$$

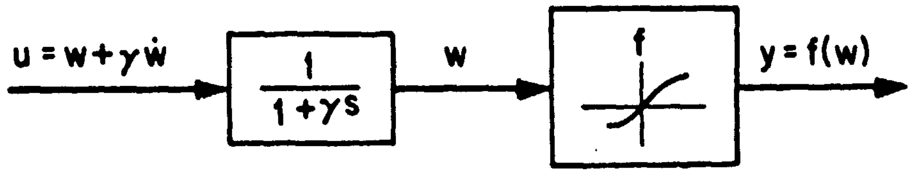


Fig. 1.32

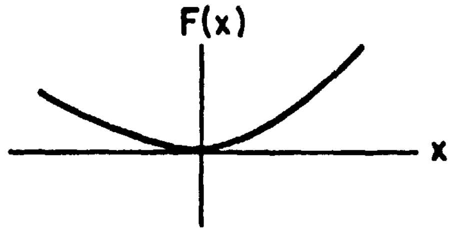


Fig. 1.33

Now

$$\int_0^T uy\ dt = \int_0^T (w+\gamma\dot{w})\ f(w)\ dt$$

$$= \int_0^T w\ f(w)\ dt + \gamma\int_0^T f(w)\ \frac{dw}{dt}\ dt$$

$$= \int_0^T w\ f(w)\ dt + \gamma\int_0^{w(T)} f(w)\ dw$$

$$= \int_0^T w\ f(w)\ dt + \gamma F(w(T)) - \gamma F(w(0))$$

$$\geqslant 0$$

since we assume $w(0) = 0$. Thus the operator of Fig. 1.32 is indeed positive.

We make two comments here about the multiplier $(1+\gamma s)$, $\gamma = \frac{1}{\beta}$. Firstly, it is easy to show that in this case this is the only such multiplier (to within a multiplicative constant) which will make $\frac{\beta s+1}{s^2+1}$ become positive real. Secondly, we could have predicted that such a multiplier exists by making use of a theorem of Brockett and J. L. Willems ([6], Theorem 2), which says that the feedback system of Fig. 1.19 is asymptotically stable for any feedback gain k satisfying $0 < k < k^*$, if and only if there exists a positive real function Z(s) such that $Z(s)\left[G(s) + \frac{1}{k^*}\right]$ is positive real.

In §1.5 (d) we summarize the application of the Positive Operator Theorem to a stability analysis of this kind.

(e) Stability by Lyapunov's Method

Now we show that the lossless second-order regulator is stable by means of the method of Lyapunov. First, we give some definitions and use them to state the Lyapunov theorem we wish to apply. Then, we make use of Dissipative System concepts, as described in Chapter 2, to obtain a Lyapunov function for the second-order regulator. Although we have already established that the regulator is stable using the Positive Operator Theorem, the use of Lyapunov's Method is important here, because of its extension to the resistive-source-impedance case, §1.7. A useful reference on Lyapunov Theory is [37].

Consider the system of equations

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t),t); \quad 0 \leq t < \infty; \quad \underline{x}(0) = \underline{x}_0; \quad \underline{f}(0,t) = \underline{0}; \quad \underline{x}(t) \in \mathbb{R}^n.$$

A real-valued function $V(\underline{x},t)$ is called a Lyapunov function for this system if

(i) $V(\underline{x},t)$ has first partial derivatives with respect to $\underline{x}$ and t which are also continuous with respect to $\underline{x}$ and t, and

(ii) $V(\underline{x},t)$ is bounded if $||\underline{x}||$ is bounded, and

(iii) $\dot{V}(\underline{x},t)$, the time rate of change of V as given by

$$\dot{V}(\underline{x},t) = \left(\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i\right) + \frac{\partial V}{\partial t}$$

satisfies

$$\dot{V}(\underline{x},t) \leq W(\underline{x}) \leq 0$$

for all $\underline{x}$, and some continuous function $W(\underline{x})$.

$V(\underline{x},t)$ is called positive definite if $V(\underline{0},t) = 0$ and if there exists a continuous increasing scalar-valued function of a scalar argument $V_1(\sigma)$, such that $V_1(0) = 0$ and

$$V(\underline{x},t) \geq V_1(||\underline{x}||) \quad .$$

$V(\underline{x},t)$ is called radially unbounded if $V_1(\sigma)$ approaches infinity as σ approaches infinity. $V(\underline{x},t)$ is called decrescent if there exists a second scalar function of a scalar argument $V_2(\sigma)$ which is continuous and nondecreasing, such that $V_2(0) = 0$ and

$$V(\underline{x},t) \leq V_2(||\underline{x}||) \quad .$$

We then have the following Lyapunov Theorem:

Theorem 1.3

If $V(\underline{x},t) = V(x,t+T)$ is a Lyapunov function for the periodic or time-invariant system of equations

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},t) = \underline{f}(\underline{x},t+T)$$

$$\underline{x}(t) \in \mathbb{R}^n; \quad 0 \leq t < \infty; \quad \underline{x}(0) = \underline{x}_0; \; f(\underline{0},t) = \underline{0},$$

and if $V(\underline{x},t)$ is positive definite, decrescent, and radially unbounded, then the system is globally asymptotically stable about $\underline{x} = \underline{0}$, provided $\dot{V}(\underline{x},t)$ is not identically zero along any nonzero solution.

For a proof of this theorem the reader is referred to the work of LaSalle [17], and also to [37].

We now show that the second-order regulator is stable using concepts from the theory of Dissipative Systems, which is closely allied with the theory of Positive Operators, as discussed in Chapter 2.

Let G be an operator (or system) with input u(t) and output y(t), $0 \leq t < \infty$, defined by the equations

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x},u) \\ y \quad = g(\underline{x}) \\ \underline{x}(0) = \underline{x}_0 \end{cases} .$$

The vector $\underline{x}(t)$ is called the <u>state vector for G</u>. Let w(u,y) be a real-valued function of u and y. Then G is said to be <u>dissipative with respect to the supply-rate w(u,y)</u> if there exists a nonnegative function $V(\underline{x})$ with $V(\underline{0}) = 0$ such that

$$\dot{V} - w \leq 0 .$$

V is called the <u>storage function for G</u>. In this thesis the only supply rate w which we consider is the product uy, so that when we refer to an operator as <u>dissipative</u> we have this supply rate in mind. In Chapter 2 we shall show that an operator is dissipative if and only if it is a positive operator.

Consider for instance a one-port RLC electrical network with input current u(t) and voltage y(t), as in Fig. 1.34. The operator G mapping u(t) into y(t) represents the impedance of the network; expressed as a transfer function G(s) it is a positive real function. The requirement

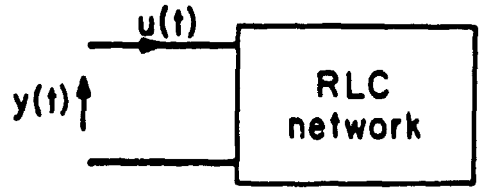


Fig. 1.34

for G to be a positive operator, i.e., $\int_0^T uy\,dt \geq 0$, says that the net energy absorbed by the network is at all times positive, the concurrent requirement being that G maps zero into zero, i.e. that no initial energy is stored in the network. If G is viewed as a dissipative operator the storage function V is the energy stored in the network, and the supply rate $w = uy$ is the rate at which energy is supplied. The requirement $\dot{V} - uy \leq 0$ says that the rate of increase of stored energy is not greater than the rate of supply, (because of dissipation within the network). This requirement is independent of the initial conditions within the network.

Now consider the feedback connection of Fig. 1.21 with $v_1 \equiv v_2 \equiv 0$. Let G_1 be a dissipative operator with state vector $\underline{x}_1$ and storage function $V_1(\underline{x}_1)$, and let G_2 be dissipative with $\underline{x}_2$ and $V_2(\underline{x}_2)$. Then

$$\begin{cases} \dot{V}_1 - u_1 y_1 \leq 0 \\ \dot{V}_2 - u_2 y_2 \leq 0 \end{cases} .$$

But $u_1 = -y_2$ and $u_2 = y_1$, so if we let $V(\underline{x}_1,\underline{x}_2) = V_1(\underline{x}_1) + V_2(\underline{x}_2)$ we obtain

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 \\ &= \dot{V}_1 - u_1 y_1 + \dot{V}_2 - u_2 y_2 \\ &\leq 0 \end{aligned}$$

Thus, a Positive Operator Theorem proof of stability provides us with a Lyapunov function for use with the Lyapunov Theorem, provided that the positive operators can be described in dissipative operator form.

Fig. 1.35 represents the second-order regulator, with $\beta = 1$ for notational convenience.

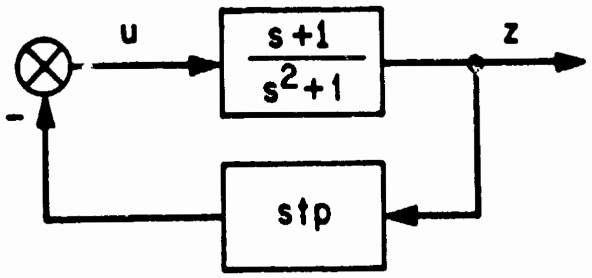


Fig. 1.35

A state-space description of this system is

$$\left\{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ z &= [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ u &= -\mathrm{stp}\ z \end{aligned}\right.$$

i.e.

$$\left\{\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{b}\,u \\ z &= \underline{g}\,\underline{x} \\ u &= -f(z) \end{aligned}\right.$$

where

$$\underline{g}(\underline{I}s-\underline{A})^{-1}\underline{b} = \frac{s+1}{s^2+1} \quad .$$

Fig. 1.36 depicts the same system with multiplier (1+s) included in the forward path and $(1+s)^{-1}$ included in the feedback path. As we saw in §1.4 (d), introducing these factors has no effect provided that the initial conditions of $\left(\frac{1}{s+1}\right)$ are correctly chosen. This means that there is a linear relationship between the initial conditions of the operator $\left(\frac{1}{s+1}\right)$ and those of $\left(1 + \frac{2s}{s^2+1}\right)$.

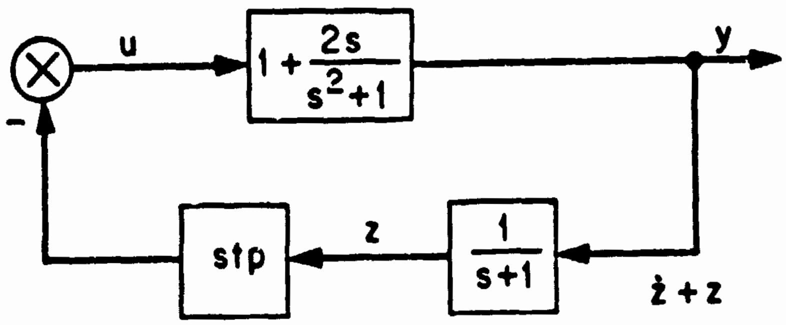


Fig. 1.36

A state-space description of the system of Fig. 1.36 is a non-minimal realization of the system of Fig. 1.35. We have

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = [2 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \\ \dot{z} = -z + y \text{ with } z(0) = x_1(0) + x_2(0) \\ u = -\text{stp } z \end{cases}$$

i.e.

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u \\ y = \underline{c}\,\underline{x} + d\,u \\ \dot{z} = -z + y \\ u = -f(z) \end{cases}$$

where

$$\underline{c}(Is-\underline{a})^{-1}\,\underline{b} + d = 1 + \frac{2s}{s^2+1} \quad .$$

Note that

$$z(t) \equiv x_1(t) + x_2(t) \quad ,$$

since

$$\dot{z} + z = y = 2x_1 + u = (\dot{x}_1+\dot{x}_2) + (x_1+x_2) \quad .$$

From Chapter 2 we know that a storage function $V_1(x)$ for $(1 + \frac{2s}{s^2+1})$ is $\frac{1}{2}\,\underline{x}'\underline{K}\,\underline{x}$ where $\underline{K} = \underline{K}'$ satisfies the matrix equation

$$\underline{K}\,\underline{A} + \underline{A}'\underline{K} + (\underline{K}\,\underline{b} - \underline{c}')(2d)^{-1}(\underline{b}'\underline{K} - \underline{c}) = \underline{0} \quad .$$

In this case this equation has a unique solution $\underline{K} = 2\underline{I}$, so that the storage function for $\left(1 + \frac{2s}{s^2+1}\right)$ is $V_1(\underline{x}) = \underline{x}'\underline{x} = x_1{}^2 + x_2{}^2$. In Chapter 2 we show that the unique storage function for the feedback operator of Fig. 1.37 is Sod z = z stp z.

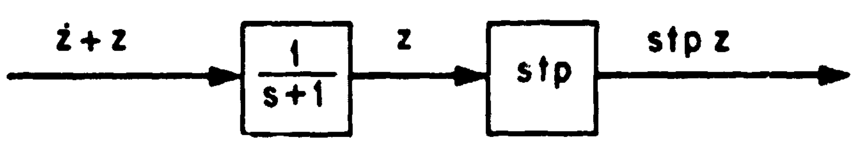


Fig. 1.37

Since $z = x_1 + x_2$, we have $V_2(\underline{x}) = \mathrm{Sod}(x_1+x_2)$. Thus a Lyapunov function for the second-order regulator described by

$$\begin{cases} \dot{x}_1 = -x_2 - \mathrm{stp}(x_1+x_2) \\ \dot{x}_2 = x_1 \end{cases}$$

is

$$V(\underline{x}) = V_1(\underline{x}) + V_2(\underline{x}) = x_1^{\ 2} + x_2^{\ 2} + \mathrm{Sod}(x_1+x_2) \quad .$$

Differentiating we get

$$\begin{aligned} \dot{V}(\underline{x}) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 + (\dot{x}_1+\dot{x}_2)\ \mathrm{stp}(x_1+x_2) \\ &= -2x_1x_2 - 2x_1\ \mathrm{stp}(x_1+x_2) + 2x_1x_2 + (x_1-x_2-\mathrm{stp}(x_1+x_2))\ \mathrm{stp}(x_1+x_2) \\ &= -(x_1+x_2)\ \mathrm{stp}(x_1+x_2) - \mathrm{stp}^2(x_1+x_2) \\ &= -\ \mathrm{Sod}(x_1+x_2) - \mathrm{stp}^2(x_1+x_2) \\ &\leq 0,\ = 0 \text{ if and only if } x_1+x_2=0. \end{aligned}$$

But

$$\begin{aligned} \dot{x}_1 + \dot{x}_2 &= x_1 - x_2 - \mathrm{stp}(x_1+x_2) \\ &= 0 \text{ on } \{x_1+x_2=0\} \text{ only at } x_1 = x_2 = 0\ , \end{aligned}$$

i.e. $\dot{V}(\underline{x})$ is identically zero only along the solution $\underline{x} = \underline{0}$. Thus, by the Lyapunov Theorem 1.3 we obtain the desired global asymptotic stability about $\underline{x} = \underline{0}$. Notice that for Fig. 1.36 we have proved stability for any initial conditions in the forward path and any initial conditions in the feedback path; this includes the case where there is a linear relationship between these two sets of initial conditions, this case being the system of Fig. 1.35.

In conclusion of this section we note that the Lyapunov method is superior to the Positive Operator Theorem method for determining stability, since with the Lyapunov method initial conditions do not require special attention, nor do we have to modify $G_1(s)$ to have poles which are off the imaginary axis. The Positive Operator Theorem is perhaps more useful for providing a Lyapunov function. We also found it necessary to modify stp x to be continuous at the origin, however the behavior at the origin is taken care of by the chattering behavior analysis of §1.4 (b), in which stp is better left as a discontinuous function. The Positive Operator Theorem and the Lyapunov method are thus used to determine that the state will reach a neighborhood of the origin from any starting point in state space.

§1.5 Fourth-Order Lossless Regulator

(a) Choice of a Feedback Law

The second member of the series of voltage regulators of Figs. 1.1 and 1.2 is shown in Fig. 1.38, where we have assumed unit values for the components. As with the second-order regulator this assumption barely limits the generality of our case: it represents all regulators of the type of Figs. 1.1 and 1.2 where the inductances are all equal and the capacitances are all equal. We want $\lim_{t \to \infty} z_4(t) = \alpha$ for some given $0 < \alpha < 1$.

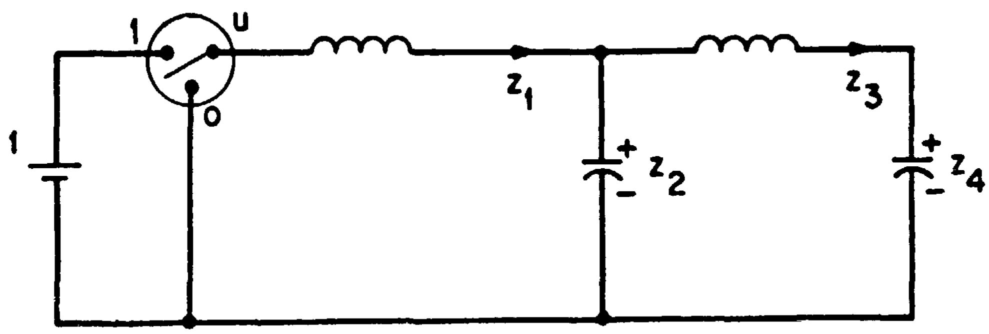


Fig. 1.38

The state evolution equations are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u .$$

Now in steady-state operation we want $z_2 = z_4 = \alpha$ and $\dot{z}_2 = \dot{z}_4 = 0$. The natural ext^nsion of the control law we chose in §1.4 (a) is thus

$$u = \begin{cases} 1 , & \text{if } \beta_1\dot{z}_2 + \beta_2(z_2-\alpha) + \beta_3\dot{z}_4 + \beta_4(z_4-\alpha) < 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta_1, \beta_2, \beta_3, \beta_4$ are suitably chosen positive constants.

(b) Stability by Total Gain Linearization

We make the change of variables $x_1 = 2z_1$, $x_2 = 2(z_2-\alpha)$, $x_3 = 2z_3$, $x_4 = 2(z_4-\alpha)$, and thus obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 2(u-\alpha) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $2(u-\alpha) = -\underset{(-2+2\alpha,2\alpha)}{\text{stp}} (\beta_1\dot{x}_2 + \beta_2x_2 + \beta_3\ddot{x}_4 + \beta_4x_4)$. By expressing x_1, x_2, x_3 in terms of $x_4 = x$ we can put this in the form

$$\ddddot{x} + 3\ddot{x} + x = -\text{stp}[\beta_1 \dddot{x} + \beta_2\ddot{x} + (\beta_1+\beta_3)\,\dot{x} + (\beta_2+\beta_4)x] .$$

The desired stability is about $x = 0$. Taking Laplace transforms we obtain the feedback representation of Fig. 1.39. This is of the form of Fig. 1.18. Using the Routh-Hurwitz Criterion it is easy to show that the associated

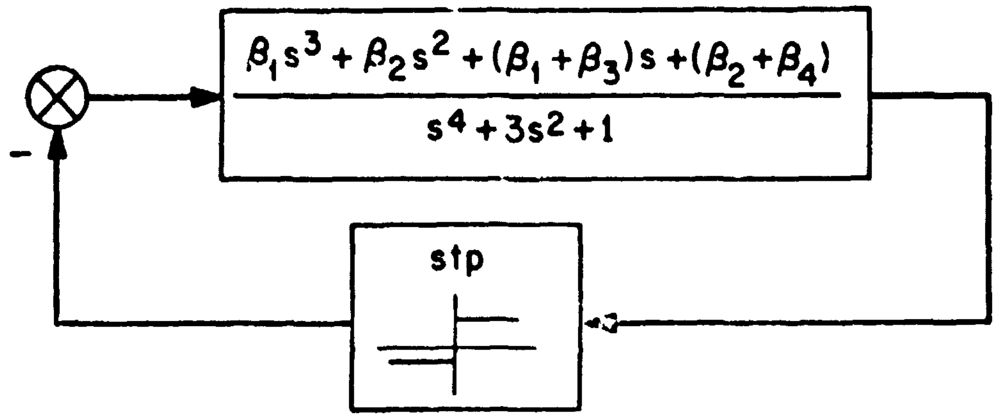


Fig. 1.39

feedback system of Fig. 1.19 is stable for all $0 < k < \infty$ if and only if

$$\begin{cases} \beta_3 \leqslant 2\beta_1 \\ \beta_3^{\,2} < \beta_1(\beta_1+\beta_3) \\ \beta_1\beta_4 \leqslant \beta_2\beta_3 \end{cases} .$$

In particular we note that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ satisfies these requirements. In §1.5 (d) we shall show that indeed Aizerman's Conjecture is correct here for $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$. This may not be too surprising, considering the nature of the Nyquist Locus of $G(s) = \frac{s^3+s^2+2s+2}{s^4+3s^2+1}$, which is shown in Fig. 1.40. The pole-zero pattern of G(s) is shown in Fig. 1.41. For design purposes the total gain linearization method appears to provide a reasonable approach for an (initial) investigation of the stability of this class of regulators; the above three inequalities for β_1, β_2, β_3, β_4 are probably the necessary and sufficient conditions for global asymptotic stability of the fourth-order regulator.

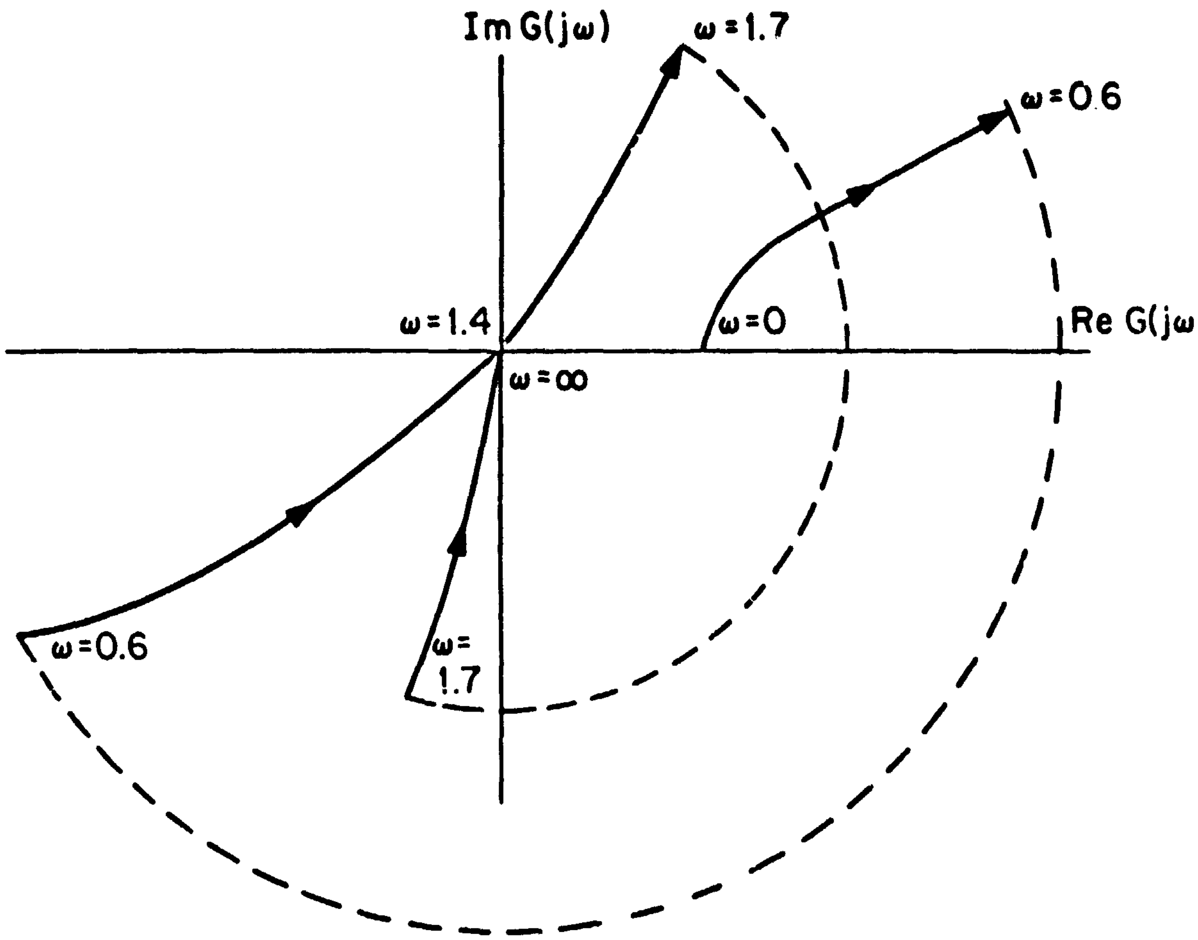
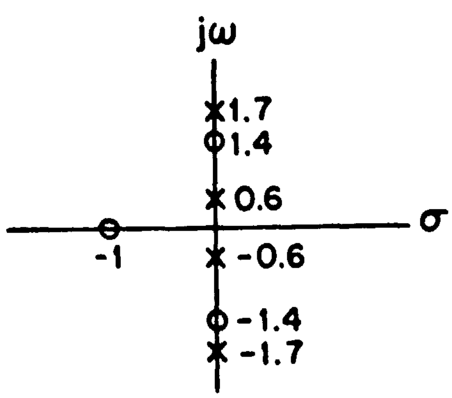


Fig. 1.40



$$G(s) = \frac{s^3 + s^2 + 2s + 2}{s^4 + 3s^2 + 1}$$

$$= \frac{(s+1)(s^2+2)}{(s^2 + 2.61)(s^2 + 0.38)}$$

Fig. 1.41

(c) Chattering Behavior

We follow the development of § 1.4(b). The system of Fig. 1.39 is of the form of that in Fig. 1.15, with

$$\underline{A} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3-\beta_1 & \beta_4 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

$$a = 2 - 2\alpha \ , \ b = 2\alpha \ .$$

The switching surface S is the hyperplane $\underline{c}\ \underline{x} = 0$,

i.e. $$\beta_1 x_1 + \beta_2 x_2 + (\beta_3-\beta_1)x_3 + \beta_4 x_4 = 0 \ ,$$

and $\underline{x} \in S$ is an endpoint if

$$-a\ \underline{c}\ \underline{b} < \underline{c}\ \underline{A}\ \underline{x} < b\ \underline{c}\ \underline{b}$$

i.e.

$$-(2-2\alpha)\beta_1 < \beta_2 x_1 + (\beta_3-2\beta_1)x_2 + (\beta_4-\beta_2)x_3 + (\beta_1-\beta_3)x_4 < 2\alpha\beta_1 \ .$$

For example when $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ we have: S is the surface $x_1 + x_2 + x_4 = 0$, and the conditions for chattering to occur at $\underline{x} \in S$ are

$$-2 + 2\alpha < x_1 - x_2 < 2\alpha$$

i.e.

$$-2 + 2\alpha < -2x_2 - x_4 < 2\alpha$$

i.e.

$$2\alpha < 2z_2 + z_4 < 2\alpha + 1 \ .$$

Motion along S is governed by $\dot{\underline{x}} = \underline{F}\ \underline{x}$ where

$$\det(\underline{I}s-\underline{F}) = \frac{s}{\beta_1}[\beta_1 s^3 + \beta_2 s^2 + (\beta_1+\beta_3)s + (\beta_2+\beta_4)] \ .$$

Now the conditions for a cubic polynomial $a_3 s^3 + a_2 s^2 + a_1 s + a_0$ to be strictly Hurwitz are

$$\begin{cases} a_0,\ a_1,\ a_2,\ a_3 > 0 \\ a_2a_1 - a_3a_0 > 0\ . \end{cases}$$

Therefore for asymptotic stability of the chattering mode on S we require

$$\begin{cases} \beta_1,\ \beta_2,\ \beta_3,\ \beta_4 > 0 \\ \beta_2\beta_3 > \beta_1\beta_4 \quad . \end{cases}$$

The second of these is a strict inequality (whereas in (b) above we had $\beta_2\beta_3 \geq \beta_1\beta_4$), and is not quite satisfied by $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$: for these values of β_i we have

$$\begin{aligned} \det(\underline{I}s-\underline{F}) &= s(s^3+s^2+2s+2) \\ &= s(s+1)(s^2+1)\ . \end{aligned}$$

However in what follows we use these borderline values for convenience.

(d) Stability by the Positive Operator Theorem

Following the development of §1.4 (d), we wish to find a multiplier Z(s) such that G(s)Z(s) is positive real, where

$$G(s) = \frac{s^3+s^2+2s+2}{s^4+3s^2+1} = \frac{(s+1)(s^2+2)}{(s^2+2.61)(s^2+0.38)} \quad .$$

We also want $Z^{-1}(s)$ followed by stp x to yield a positive operator. At this stage we encounter the difficulty of testing a function for positive realness. There are several different characterizations of positive real functions, ([16] Chapter 5), but it seems that the least difficult method to apply is to test whether Re $G(j\omega) \geq 0$ for all ω, and $p(s) + q(s)$ is strictly Hurwitz. Using this method one can show that for

$G(s) = \dfrac{s^3+s^2+2s+2}{s^4+3s^2+1}$ the required multiplier $Z(s)$ cannot be of the form $(s+a)$, nor of the form $\dfrac{s^2+as+b}{s+c}$. The simplest multiplier $Z(s)$ is therefore of the form $\dfrac{s^3+as^2+bs+c}{s^2+ds+e}$. For this we obtain

$$\mathrm{Re}\, G(j\omega)\, Z(j\omega) = \frac{(2-\omega^2)\, H(\omega)}{(\omega^2-2.61)(\omega^2-0.38)\,[(e-\omega^2)^2 + d^2\omega^2]}$$

where

$$H(\omega) = (c-a\omega^2)(e+d\omega^2-\omega^2) + \omega^2(b-\omega^2)(d-e+\omega^2) \ .$$

Thus $\mathrm{Re}\, G(j\omega)\, Z(j\omega) \geqslant 0$ for all ω if and only if

$$H(\omega) = k(2-\omega^2)(\omega^2-2.61)(\omega^2-0.38) \quad \text{for some } k > 0 \ ,$$

and this leads to the requirements

$$\begin{cases} c\,e = 2 & -(1) \\ b(d-e) + c(d-1) - a\,e = -7 & -(2) \\ b + e - d - ad + a = 5 & -(3). \end{cases}$$

To ensure that $Z^{-1}(s)$ followed by stp x yields a positive operator we make use of the following theorem of O'Shea ([26], [32], [12]):

<u>Theorem 1.4</u>

The operator F shown in Fig. 1.42 with input $u(t)$ and output $y(t)$ and $Z(s)$ rational, is a positive operator for any monotone nonlinearity $f(\sigma)$ for which $f(0) = 0$, if and only if

$$Z(s) = g_0 + \gamma s - \hat{g}(s)$$

where

$$\gamma \geqslant 0 \ , \ \hat{g}(s) = \int_0^\infty g(t)e^{-st}dt \ , \ g(t) \geqslant 0 \ , \text{ and } g_0 \geqslant \hat{g}(0) = \int_0^\infty g(t)\, dt.$$

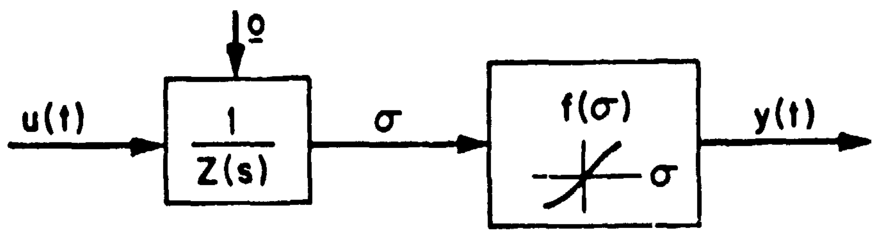


Fig. 1.42

For a proof of this theorem the reader is referred to reference [12]. Applying this theorem to our Z(s) we obtain the constraints:

$$\begin{cases} a,\ b,\ c,\ d,\ e > 0 & -(4) \\ d \leqslant a & -(5) \\ d^2 \geqslant 4e & -(6) \\ d(a-d) > b-e & -(7) \\ e(a-d) > \gamma & -(8)\ . \end{cases}$$

After some trial and error one can find a set of values for a, b, c, d, e which satisfy (1) through (8). One such set of values is

$$a = 200\ ,\ b = 96\ ,\ c = 4\ ,\ d = 1.45\ ,\ e = 0.5;$$

so that

$$Z(s) = \frac{s^3+200s^2+96s+4}{s^2+1.45s+0.5}\ .$$

We also need to show that $G(s)\ Z(s) = \frac{q(s)}{p(s)}$ has $p(s)+q(s)$ strictly Hurwitz. By following through an argument similar to that of §1.4 (d) we conclude that the fourth-order regulator of §1.5 (a) is globally asymptotically stable when $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$. The practical implications of this are that it is just as easy to stabilize a fourth-order regulator as it is to stabilize a second-order regulator.

