AN ADAPTIVE TECHNIQUE FOR A REDUNDANT-SENSOR NAVIGATION SYSTEM

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ABSTRACT

An on-line adaptive technique is developed to provide a self-contained redundant-sensor navigation system with a capability to utilize its full potentiality in reliability and performance The gyro navigation system is modeled as a Gauss-Markov process, with degradation modes defined as changes in characteristics specified by parameters associated with the model The adaptive system is formulated as a multistage stochastic process: a detection system, an identification system and a compensation system It is shown that the sufficient statistics for the partially observable process in the detection and identification system is the posterior measure of the state of degradation, conditioned on the measurement history

A suboptimal detection system in the class of linear systems of Wald's sequential analysis is developed The detection system is formulated as a combined control and decision problem by use of the concept of information value for detection and feedback of the uncertainty information of degradation indicated by the posterior measure The suboptimal control law is shown to be determined by a constant threshold number If the posterior measure is less than the threshold, the control corrects to the threshold Otherwise no control is used. The developed system is very simple for on-line implementation The system shows remarkably

close performance, expressed as mean time delay in detection under the constraint of a specified mean time between two false alarms, to that of the non-linear optimal detection system Moreover, the system can detect the degradation with simultaneous identification of its unknown polarity, which is shown to be an important piece of information for efficient isolation of the degraded instruments

It is shown that the detection process can be effectively constructed on the basis of a "design" value, specified by mission requirements, of the unknown parameter in the real system, and of a degradation mode in the form of a constant bias jump. An invariant transformation is derived to eliminate the effect of nuisance parameters in the identification system It is shown that the ambiguous multi-class identification process can be transformed into a set of pairwise disjoint simple hypothesis tests A technique of decoupled bias estimator is applied in the compensation system such that the adaptive system can be operated without any complicated reorganization

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CHAPTER 1

Introduction

1 1 Background

Adaptation, as the ability of self-adjustment in accordance with changing conditions of environment or structure, is a fundamental attribute of living organisms With the advance of technology considerable interest has been shown in the design of machines with the desirable attribute of adaptivity The concept of adaptive systems has found applicability in diverse areas of communication, control, reliability and pattern recognition The historical survey of the development of adaptive systems can be found in extensive references and will not be undertaken here Instead, a brief review will be presented from the point of view of methods of achieving adaptivity which have been discussed in the literature

A simple feedback system with a high gain in the forward path may operate quite adequately in a changing environment However, such a system may be more properly called insensitive or invariant rather than adaptive

Another approach to design the system with parameter adjustments programmed against measured operating conditions For those systems a rather comprehensive knowledge of the dynamic characteristics of the controlled process as functions of its operating environment is required This information is often difficult and expensive to obtain The programmed compensation is open-loop with respect to final system performance

More effective adaptive systems employ some measure of the system performance as a basis for adaptation The systems are provided with the capability of self-monitoring their performance, and the self-adjustment is closed-loop with respect to performance The adaptive systems in this

class may be designed either by parameter adjustments to achieve some specified characterization of its dynamic properties (e.g a specified damping ratio of the system), or by direct computer control to achieve an optimal performance criterion. The adaptive action usually consists of three phases

(1) Measurements to indicate present performance or deviations of performance from a desired or an optimal reference.

(2) Interpretation of the performance measurements to generate an indication of necessary corrective action

(3) Adjustment of system parameters to drive the actual system performance toward the desired performance

So far the concept of an adaptive system has been discussed in rather vague terms, and there is no definition at present of an adaptive system which meets with general acceptance. Tsypkin[76] suggested the following definition, which is a reasonable point of view with regard to adaptation for engineering systems.

> A process will be called adaptive when the parameters and structure (and, if necessary, the control signals) of a system are adjusted so as to use an accumulation of incoming information in such a way as to achieve some specified goal. This goal is usually some kind of optimization of a system that is initially poorly defined.

Adaptive system design is concerned with system models where a great deal of uncertainty is involved. Thus the system must be treated in such a way that one can learn about the nature of the unknown elements as the process evolves. The multistage aspects of the process must be made to compensate for the initial lack of information Secondly, the observable data will be of certain information patterns with imbedded features The problems involved in choosing useful information and in expressing this information in reasonable analytic form are usually of great subtlety and difficulty Thus the study and processing of information flow in the adaptive system is of primary importance Finally, the adaptive system must employ some criterion, usually some measure of optimality, as a basis for adaptation

The analysis of features of adaptive systems points out that the techniques of modern decision, estimation, and optimization theory can be used to effectively design the adaptive systems The theory of dynamic programming, sequential analysis, conditional Markov processes and pattern recognition can provide systematic techniques for formulating problems and effective methods of solution of adaptive system design. As a basis for further discussion, some significant results obtained by investigators in these areas will be briefly described

Formulation of the optimization problem as a multistage decision process was accomplished by Bellman [6] He applied the well-known imbedding technique of dynamic programming to derive the recursion formulas which must be satisfied by the optimal decision rule The property basic to applicability of dynamic programming is that decisions can be calculated sequentially by achieving the decomposition of the original problem into subproblems. Mitten[54] has proved that the monotonicity and separability conditions of the criterion functions are sufficient conditions for decomposition

In his development, Bellman assumes that the system state is a Markov process and that the state can be measured without error As a result of these assumptions the optimal decision becomes a function of the known state The concept of control (decision) of the system with partially observable data using the Bayesian approach has been discussed by Shiryaev [67]

The problem of designing an optimum closed-loop dual-control system is formulated by Fel'dbaum [25] The controls are necessary not only to control the system to the required goal but also to investigate the characteristics of incomplete information about the system The concept of combined optimization is fairly simple, but the required analysis is rather involved However, in some cases, the combined optimization problem can be decomposed into separable stages A well-known result is the so-called separation theorem for linear systems with quadratic criterion and Gaussian input such that the problems of estimation and control may be solved independently In many cases suboptimal algorithms must be used A commonly used suboptimal scheme is the open-loop-optimal feedback control discussed by Dreyfus[20]

Sequential analysis is a method of statistical inference whose characteristic feature is that the decision to terminate the experiment depends, at each stage, on the results of observations previously made (Blackwell and Girshick[9]). The sequential probability ratio test, devised by Wald[80], is a particular method of sequential analysis For testing a simple hypothesis against an alternative the procedure possesses the optimum property of minimizing the expected sample size for specified error probabilities Wald and Wolfowitz[83] proved the optimum character of the sequential probability ratio test based on a Bayesian formulation Some general results of sequential analysis are given by Whittle [85]

Dvoretzky et al [22], extended the discrete sequential theory of testing hypotheses to those problems about continuous stochastic processes Darling and Siegert[14] applied the solution of first passage times for homogeneous diffusion processes to problems in sequential analysis Aside from the difficulty of justifying certain limiting operations, the continuous procedure often leads to simpler analysis of the actual discrete process

While the traditional problems of sequential analysis usually assume that the unknown parameter does not change with time, the progress made in the general theory of stochastic control processes with partial observations can deal with the case when this parameter is itself a random process In this case the problem in sequential analysis can be considered as one in the theory of stochastic control with the simplest control--choice of the moment at which to stop observations, the problem of determination of optimal stopping rules.

The theory of sufficient statistics for data reduction with the preservation of the information is of particular importance for the general theory of adaptive processes A special interest in this aspect is concentrated on the case when the system can be modeled as the Markov process In study of Markov processes, if only some function of the state of a Markov process is directly measured, the fundamental entity in system optimization problems is the posterior conditional probability for the state given the observed measurement history. The study of these posterior probabilities and their evolution in time is the subject of the theory of conditional Markov process

There is extensive literature in the theory of Markov processes, for example, Doob,[18] Stratonovich[73] and Skorokhod[72] It was shown by Kolmogorov that the probability density of a continuous Markov system obeys a partial differential diffusion equation. This result was formally extended by Kushner[42] to systems with continuous noisy measurements Some results of stochastic differential equations of nonlinear filtering for Markov jump processes are given by Wonham[88] and Shiryaev[70] An important problem arises in connection with the simulation of Ito stochastic processes The problem of the convergence of ordinary integrals to the Ito stochastic integral has been treated by Wong and Zakai.[89]

For discrete systems the theory of stochastic optimal control based on the principles of dynamic programming is sufficiently established In generalizing the theory to the continuous time system it is difficult to prove the existence of the optimal control Kushner[44] and Fleming[28] considered continuous optimization problems for the case of completely observable controlled diffusion processes, and presented results on the existence, uniqueness and sufficient conditions of the optimal stochastic controls. A theory of ϵ-optimal controls for the intermediate system between cases of discrete system and continuous system has been discussed by Stratonovich[73]

The results in pattern classification have found useful application in control system design where a great deal of uncertainty is involved Ho et al[34] and Sklansky[71] present a survey and analysis of this field for applications in automatic control There are essentially two fundamental problems in pattern classification, the characterization and the abstraction problem The characterization problem can be simply but vaguely stated as finding a transformation from the original pattern to a set of features such that the feature states adequately characterize the original problem for purposes of classification but with much smaller dimension The abstraction problem is the determination of a decision function of these features based on the data of given patterns such that new samples of unknown patterns can be reliably classified

Based on the level of uncertainty of information four types of available data can be specified

(1) Functional form of the conditional density of features for the classes is given to within the specification of a set of unknown parameters

(2) The values of the parameters are also known.

(3) A set of training sample patterns of known classification is given.

(4) Sample patterns of unknown classification are given; the decision rule has to be determined only on the basis of observed but unclassified patterns.

The procedure to process the given data for calculating the decision function of classification is known as training, adaptation, or learning Depending on the types of available data, classification of algorithms can be broadly divided into two groups, probabilistically based and non-probabilistically based The main learning tool for the first group is the recursive application of Bayes' rule, while that of the latter is the iterative solution of an optimization criterion

An important and interesting application of adaptive systems is to improve reliability of complex systems. One design philosophy to improve reliability is to provide the systems with the capability for self-repair. Based on the principle of ultra-stability proposed by Ashby, Tsien[75] discussed the possibility of building into the system a certain adaptability so that incidental and unexpected malfunction will automatically be corrected for by the control system itself without external human aid Whitaker and Kezer[85] discussed the use of the model reference adaptive system combined with redundant control channels to improve the reliability of an aircraft flight control system

1.2 Description of the Problem

The primary goal of this research is to develop a technique of adaptive system design to improve reliability by detecting, identifying and compensating performance degradation of a system modeled as a Gauss-Markov process. In particular the approach attempts to illustrate the application of modern system optimization theory to adaptive system design The designed adaptive system is applied to a self-contained redundant-sensor navigation system intended for use in long duration planetary space flight missions

1.3 Summary of Contents

Chapter 2 treats the modeling of the problem The configuration of the redundant-sensor attitude system is described, and the system mission is defined In a long term free-fall space mission, the gyro bias drift rate is considered to be the most important component causing system performance error The drift rate is modeled as a Gauss-Markov process, and the operating modes are characterized by parameters associated with the model The forms of degraded modes are defined by the dominant nature of the sensor degradation characteristic A set of parity equations is generated by a linear combination of gyro outputs to eliminate the vehicle rate The residuals of parity equation outputs are the inputs to be processed by the adaptive system Each residual is transformed into a scalar state by application of the concept of linear aggregation of states The degraded gyro will be isolated by monitoring the parity equation output residuals The basic approach of the adaptive system design is then discussed. The adaptive system is formulated as a multistage stochastic process consisting of detection, identification and compensation systems

Chapter 3 presents the main results of stochastic equations of Ito form for the posterior probabilities of the partially observable process when the unobservable process is Markovian with a finite number of states The correlated measurement processes are first transformed without loss of information into corresponding asymptotically independent processes by application of the innovation principle and the Kalman filtering technique The stochastic differential equations for the posterior probabilities are then derived by application of results in the theory of conditional Markov processes It is shown that the stochastic equation in the identification system can be

considered as a degenerate case of the detection system By application of a comparison theorem for diffusion processes it is shown that the real system can be detected by choosing a "design" value on the basis of which the detection process is constructed By properly defining the instant of degradation time it is further shown that only one detection system designed for a degradation mode of constant bias jump is sufficient for degradation modes characterized by the presence of any systematic mean value

Chapter 4 deals with the detection system design A method is first reviewed in which the optimal detection problem is formulated as the solution of a Bayesian problem. It is shown that the forementioned posterior probability is the sufficient statistic for optimal detection in the class of Bayesian problems with additive risks The optimal detection problem consists of observing the evolution of the posterior probability and is reduced to the determination of the optimal boundary for the decision to stop or continue the observation at a minimum risk The problem of on-line implementation of the Ito stochastic equation of the posterior probability is discussed. To simplify the on-line implementation of the detection system, a class of linear detection systems based on Wald's sequential probability ratio test is studied It is shown that the detection performance of this system will however suffer an extra time delay in detection in comparison with the optimal system.

A suboptimal detection system is developed. The feature of the system is utilization of feedback of uncertainty information indicated by the posterior probability The solution of the suboptimal detection system is formulated as a combined optimization problem in the class of the forementioned linear detection systems In the formulation of the optimal stochastic control problem the posterior probability function is defined to be the state variable and the uncertainty feedback defined as the control variable. A suboptimal control scheme is designed to avoid the complicated computation for obtaining the solution to the combined optimization problem formulation It is found that the suboptimal control law is determined by a constant threshold. In the discrete time problem if the state variable is larger than the threshold, no feedback control is applied, if it is smaller than the threshold, the control corrects to the threshold. The difficulty in deriving the suboptimal control for the continuous time case is discussed. It is shown that the

suboptimal system can be easily modified to detect degradation with simultaneous identification of unknown polarity of mean bias, which is a useful piece of information for the identification stage and for efficient isolation of the degraded gyro.

The performance of the detection system is studied It is shown that for a sequential detection system with only a single-side boundary for degradation detection the meaningful performance criteria are the mean delay time in detection and the false alarm error probability The miss alarm error probability is not well-defined since there is no explicit decision of normal mode The detection performance of the optimal detection system and the developed suboptimal detection are evaluated and compared in detail. The performance is derived by application of the theory of first passage times for diffusion processes It is shown that under the constraint of a specified mean time between two false alarms the suboptimal system performance expressed in mean time delay in detection is remarkably close to that of the optimal system.

Simulation results are presented They consist of detection of degradation with identification of unknown polarity, a comparison of detection performance of the optimal, the suboptimal, and a linear system, and the isolation of the degraded instrument

Chapter 5 discusses the design of the identification system —Verification of the normal mode and classification of degradation modes are the decisions made in the identification stage It is shown that the determination of the likelihood function in the classification process requires not only the information of the functional form of the conditional probability density of the degradation modes but also the unknown values of the associated parameters An invariant transformation technique is designed to eliminate the effect of unknown parameters An application of the technique to the measurement residuals will then transform the original process with unknown constant and ramp mean rate into the corresponding process with a zero mean and with a mean of attenuated ramp rate. It is shown that the ambiguous three-classes identification process can now be transformed into a set of pairwise disjoint two-class identification processes Simulation results of classification of degradation modes of constant and ramp bias are presented

Chapter 6 considers the design of the compensation system. The compensation system is a linear estimator of the unknown parameter of the identified mode with knowledge of the correct model A technique of decoupled estimator is applied to estimate the unknown bias In this technique the bias estimate can be computed in terms of the residuals in the bias-free estimator During most mission periods the system will be operated in the normal mode, and a recursive filter is designed for the bias-free condition to generate the input information for detection and identification Once a degradation has been detected and identified, the filter must be modified to estimate the associated bias value With application of the decoupled estimator, the adaptive system can be operated without any complicated reorganization The problem of convergence rate of bias estimation is discussed A slow convergence rate of the bias estimate is predicted analytically and is confirmed by simulation results The long time interval required for processing information in the estimation process in comparison with that required in the decision process illustrates the justification of the multistage formulation of the thesis.

Results are summarized, discussed and related to recommended future studies in Chapter 7 Three appendices are presented.

CHAPTER 2

Model of the Problem

2 1 Description of the Redundant Sensor System

Guidance and navigation reliability requirements have led to increasing emphasis on redundancy in system design. Redundancy can be provided at the system, subsystem, and component levels, and the mechanization can be implemented either on an inertial platform or a strapdown system. In the case of a strapdown mechanization, instrument level redundancy becomes appealing since the attitude reference is accomplished within a computer Degradation checks may be performed on the redundant sensor data before it is used in the guidance and navigation loops. Such a redundant strapdown system would be compact, lightweight, and theoretically highly reliable

A redundant strapdown inertial reference unit (SIRU) [32] was developed at the M I. T. Draper Laboratory. It consists of a redundant instrument package, an electronics assembly, and a digital computation assembly The basis of the redundant concept is formulated in the redundant instrument package It employs six single-degree-of-freedom gyroscopes and six linear accelerometers, all are operated in a pulse torque to balance mode. The instruments are arrayed in a skewed configuration in which their input axes correspond to normals to the faces of a regular dodecahedron, as shown in Fig. 2. 1. The configuration allows the navigation system to be capable of operation under the condition of any three unfailed instruments, and exhibits a unique symmetry in which all instrument input axes are at a spherical angle 2ϕ ($\phi = 31^\circ\ 43'\ 2.9''$) from each other. This feature simplifies degradation detection and isolation, and minimizes geometric-error amplification by the skewed configuration in the sense that the system error ellipsoid exhibits a spherical symmetry when averaged over all possible failure cases

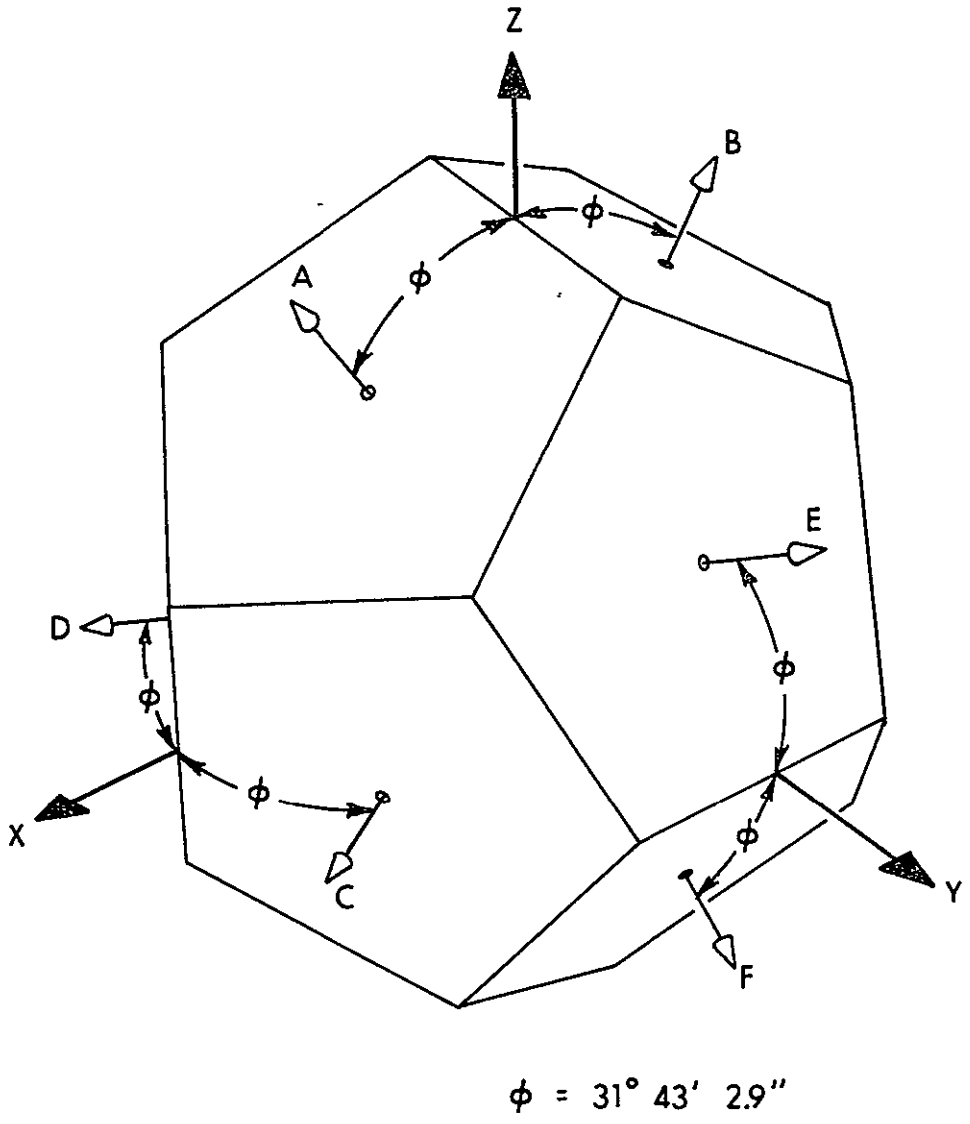


Fig. 2.1 Configuration of Redundant-Sensor Mechanization

for every given number of instrument failures. In this thesis, the adaptive system is designed for the gyro package of the redundant SIRU system.

A simplified reliability and performance analysis of the SIRU system is to be discussed. In general the reliability and performance of the redundant system are related through the adaptive operation of the failure detection, identification, and compensation systems In the following paragraphs it is assumed that the adaptive system operation is perfect and the gyros are the only sources of failure.

Let P be the probability of a given instrument failing during a specified time interval when a constant failure rate, q (or equivalently a constant mean time between failures, T_m), and a time interval τ are assumed. The probability of a given instrument failing is

$$P = 1 - e^{-q\tau} = 1 - e^{-\frac{\tau}{T_m}}$$

The probability of any m out of n instruments failing in the specified time interval is given by

$$P(m, n) = \frac{n!}{m!\,(n-m)!}\, P^m\, (1 - P)^{n-m}$$

The comparison of the reliability of the dodecahedron configuration with the reliability of various redundant configurations is illustrated in Fig. 2.2. The comparison is based on a self-contained navigation system with available external information for isolation. A marked reliability advantage for the SIRU configuration is clearly illustrated. As a further comment with respect to reliability it is important to note that redundancy itself is not a cure-all. An instrument with a mean lifetime of one-million hours does not preclude its possible failure. Thus a system with an on-line self-repair capability is of high interest.