To summarize the application of the Positive Operator Theorem to switched regulators of this kind, we see that the first step is to bring the problem to the form of determining stability of the null state for a system of the type shown in Fig. 1.15 or Fig. 1.18. We then consider the associated system of Fig. 1.19. If this is not stable for all $k > 0$ then we cannot make any conclusions in general, though instability for all $k > k_0 > 0$ will imply local instability in Fig. 1.15, and instability for all $k < k_1$ will imply instability in Fig. 1.15 for large initial conditions. If the system of Fig. 1.19 is stable for all $k > 0$ then we know ([6], Theorem 2) that there exists a class of positive real functions $\mathfrak{Z}$ such that $G(s)\ Z(s)$ is positive real for each $Z(s)$ in $\mathfrak{Z}$. Stability then follows if there is a $Z_1(s)$ in $\mathfrak{Z}$ such that $Z_1(s)$ satisfies the conditions of the O'Shea Theorem above. Behavior in the chattering mode is analyzed as outlined in §1.4 (b).

(e) Stability by Lyapunov's Method

In §1.4 (e) we saw how to obtain a Lyapunov function for the feedback system, by making use of the fact that we could express the forward and feedback positive operators in dissipative form. For the linear operator given by $G(s)\ Z(s) = \underline{c}(\underline{I}s-\underline{A})^{-1}\ \underline{b} + d$ we need to solve the matrix equation

$$\underline{K}\ \underline{A} + \underline{A}'\underline{K} + (\underline{K}\ \underline{b} - \underline{c}')(2d)^{-1}(\underline{b}'\underline{K}-\underline{c}) = \underline{0}\ .$$

This can usually be done, using suitable numerical methods if necessary. However, to obtain a storage function $V(\underline{x})$ for an operator of the form of Fig. 1.42 is not easy. In Chapter 2 we shall show that if a positive operator maps u(t) into y(t), the functions

$$V_a(\underline{x}) = \sup_{\substack{t_1 \geq 0 \\ \underline{x}(0)=\underline{x}}} -\int_0^{t_1} w(t)\, dt\,, \qquad V_r(\underline{x}) = \inf_{\substack{t_1 \geq 0 \\ \underline{x}(0)=0 \\ \underline{x}(t_1)=\underline{x}}} \int_0^{t_1} uy\, dt$$

are suitable storage functions. Unfortunately these definitions require the solution of a nonlinear optimization problem which can be solved for all initial conditions only when Z(s) is of order 1 or 2. For our fourth-order regulator problem Z(s) is third order, so we must try some other means to find a suitable $V(\underline{x})$. The most promising approach seems to be to attempt to find a realization $(\underline{A}, \underline{b}, \underline{c})$ of Z(s) which satisfies the requirements of the following theorem:

Theorem 1.5

Suppose $-\underline{A}$ is a hyperdominant matrix, $\underline{b}' = [0 \quad 0 \ldots 0 \quad \lambda]$, $\lambda > 0$, $\underline{c} = [0 \quad 0 \ldots 0 \quad 1]$, and f is any monotone nonlinearity with $f(0) = 0$. Then the operator mapping u(t) into y(t) defined by

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u \\ z = \underline{c}\,\underline{x} \\ y = f(z) \end{cases}$$

is dissipative, with storage function $V(\underline{x}) = \frac{1}{\lambda}\sum_{i=1}^{n} F(x_i)$, where

$$F(z) = \int_0^z f(\sigma)\, d\sigma\,.$$

We shall give a proof of this Theorem in Chapter 2. A hyperdominant

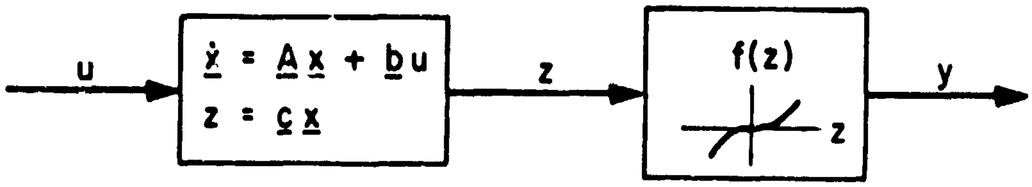


Fig. 1.43

matrix $\underline{M}$ is one for which $m_{ij} \leq 0$ whenever $i \neq j$, and $\sum_{i=1}^{n} m_{ij} \geq 0$ and $\sum_{j=1}^{n} m_{ij} \geq 0$ for all i,j.

For our fourth-order regulator we have

$$z^{-1}(s) = H(s) = \frac{s^2 + 1.45s + 0.5}{s^3 + 200s^2 + 96s + 4} \quad .$$

The standard controllable realization ([7] Chapter 17) is $H(s) = \underline{h}(\underline{I}s-\underline{F})^{-1}\underline{g}$

where

$$\underline{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -96 & -200 \end{bmatrix}, \quad \underline{g} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{h}\ [0.5 \quad 1.45 \quad 1] \quad .$$

Now let

$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & 1.45 & 1 \end{bmatrix} \quad \underline{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -1.45 & 1 \end{bmatrix}$$

so that we obtain the realization $H(s) = \underline{c}(\underline{I}s-\underline{A})^{-1}\underline{b}$ where

$$\underline{A} = \underline{P}\,\underline{F}\,\underline{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & -1.45 & 1 \\ 95.3 & 192.4 & -198.6 \end{bmatrix}, \quad \underline{b} = \underline{P}\,\underline{g} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{c} = \underline{h}\,\underline{P}^{-1} = [0 \quad 0 \quad 1] \quad .$$

Now $-\underline{A}$ is not hyperdominant, but if we choose $\underline{R}$ such that $\underline{R}\,\underline{b} = \underline{b}$ and $\underline{c}\,\underline{R}^{-1} = \underline{c}$ then $-\underline{R}\,\underline{A}\,\underline{R}^{-1}$ may be hyperdominant. Such an $\underline{R}$ is of the form

$$\underline{R} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \underline{R}^{-1} = \begin{bmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad ad - bc = 1$$

It seems to be an impossible task to find values for a, b, c, d which will make $-\underline{R}\,\underline{A}\,\underline{R}^{-1}$ hyperdominant, without the aid of a digital computer. In Chapter 2 we reconsider this question using tridiagonal realizations. We also show in Theorem 2.7 that it would even be helpful to find a, b, c, d which will make $-\underline{R}\,\underline{A}\,\underline{R}^{-1} = \underline{M}$ column dominant, i.e. $m_{jj} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n} |m_{ij}|$.

(f) Higher-Order Regulators

As a design procedure for sixth- and higher-order regulators it seems reasonable to assume that Aizerman's Conjecture holds true for the class of feedback systems obtained by using the kind of feedback control law described in §1.4 (a) and §1.5 (a). The chattering mode analysis of §1.4 (b) and §1.5 (c) shows that the numerator polynomial of the forward-path transfer function must be strict Hurwitz, but this requirement will be covered by the conditions obtained by using total gain linearization, (i.e. the system of Fig. 1.19 must be stable for all $0 < k < \infty$).

§1.6 Second-Order Regulator with Inductor Loss

In this section we consider the regulator of Fig. 1.5 with a resistance added in series with the inductor. We know in advance that this will not affect our conclusions about the stability of the second-order regulator, since "Dissipation aids stabilization". However, we wish to obtain a Lyapunov function for this case, in preparation for §1.7.

Fig. 1.44 shows the regulator under consideration, with unit values for source voltage, inductance, and capacitance. The series resistance has value r.

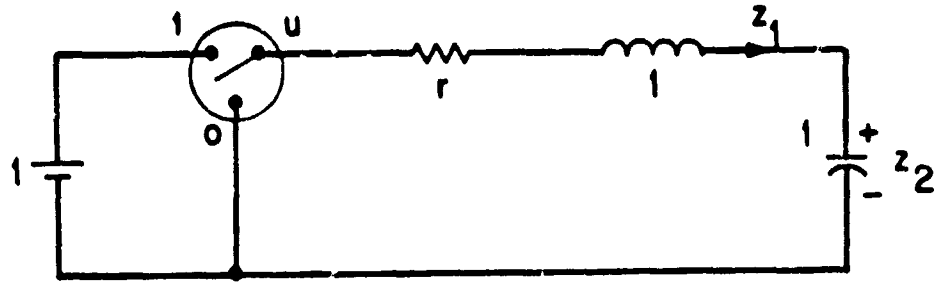


Fig. 1.44

Following the pattern of §1.4 and §1.5 we obtain the feedback representation of Fig. 1.45, which is seen to be of the form of Fig. 1.18. As in §1.4 (e)

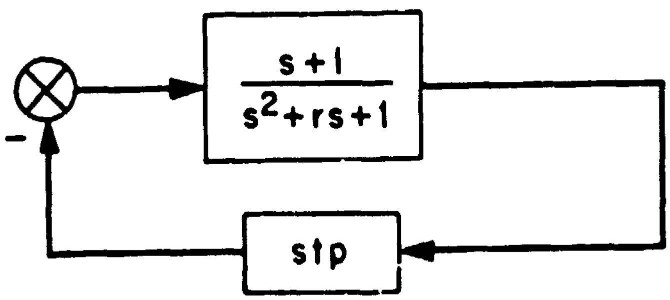


Fig. 1.45

we take $\beta=1$ for notational convenience. For $r \geq 1, G(s) = \frac{s+1}{s^2+rs+1}$ is a positive real function, so that no multiplier is necessary in such a case to prove stability using the Positive Operator Theorem. Fig. 1.46 depicts the Nyquist locus of G(s) when $r = \frac{1}{2}$, 1, and 2. However we do not make use of this positivity of G(s) for $r \geq 2$, because we want one storage function $V(\underline{x})$ for all r, for the purposes of §1.7. We must use the same

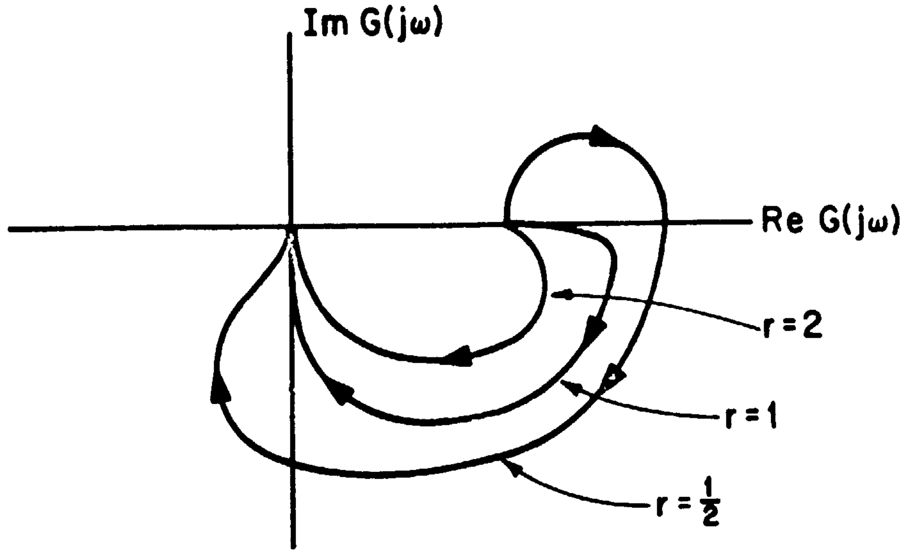


Fig. 1.46

multiplier as before, $Z(s) = s + 1$. Then $G(s)\ Z(s) = \frac{(s+1)^2}{s^2+rs+1}$, which is a positive real function for all $r \geq 0$. We have

$$G(s)\ Z(s) = 1 + \frac{(2-r)s}{s^2+rs+1} = \underline{c}(\underline{I}s-\underline{A})^{-1}\underline{b} + d$$

where

$$\underline{A} = \begin{bmatrix} -r & -1 \\ 1 & 0 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{c} = [2-r \quad 0],\ d = 1 \quad .$$

As in §1.4 (e) we therefore describe the system of Fig. 1.45 by

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u \\ y = \underline{c}\,\underline{x} + du \\ \dot{z} = -z + y \quad \text{with } z(0) = x_1(0) + x_2(0) \\ u = -\text{stp}\ z \end{cases} \quad .$$

In Chapter 2 we introduce the concept of a storage function and show that a storage function $V_1(\underline{x})$ for $(1 + \frac{2s}{s^2+1})$ is given by $\frac{1}{2}\,\underline{x}'\underline{K}\,\underline{x}$ where $\underline{K} = \underline{K}'$ satisfies the matrix equation

$$\underline{K}\,\underline{A} + \underline{A}'\underline{K} + (\underline{K}\,\underline{b} - \underline{c}')(2d)^{-1}(\underline{b}'\underline{K}-\underline{c}) = \underline{0} \quad .$$

If we let $\underline{K} = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$ then we obtain

$$\begin{cases} k_2^{\,2} - 4k_2 = 0 \\ k_2(k_1+r-2) + 2(k_3-rk_2-k_1) = 0 \\ (k_1+r-2)^2 + 4(k_2-rk_1) = 0 \end{cases}$$

from which we get the four possible solutions

$$\underline{K}_1 = \begin{bmatrix} r+2-2\sqrt{2r} & 0 \\ 0 & r+2-2\sqrt{2r} \end{bmatrix} \qquad \underline{K}_2 = \begin{bmatrix} r+2+2\sqrt{2r} & 0 \\ 0 & r+2+2\sqrt{2r} \end{bmatrix}$$

$$\underline{K}_3 = \begin{bmatrix} r+2-2\sqrt{2r-4} & 4 \\ 4 & r+2+2\sqrt{2r-4} \end{bmatrix} \quad \underline{K}_4 = \begin{bmatrix} r+2+2\sqrt{2r-4} & 4 \\ 4 & r+2-2\sqrt{2r-4} \end{bmatrix} .$$

We see that for $r < 2$ the solutions $\underline{K}_3$ and $\underline{K}_4$ are not real, so we consider only $\underline{K}_1$ and $\underline{K}_2$. Since $\underline{K}_1 < \underline{K}_2$ we see that $\frac{1}{2}\,\underline{x}'\underline{K}_1\underline{x}$ must be the available storage function, and $\frac{1}{2}\,\underline{x}'\underline{K}_2\underline{x}$ must be the required supply (see Chapter 2). Since the convex combination of these two storage functions is again a storage function, we know that $\frac{1}{2}\,\underline{x}'\underline{Q}\,\underline{x}$ is a storage function whenever $\underline{Q} = \eta\underline{K}_1 + (1-\eta)\underline{K}_2$ for some $0 \leq \eta \leq 1$. Thus, if $-1 \leq \gamma \leq 1$ we have the storage function

$$V_1(\underline{x}) = \frac{1}{2}\left(r+2+2\gamma\sqrt{2r}\right)(x_1^2+x_2^2) \quad .$$

To check that the operator mapping u into y is dissipative we evaluate

$$\dot{V} - uy = \tfrac{1}{2}(r+2+2\gamma\sqrt{2r})(x_1\dot{x}_1+x_2\dot{x}_2) - (2-r)x_1u - u^2$$

$$= -[(r+\gamma\sqrt{2r})x_1 - u]^2 - (1-\gamma^2)x_1{}^2 \text{ on substituting}$$

$$\leq 0 \text{ since } |\gamma| \leq 1.$$

We can now establish stability by noting that the storage function for the feedback operator is as it was in §1.4 (e), namely $V_2(\underline{x}) = \mathrm{Sod}(x_1+x_2)$. Thus if we describe the regulator of Figs. 1.44 and 1.45 by the equations

$$\begin{cases} \dot{x}_1 = -r\,x_1 - x_2 - \mathrm{stp}(x_1+x_2) \\ \dot{x}_2 = x_1 \end{cases}$$

we have the Lyapunov function

$$V(\underline{x}) = V_1(\underline{x}) + V_2(\underline{x}) = \tfrac{1}{2}(r+2+2\gamma\sqrt{2r})(x_1{}^2+x_2{}^2) + \mathrm{Sod}(x_1+x_2) \ .$$

Then

$$\dot{V}(\underline{x}) = (r+2+2\gamma\sqrt{2r})(x_1\dot{x}_1+x_2\dot{x}_2) + (\dot{x}_1+\dot{x}_2)\ \mathrm{stp}(x_1+x_2)$$

$$= -[(r+\gamma\sqrt{2r})x_1 + \mathrm{stp}(x_1+x_2)]^2 - 2r(1-\gamma^2)x_1{}^2 - \mathrm{Sod}(x_1+x_2).$$

By choosing $\gamma \neq \pm 1$ we have

$$\dot{V}(\underline{x}) \leq 0 \ , = 0 \text{ only at } x_1 = x_2 = 0 \ .$$

This allows us to conclude stability about $\underline{x} = \underline{0}$ using the following theorem of Yoshizawa [29], which we introduce here with a view to the time-varying situation of §1.8.

<u>Theorem 1.6</u>

If $V(\underline{x},t)$ is a Lyapunov function (as defined in §1.4 (e)) for the system of equations

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},t)$$

$$\underline{x}(t) \in \mathbb{R}^n \; ; \; 0 \leq t < \infty \; ; \; \underline{x}(0) = \underline{x}_0 \; ; \; \underline{f}(0,t) = \underline{0} \; ,$$

and if $V(\underline{x},t)$ is positive definite, decrescent, and radially unbounded, then the system is globally asymptotically stable about $\underline{x} = \underline{0}$, provided that $-W(\underline{x})$ is positive definite, where $\dot{V}(\underline{x},t) \leq W(\underline{x}) \leq 0$.

For a proof of Theorem 1.6 the reader is referred to [37].

1.7 Second-Order Regulator with Resistive Source Impedance

(a) Introductory

Inclusion of a resistance in series with the source voltage E of Fig. 1.1 leads to a much more difficult stability analysis. For the second-order regulator with source resistance, the network acts some of the time ($u = 0$) like that of Fig. 1.5, and for the rest of the time ($u = 1$) like that of Fig. 1.44. Hence, using the same control law we might expect to have stability, since both of the regulators of Figs. 1.5 and 1.44 are stable. Furthermore, stability is to be expected from the notion "Dissipation aids stabilization". In fact, we do find that these expectations are true; however, this not so easy to establish. For the second-order case we can fall back on a phase plane analysis, but for higher-order cases a Lyapunov approach seems the only way. We provide in part (c) here a Lyapunov analysis of the second-order case, which makes an elegant use of the ideas of dissipative systems. In part (b) we apply a total-gain linearization as an initial investigation of stability.

The second-order regulator is shown in Fig. 1.47.

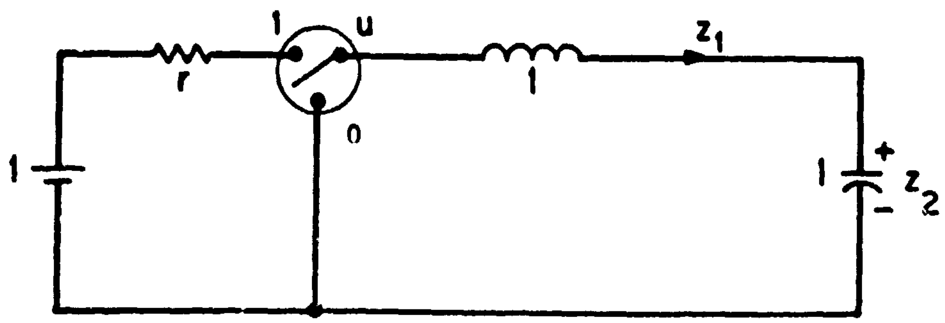


Fig. 1.47

The evolution equations are

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + u \begin{bmatrix} -r & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

i.e.

$$\dot{\underline{z}} = (\underline{A}_0 + u\underline{A}_1)\ \underline{z} + \underline{b}\ u\ .$$

We see that the control variable u enters the state evolution equations in a multiplicative way, as well as in the usual additive way. As before we let

$$u = \begin{cases} 1\ , & \dot z_2 + z_2 < \alpha \\ 0\ , & \dot z_2 + z_2 > \alpha \end{cases}$$

and $x_1 = 2z_1$, $x_2 = 2z_2 - 2\alpha$, which yields

$$\begin{cases} \dot x_1 = -x_2 - urx_1 + 2(u-\alpha) \\ \dot x_2 = x_1 \\ 2(u-\alpha) = -\mathrm{stp}_{(-2+2\alpha,2\alpha)}(x_1+x_2) \end{cases}$$

i.e.

$$\begin{cases} \dot{x}_1 = -2\alpha q x_1 - x_2 + (q x_1 - 1)\ \mathrm{stp}(x_1 + x_2) \\ \dot{x}_2 = x_1 \end{cases}$$

where $q = \frac{r}{2}$ for notational convenience. Now we can write this equation in two ways, the first way being

$$\ddot{x} + 2\alpha q \dot{x} + x = (q\dot{x} - 1)\ \mathrm{stp}(\dot{x} + x)$$

and the second way being

$$\ddot{x} + q[2\alpha - \mathrm{stp}(x + \dot{x})]\ \dot{x} + x = -\ \mathrm{stp}(\dot{x} + x)$$

i.e.

$$\ddot{x} + r\phi(\underline{x})\ \dot{x} + x = -\ \mathrm{stp}(\dot{x} + x)$$

where $\phi(\underline{x}) = \frac{1}{2}[2\alpha - \mathrm{stp}_{(-2+2\alpha,2\alpha)}(x_1 + x_2)]$. $\phi(\underline{x})$ takes on the values 0 and 1.

The first way leads us to the feedback system representation of Fig. 1.48. On comparing this with that of Fig. 1.45 we might hope that

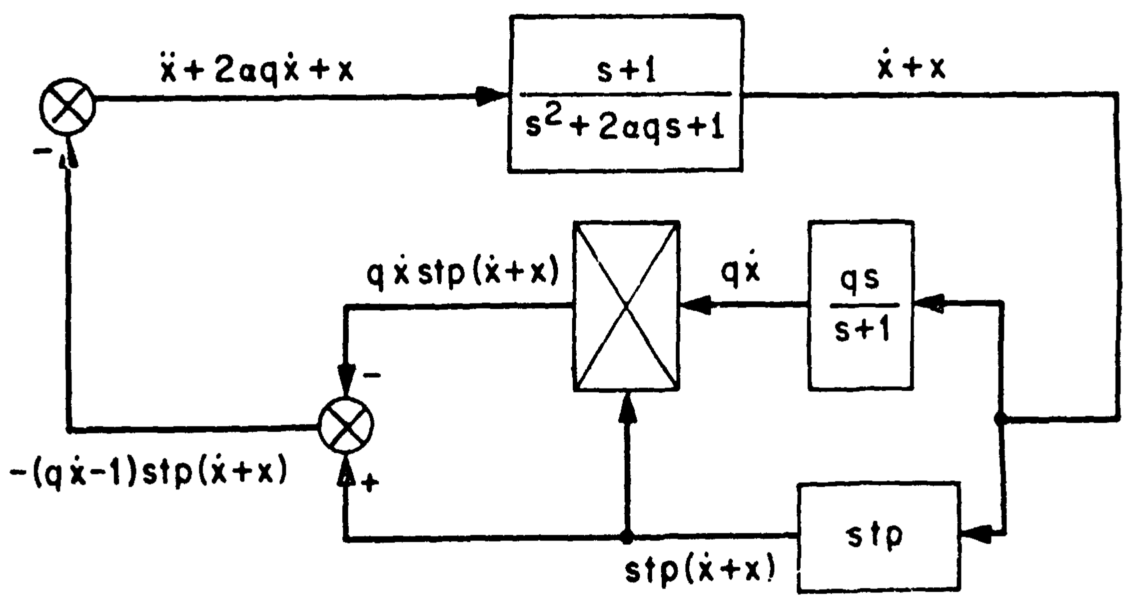


Fig. 1.48

the feedback operator in Fig. 1.48 is a positive operator. However, this is not the case. The representation of Fig. 1.48 does not seem to be helpful at all, so we look at the other way of describing this regulator. This second way in state-space form is

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -r\phi(\underline{x}) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ z = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ u = -\,\mathrm{stp}\; z \end{cases}$$

i.e.

$$\begin{cases} \dot{\underline{x}} = \underline{A}(\underline{x})\underline{x} + \underline{b}\, u \\ z = \underline{g}\, \underline{x} \\ u = -\mathrm{stp}\; z \end{cases}$$

Fig. 1.49 depicts this in feedback form:

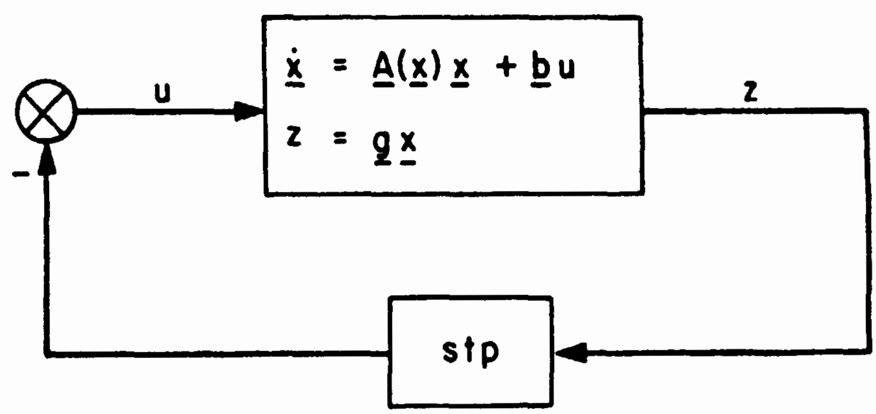


Fig. 1.49

(b) Stability by Total Gain Linearization

Following the total-gain linearization argument of §1.4 (c), and comparing Fig. 1.49 with Fig. 1.17, we expect stability for the system of Fig. 1.49 if it is stable when stp σ is replaced by $k\sigma$, for all $0 < k < \infty$. This gives us the cquations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(2\alpha q+k) & -(1+k) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} qkx_1(x_1+x_2) \\ 0 \end{bmatrix}$$

i.e.

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{f}(\underline{x})$$

For these we can use the following theorem:

Theorem 1.7

The system of equations

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}(t) + \underline{f}(\underline{x},t)$$

$$\underline{x}(t) \in \mathbb{R}^n \ ;\ 0 \leqslant t < \infty \ ;\ \underline{x}(0) = \underline{x}_0 \ ;\ \underline{f}(0,t) = \underline{0}$$

is locally asymptotically stable (i.e. if $||\underline{x}_0|| \leqslant M$ for some $0 < M < \infty$) if

$$\dot{\underline{x}}(t) = \underline{A}(t)\ \underline{x}(t) \ ,\ \underline{x}(0) = \underline{x}_0$$

is exponentially stable, and

$$\lim_{||\underline{x}|| \to 0} \frac{||\underline{f}(\underline{x},t)||}{||\underline{x}||} = 0 \ , \text{ uniformly in t.}$$

By $||\underline{x}||$ we mean the usual Euclidean norm $\sqrt{\underline{x}'\underline{x}}$; exponential stability means that $||\underline{x}(t)|| \leqslant ae^{-bt}$ for some $a,b > 0$; and by

$$\lim_{x \to 0} g(x,y) = 0 \quad \text{uniformly in y}$$

we mean that for all $\eta > 0$ there is a $\delta > 0$ which is independent of y, such that $|g(x,y)| < \eta$ whenever $|x| < \delta$. For a proof of Theorem 1.7 the reader is referred to Chapter 4 of Bellman [4].

Now we have $\det(\underline{I}s-\underline{A}) = s^2 + (2\alpha q+k)s + (1+k)$, thus $\underline{A}$ has its eigenvalues in the left half plane, so $\dot{\underline{x}} = \underline{A}\,\underline{x}$ is exponentially stable. And

$$\frac{||\underline{f}(\underline{x})||}{||\underline{x}||} = \frac{|qkx_1(x_1+x_2)|}{\sqrt{x_1^2+x_2^2}} \to 0 \text{ as } ||x|| \to 0, \text{ for all } 0 < k < \infty .$$

Thus we have local asymptotic stability for the system when stp σ is replaced by $k\sigma$.

(c) Stability by Lyapunov's Method

In attempting to find a suitable Lyapunov function we can start with that of §1.4 (e) and modify it by trial and error. Here let us take $\alpha = \frac{1}{2}$ for convenience, so that stp becomes sgn. In §1.4 (e) we had $V(\underline{x}) = \frac{1}{2}(x^2+\dot{x}^2) + \frac{1}{2}|\dot{x}+x|$. One possible approach here is to add to this a term in $|x|$. So let us try

$$V(\underline{x}) = \frac{1}{2}(x^2+\dot{x}^2) + a|x+\dot{x}| + b|x| \quad \text{where } a,b > 0.$$

Then

$$\dot{V} = x\dot{x} + \dot{x}\ddot{x} + a(\dot{x}+\ddot{x})\,\mathrm{sgn}(x+\dot{x}) + b\dot{x}\,\mathrm{sgn}\,x$$

$$= -q\dot{x}^2[1-\mathrm{sgn}(x+\dot{x})] + [(a-1)\dot{x}-ax]\,\mathrm{sgn}(x+\dot{x}) - a\,\mathrm{sgn}^2(x+\dot{x})$$

$$+ aq\dot{x}[\mathrm{sgn}^2(x+\dot{x}) - \mathrm{sgn}(x+\dot{x})] + b\dot{x}\,\mathrm{sgn}\,x .$$

Now if we take $a = \frac{1}{2}$ and $b = 2\alpha q$ we have

$$\dot{V} = -q\dot{x}^2[1-\mathrm{sgn}(x+\dot{x})] - \frac{1}{2}|x+\dot{x}| - \frac{1}{2}\,\mathrm{sgn}^2(x+\dot{x}) + \frac{1}{2}q\dot{x}[\mathrm{sgn}^2(x+\dot{x})-\mathrm{sgn}(x+\dot{x})+2\mathrm{sgn}x]$$

$$\leq W = -\frac{1}{2}|x+\dot{x}| + \frac{1}{2}\,q\dot{x}[1-\mathrm{sgn}(x+\dot{x}) + 2\,\mathrm{sgn}\,x] \quad \text{when } x + \dot{x} \neq 0.$$

By looking at the four cases given by $x \gtrless 0$, $\dot{x} \gtrless 0$, we find $W \leq 0$ (and hence $\dot{V} \leq 0$) for all $x, \dot{x}$ if and only if $q < \frac{1}{2}$. Thus we have the desired global asymptotic stability, using Theorem 1.6, provided that $r < 1$. This makes us wonder whether we might obtain instability if r is large enough.

Further study of the above V shows that we can improve on this bound on r. For, collecting terms in a different way we obtain

$$\dot{V} = -q\dot{x}^2[1-\mathrm{sgn}(x+\dot{x})] + [(a-aq-1)\dot{x}-ax]\ \mathrm{sgn}(x+\dot{x}) - a\ \mathrm{sgn}^2(x+\dot{x})$$
$$+ aq\dot{x}\ \mathrm{sgn}^2(x+\dot{x}) + b\dot{x}\ \mathrm{sgn}\ x\ .$$

Now if we let $b = aq$ and $a-aq-1 = -a$, i.e., $a = \frac{1}{2-q}$, (which requires $q < 2$ for $a > 0$), we get

$$\dot{V} = -q\dot{x}^2[1-\mathrm{sgn}(x+\dot{x})] - a|x+\dot{x}| - a\ \mathrm{sgn}^2(x+\dot{x}) + aq\dot{x}[\mathrm{sgn}^2(x+\dot{x}) + \mathrm{sgn}\ x]$$
$$\leq W = -a|x+\dot{x}| + aq\dot{x}[\mathrm{sgn}^2(x+\dot{x}) + \mathrm{sgn}\ x]\ .$$

Looking at the four cases $x \gtrless 0$, $\dot{x} \gtrless 0$, we now find that $W \leq 0$ (and hence $\dot{V} \leq 0$) for all $x, \dot{x}$ if and only if $q < 1$, so by Theorem 1.6 we now have global asymptotic stability provided $r < 2$.

We can do better than this, however, using the ideas of Dissipative Systems. We make use of the fact that for a dissipative system the convex combination of V_a and V_r is also a storage function, where V_a is the available storage and V_r is the required supply (see Chapter 2). Now in §1.6 we described the introduction of the multiplier $(s+1)$ by a nonminimal state-space representation. If we follow that method here we obtain the nonminimal state-space representation:

$$\left\{\begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -r\phi(\underline{x}) & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ \\ y = [2-r\phi(\underline{x}) \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \\ \\ \dot{z} = -z + y \qquad \text{with } z(0) = x_1(0) + x_2(0) \\ \\ u = -\,\text{stp}\ z \end{array}\right.$$

i.e.

$$\left\{\begin{array}{l} \dot{\underline{x}} = \underline{A}(\underline{x})\ \underline{x} + \underline{b}\ u \\ \\ y = \underline{c}(\underline{x})\ \underline{x} + du \\ \\ \dot{z} = -z + y \qquad \text{with } z(0) = x_1(0) + x_2(0) \\ \\ u = -\,\text{stp}\ z \quad . \end{array}\right.$$

We depict this in Fig. 1.50, which is to Fig. 1.49 as Fig. 1.36 is to Fig. 1.35. We wish to show that the forward path operator mapping u into y

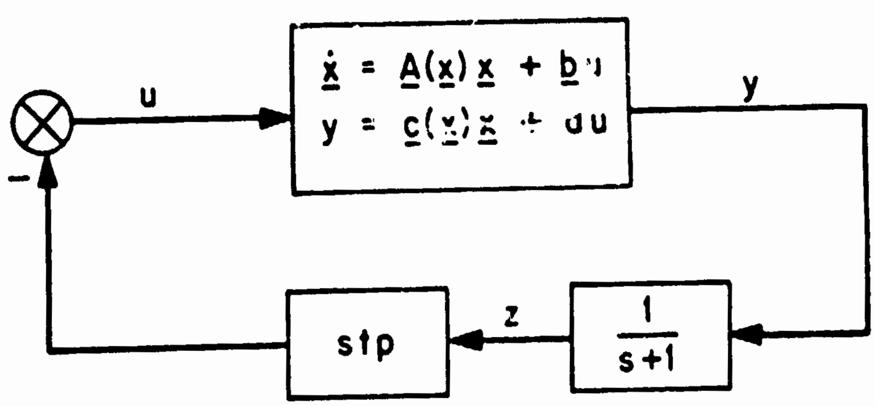


Fig. 1.50

is a dissipative operator. Let us try the storage function we obtained in §1.6, i.e.

$$V(\underset{\sim}{x}) = \tfrac{1}{2}(r+2+2\gamma\sqrt{2r})(x_1{}^2+x_2{}^2) \quad \text{where } -1 \leq \gamma \leq 1.$$

Then

$$\begin{aligned}
\dot{V} - uy &= (r+2+2\gamma\sqrt{2r})(x_1\dot{x}_1+x_2\dot{x}_2) - uy \\
&= -r\phi(r+2+2\gamma\sqrt{2r})\, x_1{}^2 + [r(1+\phi) + 2\gamma\sqrt{2r}]\, x_1 u - u^2 \\
&= \begin{cases} -[(r+\gamma\sqrt{2r})\, x_1 - u]^2 - (1-\gamma^2)\, x_1{}^2 & \text{when } \phi = 1 \\ (r+2\gamma\sqrt{2r})\, x_1 u - u^2 & \text{when } \phi = 0 \end{cases}
\end{aligned}$$

We will therefore always have $\dot{V} - uy \leq 0$ if we can choose γ so that $(r+2\gamma\sqrt{2r}) = 0$, i.e., $\gamma = -\sqrt{\frac{r}{8}}$. Since $\gamma \geq -1$ this means that our operator is dissipative for all $r \leq 8$. The storage function for the forward-path operator is therefore $V_1 = (x_1{}^2+x_2{}^2)$, and for the feedback operator of Fig. 1.50 we know the storage function is $V_2 = \text{Sod } z = \text{Sod}(x_1+x_2)$. Thus the Lyapunov function to try is $V = (x_1{}^2+x_2{}^2) + \text{Sod}(x_1+x_2)$, which, admittedly, is the one we started with. Indeed, since

$$\ddot{x} + r\phi(\underline{x})\,\dot{x} + x = -\,\text{stp}(\dot{x}+x)$$

we have

$$\begin{aligned}
\dot{V} &= 2\,\dot{x}\,\ddot{x} + 2\,x\,\dot{x} + (\ddot{x}+\dot{x})\,\text{stp}(\dot{x}+x) \\
&= -2r\phi\dot{x}^2 - [(r\phi+1)\,\dot{x} + x]\,\text{stp}(\dot{x}+x) - \text{stp}^2(\dot{x}+x) \\
&= -2r\phi\dot{x}^2 - r\phi\dot{x}\,\text{stp}(\dot{x}+x) - \text{stp}^2(\dot{x}+x) - \text{Sod}(\dot{x}+x) \\
&= -\,[\tfrac{r\phi}{2}\,\dot{x} + \text{stp}(\dot{x}+x)]^2 - \tfrac{r\phi}{4}(8-r\phi)\,\dot{x}^2 - \text{Sod}(\dot{x}+x) \\
&\leq 0 \text{ for all } x,\dot{x} \text{ if } r \leq 8, \text{ since } \phi \text{ is 0 or 1.}
\end{aligned}$$

When $\phi = 0$, $\dot{V} = 0 \Longrightarrow \dot{x} + x = 0$, so we need to use Theorem 1.3 to conclude stability, which we have since $\dot{x} + x = 0$ is not a trajectory of the system.

We have not been able to obtain a Lyapunov function which gives stability for all $r \geqslant 0$. Perhaps the next step in this direction is to try to find such a V by looking directly at the definition of a storage function.

(d) Stability by Phase Plane Analysis

Fig. 1.51 shows a set of phase-plane trajectories for the case $\alpha = \frac{1}{2}$, $r = 10$. I.e., the trajectories of Fig. 1.51 are the solutions of

$$\ddot{x} + 10\left[\frac{1 - \operatorname{sgn}(\dot{x}+x)}{2}\right]\dot{x} + x = -\operatorname{sgn}(\dot{x}+x) \ .$$

In order to obtain Fig. 1.51 we need to know the shape of the trajectories for

$$\ddot{x} + r\dot{x} + x = 0 \ .$$

These are shown in Fig. 1.52, for $r = 10$. For a discussion of phase-plane trajectories of this type, the reader is referred to Chapter 7 of reference [11] and Chapter 3 of reference [37].

From Fig. 1.51 we conclude that all trajectories will reach the chattering region of the switching line. To determine whether the resulting chattering motion is stable we need to apply the methods developed in §1.4 (b). Our system obeys

$$\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x},u) = \underline{A}(\underline{x})\ \underline{x} + \underline{b}\ u \\ u = -\operatorname{stp}_{(-a,b)}\ \underline{c}\ \underline{x} \end{cases}$$

where

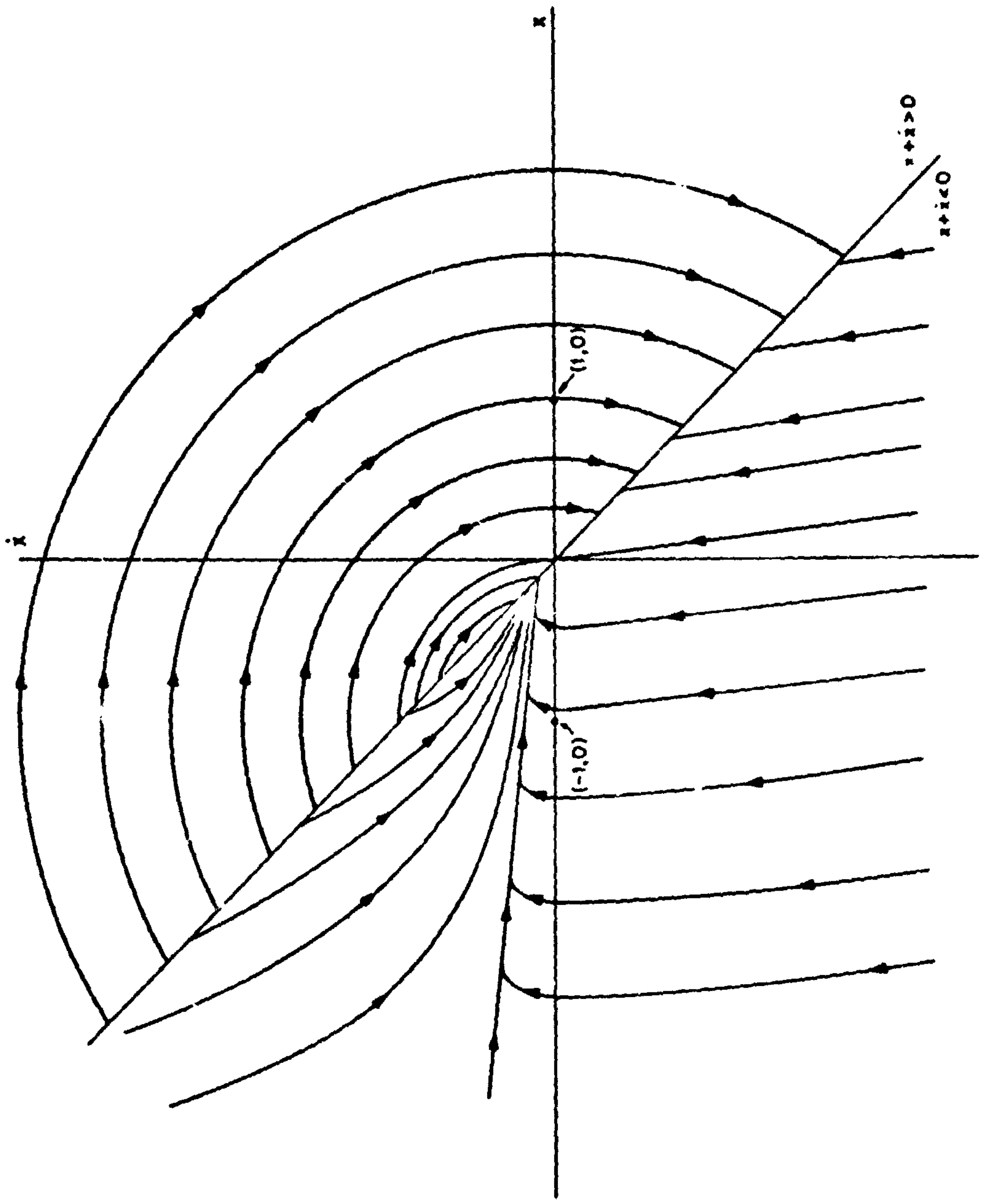


Fig. 1.51

Trajectories of $\ddot{x}+5\dot{x}+x = (5\dot{x}-1)\ \mathrm{sgn}(\dot{x}+x)$

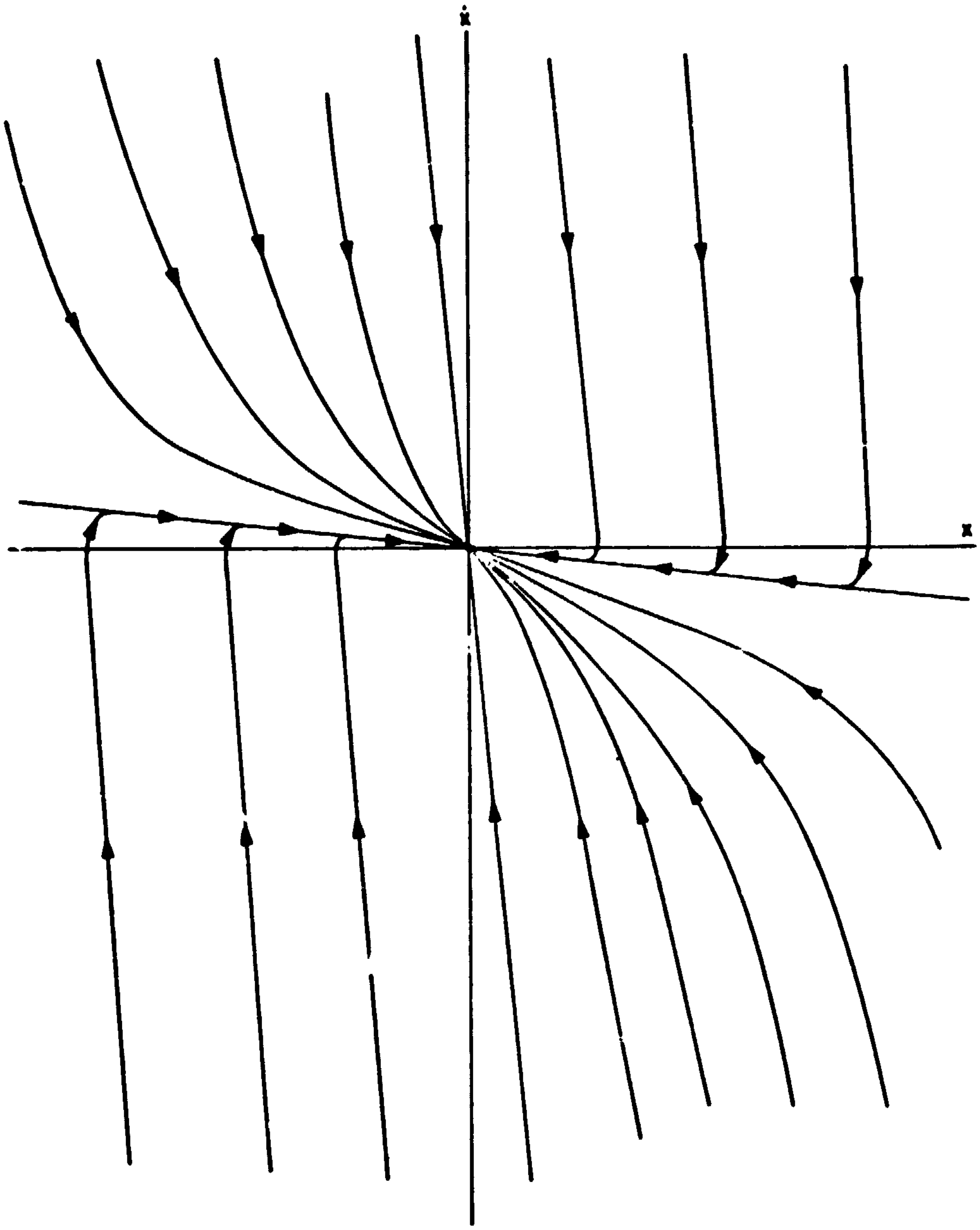


Fig. 1.52

Trajectories of $\ddot{x}+10\dot{x}+x = 0$

$$\underline{A}(\underline{x}) = \begin{bmatrix} -r\phi(\underline{x}) & -1 \\ 1 & 0 \end{bmatrix} \qquad \phi(\underline{x}) = \frac{1}{2}[2\alpha - \underset{(-2+2\alpha,2\alpha)}{\mathrm{stp}} (\beta x_1 + x_2)]$$

$$\underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \underline{c} = [\beta \quad 1] \quad a = 2 - 2\alpha \quad b = 2\alpha .$$

Let

$$\underline{A}_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \underline{A}_1 = \begin{bmatrix} -r & -1 \\ 1 & 0 \end{bmatrix} .$$

Then

$$\underline{f}^+ = \underline{A}_0\,\underline{x} - b\,\underline{b} \quad \text{and} \quad \underline{f}^- = \underline{A}_1\,\underline{x} + a\,\underline{b} \quad .$$

The switching line S is $\underline{c}\,\underline{x} = 0$, i.e. $\beta x_1 + x_2 = 0$. The first endpoint condition is $\underline{c}\,\underline{f}^+ < 0$, which on substitution becomes

$$x_2 > \frac{-2\alpha\beta^2}{\beta^2+1} \quad .$$

The second endpoint condition is $\underline{c}\,\underline{f}^- < 0$, which becomes

$$\begin{cases} x_2 < \dfrac{(2-2\alpha)\ \beta^2}{\beta+1-r\beta} & \text{if } r < \dfrac{\beta+1}{\beta} \ , \\[2ex] x_2 > \dfrac{-(2-2\alpha)\ \beta^2}{r\beta-\beta-1} & \text{if } r > \dfrac{\beta+1}{\beta} \ . \end{cases}$$

Thus, for $\alpha = \frac{1}{2}$, $\beta = 1$, $r > 2$, as in Fig. 1.51, we have chattering behavior on the half line

$$x_2 > \max(-\tfrac{1}{2}, \tfrac{-1}{r-2}) \ .$$

Motion along the chattering line obeys

$$\dot{\underline{x}} = \underline{f}^+ - (\underline{f}^- - \underline{f}^+)\ \frac{\underline{c}\,\underline{f}^+}{\underline{c}(\underline{f}^- - \underline{f}^+)}$$

which, on substituting, becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -(\tfrac{1}{\beta}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ with } \beta x_1 + x_2 = 0.$$

This is the same result as for §1.4 (b). We therefore have the desired global asymptotic stability, for all $\beta > 0$ and for all $r \geqslant 0$.

§1.8 Further Refinements

(a) Preregulation

The source impedance shown in Fig. 1.47 is the simplest kind. It does not increase the dimension of the state equations. In practice the source impedance may well be like that depicted in Fig. 1.53. The capacitor C provides a form of preregulation, that is, a smoothing of any

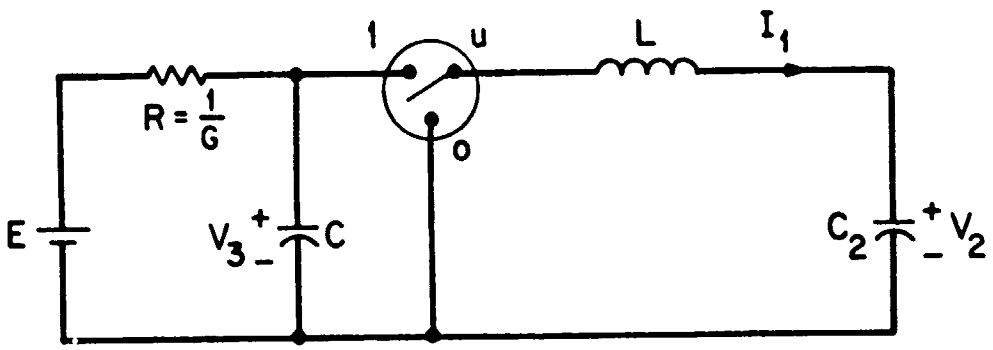


Fig. 1.53

time-variations in the source voltage. In that case the evolution equations are

$$\begin{cases} L_1\dot{I}_1 = -V_2 + uV_3 \\ C_2\dot{V}_2 = I_1 \\ C_3\dot{V}_3 = G(E-V_3) - uI_1 \end{cases} .$$

Letting $z_1 = I_1\sqrt{L_1}$, $z_2 = V_2\sqrt{C_2}$, $x_3 = I_3\sqrt{C_3}$, $\omega_2 = \dfrac{1}{\sqrt{L_1C_2}}$, $\omega_3 = \dfrac{1}{\sqrt{L_1C_3}}$, $b = \dfrac{GE}{\sqrt{C_3}}$ $g = \dfrac{G}{C_3}$, we obtain

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & -\omega_2 & 0 \\ \omega_2 & 0 & 0 \\ 0 & 0 & -g \end{bmatrix} + \begin{bmatrix} 0 & 0 & \omega_3 \\ 0 & 0 & 0 \\ -\omega_3 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} .$$

Suppose, for example, that $\omega_2 = \omega_3 = b = 1$, and that we want stability about $z_2 = \alpha$. Then, letting $x_1 = 2z_1$, $x_2 = 2z_2 - 2\alpha$, $x_3 = 2z_3 - 2$, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & u \\ 1 & 0 & 0 \\ -u & 0 & -g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2u-2\alpha \\ 0 \\ 0 \end{bmatrix} .$$

Let $2u - 2\alpha = -\mathrm{stp}_{(-2+2\alpha,2\alpha)}(\beta x_1+x_2)$, as before. Then we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & \phi(\underline{x}) \\ 1 & 0 & 0 \\ -\phi(\underline{x}) & 0 & -g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} \mathrm{stp}(\beta x_1+x_2) \\ 0 \\ 0 \end{bmatrix}$$

where

$$\phi(\underline{x}) = \frac{1}{2}[2\alpha - \mathrm{stp}_{(-2+2\alpha,2\alpha)}(\beta x_1+x_2)] .$$

We now wish to prove that this defines a motion which is asymptotically stable about $\underline{x} = \underline{0}$. As yet, we have not done this. Probably the desired stability can be obtained by extending the methods of §1.7 (b) and (c), that is, to view these equations in the form of a system like that of Fig. 1.49. Certainly it is to be expected that this system will be stable. An even more difficult stability question will be provided by replacing the resistor of Fig. 1.53 with an inductor.

(b) Time-Varying Source Voltage

So far we have only considered a fixed source voltage, and have shown that the output voltage will settle to its desired value after a change in input voltage or a change in the load resistance. In practice the source voltage will be time-varying, between limits. Usually this variation will be periodic, as for instance in rectification applications. In terms of the regulator of Fig. 1.5, this means that we have

$$E(t) = E_0 + E_1(t), \quad \text{where } E_0 > 0$$
$$\text{and } E_1(t) \geqslant 0 \text{ for all } t.$$

This modifies the evolution equations of §1.4 (a) for z_1 and z_2 to be

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1+e(t) \\ 0 \end{bmatrix} u$$

where $e(t) \geqslant 0$ for all t. Using the same feedback law we obtain, for x_1 and x_2

$$\begin{cases} \dot{x}_1 = -x_2 - \underset{(-2+2\alpha,2\alpha)}{\mathrm{stp}}(\beta x_1+x_2) + e(t)[2\alpha - \underset{(-2+2\alpha,2\alpha)}{\mathrm{stp}}(\beta x_1+x_2)] \\ \dot{x}_2 = x_1 \end{cases}$$

i.e.

$$\left\{ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} f(\underline{c}\ \underline{x},t) \\ 0 \end{bmatrix} \right.$$

where

$$f(\underline{c}\ \underline{x},t) = \begin{cases} -2(1+e(t)) + 2\alpha\ , & \beta x_1 + x_2 < 0 \\ 2\alpha & , \beta x_1 + x_2 > 0\ . \end{cases}$$

We can think of $f(\underline{c}\ \underline{x},t)$ as $\mathrm{stp}_{(-a(t),b)}(\underline{c}\ \underline{x})$. The system of Fig. 1.17 now becomes that of Fig. 1.54,

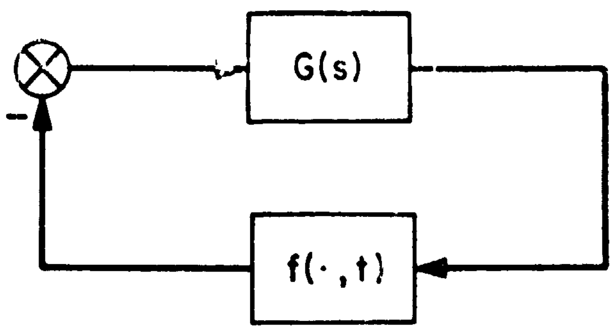


Fig. 1.54

where $G(s) = \frac{\beta s+1}{s^2+1}$. Now $f(\cdot,t)$ is a positive operator. If we have a resistance $r \geqslant \frac{1}{\beta}$ in series with the inductor, then $G(s) = \frac{\beta s+1}{s^2+rs+1}$ is a positive real function, and so by the Positive Operator Theorem we will have the required stability. As yet we have not proven stability for more general cases.

§1.9 Practical Considerations

We now discuss briefly some of the practical aspects of implementing the control laws we have been considering. Fig. 1.55 shows a schematic for one possible implementation of the type of regulator we have been discussing. The two-position control switch is effected by means of the return-path diodes D_1 and D_2, the power transistor T_1, and the driver transistor T_2. The two resistors marked R_1 are used to obtain a measure of half of the output voltage V_0. This and half the desired output voltage of αE are fed

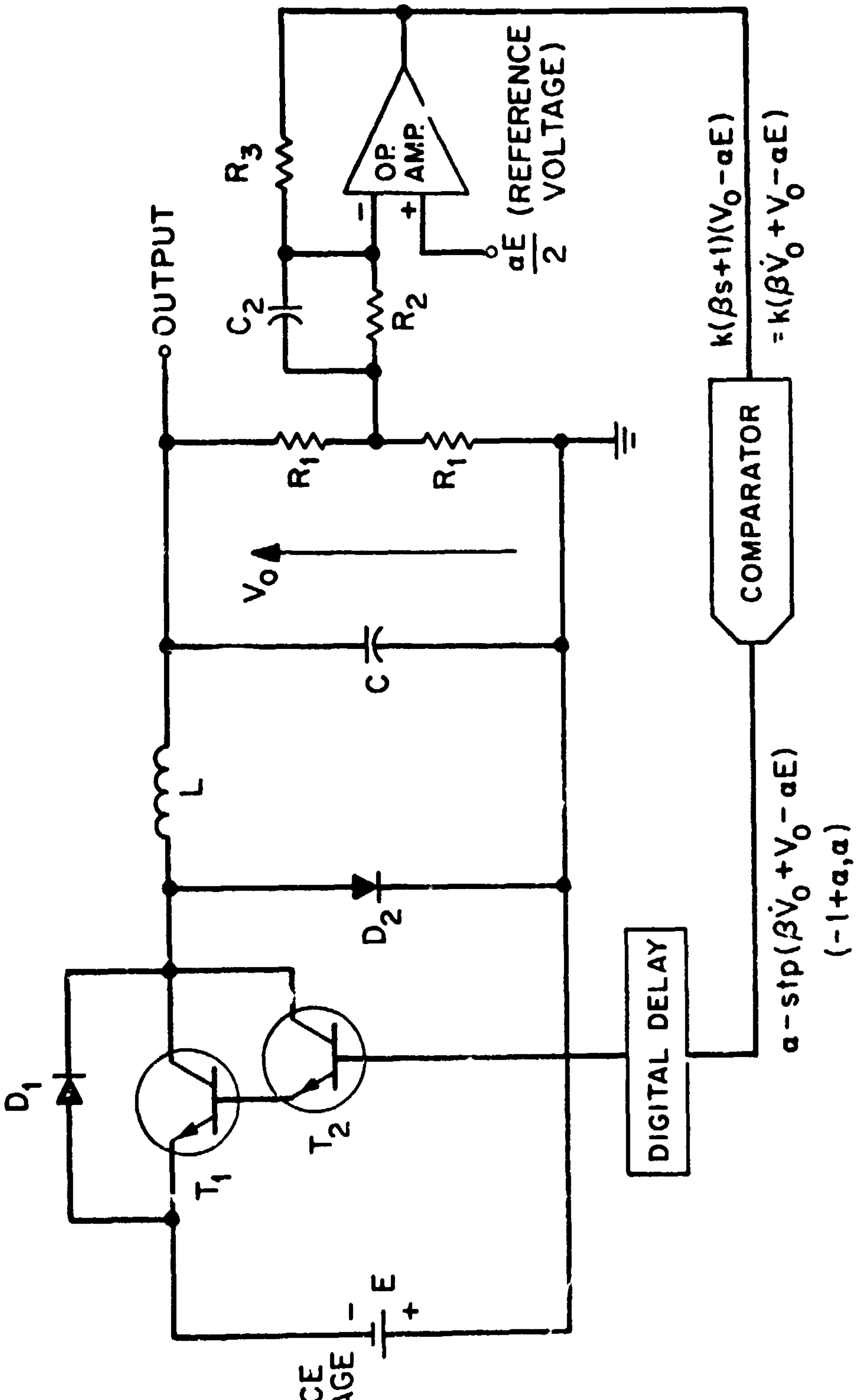


Fig. 1.55

Switching Voltage Regulator Schematic

into the operational amplifier network which gives $k(\beta s+1)(V_0-\alpha E)$, i.e. $k(\beta\dot{V}_0+V_0-\alpha E)$, where $\beta = R_2C_2$ and $k = \frac{R_3}{2R_2}$. The comparator gives a digital (0 or 1) output depending on whether the output of the operational amplifier is positive or negative. This digital signal controls the driver transistors through the digital delay, the purpose of which is to limit the switching frequency of the transistors, so that the overall efficiency remains high. The parameter β should be chosen to give a good transient response to changes in load. The easy extension of this method of implementation to the fourth-order case should be clear. Note that when a load resistance is present we are more concerned with $\dot{V}_0$ than with I_L, so there is no need to measure inductor current.

In practice the operational amplifier will involve a lag term in the measure of $(V_0-\alpha E)$, so that the voltage at its output is more accurately given by $k(\frac{1+\beta s}{1+\delta s})(V_0-\alpha E)$, where $\delta \ll \beta$. Alternatively, it may be necessary to introduce this lag $\frac{1}{1+\delta s}$ intentionally by placing a capacitor C_3 in parallel with R_3, where $C_3R_3 = \delta$, so that the operational amplifier network does not become too receptive to high frequency noise.

We must therefore re-examine the system depicted in Fig. 1.17, and ascertain whether this is stable when the forward-path transfer function is generalized to be of the form $G(s) = \frac{(\beta s+1)}{(s^2+1)(\delta s+1)}$. A short calculation shows that the associated linear feedback system of Fig. 1.19 is stable for all $k > 0$ provided $\delta \leq \beta$. We therefore know that a positive real multiplier $Z(s)$ exists such that $G(s)\,Z(s)$ is positive real. The first multiplier to try is the simplest, i.e., $Z(s) = (1+\gamma s)$, as used in §1.4 (d) and §1.6. We find

$$G(s)\ Z(s) = \frac{(\beta s+1)(\gamma s+1)}{(s^2+1)(\delta s+1)}$$

$$\text{Re}\ G(j\omega)\ Z(j\omega) = \frac{1 - (\beta\gamma-\delta\beta-\delta\gamma)\omega^2}{(1-\omega^2)(1+\delta^2\omega^2)} \quad ,$$

so Re $G(j\omega)\ Z(j\omega) > 0$ for all ω if and only if $\beta\gamma-\delta\beta-\delta\gamma = 1$, i.e. $\gamma = \frac{1+\delta\beta}{\beta-\delta}$; and $\gamma > 0$ if and only if $\delta < \beta$. We thus find that the same type of multiplier as considered for the simple second order case can be used here to prove stability. The reason for this is, essentially, that in that case we had Re $G(j\omega)\ Z(j\omega) = 1$ for all ω , i.e. we had some "room to spare". The important conclusion here is that since the lag term (($\delta s+1$) in the denominator of $G(s)$) will always be present in practice, some form of phase advance (($\beta s+1$) in the numerator of $G(s)$) is necessary for stability, with $\beta > \delta$. Indeed, an analog computer simulation shows that if β is reduced below δ, an oscillation will occur, at a frequency of the order of $\frac{1}{2\pi\sqrt{LC}}$.