Performance is not generally a factor in establishing a redundancy configuration However the nature of the skewed redundancy alternate is such that performance must be included This need arises from "geometrical amplification" of gyro error that occurs with skewed gyro emplacement, the extent of this amplification is directly related to the particular emplacement orientation

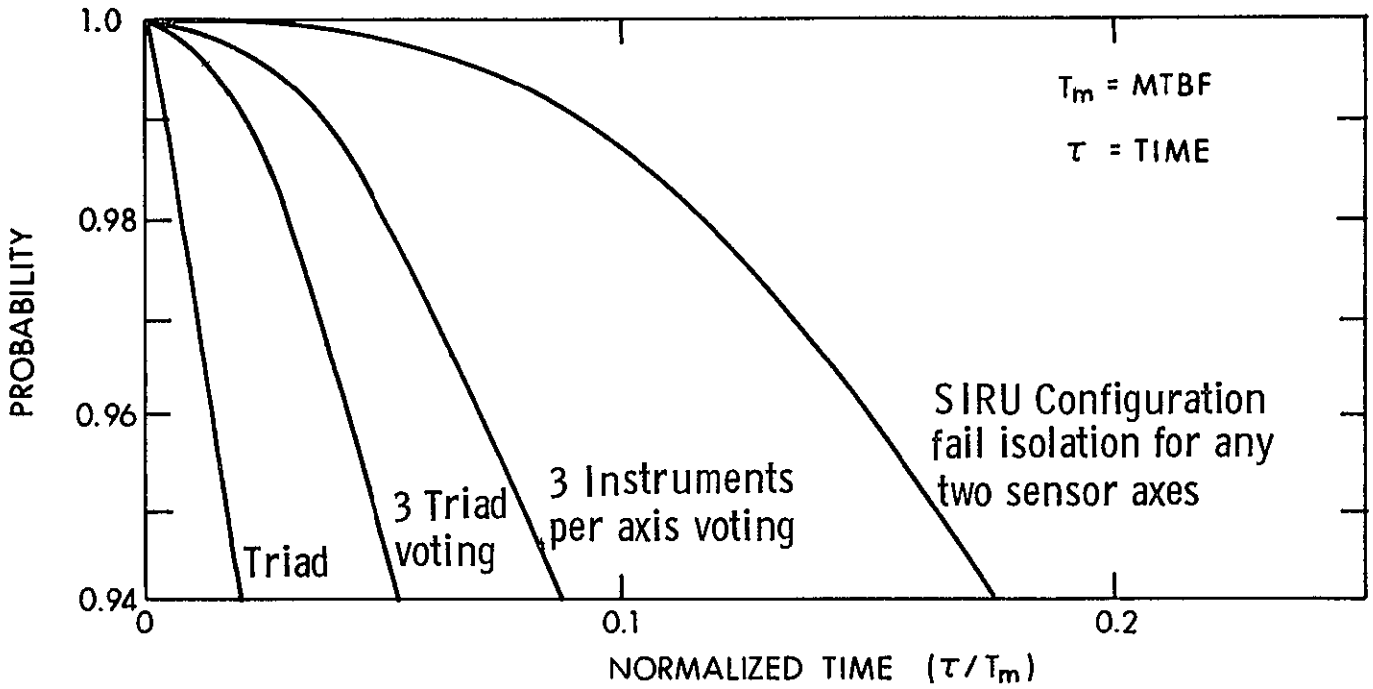


Fig. 2.2 Mission Success Probability

The performance of the SIRU configuration for various failure conditions are represented by comparing a SIRU reference-triad error covariance to that of the triad system in the orthogonal configuration In both cases, the errors of all instruments are assumed to be statistically independent, with zero mean and unit variance under unfailed condition. The triad error in the estimate of the sensed rate is obtained by a least square solution of the sensed rate from the subset of unfailed instruments of the SIRU system Table 2.1 presents the standard deviations of the errors and the average errors of the SIRU system:

Table 2.1

SIRU Performance vs. Failures

No. Instruments Failed	Error Coordinate System Axes A	B	C	Average Error
0	0.707	0.707	0.707	0.707
1 (6 cases)	1.000	0.707	0.707	0.816
2 (15 cases)	1.345	0.831	0.707	1.000
3(a) (8 cases)	1.345	1.345	0.727	1.581
3(b) (12 cases)	3.078	0.831	0.831	

In Table 2 1, the performance is expressed in the error coordinate system For a given number of failed instruments the system errors, if expressed in terms of the body coordinate system, will differ for each possible case The error coordinate system, in which errors are uncorrelated in the error-covariance matrix through a diagonalized transformation, can be related to the configuration of the failed and unfailed instruments to provide a physical picture of its orientation. For example, axis A is along the input axis of the failed gyro for the one failed gyro case For two gyros failed, their input axes form two acute and two obtuse angles. Axis A bisects the acute angles Axis B bisects the obtuse angles Axis C is perpendicular to both input axes A detailed analysis can be found in Evans and Wilcox[24]

Average errors in Table 2 1 are computed by adding together the covariance matrices (without diagonalization) of all possible cases for a given number of instruments failed and dividing by the number of matrices The result turns out to be equal to the identity matrix multiplied by a scalar, the square root of which is called the average error Thus when averaged

over all possible failure cases the errors are spherically symmetric It can be seen from the average errors that the performance of a set of four instruments in the dodecahedron configuration is roughly equivalent to that of three instruments in the orthogonal configuration

2.2 Definition of Degradation Modes

There are a large number of error sources which result in performance degradation of gyro sensors. Sensors in a strapdown package are subjected to a severe environment including high angular rates, angular vibrations and other error dynamic error sources. However, for inertial navigation systems which operate over long-duration mission such as planetary space flight, there are no significant random disturbances to the spacecraft except for short-interval maneuver and thrusting phases. Thus the gyro drift will provide the main source of error. The choice of a model for gyro drift depends to a great extent on the desired degree of accuracy. Here a mathematical model for gyro drift will be reviewed. The error torque M_e physically present on the float is the resultant of many independent torque-producing errors. From the basic error physical mechanisms it can be functionally expressed of variables as specific force (g), float motion (r, θ, $\dot{r}$, $\dot{\theta}$), temperature (T), and power (W):

$$M_e = f(g_x, g_y, g_z, x, y, z; \theta_x, \theta_y, \theta_z, \dot{x}, \dot{y}, \dot{z}; \dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z, T, W)$$

All the disturbance sources can be expected to vary over only small ranges for a normal operating gyro, with the exception of components g_I, g_S, g_O of g along the input axis (IA), spin reference axis (SRA), and output axis (OA) respectively, which in some applications vary quite widely. It is therefore logical to regard g as the primary source of disturbance. M_e can be expanded as a series in g_I, g_S, g_O about g = 0 as follows:

$$M_e = M_e\Big|_{g=0} + \frac{\partial f}{\partial g_I}\Big|_{g=0} g_I + \frac{\partial f}{\partial g_S}\Big|_{g=0} g_S + \frac{\partial f}{\partial g_0}\Big|_{g=0} g_0 + \cdots$$

The linearized model to be studied here consists of terms through first order

$$M_e = D_G + D_I \cdot g_I + D_S g_S + D_O g_O$$

The four coefficients are known as the bias drift coefficient, and the acceleration sensitive drift coefficient along IA, SRA, and OA axes respectively The general coefficient D_i (i = G, I, S, O) can be written as a function of disturbances.

$$\cdot D_i = f_i(x, y, z, \theta_x, \theta_y, \theta_z, \ldots T, \ldots W)$$

The physical mechanism of these disturbance variables may be associated with mechanical structure, flotation fluid, electronics and thermal controls

Lorenzini [49] in investigation of error mechanisms concluded that many causes of gyro drift rate variation can be classified as deterministic if a mathematically describable physical process which relates these disturbance errors to the gyro drift rate can be derived He found the relative motion of the gyro float with respect to the case and temperature gradients to be major contributors to the total drift rate error observed He suggested that there is much room for improvement in gyro performance by reducing their effects through output signal compensation if those disturbance errors can be monitored and measured This approach for a more complete and accurate error model however requires a complicated theoretical or experimental analysis of the instrument and its associated equipment, and a great deal of information concerning actual degradation in the operational environment

Other proposed models (Wilkinson [87], Dushman [21]) regard each drift coefficient D_i as a stochastic process In their formulation emphasis is put on the lumped behavior of the gyro output signal rather than the detailed investigation of the gyro output as a function of various error sources It is modeled as the sum of two components a stationary first-order Gauss-Markov process and a non-stationary random walk process An investigation of the physical noise processes in the gyro, such as the effects of Brownian movement in the damping fluid, electrical circuit noise,

creep in the flex leads, and bearing wear etc. suggests that the formulated process is a reasonable model

From the system point of view, the output of the instrument is the most significant state. The instrument has degraded when and only when its output error exceeds the specification The approach taken in this thesis is that the operating modes of the gyros will be characterized by their output signals. The drift rate will be modeled as a stochastic process, yet it is recognized that a portion of the uncertainty observed in the output signal of a gyro stems from deterministic error sources. These error sources will reflect systematic values to the output signal of the gyro.

The operating modes of gyros are characterized by the parameters associated with the stochastic process of the mathematical model The gyro is defined to be operating in the normal mode if the values of parameters associated with the driving noise of the model are of zero mean and normal variance The degradation modes of gyros are characterized by either the presence of a systematic mean value or an increase in the variance of the driving noise of the model

The types of characteristic change of the mean drift rate may be of two forms: a jump shift of a constant mean drift rate or a ramp of the mean drift rate. To define the degradation modes of the mean drift rate in these two simple forms permits the solution to be analytically tractable and the degradation to be practically compensable, and preserves the dominant nature of the real degradation characteristic A typical illustration of the gyro degradation mode is shown in Fig 2 3 It has been observed that a degradation in mean drift rate is mostly of a jump nature

The gyro is defined to be operating in a degraded mode if the value of one of the parameters of the mean drift rate exceeds a "design" threshold, specified by mission performance requirements. The performance requirements to insure successful completion for the Apollo Mission are illustrated in Fig. 2 4 [53] The definition of "good" performance means that the gyros will contribute errors to various phases of the mission less than that indicated As an illustration, a change of less than 20 meru in bias drift rate D_G of all gyros will assure a probability of one that the

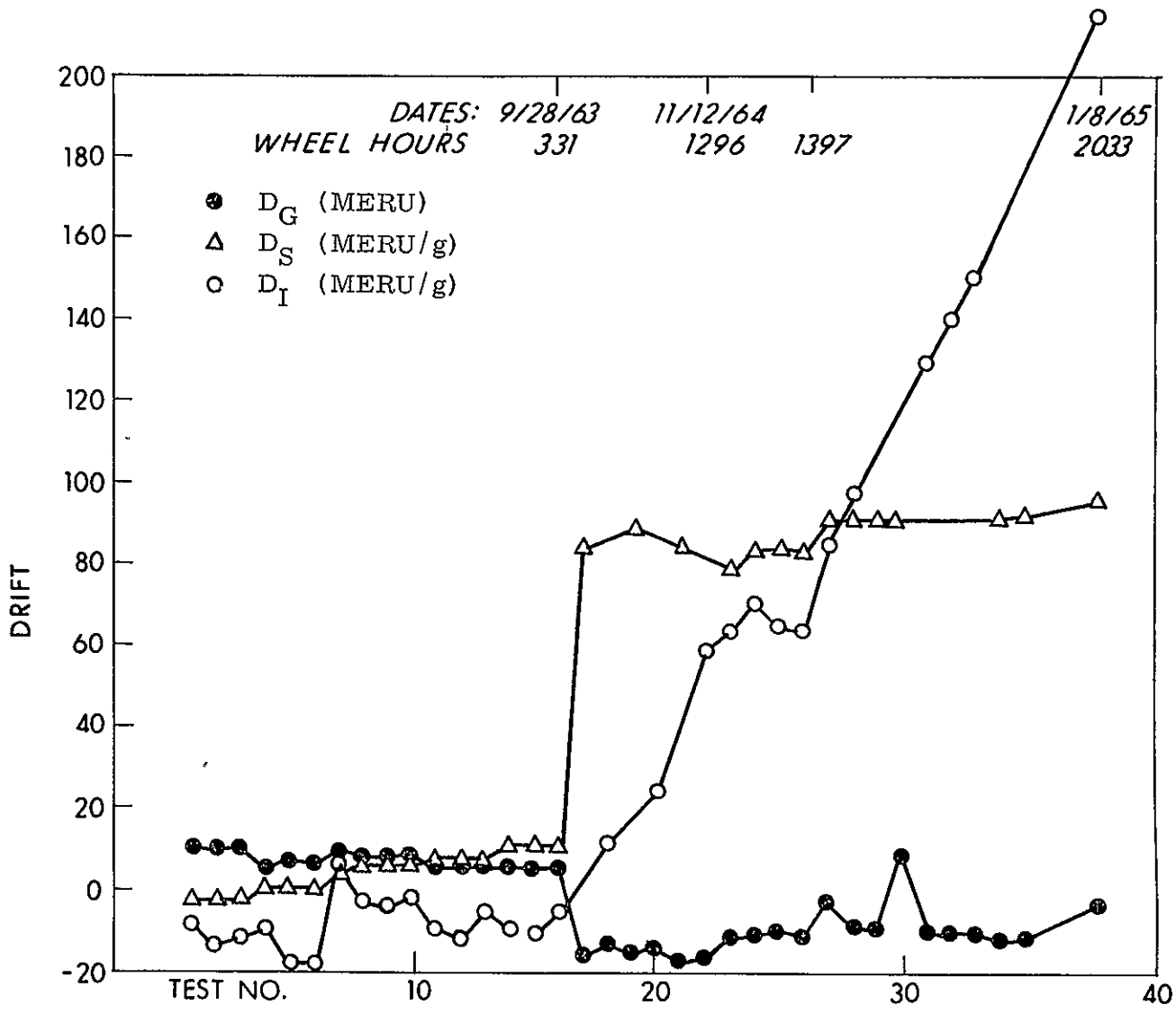


Fig. 2.3 Gyro Drift History

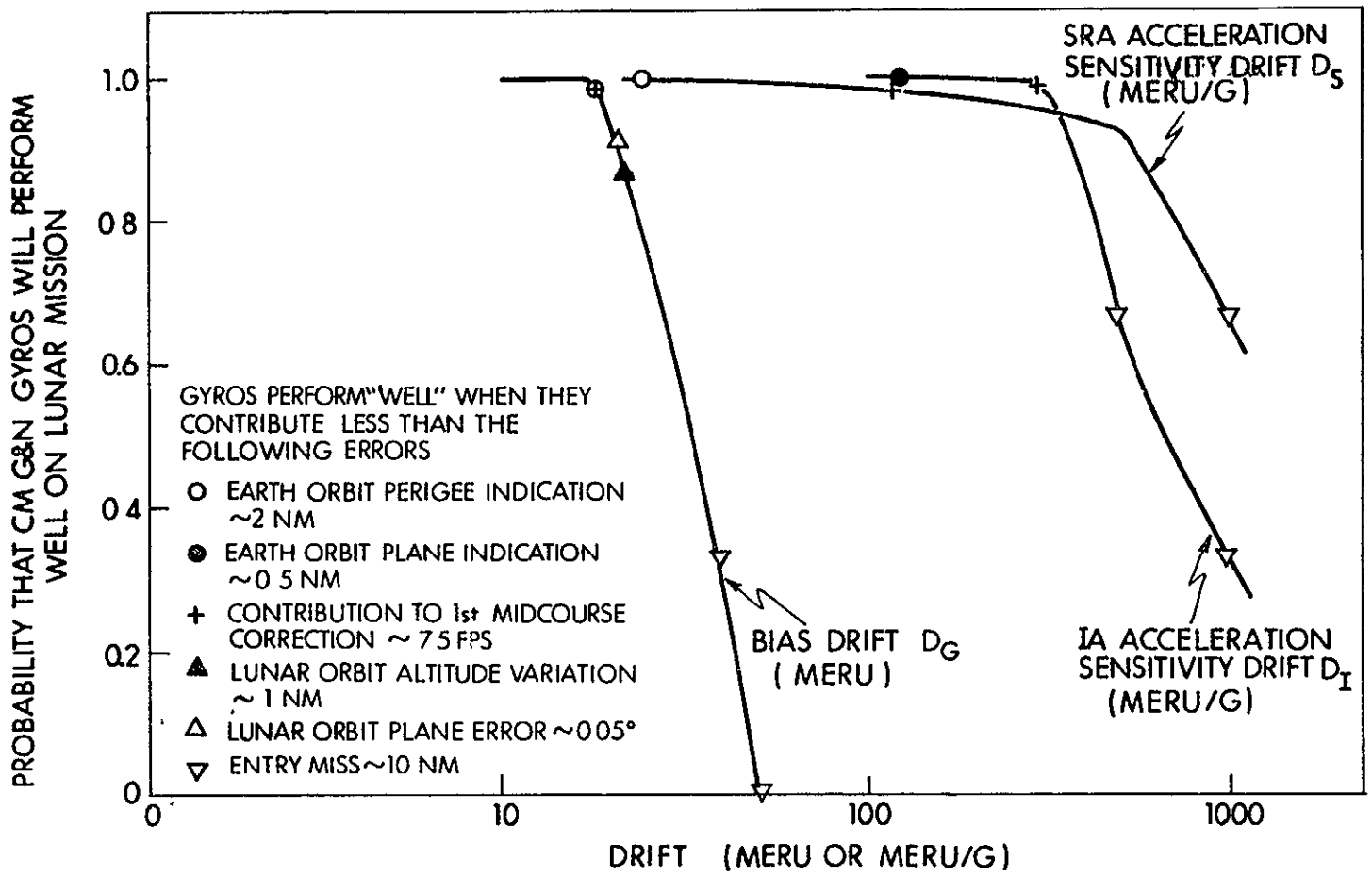


Fig. 2.4 Mission Gyro Performance Requirements

error contributed by the gyros to the entry miss would be less than 10 nautical miles. If a certain gyro indicates a change of 40 meru in D_G, the probability to ensure "good" performance would be 0.33. This probability will be reduced to zero if any gyro indicates a change of 60 meru in D_G. As can be seen from Fig. 2.4, the bias drift rate is more sensitive to mission performance than the acceleration sensitive drift parameters. A change in acceleration sensitive drift coefficient D_I can be an order of magnitude greater than that in D_G to ensure the same mission performance.

Before the model of degradation modes will be defined, an analysis of various components of the gyro drift rate on the performance degradation of gyros will be presented. In long term free fall space flight missions the most important component is the bias drift rate $D_G(t)$ caused by bias torques independent of specific forces. Error signals due to acceleration sensitive coefficients $D_I(t)$ (i = I, S, O) will appear only in short duration during thrust phases. It is a well recognized fact that the coefficient $D_I(t)$ is usually the most noisy component of the gyro drift rate and is the most sensitive indicator of performance degradation in a g environment. However, in the case of long duration space flight missions their effects on mission success will be insignificant in comparison with the bias drift rate. This is illustrated in Fig. 2.4 in the analysis of the lunar mission.

For modeling the acceleration sensitive drift rate components, each coefficient $D_I(t)$ should theoretically be treated as a stochastic process. In theory, this approach can be treated without difficulty by a simple extension of the dimension of the drift rate process. But the approach is impractical due to the following considerations. First, during thrust phases in a short duration of minutes, the limited data just do not give enough information for any statistical significance. Second, unless there is a change in orientation and/or magnitude of the specific force during the thrust phase, it can be verified that the system of the extended drift rate process is unobservable, i.e. the acceleration-induced $\ddot{d}$rift is indistinguishable from a shift in bias drift.

Therefore it seems doubtful that one could justify the more complicated computation load resulting from extending the state vector to a dimension of 24 components (6 gyros, each with 4 drift coefficient components). In this thesis the degradation modes of acceleration sensitive drift rate coefficients will be modeled only as a mean bias shift. This is a reasonable assumption if one considers that these error sources will only be active for a short duration and their effects on mission performance will be serious only when there is an appreciable mean value.

Gyros in a strap-down package will experience the full vehicle angular rates, so a scale factor change ΔSF will result in a large equivalent drift rate change. For example, a change of 1 ppm in the nominal scale factor SF with a vehicle rate of 0.1 rad/sec. will give an equivalent drift rate of 1.4 MERU. In the Apollo mission, the spacecraft is given a continuous slow rate of 0.3^{o}/sec to equalize the heat exposure (Barbecue Mode). In this case a change ΔSF of 10 ppm will give an equivalent drift rate of 0.6 MERU. This drift rate will degrade the performance over the whole mission duration. However, as long as the spacecraft rotates at a constant rate, this degradation mode will reflect at the gyro output signal as the presence of a systematic mean value of the bias drift rate $D_G(t)$. These two error sources are inseparable, and indeed need not be separated for compensation from the point of view of system performance. This degradation mode can be lumped into and treated as the corresponding form of the bias drift rate. Vehicle rates other than the roll rate associated with this Barbecue Mode will appear only in short durations during maneuver phases. The problem of scale factor change will be treated by the same approach as that used for the acceleration sensitive coefficients. The physical mechanism of ΔSF change has not been studied well enough to define a reliable model, but some error sources have been reported. A SF non-linearity has been observed which depends on both input rate and direction. This is derived from the inherent characteristics of the instrument. This error source will be assumed to be corrected by on-line compensation based on vehicle rate information. It has been observed over a long-term period of months, that the scale factor for a fixed rate and direction show a change ΔSF of $100 \sim 200$ ppm. Fig. 2.5 illustrates a change of nominal scale factor ΔSF in the form of a ramp rate.

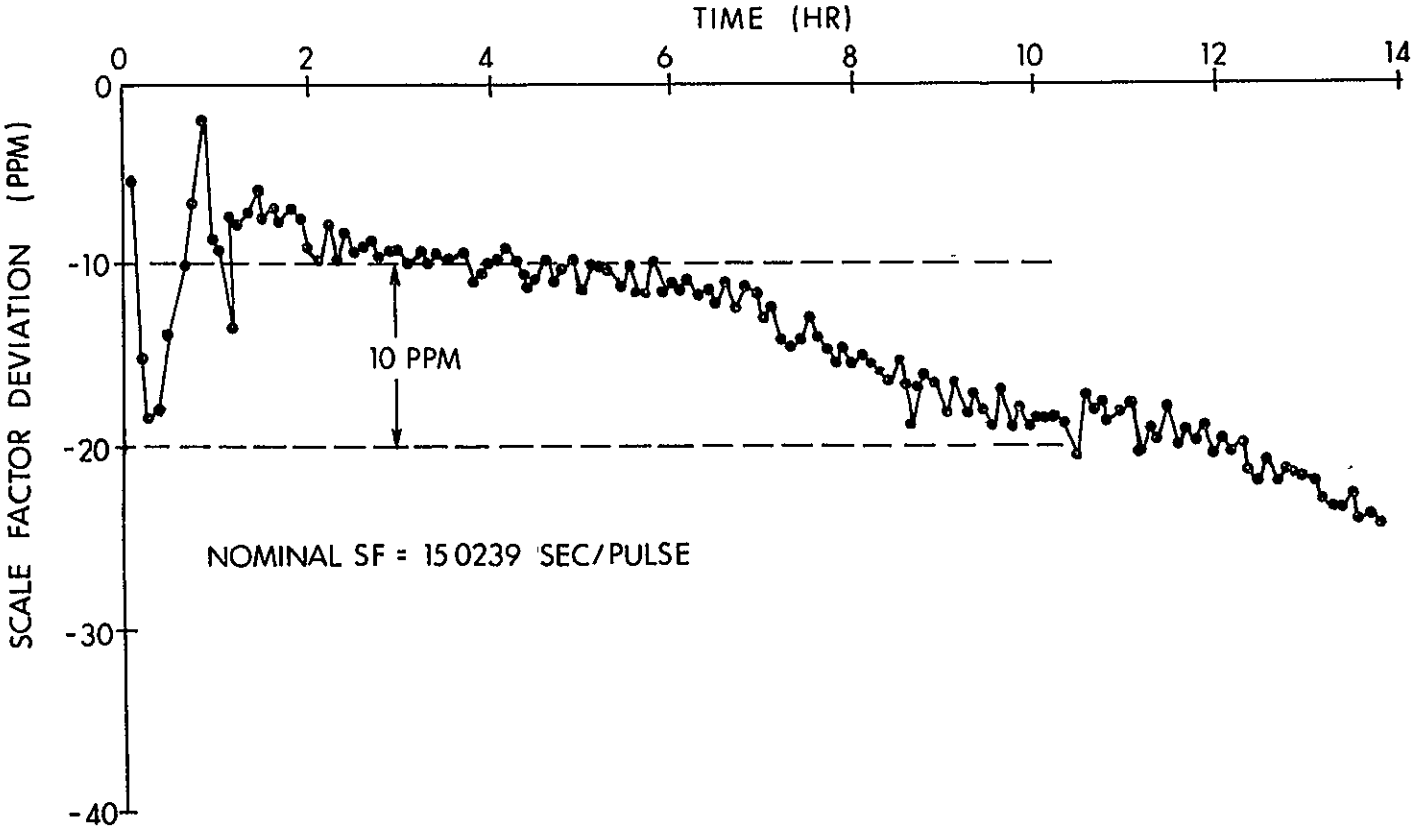


Fig. 2.5 Scale Factor Deviation

In the strap-down system, the misalignment error is practically the constant mechanical misalignment value It will be assumed that this error source has been corrected in precalibration and will not be considered here.

Inherent in measurements of gyro pulses are quantization errors. The quantization error is modeled as white noise and will be considered as the measurement noise Although the quantization error is uniformly distributed, the measurement noise is assumed to be Gaussian with zero mean.

There are two different physical mechanisms for non-stationary processes. For "normal" gyros, there is a random-walk process, and for "degraded" gyros a non-stationary process may result from some macroscopic change (e.g., bearing wear) resulting in an increase of noise variance. The random-walk component of the gyro drift rate is proposed to be treated as an unknown mean of arbitrary form to be matched to one of the two given classes, a constant or a ramp mean bias, and will not be separately discussed.

A set of real data describing the variance increase of the gyro degradation history is illustrated in Fig. 2.6, where a computed root mean square standard deviation is used as a criterion of variance change. While the drift rate coefficient D_I shows an increase in variance after failure over a period of 2000 wheel hours, only a small and gradual increase of variance has been observed in the bias drift rate D_G over the same period. Thus the degradation mode of variance increase will probably be of little significance on system performance degradation in a space flight mission in contrast with the characteristic of an appreciable jump in mean bias change. Moreover, it is to be noted that the above data refer to gyros with wheels supported by ball bearings. There are indications and physical reasons to believe that there will be no appreciable wear in gas bearings. In this thesis the discussion will be restricted to the case of degradation modes characterized by the presence of mean drift rate but with normal variance However, some discussion of the degradation mode arising from variance increase and of the treatment of the random walk process will be presented in Chapter 7.