References [25] and [14] are recent publications summarizing state-of-the-art techniques used in practice in the design of solid-state power supplies. Both give actual design examples of second-order switching voltage regulators which are similar to that considered here; ([25], Vol. 2, p. 165; [14], p. 196). The circuit given in [25] is of the type shown in Fig. 1.56, which will be seen to be very similar to that shown in Fig. 1.55. Frequency limiting is effected by the hysteresis in the Schmitt trigger, the circuit of which is shown in detail in Fig. 1.57. The Schmitt trigger can be approximated (very roughly, but adequately for our purposes) as being equivalent to a linear system with transfer function ($\beta s+1$),

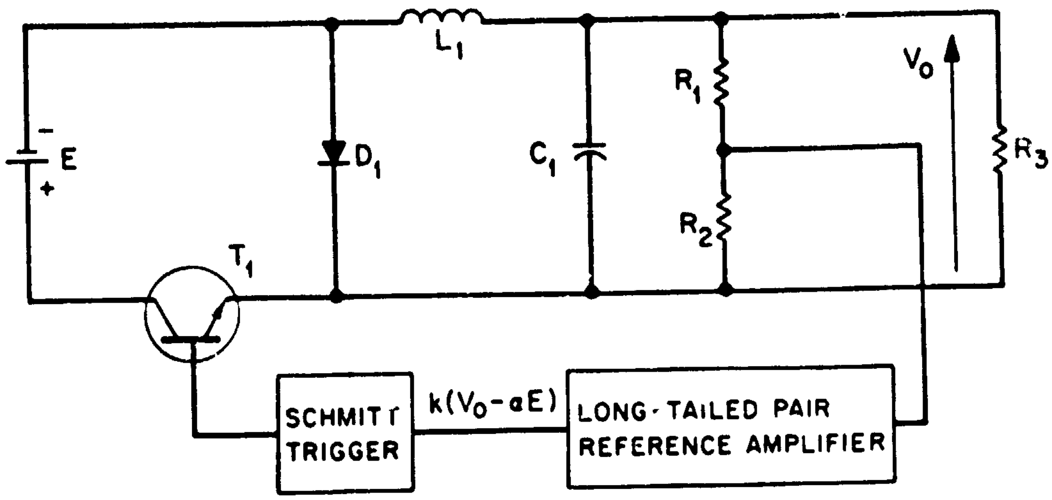


Fig. 1.56

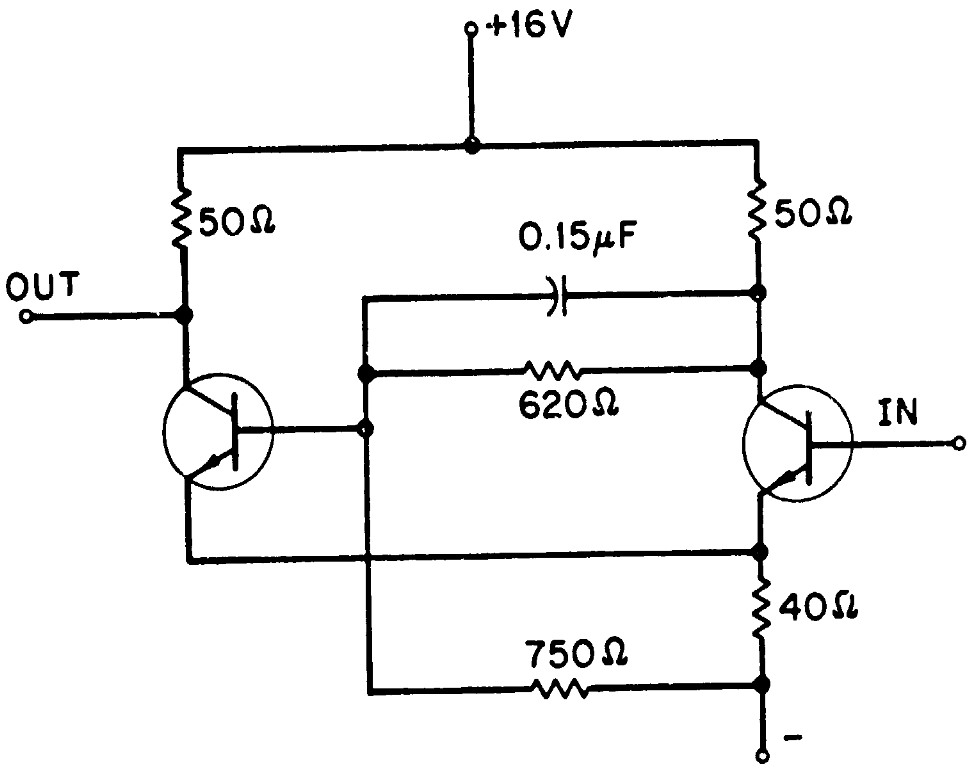


Fig. 1.57

followed by a hysteresis device with two-state output. The important point is that since our analysis has shown that β must be nonzero, we see that the 0.15 μF "commutating" capacitor in the Schmitt trigger circuit plays a crucial role in determining stability and ensuring a reasonable transient response, since it is responsible for the phase advance term (βs+1). Both authors [25], [14] give considerable attention to other less important design features, but do not even mention stability considerations, let alone explain the presence of this capacitor. Furthermore, neither author considers the possibility of using a fourth- or higher-order filter, which we have shown to be just as easy to implement and stabilize, while having superior design advantages.

Reference [15] also considers a second-order switching regulator, in which the Schmitt trigger is considered to have no dynamics, i.e. it cannot introduce a stabilizing phase advance factor (βs+1). The regulators considered in [15] are assumed to have a small resistance in series with the output filter capacitor. This will introduce the desired phase advance term in the forward-path transfer function. We might call this an "accidental" stabilization, which is another example of dissipation aiding stabilization. Clearly there is no control of the transient response in this design.

In conclusion we note that all of the various methods of stability analysis which we have considered in Chapter 1 are needed for a full understanding of systems of the type we have been considering.

CHAPTER 2

POSITIVE OPERATORS AND DISSIPATIVE SYSTEMS

§ 2.1 Introduction

The intent of this chapter is to outline the relevant background material for Chapter 1. In § 2.2 we follow the development of reference [34] in providing a simple proof of the Positive Operator Theorem, which we believe is a useful theorem, though so far has been applied relatively little. For more extensive and rigorous treatments, the reader is referred to [34] and [42]. In § 2.3 we discuss concepts of Dissipative System Theory following the development in the recent two-part paper by Willems, [35], [36]. In § 2.4 we address the problem of obtaining state-space realizations for transfer functions of the O'Shea type, as introduced in § 1.5(d) and (e), Theorems 1.4 and 1.5.

§ 2.2 Positive Operators

(a) Operators, and functions of time

We consider functions of time on the interval $0 \leq t < \infty$. The functions will all be real-valued, though our statements and theorems are easily generalized to the case of vector-valued functions of time. An operator F maps a function of time $x(t)$ into another function of time $y(t)$; we write $y = Fx$. Usually we think of these two functions x and y as the input and the output, and of the operator as an input-output system. An operator may be specified by a characteristic graph, a convolution integral, a transfer function, or other means. We assume that all operators considered are causal, i.e. present and past values of the outputs do not depend on future values of the inputs. If F_1 and F_2 are operators and α is a real number then the operators $F_1 + F_2$ and αF_1 are defined by $(F_1 + F_2)x = F_1x + F_2x$ and $(\alpha F_1)x = \alpha(F_1x)$ respectively. The operator F_2F_1 is called the composition of F_2 with F_1 and is defined by $(F_2F_1)x = F_2(F_1x)$. In general $F_2F_1 \neq F_1F_2$. If $F_2F_1 = F_1F_2$ then we say that F_1 and F_2 commute. An operator F is linear if $F(\alpha_1x_1 + \alpha_2x_2) = \alpha_1Fx_1 + \alpha_2Fx_2$ holds for all x_1, x_2. The identity operator I is defined by $1x = x$, and the zero operator 0 is defined by $0x = 0$. An operator F is invertible if there is another operator F^{-1} such that $FF^{-1} = F^{-1}F = 1$. $(F_2F_1)^{-1} = F_1^{-1}F_2^{-1}$. An operator of importance to us is the truncation operator P_T, (more usually called a projection operator), defined by

$$(P_Tx)(t) = \begin{cases} x(t) & \text{for } 0 \leq t \leq T < \infty \\ 0 & \text{otherwise} . \end{cases}$$

An operator F is causal if P_TF commutes with P_T for all T, i.e. $P_TFP_T = P_TF$ for all T. (Note that $P_T^2 = P_T$). This is equivalent to requiring that $P_Tx_1 = P_Tx_2 \Rightarrow P_TFx_1 = P_TFx_2$ for all T, (provided $F0 = 0$).

We assume that the reader is familiar with the concept of a vector space. A vector space V is called a normed vector space if a map, called the norm and denoted by $|| \ ||$, from V into the real numbers $\mathbb{R}$ is defined on V, such that for all $x, y \in V$ and $\alpha \in R$,

(1) $\|x\| \geq 0$, $= 0$ if and only if $x = 0$

(2) $\|\alpha x\| = |\alpha| \; \|x\|$

(3) $\|x+y\| \leq \|x\| + \|y\|$ (the triangle inequality).

A real inner product space is a vector space V with a map, denoted by $<,>$ and called the inner product, from V x V into $\mathbb{R}$ such that for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{R}$

(1) $<x,y> = <y,x>$

(2) $<\alpha x+by,z> = \alpha<x,z> + \beta<y,z>$

(3) $<x,x> \geq 0$, $= 0$ if and only if $x = 0$.

An inner product space is a normed vector space, with $\|x\| = \sqrt{<x,x>}$. The Cauchy-Schwartz inequality states that

$$|<x,y>| < \|x\| \cdot \|y\| .$$

The inner-product space which we consider is the space of real-valued square-integrable functions of time, $L_2[0,\infty]$. A function $x(t)$ is in L_2 if $\int_0^\infty x^2 \, dt < \infty$. For $x, y \in L_2$ the inner product is defined by $<x,y> = \int_0^\infty xy \, dt$, so that $\|x\| = \sqrt{\int_0^\infty x^2 \, dt}$.

There are many functions of interest to us which are not in L_2, for example the constant functions, or functions such as t and e^t. In order to be able to handle these we introduce the extended space L_{2e}, which consists of all those functions $x(t)$ for which $P_T x \in L_2$ for all finite T. This includes all functions $y(t)$ for which $t < \infty \Rightarrow |y(t)| < \infty$.

We assume henceforth that all operators considered satisfy $F0 = 0$. The operator F is said to be bounded if

$$\sup_{\substack{x \in L_2 \\ x \neq 0}} \frac{||Fx||}{||x||} < \infty .$$

This supremum will be called the gain of F, and denoted $||F||$. It is easy to show that $||F_2 F_1|| \leq ||F_2|| \cdot ||F_1||$, that $||P_T x|| \leq ||x||$ for all T, and that $||P_T F|| \leq ||F||$ for all T. F is said to be Lipschitz continuous if

$$\sup_{\substack{x,y \in L_2 \\ x \neq y}} \frac{||Fx-Fy||}{||x-y||} < \infty .$$

This supremum will be called the Lipschitz constant of F, and denoted $||F||_\Delta$; it satisfies the inequalities just given for $||F||$.

(b) Positive Operators

Let $F: L_2 \to L_2$, i.e. F maps L_2 into L_2. Then F is said to be positive on L_2 if for all $x \in L_2$,

$$\langle x, Fx \rangle \geq 0 ,$$

i.e. $\int_0^\infty xy \, dt \geq 0$ where $y = Fx$.

C-2

F is said to be <u>strictly positive</u> on L_2 if $F - \eta I$ is positive on L_2 for some $\eta > 0$, i.e. $\langle x,Fx\rangle \geqslant \eta||x||^2$.

Suppose now that $F:L_{2e} \to L_{2e}$. Then F is said to be positive on L_{2e} if for all $x \in L_{2e}$ and all $T < \infty$, $\langle P_T x, P_T Fx\rangle \geqslant 0$, i.e. $\int_0^T xy\,dt \geqslant 0$ for all T, where $y = Fx$. F is strictly positive on L_{2e} if $F - \eta I$ is positive on L_{2e} for some $\eta > 0$. The relationship between positivity of an operator on L_2 and L_{2e} is simple:

<u>Lemma</u> If $F:L_{2e} \to L_{2e}$ and $F:L_2 \to L_2$, then F is positive on L_{2e} if and only if it is positive on L_2.

<u>Proof</u> Since F is causal, for all $x \in L_{2e}$ we have

$$\langle P_T x, P_T Fx\rangle = \langle P_T x, P_T F P_T x\rangle = \langle P_T x, F P_T x\rangle ,$$

which shows that positivity on L_2 implies positivity on L_{2e}. Now assume that F is positive on L_{2e}, but that for some $x \in L_2$, $\langle x,Fx\rangle < 0$. Since $\lim_{T\to\infty} P_T x = x$, this implies by continuity of $\langle , \rangle$ that for some T, $\langle P_T x, Fx\rangle = \langle P_T x, P_T Fx\rangle < 0$, which yields the contradiction. Q.E.D.

Note that the lemma still holds if "positive" is replaced by "strictly positive".

If F_1 and F_2 are positive and α_1 and α_2 are nonnegative real numbers, then $\alpha_1 F_1 + \alpha_2 F_2$ is positive. If F^{-1} exists and F is positive then F^{-1} is also positive, (since if $y = Fx$ then $\langle F^{-1}y,y\rangle = \langle x,Fx\rangle \geqslant 0$). Note that the inequality $\int_0^T xy\,dt \geqslant 0$ can be viewed as a statement about the correlation between the functions of time x and y, a point of view which is utilized in the proof of the O'Shea theorem (Theorem 1.4), [12].

Examples of positive operators are (i) A positive linear gain, $(Fx)(t) = kx(t)$ where $k \geqslant 0$, (ii) A first- and third-quadrant nonlinearity, $(Fx)(t) = f(x(t))$ where $\sigma f(\sigma) \geqslant 0$, e.g. the functions stp σ, sgn σ, sat k σ, defined in Chapter 1, (iii) A time-varying nonlinearity, $(Fx)(t) = f(x(t),t)$ where $\sigma f(\sigma,t) \geqslant 0$, as for example in § 1.8. All of these examples are memoryless operators; that is, present values of the outputs depend only on present values of the inputs. An operator which is not memoryless is said to be dynamic. An important class of dynamic operators is the convolution operators, i.e. those for which $Fx = y$ where $y(t) = \int_0^t g(t-\tau)x(\tau)d\tau$. Usually these are defined by a rational transfer function $G(s)$, which is a function of the complex frequency variable s, representing a differential equation relating $y(t)$ and $x(t)$, and given by $G(s) = \int_0^\infty e^{-st}g(t)dt$. Such an operator will be positive on L_{2e} if and only if $G(s)$ is a positive real function, as discussed at the beginning of § 1.4(d). This requires that $\text{Re}\, G(j\omega) \geqslant 0$ for all ω, and strict positivity requires that $\text{Re}\, G(j\omega) \geqslant \eta > 0$ for all ω, for some $\eta > 0$.

To indicate why this is true we make use of Parseval's equality of Fourier transform theory, ([34] section 1.3, [29]). We extend our time interval of definition to $(-\infty,\infty)$ by letting $x(t) = 0$ for $-\infty<t<0$. Now if $x(t) \in L_2(-\infty,\infty)$ let $X(j\omega) = \lim_{T\to\infty} \int_{-T}^{T} x(t)e^{-j\omega t}dt$. Then $X(j\omega) \in L_2(-\infty,\infty)$, and $X(j\omega)$ is called the limit-in-the-mean Fourier transform of $x(t)$. We have the inverse transform $x(t) = \frac{1}{2\pi} \lim_{W \to \infty} \int_{-W}^{W} X(j\omega) e^{j\omega t} d\omega$. Parseval's equality states that $\int_{-\infty}^{\infty} u(t)\bar{y}(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega)\bar{Y}(j\omega)d\omega$, where the overbar denotes complex conjugation.

Suppose now that $y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$, so that

$$Y(j\omega) = G(j\omega)U(j\omega).$$

Then
$$\begin{aligned}
\int_{-\infty}^{\infty} u(t)y(t)\,dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega)\bar{Y}(j\omega)\,d\omega \qquad \text{by Parseval} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega)\bar{G}(j\omega)\bar{U}(j\omega)\,d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |U(j\omega)|^2 \bar{G}(j\omega)\,d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |U(j\omega)|^2 \mathrm{Re}G(j\omega)\,d\omega \text{ since } u(t) \text{ and } \\
&\qquad\qquad y(t) \text{ are real.}
\end{aligned}$$

From this it follows that $\int_{-\infty}^{\infty} uy\,dt \geq 0$ for all $u \in L_2[0,\infty)$ if and only if $\mathrm{Re}\, G(j\omega) \geq 0$ for all ω.

Finally, another important class of positive operators are those obtained by the composition of a monotone nonlinearity F with a transfer function $G(s)$. For a specific nonlinearity F, such as $f(\sigma) = \sigma^3$, it is not known at present how to determine conditions on $G(s)$ for FG to be a positive operator. However, if we require FG to be positive for any monotone nonlinearity f with $f(0) = 0$, then the theorem of O'Shea (our Theorem 1.4) provides the answer.

(c) The Positive Operator Theorem

Consider the feedback system of Fig. 2.1.

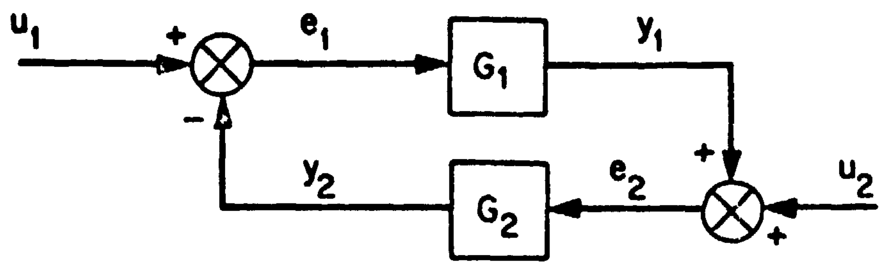


Fig. 2.1

The governing equations for this system are

$$\begin{cases} e_1 = u_1 - y_2 \\ e_2 = u_2 + y_1 \\ y_1 = G_1 e_1 \\ y_2 = G_2 e_2 \end{cases} .$$

We assume that the operators G_1 and G_2 satisfy $G_i\ 0 = 0$, $(i = 1, 2)$. We call u_1, u_2 the inputs; e_1, e_2 the errors; and y_1, y_2 the outputs. The inputs may represent driving functions, driving noise, or initial condition responses. Assume that for any $u_1, u_2 \in L_{2e}$, solutions e_1, e_2, y_1, y_2 in L_{2e} exist for the above equations, and depend on u_1, u_2 in a causal way; this assumption is called well-posedness.

We would like to know when the feedback system is stable in the sense that bounded inputs yield bounded errors and outputs.

<u>Definition</u> The feedback system of Fig. 2.1 is <u>finite-gain stable</u> if any inputs u_1, $u_2 \in L_2$ yield e_1, e_2, y_1, $y_2 \in L_2$, and there exist constants ρ_1, $\rho_2 < \infty$ such that for any u_1, $u_2 \in L_2$

$$||e_1||,\ ||e_2||,\ ||y_1||,\ ||y_2|| \quad \leq \rho_1\,||u_1|| + \rho_2\,||u_2|| \;.$$

<u>Theorem 2.1</u> ([34], Chapter 4). Suppose that G_2 is bounded and either G_1 is bounded or $||P_TG_2x|| \geq \eta||P_Tx||$ for all $x \in L_{2e}$, for some $\eta > 0$. Then the system of Fig. 2.1 is finite-gain stable if and only if $||(I + G_2G_1)^{-1}|| < \infty$ on L_2.

The theorem holds if the roles of G_2 and G_1 are reversed.

<u>Proof</u> Since $e_1 = u_1 - G_2e_2 = u_1 - G_2(G_1e_1+u_2)$, $(I+G_2G_1)e_1 = u_1 + G_2G_1e_1 - G_2(G_1e_1+u_2)$, so $(I+G_2G_1)^{-1}$ exists on L_{2e} and is causal, by well-posedness. Let u_1, $u_2 \in L_2$ be given. Then

$$\begin{aligned}
||P_T(I+G_2G_1)e_1|| &= ||P_Tu_1 + P_TG_2G_1e_1 - P_TG_2(G_1e_1+u_2)|| \\
&\leq ||P_Tu_1|| + ||P_TG_2P_TG_1e_1 - P_TG_2P_T(G_1e_1+u_2)|| \\
&\leq ||P_Tu_1|| + ||P_TG_2||_\Delta\, ||P_TG_1e_1 - P_T(G_1e_1+u_2)|| \\
&\leq ||u_1|| + ||G_2|| \cdot ||u_2|| \;.
\end{aligned}$$

Now $$e_1 = (I+G_2G_1)^{-1}(I+G_2G_1)e_1$$

So $$\begin{aligned}
P_Te_1 &= P_T(I+G_2G_1)^{-1}(I+G_2G_1)e_1 \\
&= P_T(I+G_2G_1)^{-1}P_T(I+G_2G_1)e_1 \text{ since } (I+G_2G_1)^{-1} \text{ is causal.}
\end{aligned}$$

$$\begin{aligned}\text{Thus}\quad ||P_Te_1|| &= ||P_T(I+G_2G_1)^{-1}.\ P_T(I+G_2G_1)e_1|| \\ &\leq ||P_T(I+G_2G_1)^{-1}||.\ \ ||P_T(I+G_2G_1)e_1|| \\ &\leq ||(I+G_2G_1)^{-1}||.\ \ ||P_T(I+G_2G_1)e_1|| \ .\end{aligned}$$

Therefore, $||P_Te_1|| \leq ||(I+G_2G_1)^{-1}||.||u_1|| + ||(I+G_2G_1)^{-1}||.||G_2||.||u_2||$, for all T. Thus $e_1 \in L_2$, and the finite-gain condition is satisfied. Since $y_2 = u_1-e_1$, y_2 also satisfies these conditions. Now if $||G_1|| < \infty$, we see that y_1. e_2 also satisfy these conditions. Alternatively, if $\eta||P_Te_2|| \leq ||P_Ty_2||$ then $e_2 \in L_2$ with $||e_2|| \leq \eta^{-1}||y_2||$, and $y_1 \in L_2$ also, since $y_1 = e_2 - u_2$. The necessity of $||(I+G_2G_1)^{-1}|| < \infty$ for finite-gain stability follows from

$$e_1 = (I+G_2G_1)^{-1}u_1 \ , \text{ if we take } u_2 = 0. \qquad \text{QED}$$

Theorem 2.2. Positive Operator Theorem ([42],[36] Chapter 4). The system of Fig. 2.1 is finite-gain stable if G_1 and G_2 are positive on L_{2e}, and one of them is strictly positive and Lipschitz continuous on L_2.

Proof. Suppose G_2 is strictly positive and Lipschitz continuous. We shall show that the conditions of Theorem 2.1 are met. For any $x \in L_{2e}$ we have

$$\begin{aligned}\eta||P_Tx||^2 &\leq \langle P_TG_2x, P_Tx\rangle \text{ by strict positivity of } G_2 \\ &\leq ||P_TG_2x||.\ \ ||P_Tx|| \text{ by the Cauchy-Schwartz inequality.}\end{aligned}$$

Thus, $||P_TG_2x|| \geq \eta||P_Tx||$ for all T. We therefore only need to show that $||I+G_2G_1)^{-1}|| < \infty$ on L_2. So let $x \in L_2$ and suppose $(I+G_2G_1)y = x$. Then $y \in L_{2e}$ by assumption, and $P_Ty + P_TG_2G_1y = P_Tx$.

Now $< P_T y + P_T G_2 G_1 y, P_T G_1 y > = < P_T G_1 y, P_T y > + < P_T G_2 G_1 y, P_T G_1 y >$

$$\geq 0 + \eta ||P_T G_1 y||^2$$

since G_1 is positive and G_2 is strictly positive.

Thus $\eta ||P_T G_1 y||^2 \leq < P_T x, P_T G_1 y > \leq ||P_T x|| . \; ||P_T G_1 y||$ by the Cauchy-Schwartz inequality,

so $||P_T G_1 y|| \leq \eta^{-1} ||P_T x|| \leq \eta^{-1} ||x||$.

Therefore, $G_1 y \in L_2$ and $||G_1 y|| < \eta^{-1} ||x||$. Furthermore,

$||G_2 G_1 y|| \leq ||G_2|| . ||G_1 y|| \leq \eta^{-1} ||G_2|| . \; ||x||$. Now $y = -G_2 G_1 y + x$,

so $y \in L_2$ and
$$\begin{aligned} ||y|| &= ||-G_2 G_1 y + x|| \\ &\leq ||G_2 G_1 y|| + ||x|| \\ &\leq \eta^{-1} ||G_2|| . ||x|| + ||x|| . \end{aligned}$$

Thus, since $y = (I+G_2G_1)^{-1} x$, we have $||(I+G_2G_1)^{-1}|| \leq \eta^{-1}||G_2|| + 1 < \infty$, and the conditions of Theorem 2.1 are satisfied. QED.

It is interesting to note that if in addition G_1^{-1} exists, then $G_1(I+G_2G_1)^{-1}$ turns out to be itself strictly positive and bounded on L_2 ([34], p. 39). Since $y_1 = G_1(I+G_2G_1)^{-1}u_1$ when $u_2 = 0$, this says that the closed-loop system itself is a positive operator. We can interpret this in terms of passive electrical networks, [42]: Let G_1 be the driving-point impedance of a passive two-terminal network, and let G_2 be a passive driving-point admittance. Then $G_1(I+G_2G_1)^{-1} = G_1G_1^{-1}(G_1^{-1}+G_2)^{-1} = (G_1^{-1}+G_2)^{-1}$ is the driving-point impedance of the series connection, as shown in Fig. 2.2, and this is passive if G_1 and G_2 are passive.

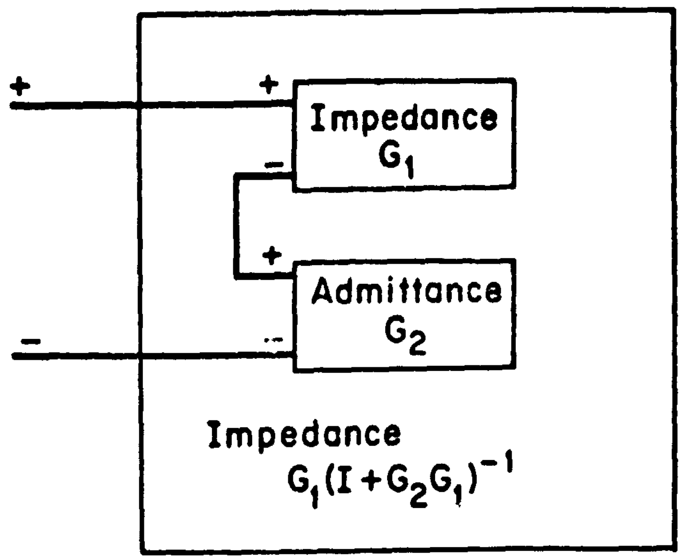


Fig. 2.2

It is also interesting to note that the requirements of Theorem 2.2 can be altered to strict positivity of G_1 and G_2, but no boundedness.

§ 2.3 Dissipative Systems

In Chapter 1, § 1.4(e), § 1.5(e), § 1.6, and § 1.7(c) we introduced and made use of the idea of a dissipative system. Dissipative systems are of interest in engineering and physics; typical examples are passive electrical networks in which part of the energy is dissipated in resistors as heat, viscoelastic systems where energy is lost through viscoelastic friction, and thermodynamic systems for which the second law of thermodynamics postulates a form of dissipation leading to an increase in entropy. We use the term dissipative as a generalization of the concept of passivity, and the term storage function as a generalization of the concept of stored energy or entropy.

Let G be an operator, or system, with input $\underline{u}(t)$ and output $\underline{y}(t)$, $0 \leq t < \infty$, defined by the equations

$$\begin{cases} \underline{\dot{x}}(t) = \underline{f}(\underline{x},\underline{u},t) \\ \underline{y} = \underline{g}(\underline{x}) \\ \underline{x}(0) = \underline{x}_0 \end{cases} .$$

The vector $\underline{x}(t)$ is called the state vector for G; the space X to which $\underline{x}(t)$ belongs is called the state space for G. Let $w(\underline{u},\underline{y})$ be a real-valued function of u(t) and y(t). Then G is said to be dissipative with respect to the supply-rate $w(\underline{u},\underline{y})$ if there exists a nonnegative function $V(\underline{x})$ with $V(\underline{0}) = 0$ such that

$$\dot{V} - w \leq 0 .$$

V is called the storage function for G. The inequality $\dot{V} - w \leq 0$ is called the dissipation inequality, and it is easy to show (using for instance Theorems 6.10 and 6.15 of reference [27] that this is equivalent to the inequality

$$V(\underline{x}_0) + \int_{t_0}^{t_1} w(t)\, dt \geq V(\underline{x}_1)$$

where $\underline{x}(t_0) = \underline{x}_0$, $\underline{x}(t_1) = \underline{x}_1$.

Now we introduce a quantity called the available storage; it is a generalization of the concept of "available energy" or "recoverable work". For the network of Fig. 1.34 it represents the maximum possible energy available at the terminals, starting from some given initial condition. The available storage V_a of the system G in state $\underline{x}_0$ is defined by

$$V_a(\underline{x}_0) = \sup_{\substack{t_1 \geq 0 \\ \underline{x}(0) = \underline{x}_0}} - \int_0^{t_1} w(t)\, dt$$

The supremum is taken over all motions starting in state $\underline{x}_0$ at time 0, and all possible $\underline{u}(t)$. The available storage is an important function in determining whether or not a system is dissipative, as is shown by the following theorem:

Theorem 2.3. [35] The available storage V_a is finite for all $\underline{x}_0$ if and only if G is dissipative. Moreover, $0 \leq V_a \leq V$ for any storage function V, and V_a itself satisfies the dissipation inequality.

The reader is referred to [35] for a proof.

The state space X of the system G is said to be reachable from $\underline{x}_0 \in X$ if for any $x_1 \in X$ there is an input function $u_1(t)$ which will transfer the state of G from $\underline{x}_0$ to $\underline{x}_1$. We shall assume henceforth that all storage functions V have the property $V(\underline{0}) = 0$, i.e. $\underline{0}$ is the point of minimum storage for G in X.

Next we introduce another quantity, called the required supply; this is the minimum amount of supply which must be delivered to the system in order to transfer it from its state of minimum storage (the zero state) to some other given state. The required supply V_r for the state $\underline{x}_1$ of the system G is defined by

$$V_r(\underline{x}_1) = \inf_{\substack{t_1 \geq 0 \\ \underline{x}(0)=\underline{0} \\ \underline{x}(t_1)=\underline{x}_1}} \int_0^{t_1} uy\, dt \quad .$$

The infimum is taken over all possible motions starting in state $\underline{x}(0) = \underline{0}$ and terminating at time t_1 in state $\underline{x}_1$. We now have:

Theorem 2.4 [35] Let G be dissipative with storage function V for which $V(\underline{0}) = 0$. Then $0 \leq V_a \leq V \leq V_r$. Moreover, if the state space X of G is reachable from $\underline{0}$ then $V_r < \infty$ and V_r is itself a possible storage function.

For a proof of this theorem the reader is again referred to [35]; we show that V_r satisfies the dissipation inequality in the proof of Theorem 2.5.

From Theorem 2.4 we see that the storage function V of a dissipative system always satisfies the inequality $V_a \leq V \leq V_r$, i.e. a dissipative system can only supply to the outside a fraction of what it has stored and can store only a fraction of what has been supplied. V_a and V_r themselves always satisfy the dissipation inequality, and hence are storage functions. However, not every function $V_1(\underline{x})$ which satisfies $V_a \leq V_1 \leq V_r$ will be a storage function. It appears to be difficult to state other general properties of the set of possible storage functions, except for its convexity: If V_1 and V_2 are storage functions, then s[illegible]s $\alpha V_1 + (1-\alpha)V_2$ for any $0 \leq \alpha \leq 1$. This follows immediately from [illegible]issipation inequality. In particular, if the state space is reachable from $\underline{0}$ then $\alpha V_a + (1-\alpha)V_r$ is a storage function for any $0 \leq \alpha \leq 1$.

As a consequence of the normalization $V(\underline{0}) = 0$ we obtain the following expected relationship between positive operators and dissipative systems:

Theorem 2.5 Let an operator G with input u(t) and output y(t) be defined by

$$\begin{cases} \underline{x} = \underline{f}(\underline{x},u,t) \\ y = g(\underline{x}) \\ 0 \leq t < \infty \\ \underline{x}(0) = \underline{x}_0,\ \underline{f}(\underline{0}) = \underline{0},\ g(\underline{0}) = 0 \end{cases}$$

and assume that the state space of G is reachable from $\underline{0}$. Then when $\underline{x}_0 = \underline{0}$ G is a positive operator if and only if G is dissipative with respect to the supplv rate uy with a storage function $V(\underline{x}) \geq 0$ for which $V(\underline{0}) = 0$.

Proof. If G is dissipative then when $\underline{x}_0 = \underline{0}$ we have

$$\int_0^T uy\,dt \geq \int_0^T \dot{V}\,dt = V(\underline{x}(T)) - V(\underline{x}(0))$$
$$= V(\underline{x}(T)) \geq 0,$$

and thus G is a positive operator. Suppose now that G is a positive operator whenever $\underline{x}_0 = \underline{0}$. Then

$$V(\underline{x}_1) = \inf_{\substack{t_1 \geq 0 \\ \underline{x}(0)=\underline{0} \\ \underline{x}(t_1)=\underline{x}_1}} \int_0^{t_1} uy\,dt \quad \text{which is the required supply.}$$

Then $V(\underline{x}_1) \geq 0$ since $\int_0^{t_1} uy\,dt \geq 0$, and $V(\underline{0}) = 0$ by taking $t_1 = 0$.

To show that $\dot{V} - uy \leq 0$ we observe that

$$V(\underline{x}_0) + \int_{t_0}^{t_1} uy\,dt \geq V(\underline{x}_1) \qquad \text{since}$$

$$\inf_{\substack{t_0 \geq 0 \\ t_0 \leq t_1 \\ \underline{x}(0)=\underline{0} \\ \underline{x}(t_0)=\underline{x}_0}} \int_0^{t_0} uy\,dt + \int_{\substack{t_0 \\ \underline{x}(t_0)=\underline{x}_0 \\ \underline{x}(t_1)=\underline{x}_1}}^{t_1} uy\,dt \geq \inf_{\substack{t_1 \geq 0 \\ \underline{x}(0)=\underline{0} \\ \underline{x}(t_1)=\underline{x}_1}} \int_0^{t_1} uy\,dt \quad .$$

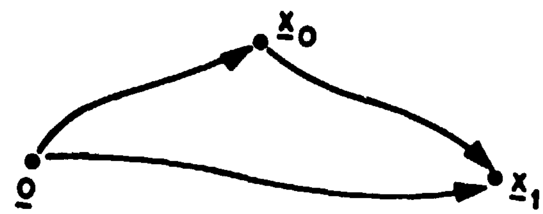


Fig. 2.3

Fig. 2.3 illustrates this last inequality, which we see is precisely the dissipation inequality. QED.