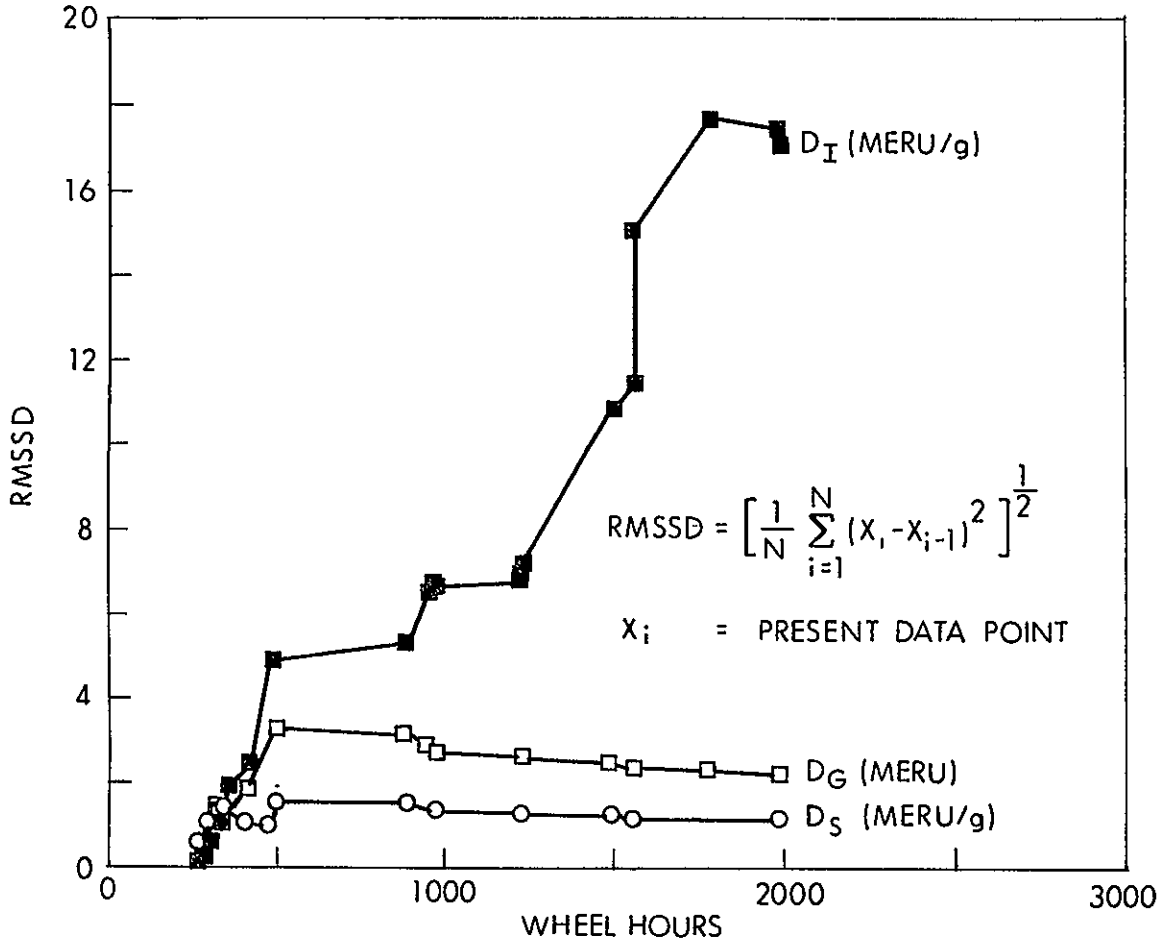


Fig. 2.6 RMSSD vs Gyro Wheel Hours

The mathematical model of the gyro drift rate process will now be defined. The gyros are operated in a pulse-torque-to-balance mode, the signal outputs correspond to an accumulated angle counted over some interval. Since the gyro error is modeled on drift rate rather than angle, the state variable is derived by averaging measurements over a sampling period. The sampling period is chosen to get a reasonable resolution of quantization errors. The state equations of the stochastic drift rate processes in continuous time are presented as follows

(1) Gyro in normal operating mode (denoted as H_o)
The state equation is defined as

$$\dot{D}_G(t) = - \omega \cdot D_G(t) + g_G(t) \qquad (2\text{-}1)$$

The measurement equation is represented by

$$D(t) = D_G(t) + v_G(t)$$

where the notations are defined as

$g_G(t)$ = normally distributed drift rate driving noise with known statistics given by $E[g_G(t)] = 0$, and
$E\left[g_G(t)g_G(\tau)\right] = W_G\delta(t-\tau)$

$v_G(t)$ = measurement noise of quantization error with given statistics $E[v_G(t)] = 0$, $E[v_G(t)v_G(\tau)] = V_G\delta(t-\tau)$

ω = system dynamics coefficient.

(2) Gyro in degradation mode in the form of constant mean bias jump (denoted as H_1).
The state equation remains unchanged, while the measurement equation is represented by $D(t) = D_G(t) + m + v_G(t)$ (2-2)

The degradation state m may be caused by a jump of bias drift rate component, of acceleration sensitive drift rate component in thrusting phase, or of scale factor in presence of vehicle rate.

(3) Gyro in degradation mode in the form of a ramp rate (denoted as H_2),

The measurement equation can be found to be

$$D(t) = D_G(t) + nt + v_G(t) \tag{2-3}$$

where n is the constant ramp rate. The degradation state n for the ramp case may be caused by a ramp bias mean drift rate or a ramp change of scale factor in case of continuous maneuver with a constant rate.

The corresponding state equation of the drift rate process in the discrete time system is expressed as

$$D_G(k+1) = F \cdot D_G(k) + q_G(k) \tag{2-4}$$

The following notations are employed:

$F = \Phi[(k+1)\Delta, k\Delta] = \text{Exp}\{-\omega\cdot\Delta\}$

Δ = sampling period

The normally distributed random variable $q_G(k)$ is defined as

$$q_G(k) = \int_{k\cdot\Delta}^{(k+1)\cdot\Delta} \Phi[(k+1)\Delta, \tau]\, g_G(\tau)\, d\tau \tag{2-5}$$

with known statistics given by

$$E[q_G(k)] = 0$$

$$E[q_G(k) \cdot q_G(\ell)] = Q_G \cdot \delta_{k\ell}$$

$$Q_G = \int_{k\Delta}^{(k+1)\Delta} \Phi^2[(k+1)\Delta, \tau]\, W_G \cdot dt$$

where the statistic W_G has been defined in (2-1).

The measurement equation for mode H_o is defined as

$$D(k) = D_G(k) + r_G(k) \tag{2-6}$$

where $r_G(k)$ is a normally distributed random variable with known statistics given by

$$E[r_G(k)] = 0, \quad E[r_G(k)\, r_G(\ell)] = R_G \cdot \delta_{k\ell} = \frac{V_G}{\Delta}\delta_{k\ell}$$

where the statistic V_G has been defined in (2-1).

The measurement and state equations of the discrete time system of mode H_1 and H_2 can be similarly defined.

2.3 The Parity Equations

In order to isolate the sensor error from the desired vehicle rate information, a set of parity equations can be implemented by a simple deterministic concept using direct comparison of instrument outputs. Each parity equation is generated by a linear combination of gyro outputs such that the vehicle rate, if it is present, will be cancelled out. Two sets of parity equations have been studied. Since there are three independent components of vehicle rate, four instruments are the minimum number to form a parity equation. By simple geometrical projection on the body coordinate axes, a set of fifteen parity equations can be formed as shown in Table 2.2 (denoted as Set A). The other set is formed by a linear combination of six gyro outputs[33]. By projection on each of the gyro input axes, a set of six parity equations can be generated as shown in Table 2.3 (denoted as Set B).

The residuals of parity equation outputs provide the only available information reflecting gyro errors in the real system. It will be shown that each residual as a combination of four instrument signal outputs can be transformed into a scalar state by application of the concept of the linear aggregation of states. As an illustration, the first parity equation residual z in Table 2.2 will be examined under operating mode H_o.

For the normal mode H_o the state equations of the gyro bias drift rate are represented by Eq. (2-1)

$$\dot{D}_{Gi}(t) = -\omega_i D_{Gi}(t) + g_{Gi}(t) \qquad i = A, B, C, D$$

The measurement equation is the parity equation residual. By definition of the parity equation the residual can be simplified to represent only the gyro error sources:

Table 2.2

Parity Equation Set A

(For Mechanized SIRU System)

No.	Instruments	Equation
1	ABCD	$(\Omega_A - \Omega_B)C + (\Omega_C + \Omega_D)S = z$
2	ABCE	$(\Omega_B - \Omega_C)C - (\Omega_A + \Omega_E)S = z$
3	ABCF	$-(\Omega_A + \Omega_C)C + (\Omega_B + \Omega_F)S = z$
4	ABDE	$-(\Omega_A + \Omega_D)C + (\Omega_B + \Omega_E)S = z$
5	ABDF	$(\Omega_B - \Omega_D)C - (\Omega_A + \Omega_F)S = z$
6	ABEF	$(\Omega_E + \Omega_F)C - (\Omega_A + \Omega_B)S = z$
7	ACDE	$(\Omega_D - \Omega_E)C + (\Omega_A - \Omega_C)S = z$
8	ACDF	$(\Omega_C - \Omega_F)C + (\Omega_A - \Omega_D)S = z$
9	ACEF	$(\Omega_A - \Omega_F)C + (\Omega_C - \Omega_E)S = z$
10	ADEF	$(\Omega_A - \Omega_E)C + (\Omega_D - \Omega_F)S = z$
11	BCDE	$(\Omega_C + \Omega_E)C + (\Omega_B + \Omega_D)S = z$
12	BCDF	$-(\Omega_D + \Omega_F)C + (\Omega_B + \Omega_C)S = z$
13	BCEF	$(\Omega_B - \Omega_E)C - (\Omega_C + \Omega_F)S = z$
14	BDEF	$(\Omega_B - \Omega_F)C - (\Omega_D + \Omega_E)S = z$
15	CDEF	$(\Omega_C - \Omega_D)C + (\Omega_E - \Omega_F)S = z$

$C = \cos(\phi)$ $S = \sin(\phi)$ z = Parity Equation Residuals

Table 2.3

Parity Equation Set B

(For Mechanized SIRU System)

No.	Equation
1	$\Omega_A - \cos 2\phi \cdot (\Omega_B - \Omega_C - \Omega_D + \Omega_E + \Omega_F) = z$
2	$\Omega_B - \cos 2\phi \cdot (\Omega_A + \Omega_C + \Omega_D + \Omega_E + \Omega_F) = z$
3	$\Omega_C - \cos 2\phi \cdot (-\Omega_A + \Omega_B + \Omega_D - \Omega_E + \Omega_F) = z$
4	$\Omega_D - \cos 2\phi \cdot (-\Omega_A + \Omega_B + \Omega_C + \Omega_E - \Omega_F) = z$
5	$\Omega_E - \cos 2\phi \cdot (\Omega_A + \Omega_B - \Omega_C + \Omega_D - \Omega_F) = z$
6	$\Omega_F - \cos 2\phi \cdot (\Omega_A + \Omega_B + \Omega_C - \Omega_D - \Omega_E) = z$

$$z(t) = [\,c \;\; -c \;\; s \;\; s\,] \begin{bmatrix} D_{GA}(t) \\ D_{GB}(t) \\ D_{GC}(t) \\ D_{GD}(t) \end{bmatrix} + v(t) \qquad (2\text{-}7)$$

where $c = \cos\phi = \sqrt{\frac{5+\sqrt{5}}{10}} = 0.85065$

$$s = \sin\phi = \sqrt{\frac{5-\sqrt{5}}{10}} = 0.52574$$

v(t) denotes the measurement noise after the first parity equation. It is observed that there is a distinction of definition of measurement noise of gyros before and after the parity equation. Since all gyros sense the same disturbance source, the measurement noises between gyros before the parity equation are highly correlated. However, by definition of the parity equations, the correlated part of the measurement noises will be cancelled out after the parity equations. This will make detection of subtle degradation of gyro performance more effective due to the reduced background noise level. Thus the measurement noise v(t) can be defined as if no environment disturbance existed, and v(t) is essentially the quantization noises $v_{Gi}(t)$

$$v(t) = [\,c \;\; -c \;\; s \;\; s\,] \begin{bmatrix} v_{GA}(t) \\ v_{GB}(t) \\ v_{GC}(t) \\ v_{GD}(t) \end{bmatrix}$$

Since it can be assumed that V_{Gi}'s are independent of one another, the statistics of v(t) can be easily evaluated

$$E[v(t)] = 0, \quad E[v(t)\,v(\tau)] = 2V_G\,\delta(t-\tau) \triangleq V \cdot \delta(t-\tau) \qquad (2\text{-}8)$$

For the construction of computationally efficient algorithms associated with dynamic systems of high dimension, a method is applied to aggregate the original system state vector into a lower-dimensional vector (Aoki)[2].

Consider the dynamic system defined by the vector equation:

$$\frac{d\ \underline{D}_G(t)}{dt} = A\ \underline{D}_G(t) + B\underline{g}_G(t)$$

Define a transformation of $\underline{D}_G$ (t) to a scalar x(t)

$$x(t) \triangleq C\ \underline{D}_G(t)$$

The statement that x(t) satisfies the differential equation,

$$\frac{d\ x(t)}{dt} = F\ x(t) + G\ \underline{g}_G \qquad\qquad x(0) = C\ \underline{D}_G(0),$$

is equivalent to the condition that F and G are related to A and B by

$$\begin{aligned} G &= CB \\ FC &= CA \end{aligned} \tag{2-9}$$

This can be easily seen by a comparison with the equation

$$\frac{d\ (C\ \underline{D}_G)}{dt} = CA\ \underline{D}_G + CB\underline{g}_G$$

Designers in general have some freedom in choosing the aggregation matrix C subject to some constraints imposed by the problems. For example, the choice of C is to be made in such a way that the error in modeling the original system by the system of lower dimension is minimized in some sense taking into account the performance index for the original system.

In the specific case considered, the choice C is seen by geometry to be the measurement matrix.

$$C = [c \quad -c \quad s \quad s]$$

and

$$A = \begin{bmatrix} -\omega_A & & & 0 \\ & -\omega_B & & \\ & & -\omega_C & \\ 0 & & & -\omega_D \end{bmatrix} \qquad (2\text{-}10)$$

where the ω_1's are the system dynamics coefficients for the respective gyro drift rate models.

Substituting (2-10) into (2-9) produces

$$F \cdot [c - c \quad s \quad s] = [-c\omega_A \quad c\omega_B - s\omega_C - s\omega_D]$$

For the state x to be a scalar, F must be a scalar. The above equation is incompatible unless the following is true,

$$\omega_A = \omega_B = \omega_C = \omega_D = \omega$$

and $F = -\omega$.

But it is reasonable to assume that gyros of same type will be characterized by the same system dynamics coefficient ω. Thus the dynamic system associated with each parity equation can be reduced to a scalar system:

$$\begin{aligned} \dot{x}(t) &= -\omega \cdot x(t) + g(t) \\ z(t) &= x(t) + v(t) \end{aligned} \qquad (2\text{-}11)$$

The statistics of the scalar random process x(t) can be evaluated easily:

$$E[g(t)] = 0, \quad E[g(t)\,g(\tau)] = 2\,W_G\,\delta(t-\tau) = W \cdot \delta(t-\tau)$$

A remark of interest need be mentioned. Consider the observability matrix of the original system·

$$Q = [C, (CA)^T, (CA^2)^T, (CA^3)^T]$$

It can be seen that Q is of rank 1 for $\omega_1 = \omega$, and of rank 4 if $\omega_A \neq \omega_B \neq \omega_C \neq \omega_D$. It means that the observable system cannot be

aggregated. The cases for modes H_1 and H_2 can be treated in the same way. It is easy to show that each parity equation of the set B in Table 2.3 can be similarly aggregated into a scalar dynamic system. In order to make a comparison of merits between two sets, a sensitivity analysis will be made for the case of mode H_1.

For both set A and set B, the state and measurement equations are of the same form

$$\begin{aligned} \dot{x}(t) &= -\omega \cdot x(t) + g(t) \\ z(t) &= x(t) + m + v(t) \end{aligned} \tag{2-12}$$

where the state x(t) is defined as

$$x(t) = \cos\phi \cdot D_{GA} - \cos\phi \cdot D_{GB} + \sin\phi \; D_{GC} + \sin\phi \; D_{GF})$$

for set A, and is defined as

$$x(t) = D_{GA} \; -\cos 2\phi \; (D_{GB} - D_{GC} - D_{GD} + D_{GE} + D_{GF})$$

for set B.

With these definitions, the statistics of g(t) and v(t) can be seen to be the same for both sets A and B of parity equations.

$$E[\, g(t)] = 0, \quad E[\, g(t)\, g(\tau)] = W \cdot \delta(t-\tau)$$

$$E[\, v(t)] = 0, \quad E[\, v(t)\, v(\tau)] = V \cdot \delta(t-\tau)$$

Suppose that gyro A has been degraded with a shift of mean bias of magnitude m_1. Then the parity residual z(t) will reflect an expected value:

$$E[\, z(t)] = \cos\phi \cdot m_1 \,, \qquad \text{for set A}$$

$$E[\, z(t)] = m_1 \qquad \text{for set B}$$

Now E [z(t)] can be regarded as the signal used for detecting the degradation mode. Then as far as this single parity equation is concerned, the set B formulation will have the better sensitivity in the sense that it gives a larger signal to noise ratio. But this conclusion is not true if the

whole set of parity equations is taken into consideration. In parity equation set B, five other equations will also give some indication of degradation, with an attenuation factor of cos 2ϕ. But in parity equation set A, the parity equations (no. 11-15) will give null indication of degradation. So these equations will give a perfect capability for isolating degraded gyro A, if all other gyros are in the normal mode. In this sense the parity equation set A will have a better isolation capability.

In this thesis it will be assumed that there will not be the case of dual instruments degraded during the same interval. This will be a reasonable assumption if the instruments have a long mean life time and the instruments are operated in a system with a capability to correct the performance degradation. To make a quantitative analysis, the distribution of the instrument degradation is assumed to be exponential

$$P(\theta \leqq t) = 1 - e^{-q \cdot t}$$

where θ is the instant that degradation occurred, q is the constant failure rate.

It is well known that the exponential distribution has the so-called strong Markov property:

$$P(\theta \leqq s + t \mid \theta > s) = 1 - e^{-q \cdot t}$$

i.e. the distribution is independent of s, even if s is a random variable.

Now suppose that an instrument has been detected in the degraded mode at t = s, and the system will take an additional time T to identify the degradation mode and compensate for the performance degradation. Then the liklihood of failure of two instruments is given by the probability

$$\begin{aligned} P(DF) &\triangleq P(\theta \leqq s + T \mid \theta > s) \\ &= 1 - e^{-q \cdot T} \end{aligned} \tag{2-13}$$

A reasonable mean time between failures T_m of the gyroscope instruments is 1.7×10^5 hours. To make an estimate of the time interval T required for compensation, consider a simplified case. Let a random

process x(t) have a mean m and a correlation function defined as

$$\psi_{xx}(\tau) = \sigma_x^2 \cdot e^{-\frac{|\tau|}{\tau_x}} + m^2$$

The time interval T can be viewed as that required for an accurate measurement of the mean m. Consider the simple measurement scheme by integration.

The measured mean M is defined as

$$M = \frac{1}{T} \int_0^T x(t)\, dt$$

It can be shown that M is an unbiased estimate of mean m with a variance:

$$\sigma_M^2 = \frac{2}{T} \int_0^T (1 - \frac{\tau}{T})\ (\psi_{xx}(\tau) - m^2) d\tau$$

Some simple manipulation yields

$$\frac{\sigma_M^2}{\sigma_x^2} = 2 \cdot \{ \frac{\tau_x}{T} - \frac{\tau_x^2}{T^2} + \frac{\tau_x^2}{T^2}\ e^{-\frac{T}{\tau_x}} \}$$

Suppose an estimate of m with a good quality of $\sigma_M = 0.1\ \sigma_x$ is required, the required time interval T can be found to be

$$\frac{T}{\tau_x} \cong 200$$

With a typical time constant of gyro drift rate process $\tau_x = 0.5$ hr., a time interval T of 100 hours is required.

On substituting T into (2-13), the probability of dual failure can now be estimated:

$$P(DF) = 1 - e^{-\frac{T}{T_m}}$$

$$= 1 - e^{-(100/170,000)} \approx 0.06\%$$

It can be seen that the case of dual failure can indeed be neglected.

In this case it would not be necessary to inspect all fifteen equations of parity equation set A at each parity test time. A set of six out of the fifteen parity equations is selected to be processed for a single degraded instrument isolation. If a dual failure occurs, one may then resort to the full equation set. The parity equation set C to be actually implemented is shown in Table 2.4. The choice is based on geometrical symmetry for a capability of unique isolation of a single instrument failure. From the analysis of geometrical error amplification it can be seen that the set will give an equal likelihood of error amplification in all three reference axes.

A simple decision function based on the list of gyros which enter each parity equation will be applied to isolate the degraded gyro by monitoring the six parity equation output residuals. It is noted that the parity equation residuals in Table 2.4 will reflect an attenuated instrument signal output by either a factor $\sin\phi = 0.52574$ or $\cos\phi = 0.85065$. A large signal-noise ratio will be desirable for the efficient detection of a probable instrument degradation. Thus for isolation of instrument A, an efficient set will be the equations 1(ABCD), 4(ADEF), 5(BCDE) and 6(CDEF). For example, the parity equation 3(ABEF) will give a lower signal-noise ratio of $\frac{s^2}{c^2} \approx \frac{1}{2.5}$ in comparison with the equation 1(ABCD). But in order to give a reliable isolation capability from four parity equations additional polarity information of the parity equation residuals must be identified. This will be discussed in Chapter 4.

2.4 The Basic Approach

The design philosophy for improvement of the navigation system reliability and performance in this thesis is to provide the system with the adaptive capability for self-repair of certain failure modes. When a part of the system degrades, the effect of the degradation is automatically compensated for by making some subtle adjustment. Implied in this activity are several distinct steps which must be mechanized: the existence of a degradation must be detected, its nature must be determined, and finally repair activity must be initiated. Depending on the available information in the navigation system the design of the adaptive system can be accomplished in different ways. This thesis is the design of a self-contained adaptive system without external

Table 2.4

Parity Equation Set C

(For Mechanized SIRU System)

No.	Instrument	Equation
1	ABCD	$(\Omega_A - \Omega_B)C + (\Omega_C + \Omega_D)S = z$
2	ABCF	$-(\Omega_A + \Omega_C)C + (\Omega_B + \Omega_F)S = z$
3	ABEF	$(\Omega_E + \Omega_F)C - (\Omega_A + \Omega_B)S = z$
4	ADEF	$(\Omega_A - \Omega_E)C + (\Omega_D - \Omega_F)S = z$
5	BCDE	$(\Omega_C + \Omega_E)C - (\Omega_B + \Omega_D)S = z$
6	CDEF	$(\Omega_C - \Omega_D)C + (\Omega_E - \Omega_F)S = z$

$C = \cos\phi, \quad S = \sin\phi$

aiding information and without monitoring sensors of the internal states of the instruments. Redundant instruments are assumed which makes it possible to indicate instrument errors.

The basic approach in the thesis is the application of modern system theory to the design of an adaptive system. The adaptive system is formulated as a multistage statistical decision and estimation process

(1) a detection system to detect and isolate a degradation in sensors in the shortest possible time so as not to degrade the overall navigation performance by an appreciable amount.

(2) an identification system to verify a degradation of the isolated sensor and to classify the degradation mode of the degraded sensor with small error probability.

(3) a compensation system to estimate with high accuracy the unknown parameter associated with the identified degradation mode, and to recover the degraded sensor performance by compensation

The emphasis to formulate the detection and identification systems as a statistical decision problem is motivated by two considerations.

First, the formulation of the detection system as an optimal decision problem will show very efficient performance (expressed in time delay) in detecting a degradation compared with the formulation as an estimation problem to estimate the probable parameter change. The operating modes of gyros are characterized by the parameters associated with the gyro model. The detection of the characteristic change is a problem in statistical decision theory. The statistical decision theory determines from which of several hypothesized probability distributions (with specified statistics) a set of samples comes, to within set error probabilities. The confidence level is specified by the error probabilities with respect to decision reliability with no concern about the accuracy of the true value of the parameter. The estimation process, on the other hand, is concerned with the accuracy of the estimated value. The confidence level is specified such that the estimated value will approximate the true value within a small deviation. In comparison with the estimation process, the decision process will thus require "coarser" information and allow greater uncertainty about the opera-

ting system in making a decision. The measurement time can be much shorter.

Second, the degraded system may be operating in any of many possible degradation modes. The observed input data are unclassified, bringing in additional uncertainty to complicate the estimation process. One will then pay for the uncertainty about the classification of the given data in terms of slower rate of estimation (learning). Moreover, the recursive computational procedure for estimation is more difficult than the counterpart with data of given classification. This problem has been discussed in the pattern recognition field as the self-learning or "learning without teacher" problem (Ho et al.).[34] By application of statistical classification techniques the most probable mode will first be identified and then the unknown parameter will be estimated under the condition of an identified mode. The estimate will be more reliable and the computation will be more efficient.

A block diagram of the adaptive system is shown in Fig. 2.7. It is well-known in the identification process that there is an uncertainty relation between measurement time and identification accuracy. To solve this dilemma the detection and identification systems are designed as two separate stages. In the detection stage the performance criterion is the shortest time delay in detection of a degradation under the constraint of a tolerable false alarm error probability. After a degradation has been detected, the degraded instrument will then be isolated. Because of availability of redundant sensors, the "degraded" instrument can be switched off for navigation information (the instruments will always remain in operation in the adaptive system). More time is then allowed to make a reliable identification with small error probability. The identification consists of making one of the following three decisions. a verification of normal mode H_0 due to a possible false detection or a possible requalification of the instrument, an identification of degradation mode H_1, or an identification of degradation mode H_2. The compensation system is an estimator of the unknown parameter of the identified mode with knowledge of the accurate model. The estimated parameter is used to recover the degraded sensor performance by compensation.

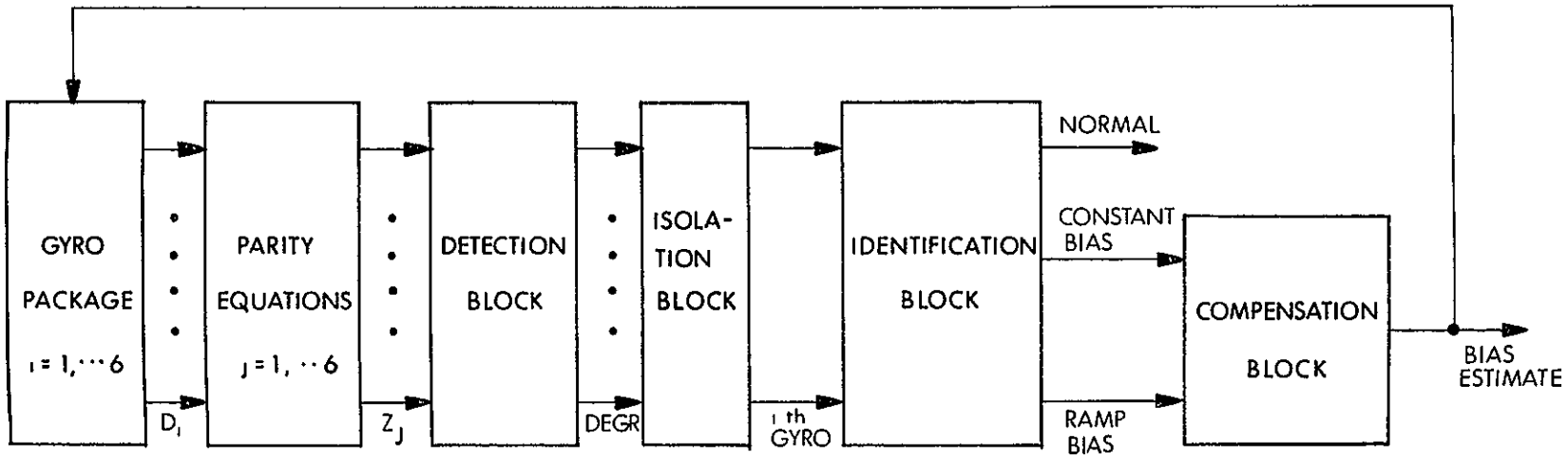


Fig. 2.7 Block Diagram of Adaptive System

CHAPTER 3

Stochastic Differential Equations

3.1 General Discussion

The purpose of this chapter is to derive the stochastic differential equations for the detection and identification systems The dynamical equation in the detection and identification system is formulated as a two-dimensional partially observable* process, in which the process describing the state of degradation is inaccessible to direct observation. The problem with partially observable information is approached by deriving an expression for the posterior probability of the unobservable component, conditioned on all a priori information and the measurement history up to the current time. It will be shown that the derivation of the posterior probability is simplified by a transformation of the correlated measurement process into a corresponding independent process The approach of the transformation is based on the concept of the innovation process. A simplified Kalman filtering technique is proposed to implement the transformation process. It will be shown that the transformation preserves the statistical information. Based on the transformed measurement process the stochastic differential equations of the posterior probability will be derived. The derivation is based on the application of a representation theorem due to Doob concerning the conditions when a process with continuous trajectories can be obtained as a solution of some Ito|stochastic differential equation. It is shown that the derivation of the posterior probability in the identification system can be considered as a degenerated case in the detection system.

*Note: The term "observable" is defined in this thesis as the meaning "inaccessible to direct measurements".

Finally, it will be shown that only one detection system, designed for degradation in the form of a jump of constant bias, is sufficient for all degradation modes characterized by a systematic mean change Moreover the detection system can be constructed on the basis of a "design" value, specified by the mission performance, of the unknown parameter in the real system. The approach is justified by application of a comparison theorem for diffusion processes.

3.2 Problem Statement

Let $(\theta(t), z(t))$ be a two-dimensional partially observable Markov process. The observable component is the measurement process z(t) of the parity equation residuals. The measurement process is represented in each operating mode as follows:

A. Normal Mode H_0:
$z(t) = x(t) + v(t)$

B. Degradation Mode H_1:
$z(t) = x(t) + m + v(t)$ (3-1)

C. Degradation Mode H_2:
$z(t) = x(t) + nt + v(t)$

where m is an unknown and deterministic constant bias parameter, and n is an unknown and deterministic constant ramp rate parameter. The scalar state x(t) is modeled as a first-order Gauss-Markov process given by (2-11):

$$\dot{x}(t) = -\omega \cdot x(t) + g(t) \tag{3-2}$$

The driving noise g(t) and measurement noise v(t) are white-noise processes with zero mean and known correlation:

$$E[\, g(t)g(\tau)\,] = W \cdot \delta(t-\tau)$$
$$E[\, v(t)v(\tau)\,] = V \cdot \delta(t-\tau) \tag{3-3}$$

During the mission the system will be degraded at an unknown instant θ to one of the degradation modes The observable process z(t) will thus reflect a jump transition from H_0 to H_1 (or H_0 to H_2) at the instant θ in a

way described by the unobservable component process $\theta(t)$.

The process $\theta(t)$ is expressed as a stationary Markov jump process with two states 0 (Mode H_0) and 1 (Mode H_1 or H_2), and with a unique transition from state 0 to state 1 In the formulation it is assumed that the parameter θ, the instant at which the process $\theta(t)$ makes a transition, is defined by an a priori exponential distribution:

$$P(\theta > t) = e^{-qt} \tag{3-4}$$

where q is the constant failure rate of the instrument

Using these definitions the problem can now be stated as follows Let $\pi(t) = P\{\theta \leq t \mid z(\tau), \tau \leq t\}$ be the posterior probability of degradation up to time t and conditioned on all a priori information and the measurement history up to time t Derive a stochastic differential equation for the process $\pi(t)$ This chapter deals with the continuous time system Some results of the discrete time system will be mentioned in Section 3 3 The corresponding formulation of the posterior probability in the discrete time system will be found in Section 4.3 of Chapter 4.

3 3 A Transformation of the Measurement Process

To simplify the derivation of the posterior probability the correlated measurement process is transformed into a corresponding independent process in this section The approach is based on the concept of the innovation process and the Kalman filtering technique is applied for implementation of the process If the filter is implemented without modeling error of the real dynamic system, then it is well known that the innovation process is a Gaussian white noise process However, the correct model, i e., the actual operating mode of the real system is only known with some degree of uncertainty in the adaptive system To simplify the design of the adaptive system and to reduce the computation required by the filtering algorithm it is proposed in this section to design the filter only on a model of the normal mode H_0 It can be shown that the original process in the degraded mode can be transformed into a white noise process superimposed with a corresponding systematic mean with its value modified by a known attenuation factor Moreover the transformed and the original processes

contain the same statistical information The transformation is developed for the continuous-time system. The corresponding results of the discrete-time system which are useful for actual implementation will also be derived

The innovation approach is to first convert the observation process to a white noise, called the innovation process, and then treat the simplified problem based on derived white noise observations This whitening filter approach was used by Bode and Shannon in the solution of the Weiner filtering problem for stationary processes over a semi-infinite interval Kailath[38] extended the technique to handle the non-stationary continuous time process over a finite time interval, and give a simple derivation of the Kalman-Bucy recursive filtering formulas.

Consider the continuous observation process of the system defined in the normal mode H_0.

$$z(t) = x(t) + v(t) \tag{3-5}$$

Assume that the process x(t) obeys the finite-expected energy requirement, i e

$$\int_0^T E[x(t)^2]\, dt < \infty$$

and that x(t) and v(t) are completely independent, v(t) is the white Gaussian noise process

Define

$$\tilde{x}(t) = E[x(t) \mid z(\tau),\ \tau \leqq t,\ H_0] \tag{3-6}$$

= the conditional mean of x(t), given observations up to the instant t and assuming that $\{z(\tau) = x(\tau) + v(\tau)\}$

Then the innovation process of x(t) defined by

$$\begin{aligned} \nu_0(t) &\triangleq z(t) - \tilde{x}(t) \\ &= x(t) - \tilde{x}(t) + v(t) \\ &= \tilde{x}'(t) + v(t) \end{aligned} \tag{3-7}$$

is a white Gaussian noise process with the same statistics as the original measurement noise process v(t).

The proof essentially rests on a martingale theorem of Doob, and can be found in Kailath[39]. Thus for the normal mode H_0, the original correlated process z(t) (3-5) can be converted into the conditionally known signal process $z(t) = \tilde{x}(t) + \nu_0(t)$. For a Gauss-Markov process x(t) the transformation can be implemented by the well-known Kalman filter

A heuristic explanation of the result will be given First, since $\tilde{x}'(t)$ is the protion of x(t) that cannot be predicted from past z(·) since v(t), being white noise, is completely unpredictable from past z(·), the quantity $\nu_0(t)$ may be regarded as describing the "new information" brought by the current observation z(t), being given all the past observations z(s), $s < t$, and the old information deduced therefrom Therefore the name innovation process of x(t) came into being. Physically it represents the measurement residual process, a familiar term in estimation theory Second, the innovation process $\nu_0(t)$ in the continuous-time case will have the same statistics as that of the measurement noise v(t). Since the processes $\nu_0(t)$ and v(t) have the same finite-dimensional distribution, they are thus statistically indistinguishable from each other, though they have different sample functions. However, it will be shown that this result will not be true in the discrete time case

It is to be noted that the processes z(t) and $\nu_0(t)$ contain the same statistical information, i e , they are equivalent The proof of the equivalence is obvious if the Kalman-Bucy filtering formula is assumed Since $\nu_0(t) = z(t) - \tilde{x}(t)$ and $\tilde{x}(t)$ can be calculated from z(s), $s \leqq t$, therefore $\nu_0(t)$ is completely determined by z(s), $s \leqq t$ Conversely, assuming the dynamic system equation of the state x, the Kalman-Bucy formula

$$\dot{\tilde{x}}(t) = \omega\tilde{x}(t) + K(t)\,[\,z(t) - \tilde{x}(t)]$$

$$\tilde{x}(0) = 0$$

shows that $\tilde{x}(t)$ is determined if $\{\nu_0(s),\ s \leqq t\}$ is known, and then z(t) can be obtained as $z(t) = \tilde{x}(t) + \nu_0(t)$, since by (3-7)

$$\tilde{x}(t) + \nu_0(t) = \tilde{x}(t) + \tilde{x}'(t) + v(t)$$
$$= x(t) + v(t) \triangleq z(t)$$

Therefore $\nu_0(t)$ and $z(t)$ can each be obtained from the other by means of a casual and casually invertible linear operation, or each can be considered to be a deterministic function of the other Therefore they are equivalent

Now consider the continuous observation process of the system defined in the general degraded mode H:

$$\dot{z}(t) = x(t) + m(t) + v(t) \qquad (3\text{-}8)$$

where $m(t)$ is defined as

$m(t) = m$ for system in mode H_1

$m(t) = nt$ for system in mode H_2

It can be seen that this process can be similarly treated by a simple extension of the innovation transformation The approach is to regard the unknown bias parameters m or n as random variables, the observation processes can be then be whitened by subtracting out the estimates of the state $x(t)$ and the bias process $m(t)$ The transformation can be implemented by the Kalman filter with an augmented bias state $m(t)$. The observation process $z(t)$ (3-8) can be similarly converted into the conditionally known signal processes $\hat{x}(t)$ and $\hat{m}(t)$ without loss of information.

$$z(t) = \hat{x}(t) + \hat{m}(t) + \nu_1(t) \qquad (3\text{-}9)$$

where $\nu_1(t)$ is the innovation process In this approach the state in the observation process (3-9) can be viewed as the vector form of the process (3-5) with an augmented bias state Thus the conclusion about the properties of the $\nu_0(t)$ process (3-7) can be formally extended to apply to the $\nu_1(t)$ process under the corresponding assumptions Then the process $\nu_1(t)$ is a white noise process with the same statistics as the $v(t)$ - process. Again the $\nu_1(t)$ - process will have a different variance from that of either the $v(t)$ - process or the $\nu_0(t)$-process in the discrete-time case.

Now consider the formulation of the following basic decision problem which is of interest in this thesis. Given observations $\{z(\tau),\ 0 \leqq \tau \leqq t\}$, determine, so as to minimize an expected risk function, which of the following modes is true:

$$\begin{aligned} H_0 &: z(t) = x(t) + v(t) \\ H &: z(t) = x(t) + m(t) + v(t) \end{aligned} \tag{3-10}$$

By application of the innovation approach, the original optimal decision problem (3-10) can be converted into the "conditionally known" signal problems:

$$\begin{aligned} H_0 &: z(t) = \tilde{x}(t) + \nu_0(t) \\ H &\quad z(t) = \hat{x}(t) + \hat{m}(t) + \nu_1(t) \end{aligned} \tag{3-11}$$

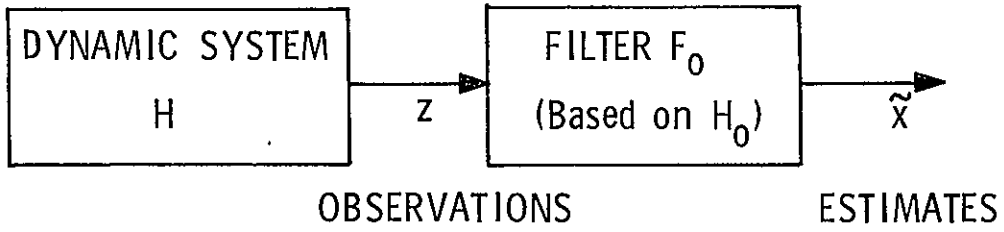
Since the processes $\tilde{x}(t)$, $\hat{x}(t)$, and $\hat{m}(t)$ are conditionally known, the problem (3-11) can be further reduced to

$$\begin{aligned} H_0 &: z(t) - \tilde{x}(t) = \nu_0(t) \\ H &: z(t) - \hat{x}(t) - \hat{m}(t) = \nu_1(t) \end{aligned} \tag{3-12}$$

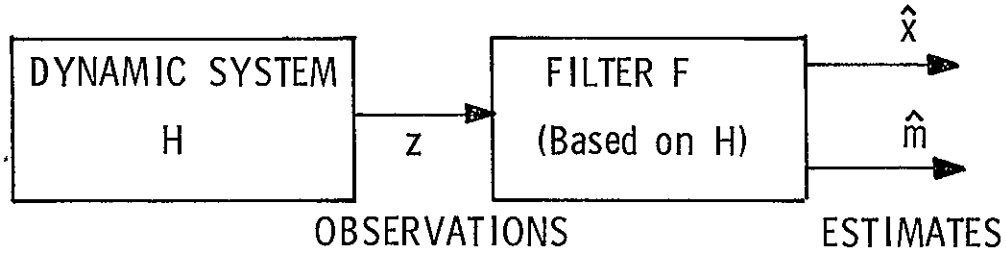
It is to be emphasized that this linear transformation (3-11) preserves the statistical information by its casual and casually invertible property, and, moreover, provides the optimal information for decision by its orthogonal projection property

However, the approach requiring m(t) estimation is not an appealing one. First, there is no information about the actual operating mode (mode H_1 or H_2) for accurately modeling the process m(t) Second, the estimation of m(t) will unnecessarily complicate the computation since the system is operated in the normal mode during most of the mission A different approach will be considered In this approach the observation process of the system defined in the degraded mode H will be treated by modeling the bias process m(t) as a deterministic process with unknown constant parameters m or n.

Consider the following configurations of filter modeling (Fig 3. 1).



(A)



(B)

Fig. 3.1 Configurations of Filter Modeling.

In configuration A the filter is constructed on the model of the normal system H_0, while the true dynamic system is of model H. In configuration B, the filter is correctly constructed on the model of the real system H

Then it will be shown that the observation process z(t) can be asymptotically transformed into the following form.

$$H_1: \; z(t) = x(t) + \tilde{a}_1 \cdot m + \nu_0(t) \tag{3-13}$$

for z(t) defined in degraded mode H_1, and into the approximate form

$$H_2: \; z(t) \simeq \tilde{x}(t) + \tilde{a}_1 \cdot n \cdot t + \nu_0(t) \tag{3-14}$$

for z(t) defined in degraded mode H_2. Here $\tilde{a}_1$ is a constant parameter to be determined. The process $\tilde{x}(t)$ is the bias-free estimate of the state x(t), given observations up to the instant t and conditioned on H_0 that $\{z(t) = x(t) + v(t)\}$. The process $\nu_0(t)$ is a pseudo-innovation process with the same statistics as the process (3-7) In the following discussion equation (3-13) for mode H_1 will first be derived, the result will then be extended to discuss the approximate form (3-14) for mode H_2. For convenience of presentation the notation m(t) of the generalized bias process will be used

The expression (3-12) can be derived as follows. Let the process z(t) be defined as,

$$z(t) = x(t) + m(t) + v(t)$$

and a Kalman filter F_0 is constructed on the model of the normal system H_0 as in configuration A of Fig. 3.1. Since the Kalman filter is a linear operation, then by an application of the law of superposition the transformation of the z(t) - process can be derived as a summation of two pseudo-processes $z_0(t) = x(t) + v(t)$ and m(t).

The dynamic system and the filter equation for the pseudo-process $z_0(t)$ can be written as (Fig 3.2):

$$\dot{x}(t) = -\omega \, x(t) + g(t)$$

$$\dot{\tilde{x}}_0(t) = -\omega \, \tilde{x}_0(t) + K(t) \cdot [z_0(t) - \tilde{x}_0(t)]$$

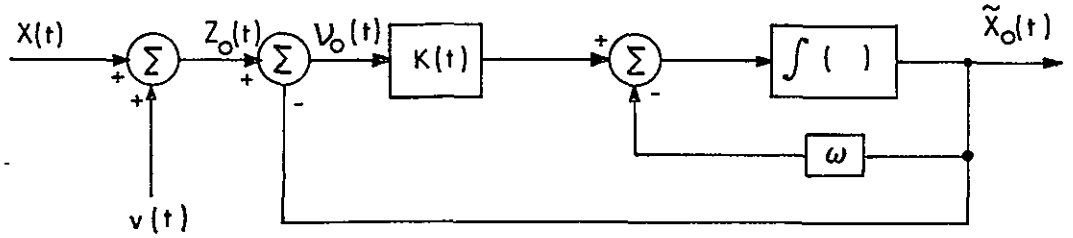


Fig. 3.2. Filter for the Pseudo Measurement Process

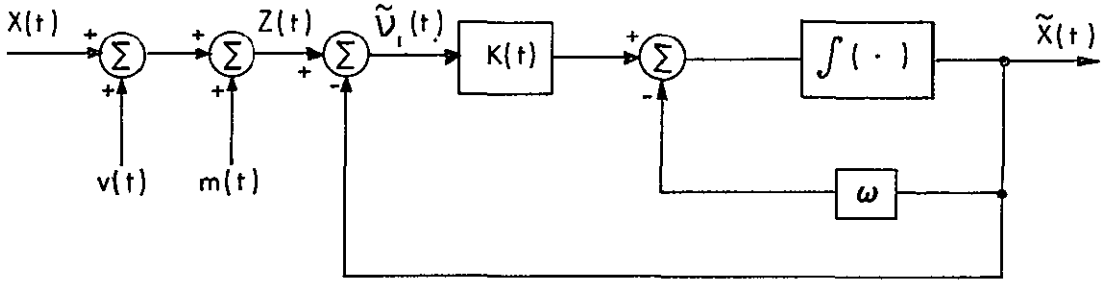


Fig. 3.3. Filter for the Measurement Process

$$K(t) = \tilde{P}_x(t)/V$$
$$\nu_0(t) = z_0'(t) - \tilde{x}_0(t) \tag{3-15}$$

where

$\tilde{P}_x(t)$ = the variance of the error in the pseudo-estimate $\tilde{x}_0(t)$ at time t.

Consider the pseudo-bias process $\tilde{m}(t)$ as the input to the filter F_0, the output process $\tilde{m}(t)$ satisfies the filter equation

$$\dot{\tilde{m}}(t) = -\omega\tilde{m}(t) + K(t)\,[m(t) - \tilde{m}(t)] \tag{3-16}$$

The process z(t) can then be expressed as:

$$z(t) = \tilde{x}(t) + \tilde{\nu}_1(t) \tag{3-17}$$

where the processes $\tilde{x}(t)$ and $\tilde{\upsilon}_1(t)$ are defined as

$$\tilde{x}(t) = \tilde{x}_0(t) + \tilde{m}(t)$$
$$\tilde{\nu}_1(t) = \nu_0(t) + (m(t) - \tilde{m}(t)) \tag{3-18}$$

It is to be noted that the representation in Fig. 3.3 illustrates the implementation of configuration A in Fig. 3.1. The process $\tilde{x}(t)$ is the bias free estimate of the state x(t) defined in (3-13) and is the only conditionally known signal of the configuration A.

Now (3-16) of the pseudo process $\tilde{m}(t)$ can be rewritten as

$$\dot{\tilde{m}}(t) = -(\omega + K(t))\,\tilde{m}(t) + K(t) \cdot m(t) \quad \tilde{m}(0) = 0 \tag{3-19}$$

where $m(t) \triangleq m$ for degraded mode H_1.

Equation (3-19) is a first-order ordinary differential equation with the time-variant coefficients K(t) If the time-variant effect of K(t) on the solution m(t) during the transient period is to be neglected, and only the stationary Kalman filter is implemented, then the solution $\tilde{m}(t)$ can be shown to be

$$\tilde{m}(t) = \left(\frac{\tilde{K}_\infty}{\tilde{K}_\infty + \omega} \right) \quad m \{1 - \exp [- (\tilde{K}_\infty + \omega) t] \} \qquad (3\text{-}20)$$

where $\tilde{K}_\infty = (\tilde{P}_\infty / V)$ is the steady state gain of the implemented Kalman filter F_0 The asymptotic solution of $\tilde{m}(t)$ can be further simplified to the form

$$\lim_{t \to \infty} \tilde{m}(t) = -V_x \cdot m$$

where $$V_x = - \left(\frac{\tilde{K}_\infty}{\tilde{K}_\infty + \omega} \right) \qquad (3\text{-}21)$$

The negative sign before V_x is employed to be consistent with the notation in Chapter 6 Substituting (3-21) into (3-17) and (3-18) produces

$$\begin{aligned} \lim_{t \to \infty} z(t) &= \lim_{t \to \infty} \{ \tilde{x}(t) + (m(t) - \tilde{m}(t)) + \nu_0(t) \} \\ &= \lim_{t \to \infty} \{ \tilde{x}(t) + (1 + V_x) \cdot m + \nu_0(t) \} \end{aligned} \qquad (3\text{-}22)$$

in the form of (3-13) The factor $\tilde{a}_1$ is defined as the coefficient $(1 + V_x)$

In summary the following statement has been proved
Given observations $\{ z(t),\ 0 \leqq \tau \leqq t \}$, it is required to determine, so as to minimize an expected risk function, discrimination between two modes defined as

$$\begin{aligned} H_0 &: z(t) = x(t) + v(t) \\ H_1 &: z(t) = x(t) + m + v(t) \end{aligned} \qquad (3\text{-}23)$$

This optimal decision problem can be transformed into the equivalent discrimination problem as follows

$$
\begin{aligned}
&H_0 \;.\; z(t) = \tilde{x}(t) + \nu_0(t) \\
&H_1 : z(t) = \tilde{x}(t) + \tilde{\nu}_1(t) \\
&\qquad\quad = \tilde{x}(t) + (m - \tilde{m}(t)) + \nu_0(t)
\end{aligned}
\tag{3-24}
$$

Moreover, the problem can be further converted asymptotically into the equivalent problem of discrimination between two modes

$$
\begin{aligned}
&H_0 \cdot z(t) - \tilde{x}(t) = \nu_0(t) \\
&H_1 : z(t) - \tilde{x}(t) = (1 + V_x)\; m + \nu_0(t)
\end{aligned}
\tag{3-25}
$$

The equivalence of the discrimination problems (3-23) and (3-25) can be stated as follows. Reasoning as the proof that the processes z(t) and $\nu_0(t)$ contain the same statistical information, it can be shown that the processes $\tilde{\nu}_1(t)$ in (3-24) and z(t) are equivalent, since each can be obtained from the other by a casual and casually invertible linear operation More-over, the transformation (3-25) gives asymptotically the optimal information for decision by an operation of the orthogonal projection The last point can be seen as follows It will be shown in Chapter 6 that the following asymptotic relations (6-30) and (6-31) can be proved.

$$\hat{x}(t) = \tilde{x}(t) + V_x \cdot m$$

$$\hat{m} = m$$

Substituting the above relations into the formulation of the problem H_1 in (3-12)

$$H_1 : z(t) - \hat{x}(t) - \hat{m}(t) = \nu_1(t)$$

produces the asymptotically equivalent problem:

$$H_1 : z(t) - \tilde{x}(t) = (1 + V_x) \cdot m + \nu_1(t)$$

Since the processes $\nu_1(t)$ and $\nu_0(t)$ are statistically indistinguishable from each other, then the representations (3-12) and (3-25) are asymptotically equivalent

Define the transformed observation process y(t) as

$$\dot{y}(t) = \tilde{r}(t)$$

where $\tilde{r}(t)$ is the measurement residual of bias-free estimation, or

$$\tilde{r}(t) \triangleq z(t) - \tilde{z}(t) = z(t) - \tilde{x}(t)$$

The transformed observation process y(t) can be formally represented in the equivalent stochastic differential equation for the system defined in the mode H_1

$$dy(t) = a\ dt + \sigma\ dw(t) \tag{3-26}$$

where

$$a = (1 + V_x) \cdot m$$

The parameter σ^2 in (3-26) is the statistic of the innovation process $\nu_0(t)$, and has been shown to be equal to the parameter V of the measurement noise process v(t). The w(t) process is the standard Weiner process with unit variance rate.