Note that we could also make use of the available storage to define V, rather than the required supply. The advantage of the positive operator concept is that it is an input-output concept which does not involve introduction of state space notions. However, in this approach we are restricted to considering operators which map 0 into 0, which though not a serious restriction, does in fact mean that we know something about the internal properties of such an operator, i.e. it must start from a state of minimum internal storage; the dissipative property of a system is independent of its initial condition. In applications both viewpoints are essential. For instance, in Chapter 1 we saw that for the second-order systems a dissipative system characterization of operators was superior since it led to Lyapunov functions, while for the fourth-order systems we had to fall back on the positive-operator methods.

If we write $d = \dot{V} - w$ then d is called the dissipation rate. When $d \equiv 0$ the system G is called lossless. An example would be the driving-point impedance of a network containing only inductors and capacitors. Energy storage and retrieval is 100% efficient for such a system. In this case the dissipation inequality becomes an equality, and the storage function is defined uniquely by $V_a = V = V_r = \int w\,dt$.

It may happen that $V_a = V_r$ for a system which is not lossless; such a system is called <u>quasi-lossless</u> since it can be transferred between states with arbitrarily small dissipation if the input is suitably chosen. All first-order systems (such as that of Fig. 1.32) are quasi-lossless. As an example consider the system of Fig. 2.4, where $f(\sigma)$ is a function with $f(0) = 0$ and $\sigma f(\sigma) > 0$. Let $F(x) = \int_0^x f(\sigma)\,d\sigma$.

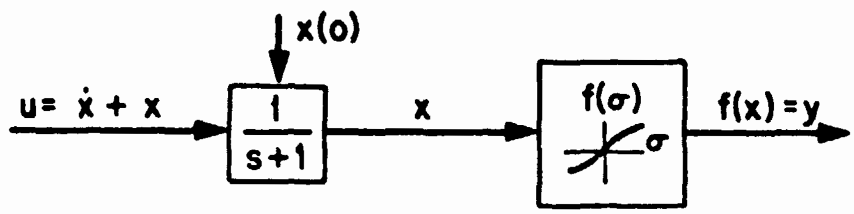


Fig. 2.4

If $w = uy = (x+\dot{x})f(x) = xf(x) + f(x)\dot{x}$ we have

$$V_a(x_0) = \sup_{\substack{t_1 \geq 0 \\ x(0)=x_0}} - \int_0^{t_1} \left[xf(x) + f(x)\frac{dx}{dt} \right] dt$$

$$= \sup_{\substack{t_1 \geq 0 \\ x(0)=x_0}} \left[-\int_0^{t_1} xf(x)\,dt - F(x(t_1)) + F(x(0)) \right] .$$

Now we can always choose u so that $xf(x)$ is bounded. Thus, letting $x(t_1) = 0$ and $t_1 \to 0$ we obtain $V_a(x_0) = F(x_0)$. Similarly

$$V_r(x_1) = \inf_{\substack{t_1 \geq 0 \\ x(0)=0 \\ x(t_1)=x_1}} \left[\int_0^{t_1} xf(x)\,dt + F(x_1) \right] = F(x_1) \ .$$

Let us consider an electrical interpretation of this when $f(\sigma) = \sigma$. In Fig. 2.5 u is the applied input voltage to a resistor-inductor network, and x is the resulting current, given by $\dot{x} + x = u$. If $x(0) = 0$ then the energy supplied to the network in the interval $0 \leq t \leq t_1$ is

$$E_1 = \int_0^{t_1} ux\,dt = \int_0^{t_1} x^2 dt + \frac{1}{2} x_1^2 \ , \text{ where } x_1 = x(t_1) \ .$$

The energy stored in the inductor at time t_1 is $E_2 = \frac{1}{2} x^2(t_1)$. In order to demonstrate the quasi-lossless property of this network we must exhibit a $u(t)$ for $0 \leq t \leq t_1$ which will make the difference $E_1 - E_2 = \int_0^{t_1} x^2\,dt$ arbitrarily small. One such $u(t)$ is the constant function $u(t) = \left(\frac{x_1}{1 - e^{-t_1}}\right)$ for $0 \leq t \leq t_1$, which gives $x(t) = \dfrac{x_1(1 - e^{-t})}{1 - e^{-t_1}} \leq x_1$, so that $\int_0^{t_1} x^2\,dt < t_1 x_1^2$, which approaches zero as $t_1 \to 0$. Thus, the

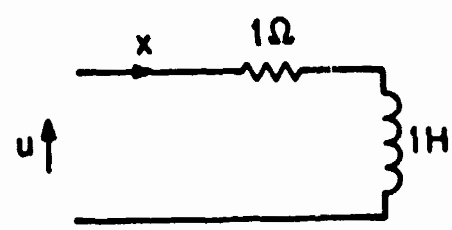


Fig. 2.5

most efficient driving function for this network is an impulse, i.e. for high efficiency the energy should be delivered to the network in as short a time as possible.

Consider now the linear system G described by

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \\ \underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \end{cases}$$

where $\underline{x}(t) \in \mathbb{R}^n$; $u(t)$, $y(t) \in \mathbb{R}^m$, and $\underline{A},\underline{B},\underline{C},\underline{D}$, are constant matrices of appropriate dimensions. Assume that $(\underline{A},\underline{B},\underline{C},\underline{D})$ is minimal. We wish to know when this system is dissipative with respect to the supply rate $w = \underline{u}'\underline{y}$. If $\underline{D} + \underline{D}'$ is invertible the evaluation of the available storage V_a and the required supply V_r reduces to an optimal control problem which may be solved by considering appropriate solutions of the matrix equation

$$\underline{K}\,\underline{A} + \underline{A}'\underline{K} + (\underline{K}\,\underline{B}-\underline{C}')\,(\underline{D} + \underline{D}')^{-1}\,(\underline{B}'\underline{K}-\underline{C}) = \underline{0}.$$

It can be shown [36] that this has a real symmetric positive definite solution if and only if G is dissipative. Then $V_a(\underline{x}) = \frac{1}{2}\,\underline{x}'\underline{K}^-\underline{x}$ and $V_r(\underline{x}) = \frac{1}{2}\,\underline{x}'\underline{K}^+\underline{x}$, where $\underline{K}^+$ and $\underline{K}^-$ are solutions of the above matrix equation. If $\underline{D} + \underline{D}'$ is not invertible then $\underline{K}^+$ and $\underline{K}^-$ are given by the limits as $\eta \to 0$ of $\underline{K}^+_\eta$ and $\underline{K}^-_\eta$, which are solutions of the above matrix equation obtained by replacing $\underline{D}$ by $(\underline{D} + \eta\underline{I})$. We also have that G is dissipative with respect to $w = \underline{u}'\underline{y}$ if and only if there is a real matrix $\underline{Q} = \underline{Q}' \geq \underline{0}$ which satisfies

$$\left[\begin{array}{c|c} \underline{A}'\underline{Q} + \underline{Q}\,\underline{A} & \underline{Q}\,\underline{B}-\underline{C}' \\ \hline \underline{B}'\underline{Q}-\underline{C} & -\underline{D}-\underline{D}' \end{array}\right] \leq \underline{0}$$

Moreover, $\frac{1}{2}\,\underline{x}'\underline{Q}\,\underline{x}$ is a storage function if and only if $\underline{Q}$ satisfies this inequality. In particular $\underline{K}^+$ and $\underline{K}^-$ satisfy it, and every solution $\underline{Q}$ satisfies $\underline{0} < \underline{K}^- \leq \underline{Q} \leq \underline{K}^+$.

§ 2.4 Realizations for O'Shea Functions

Suppose we have an operator F with input u(t) and output y(t) defined on $0 \leq t < \infty$ by the equations

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u \\ z = \underline{c}\,\underline{x} \\ y = f(z) \\ \underline{x}(0) = \underline{0} \end{cases}$$

where $f(\sigma)$ is a monotone function with $f(0) = 0$, as depicted in Fig. 2.6. As stated in Theorem 1.4, we know that F is a positive operator if and only if $Z(s) = [\underline{c}(\underline{I}s-\underline{A})^{-1}\underline{b}]^{-1} = g_0 + \gamma s - \hat{g}(s)$, where $\gamma \geq 0$, $\hat{g}(s) = \int_0^\infty g(t)e^{-st}dt$, $g(t) \geq 0$, and $g_0 \geq \hat{g}(0) = \int_0^\infty g(t)dt$. If we also know that $f(\sigma)$ is odd, i.e. $f(\sigma) = -f(-\sigma)$, then we can relax the conditions on $Z(s)$ to require that $Z(s) = g_0 + \gamma s - \hat{g}(s)$ where $\gamma \geq 0$, $\hat{g}(s) = \int_0^\infty g(t)e^{-st}dt$, and $\int_0^\infty |g(t)|dt \leq g_0$. These results are due to O'Shea, [26],

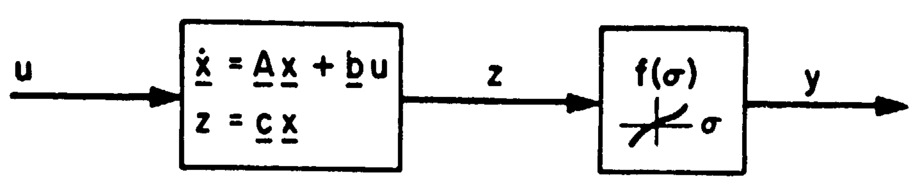


Fig. 2.6

[32], [12], and we call a transfer function $z^{-1}(s)$ for which $z(s)$ meets these conditions an <u>O'Shea function</u>.

Given an O'Shea function $z^{-1}(s)$ we would like to be able to construct a realization $(\underline{A},\underline{b},\underline{c})$ of $z(s)$, as in Fig. 2.6, which will allow us to write down a storage function for F as a function of the state vector $\underline{x}$. As yet a general solution is not known to this problem, however, we have the following theorem, (cf. Theorem 1.5):

<u>Theorem 2.6</u> Suppose that $-\underline{A}(t)$ is a hyperdominant matrix for all t, $\underline{b}' = [0\ 0\ \cdots 0\ \gamma]$, $\gamma > 0$, $\underline{c} = [0\ 0\ \cdots\ 0\ 1]$, and f is any monotone non-linearity with $f(0) = 0$. Then the operator mapping $u(t)$ into $y(t)$ defined by

$$\begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u \\ z = \underline{c}\,\underline{x} \\ y = f(z) \end{cases}$$

is dissipative with respect to the supply rate uy; with storage function $V(\underline{x}) = \frac{1}{\lambda}\sum_{i=1}^{n} F(x_i)$, where $F(z) = \int_0^z f(\sigma)\,d\sigma$. Furthermore, if f is odd, then $-\underline{A}(t)$ need only be dominant.

A matrix $\underline{M}$ whose ijth element is m_{ij} is said to be <u>dominant</u> if $m_{ii} \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{ij}|$ and $m_{jj} \geq \sum_{\substack{i=1 \\ i\neq j}}^{n} |m_{ij}|$ for all i,j; that is, for each row and column the on-diagonal elements are larger than the sum of the moduli of the off-diagonal elements. $\underline{M}$ is said to be <u>hyperdominant</u> if it is dominant and all the off-diagonal elements are negative; i.e. $\underline{M}$ is hyperdominant if $m_{ij} \leq 0$ when $i \neq j$, and $\sum_{i=1}^{n} m_{ij} \geq 0$ and $\sum_{j=1}^{n} m_{ij} \geq 0$ for all i,j.

Proof of Theorem 2.6:

$$\dot{V}(\underline{x}) = \frac{1}{\lambda}\sum_{i=1}^{n} \frac{dF}{dx}\dot{x}_i = \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\dot{\underline{x}} \quad \text{say}$$

where $$\underline{V}_{\underline{x}} = [f(x_1) \quad f(x_2) \quad \cdots \quad f(x_n)].$$

Thus
$$\begin{aligned}\dot{V}(\underline{x}) &= \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\underline{\Lambda}\,\underline{x} + \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\underline{b}u \\ &= \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\underline{\Lambda}\,\underline{x} + uf(x_n) \\ &= \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\underline{\Lambda}\,\underline{x} + uy\end{aligned}$$

Therefore $\dot{V} - uy = \frac{1}{\lambda}\,\underline{V}_{\underline{x}}\,\underline{\Lambda}\,\underline{x}$.

Now, it can be shown ([33],[34] Chapter 3) that if $f(\sigma)$ is any monotone function with $f(0) = 0$ then $\underline{V}_{\underline{x}}\underline{\Lambda}\,\underline{x}$ is negative for all $\underline{x}$ if and only if $-\underline{\Lambda}$ is hyperdominant, and that if $f(\sigma)$ is any odd monotone function then $\underline{V}_{\underline{x}}\underline{\Lambda}\,\underline{x}$ is negative for all $\underline{x}$ if and only if $-\underline{\Lambda}$ is dominant. QED

Necessary and sufficient conditions on $\underline{\Lambda}$ for $\underline{V}_{\underline{x}}\underline{\Lambda}\,\underline{x}$ to be negative for all $\underline{x}$ when f is a particular given function are not known in general. However there is one function f of interest (particularly with regard to applications of Chapter 1) for which these conditions are known: this is the sgn function. Let a row-dominant matrix $\underline{M}$ be one for which

$m_{ii} \geq \sum_{j\neq i} m_{ij}$ for all i,

and if $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ then let $\underline{\text{sgn}}\ \underline{x} = \begin{bmatrix} \text{sgn } x_1 \\ \text{sgn } x_2 \\ \cdot \\ \cdot \\ \cdot \\ \text{sgn } x_n \end{bmatrix}$ and $\underline{\text{stp}}\ \underline{x} = \begin{bmatrix} \text{stp } x_1 \\ \text{stp } x_2 \\ \cdot \\ \cdot \\ \cdot \\ \text{stp } x_n \end{bmatrix}$

we have:

Theorem 2.7 $\underline{x}'\underline{M}\ \underline{\text{sgn}}\ \underline{x} \geq 0$ for all $\underline{x}$ if and only if $\underline{M}$ is row-dominant, and $(\underline{\text{sgn}}\ \underline{x})'\ \underline{M}\ \underline{x} \geq 0$ for all $\underline{x}$ if and only if $\underline{M}$ is column-dominant.

Proof. Sufficiency:
$$\begin{aligned}\underline{x}'\underline{M}\ \underline{\text{sgn}}\ \underline{x} &= x_1(m_{11}\ \text{sgn}\ x_1 + m_{12}\ \text{sgn}\ x_2 + \cdots + m_{1n}\text{sgn}\ x_n) \\ &\quad + x_2(m_{21}\ \text{sgn}\ x_1 + \cdots \qquad\qquad) \\ &\quad + \cdots \\ &= |x_1|\ [m_{11}(\text{sgn}^2 x_1) + m_{12}(\text{sgn}\ x_1)(\text{sgn}\ x_2) + \cdots + m_{1n}(\text{sgn}\ x_1)(\text{sgn}\ x_n)] \\ &\quad + \cdots \qquad\qquad (\text{assuming } x_1 \neq 0) \\ &\geq 0 \text{ if } \underline{M} \text{ is row-dominant}.\end{aligned}$$

Necessity: Suppose row 1 has $m_{11} - \sum_{j\neq 1} |m_{ij}| < 0$, (the proof for the case $m_{ii} - \sum_{j\neq i} |m_{ij}| < 0$ being similar). Let $\underline{x} = (1, -\eta\ \text{sgn}\ m_{12}, \cdots, -\eta\ \text{sgn}\ m_{1n})'$.

Then $\underline{\text{sgn}}\ \underline{x} = (1, -\text{sgn}\ m_{12}, -\text{sgn}\ m_{13}, \cdots, -\text{sgn}\ m_{1n})'$.

Thus, $\underline{x}'\underline{M}\ \text{sgn}\ \underline{x} = m_{11} - \sum_{j=2}^{n} |m_{ij}| - \eta\ \text{sgn}\ m_{12}(\pm m_{12} \pm \cdots \pm m_{2n})$

< 0 for η sufficiently small. QED.

One can also show in a similar way that a sufficient (but not necessary) condition for $\underline{x}'\underline{M}\ \underline{\text{stp}}_{(-a,b)}\ \underline{x}$ to be nonnegative for all $\underline{x}$ is that $m_{ii} \geq \mu \sum_{j\neq i} |m_{ij}|$ for each i, where μ is the larger of $\frac{b}{a}$ and $\frac{a}{b}$.

Let us call a realization $(\underline{A},\underline{b},\underline{c})$ of H(s) which meets the conditions of Theorem 2.6 a <u>dissipative (hyper)dominant realization</u>. Though it is not known what the necessary and sufficient conditions are for a given H(s) to have a dissipative (hyper)dominant realization, clearly it is necessary for H(s) to be an O'Shea function: $H(s) = (g_0 + \gamma s - \hat{g}(s))^{-1}$ with $\gamma \neq 0$. I conjecture that this is also a sufficient condition; i.e. the conjecture is that every O'Shea function of the type $(g_0 + \gamma s - \hat{g}(s))^{-1}$ with $\gamma \neq 0$

has a time-invariant dissipative (hyper)dominant realization. Let us further discuss this question.

Firstly, it is easy to show that the conjecture is true for a second-order O'Shea function, and that such a function need not have real poles. Secondly, it is always possible to find a realization $(\underline{A},\underline{b},\underline{c})$ with $\underline{b}' = [0 \cdots 0\ \lambda]$ and $\underline{c} = [0 \cdots 0\ 1]$ for a given transfer function H(s) provided that $\lim_{s\to\infty} s\,H(s) = \lambda \neq 0$. For, let

$$H(s) = \frac{q_{n-1} s^{n-1} + q_{n-2} s^{n-2} + \cdots + q_0}{s^n + p_{n-1} s^{n-1} + \cdots\ p_0} \quad \text{where } q_{n-1} = \lambda .$$

Then we have the sta ..rd controllable realization ([7] Chapter 17) $H(s) = \underline{h}(\underline{I}s-F)^{-1}\underline{g}$ where

$$\underline{F} = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & & \\ & & & & 1 \\ -p_0 & -p_1 & \cdots & & -p_{n-1} \end{bmatrix}, \qquad \underline{g} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \qquad \underline{h} = [q_0\ q_1 \cdots q_{n-1}]$$

Now let

$$\underline{P} = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & \ddots & & & \\ & & & & 1 & 0 \\ q_0 & q_1 & \cdots & \cdots & q_{n-2} & q_{n-1} \end{bmatrix}, \quad \underline{P}^{-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ \frac{-q_0}{q_{n-1}} & \frac{-q_1}{q_{n-1}} & \cdots & \frac{-q_{n-2}}{q_{n-1}} & \frac{1}{q_{n-1}} \end{bmatrix}.$$

Then we have the realization $H(s) = \underline{c}(\underline{I}s-\underline{A}_1)^{-1}\underline{b}$ where

$$\underline{A}_1 = \underline{P}\underline{F}\underline{P}^{-1}\ ,\ \underline{b} = \underline{P}\ \underline{g} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ q_{n-1} \end{bmatrix} \quad \underline{c} = \underline{h}\ \underline{P}^{-1} = [0\cdots0\ 1] \quad .$$

Now A_1 thus obtained will in general not be hyperdominant; the conjecture is, however, that if H(s) is O'Shea then there exists an $\underline{R}$ such that $-\underline{R}\ \underline{A}_1\underline{R}^{-1} = -\underline{A}_2$ is hyperdominant, $\underline{R}\ \underline{b} = \underline{b}$, and $\underline{c}\ \underline{R}^{-1} = \underline{c}$. Note that if $H(s) = (g_0 + \gamma s-\hat{g}(s))^{-1}$ with $\gamma \neq 0$ then $\lim_{s \to \infty} sH(s) = \gamma^{-1}$. For $\underline{R}$ to satisfy $\underline{R}\ \underline{b} = \underline{b}$ and $\underline{c}\ \underline{R}^{-1} = \underline{c}$ we must have

$$\underline{R} = \left[\begin{array}{c|c} \underline{R}_{11} & \underline{0} \\ \hline \underline{0} & 1 \end{array}\right] \quad .$$

In § 1.5(e) we considered an example of this approach for a third-order O'Shea function.

Now let us consider another approach. An interesting subclass of the O'Shea functions is the class of RC impedance functions Z(s) which have $Z(\infty) = 0$. A general RC impedance function Z(s) can always be expressed ([2],[16],[42]) as $Z(s) = g_0 + \sum_{i=1}^{n} \frac{g_i}{s+s_i}$ where $g_i,\ s_i \geq 0$ for all i. If $g_0 = 0$ then $Z(\infty) = 0$. An RC impedance has poles and zeros alternating on the negative real axis, and a Nyquist locus which lies inside a circle in the right half plane. It is proven in reference [42] that such an RC impedance is an O'Shea function. Now the four canonical RC network realizations of an RC impedance are the Cauer I, Cauer II, Foster I, and Foster II methods. Of these the Cauer I method is of interest to us here, because it leads to a tridiagonal realization which in some cases is hyperdominant. A tridiagonal realization $(\underline{A},\underline{b},\underline{c})$ is one for which the matrix $\underline{A}$ is tridiagonal, i.e. the only nonzero entries are on the diagonal and immediately above and below it: $a_{ij} = 0$ unless $j \in \{i-1,i,i+1\}$. Consider the network of Fig. 2.7 with input current I and resulting terminal voltage V.

We have

$$\left\{\begin{array}{rcl} C_1\dot{V}_1 & = & \dfrac{V_2-V_1}{R_1} - \dfrac{V_1}{R_0} \\[2ex] C_2\dot{V}_2 & = & \dfrac{V_3-V_2}{R_2} - \dfrac{V_2-V_1}{R_1} \\[6ex] C_{n-1}\dot{V}_{n-1} & = & \dfrac{V_n-V_{n-1}}{R_{n-1}} - \dfrac{V_{n-1}-V_{n-2}}{R_{n-2}} \\[2ex] C_n\dot{V}_n & = & \dfrac{-V_n-V_{n-1}}{R_{n-1}} + I \end{array}\right.$$

Letting $x_i = V_i\sqrt{C_i}$, $u = \dfrac{I}{\sqrt{C_n}}$, $y = V\sqrt{C_n}$, we obtain

$$\left\{\begin{array}{rcl} \underline{\dot{x}} & = & \underline{A}\,\underline{x} + \underline{b}u \\ y & = & \underline{c}\,\underline{x} + du \end{array}\right.$$

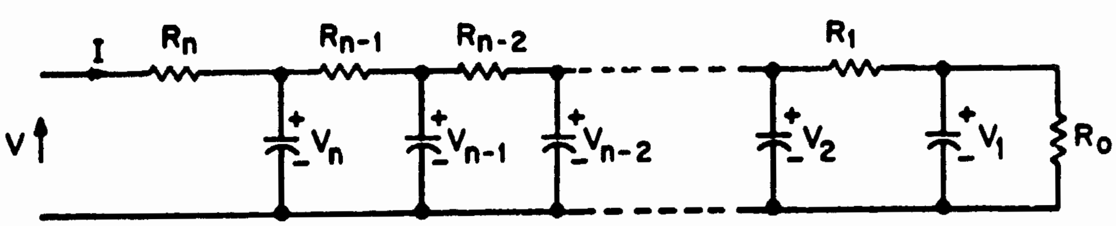


Fig. 2.7

where

$$\underline{A} = \begin{bmatrix} \frac{-1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_0}\right) & \frac{1}{R_1\sqrt{C_1C_2}} & 0 & \cdots & & \\ \frac{1}{R_1\sqrt{C_1C_2}} & -\frac{1}{C_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & \frac{1}{R_2\sqrt{C_2C_3}} & & & \\ 0 & \frac{1}{R_2\sqrt{C_2C_3}} & -\frac{1}{C_3}\left(\frac{1}{R_2}+\frac{1}{R_3}\right) & & & \\ \vdots & & & \ddots & & \\ & & & & -\frac{1}{C_{n-1}}\left(\frac{1}{R_{n-2}}+\frac{1}{R_{n-1}}\right) & \frac{1}{R_{n-1}\sqrt{C_{n-1}C_n}} \\ & & & & \frac{1}{R_{n-1}\sqrt{C_{n-1}C_n}} & \frac{-1}{R_{n-1}C_n} \end{bmatrix}$$

$\underline{b}' = [0\ 0\ \ldots\ 0\ 1]$, $\underline{c} = [0\ 0\ \ldots\ 0\ 1]$, $d = R_nC_n$. Now $-\underline{A}$ is hyperdominant if and only if $C_i = C$ for all i. Thus, if Z(s) is an RC impedance with $Z(\infty)=0$ which can be realized by a Cauer I network with all capacitances equal, then Z(s) has a tridiagonal dissipative hyperdominant realization. However, a general RC impedance cannot be realized by a Cauer I network with all capacitances equal. (For example $\frac{s+1}{s^2+3s+1}$ cannot). For $n \leq 3$ it appears that every RC impedance does indeed have a tridiagonal dissipative hyperdominant realization; furthermore one can show that every such realization for $n \leq 3$ is an RC impedance.

In obtaining a Cauer I network realization of a given Z(s), Z(s) is expressed as a continued fraction expansion. For instance, for

$Z(s) = \frac{s+1}{s^2+3s+1}$ we write

$$z(s) = \cfrac{1}{s + \cfrac{1}{\frac{1}{2} + \cfrac{1}{4s + \frac{1}{(1/2)}}}}$$

This gives the network of Fig. 2.8.

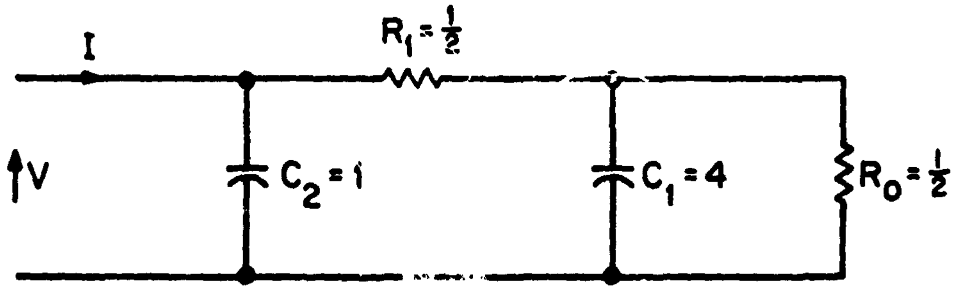


Fig. 2.8

To obtain realizations for O'Shea functions in tridiagonal form we make use of analog computation symbols as defined in Fig. 2.9. (We leave initial conditions unspecified).

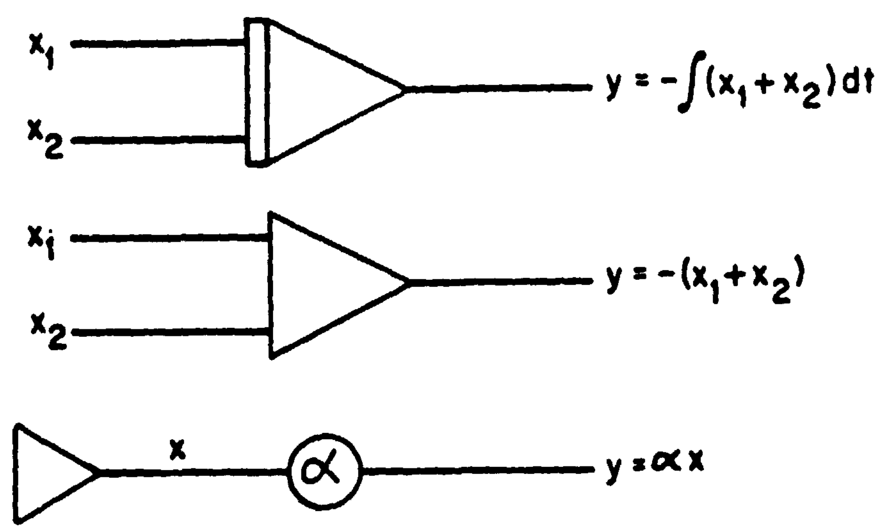


Fig. 2.9

Now a transfer function $G_1(s) = \frac{1}{as+f(s)}$ with input u and output y can be realized by the configuration of Fig. 2.10. Similarly a transfer function $G_2(s) = \frac{1}{b+f(s)}$ can be realized by the configuration of Fig. 2.11.

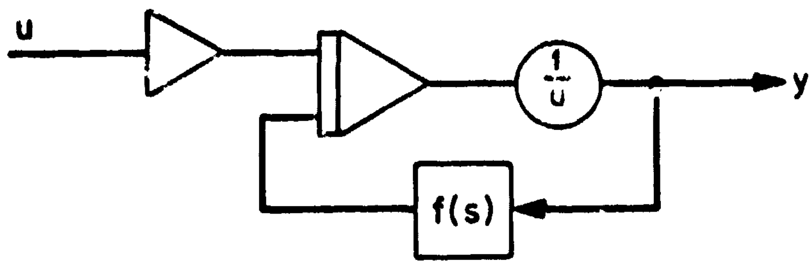


Fig. 2.10

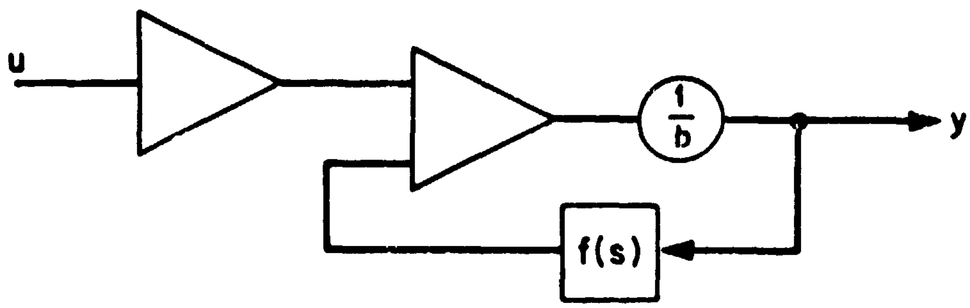


Fig. 2.11

Now given any transfer function $Z(s)$ with $\lim_{s\to\infty} sZ(s) \neq 0$ we can always obtain a tridiagonal realization $(\underline{A},\underline{b},\underline{c})$ with $\underline{b}' = [0\cdots0\lambda]$ and $\underline{c} = [0\cdots01]$ by expressing $Z(s)$ as a continued fraction expansion and representing this expansion by a succession of connections as in Figs. 2.10 and 2.11. To illustrate this method consider the RC impedance

$$G(s) = \frac{1}{3}\left(\frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}\right)$$

$$= \frac{s^2 + 4s + \frac{11}{3}}{s^3 + 6s^2 + 11s + 6}$$

$$= \cfrac{1}{\frac{s}{3} + \cfrac{1}{\frac{3}{2} + \cfrac{1}{2s + \cfrac{1}{\frac{3}{10} + \cfrac{1}{\frac{50s}{3} + 30}}}}}$$

Using the connections of Figs. 2.10 and 2.11 we now obtain the representation of Fig. 2.12, in which we have made the three integrator outputs proportional to the state variables x_1, x_2, x_3. Such a labelling of integrator outputs gives rise to a tridiagonal realization.