The result will be extended to the approximate form (3-14) for the observation process z(t) defined in the mode H_2 In this case the bias process m(t) is defined as m(t) = nt, and the solution of $\tilde{m}(t)$ in (3-19) can be shown to be

$$\tilde{m}(t) = \frac{\tilde{K}_\infty}{(\tilde{K}_\infty + \omega)^2}\ n\ \{(\tilde{K}_\infty + \omega)t - 1 + \exp\ [-(\tilde{K}_\infty + \omega)\ t]\} \tag{3-27}$$

The asymptotic solution of $\tilde{m}(t)$ can be shown to be

$$\lim_{t\to\infty} \tilde{m}(t) = \left(\frac{\tilde{K}_\infty}{\tilde{K}_\infty + \omega}\right) nt - \frac{\tilde{K}_\infty}{(\tilde{K}_\infty + \omega)^2} n \simeq \left(\frac{\tilde{K}_\infty}{\tilde{K}_\infty + \omega}\right) nt \tag{3-28}$$

if the constant term is neglected for large t. Thus the z(t)-process defined in mode H_2 can be asymptotically transformed into the approximate form (3-14) by substituting (3-28) into (3-17) and (3-18)

$$z(t) \simeq \tilde{x}(t) + (1 + V_x)\, nt + y_0(t) \tag{3-29}$$

where V_x has been defined in (3-21)

The transformed observation process y(t) for the system defined in the mode H_2 for the asymptotic case can then be formally represented in the stochastic differential equation

$$dy \simeq bt\ dt + \sigma dw(t) \tag{3-30}$$

in the sense of the approximation in (3-28) Here, the parameter σ has been defined in (3-26), and the parameter b is defined as

$$b = (1 + V_x) \cdot n \tag{3-31}$$

A discussion about the factor $(1 + V_x)$ is required It is noted that the magnitude of the factor a (or b) represents the equivalent signal amplitude that is used for detection However, one should not conclude that the transformation will impair the detection performance by an attenuation of the magnitude m (or n) From the definition V_x in (3-21), one will notice that the factor V_x depends essentially on the parameter ω of the dynamic system, or on the correlation of the system state model. The factor $(1 + V_x)$ is induced by the transformation which converts the correlated process z(t) to a white noise process such that every transformed data is now a piece

of new information In case the coefficient ω is zero, then the factor $(1 + V_x)$ will be zero The signals a (or b) will be attenuated to zero But this means that the system state process is the Brownian motion process In this case the constant bias term cannot be distinguished from the system state

It should be emphasized that the simplification of the proposed system is made possible under the condition that no information about the estimates of parameters m and n has been obtained The parameters m and n are unknown constants that have not been estimated. The detection process however can be constructed on the basis of "design" values of these unknown parameters specified by the mission requirements. The approach is justified by the application of a comparison theorem to be discussed in Section 3. 5.

The transformation of the measurement process in the discrete-time case will now be presented. The discrete-time case will be useful for actual implementation.

The following notation is employed The measurement process is represented in one of the operating modes defined as follows·

A · Normal Mode H_0·

$$z(k) = x(k) + r_1(k)$$

B Degradation Mode H_1: (3-32)

$$z(k) = x(k) + m + r_1(k)$$

C. Degradation Mode H_2·

$$z(k) = x(k) + n \cdot k + r_1(k)$$

The scalar state x(k) is modeled as a first order Gauss-Markov sequence given by

$$x'(k + 1) = F\, x(k) + q_1(k) \qquad (3\text{-}33)$$

The driving noise $q_1(k)$ and measurement noise $r_1(k)$ are white random sequences with known statistics.

$$E[q_1(k)] = 0, \quad E[q_1(k)\ q_1(\ell)] = Q \cdot \delta_{k\ell}$$

$$E[r_1(k)] = 0, \quad E[r_1(k)\ r_1(\ell)] = R \cdot \delta_{k\ell}$$

The state x(k) in the discrete-time system is similarly generated by the linear aggregation of the gyro drift rate states. The gyro drift rate state $D_G(k)$ in the discrete time case has been defined by (2-4) in Chapter 2. The relation between the strength of the noises Q, R in the discrete-time and of the noises W, V in the continuous time cases can be expressed as

$$Q = \int_{k\Delta}^{(k+1)\Delta} \Phi^2[(k+1)\Delta, \tau]\ W\ d\tau$$
$$R = \frac{V}{\Delta} \tag{3-34}$$

where Δ = sampling period

$$F = \Phi[(k+1)\Delta, k\Delta] = \text{Exp}\{-\omega \cdot \Delta\}$$

For the discrete-time system similar results for the innovation process will be obtained with some modification. the innovation process is a white random sequence but with a different variance from that of the observation noise. Consider the observation process of the system in the normal mode H_0:

$$z(k) = x(k) + r_1(k)$$

In this case the innovation process is defined as

$$\begin{aligned} \nu_0(k) &= z(k) - \tilde{x}(k|k-1) \\ &= \tilde{x}'(k|k-1) + r_1(k) \end{aligned} \tag{3-35}$$

where

$$\tilde{x}(k|k-1) = E[x(k) \mid z(\ell),\ 0 \leqq \ell \leqq k-1] \tag{3-36}$$

Then it can be shown that

$$E[\nu_0(k)] = 0 \quad .$$

$$E[\nu_0(k)\ \nu_0(\ell)] = [\tilde{P}'(k) + R]\ \delta_{k\ell} \qquad (3\text{-}37)$$

where $\tilde{P}'(k)$ = variance of the error in the estimate $\tilde{x}'(k \mid k-1)$

$$= E[x(k) - \tilde{x}(k|k-1)]^2$$

The above result is well-known in the discrete-time Kalman Filter solution. The variance of the $\{\nu_0(k)\}$ sequence contains an additional term, which effect becomes negligible in the continuous-time case. The continuous-time case can be considered to be approached by a limiting procedure in which R becomes indefinitely large while $\tilde{P}'(k)$ remains finite so that the variance of $\nu_0(t)$ and v(t) become essentially the same.

Now consider the observation process of the system defined in the degraded mode. Let the discrete bias free Kalman filter based on the model of the normal mode H_0 be denoted as F_0 as shown in Fig 3 1. Define the measurement residual sequence $\tilde{r}(k)$ as

$$\tilde{r}(k) = z(k) - \tilde{x}(k|k-1) \qquad (3\text{-}38)$$

where $\tilde{x}(k \mid k-1)$ is the bias-free estimate defined in (3-36). Then it will be shown that the following asymptotic relations can be established.

A. For the real system defined in mode H_1

$$\lim_{k\to\infty} \tilde{r}(k) = \lim_{k\to\infty} \nu_0(k) + a \qquad (3\text{-}39)$$

where the factor a in the discrete time case is defined as (in notations analogous to that of the continuous time case)

$$a = (1 + F_i \cdot V_x) \cdot m$$

B. For the real system defined in mode H_2:

$$\lim_{k\to\infty} \tilde{r}(k) \simeq \lim_{k\to\infty} \nu_0(k) + b \cdot k \qquad (3\text{-}40)$$

in the sense of an approximation analogous to that defined in the continuous time case. The factor b is defined as

$$b = (1 + F \cdot V_x) \cdot n$$

The derivation will be given only for the system defined in the degraded mode H_1 The result can be extended to the approximate form (3-40) in an analogous way as in the continuous time case, and will not be discussed. The measurement process in the discrete time case will be expressed as

$$z(k) = x(k) + m(k) + r_1(k) \qquad (3\text{-}41)$$

where m(k) is defined as m for the mode H_1

Define a pseudo process $z_0(k)$ as in the continuous-time case.

$$z_0(k) = x(k) + r_1(k) \qquad (3\text{-}42)$$

The pseudo innovation process $\underset{\sim}{\nu}_0(k)$ can be implemented by the discrete Kalman filter F_0 based on the dynamic system defined in the normal mode H_0.

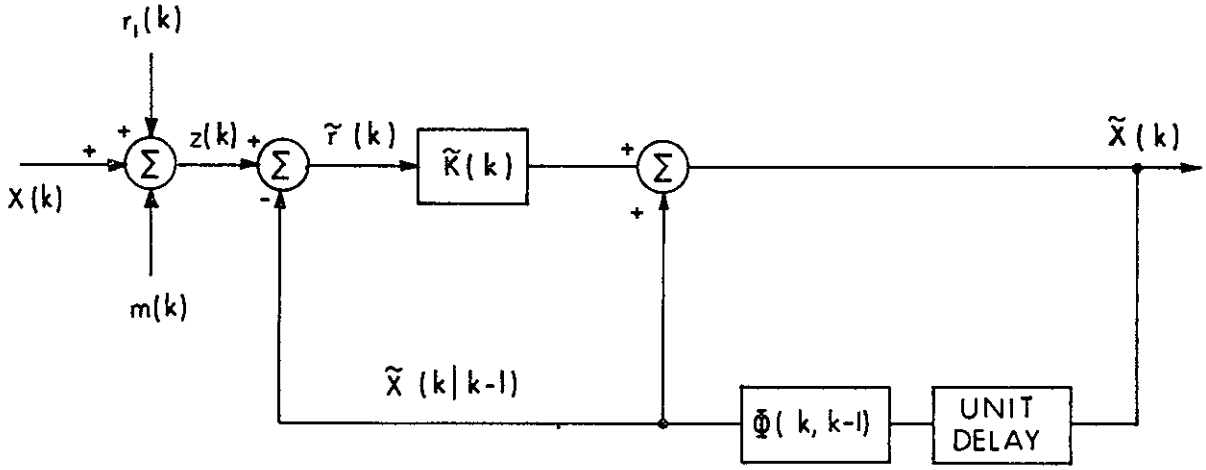


Fig. 3.4. Discrete Filter of Measurement Process.

The following relations can be established as in the continuous time case:

$$\tilde{r}(k) = z(k) - \tilde{x}(k|k-1) \tag{3-43}$$

where $\tilde{x}(k|\,k-1)$ is the bias-free estimate.

By application of the law of superposition, let

$$\tilde{x}(k|k-1) = \tilde{x}_0(k|k-1) + \tilde{m}(k|k-1) \tag{3-44}$$

then the pseudo process $\tilde{m}(k)$ satisfies the filter equation

$$\tilde{m}(k) = F \cdot \tilde{m}(k-1) + \tilde{K}(k)\,\{m(k) - F \cdot \tilde{m}(k-1)\},\ \tilde{m}(0) = 0 \tag{3-45}$$

Define
$$\nu_0(k) = z_0(k) - \tilde{x}_0(k|k-1) \tag{3-46}$$

Substituting (3-41), (3-42), (3-44), and (3-46) into (3-43) yields

$$\tilde{r}(k) = \nu_0(k) + m(k) - \tilde{m}(k|k-1) \tag{3-47}$$

where
$$m(k) \triangleq m \quad \text{for the mode } H_1$$

let
$$\tilde{m}(k) \triangleq - m \cdot V_x(k) \tag{3-48}$$

Again the negative sign in (3-48) is used to be consistent with the notation in Chapter 6. With the above notation, equation (3-47) can be rewritten as

$$\tilde{r}(k) = \nu_0(k) + [1 + F \cdot V_x(k-1)] \cdot m \tag{3-49}$$

On substituting (3-48) into (3-45), the factor $V_x(k)$ can be shown to satisfy

$$V_x(k) = F \cdot (1 - \tilde{K}(k))\ V_x(k-1) - \tilde{K}(k),\ V_x(0) = 0 \tag{3-50}$$

The coefficient $V_x(k)$ can be recursively computed. The factor $\tilde{K}(k)$ is the discrete bias-free Kalman filter gain. It is easy to solve the asymptotic solution of $V_x(k)$

$$\lim_{k\to\infty} V_x(k) = -\frac{\tilde{K}}{1-F(1-\tilde{K})} \triangleq V_x \tag{3-51}$$

where $\tilde{K}$ is the asymptotic steady-state gain of $\tilde{K}(k)$. It is to be noted that although the same notation is used for the factor V_x in both discrete and continuous time cases, they are of different computed value. Again the factor V_x is of negative polarity. By taking limits on (3-49) and using (3-51) the final results of (3-39) can be established.

3.4 The Stochastic Differential Equation

In this section the stochastic differential equation of the process $\pi(t)$ for the case of degradation mode H_1 with constant bias will be derived. The problem of the treatment of the case of degradation mode H_2 with ramp rate will be discussed in Section 3.5.

With the results of Section 3.3, the problem can be restated as follows:

Let $\{a, y\}$ be a two-dimensional partially observable Markov process. The observation process y(t) is represented by the stochastic differential equation (3-26)

$$dy(t) = a(t)dt + \sigma\, dw(t) \tag{3-52}$$

The unobservable process $\{a(t),\ t \geqq 0\}$ is a Markov jump process of two states with transition probability defined as

$$P[a(t+h) = a_1 \,|\, a(t) = a_0] = 1 - e^{-qh} \tag{3-53}$$

where a(t) physically represents the unknown bias parameter of the transformed measurement residual process. The parameter a_1 is the transformed design value specified for the degraded mode, and a_0 is the corresponding value defined for the normal mode. The asymptotic relation between the bias parameter of the measurement residual process and the bias parameter m of the drift rate process has been shown in (3-26) to be

$$a = (V_x + 1)\, m$$

It is required to derive the posterior probability defined as

$$\pi(t) \triangleq P\ \{\theta < t \,|\, y(s),\ 0 \leqq s \leqq t\} \qquad (3\text{-}54)$$
$$= P\ \{a \geqq a_1 \,|\, y(s),\ 0 \leqq s \leqq t\}$$

The last equality comes from the definition of instant of degradation illustrated in Fig 3 5

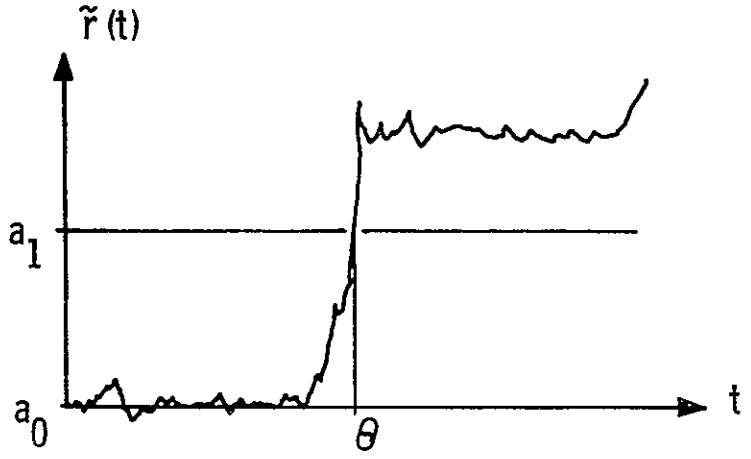


Fig. 3.5. Definition of Degradation of Bias Change.

It is noted that in the real system the state space of the jump process a(t) is continuous, and the derivation of the posterior probabilty (3-54) for a continuous state space has been discussed heuristically in the reference[88]. However, the result cannot readily be interpreted as specifying the dynamics of a practically realizable filter for generating the probability from the input data y Therefore the presentation will be derived on the assumption that the state space of the a(t)-process is a finite set In particular, for the case discussed in the thesis, a(t) will take only two distinct values a_0 and a_1 such that the state space $S = \{a_0, a_1\}$. Denote A as the event that $a(\omega) \geqq a_1$, then the following relation can be established.

$$\Pr\ \{A\} = \sum_{a_i \in A \cap S} \Pr\ (\{\omega \cdot a(\omega) = a_i\})$$

$$= \Pr\ \{\omega : a(\omega) = a_1\}$$

In this case, the definition (3-54) will be redefined as

$$\pi(t) = P\{ a = a_1 \mid y(s),\ o \leqq s \leqq t\} \qquad (3\text{-}55)$$

This definition will be used throughout the thesis

The stochastic differential equation of the process $\pi(t)$ has been studied by Shiryaev and Wonham In this section the main result will be presented and the derivation will be discussed To simplify the presentation the approach will follow Wonham[88]. A derivation of the more generalized case can be found in Shiryaev[70] For purpose of reference, it is convenient to introduce the following terminology

A probability space $(\Omega, \mathcal{F}, P)$ is defined by the specification of a nonempty set Ω (the sample space), a σ-field (Borel-field) $\mathcal{F}$ of subsets of Ω (events) and a probability P defined in $\mathcal{F}$ A σ-field of subsets of a set Ω is a class of subsets of Ω which contains ϕ and Ω and is closed (generated) under the operations of complementation, countable union, and countable intersection. The specification of the σ-field as the domain of the probability measure is of important significance With this specification measurability is preserved under practically all ordinary processes of analysis An extremely important property is that the limit function of any convergent sequence of measurable functions is measurable. i e. Given a sequence $\{f_n(x)\}$ of measurable functions and given $\lim f_n(x) = f(x)$. The $f(x)$ will be measurable. This property will be of use in derivation of the stochastic differential equation of the process $\pi(t)$. A heuristic exposition of the measure theory can be found in Wernikoff[84]

The derivation of the process $\pi(t)$ is formulated in a more general form. Let $\{a(t),\ t \geqq 0\}$ be a stationary Markov jump process with a finite number of states (distinct step levels) $a_1, \ldots, a_K$ and let the initial distribution of $a(0)$ be $\{\pi_j(0), j = 1, \ldots, K\}$. Denote the transition probabilities by

$$P_{ij}(h) \triangleq P\{a(t+h) = a_j \mid a(t) = a_i\}$$

$$P_{ij}(h) = \begin{cases} 1 - \nu_i h + 0(h) & j = i \\ \nu_{ij} h + 0(h) & j \neq i \end{cases} \qquad (3\text{-}56)$$

where the $\nu_{ij} \geqq 0$ are constants and

$$\nu_i = \sum_{\substack{j=1 \\ j \neq i}}^{K} \nu_{ij} \qquad i = 1, \ldots, K$$

The observed process $\{y(t),\ t>0\}$ is defined by

$$dy(t) = a(t)\ dt + \sigma(t)\ dw(t) \qquad y(0) = 0 \qquad (3\text{-}57)$$

where $\{w(t),\ t \geq 0\}$ is a standard Wiener process with $P\{w(0) = 0\} = 1$ It is required to derive a system of stochastic differential equations for the process $\pi_j(t)$ defined as

$$\pi_j(t) = P\{a(t) = a_j \mid y(s),\ 0 \leq s \leq t\},\ j = 1, \ldots, K$$

The basic points of the derivation can be outlined in three steps the derivation of an expression for the posterior conditional probabilities $\pi_j(t)$, the generation of the $\pi_j(t)$-processes as solutions of stochastic differential equations by a representation theorem due to Doob, and the identification of the derived stochastic equations of the $\pi_j(t)$-processes with that of the input y(t)-process

A formula for $\pi_j(t)$ is first evaluated The derivation proceeds by considering the finite difference model (to be defined in (3-58)) of equation (3-57) and subsequently taking limits To simplify the derivation it is convenient to augment to the probability space of the $\{a(t),\ w(t)\}$ process, a dummy step process $\{\tilde{a}(t),\ t > 0\}$ defined to have the same range, initial distribution and transition probabilities as the a(t) process, but independent of $\{a(t)\}$ and $\{w(t)\}$

Let the double indexed sequence $s_{rn} = rt/n$ $(r = 0, 1, \ldots n, n = 1, 2, \ldots)$ and define

$$\tilde{\xi}_{rn} = \int_{(r-1)t/n}^{rt/n} \tilde{a}(s)\ ds$$

$$\xi_{rn} = \int_{(r-1)t/n}^{rt/s} a(s)\ ds$$

Then

$$\eta_{rn} \triangleq y(s_{rn}) - y(s_{r-1,n})$$
$$= \xi_{rn} + \int_{(r-1)\,t/n}^{rt/n} \sigma(s)\ dw(s) \qquad (3\text{-}58)$$

For each fixed n, the random variable $(\eta_{rn} - \xi_{rn})$ $(r = 0, 1, \ldots, n)$ are independent and Gaussian with zero mean and variance

$$v_{rn} = \int_{(r-1)t/n}^{rt/n} \sigma^2(s)\ ds .$$

Define

$$\pi_j^{(n)}(t) \triangleq P\ \{a(t) = a_j \mid y(s_{rn}),\ r = 0, \ldots, n\}$$
$$= P\ \{a(t) = a_j \mid \eta_{rn},\ r = 1, \ldots, n\}$$

The last equality comes from the fact that the same σ-field is generated by the random sequence $\{y(s_{rn})\}$ or $\{\eta_{rn}\}$.

Then an expression for $\pi_j(t)$ can be evaluated using the transition probability of the Markov process and the Gaussian distribution of the random sequence $\{\eta_{rn}\}$

$$\pi_j(t) = \lim_{n\to\infty} \pi_j^{(n)}(t)$$
$$= N \sum_{i=1}^{K} \pi_i(0)\ P_{ij}(t) \cdot E\ \{e^{\theta(t)} \mid \tilde{a}(0) = a_i,\ \tilde{a}(t) = a_j;\ y(s),\ 0 \leqq s \leqq t\} \qquad (3\text{-}59)$$

where

$$\theta(t) = \int_0^t \sigma(s)^{-2}\ \tilde{a}(s)\ dy(s) - 1/2 \int_0^t \sigma(s)^{-2} \cdot \tilde{a}(s)^2 ds$$

and N is the normalizing factor for probability Let

$$\psi(\tau, t) = \exp\ [\int_\tau^t \sigma(s)^{-2} \tilde{a}(s)\ dy\ (s) - 1/2 \int_\tau^t \sigma(s)^{-2}\ \tilde{a}(s)^2\ ds\ ]$$
$$(0 \leqq \tau \leqq t)$$

By using the fact that the $\tilde{a}(t)$ process is Markov and is independent of the y(t) process, it can be shown

$$\pi_j(t + h) = N \sum_{i=1}^{K} \pi_i(t) \; P_{ij}(h)$$

$$E \{ \psi(t, t+h) \Big| \tilde{a}(t) = a_i, \tilde{a}(t+h) = a_j, y(s), t \leqq s \leqq t+h \} \qquad (3\text{-}60)$$

It follows that the joint process $\{ a(t), \pi(t), t \geqq 0 \}$ is Markov

The expression for $\pi(t)$ is useful in applications only when t is fixed Since the observation data is coming in continuously, the second step therefore is to derive a stochastic differential equation which generates the posterior distribution when the input data is continually observed The basis of the derivation of the stochastic differential equation is an application of a representation theorem due to Doob(or a related formula of Dynkin) concerning conditions when a process with continuous trajectories can be obtained as a solution of some Ito stochastic differential equation. The Dynkin formula is more easy to apply if the process $\pi(t)$ can be represented as an explicit functional of the observed process y(t) However, a representation of the derived $\pi(t)$ expression as an explicit functional of the input process is unavailable, the second step in derivation of the stochastic process $\pi(t)$ will then be to apply the Doob theorem[18] The Doob treatment will be extended to the multi-dimensional generalization It is to be shown that the process $\pi(t)$ can be represented in the form of the Ito stochastic differential equations

$$d\pi_j(t) = m_j(a(t), \pi(t))dt + b_j(t, \pi(t)) \; d\tilde{w}_j(t) \quad j = 1, \ldots, K \qquad (3\text{-}61)$$

where $\tilde{w}_j(t)$ are the standard Wiener processes

In the application of the Doob theorem, the verification of the assumptions in the statement of the theorem is discussed in the references[70]. The coefficients m_j and b_j are calculated from the derived $\pi_j(t)$ expressions (3-57) by

$$m_j(t,\pi,a) = \lim_{h\to 0} E\left\{\frac{\pi_j(t+h) - \pi_j(t)}{h}\,\middle|\, a(t) = a, \quad (t) = \pi\right\} \tag{3-62}$$

$$b_j^2(t,\pi,a) = \lim_{h\to 0} E\left\{\frac{[\pi_j(t+h) - \pi_j(t)]^2}{h}\,\middle|\, a(t) = a, \ (t) = \pi\right\} \tag{3-63}$$

After some rather tedious calculations it can be shown

$$m_j(t,a,\pi) = -\nu_j \pi_j + \sum_{\substack{i=1\\ i\neq j}}^{K} \nu_{ij}\pi_i + \frac{(a-\bar{a})(a_j-\bar{a})}{\sigma^2}\pi_j$$

$$b_j^2(t,a,\pi) = \left(\pi_j \frac{a_j-\bar{a}}{\sigma}\right)^2 \tag{3-64}$$

where $\bar{a} = \sum_{i=1}^{K} a_i \pi_i$

In order to show that $\{\pi_j(t),\ j = 1, \ldots, K\}$ can be represented in the form (3-61) the sample functions of the $\pi_j(t)$-process must be proved to be continuous On using the $\pi_j(t)$ expression (3-59) the following relation can be established.

$$E\{|\pi_j(t+h) - \pi_j(t)|^4 \,|\, \mathcal{F}_t\} \leq \text{const. } h^{1+\rho} \qquad \rho > 0$$

where $\mathcal{F}_t$ is the smallest σ-field with respect to which the random processes $\{a(s), \pi(s),\ 0 < s < t\}$ are measurable

By application of Kolmogorov's criterion on the continuity of random functions it follows that the process $\pi_j(t)$ is continuous with probability 1 The representation theorem of Doob will then be generalized to allow for the fact that only the $\pi(t)$ components of the $\{a(t),\ \pi(t)\}$ process are continuous This depends on the application of properties of the martingale process

Let $\mathcal{F}_t$ be the smallest σ-field with respect to which the random processes $\{a(s), \pi(s), 0 \leqq s \leqq t\}$ are measurable Then it can be shown that each process

$$\{\tilde{\pi}_j, \mathcal{F}_t, t \geqq 0\} = \{\pi_j(t) - \pi_j(0) - \int_0^t m_j(s, a(s), \pi(s))\, ds \mid \mathcal{F}_t, t \geqq 0\} \tag{3-65}$$

is a martingale process By application of a theorem of Doob to the martingale process $\{\tilde{\pi}_j, \mathcal{F}_t, t \geqq 0\}$, a standard Wiener process $\tilde{w}_j(t)$ can be found such that the following results will be true

$$d\pi_j(t) = \{\pi_j(t) \frac{[a(t) - \bar{a}(t)\ (a_j - \bar{a}(t))]}{\sigma^2(t)} - \nu_j \pi_j(t) + \sum_{\substack{i=1 \\ i \neq j}}^{K} \nu_{ij} \pi_i(t)\}\, dt \tag{3-66}$$

$$+ \pi_j(t) \frac{(a_j - \bar{a}(t))}{\sigma(t)}\, d\tilde{w}_j(t) \qquad j = 1, \ldots, K$$

The process $\tilde{w}_j(t)$ is constructed as follows:

$$\tilde{w}_j(t) = \int_0^t [b_j(s, a(s), \pi(s)]^{-1}\, d\pi_j(s) \tag{3-67}$$

Finally, it remains to identify each $\tilde{w}_j(t)$ process with the $w(t)$ process of the observation process $y(t)$ such that the derived stochastic equations of the $\pi(t)$-process (3-66) can be implemented. The observation process is represented by

$$dy(t) = a(t)\, dt + \sigma(t)\, dw(t)$$

Let

$$\tilde{y}(t) = y(t) - \int_0^t a(s)\, as$$

$$= \int_0^t \sigma(s)\, dw(s) \tag{3-68}$$

It can be shown that the process $\{\tilde{y}(t), \mathcal{F}_t, t \geq 0\}$ is a martingale Furthermore

$$w(t) = \int_0^t \sigma(s)^{-1} \quad d\tilde{y}(s) \tag{3-69}$$

Using the relation

$$\lim_{h \to 0} h^{-1} E\{[\tilde{y}(t+h) - \tilde{y}(t)] \quad [\tilde{\pi}_j(t+h) - \tilde{\pi}_j(t)] \mid \mathcal{F}_t\}$$

$$= \sigma(t)\, b_j[t, a(t), \pi(t)] \tag{3-70}$$

it follows that

$$E\{[w(t) - w(s)] \; [\tilde{w}_j(t) - \tilde{w}_j(s) \mid \mathcal{F}_s\} = t-s, \quad 0 < s < t \tag{3-71}$$

In view of the conditions that the processes w(t) and $\tilde{w}_j(t)$ are standard Wiener processes, it can be verified that

$$E\{[w(t) - \tilde{w}_j(t)]^2\} = 0 \quad \text{for } t > 0$$

Hence for each $t > 0$, $w(t) = \tilde{w}_j(t)$ with probability 1 From continuity of Wiener processes it follows that with probability one, these processes are essentially identical. Thus the following main results have been established:

$$d\pi_j(t) = \{[-\nu_j \pi_j(t) + \sum_{\substack{i=1 \\ i \neq j}}^{K} \nu_{ij} \pi_i(t)] - \frac{\bar{a}(t)\,(a_j - \bar{a}(t))}{\sigma(t)^2} \pi_j(t)\}\, dt$$

$$+ \frac{[a_j - \bar{a}(t)]\, \pi_j(t)}{\sigma(t)^2} dy(t) \qquad j = 1, \ldots, K \tag{3-72}$$

The following remarks are of interest

(1) It follows from the above equation that the results can be split into two parts The term $[-\nu_j \pi_j(t) + \sum_{\substack{i=1 \\ i \neq j}}^{K} \nu_{ij} \pi_i(t)]$ corresponds

to the a priori evolution of the $\pi_j(t)$ when the process y(t) is not observed or when $\sigma(t) = \infty$. On the other hand if $\nu_{ij}(t) \equiv 0$ the process a(t) is degenerate in the sense that there is no transition among states a_i. The process corresponds to an unknown but unchanged state a_i. The stochastic differential equation is then simplified to

$$d\pi_j(t) = \pi_j(t) \frac{(a_j - \bar{a}(t))}{\sigma(t)^2} (dy - \bar{a}(t)\, dt) \tag{3-73}$$

(2) The observation process y(t) can be represented by the stochastic equation·

$$dy(t) = \bar{a}(t)\, dt + \sigma(t)\, dw(t) \tag{3-74}$$

where

$$\bar{a}(t) = \sum_{i=1}^{K} a_i \pi_i(t)$$

The proof essentially rests on the martingale theorem of Doob Define a process $\tilde{y}(t)$ such that for $t \in [0, T]$

$$\begin{aligned} d\tilde{y}(t) &= dy(t) - \bar{a}(t)\, dt \\ &= (a(t) - \bar{a}(t))\, dt + \sigma(t)\, dw(t) \end{aligned} \tag{3-75}$$

Write $\mathcal{F}_t = \sigma\{ y(\tau),\ 0 \leq \tau \leq t\}$

$$\begin{aligned} \bar{a}(t) &= E\{a(t) \mid y(\tau),\ 0 \leq \tau \leq t\} \\ &= E\{a(t) \mid \mathcal{F}_t\} \end{aligned}$$

where $\mathcal{F}_t$ is defined as the smallest Borel field (σ-field) generated by the random variables $\{y(\tau),\ 0 \leq \tau \leq t\}$. Then $\tilde{y}(t)$ is $\mathcal{F}_t$ measurable The relation

$$\tilde{y}(t) = \tilde{y}(s) + \int_0^t \left(a(\tau) - \bar{a}(\tau)\right) d\tau + \int_s^t \sigma(t)\, dw(t)$$

$$0 \leqq s \leqq t \leqq T \tag{3-76}$$

implies that

$$E\{\tilde{y}(t) \mid \mathcal{F}_s\} = \tilde{y}(s) + \int_s^t E[a(\tau) - \bar{a}(\tau) \mid \mathcal{F}_s]\, d\tau \tag{3-77}$$

Using a smooth property of successive conditional expectation:

$$E\{E\{f(\omega) \mid \mathcal{F}_\tau\} \mid \mathcal{F}_s\} = E\{f(\omega) \mid \mathcal{F}_s\} \tag{3-78}$$

if $\mathcal{F}_s \leqq \mathcal{F}_\tau$. This means that every set in $\mathcal{F}_s$ is also a set in $\mathcal{F}_\tau$, or equivalently every set in $_s$ is the union of sets in $\mathcal{F}_\tau$. Thus $\mathcal{F}_s$ is coarser, it contains fewer sets or "less information" On using (3-78), (3-77) can be written as

$$E\{\tilde{y}(t) \mid \mathcal{F}_s\} = \tilde{y}(s) + \int_s^t E\{a(\tau) - \bar{a}(\tau)\} \mid \mathcal{F}_\tau\} \mid \mathcal{F}_s\}\, dt \tag{3-79}$$

$$= \tilde{y}(s)$$

This proves that $\{\tilde{y}(t), \mathcal{F}_t,\ t > 0\}$ is a martingale process Reasoning as in (3-66) the representation (3-74) can be established

(3) The stochastic differential equation $\pi(t)$ is non-linear and moreover there are multiplicative factors of a diffusion nature associated with the term dy. This will complicate on-line implementation A simpler expression can be obtained by a transformation of $\pi(t)$ into a likelihood ratio function $\lambda(t)$. Consider a special case with $j = 1, 2$, and define

$$\lambda(t) = \ell n \left[\frac{\pi(t)}{1-\pi(t)}\right] \tag{3-80}$$

where $\pi(t) \triangleq \pi_2(t), \quad 1 - \pi(t) \triangleq \pi_1(t)$

Then the equation (3-72) is simplified to:

$$d\pi = \pi(1-\pi)\frac{(a_1 - a_0)}{\sigma^2}\{\, dy - [a_1\pi + a_0(1-\pi)]\,dt\} + q(1-\pi)\,dt \qquad (3\text{-}80)$$

$$\pi(0) = \pi_0$$

where $q = \nu_{12}$ from (3-53) and (3-56) and $\nu_{21} = 0$

The stochastic equation of the $\lambda(t)$-process can be established by an application of the Dynkin formula[23]. The main result of Dynkin's formula for the one-dimensional case can be stated as follows. Let the process $\{z(t),\ t > 0\}$ satisfy the stochastic differential equation

$$dz(t) = m[t, z(t)]\ dt + \sigma[t,\ z(t)]dw(t)$$

Let $\phi = \phi(t, \xi)$ be a numerical function twice continuously differentiable in (t, ξ) for $t > 0$ and ξ in R. Put $\psi(t) = \phi[t, z(t)]$. Then the process $\{\psi(t),\ t > 0\}$ satisfies the stochastic differential equation:

$$d\psi(t) = \tilde{m}[t, z(t)]\,dt + \tilde{\sigma}[t, z(t)]\ dw(t)$$

where functions $\tilde{m}$ and $\tilde{\sigma}$ are given by

$$\tilde{m}(t,\xi) = \frac{\partial\phi(t,\xi)}{\partial t} + \frac{\partial\phi(t,\xi)}{\partial\xi}\ m(t,\xi)$$

$$+\frac{1}{2}\frac{\partial^2\phi(t,\xi)}{\partial\xi^2}\cdot\sigma^2(t,\xi)$$

$$\tilde{\sigma}(t,\xi) = \frac{\partial\phi(t,\xi)}{\partial\xi}\cdot\sigma(t,\xi)$$

This equation of the $\psi(t)$-process can also be written

$$d\psi(t) = \left\{ \frac{\partial\phi(t,\xi)}{\partial t}\bigg|_{\xi = z(t)} + \frac{1}{2}\frac{\partial^2\phi(t,\xi)}{\partial\xi^2}\cdot\sigma^2(t,\xi)\bigg|_{\xi = z(t)} \right\} dt$$

$$+ \frac{\partial\phi(t,\xi)}{\partial\xi}\bigg|_{\xi = z(t)} dz(t) \qquad (3\text{-}81)$$

By identifying $\lambda(t)$ and $\pi(t)$ with $\psi(t)$ and $z(t)$ in (3-81) the stochastic equation

$$d\lambda = \frac{(a_1 - a_0)}{\sigma^2} \left[dy - \frac{1}{2}(a_1 + a_0)dt\right] + q(e^{-\lambda} + 1)\,dt, \quad \lambda(0) = \lambda_0 \qquad (3\text{-}82)$$

can be established. The stochastic equation is non-linear, but the multiplication factor of a diffusion nature was eliminated.

3.5 A Comparison Theorem:

As mentioned in the discussion of the system model, the actual system is known only to be in one of the operating modes, with a given functional form of the conditional probability density of the modes but with unknown associated parameters. The difficulty of unknown parameter estimation is to be avoided by choosing a "design" value on the basis of which the decision process is constructed. The performance criteria, being either the error probabilities or time delay in detection, are specified for this design value. It will be shown that the real system can be detected at least as fast as that achieved in the case of the design value, when the magnitude of the actual value of the unknown parameter is larger than that of the specified design value.

In this thesis the time of degradation due to a ramp rate is defined as the instant when the ramp rate will result in an equivalent intolerable degradation in performance corresponding to the "design" value of a constant bias jump in drift rate. With this definition for the degradation mode of ramp change, it will be shown that only one detection system designed for a degradation mode of constant bias change is sufficient for both degradation modes of mean change. The proof of these results is based on a comparison theorem for diffusion processes The statement of the theorem can be expressed as follows (Skorokhod).[72]

Suppose that $a_1(t, x)$, $a_2(t, x)$ and $\sigma(t, x)$ satisfy the following conditions:

(1) $a_1(t, x)$, and $a_2(t, x)$, and $\sigma(t, x)$ are continuous in their variables for $t \in [t_0, T]$, $x \in (-\infty, \infty)$

(2) $\sigma(t, x) > 0$, and for every $c > 0$, there exists $\alpha > \frac{1}{2}$ and $L > 0$ such that for $|x| \leq c, |y| \leq c$.

$$|\sigma(t, x) - \sigma(t, y)| \leq L\,|x - y|^{\alpha}$$

Suppose further that $\xi_1(t)$ and $\xi_2(t)$ are with probability 1 continuous solutions of the equation

$$\xi_i(t) = \xi_i(t_0) + \int_{t_0}^{t} a_i(s, \xi_i(s))ds + \int_{t_0}^{t} \sigma(s, \xi_i(s))\, dw(s) \qquad i = 1, 2$$

Under these conditions, if $a_1(t, x) < a_2(t, x)$ for every t and x and if $P\{\xi_1(t_0) \leqq \xi_2(t_0)\} = 1$ then $\xi_1(t) \leqq \xi_2(t)$ for every t with probability 1.

Loosely speaking, under the above assumptions, a diffusion process is a monotonic function of the transition coefficient

The comparison theorem will now be applied to prove the following statement

Let the observation process be defined

$$dy(t) = a(t)dt + \sigma\, dw(t) \tag{3-82}$$

Suppose that the stochastic differential equation of the process $\pi(t)$ (3 - 81) is designed on the basis of the "design" value a_1 of the parameter a such that,

$$d\pi = \pi(1-\pi)\frac{(a_1-a_0)}{\sigma^2}\{dy - [a_1\pi + a_0(1-\pi)]\,dt\} \tag{3-83}$$

$$+ q(1-\pi)dt$$

Then if the unknown parameter has an actual value $a > a_1$, the following relation is true

$$\pi(a, t) \geqq \pi(a_1, t) \qquad \text{for all } t \tag{3-84}$$

The proof is a straightforward application of the comparison theorem. Substituting (3-31) into (3-83) produces

$$d\pi(a, t) = \pi(1-\pi)\frac{(a_1-a_0)}{\sigma^2}\{a - [a_1\pi + a_0(1-\pi)]\}\,dt$$

$$+q(1-\pi)dt + \pi(1-\pi)\frac{(a_1-a_0)}{\sigma}\,dw(t)$$

$$d\pi(a_1,t) = \pi(1-\pi)\,\frac{(a_1-a_0)}{\sigma^2}\cdot\{a_1 - [a_1\pi + a_0(1-\pi)]\}dt$$
$$+ q(1-\pi)dt + \pi(1-\pi)\,\frac{(a_1-a_0)}{\sigma}\,dw(t)$$

Since the initial conditions can be assumed to be the same, the following relation follows:

$$\pi(a,t) \geqq \pi(a_1,t) \qquad \text{for all } t \tag{3-85}$$

It will be shown in Chapters 4 and 5 that the optimal decision (detection or identification) problem consists of observing the posterior probability process $\pi(t)$, and evaluating it against a specified threshold to determine the decision. Since the process $\pi(t)$ is a diffusion process with a continuous trajectory almost surely, then the condition (3-85) implies that the process $\pi(a,t)$ will cross the threshold earlier. This proves the result.

On the other hand if the value of the parameter a and the decision value a_1 are both of negative polarity, then the condition $-a < -a_1$ implies that

$$\pi(-a,t) \leqq \pi(-a_1,t) \qquad \text{for all } t \tag{3-86}$$

This means that in this case the process $\pi(-a,t)$ is of larger magnitude but with negative polarity. Since a threshold of negative polarity will be specified in this case, this means that the process $\pi(-a,t)$ will also cross the threshold earlier

The following statement can be proved in a similar way. If the unknown parameter of the real system has an actual value $a < a_1$, then the process $\pi(t)$ designed on the basis of the "design" value a_1 over a fixed mission duration will have less chances to cross the specified threshold for decision, i.e., the performance of the actual system will have a smaller value of false alarm error probability than the computed performance based on the design value.

The problem of the treatment of the degradation mode of ramp change will now be discussed. The instant of degradation in the ramp case can be defined in two ways as shown in Fig 3 6

(1) $t = \theta^*$, when a ramp rate of an intolerable magnitude appears.

(2) $t = \theta$, when the ramp will result in an equivalent intolerable constant bias drift rate.

The latter definition will be adopted in this thesis. It will be shown that in this case only one detection system designed for the constant bias case is sufficient for both cases of degradation modes H_1 and H_2.

The measurement equations (3-31) and (3-32) can be rewritten as

$$\begin{aligned} H_1: \quad & dy = a\,dt + \sigma\, dw(t) \\ H_2: \quad & dy \cong bt\, dt + \sigma\, dw(t) \\ & \quad = a\,dt + (bt-a)\, dt + \sigma\, dw(t) \end{aligned} \tag{3-87}$$

Substituting (3-87) into the stochastic equation $\pi(t)$, it can be easily established by the comparison theorem that for $t \geq T + \theta^*$,

$$\pi(t, H_2) > \pi(t, H_1) \qquad \text{with probability 1} \tag{3-88}$$

the notations $\pi(t, H_2)$ and $\pi(t, H_1)$ are defined for the process $\pi(t)$ conditioned on modes H_2 and H_1, respectively The value T is determined by the following relation

$$\begin{aligned} (bT-a) &= 0 \\ T &= \frac{a}{b} \end{aligned} \tag{3-89}$$

where the parameters a and b have been defined in (3-31) and (3-35) As illustrated in Fig. 3 6, with the employed definition of the degradation instant, the degradation mode H_2 can be detected at least as fast as is achieved in the case of the design value by the detection system based on the constant bias degradation mode H_1.

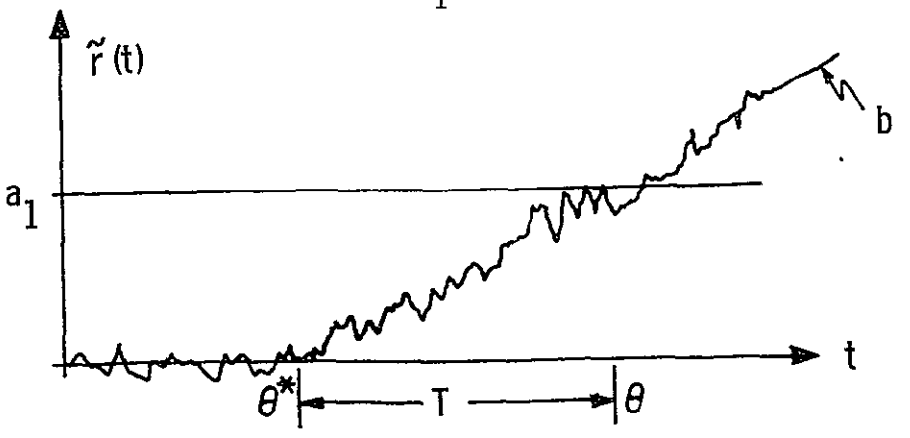


Fig. 3.6. Detection of Ramp Change by Bias Threshold.

CHAPTER 4

The Detection System

4.1 General Discussion

In this chapter the optimal detection problem is formulated as the solution of a Bayesian problem. It is shown that the posterior probability is the sufficient statistic for optimal detection for the class of Bayesian problems with Markovian additive risk function The optimal detection problem consists of observing the posterior probability process and is reduced to the determination of the optimal boundary for the decision to stop or continue the observation at a minimum risk The problem of on-line implementation of the Ito stochastic equation of the posterior probability is discussed.

A suboptimal detection system in the class of Wald's sequential test process is developed The solution of the suboptimal detection system is formulated as a combined stochastic problem by utilization of feedback of uncertainty information. The suboptimal control law in the discrete time system has been found to be determined by a constant threshold. The extension of the optimal control into the continuous time system is discussed It is shown that the suboptimal system can be simply modified to detect the degradation mode with simultaneous identification of polarity with little additional computation

The performance of the detection system is studied. It is shown for a sequential detection system with only a single-side boundary for degradation detection the meaningful performance criteria are the mean delay time in detection and the false alarm error probability. The detection performance of the optimal detection system and the developed suboptimal detection are evaluated by application of the theory of first passage times for diffusion processes. It is shown that under the constraint of a specified mean time between two false alarms the suboptimal system performance expressed in mean time delay in detection is remarkably close to

that of the optimal system. Simulation results are presented.

4.2 Decision and Sufficient Statistic.

The detection system is characterized by the feature that information about the state of interest is not directly accessible in the observation process In Chapter 3, the posterior probability of the unobservable state describing degradation conditioned on the past history of observations has been derived, and has been shown to be Markovian The posterior probability gives the most complete description of the information about degradation of the system. This transformation of the original system into the posterior probability is the feature of the treatment of the partially observable process. Based on this information the problem is transformed into process with complete information. The optimal detection problem will then be formulated as a special case of the optimal control problem in which the optimal control is the decision to continue or stop the observation at a minimum risk. Under the general case of Markovian additive problems, it will be shown that the posterior probability is the sufficient statistic for determining the optimal decision.

The concept of sufficient statistics is developed in the theory of stochastic optimal control. In the stochastic optimal control problem, control solutions are chosen from the set of admissible controls on the basis of the observable measurement history up to the current instant. If there exists a finite set of sufficient statistics, which determine the expected cost to complete the process, then the optimal control becomes a function of these sufficient statistics. In particular the statistics which are to replace the measurement history must be sufficient in the following respects. They must be sufficient to define the expected cost to complete the process; they must be sufficient to determine the selection of controls from the admissible controls; and they must be sufficient to determine their own future evolution.

It has been shown in Chapter 3 that the posterior probability will satisfy the third requirement. In this section it will be shown that the posterior probability will be sufficient for the problems of the optimal decision in the class of Markovian additive problems. The problem of determining the sufficient statistic is approached by deriving an expression for the re-

current formula of the expected risk function via the method of dynamic programming. The posterior probability will then be shown to be sufficient to define the expected risk to complete the process.

Consider the system in the discrete-time case, with $n \in [0, N]$ taking $0,1,\ldots N < \infty$, and N is fixed. Let $\psi(n)$ denote a generic symbolic function. The following notations will be employed.

$$\psi_n = \psi(n), \quad \psi^n = \{\psi(i), \ 0 < i \leqq n\},$$

$$\psi_s^n = \{\psi(i), \ s < i \leqq n\}$$

Let $d = \{d_n\}$, $\theta = \{\theta_n\}$ and $y = \{y_n\}$ be the random sequence generated by the decision, unobservable and observable processes of the investigated system respectively. The corresponding realizations and sample spaces are denoted as $\{\delta, \beta, \eta\}$ and $\{D, \Theta, \Gamma\}$ respectively. Let σ_n denote the triple state $\{d_n, \theta_n, y_n\}$ of the system. Define the loss function $W(\cdot) \geqq 0$ as follows: $W(N, \sigma^N)$ = total loss from the initial time i = 0 to the terminal time i = N, using the admissible decision function $\{d_n\}$, n = 1, ... N, which leads to the value σ^N.