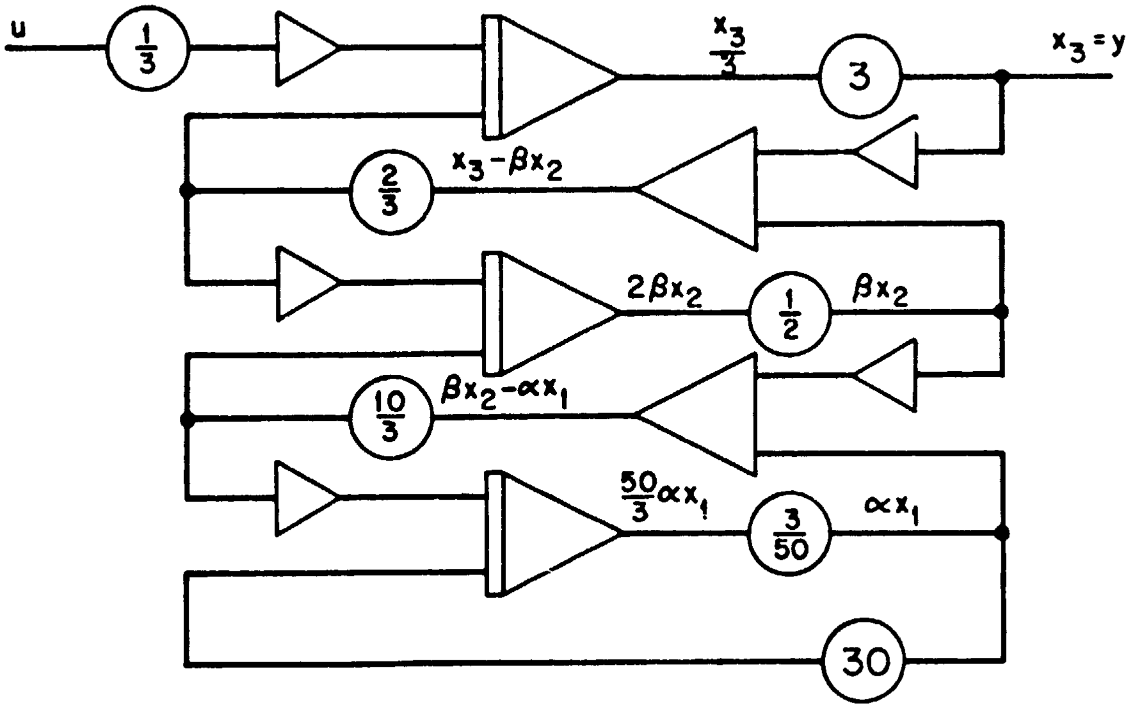


Fig. 2.12

We have

$$\begin{cases} \frac{50}{3}\alpha\dot{x}_1 = \frac{10}{3}(\beta x_2-\alpha x_1) - 30\alpha x_1 \\ 2\beta\dot{x}_1 = \frac{2}{3}(x_3-\beta x_2) - \frac{10}{3}(\beta x_2-\alpha x_1) \\ \frac{\dot{x}_3}{3} = \frac{u}{3} - \frac{2}{3}(x_3-\beta x_2) \end{cases}$$

i.e.

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & \frac{\beta}{5\alpha} & 0 \\ \frac{5\alpha}{3\beta} & -2 & \frac{1}{3\beta} \\ 0 & 2\beta & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

The conditions which α and β must satisfy for this to be a hyperdominant realization are

$$\begin{cases} \alpha,\beta > 0 \\ \frac{\beta}{10(1-\beta)} \leq \alpha \leq \frac{6\beta-1}{5} \end{cases}$$

The allowable values for α and β are depicted in Fig. 2.13. For example, if we pick $\alpha = \frac{1}{5\sqrt{2}} = .14$, and $\beta = \frac{1}{\sqrt{6}} = .41$, then $\underline{A}$ is symmetric:

$$\underline{A} = \begin{bmatrix} -2 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -2 & \sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & -2 \end{bmatrix}$$

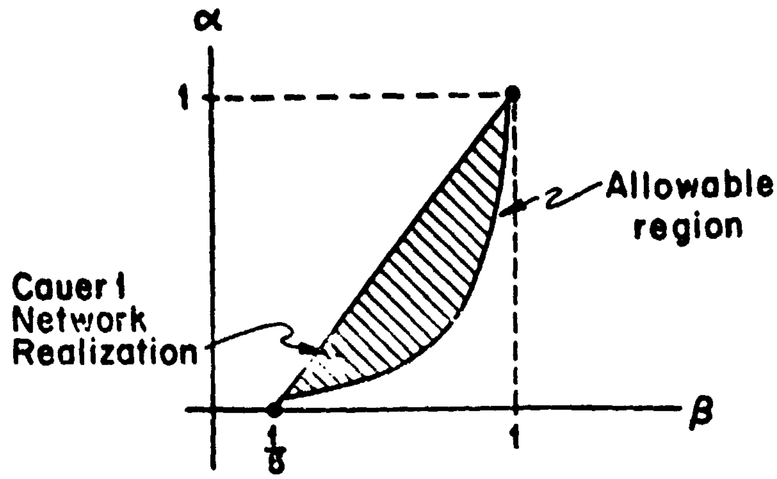


Fig. 2.13

This corresponds to the Cauer I network realization of Fig. 2.14.

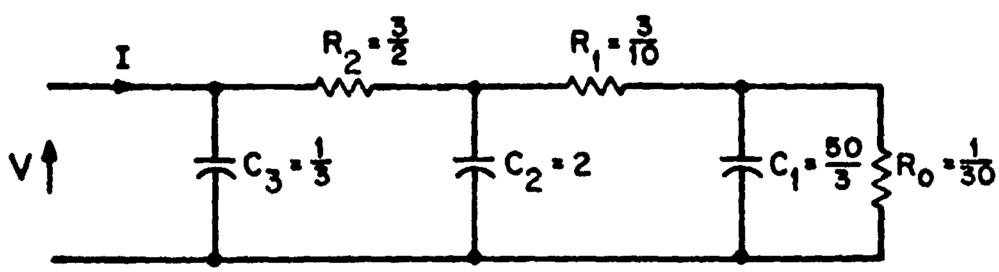


Fig. 2.14

In a realization of a given Z(s) by this method, if we let $x_i = k_i w_i$ where w_i is the output of integrator i and k_i is some positive constant, then the $\underline{A}$ matrix obtained will have diagonal elements negative and off-diagonal elements positive if and only if the coefficients in the continued fraction expansion are all positive. If this is so then Z(s) can be realized as a (Cauer I) RC network impedance.

Consider now the O'Shea function which we obtained in § 1.5(d):

$$G(s) = \frac{s^2 + 1.45s + 0.5}{s^3 + 200s^2 + 96s + 4}$$

$$= \cfrac{1}{s + \cfrac{1}{\cfrac{1}{198.6} + \cfrac{1}{205s + \cfrac{1}{-0.326 + \cfrac{1}{-1.68s + 2.24}}}}}$$

This gives us the representation of Fig. 2.15, from which we obtain the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.49 & -1.82\,\frac{\beta}{\alpha} & 0 \\ 0.015\,\frac{\alpha}{\beta} & -0.96 & \frac{0.97}{\beta} \\ 0 & 198.6\beta & -198.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This realization cannot be made hyperdominant for any values of α and β; we need to label the integrator outputs of Fig. 2.15 with a general linear combination, i.e. we must label the outputs of the lower two integrators $(\alpha x_1 + \beta x_2)$ and $(\gamma x_1 + \delta x_2)$, then look for suitable values of $\alpha,\beta,\gamma,\delta$. This, however, is essentially the same task as finding a,b,c,d for the matrix R of § 1.5(e).

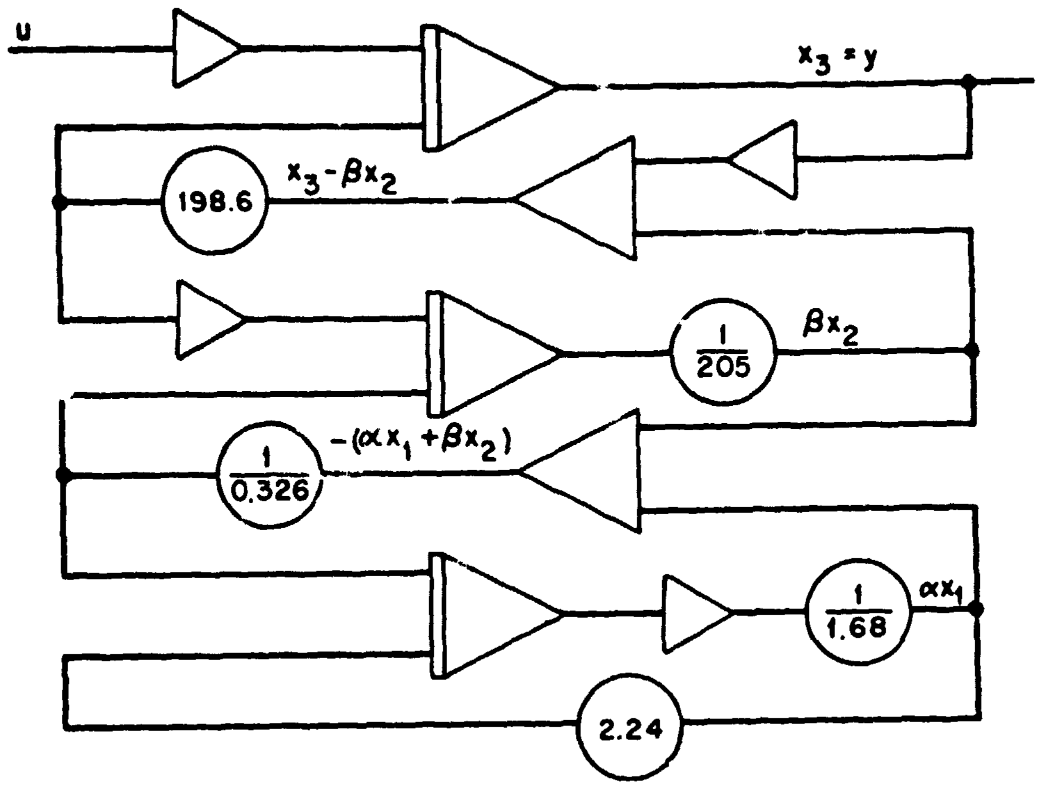


Fig. 2.15

CHAPTER 3

SWITCHED ELECTRICAL NETWORKS AND BILINEAR EQUATIONS

§ 3.1 Introduction

A linear dynamical system can be described by a set of equations of the form

$$(1)\quad \begin{cases} \dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \\ \underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \end{cases}$$

where $\underline{u}$ is the vector of inputs or controls, $\underline{y}$ is the vector of outputs, and $\underline{x}$ is the state vector for the system. We saw in Chapter 1 that equations of this form describe the behavior of certain types of electrical networks: in Chapter 1 these were voltage-conversion networks operating from a battery with zero internal impedance, and we were able to exploit the fact that the resulting evolution equations were linear, for this led to a feedback system with a linear operator in the forward path. However, the class of systems describable by linear equations is a restricted one; most of the systems encountered in the field of electrical power processing (i.e. DC-DC conversion, DC-AC conversion, etc.) cannot be described by linear equations. In this Chapter we address ourselves to the equation "What kinds of state question arise in the description of power processing systems?" Having answered this question in § 3.2, we then ask in § 3.3 "What statements can we make about classifying such systems: what canonical forms for the state equations do we have?" In § 3.4 we outline the role played by Lie groups and Lie algebras in characterizing these systems, and in § 3.5 we give network examples.

§ 3.2 Bilinear Equations

Consider an electrical network composed of resistors, inductors, capacitors, transformers, batteries, current sources, and ideal switches. With these components, we can model the essential features of power conversion networks. The state of this network will be a vector $\underline{x}(t)$ in $\mathbb{R}^n$ whose instantaneous value represents the inductor currents and capacitor voltages; usually we take the state variables x_i to be scalar multiples of these currents and voltages.

If the topology of the network is fixed, that is, the switches are all held in one set of positions, the state evolution equations will take the linear form (1) given in § 3.1. The reason for this is that capacitor voltages and inductor currents obey linear first-order differential equations. The matrix $\underline{A}$ of § 3.1 will be a constant matrix whose eigenvalues represent the natural frequencies of the circuit. These natural frequencies are determined by the component values and the topology of the network, which will be changed if the switch positions are changed.

Consequently, if we have a network in which the switches are considered as controls, with the position of switch i being given by u_i where $u_i = 0$ or 1, then the matrix $\underline{A}$ will be a function of the u_i's. The resulting state evolution equations take the form

$$(2) \quad \left\{ \dot{\underline{x}} = (\underline{A}_0 + u_1\underline{A}_1 + u_2\underline{A}_2 + \cdots + u_m\underline{A}_m)\underline{x} + (\underline{b}_1u_1 + \underline{b}_2u_2 + \cdots + \underline{b}_m u_m). \right.$$

As a simple example consider the regulator of § 1.7, shown in Fig. 3.1. We have

$$\left\{ \begin{aligned} L_1\dot{I}_1 &= -V_2 + u(b-rI_1) \\ C_1\dot{V}_2 &= I_1 \end{aligned} \right. .$$

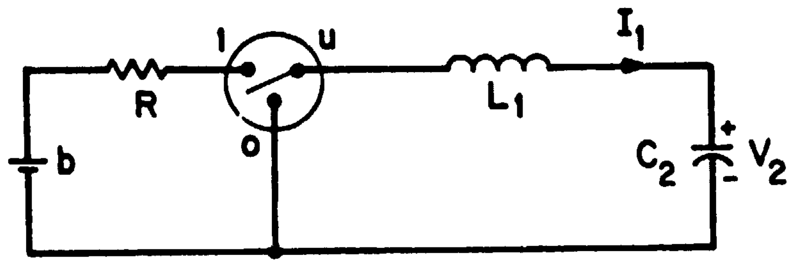


Fig. 3.1

Thus, letting $x_1 = I_1 \sqrt{L_1}$ and $x_2 = V_2 \sqrt{C_2}$ (so that $\frac{1}{2}\underline{x}'\underline{x}$ is the stored energy in the network) and $\omega = \dfrac{1}{\sqrt{L_1 C_2}}$, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} + u \begin{bmatrix} -\frac{R}{L_1} & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{\sqrt{L_1}} \\ 0 \end{bmatrix} u$$

i.e. $\underline{\dot{x}} = (\underline{A}_0 + u\underline{A}_1)\underline{x} + \underline{b}u$.

When R = 0 then $\underline{A}_1 = \underline{0}$ and the state equation is linear; in terms of the network we see that when R = 0 the natural frequency is unchanged by the switch position, since the dynamic impedance of the loop including L_1 and C_2 is the same for both switch positions. This example is of the simplest kind, though of considerable practical significance. In § 1.8(a) we briefly considered higher-order regulators of this type.

Now any system which can be realized by a set of equations of the form (2), i.e.

$$(3) \qquad \begin{cases} \underline{\dot{z}} = \left(\underline{F}_0 + \sum_{i=1}^{m} u_i \underline{F}_i \right) \underline{z} + \sum_{i=1}^{m} u_i \underline{g}_i \\ \underline{y} = \underline{H}\,\underline{z} \end{cases}$$

can be realized by a set of equations of the form

$$(4)\quad \begin{cases} \dot{\underline{x}} = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i\, \underline{A}_i\right) \underline{x} \\ \underline{y} = \underline{C}\,\underline{x} \end{cases}$$

To see this we let $\underline{A}_i$ be obtained by adding an extra row and column to $\underline{F}_i$, thus:

$$\underline{A}_0 = \begin{bmatrix} \underline{F}_0 & \underline{0} \\ \underline{0} & 0 \end{bmatrix} \qquad \underline{A}_i = \begin{bmatrix} \underline{F}_i & \underline{g}_i \\ \underline{0} & 0 \end{bmatrix},$$

and letting $\underline{x} = \begin{bmatrix} \underline{z} \\ 1 \end{bmatrix}$ and $\underline{C} = [\underline{H} \;\; \underline{0}]$.

We are therefore interested in equations in the form of (4). This is called the bilinear form. This form is even more general than we have just shown it to be, for it is shown in reference [9] that if $\underline{P}(\underline{x}) = \underline{P}(x_1, x_2, \cdots, x_n)$ is a multinomial expression in the variables $x_1, \cdots, x_n$, then any system of the form

$$(5)\quad \begin{cases} \dot{\underline{x}} = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i\, \underline{A}_i \right) \underline{x} \\ \underline{y} = \underline{P}(\underline{x}) \end{cases}$$

can also be realized in the form of (4). Thus nonlinear output maps can be reduced to linear forms provided that they are of the finite power series type. Again, this is done by extending the dimension of the state vector. For example, the equations

$$\begin{cases} \dot{x} = -x + u \\ y = x^3 \end{cases}$$

can be written as

$$\left\{\begin{array}{l} \dfrac{d}{dt}\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix} = \left\{\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&-2&0\\0&0&0&-3\end{bmatrix} + u\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&2&0&0\\0&0&3&0\end{bmatrix}\right\}\begin{bmatrix}1\\x\\x^2\\x^3\end{bmatrix} \\ \quad y = \begin{bmatrix}0&0&0&1\end{bmatrix}\underline{x} \end{array}\right. .$$

This technique for handling nonlinear combinations of the state variables is of more than passing interest to us. Consider, for instance, the voltage regulator we studied in § 1.4.

There we had

$$\left\{\begin{array}{l} \ddot{x} + x = u \\ y = \beta\dot{x} + x \\ u = -\text{stp}\ y \end{array}\right. .$$

We found that once the state reached the chattering region of the switching line near the origin, the settling time was proportional to β. Thus, near the origin we want β small, while for large values of $||\underline{x}||$ we do not. In order to obtain an improved overall transient response we might therefore try a feedback strategy of the form $y = \dot{x}^3 + x$. We can now put the equations

$$\left\{\begin{array}{l} \ddot{x} + x = u \\ y = \dot{x}^3 + x \\ u = -\text{stp}\ y \end{array}\right.$$

in the bilinear form (4). It must be pointed out that this technique will lead to state equations of large dimension; in this regulator example the state is 10-dimensional:

$$\underline{x}' = \left[1 \quad x \quad \dot{x} \quad x^2 \quad x\dot{x} \quad \dot{x}^2 \quad x^3 \quad x^2\dot{x} \quad x\dot{x}^2 \quad \dot{x}^3\right].$$

For a power conversion network with state evolution equations in the form of (2) or (4), it will sometimes be the case that some of the variables u_i are restricted to being functions of the state. Such a situation may arise if there are diodes in the network, as for example in Fig. 3.2, for which

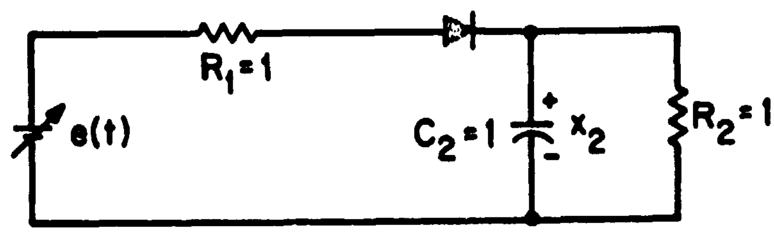


Fig. 3.2

we have

$$\dot{x}_2 = -x_2 + u(e-x_2)$$

where $$u = \begin{cases} 1 & , x_2 < e \\ 0 & , x_2 \geq e \end{cases} .$$

In this thesis the only type of switch that we consider is the ideal two-position switch depicted in Fig. 3.3.

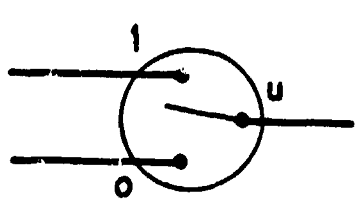


Fig. 3.3

We use this switch in obtaining "well-posed" network models for power conversion systems; by a well-posed network we mean one for which capacitor voltages and inductor currents are continuous functions of time. Fig. 3.4 is an example of an ill-posed network (cf. Fig. 1.5). This is not well-

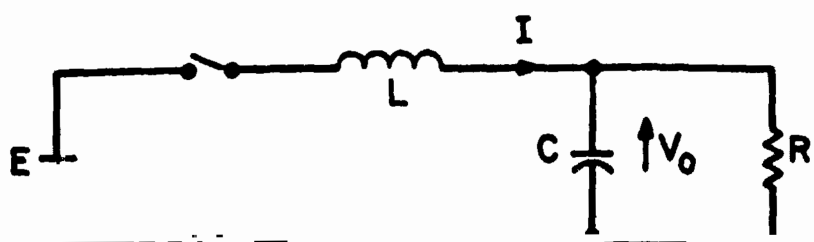


Fig. 3.4

posed because an infinite voltage would be developed across the switch if it were to be opened when the inductor current I was nonzero. By providing an alternate path for the inductor current when the switch position changes, as in Fig. 1.5, the current will be a continuous function of time.

Implementation of an ideal two-position switch will in general require the use of two transistors (or thyristors) and two "freewheeling" diodes, as for example in Fig. 3.5 which shows a scheme for implementing

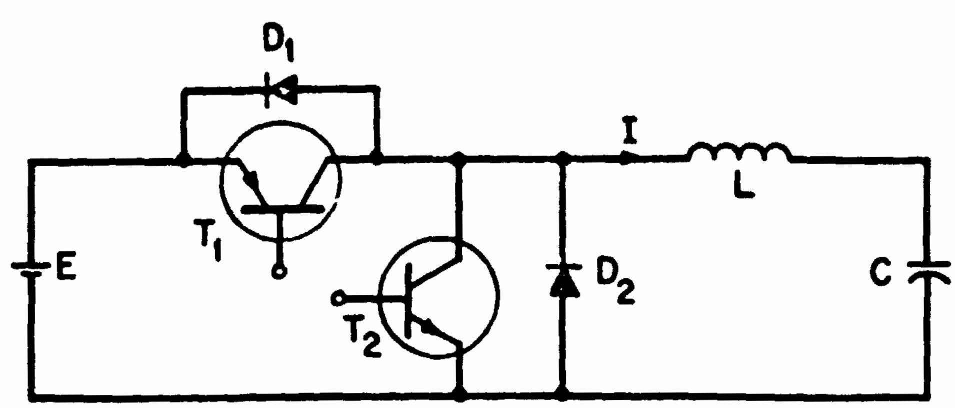


Fig. 3.5

the regulator of Fig. 1.5 (or of Fig. 3.1 with $R = 0$). When $u = 1$ the transistor T_1 is turned "on" (i.e. acts as a short circuit) while the transistor T_2 is turned "off" (i.e. acts as an open circuit). In this condition, the network of Fig. 3.5 is equivalent to that of Fig. 3.6, in which the inductor current I flows through T_1 when $I > 0$ and back through

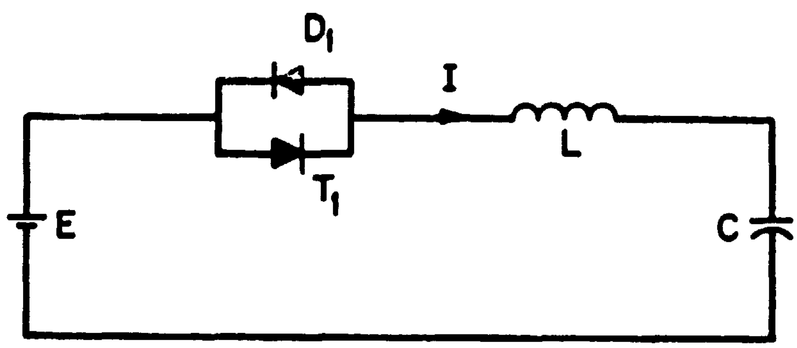


Fig. 3.6

D_1 whenever $I < 0$. When $u = 0$, T_1 is "off" and T_2 is "on", and then the network is equivalent to that of Fig. 3.7, in which the inductor current flows through D_2 when $I > 0$ and through T_2 when $I < 0$.

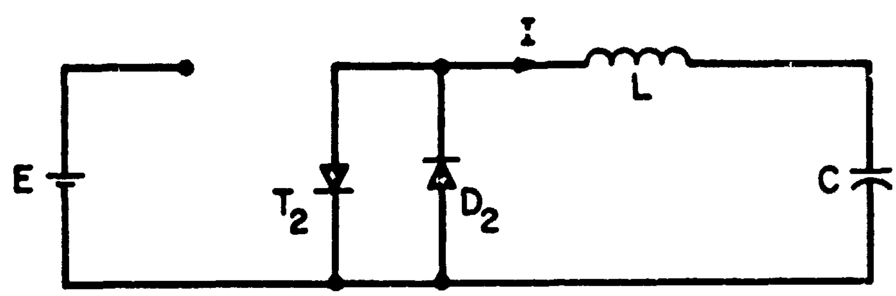


Fig. 3.7

In practice, when constructing a regulator of this type, it is usual to omit the transistor T_2 (as in Figs. 1.55 and 1.56). This still yields a well-posed network, however the analysis of Chapter 1 will apply only if the condition $\{u = 0,\ I < 0\}$ never arises, which will be the case if the load current is large enough to ensure that at all times $I > 0$.

In conclusion, we see that the state equations for power conversion networks will be of the bilinear form (4), with some of the u_i's restricted to being functions of the state, while others can be chosen freely.

§ 3.3 Canonical Forms and Equivalent Systems

Since a wide variety of complex electric power conversion networks have state equations of the form of (4), we are particularly interested in classifying bilinear systems. We would like to be able to determine when two electrical networks which are topologically different have similar dynamical characteristics; not only would this be conceptually helpful, but a feedback law which was found to be suitable for one network could be translated into a suitable feedback law for the other. Reference [9] describes some recent results which answer the question of when two bilinear systems are dynamically similar. We shall now briefly outline these.

Suppose we have two systems of bilinear equations:

$$(4)\quad \begin{cases} \dot{\underline{x}} = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i \underline{A}_i\right) \underline{x} \\ \underline{y} = \underline{C}\,\underline{x} \end{cases}$$

$$(6)\quad \begin{cases} \dot{\underline{z}} = \left(\underline{F}_0 + \sum_{i=1}^{m} u_i \underline{F}_i\right) \underline{z} \\ \underline{y} = \underline{H}\,\underline{z} \end{cases}$$

Let $\underline{u} = (u_1, u_2, \cdots u_m)$ be the input vector, and $\underline{y}$ the output vector. We say that (4) and (6) are equivalent if there exists a nonsingular matrix $\underline{P}$ such that $\underline{F}_i = \underline{P}\,\underline{A}_i\,\underline{P}^{-1}$ for $0 \leq i \leq m$ and $\underline{H} = \underline{C}\,\underline{P}^{-1}$. Our reason for this definition is that if we are given (4), and we let $\underline{z} = \underline{P}\,\underline{x}$, then we obtain (6) with $\underline{F}_i = \underline{P}\,\underline{A}_i\,\underline{P}^{-1}$ and $\underline{H} = \underline{C}\,\underline{P}^{-1}$; in this case (4) and (6) are realizations of the same input-output map. We would like to know when the condition that (4) and (6) realize the same input-output map implies that they are equivalent. We shall answer this question below in Theorem 3.1, which is similar to the well-known result on the equivalence of realizations of a linear system ([7], section 18).

We call a realization in the form of (4) irreducible if there is no invertible matrix $\underline{P}$ such that for $0 \leq i \leq m$

$$\underline{P}\,\underline{A}_i\,\underline{P}^{-1} = \begin{bmatrix} \underline{B}^i_{11} & \underline{0} \\ \underline{B}^i_{21} & \underline{B}^i_{22} \end{bmatrix}$$

where the $\underline{B}^i_{11}$ are square matrices, all of the same dimension. That is, for no choice of basis is the realization in block triangular form. Otherwise it is called reducible. It is a fact ([9], Theorem 3) that every bilinear realization (4) is equivalent to one in which the matrices $\underline{A}_i$ are in block triangular form, with the diagonal blocks being irreducible, thus:

$$\underline{A}_i = \begin{bmatrix} \underline{A}^i_{11} & \underline{0} & \underline{0} & \cdot & \cdot \\ \underline{A}^i_{21} & \underline{A}^i_{22} & \underline{0} & \cdot & \cdot \\ \underline{A}^i_{31} & \underline{A}^i_{32} & \underline{A}^i_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} .$$

We shall call $\underline{x}_0$ an <u>equilibrium state</u> of the bilinear system (4) if $\underline{A}_0\underline{x}_0 = \underline{0}$. This is the same as requiring that $\underline{x}_0$ be an equilibrium solution of (4) when $u_i = 0$ for all i.

We have

<u>Theorem 3.1</u> ([9], Theorem 8): Suppose that we are given two realizations of the same input-output map

$$(4)\quad \begin{cases} \dot{\underline{x}}(t) = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i\underline{A}_i\right)\underline{x} \\ \underline{y} = \underline{C}\,\underline{x} \quad ; \quad \underline{x}(0) = \underline{x}_0 \end{cases}$$

$$(6)\quad \begin{cases} \dot{\underline{z}}(t) = \left(\underline{F}_0 + \sum_{i=1}^{m} u_i\underline{F}_i\right)\underline{z} \\ \underline{y} = \underline{H}\,\underline{z} \quad ; \quad \underline{z}(0) = \underline{z}_0 \quad . \end{cases}$$

Let $\underline{x}_0$ and $\underline{z}_0$ be equilibrium states. Suppose that both realizations are controllable in the sense that the set of states reachable from $\underline{x}_0$ or $\underline{z}_0$ is not confined to a proper linear subspace of the state space for $\underline{x}$ or $\underline{z}$. Suppose also that both systems are observable in the sense that any two starting states (not necessarily $\underline{x}_0$ and $\underline{z}_0$) can be distinguished by means of the measurement of $\underline{y}$, for a suitable choice of $\underline{u}$. Then the two realizations are equivalent.

§ 3.4 <u>The Nature of Solutions for Bilinear Equations</u>

For the linear equations (1) given in § 3.1 it is well known that the explicit solution for $\underline{x}(t)$ in terms of $\underline{u}(t)$ is given, for all t, by

$$(7) \quad \left\{ \quad \underline{x}(t) = e^{\underline{A}t}\underline{x}(0) + \int_0^t e^{\underline{A}(t-\tau)}\underline{B}\,\underline{u}(\tau)d\tau \right. .$$

This formula for $\underline{x}(t)$ is sometimes called the Variation of Constants Formula, and is useful not only for explicit calculation of solutions, but also for determining properties of these solutions.

It is not possible to write down an analogous explicit solution to equation (4) for all t. However, as shown in [8] we can study the intrinsic properties of the solutions of (4) by using the tools provided by the theory of matrix Lie Groups and Lie Algebras ([20], [28], [8], [3], [30], [10]). To this end, we now introduce these concepts.

Let $\mathbb{R}^{nxn}$ denote the set of real nxn matrices; it is a vector space of dimension n^2. A Lie algebra L in $\mathbb{R}^{nxn}$ is a vector subspace of $\mathbb{R}^{nxn}$ which has the property that if $\underline{X}$ and $\underline{Y}$ belong to L, then so does $[\underline{X},\underline{Y}] = \underline{X}\underline{Y}-\underline{Y}\underline{X}$. We call $[\underline{X},\underline{Y}]$ the commutator or Lie product of $\underline{X}$ and $\underline{Y}$. As examples of matrix Lie algebras we have:

Example 1: The set of all real nxn matrices $\mathbb{R}^{nxn}$ is itself a Lie algebra, sometimes called the general linear Lie algebra and denoted $g\ell(n,\mathbb{R})$.

Example 2: The set of all nxn real matrices with zero trace is called the special linear Lie algebra, and denoted $sl(n,\mathbb{R})$.

Example 3: The set of all nxn real skewsymmetric matrices, i.e. those which satisfy $\underline{X}' + \underline{X} = \underline{0}$, is called the orthogonal Lie algebra and denoted $o(n,\mathbb{R})$.

Example 4: The set of all 2n x 2n real symplectic matrices, i.e. those which satisfy $\underline{X}'\underline{J} + \underline{J}\,\underline{X} = \underline{0}$ where

$$\underline{J} = \begin{bmatrix} \underline{0} & -\underline{I} \\ \underline{I} & \underline{0} \end{bmatrix}$$, is called the symplectic Lie algebra and denoted sp(n,$\mathbb{R}$).

Example 5: Consider the three matrices

$$\underline{R}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \underline{R}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \underline{R}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These are a basis for o(3,$\mathbb{R}$), and we find that

$$[\underline{R}_y, \underline{R}_z] = \underline{R}_x, \quad [\underline{R}_z, \underline{R}_x] = \underline{R}_y, \quad [\underline{R}_x, \underline{R}_y] = \underline{R}_z.$$

Example 6: The affine algebra of the line, aff(1), consists of all real matrices of the form $\begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix}$. The two matrices $\underline{X} = \begin{bmatrix} 1 & C \\ 0 & 0 \end{bmatrix}$ and $\underline{Y} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ form a basis, and we find that $[\underline{X},\underline{Y}] = \underline{Y}$.

Given an arbitrary subset of $\mathbb{R}^{nxn}$ we can add additional elements to it so as to imbed it in a Lie algebra. To obtain the smallest Lie algebra which contains a given set N we first add to N all linear combinations of elements of N so as to obtain a vector space N_1. Then we form all possible Lie products of elements in N_1 to obtain a set $[N_1,N_1]$. We add this to N_1 and obtain $N_2 = N_1 + [N_1,N_1]$. If N_2 is not contained in N_1 we $N_3 = N_2 + [N_2, N_2]$, etc. This process must stop in a finite number of steps since otherwise at each stage we increase the dimension of the vector space by at least one, and the dimension cannot exceed n^2. We call the Lie algebra thus obtained the Lie algebra generated by N, and we denote it $\{N\}_A$. For example, in Example 5 above the Lie algebra generated by any two of $\underline{R}_x, \underline{R}_y, \underline{R}_z$ is $o(3,\mathbb{R})$.

Let L be a Lie algebra. The set [L,L] of all possible Lie products of elements in L is called the <u>derived algebra</u> of L and is denoted L'. The derived algebra of L' is denoted L". Continuing, we have the <u>derived series</u>:

$$L \supset L' \supset L'' \supset \cdots L^{(m)} \supset L^{(m+1)} \supset \cdots .$$

The Lie algebra L is said to be <u>solvable</u> if $L^{(m)}$ is zero for some m. We shall call L <u>simple</u> if L' = L. For example o(3) in Example 5 above is simple, while if L = aff(1) as in Example 6, then L' is spanned by Y and L" is zero, so that L is solvable. It is a fact that any Lie algebra can be decomposed into the semidirect sum of simple and solvable parts, [28].

For each $\underline{X}$ in the Lie algebra L we define the operator $ad_{\underline{X}}$ by

$$ad_{\underline{X}}\, \underline{Y} = [\underline{X},\underline{Y}] \text{ for all } \underline{Y} \in L.$$

Powers of $ad_{\underline{X}}$ are defined by

$$ad_{\underline{X}}^n\, \underline{Y} = \underbrace{[\underline{X},[\underline{X},\cdots,[\underline{X},\underline{Y}]\cdots]]}_{n \text{ times}} .$$

Using this notation, we can state the following result, which we use in § 3.5:

<u>Lemma</u> (Baker-Hausdorff): If $\underline{X},\underline{Y}$ are elements of the Lie algebra L then $e^{\underline{X}}\,\underline{Y}e^{-\underline{X}} \in L$ and $e^{\underline{X}}\underline{Y}e^{-\underline{X}} = \left(e^{ad_{\underline{X}}}\right)\underline{Y} = \underline{Y} + [\underline{X},\underline{Y}] + \frac{1}{2!}[\underline{X},[\underline{X},\underline{Y}]]$

$$+ \frac{1}{3!}[\underline{X},[\underline{X},[\underline{X},\underline{Y}]]]+\cdots .$$

For a proof of this the reader is referred to [19]. (To see how the series is obtained let

$$f(s) = e^{s\underline{X}}\underline{Y}e^{-s\underline{X}}$$

Then $$f'(s) = e^{s\underline{X}}\underline{X}\,\underline{Y}\,e^{-s\underline{X}} - e^{s\underline{X}}\underline{Y}\,\underline{X}\,e^{-s\underline{X}}$$

$$= e^{s\underline{X}}[\underline{X},\underline{Y}]e^{-s\underline{X}}$$

Similarly $$f^{(n)}(s) = e^{s\underline{X}}\,\mathrm{ad}^n_{\underline{X}}\,\underline{Y}\quad e^{-s\underline{X}}$$

Then $$f(s) = f(0) + sf'(0) + \frac{s^2}{2!}f''(0) + \cdots$$

$$= \underline{Y} + s[\underline{X},\underline{Y}] + \frac{s^2}{2!}[\underline{X},[\underline{X},\underline{Y}]] + \cdots$$

and the result follows on putting $s = 1$.)

Having introduced matrix Lie algebras we next introduce the concept of a matrix Lie Group. If M is a set of nonsingular matrices in $\mathbb{R}^{n\times n}$, we let $\{M\}_G$ denote the multiplicative matrix group generated by M, i.e. it is the smallest set of matrices in $\mathbb{R}^{n\times n}$ which contains M and which is closed under multiplication and inversion. If N is a linear subspace of $\mathbb{R}^{n\times n}$, then let P be the set of all matrices of the form $e^{\underline{X}_1} e^{\underline{X}_2} \cdots e^{\underline{X}_p}$ where $\underline{X}_i \in N$ for each i and $p = 0, 1, 2, \cdots$. P contains no singular matrices since for any matrix $\underline{X}_i$, $\det(e^{\underline{X}_i}) = e^{\mathrm{tr}\,\underline{X}_i} > 0$, ([7], section 4). Since it is closed under multiplication and inversion, P is a group, and we write $P = \{\exp N\}_G$. It is an interesting and useful fact that $\{\exp N\}_G = \{\exp \{N\}_A\}_G$ ([8] Theorem 1.)

If L is a Lie algebra then we call $\{\exp L\}_G$ the <u>Lie group associated with L</u>. For a full treatment of the relationship between Lie groups and Lie algebras the reader is referred to [30] and [10]. We now give the

Lie groups which are associated with the six Lie algebras given above:

Example 1: The general linear group $GL(n,\mathbb{R})$ consists of all real nxn invertible matrices.

Example 2: The special linear group $SL(n,\mathbb{R})$ consists of all real nxn matrices with determinant 1.

Example 3: The orthogonal group $O(n,\mathbb{R})$ consists of the real nxn matrices which satisfy $\underline{X}'\underline{X} = \underline{I}$. The special orthogonal group $SO(n)$ consists of all matrices in both $O(n)$ and $SL(n,\mathbb{R})$.

Example 4: The symplectic group $Sp(n,\mathbb{R})$ consists of the real 2n x 2n matrices which satisfy $\underline{X}'\underline{J}\,\underline{X} = \underline{I}$ where

$$\underline{J} = \begin{bmatrix} \underline{0} & -\underline{I} \\ \underline{I} & \underline{0} \end{bmatrix}$$

Example 5: The group $SO(3)$ consists of all real 3 x 3 matrices which have determinant +1 and satisfy $\underline{X}'\underline{X} = \underline{I}$.

Example 6: The affine group of the line consists of all real 2x2 matrices whose second row is [0 1]. Any such matrix is of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, and represents a transformation $y = ax+b$ of the real line.

Now we can discuss the question of obtaining solutions to the equation

$$(8) \quad \left\{ \; \dot{\underline{x}} = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i \underline{A}_i \right) \underline{x} \; , \; \underline{x}(0) = \underline{x}_0 \in \mathbb{R}^n \; . \right.$$

It is well known ([7], section 3) that the solution to (8) is given by

$$(9) \quad \left\{ \; \underline{x}(t) = \underline{\Phi}(t)\underline{x}_0 \right.$$

where $\underline{\Phi}(t)$ is an nxn matrix, called the transition matrix for (8), which is the solution of

$$(10) \quad \left\{ \quad \dot{\underline{\Phi}}(t) = \left(\underline{A}_0 + \sum_{i=1}^{m} u_i \underline{A}_i \right) \underline{\Phi} \quad , \quad \underline{\Phi}(0) = \underline{I} \; . \right.$$

We are, therefore, interested in solutions of equation (10). Now it can be shown ([8], Theorem 5) that for all t, $\underline{\Phi}(t)$ belongs to the matrix Lie Group $\{\exp\{\underline{A}_i\}_A\}_G$, that is, the matrix $\underline{\Phi}(t)$ will evolve on the Lie Group associated with the Lie Algebra generated by the coefficient matrices $\underline{A}_0, \underline{A}_1, \cdots, \underline{A}_m$.

There are two important consequences of this fact for power conversion networks described by bilinear equations. The first of these stems from the fact that in addition to having the properties of a group, a Lie Group has the properties of a manifold, that is, a subset of Euclidean space $\mathbb{R}^d$ with special geometrical characteristics. The two-dimensional surface of a sphere in three-dimensional space is an example of a manifold. The geometrical characteristics of the particular manifold on which the state of a bilinear system evolves will play a fundamental role in determining the nature of the behavior of the system. For example, if the Lie group is compact, i.e. closed and bounded as a subset of $\mathbb{R}^{nxn}$, then we know that the state of the system is bounded, i.e. the amount of energy stored in the inductors and capacitors will be finite. For instance, the Lie group SO(n) is bounded, and the group Sp(n) is not. Thus, given a bilinear system of equations, the natural first question we ask is "On what Lie group does its state transition matrix evolve?"

The second consequence is that questions about the system (such as controllability, reachability, observability, and stabilizability) can be reduced to questions about the Lie algebra generated by the coefficient matrices. In many cases it is possible to arrive at conditions which are easily visualized and tested, as for example our results of Chapter 4, or the results of reference [8]. Of relevance here is the classification result from the theory of Lie algebras, by which any simple Lie algebra is shown to be equivalent to one of a short list of canonical algebras, ([28] Chapter 2, [24]). For solvable algebras there is no such complete list of all possibilities, although a good many facts are known, [28]. Note that if $\underline{P}$ is a nonsingular matrix, equation (8) is unchanged by the change of variable $\underline{x} \to \underline{x}\,\underline{P}$, and equation (10) is unchanged by $\underline{\Phi} \to \underline{\Phi}\,\underline{P}$.

Now let us further discuss the problem of obtaining a solution to equation (10). Let $\{\underline{X}_1, \underline{X}_2, \cdots \underline{X}_k\}$ be a basis for the Lie algebra L generated by $\{\underline{A}_0, \underline{A}_1, \cdots, \underline{A}_m\}$. In reference [19] Magnus showed that there exists a t_0 such that for $0 \leqslant t < t_0$ the solution to (10) can be expressed in the form

$$(11) \quad \left\{ \quad \underline{\Phi}(t) = e^{b_1(t)\underline{X}_1 + b_2(t)\underline{X}_2 + \cdots + b_k(t)\underline{X}_k} \right.$$

where $b_1, \cdots, b_k$ are scalar functions of time satisfying differential equations which depend on L and $u_1, \cdots, u_m$. The difficulties with this approach to obtaining a solution are that it is difficult to derive and solve the differential equations for $b_1, \cdots, b_k$, and that only in severely restricted cases [31] is the representation (11) valid for all t in $[0, \infty)$.

Wei and Norman [31] showed that it is often preferable to look for a solution in the form of a product of exponentials. In fact they showed that there exists a t_0 such that for $0 \leqslant t < t_0$ the solution to (10) can be expressed in the form

$$(12)\ \left\{\ \underline{\Phi}(t) = e^{g_1(t)\underline{X}_1}\ e^{g_2(t)\underline{X}_2}\ \cdots\ e^{g_k(t)\underline{X}_k}\right.$$

where $g_1, g_2, \cdots, g_k$ are scalar functions of time satisfying differential equations which depend on L and on $u_1, \cdots u_m$. Moreover, if L is solvable, or if the matrices are 2x2, then this representation is global, i.e. it holds for all $0 \leqslant t < \infty$. In § 3.5(c) we give an example of such a global representation.

In conclusion we note that while in some cases it may be possible to analyze the behavior of electrical power conversion networks with piecewise-linear models, concepts from the theory of Lie groups and Lie algebras are useful in characterizing the inherent dynamical features of such systems, especially since the methods and conclusions are basis-free, i.e. they do not depend on the particular basis chosen for the state space. We give examples in § 3.5.

§ 3.5 Network Examples

(a) An SO(3) Network

Fig. 3.8 shows a simple network in which charge stored on one capacitor can be transferred by means of the inductor to the other capacitor. If the capacitances are different a voltage conversion will be effected. A transferral cycle might be executed as follows. Starting from $V_1(0) = V_{10}$, $I_3(0) = 0$, $V_2(0) = 0$, the switch is held in the u = 0 position until $V_1 = 0$ and all the energy is stored in L_3, at which time the switch is changed to the u = 1 position. It is held there until I_3 again becomes zero, at which time the switch is reverted to the u = 0 position. All the

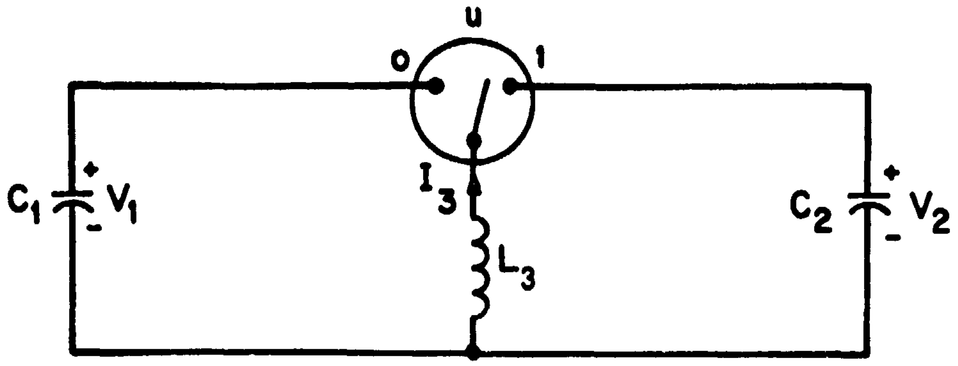


Fig. 3.8

initial energy is now stored in C_2, with $V_2 = -\frac{C_1}{C_2} V_{10}$. While this network is too simple to model a complete DC-DC convertor, it may very well model the conversion portion of a convertor in which the charge on C_1 is replenished from an external supply during the u = 1 portion of the cycle, and a load current is drawn from C_2 during the u = 0 portion. The equations for this network are

$$\begin{cases} C_1\dot{V}_1 = (1-u)I_3 \\ C_2\dot{V}_2 = u\,I_3 \\ L_3\dot{I}_3 = -(1-u)V_1 - uV_2 \end{cases} .$$

Letting $V_i = \frac{x_i}{\sqrt{C_i}}$ for i = 1, 2 and $I_3 = \frac{x_3}{\sqrt{L_3}}$ so that $\frac{1}{2}(\underline{x}'\underline{x})$ is the total stored energy, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ -\omega_1 & 0 & 0 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & -\omega_1 \\ 0 & 0 & \omega_2 \\ \omega_1 & -\omega_2 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $\omega_1 = \dfrac{1}{\sqrt{L_3C_1}}$, $\omega_2 = \dfrac{1}{\sqrt{L_3C_2}}$,

i.e. $$\dot{\underline{x}} = \left[(1-u)\underline{A}_1 + u\underline{A}_2\right]\underline{x}$$

where $$\underline{A}_1 = \begin{bmatrix} 0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ -\omega_1 & 0 & 0 \end{bmatrix} \qquad \underline{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_2 \\ 0 & -\omega_2 & 0 \end{bmatrix} .$$

We see that $\underline{A}_1$ and $\underline{A}_2$ are just scalar multiples of $\underline{R}_y$ and $\underline{R}_x$ of the Lie algebra Example (5) in § 3.4. Thus the Lie algebra generated by $\underline{A}_1$ and $\underline{A}_2$ is $o(3)$, and the transition matrix $\underline{\Phi}(t)$ for this network evolves on the Lie group SO(3). The state vector $\underline{x}(t)$ evolves on the 2-sphere S^2.

(b) Simple and Solvable Parts

If we add a current sink in parallel with C_2 of Fig.3.8, to represent a load for instance, we obtain the network of Fig. 3.9.

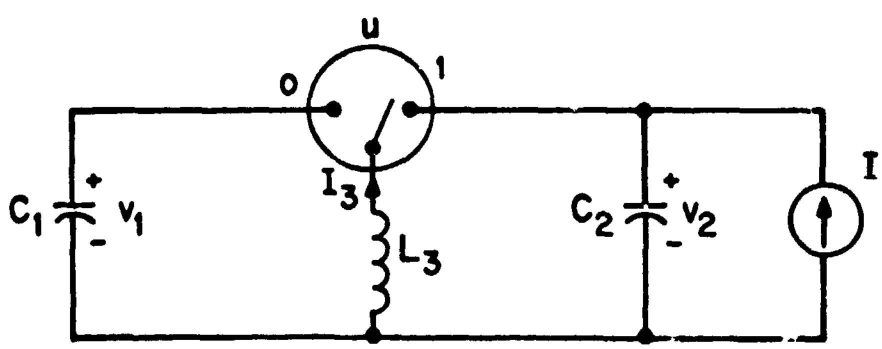


Fig. 3.9

The state equations are now

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ -\omega_1 & 0 & 0 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & -\omega_1 \\ 0 & 0 & \omega_2 \\ \omega_1 & -\omega_2 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix}$$

where $\gamma = \dfrac{I}{\sqrt{C_2}}$.

Letting $x_4 = 1$ we can use the method of § 3.2 to put this in bilinear form, thus:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & -\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + u \begin{bmatrix} 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & \omega_2 & 0 \\ \omega_1 & -\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

i.e. $\quad \underline{\dot{x}} = (\underline{B}_0 + u\underline{B}_1)\underline{x}$.

The Lie algebra L generated by $\underline{B}_0$ and $\underline{B}_1$ is six-dimensional and has as a basis the following matrices:

$$\underline{X}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{X}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{X}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{X}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underline{X}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \underline{X}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that $\underline{X}_1$, $\underline{X}_2$, $\underline{X}_3$ are a reducible representation of o(3), while $\underline{X}_4$, $\underline{X}_5$, $\underline{X}_6$ represent a solvable Lie algebra. Thus L is decomposable into simple and solvable parts, with the simple part determined by the "natural dynamics" of the network, i.e. the interconnection of its inductors and capacitors, while the solvable part is contributed by the "driving forces", i.e. the batteries and current sources. This type of decomposition is a general characteristic of the types of networks we are considering, (as introduced at the beginning of § 3.2).

(c) A Transformerless DC-DC Convertor

If we now replace the capacitor C_1 of Fig. 3.9 by a battery and allow the switch to take a third position in which the inductor is unconnected, we obtain a model of a simple DC-DC convertor, as shown in Fig. 3.10.

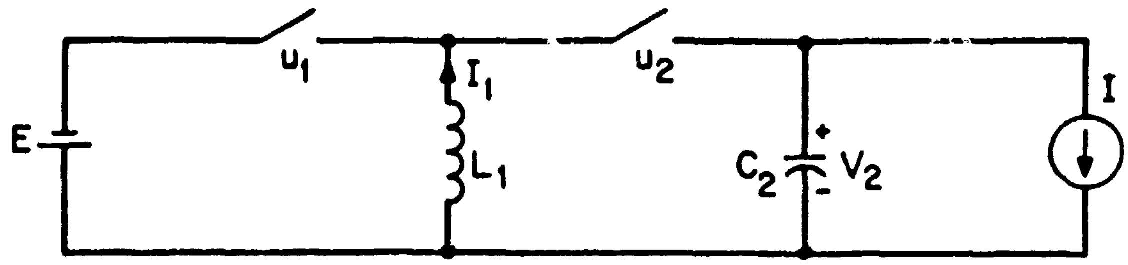


Fig. 3.10

We have used two switches in Fig. 3.10 to represent the switching action; $u_i = 1$ denotes that switch i is closed, and $u_i = 0$ denotes that switch i is open, where i = 1, 2. To ensure that the model is well-posed (as discussed in § 3.2) we require (a) that $u_1(t)u_2(t) = 0$ for all t, (i.e. both switches cannot be closed simultaneously), and (b) that if one switch is open, the other cannot be opened unless $I_1 = 0$. The conversion cycle we envisage is similar to that described in example (a) above, i.e. first we set $u_1 = 1$ until I_1 reaches some predetermined desired value, then we let $u_2 = 1$ until I_1 is again zero. In order to obtain a smoother output voltage it may be desirable to use a low-pass filter at the output, as depicted in Fig. 3.11.

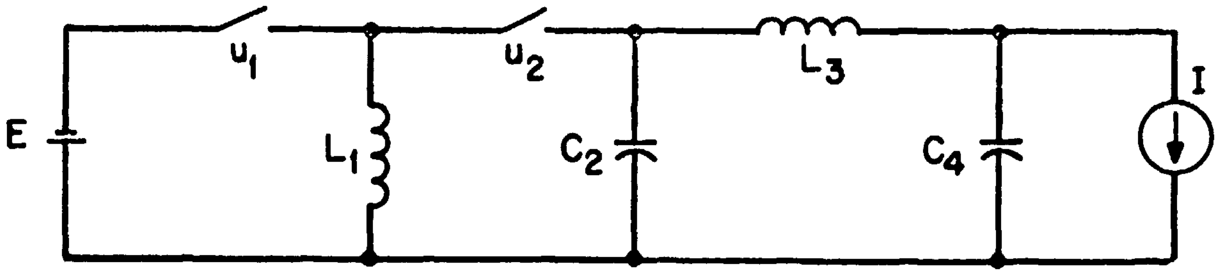


Fig. 3.11

The network of Fig. 3.10 might be implemented with a scheme such as that shown in Fig. 3.12, in which $u_1 = 1$ when transistor T_1 is turned on, and $u_2 = 1$ when the current in diode D_1 is nonzero.

We shall now illustrate the method of Wei and Norman for obtaining solutions to the state-evolution equations of the network of Fig. 3.10. These are:

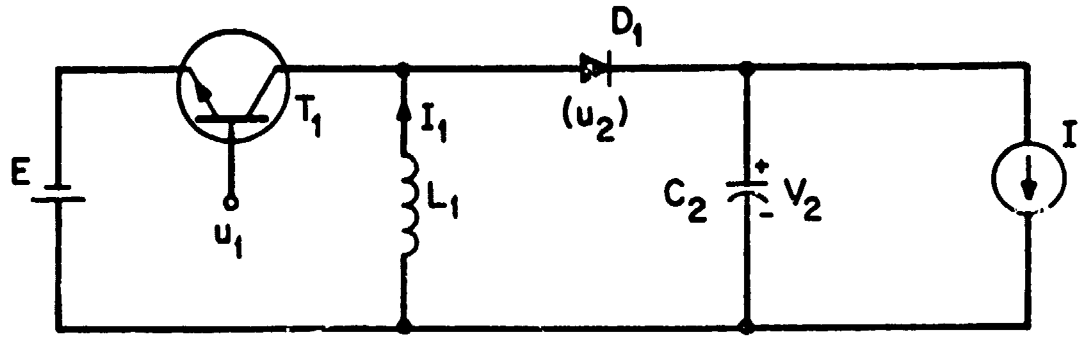


Fig. 3.12

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma \\ 0 & 0 & 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $x_1 = I_1\sqrt{L_1}$, $x_2 = V_2\sqrt{C_2}$, $x_3 = 1$, $\alpha = \dfrac{1}{\sqrt{L_1C_2}}$, $\beta = \dfrac{E}{\sqrt{L_1}}$, $\gamma = \dfrac{I}{\sqrt{C_2}}$,

i.e. $\dot{\underline{x}} = (\underline{A}_0 + u_1\underline{A}_1 + u_2\underline{A}_2)\underline{x}$.

We find that $[\underline{A}_0, \underline{A}_1] = \underline{0}$, $[\underline{A}_1, \underline{A}_2] = \dfrac{\alpha\beta}{\gamma}\underline{A}_0$, $[\underline{A}_2, \underline{A}_0] = \dfrac{\alpha\gamma}{\beta}\underline{A}_1$. Thus the Lie algebra generated by $\underline{A}_0$, $\underline{A}_1$, $\underline{A}_2$ is solvable and has $\underline{A}_0$, $\underline{A}_1$, $\underline{A}_2$ as a basis. By the Wei-Norman result we therefore know that there exist functions $g_0(t)$, $g_1(t)$, $g_2(t)$ such that for all t in $[0, \infty)$,

$$\underline{\Phi}(t) = e^{g_0\underline{A}_0} e^{g_1\underline{A}_1} e^{g_2\underline{A}_2}$$

where $\dot{\underline{\Phi}} = (\underline{A}_0 + u_1\underline{A}_1 + u_2\underline{A}_2)\underline{\Phi}$ and $\underline{\Phi}(0) = \underline{I}$.

To obtain the differential equations satisfied by g_0, g_1, g_2 we look at

$$\dot{\underline{\Phi}} = \dot{g}_0\underline{A}_0 e^{g_0\underline{A}_0} e^{g_1\underline{A}_1} e^{g_2\underline{A}_2} + e^{g_0\underline{A}_0}\dot{g}_1\underline{A}_1 e^{g_1\underline{A}_1} e^{g_2\underline{A}_2} + e^{g_0\underline{A}_0} e^{g_1\underline{A}_1}\dot{g}_2\underline{A}_2 e^{g_2\underline{A}_2}$$

$$= \dot{g}_0\underline{A}_0\,\underline{\Phi} + \dot{g}_1\left(e^{g_0\underline{A}_0}\underline{A}_1 e^{-g_0\underline{A}_0}\right)\underline{\Phi} + \dot{g}_2\left(e^{g_0\underline{A}_0} e^{g_1\underline{A}_1}\underline{A}_2 e^{-g_1\underline{A}_1} e^{-g_0\underline{A}_0}\right)\underline{\Phi} \quad .$$

We now make use of the Baker-Hausdorff lemma (§ 3.4) to obtain:

$$e^{g_0\underline{A}_0}\underline{A}_1 e^{-g_0\underline{A}_0} = \underline{A}_1 + g_0[\underline{A}_0,\underline{A}_1] + \frac{g_0^2}{2}[\underline{A}_0,[\underline{A}_0,\underline{A}_1]] + \cdots$$

$$= \underline{A}_1 \quad .$$

Similarly

$$e^{g_1\underline{A}_1}\underline{A}_2 e^{-g_1\underline{A}_1} = \underline{A}_2 + \left(\frac{\alpha\beta}{\gamma}\right) g_1\underline{A}_0$$

and
$$e^{g_0\underline{A}_0}\underline{A}_2 e^{-g_0\underline{A}_0} = \underline{A}_2 - \left(\frac{\alpha\gamma}{\beta}\right) g_0\underline{A}_1 \quad .$$

We therefore obtain

$$\dot{\underline{\Phi}} = \left[\left(\dot{g}_0 + \frac{\alpha\beta}{\gamma} g_1\dot{g}_2\right)\underline{A}_0 + \left(\dot{g}_1 - \frac{\alpha\gamma}{\beta} g_0\dot{g}_2\right)\underline{A}_1 + \dot{g}_2\underline{A}_2\right]\underline{\Phi}$$

which, on comparison with the defining equation for $\underline{\Phi}$, yields

$$\begin{cases} \dot{g}_0 + \frac{\alpha\beta}{\gamma} g_1 \dot{g}_2 = 1 \\ \dot{g}_1 - \frac{\alpha\gamma}{\beta} g_0 \dot{g}_2 = u_1 \\ \dot{g}_2 = u_2 \end{cases} .$$

Since $\underline{\Phi}(0) = \underline{I}$ we have $g_0(0) = g_1(0) = g_2(0) = 0$. We can therefore write the defining differential equations for g_0, g_1, g_2 as

$$\begin{bmatrix} \dot{g}_0 \\ \dot{g}_1 \\ \dot{g}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-\alpha\beta}{\gamma} g_1 \\ 0 & 1 & \frac{\alpha\gamma}{\beta} g_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} g_0(0) \\ g_1(0) \\ g_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

(d) Two Fourth-Order Lossless Networks

Lossless electrical networks are of interest to us since they may define that part of a power conversion network corresponding to the simple part of the Lie algebra, as discussed in example (b) above. Here we show how a small change in network topology yields a fundamental change in the associated Lie algebra. Consider the network of Fig. 3.13, where we assume that the two switches are operated synchronously, i.e. the single control variable u denotes the state of both switches. As usual we let $x_1 = I_1\sqrt{L_1}$, $x_2 = V_2\sqrt{C_2}$, etc., and we let $\alpha = \frac{1}{\sqrt{L_1C_2}}$, $\beta = \frac{1}{\sqrt{L_3C_4}}$, $\gamma = \frac{1}{\sqrt{L_1C_4}}$, $\delta = \frac{1}{\sqrt{L_3C_2}}$.

Then we obtain

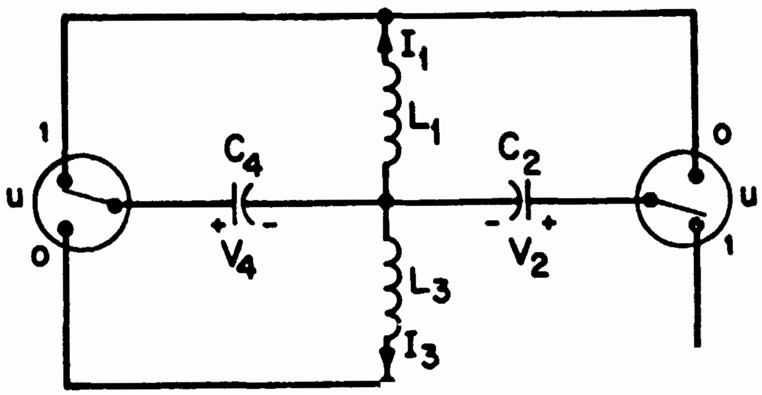


Fig. 3.13

$$\dot{\underline{x}} = (\underline{A}_0 + u\underline{A}_1)\underline{x}$$

where

$$\underline{A}_0 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{bmatrix} \quad \underline{A}_1 = \begin{bmatrix} 0 & 0 & 0 & -\gamma \\ 0 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{bmatrix} .$$

Now the Lie algebra generated by $\underline{A}_0$ and $\underline{A}_1$ is six-dimensional, except when $\alpha = \beta$ or $\gamma = \delta$ when it is four-dimensional. If $\alpha = \beta$ and $\gamma = \delta$ then $\alpha = \beta = \gamma = \delta$ and the Lie algebra is two-dimensional, with basis $\underline{A}_0$, $\underline{A}_1$. Now the network of Fig. 3.13 can be redrawn as shown in Fig. 3.14.

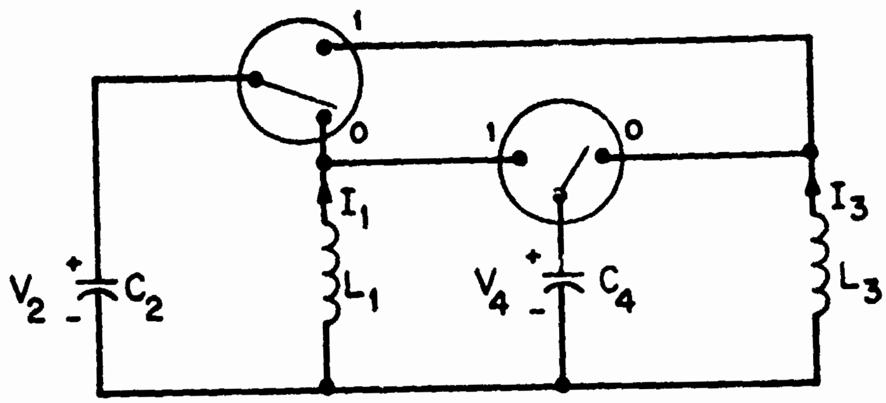


Fig. 3.14

We can modify this network slightly by adding another synchronized switch which has the effect of reversing the polarity of C_4. Fig. 3.15 depicts this situation.

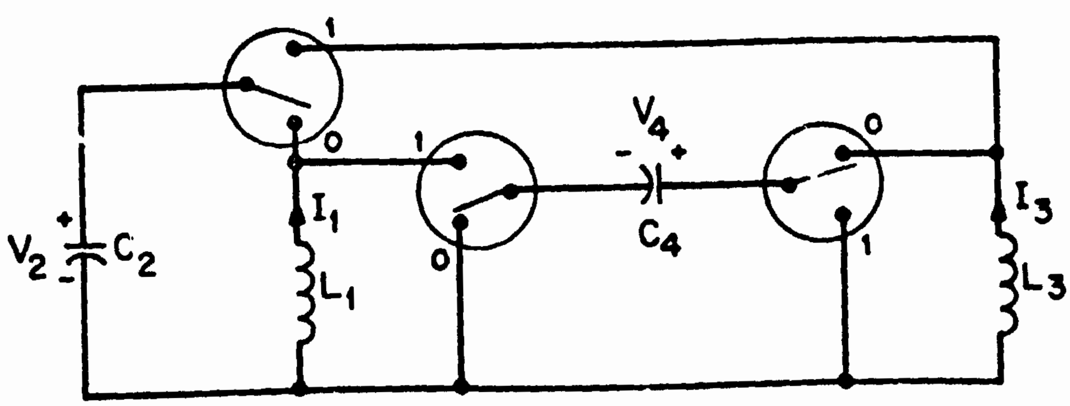


Fig. 3.15

For this we obtain

$$\dot{\underline{x}} = (\underline{A}_0 + u\underline{A}_1)\underline{x}$$

where

$$\underline{A}_0 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{bmatrix} \quad \underline{A}_1 = \begin{bmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 0 \\ -\gamma & 0 & 0 & 0 \end{bmatrix}$$

with α, β, γ, δ defined as before. Again we find that the Lie algebra generated by $\underline{A}_0$ and $\underline{A}_1$ is six-dimensional, except when $\alpha = \beta$ or $\gamma = \delta$ when it is four-dimensional. This time however, if $\alpha = \beta = \gamma = \delta$ then the Lie algebra is three-dimensional, and in fact is a representation for the Lie algebra o(3). Thus, when $\alpha = \beta = \gamma = \delta$ the state transition matrix for the network of Fig. 3.15 evolves on the Lie group SO(3).