It is assumed that at any instant n the decision space depends only on the past history $\chi^{n-1} = (d^{n-1}, y^{n-1})$ without anticipation. Furthermore, the past outcomes of observations χ^{n-1} will impose certain restrictions on the admissible decision space from which the decision function d_n can be chosen, i.e. the maximal admissible subspace $D_n = D_n(\chi^{n-1}) \subseteq \mathcal{D}_n$, where $\mathcal{D}_n$ is the admissible space with no restrictions. A decision rule is defined by a sequence of functionals $\{\delta_n\}$:

$$\delta_n = \delta_n(D_n, d^{n-1}, y^{n-1}), \qquad n = 1, \ldots N$$

where δ_n is the decision to be made at the time step $I \in (n-1, n]$

For each decision rule δ an average risk is defined

$$R^N(\delta) = E_\delta W(N, \sigma^N) \tag{4-1}$$

where the mathematical expectation is taken with respect to a probability measure $P_\delta(\cdot)$ generated by the given decision rule δ. A decision rule δ

will be defined to be optimal, if

$$\rho^N = \inf_{\delta} R^N(\delta) = R^N(\delta^*) \tag{4-2}$$

Consider the special case with the following assumptions.

A. The loss function is an additive function:

$$W(n, \sigma^n) = \sum_{r \leqq n} W(r, \sigma_r)$$

B. The unobservable controlled process $\{\theta\}$ is a Markov process:

$$\beta_n(\theta, d^n, \theta^{n-1}) = \beta_n(\theta; d_n, \theta_{n-1})$$

C. The observable process $\{y\}$ is a sequence of conditionally independent random variables. For a fixed realization of the sequence $\{\delta_n\}$ and $\{\beta_n\}$, the random variable y_n is independent of the history y^{n-1}:

$$\eta_n(\Gamma, d^n, \theta^n, y^{n-1}) = \eta_n(\Gamma, d_n, \theta_n)$$

D. The admissible decision space D depends on the past data in the Markovian manner:

$$D_n(\theta^{n-1}, y^{n-1}) = D_n(\theta_{n-1}, y_{n-1})$$

Under assumptions A-D, the problem of sufficient statistics for the partially observable system will be discussed. Later, for the sequential decision problem, the assumptions will be further simplified. In particular the unobservable process is not a function of the decision d_n, and the preceding outcomes of observations impose no restrictions on the admissible decisions.

First the basic recurrent equation which must be satisfied by the optimal risk for decision problems is to be derived. Let

$$\rho_n^N(\chi^n) = \inf_{\{\delta_{n+1}, \ldots, \delta_N\}} E\{\sum_{\tau=n+1}^{N} W(\tau, \sigma_\tau) \mid \chi^n\} \tag{4-3}$$

Then $\rho_n^N(\chi^n)$ is the minimum expected cost to complete the process from time n, given the observation history y^n, using some admissible decision function $\{d_i\}$ i = 1, ..., n, and the optimal decision function $\{d_i^*\}$ in the interval $(n+1) \leqq i \leqq N$. It follows from the "principle of optimality" (Bellman[6]) that

$$\rho_n^N(\chi^n) = \inf_{\delta_{n+1}} E\{[W(n+1, \sigma_{n+1}) + \rho_{n+1}^N(\chi^{n+1})] \mid \chi^n\} \tag{4-4}$$

where $W(n+1, \sigma_{n+1}) \triangleq W^{\delta_{n+1}}(n, \sigma_n)$ is the incremental loss generated by δ_{n+1} at time step $I \in (n, n+1]$.

The problem of deriving sufficient statistics is reduced to the study of the measurability of the risk in the recurrent formula (4-4), or to the study of the minimum number of statistics which are sufficient to determine the expected risk function to complete the process.

For the last stage n = N-1,

$$\rho_{N-1}^N(\chi^{N-1}) = \inf_{\delta_N} E\{W(N, \sigma_N) \mid \chi^{N-1}\} \tag{4-5}$$

Let an admissible decision function d_N be chosen, then by assumption D, $d_N \in D_N(\chi_{N-1})$. The expected value of the function (4-5) can be explicitly expressed as:

$$\rho_{N-1}^N(\chi^{N-1}) = \inf_{d_N} \{\int W(N, \beta, \eta, d_N)\, dP_{\theta_N, y_N}(\beta, \eta \mid d_N, \chi^{N-1})\} \tag{4-6}$$

By application of the Bayes rule, the probability P(·) can be expressed as·

$$P_{\theta_N, y_N}(\beta, \eta \mid d_N, \chi^{N-1})$$

$$= \int_{\Theta} P_{y_N}(\eta \mid d_N, \chi^{N-1}, \beta)\, P_{\theta_N}(\beta \mid \tilde{\beta}, \chi^{N-1}, d_N)\, dP_{\theta_{N-1}}(\tilde{\beta} \mid \chi^{N-1} d_N) \tag{4-7}$$

By assumption C, since y_N is independent of the past history for fixed

d_N and θ_N, therefore

$$P_{y_N}(\eta \mid d_N, \chi^{N-1}, \beta) = P_{y_N}(\eta \mid d_N, \beta) \tag{4-8}$$

The probability $P_{\theta_{N-1}}(\tilde{\beta} \mid \chi^{N-1}, d_N)$ does not depend on the future decision d_N,

$$P_{\theta_{N-1}}(\tilde{\beta} \mid \chi^{N-1}, d_N) = P_{\theta_{N-1}}(\tilde{\beta} \mid \chi^{N-1}) \tag{4-9}$$

since the processes $\{d\}$, $\{\theta\}$ and $\{y\}$ are defined as processes without anticipation. Define

$$\pi_{N-1}(\tilde{\beta}) = P_{\theta_{N-1}}(\tilde{\beta} \mid \chi^{N-1}) \tag{4-10}$$

and notice that it is sufficient to know θ_N from θ_{N-1} because of the Markovian property of the process $\{\theta\}$ by assumption B. Then Eq. (4-7) can be rewritten as

$$\begin{aligned} & P_{\theta_N, y_N}(\beta, \eta \mid d_N, \chi^{N-1}) \\ &= \int_{\Theta} P_{y_N}(\eta \mid d_N, \beta) \cdot P_{\theta_N}(\beta \mid \tilde{\beta}, d_N)\, d\, \pi_{N-1}(\tilde{\beta}) \\ &= \int_{\Theta} P_{y_N, \theta_N}(\eta, \beta \mid \tilde{\beta}, d_N)\, d\pi_{N-1}(\tilde{\beta}) \end{aligned} \tag{4-11}$$

Denote $$P(\cdot \mid d_N, \pi_{N-1}) = \int_{\Theta} P(\cdot \mid d_N, \tilde{\beta})\, d\pi_{N-1}(\tilde{\beta}) \,. \tag{4-12}$$

Since the true state of the process θ is unknown, its best information can be represented only by a set of posterior probabilities. π_{N-1} is the set of random variables $\{\pi_{N-1}(\tilde{\beta}),\ \tilde{\beta} \in \Theta\}$, where $\pi_{N-1}(\tilde{\beta})$ represents the posterior probability of an element of the given sample space Θ. The expression $P(\cdot \mid d_N, \pi_{N-1})$ can be interpreted as the expected value of $P(\cdot \mid d_N, \theta)$ against the posterior probability π_{N-1} based on given observa-

tions. $P(\cdot \mid d_N, \pi_{N-1})$ is not a conditional distribution in the usual sense of conditioning on a given value of the random variable π_{N-1}. The symbol just means that the distribution π_{N-1} according to the defining expression (4-12).

If the sample space Θ is discrete with a finite set of states denoted by $\{\beta_i\}$, where i = 1, ..., m, then the expression can be written as:

$$P(\cdot \mid d_N, \pi_{N-1}) = \sum_{i=1}^{m} P_1(\cdot \mid d_N)\, \pi^1_{N-1}$$

$$\pi_{N-1} = \{\pi^1_{N-1}, \ldots, \pi^m_{N-1}\} \tag{4-13}$$

where the $\pi^i_{N-1} \triangleq \pi_{N-1}(\beta_i)$ represents the posterior probability of the state β_i based on observations up to (N-1) and $P_1(\cdot \mid d_N) \triangleq P(\cdot \mid d_N, \beta_i)$ represents the probability of an event conditioned on the event that the state β_i is true. Therefore (4-11) can be expressed as

$$P_{\theta_N, y_N}(\beta, \eta \mid d_N, \chi^{N-1}) = P_{\theta_N, y_n}(\beta, \eta \mid d_N, \pi_{N-1}) \tag{4-14}$$

Substituting (4-14) into (4-6) produces

$$\rho^N_{N-1}(\chi^{N-1}) = \inf_{d_N} \{ \int W(N, \beta, \eta, d_N) \cdot d\, P_{\theta_N, y_N}(\beta, \eta \mid \pi_{N-1}, d_N) \}. \tag{4-15}$$

Therefore the conditional expected risk $E\{W(N, \sigma_N) \mid \chi^{N-1}\}$ has been shown to be defined by the posterior probability π_{N-1}.

Thus the choice of an optimal decision δ_N at interval $I \in (N-1, N]$ is defined, not by the past history χ^{N-1}, but only by the posterior probability π_{N-1}, and $\chi_{N-1} = (d_{N-1}, y_{N-1})$ due to restrictions imposed on the admissible space $D_N(\chi_{N-1})$. Similarly the function $\rho^N_{N-1}(\chi^{N-1})$ can be denoted as $\rho^N_{N-1}(\pi_{N-1}, \chi_{N-1})$. The dependence of $\rho^N_{N-1}(\cdot)$ on χ_{N-1} again stems from the restrictions imposed by χ_{N-1} on the admissible decision space. Consider the relation (4-4) for n = N- 2:

$$\rho^N_{N-2}(\chi^{N-2}) = \inf_{\delta_{N-1}} E\{W(N-1,\sigma_{N-1}) + \rho^N_{N-1}(\chi^{N-1}) \mid \chi^{N-2}\} \qquad (4\text{-}16)$$

Using the relation

$$\rho^N_{N-1}(\chi^{N-1}) = \rho^N_{N-1}(\pi_{N-1}, \chi_{N-1})$$

equation (4-16) can be rewritten as

$$\rho^N_{N-2}(\chi^{N-2}) = \inf_{\delta_{N-1}} E\{W(N-1,\sigma_{N-1}) + \rho^N_{N-1}(\pi_{N-1}, \chi_{N-1}) \mid \chi^{N-2}\}$$

$$= \inf_{d_{N-1}} \{ E[W(N-1, \sigma_{N-1}) \mid \chi^{N-2}, d_{N-1}] \qquad (4\text{-}17)$$

$$+ E[\rho^N_{N-1}(\pi_{N-1}, \chi_{N-1}) \mid \chi^{N-2}, d_{N-1}] \}$$

For every fixed $d_{N-1} \in D_{N-1}(\chi_{N-2})$, the first term in (4-17) can be shown to depend only on χ_{N-2} and π_{N-2} as in (4-15). The second term in (4-17) can be treated as follows:

$$E[\rho^N_{N-1}(\pi_{N-1}, \chi_{N-1}) \mid \chi^{N-2}, d_{N-1}]$$

$$= \int \rho^N_{N-1}(\pi_{N-1}, d_{N-1}, \eta)\, d P_{y_{N-1}}(\eta \mid \chi^{N-2}, d_{N-1}) \qquad (4\text{-}18)$$

By assumptions B-C, the process π_n has been shown to be a Markov diffusion process in Chapter 3. Thus the evolution of the conditional probability π_{N-1} in (4-18) depends only on π_{N-2} and χ_{N-1}, and does not depend on the history χ^{N-1}. Furthermore the probability $P(\cdot)$ can be evaluated as (4-7)

$$P_{y_{N-1}}(\eta \mid \chi^{N-2}, d_{N-1})$$

(equation continued)

$$= \int P_{y_{N-1}}(\eta|\beta, \chi^{N-2}, d_{N-1})\, d\, P_{\theta_{N-1}}(\beta|\chi^{N-2}, d_{N-1})$$

$$= \int P_{y_{N-1}}(\eta|\beta, d_{N-1})\, d\, P_{\theta_{N-1}}(\beta|\chi^{N-2}, d_{N-1}) \tag{4-19}$$

and

$$P_{\theta_{N-1}}(\beta|\chi^{N-2}, d_{N-1}) = \int P_{\theta_{N-1}}(\beta|\tilde{\beta}, d_{N-1}) \cdot d\, P_{\theta_{N-2}}(\tilde{\beta}|\chi^{N-2}, d_{N-1}) \tag{4-20}$$

Reasoning as before, (4-18) can be considered to be dependent on π_{N-2},

$$E\,[\rho^{N}_{N-1}(\pi_{N-1}, \chi_{N-1}) \;|\chi^{N-2}, d_{N-1}]$$

$$= \int \rho^{N}_{N-1}(\pi_{N-1}, d_{N-1}, \eta) \cdot d\, P_{y_{N-1}}(\eta|\pi_{N-2}, d_{N-1}) \tag{4-21}$$

Since the evolution of the probability π_{N-1} depends only on π_{N-2} and χ_{N-1}, thus for each fixed value of d_{N-1}, the expressions in the braces of (4-17) depend on only π_{N-2} and χ_{N-2}. Consequently the optimal decision function at the instant N-1 depends only on π_{N-2} and χ_{N-2}, and moreover,

$$\rho^{N}_{N-2}(\chi^{N-2}) = \rho^{N}_{N-2}(\pi_{N-2}, \chi_{N-2}) \tag{4-22}$$

The conclusion can be similarly shown to be true for n = N-3, ... up to the final step:

$$\rho^{N} = \inf_{d_1} E\,\{W(1, \sigma_1) + P^{N}_{1}(\pi_1, \chi_1)\,|\,\pi_0\} \tag{4-23}$$

where π_0 is the a priori probability that a degradation has occurred. ρ^{N} gives the minimum possible risk in (4-2).

The following general statement has thus been proved. If the conditions of assumptions A-D are satisfied, then the optimal decision function δ_n at the time instant n($n \geq 1$) depends on the past history $\chi^{n-1} = (y^{n-1}, d^{n-1})$

only through the posterior probability π_{n-1} and χ_{n-1}. In other words π_{n-1} and χ_{n-1} are the sufficient statistics for determination of the optimal decision function.

Consider the case of a sequential decision process such that the observations can be terminated before the end of the time N allotted for observation, and assume that the unobservable process is not a controlled process. The decision procedure at any instant n consists of two components: a stopping rule for the choice to continue or to terminate the process of observations, and a decision rule at the termination stage. The decision space for this case can be defined as $D = D^0 \cup D^1$, where D^0 is the decision space at the termination stage and D^1 is the decision space for continuation of observations. If a decision has been made at n to continue the observation, then the decision space $D_n = D^0 \cup D^1$. The decision space $D_n = D^0$ if a decision is made to terminate the process of observations. In both cases the previous observations and decisions already made impose no restrictions on the subsequent admissible decision spaces. Since the dependence of the risk function $\rho_n^N(\pi_n, \chi_n)$ on χ_n is only through the restrictions imposed by χ_n on the admissible decision space, the risk function $\rho_n^N(\pi_n, \chi_n)$ depends thus only on the posterior probability π_n and not on the statistic χ_n, and the sufficient statistic is reduced only to the posterior probability. The minimum risk can be represented by

$$\rho_n^N(\pi_n) = \text{Min}\,[\rho_{n,0}^N(\pi_n),\ \rho_{n,1}^N(\pi_n)] \tag{4-24}$$

where $\rho_{n,0}^N(\pi_n)$ is the minimal risk obtained for the terminal decision taken at the moment n, and $\rho_{n,1}^N(\pi_n)$ is the minimal risk for continuation of observations. Since the unobservable process in the sequential decision process is independent of the decision function, the posterior probability π_n is redefined as

$$\pi_n(\beta) = P_{\theta_n}(\beta \mid y^n)$$

where y^n is the past history of the observed process $\{y\}_i$.

It is convenient to represent the total loss function in this case as

follows:

$$W(n,\sigma^{n}) = \sum_{i=0}^{n-1} W(i,\sigma_i) + K(n,\sigma_n), \quad n \geq 1 \tag{4-25}$$

For notational convenience, $W(i,\sigma_i)$ is defined as $W^{\delta_{i+1}}(i,\sigma_i)$ where $\delta_{i+1} \in D^1$. It is interpreted as a loss for a certain step $I \in [i, i+1)$. Here $K(n,\sigma_n)$ is defined as $W^{\delta_{n+1}}(n,\tau_n)$ where $\delta_{n+1} \in D^0$. It is defined as the terminal loss if the observations are discontinued at the instant of time n.

The minimal risk $\rho^{N}_{n,1}(\pi_n)$ for continuation of observations satisfies the recurrent relation:

$$\rho^{N}_{n,1}(\pi_n) = \inf_{d_{n+1} \in D^1} E\{W(n,\sigma_n) + \rho^{N}_{n+1,1}(\pi_{n+1}) \mid \pi_n\} \tag{4-26}$$

where π_{n+1} is the posterior probability of π_n after a new observation has taken.

The minimal risk $\rho^{N}_{n,0}(\pi_n)$ obtained for the terminal decision taken at the moment n is

$$\rho^{N}_{n,0}(\pi_n) = \inf_{d_k \in D^0} E[K(n,\sigma_n) \mid \pi_n] \tag{4-27}$$

where the terminal decision d_k is taken over all possible classes to minimize the terminal cost. In the discrete case where the terminal decision space D^0 consists of m finite decision classes $\{\beta_i\}$, i=1, ..., m. Then (4-27) can be expressed as

$$\rho^{N}_{n,0}(\pi_n) = \inf_{d_k \in D^0} \left\{ \sum_{k=1}^{m} K(\beta_i, d_k) \cdot \pi^{k}_{n} \right\} \tag{4-28}$$

where $K(\beta_i, d_k)$ is the terminal cost from taking the decision d_k when the true class is β_i, and π^{k}_{n} represents the posterior probability of the class β_k.

C 12

The explicit dependence on time n will be discussed. In general, if the loss function $W(n, \sigma_n)$ at each time step is a constant c; if the θ_n-process is a homogeneous Markov process, and if the y_n-process is a conditionally independent sequence of identical distribution (up to and after the degradation), then the sufficient statistics will not explicitly depend on time n. However, for the decision process truncated at stage N the risk function is time-variant such that the optimal decision boundary for the statistic π_n for termination of observations will in general depend on time n. For the untruncated decision process with $N \to \infty$ the minimal risk $\rho^N_{n,1}(\pi_n)$ for continuation of observations can be shown to converge to a limit $\rho_1(\pi_n)$. The proof is based on the boundedness and monotonicity property of the $\{\rho^N_{n,1}\}$ sequence, and can be found in references (Blackwell[9], DeGroot[16]). The optimal risk function $\rho(\pi_n)$ in this case satisfies

$$\rho(\pi_n) = \text{Min}\ \{\rho_0(\pi_n),\ \rho_1(\pi_n)\} \tag{4-29}$$

where (4-26) and (4-27) can be written as

$$\rho_0(\pi_n) = \inf_{d_k \in D^0} E\,[K(\sigma_n) \mid \pi_n] \tag{4-30}$$

and

$$\rho_1(\pi_n) = \inf_{d_{n+1} \in D^1} E\{c + \rho_1(\pi_{n+1}) \mid \pi_n\} \tag{4-30}$$

An intuitive interpretation of the asymptotic limit is the following. Consider the truncation time N as the starting time, and $\rho^N_{N-1,1}(\cdot)$ as the initial condition for a backward propagation of the risk function. As $N \to \infty$, the transient due to the "initial condition" dies down, and the asymptotic interval grows without bound. In other words, the finite "transient interval" tends to infinity and the "steady-state interval" occupies all finite time.

The conclusion regarding sufficient statistics will be different if the loss function is non-additive. Shiryaev[66] discussed the problem of

Markov sufficient statistics in non-additive Bayes decision problems, and derived new sufficient statistics other than the posterior probability.

4.3 The Optimal Detection System

In this section the main results of an optimum formulation of a quickest detection problem by Shiryaev[64] (also Stratonovich[73]) will be presented, and the problems of the optimum system implementation will be discussed.

Shiryaev considered the following problem. Suppose the observed process $y(t)$, $t \geq 0$ satisfies the stochastic differential equation

$$dy(t) = a(t - \theta) \cdot dt + \sigma dw(t) \tag{4-31}$$

where $w(t)$ is a standard Wiener process, $w(0) = 0$,

$$E(\Delta w) = 0, \quad E(\Delta w)^2 = \Delta t$$

and

$$a(\tau) = \begin{cases} a_1 & \tau \geq 0 \quad \text{(denoted as state } H_1\text{)} \\ 0 & \tau < 0 \quad \text{(denoted as state } H_0\text{)} \end{cases}$$

The moment θ, signaling the appearance of a degradation, is unknown. It is required to derive an optimum system of detecting the arrival of a degradation in the observed stochastic process $y(t)$, as quickly as possible subject to a limitation on the number of false alarms.

A simple model of the process $\theta(t)$ is considered in both discrete and continuous cases. In the discrete case, the degradation occurs at a discrete instant θ according to a geometrical distribution with parameter p.

$$\begin{aligned} &P(\theta = 0) = \pi \\ &P(\theta = n \mid \theta > 0) = p(1 - p)^{n-1} \qquad n \geq 1 \end{aligned} \tag{4-32}$$

In the continuous time version of the model, the a priori distribution of θ is exponential:

$$P(\theta > t) = e^{-qt} \qquad (t \geqq 0) \qquad (4\text{-}33)$$

with the given constant rate q. The exponential distributions (geometrical distributions in discrete-time case) have the useful property that

$$P(\theta > t + s \mid \theta > s) = P(\theta > t \mid \theta > 0)$$

In both cases it may be decided at any instant t to make a decision about the system mode. This theory assumes that the correctness of this decision can then somehow be determined. This is to be done by the identification process described in Chapter 5. Then, if it is found that $\theta < t$ the detection process terminates; but observation must be resumed immediately after a false alarm. For the case of exponential and geometrical distributions of θ, the posterior distribution of θ at any time does not depend on anything which took place before the last false alarm. Then, after carrying out one stage of the observation (up to the stopping time ν at which a disorder is detected) and after ascertaining that it is necessary to continue the observation, one has the same situation (in the sense of the distribution for the unknown parameter θ) as before the observation began, the only difference being that one erroneous signal has been produced.

The optimum observation method will first be constructed for processes with discrete time as the Bayesian solution to problems involving the minimization of specified risks. The method of constructing Bayes solutions is based on the use of a recurrent equation for the risk, and it turns out that the Bayes solution is based on the posterior probability. This is in fact an application of the general result on the sufficient statistic developed in the section 4.2.

Define a risk function associated with a given a priori probability π and each decision function δ:

$$R(\pi, \delta) = E_{\delta} W(\nu, \theta)$$

Let the non-negative loss function W(t, s) be such that

$$W(t, s) = \begin{cases} c_1 & t < s \\ c_2 \cdot (t-s) & t \geq s \end{cases} \qquad (4\text{-}34)$$

where c_1 and c_2 are specified positive constants, and $W(0, s) < \infty$ for $s < \infty$. The constants can be viewed as penalties, with c_1 as the terminal penalty for false alarms, and c_2 the penalty for delay in detecting a degradation.

Let $D = \{D^0, D^1\}$ be the set of all decision functions with D^0 the set of terminal decisions about the presence of degradation, D^1 the set of continuation of observation. Define

$$\rho(\pi) = \inf_{\delta \in D} R(\delta, \pi)$$

$$\rho_0(\pi) = \inf_{\delta \in D^0} R(\delta, \pi) \qquad (4\text{-}35)$$

$$\rho_1(\pi) = \inf_{\delta \in D^1} R(\delta, \pi)$$

Then

$$\rho(\pi) = \text{Min}\,(\rho_0(\pi), \rho_1(\pi))$$

Define

$$\pi_n = P(\theta \leqq n \mid y_1, \ldots, y_n) \qquad (4\text{-}36)$$

where $y_1, \ldots, y_n$ are observations taken up to the instant n. It is to be noted that the measurement actually observed is not the function y(t) defined in the stochastic equations (3-26) and (3-30), but rather some approximation to the "function" $\dot{y}(t)$. Thus in the discrete time case the notation $\{y_n\}$ stands for the measurement residuals $\{\tilde{r}(n)\}$ defined in (3-39) and (3-40), and represents a sequence of conditionally independent random variables. However the posterior probability (4-36) can be defined to be conditioned either on the sequence $\{\tilde{r}(n)\}$ or on their summation (equivalently on the integral y(t) in the continuous time case), since both generate the same Borel-field. For notational convenience the

"generalized" notation y will be employed to denote the observation throughout the rest of the thesis without further explanation.