(e) Higher-Order SO(3) Networks

The abstract Lie algebra o(3) is defined by the relationships

$$[\underline{S}_y, \underline{S}_z] = \underline{S}_x, \ [\underline{S}_z, \underline{S}_x] = \underline{S}_y, \ [\underline{S}_x, \underline{S}_y] = \underline{S}_z \quad .$$

A representation of this abstract Lie algebra is a set of three matrices which satisfy these relationships. A representation is said to be irreducible if its component matrices cannot simultaneously be put in block triangular form, as in § 3.3. Now it can be shown ([28], Chapter 1) that, over the complex field, all irreducible nxn irreducible representations of o(3) are equivalent to the following representation, where $J = \frac{n-1}{2}$ and $\mu_i = i(n-i)$:

$$\underline{S}_x = -\frac{1}{2}\sqrt{-1}\begin{bmatrix} 0 & \sqrt{\mu_1} & 0 & & & \\ \sqrt{\mu_1} & 0 & \sqrt{\mu_2} & & & \\ 0 & \sqrt{\mu_2} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{\mu_{n-1}} \\ & & & & \sqrt{\mu_{n-1}} & 0 \end{bmatrix}$$

$$\underline{S}_y = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{\mu_1} & 0 & & & \\ -\sqrt{\mu_1} & 0 & \sqrt{\mu_2} & & & \\ 0 & -\sqrt{\mu_2} & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \sqrt{\mu_{n-1}} \\ & & & & -\sqrt{\mu_{n-1}} & 0 \end{bmatrix}$$

$$\underline{S}_z = \sqrt{-1}\begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & J-2 & & \\ & & & \ddots & \\ & & & & -J \end{bmatrix}.$$

An nxn complex representation of o(3) can be made into a 2n x 2n real representation by identifying a given complex matrix ($\underline{R} + \sqrt{-1}\,\underline{Q}$) with the real matrix

$$\begin{bmatrix} \underline{R} & \underline{Q} \\ -\underline{Q} & \underline{R} \end{bmatrix} .$$

For instance, when n = 3 we obtain the 6 x 6 real representation of o(3) given by

$$\underline{A}_x = \frac{1}{\sqrt{2}}\left[\begin{array}{ccc|ccc} & & & 0 & -1 & 0 \\ & & & -1 & 0 & -1 \\ & & & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & & & \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \end{array}\right], \quad \underline{A}_y = \frac{1}{\sqrt{2}}\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ -1 & 0 & 1 & & & \\ 0 & -1 & 0 & & & \\ \hline & & & 0 & 1 & 0 \\ & & & -1 & 0 & 1 \\ & & & 0 & -1 & 0 \end{array}\right]$$

$$\underline{A}_z = \left[\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right] .$$

Suppose now that we would like to find a sixth-order network which has the state equations

$$\dot{\underline{x}} = \left[u\underline{A}_x + (1-u)\underline{A}_y\right]\underline{x} \quad .$$

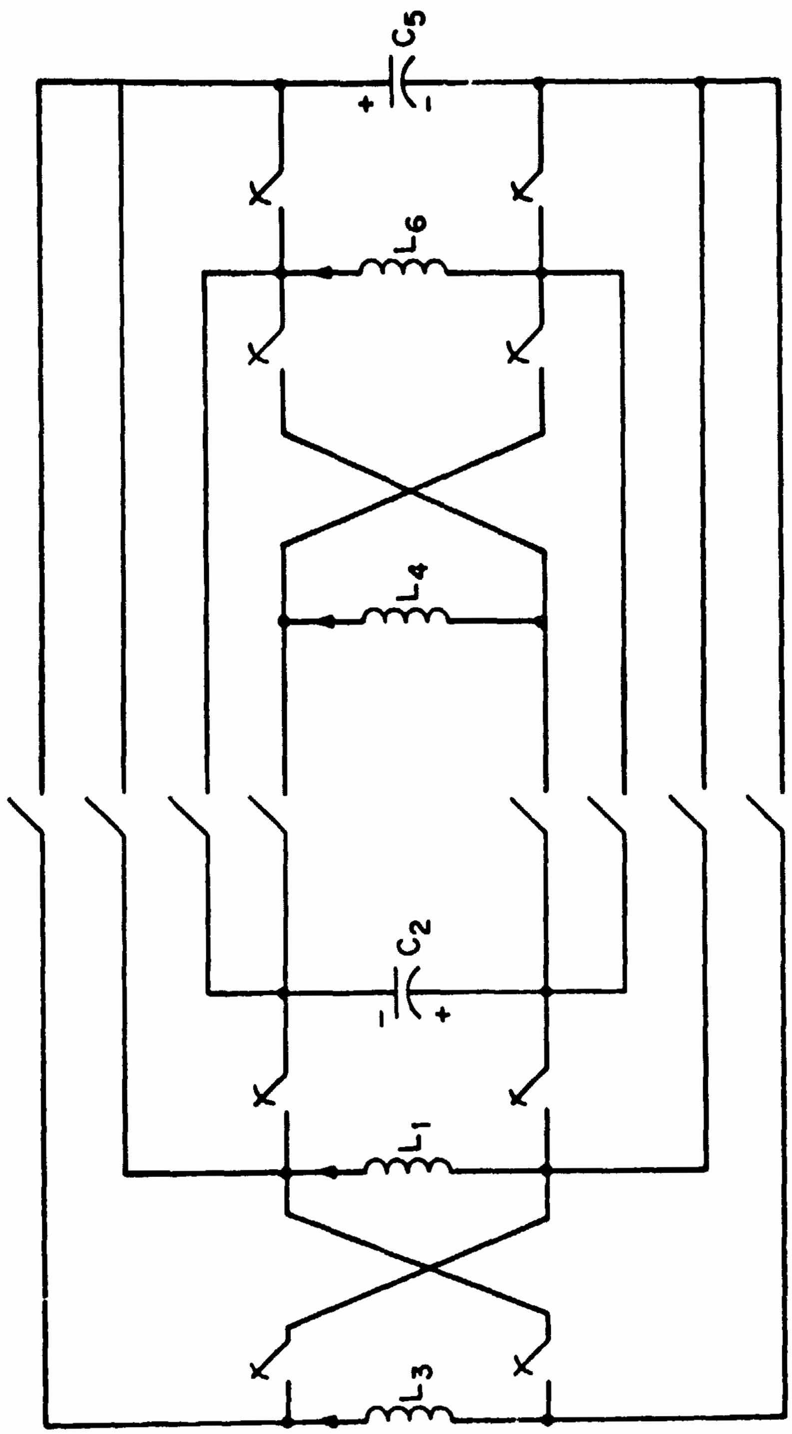


Fig. 3.16

Sixth Order SO(3) Network

To do this we examine $\underline{A}_x$ and $\underline{A}_y$ and reverse the process by which we normally obtain the state equations for a network. Assume that x_1 is an inductor current. We see from $\underline{A}_x$ that when $u = 1$ we need element 5 to be a capacitor C_5 and element 3 to be an inductor L_3, with L_1 and L_3 connected in parallel with C_5. From $\underline{A}_y$ we see that when $u = 0$ we need L_1 and L_3 in parallel with a capacitor C_2, and C_5 in parallel with inductors L_4 and L_6. The resulting network obtained by this process is shown in Fig. 3.16 in which the switches all operate synchronously: the switches denoted are closed when $u = 1$ and open when $u = 0$, while those denoted are closed when $u = 0$ and open when $u = 1$.

By a similar process we can construct a network of order 2n for any $n \geq 2$ whose state transition matrix evolves on the Lie group SO(3).

(f) A Reducible Network

Our final example shows how reducibility of the Lie algebra can correspond to reducibility of the network. In Fig. 3.17 we wish to transfer energy from the battery E to the output capacitor C_2. We might ask whether an unlimited amount of energy can be extracted from the battery. The two switches are operated synchronously. In a similar manner as before we obtain the state evolution equations as

$$\dot{\underline{x}} = (\underline{A}_0 + u\underline{A}_1)\ \underline{x}$$

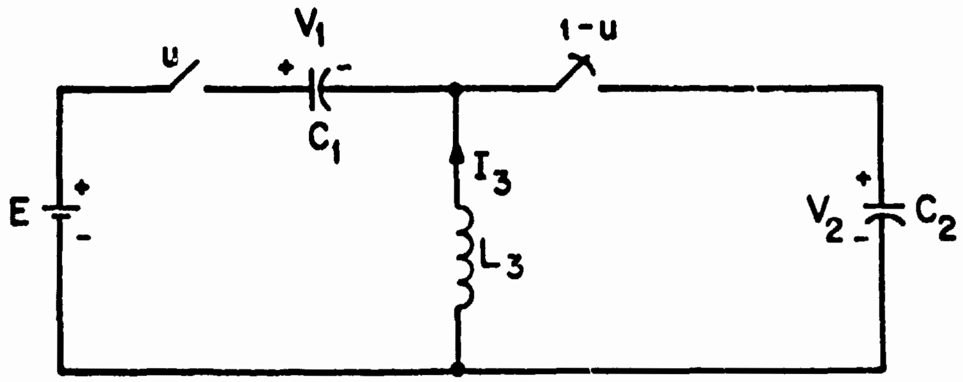


Fig. 3.17

where

$$\underline{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ 0 & -\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{A}_1 = \begin{bmatrix} 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & -\omega_2 & 0 \\ \omega_1 & \omega_2 & 0 & -\gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\omega_1 = \dfrac{1}{\sqrt{L_3C_1}}$, $\omega_2 = \dfrac{1}{\sqrt{L_3C_2}}$, $\gamma = \dfrac{E}{\sqrt{L_3}}$.

The Lie algebra generated by $\underline{A}_0$ and $\underline{A}_1$ is o(3), and thus we conclude that this network is dynamically similar to that of Fig. 3.8 and that only a finite amount of energy can be stored in C_1, C_2, L_3 since the associated Lie group is bounded. This may at first seem surprising, since the network of Fig. 3.17 has a battery in it. In the case $\omega_1 = \omega_2 = 1$ the Lie algebra has as a basis

$$\underline{X}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{X}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underline{X}_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $[\underline{X}_2,\underline{X}_3] = \underline{X}_1$, $[\underline{X}_3,\underline{X}_1] = \underline{X}_2$, $[\underline{X}_1,\underline{X}_2] = \underline{X}_3$.

Examination of the network of Fig. 3.17 shows that we can reduce it to a simpler form. The network is unchanged if the switch u and capacitor C_1 are interchanged, giving the network of Fig. 3.8 but for a battery in series with C_1. Now the total energy which can be extracted from the combination of Fig.3.18 is $\frac{1}{2}CE^2$. (This is an interesting singular

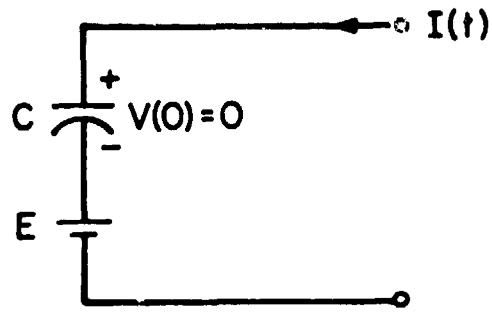


Fig. 3.18

optimal control problem which may be solved by observing that the arrangement of Fig. 3.18 is externally identical with that of Fig. 3.19, since both are governed by

$$V(t) = E + \frac{1}{C}\int_0^t I(\tau)\,d\tau \quad . \qquad)$$

In fact we see that the network of Fig. 3.17 is essentially the network of Fig. 3.8, but with the capacitor C_1 being given an extra initial voltage

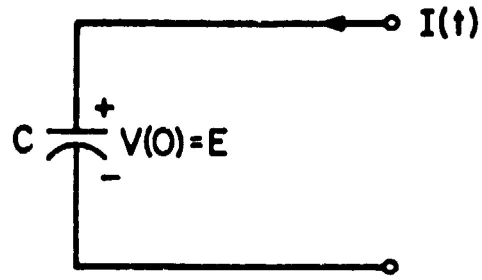


Fig. 3.19

of E. The matrices $\underline{X}_1$, $\underline{X}_2$, $\underline{X}_3$ given here are a reducible representation of the Lie algebra o(3): they can be reduced by eliminating from each one the last row and column, which represents the contribution of the batteries and current sources in the network, as we saw in example 3.5(b).

CHAPTER 4

FEEDBACK STABILIZATION OF BILINEAR SYSTEMS

In Chapter 3 we showed that bilinear systems arise naturally in the study of power conversion networks. Since it will in many cases be desired to stabilize such systems by means of state feedback, we are interested in the general study of feedback stabilization for bilinear systems. In Chapter 1 we discussed this question for a certain class of DC-DC conversion networks. In Chapter 4 we shall address ourselves to the question of feedback stabilization of systems which evolve on the (n-1) sphere S^{n-1}, i.e. systems whose state vector $\underline{x} \in \mathbb{R}^n$ satisfies $\underline{x}'\underline{x}$ = constant. We have in mind two specific questions concerning the feedback stabilization of a general bilinear system: (i) If the system is controllable (in some approximately defined sense), can we find a feedback law such that the closed-loop system is asymptotically stable about a particular point? (ii) If the system is controllable, can we find a feedback law such that a given oscillation is stabilized for the closed-loop system?

<u>Theorem 4.1</u> Consider the system of equations

$$\dot{\underline{x}} = \sum_{i=1}^{m} u_i \underline{A}_i \underline{x} \quad , \quad \underline{x}(t) \in \mathbb{R}^n \quad , \quad \underline{x}'(0)\underline{Q}\,\underline{x}(0) = 1,$$

where $\underline{Q}\,\underline{A}_i + \underline{A}_i'\underline{Q} = \underline{0}$ for $1 \leq i \leq m$ and $\underline{Q}' = \underline{Q} > \underline{0}$. Given some $\underline{x}_0$ which satisfies $\underline{x}'_0\underline{Q}\,\underline{x}_0 = 1$, suppose that the matrix whose columns are $\underline{Q}\,\underline{A}_1\underline{x}_0$, $\underline{Q}\,\underline{A}_2\underline{x}_0, \cdots, \underline{Q}\,\underline{A}_m\underline{x}_0$ has rank n - 1. Let $\underline{u} = [u_1\ u_2\ \cdots\ u_m]$. Then there exists a feedback control law $\underline{u} = \underline{u}(\underline{x})$ such that the closed-loop feedback system is asymptotically stable from any starting point $\underline{x}(0)$ other than $-\underline{x}_0$.

<u>Proof</u>. Let M denote the set of points $\underline{x}$ satisfying $\underline{x}'\underline{Q}\underline{x} = 1$; when $\underline{Q} = \underline{I}$ then M is the (n-1) sphere S^{n-1}. First, note that we have $\underline{x}'(t)\underline{Q}\,\underline{x}(t)=1$ for all t in $[0,\infty)$ if $\underline{x}(0) \in M$, since $\frac{d}{dt}\underline{x}'\underline{Q}\underline{x} = \dot{\underline{x}}'\underline{Q}\underline{x} + \underline{x}'\underline{Q}\dot{\underline{x}} = 0$ since $\underline{Q}\underline{A}_i + \underline{A}_i'\underline{Q} = \underline{0}$ for each i. Next, consider the Lyapunov function $V = \frac{1}{2}(\underline{x} - \underline{x}_0)'\underline{Q}(\underline{x} - \underline{x}_0)$. Since $\underline{Q} > \underline{0}$ we know from the Lyapunov theorem of §1.4(e) that if in some subset N of M with $\underline{x}_0 \in N$ we have $\dot{V} \leqslant 0$ with $\dot{V} = 0$ only along the trajectory $\underline{x}(t) \equiv \underline{x}_0$, then the desired asymptotic stability about $\underline{x}_0$ will be obtained.

Now since $\underline{x}(t) \in M$ for all t we have

$$\dot{V} = \frac{1}{2}\dot{\underline{x}}'\underline{Q}(\underline{x} - \underline{x}_0) + \frac{1}{2}(\underline{x} - \underline{x}_0)'\underline{Q}\dot{\underline{x}}$$

$$= -\underline{x}_0'\underline{Q}\dot{\underline{x}}$$

thus the desired stability about $\underline{x}_0 \in N \subset M$ will ensue if $\underline{x}_0'\underline{Q}\dot{\underline{x}} \geqslant 0$ in N, with $\underline{x}_0'\underline{Q}\dot{\underline{x}} = 0$ only at $\underline{x}_0$.

Now let $f(\sigma)$ be any odd function on the real line for which $f(\sigma) = 0$ only at $\sigma = 0$. Let $u_i = f(\underline{x}_0'\underline{Q}\underline{A}_i\underline{x})$ for $1 \leq i \leq m$. Then

$$\underline{x}_0'\underline{Q}\dot{\underline{x}} = \underline{x}_0'\underline{Q}\sum_{i=1}^{m} f(\underline{x}_0'\underline{Q}\underline{A}_i\underline{x})\underline{A}_i\underline{x}$$

$$= \sum_{i=1}^{m} (\underline{x}_0'\underline{Q}\underline{A}_i\underline{x})f(\underline{x}_0'\underline{Q}\underline{A}_i\underline{x})$$

$$= \sum_{i=1}^{m} (\underline{x}'\underline{A}_i'\underline{Q}\underline{x}_0)f(\underline{x}'\underline{A}_i'\underline{Q}\underline{x}_0)$$

$$= \sum_{i=1}^{m} (\underline{x}'\underline{Q}\underline{A}_i\underline{x}_0)f(\underline{x}'\underline{Q}\underline{A}_i\underline{x}_0) \quad \text{since } \underline{Q}\underline{A}_i + \underline{A}_i'\underline{Q} = \underline{0} \text{ and f is odd.}$$

Thus $\underline{x}'_0 \underline{Q}\,\dot{\underline{x}} \geqslant 0$ for all $\underline{x} \in M$, and $\underline{x}'_0 \underline{Q}\,\dot{\underline{x}} = 0$ if and only if $\underline{x}'\underline{Q}\,\underline{A}_i \underline{x}_0 = 0$ for $1 \leqslant i \leqslant n$. Now since $[\underline{Q}\,\underline{A}_1\underline{x}_0, \underline{Q}\,\underline{A}_2\underline{x}_0, \cdots, \underline{Q}\,\underline{A}_m\underline{x}_0]$ is of rank n-1 and $\underline{x}_0$ is in the one-dimensional subspace perpendicular to $\{\underline{Q}\,\underline{A}_1\underline{x}_0, \cdots, \underline{Q}\,\underline{A}_m\underline{x}_0\}$, we must have that

$$\underline{x}'\underline{Q}\,\underline{A}_i\underline{x}_0 = 0 \text{ for all } i \implies \underline{x} = k\underline{x}_0$$

for some real k. From $\underline{x}'\underline{Q}\,\underline{x} = 1$ we obtain $k = \pm 1$, and the result follows.

QED

The following theorem was proved jointly by Professor R. W. Brockett and myself.

<u>Theorem 4.2</u>. Consider the system of equations

$$\dot{\underline{x}} = (\underline{A} + u\underline{B})\underline{x} \quad , \quad \underline{x}(t) \in \mathbb{R}^n, \ \underline{x}'(0)\underline{x}(0) = 1,$$

where $\underline{A}' + \underline{A} = \underline{B}' + \underline{B} = \underline{0}$. Suppose that $\underline{x}'_0\underline{x}_0 = 1$ and that $\underline{A}\,\underline{x}_0 = \underline{0}$ and $\underline{B}\,\underline{x}_0 \neq \underline{0}$. Let $u = f(\underline{x}'_0\underline{B}\,\underline{x})$ where $f(\sigma)$ is any function on the real line satisfying $\sigma f(\sigma) \geqslant 0$, with $f(\sigma) = 0$ only at $\sigma = 0$. Then the resulting closed-loop system is asymptotically stable about $\underline{x}_0$ in a neighborhood N of $\underline{x}_0$ if and only if the pair $(\underline{x}'_0\underline{B}, \underline{A})$ is observable (in the linear system sense). Moreover, a sufficient but not necessary condition for this is that the set of matrices $\{Ad^k_{\underline{A}}\underline{B}, \underline{A}\}$ span the space of skewsymmetric matrixes, where $k = 0,1,2,\cdots$, and $Ad^k_{\underline{A}}\underline{B}$ is as defined in §3.4.

<u>Proof</u>. Since $\dot{\underline{x}} = \underline{A}\,\underline{x} + f(\underline{x}'_0\underline{B}\,\underline{x})\underline{B}\,\underline{x}$ we have $\underline{x}'_0\dot{\underline{x}} = (\underline{x}'_0\underline{B}\,\underline{x})f(\underline{x}'_0\underline{B}\,\underline{x}) \geqslant 0$ for all $\underline{x}$. Now $\underline{x}'_0\dot{\underline{x}} = 0$ for all t if and only if $\underline{x}'_0\underline{B}\,\underline{x} = 0$ for all t, i.e. if and only if $\underline{x}'_0\underline{B}\, e^{\underline{A}t}\underline{x}_1 = 0$ for all t for some $\underline{x}_1$. Thus, by the Lyapunov method in the proof of Theorem 4.1, we conclude that asymptotic stability in N is equivalent to the requirement that there does not exist an $\underline{x}_1 \neq \underline{x}_0$ in N such that $\underline{x}'_0\underline{B}\, e^{\underline{A}t}\underline{x}_1 = 0$ for all t. But this is equivalent

([7], section 13) to the requirement that there is no $\underline{x}_1$ in N such that $\underline{x}'_0 \underline{B}\,\underline{A}^i\,\underline{x}_1 = 0$ for $i = 0,1,2,\cdots$, which is equivalent to the requirement that $(\underline{x}'_0\underline{B},\underline{A})$ be observable ([7], section 14). Thus, asymptotic stability ensues if and only if $(\underline{x}'_0\underline{B},\underline{A})$ is observable. Now if $\underline{x}'_0\underline{A} = \underline{0}$, it is straightforward to show by induction that $\underline{x}'_0 \mathrm{Ad}^m_{\underline{A}}\underline{B} = (-1)^m\, \underline{x}'_0\underline{B}\,\underline{A}^m$ for $m = 0,1,2,\cdots$. (E.g: $\underline{x}'_0\mathrm{Ad}_{\underline{A}}\underline{B} = \underline{x}'_0(\underline{A}\,\underline{B} - \underline{B}\,\underline{A}) = -\,\underline{x}'_0\underline{B}\,\underline{A}$). Thus asymptotic stability in N ensues if there is no $\underline{x}_1$ in N such that $\underline{x}'_0\mathrm{Ad}_{\underline{A}}{}^k\underline{B}\,\underline{x}_1 = 0$ for all k and $\underline{x}'_0\,\underline{A}\,\underline{x}_1 = 0$. But $\underline{x}'_0\,\mathrm{Ad}_{\underline{A}}{}^k\underline{B}\,\underline{x}_1 = \mathrm{tr}\,\mathrm{Ad}_{\underline{A}}{}^k\underline{B}\,\underline{x}_1\,\underline{x}'_0$, and $\underline{x}'_0\underline{A}\,\underline{x}_1 = \mathrm{tr}\,\underline{A}\,\underline{x}_1\underline{x}'_0$. Now if $\mathrm{tr}\,\underline{X}\,\underline{Y}' = 0$ for all $\underline{X}$ satisfying $\underline{X} + \underline{X}' = \underline{0}$, then $\underline{Y} = \underline{Y}'$; and if $\underline{x}\,\underline{y}'$ is symmetric then $\underline{y} = c\,\underline{x}$ for some $c \in \mathbb{R}$. Hence, if the matrices $\{\mathrm{Ad}_{\underline{A}}{}^k\underline{B},\underline{A}\}$ span the skewsymmetric matrices then $\underline{x}'_0\mathrm{Ad}_{\underline{A}}{}^k\underline{B}\,\underline{x}_1 = \underline{x}'_0\,\underline{A}\,\underline{x}_1 = 0$ will imply that $\underline{x}_1 = \pm\,\underline{x}_0$ (assuming that $\underline{x}_1, \underline{x}_0 \in S^{n-1}$). i.e. If $\{\mathrm{Ad}_{\underline{A}}{}^k\underline{B},\underline{A}\}$ spans the space of skewsymmetric matrices then asymptotic stability ensues, from any starting point in S^{n-1} other than $-\underline{x}_0$. To show that this condition is not necessary, consider the case where

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Then the Lie algebra generated by $\underline{A}$ and $\underline{B}$ is the space of 5 x 5 skewsymmetric matrices, but $\{\mathrm{Ad}_{\underline{A}}{}^k\underline{B},\underline{A}\}$ is the set of all matrices of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & d \\ -1 & 0 & 1 & 0 & c \\ 0 & -1 & 0 & 1 & b \\ 0 & 0 & -1 & 0 & a \\ -d & -c & -b & -a & 0 \end{bmatrix}$$

which is not the whole space of 5 x 5 skewsymmetric matrices. Now the only $\underline{x}_0$ in S^{n-1} satisfying $\underline{A}\,\underline{x}_0 = \underline{0}$ is

$$\underline{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix} .$$

From this we find that $(\underline{x}_0'\,\underline{B},\,\underline{A})$ is observable, and the result follows.

QED

In general, it is not possible to obtain a global stabilization on S^{n-1} for systems of the type considered in Theorems 4.1 and 4.2 when the feedback control law $\underline{u}(\underline{x})$ is restricted to being a continuous function. (Cf. Hopf's Theorem concerning the number of singular points of a smooth vector field on a manifold without boundary). In practice the fact that there will always be a "deadpoint" $\underline{x}_1 \neq \underline{x}_0$ such that $\dot{\underline{x}}\big|_{\underline{x}_1} = \underline{0}$ would probably be of little concern. However, one might ask the question "Can the point $\underline{x}_1$ be chosen to be any other point on S^{n-1}?" We would expect that it can, since this amounts to a smooth topological deformation of the vector field obtained in Theorems 4.1 and 4.2. As an example, consider the system on S^1 given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = u \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Suppose that it is desired to stabilize this about the point (a,b) where

$a^2+b^2=1$ and $a,b > 0$. Then let $u = a - x_1$. One can show that asymptotic stability about (a,b) ensues, from any starting point in S^1 other than $(a,-b)$. The vector field is as shown in Figure 4.1.

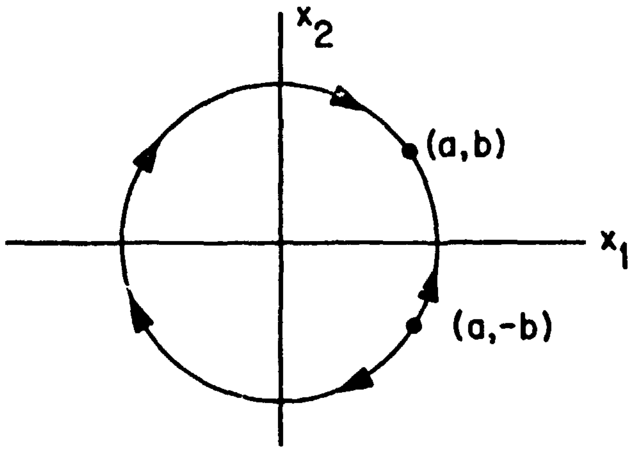


Fig. 4.1

A similar example can be given on S^2. In such cases one proves stability in $S^{n-1} - \{\underline{d}\}$ about $\underline{c}$ by showing that there is a neighborhood N_1 about $\underline{c}$ with $\underline{c}'\dot{\underline{x}} \geq 0$ in N_1, and that there is a neighborhood N_2 about $\underline{d}$ with $\underline{d}'\dot{\underline{x}} \leq 0$ in N_2, such that $N_1 \cup N_2 = S^{n-1} - \{\underline{d}\}$.

Finally, we consider the problem of stabilizing an oscillation on S^{n-1}. Given $\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u})$ one should choose, if possible, $\underline{u}(\underline{x})$ so that $\underline{x} \to E = \{\underline{x} | \dot{V}(\underline{x}) = 0\}$ where V is a suitable Lyapunov function and E is the set of points in the desired orbit. One should also choose $\underline{u}(\underline{x})$ so that on E $\underline{x}$ follows the orbit cyclically.

We consider first an example on S^2. Suppose

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \left\{ u_1\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + u_2\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}\right\}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

and that it is desired to stabilise an oscillation around the set $\{x_1 = a\}$, for some $0 \leqslant a < 1$.

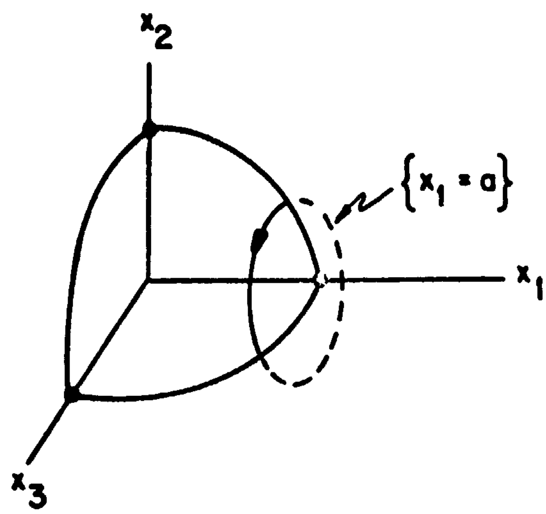


Fig. 4.2

Let $u_2 = 1$, so that when $u_1 = 0$ on $\{x_1 = a\}$ we have the simple harmonic oscillation given by

$$\begin{bmatrix}\dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ x_3\end{bmatrix} .$$

It then remains to choose $u_1(\underline{x})$ so that $x_1 \to a$.

We shall make use of the Invariance Principle of LaSalle, [17], [37]. Consider the periodic or time-invariant system of equations

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x},t) = \underline{f}(\underline{x},t+T) \\ \underline{x}(t) \in R^n \ ; \ 0 \leq t < \infty; \qquad \underline{x}(0) = \underline{x}_0 \ . \end{cases}$$

Let $V(\underline{x},t) = V(\underline{x},t+T)$ be a Lyapunov function (not necessarily positive definite) on a closed bounded set G for this system of equations, as defined in § 1.4(e). Let $E = \{\underline{x} | W(\underline{x}) = 0, \ \underline{x} \in G\}$ where $\dot{V}(\underline{x}, t) \leq W(\underline{x}) \leq 0$. Then the Invariance Principle of LaSalle (from which our Theorem 1.3 follows) states that any solution of the above equations which remains in G for all $t \geq 0$ approaches a subset of E which is the union of all the invariant sets which lie entirely within E. (An invariant set H is one for which $\underline{x}(t_1) \in H \Rightarrow \underline{x}(t_2) \in H$ for all $t_2 \geq t_1$).

Now since $\dot{x}_1 = u_1 x_2$, let us try $u_1 = (a-x_1)x_2$. Then $\dot{x}_1 = (a-x_1)x_2^2$. Consider the closed bounded subset N_1 of S^2 defined by $N_1 = \{\underline{x} \in S^2 | x_1 \leq a\}$. Let $V_1 = -\underline{a}'\underline{x}$ where $\underline{a} = (a, 0, 0)$.

$$\begin{aligned} \text{Then } \dot{V}_1 &= -\underline{a}'\dot{\underline{x}} \\ &= -a(a-x_1)x_2^2 \\ &\leq 0 \text{ in } N_1. \end{aligned}$$

Now $x_1 = a$ defines a trajectory for the system, thus no other trajectory can cross $\{x_1 = a\}$, i.e. Any motion starting in N_1 at t=0 remains in N_1 for all $t \geq 0$.

$$\begin{aligned} \text{Let} \qquad E_1 &= \{\underline{x} \in N_1 | \dot{V}_1(\underline{x}) = 0\} \\ &= \{\underline{x} \in S^2 | x_1 \leq a \text{ and either } x_2 = 0 \text{ or } x_1 = a\}. \end{aligned}$$

On $\{x_2 = 0\}$ we have $\dot{x}_2 = -x_3 = 0$ on E_1 only at $(0, 0, -1)$. Thus the union of the invariant sets in E_1 is

$$\{\underline{x} \in N_1 | x_1 = a \text{ or } \underline{x}' = (0, 0, -1)\}.$$

Furthermore a motion starting from any point in N_1 other than $(0, 0, -1)$ cannot approach $(0, 0, -1)$, since $\dot{x}_1 \geqslant 0$. Thus, by the Invariance Principle, we conclude that the desired oscillation is stabilized from any starting point in N_1 other than $(0, 0, -1)$.

Considering the subset N_2 of S^2 defined by $N_2 = \{\underline{x} \in S^2 | x_1 \geqslant a\}$ together with $V_2 = \underline{a}'\underline{x}$, we conclude by a similar argument that the oscillation is stabilized from any starting point in N_2 other than $(0, 0, 1)$.

Thus, we have stabilized a circular (simple harmonic) oscillation on the sphere S^2 around $x_1 = a$ from any starting point other than $(0, 0, \pm 1)$. Furthermore this means that we have stabilized such an oscillation around any circular orbit on S^2, since bilinear evolution equations are invariant under the transformation $\underline{x} \to \underline{x}\,\underline{P}$, as mentioned in § 3.4.

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