Shiryaev proved the following statement. The Bayes solution to the problem of minimizing the risk consists in observing the process $\pi(n)$ $(\pi(0) = \pi)$ until the first moment ν at which $\pi(\nu) \geqq \pi'$, where π' is the root of the equation $\rho_0(\pi') = \rho_1(\pi')$.

The optimal risk function $\rho(\pi_n)$ satisfies

$$\rho(\pi_n) = \text{Min}\,[\rho_0(\pi_n),\ \rho_1(\pi_n)]$$

where the terminal risk $\rho_0(\pi_n)$ can be derived from (4-34) and (4-28) in section 4.2:

$$\rho_0(\pi_n) = (1-\pi_n)\, c_1 \tag{4-37}$$

The risk $\rho_1(\pi_n)$ satisfies the recurrent relation[64]:

$$\rho_1(\pi_n) = \int_{-\infty}^{\infty} \rho_1(\pi_{n+1}(\eta))\, dP_y(\eta \mid \pi_n) + c_2\pi_n \tag{4-38}$$

where $P_y(\eta \mid \pi_n)$ denotes the probability that $y \leqq \eta$ conditioned on the posterior probability π_n (the random variable π_n represents in turn the a priori probability for the observation y at instant (n+1)). An explicit expression for the term $P_y(\eta \mid \pi_n)$ and the probability $\pi_{n+1} = \pi_{n+1}(\eta)$ can be found later in equations (4-73) and (4-72).

Shiryaev studied the optimal observation method in a different version. Among all decision rules D_α with a specified probability of false alarm $\alpha = P_\delta(\nu < \theta)$, find an optimal rule δ^* which gives the minimum mean time of delay for detection of a disorder:

$$E_{\delta^*}(\nu-\theta \mid \nu \geqq \theta) = \inf_{\delta \in D_\alpha} E_\delta(\nu-\theta \mid \nu \geqq \theta) \tag{4-39}$$

This problem can be transformed into the Bayes problem by defining the risk function.

$$R(\delta, \pi, c_1) = c_1 P(\nu < \theta) + c_2 E(\nu-\theta \mid \nu \geqq \theta) \tag{4-40}$$

where the loss function is now defined as

$$W(t, s) = \begin{cases} c_1 & t < s \\ c_2(t-s) & t > s \end{cases} \qquad (4\text{-}41)$$

The difference is that the constant c_1 is now the Lagrange multiplier, which is to be evaluated through the specified constraint α, i.e. $c_1 = c_1(\alpha)$.

For this choice of the risk, set

$$\rho(\pi, c_1) = \inf_{\delta \in D_\alpha} R(\delta, \pi, c_1)$$

$$\rho_0(\pi, c_1) = \inf_{\delta \in D^0} R(\delta, \pi, c_1) = c_1(1-\pi)$$

$$\rho_1(\pi, c_1) = \inf_{\delta \in D^1} R(\delta, \pi, c_1)$$

The optimum decision rule in this case can be stated as follows. For any $\alpha = P(\nu < \theta)$ the optimum method in the problem of minimizing the mean time of delay in detecting degradation $\tau = \tau(\alpha)$ consists in observing the process π_n until a certain level $\pi' = \pi'(\alpha)$, $0 \leqq \pi' \leqq 1$ is attained for the first time.

The fact that the posterior probability $\pi_n = P(\theta \leqq n \mid Y_1, \ldots, Y_n)$ is the only sufficient statistic for the optimum detection system can be viewed as an application of the general result of section 4.2. The loss function in (4-25) corresponds to:

$$W(i, \sigma_i) = c_2 \qquad \text{for all } i$$

$$K(n, \sigma_n) = c_1$$

The loss function satisfies the additive property, and the loss over each step is a constant. The assumptions B-D in section 4.2 are obviously satisfied. Furthermore the distribution of θ is a homogeneous Markov process, i.e. $P(\theta = n+1 \mid \theta > n) = P(a_{n+1} = a_1 \mid a_n = 0) = p$ for any n. Since only the untruncated decision rule with $N = \infty$ is considered, the sufficient statistic therefore can be simplified to the posterior probability π_n.

Similar results are formally extended to the continuous time case by Shiryaev. Some exact formulae and explicit calculations have been discussed[68]. (Also Stratonovich[73]). In chapter 3 it was shown that $\pi_t = \pi(t)$ is a diffusion process which satisfies the stochastic equation:

$$d\pi_t = (1-\pi_t) \cdot (q - \frac{a_1^2}{\sigma^2}\pi_t^2)dt + \frac{a_1}{\sigma^2}\pi_t(1-\pi_t)\, dy_t$$

and

$$dy_t = a(t-\theta) \cdot dt + \sigma\, d\, w_t \tag{4-42}$$

From (4-42) the following statistics can be computed:

$$\begin{aligned} E[\, d\pi \mid \pi_t] &= q(1-\pi_t)dt \\ E[\, d\pi^2 \mid \pi_t] &= \frac{a_1^2}{\sigma^2}[\pi_t \cdot (1-\pi_t)]^2\, dt + 0(dt) \end{aligned} \tag{4-43}$$

At any moment of time one can either stop or continue the observation,

$$\rho\,(\pi_t) = \text{Min}\,[\rho_0(\pi_t),\ \rho_1(\pi_t)]$$

where $\rho_0(\pi_t)$ is the optimum risk in terminating the observation at time t if $\pi(t) = \pi_t$, and $\rho_1(\pi_t)$ is the risk from continuing the observation at time t when $\pi(t) = \pi_t$. In the continuation region, where observations are made, denote $\rho_1(\pi_t) = \rho(\pi_t)$. The discrete recurrence relation (4-38) can be expanded to

$$\rho\,(\pi(t)) = c_2 \cdot \pi(t) \cdot \Delta t + \int \rho\,(\pi(t) + \Delta\pi) \cdot d\,P(\Delta\pi \mid \pi(t)) + 0\,(\Delta t) \tag{4-44}$$

Considering the class of decision rules for which a bounded second derivative $\rho''(\pi_t)$ exists and using (4-43) it can be shown that $\rho(\pi_t)$ satisfies

the differential equation:

$$\frac{a_1^2}{2\sigma^2}\pi_t^2 (1-\pi_t)^2 \rho''(\pi_t) + q(1-\pi_t)\rho'(\pi_t) = -c_2\pi_t \qquad (4\text{-}45)$$

in the continuation region $\pi_t \in [0, \gamma]$, where γ is the boundary to be determined. Since the risk function is not an explicit function of time for untruncated processes, the equation of $\rho(\pi_t)$ is in fact reduced to an ordinary differential equation. The optimal detection system thus consists in observing the process π_t (or π_n in the discrete time case) and is reduced to the determination of the optimal boundary for the decision to stop or continue the observation at a minimum risk. For on-line mechanization of the optimal detection system, only the stochastic differential equation π_t need be implemented and the optimal boundary for termination of observations need be determined. These problems are now to be discussed.

In general, calculation of the optimal boundary γ involves solution of the recursive backward equation (4-45) of the minimum risk. The boundary conditions for optimal decision are established by the following relations:

$$\begin{aligned} \rho_0(\gamma) &= \rho_1(\gamma) \\ \left.\frac{d\rho_0}{d\pi}\right|_{\gamma} &= \left.\frac{d\rho_1}{d\pi}\right|_{\gamma} \\ \left.\frac{d\rho_1}{d\pi}\right|_{\pi=0} &= 0 \end{aligned} \qquad (4\text{-}46)$$

The three boundary conditions are sufficient to find the expression for the risk function $\rho_1(\pi_t)$ in the region $[0, \gamma]$ and the unknown boundary γ. The first boundary condition is obvious, and the third boundary condition is derived from the assumption concerning the boundedness of the second derivative $\rho''(\pi_t)$. The derivation of the second condition will be discussed in section 4.4.2.

In the version (4-39) of the optimal detection system the optimal boundary γ can be solved by the optimal boundary conditions (4-46). The additional unknown Lagrange multiplier c_1 associated with the terminal cost $\rho_0(\pi_t)$ will have to be removed by the specified constraint on false alarms. However a simple relation for the boundary γ in terms of the constraint of the specified false alarm error probability α can be solved by application of the theory of the first passage problem of diffusion processes. The result can be stated as follows.

$$\gamma = 1 - \alpha(1-\pi) \tag{4-47}$$

where π is the a priori probability of degradation defined in (4-32). It is noted that γ does not depend on the parameters a_1, σ and q. The derivation is based on the straight-forward solution to the Kolmogorov backward equation associated with the diffusion process of the posterior probability π_t defined in the normal mode. The detailed solution can be found in Shiryaev[68] or Stratonovich[73]. The theory of first passage times of diffusion processes will be discussed in detail in section 4.5 for derivation of the detection performance.

The on-line implementation of the Ito stochastic differential equation $\pi(t)$ will now be discussed The Ito integral has certain special properties which derive basically from the fact that the increments dw(t) of a Wiener process are of the order of $(dt)^{1/2}$ and not of (dt) as they would be for a smoothed random process This means that second-order terms $(dw)^2$ cannot be neglected in the Ito calculus

Consider the differential equation (4-42), of the Ito stochastic process $\pi(t)$

$$d\pi_t = (1-\pi_t) \cdot (q - \frac{a_1^2}{\sigma^2}\pi_t^2)\, dt + \frac{a_1}{\sigma^2}\pi_t(1-\pi_t)\, dy_t$$

and the observed process y(t) is defined as (4-42)

$$dy_t = a\, dt + \sigma\, dw_t$$

The process $y(t)$ is a nondifferentiable process with probability 1, and it represents an idealization of the actual process $y_H(t)$. In a practical system the observed process is not the process $y(t)$, but some smoothed band-limited process $y_H(t)$ which passes the frequencies $[0, H]$ only. The derivative of y_H exists with probability 1, and satisfies the equation

$$\dot{y}_H(t) = \dot{a} + \dot{w}_H(t) \tag{4-48}$$

Consider the following terms

$$I(t) = \int_0^t \pi(s)\,[1-\pi(s)]\; dy(s) \tag{4-49}$$

and

$$I_H(t) = \int_0^t \pi_H(s)\,[1-\pi_H(s)]\; dy_H(s) \tag{4-50}$$

It is natural to expect that for $H \to \infty$, the value of $I_H(t)$ converges to $I(t)$ However, it is well known that such convergence will not exist The reason is that for the smoothed process $w_H(t)$

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} [w_H(s_{i+1}) - w_H(s_1)]^2 = 0$$

for $0 = s_0 < s_1 \ldots < s_{m-1} = t$, while for the limiting process $w(t)$, this limit is t Thus the equation (4-49) in the Ito form does not have a physical analogy and cannot be implemented on-line

The problem of modeling ordinary scalar differential equations by stochastic differential equations is discussed by Wong and Zakai[89] Let the realizable process $y_H(t)$ be defined by

$$\dot{y}_H(t) = f(y_H(t),\ t) + g(y_H(t),\ t) \cdot \dot{w}_H(t) \tag{4-51}$$

Then the process $y_H(t)$ can be modeled by the Ito stochastic differential equation

$$dy(t) = [f(y(t),t) + \frac{1}{2} \frac{\partial g(y(t),t)}{\partial y(t)} g(y(t),t)]\, dt + g(y(t),t)\, dw(t) \qquad (4\text{-}52)$$

in the sense that

$$\lim_{H\to\infty} y_H(t) = y(t)$$

whenever

$$\lim_{H\to\infty} [w_H(t) = \int_0^t \dot{w}_H(s)\, ds] = w(t)$$

Thus the stochastic equation (4-42) for on-line implementation should be replaced by the following equation·

$$\pi_H(t) = \pi(0) + \int_0^t [1 - \pi_H(s)]\, [q - \frac{a_1^2}{\sigma^2}\pi_H^2(s)]\, ds$$

$$- \frac{1}{2}\frac{a_1^2}{\sigma^2} \int_0^t [1 - 2\pi_H(s)] \cdot \pi_H(s)\, [1 - \pi_H(s)]\, ds \qquad (4\text{-}53)$$

$$+ \frac{a_1}{\sigma^2} \int_0^t \pi_H(s)\,(1-\pi_H(s))\, \dot{y}_H(s)\, ds$$

where $y_H(s)$ is the actual band-limited observed process

$$\dot{y}_H(s) = a + \sigma\, \dot{w}_H(s)$$

The equation (4-53) can be handled as an ordinary differential equation This subtle point, which must be taken into account when implementing $\pi(t)$ can be avoided if an equivalent process $\lambda(t)$ is implemented in place of $\pi(t)$·

$$\lambda(t) = \ell n\left[\frac{\pi(t)}{1-\pi(t)}\right]$$

From (3-82) the process $\lambda(t)$ satisfies the integral equation

$$\lambda(t) = \lambda(0) + \frac{a_1}{\sigma^2}\int_0^t dy(s) + \int_0^t (q(1+e^{-\lambda(s)}) - \frac{1}{2}\frac{a_1^2}{\sigma^2})ds \qquad (4-54)$$

In comparison with the formula for $\pi(t)$ it is noted that in the process $\lambda(t)$ there are no multiplicative factors of a diffusion nature associated with the term dy(t). The stochastic integral can be computed in the same manner as for the smoothed process. Define

$$\lambda_H(t) = \ell n\left[\frac{\pi_H(t)}{1-\pi_H(t)}\right]$$

Then the process $\lambda_H(t)$ defined as

$$\lambda_H(t) = \lambda(0) + \frac{a_1}{\sigma^2}\int_0^t \dot{y}_H(s)\, ds + \int_0^t [q(1+e^{-\lambda_H(s)}) - \frac{1}{2}\frac{a_1^2}{\sigma^2}]\, ds \qquad (4-55)$$

will be the correct model for implementation of the Ito stochastic differential equation (4-54). The above equation for $\lambda(t)$ is simpler, but there exists still a non-linear element for implementation In the next section a class of linear system formulations will be discussed.

4 4 A Suboptimal Detection System

4 4.1 A Linear Detection System With Uncertainty Feedback

A study of the stochastic differential equation $\lambda(t)$ (4-54) reveals that the process $\lambda(t)$ will be transformed to a linear system for $q = 0$

$$d\lambda(t) = \frac{a_1}{\sigma^2}[dy - \frac{1}{2}a_1 \cdot dt] \qquad (4-56)$$

Physically this means that the system is defined to be operated in an unknown yet unchanged mode There is no transition between operating modes during the mission, and the problem is reduced to that of discrimination between simple hypotheses This is the formulation of the well-known Wald's sequential analysis in the continuous-time case[22] The Wald optimal decision rule can be simply described as follows. Let the observed process be defined as

$$dy(t) = a\,dt + \sigma\,dw(t) \tag{4-57}$$

Let α and β denote the probabilities of false decisions under the hypotheses $H_0(a=0)$ and H_1 $(a=a_1)$ Then one observes the likelihood ratio function $\lambda(t)$ (4-56) with $\lambda(0) = 0$, until the first realization of one of the equalities $\lambda(t) = A$ or $\lambda(t) = B$, where the boundaries A and B $(B > 0 \geqq A)$ are defined by the error probabilities

$$A = \ell n\left(\frac{\beta}{1-\alpha}\right)$$
$$B = \ell n\left(\frac{1-\beta}{\alpha}\right) \tag{4-58}$$

In the first case one accepts the hypothesis H_0, and in the second, the hypothesis H_1. A sketch of the sequential probability ratio test is illustrated in Fig 4 1. It is well known that the Wald SPRT has the optimal property of being the test that, on average, requires the minimum number of samples of all tests to come to a decision for some specified error probabilities Wald's sequential analysis will be discussed in detail in Chapter 5

The distinct feature of a transition of operating modes in the degration process distinguishes the problem from Wald's formulation. In general, Wald's sequential procedure is a single stage decision in the sense that the process terminates whenever acceptance of either class H_0 or H_1 has been decided However in the detection system one is only interested in detecting the degradation as it occurs As will be shown in section 4 5, the definitions of stages and the error probabilities are different in these two formulations In order to use Wald's

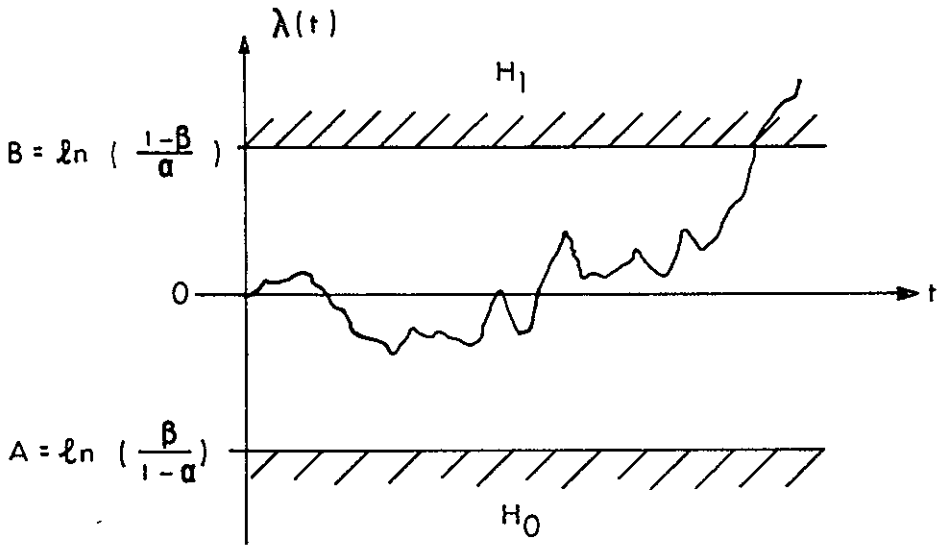


Fig. 4.1. Sequential Probability Ratio Test.

procedures for the detection system, a modification is proposed (Shiryaev[65]): When the lower boundary $A \leq 0$ is first reached, the observation process $\lambda(t)$ starts anew. This is equivalent to having a reset of the process from the boundary A to zero.

It will be shown however, that the detection performance of this modified Wald's procedure will suffer an extra time delay in detection in comparison with the optimal system. Consider the case when the system is operated in the normal mode with $a = 0$ In this case equation (4-57) can be represented by

$$dy(t) = \sigma\, dw(t) \tag{4-59}$$

Substituting (4-59) into (4-56) produces

$$d\lambda(t) = -\frac{a_1^2}{2\sigma^2} dt + \frac{a_1}{\sigma} dw(t) \tag{4-60}$$

It can be seen that the process $\lambda(t)$ for the system defined in the normal mode H_0 will show a negative average drift rate. Similarly the process $\lambda(t)$ will show a positive average drift rate when the system is defined in the degraded mode H_1.

$$d\lambda(t) = \frac{a_1^2}{2\sigma^2} dt + \frac{a_1}{\sigma} dw(t)$$

Suppose a degradation occurs at $t = \theta$, as shown in Fig 4 2, then the likelihood ratio function $\lambda(t)$ will take an extra time delay τ, to compensate for the negative quantity of the likelihood ratio function accumulated before the occurrence of the degradation. Thus the linear detection system must be modified in order to give a better performance, and it will be shown how this can be done by the feedback of an uncertainty information of the operating system.

The motivation of the uncertainty information feedback can be presented as follows The posterior probability function $\pi(t)$ and the likelihood ratio function $\lambda(t)$ have their physical meaning By definition they indicate the current state of uncertainty concerning the system being operated in mode H_1 or H_0 conditioned on the history of the available observations. From the a priori information it is known that the system is initially operated in the normal mode and characterized by a unique transition from the normal mode to the degraded mode at an unknown instant Thus one is mainly interested in detecting the degradation mode rather than the verification of the normal mode It has been shown that the detection process will take an extra time delay in detecting a degradation if both decisions are to be made. Now if the uncertainty information is fed back in some optimal way, such that the decision works in cooperation with the available a priori information, instead of working in complete ignorance of what the current state of information is, as it does in the absence of uncertainty feedback, one expects that the system will improve the performance of detection This is the motivation of uncertainty information feedback. Since there is a deterministic relation between $\lambda(t)$ and $\pi(t)$, i e , $\lambda(t) = \ell n[\frac{\pi(t)}{1-\pi(t)}]$, either $\lambda(t)$ or $\pi(t)$ will be used as the uncertainty information whenever appropriate

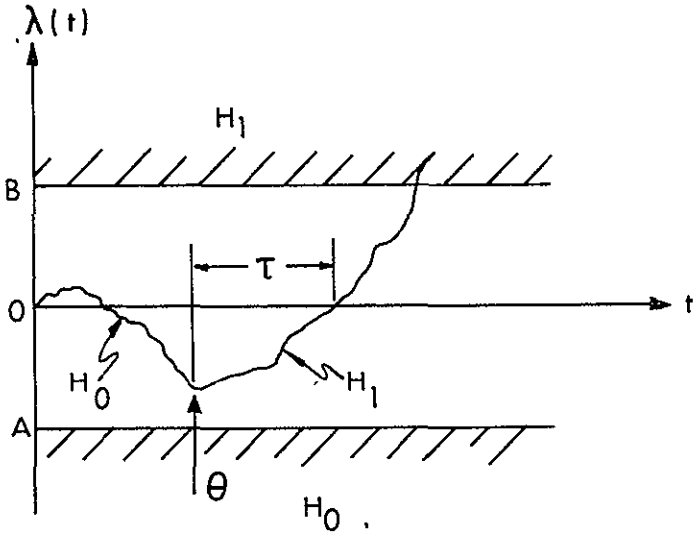


Fig. 4.2. Illustration of Extra Delay for Detection in SPRT.

4.4.2 Some Bayesian Results of Wald's Sequential Analysis

In this section Bayes solutions of Wald's sequential analysis of testing two hypotheses will be discussed Let a be and unknown constant assuming two values 0 (denoted as state H_0) and a_1 (denoted as state H_1) with a priori probability $(1-\pi)$ and π respectively Suppose that the random process $y(t) \triangleq y_t$ is observed such that

$$dy(t) = a\ dt + \sigma\ d\ w(t) \qquad (4\text{-}62)$$

where w(t) is the standard Wiener process The problem considered consists of testing between two hypotheses H_0 against H_1 from the results of observations of the process $y^t = \{y(s),\ 0 < s \leqq t\}$

Let $(\Omega, \mathcal{F}, P)$ be the probability space, and ω be the element of the set generated by the random process y^t The probability distribution generated by the process y^t and based on the a priori probability π is represented by

$$P_\pi(\omega) = \pi \cdot P_1(\omega) + (1-\pi) \cdot P_0(\omega) \qquad (4\text{-}63)$$

Here $P_i(\cdot)$ denotes the conditional probability generated by the process y^t conditioned on state H_i, i.e.,

$$P_i(\omega) = P(\omega \mid H_i) \qquad i = 0, 1$$

It is to be noted that $P_i(\omega)$ represents the joint distribution. In particular, for the discrete case $P_i(\omega)$ denotes the conditional joint distribution of the observations $\{y_1, y_2, \quad , y_t\}$ generated by the random process $\{y(s),\ 0 < s \leqq t\}$.

Let $\delta(\omega) = \{\tau(\omega),\ d(\omega)\}$ be some decision function from the results of observations on the process $\{y(s),\ s \leqq \tau\}$. Here $\tau(\omega)$ is the random stopping time for terminating the observation, $d(\omega)$ is the terminal decision rule The terminal decision space D^0 consists of two points d_0 and d_1. The decision d_i will be interpreted as the decision that the hypotheses H_i is accepted Let $\alpha(\delta) = P\{d(\omega) = d_1 \mid H_0\}$ and $\beta(\delta) = P\{d(\omega) = d_0 \mid H_1\}$ be the false and miss alarm error probabilities at the terminal decision for the decision function $\delta(\omega)$. Then the risk function associated with $\delta(\omega)$ is defined as

$$R(\delta, \pi) = \pi \{c\, E_1(\tau) + \tilde{a} \cdot \beta(\delta)\} + (1-\pi) \{c\, E_0(\tau) + \tilde{b} \cdot \alpha(\delta)\} \qquad (4\text{-}64)$$

Notice that $R(\delta, \pi)$ denotes the total expected cost starting from the initial state π to the terminal decision based on the decision function δ. For the case of specified α and β, $\tilde{a}$ and $\tilde{b}$ are the Lagrange multipliers. The coefficient c is the sampling cost per unit time. The decision δ^* is called Bayesian if

$$R(\delta^*, \pi) = \inf_{\delta} R(\delta, \pi) \qquad \text{for all } 0 \leqq \pi \leqq 1 \qquad (4\text{-}65)$$

where $R(\delta^*, \pi)$ is the Bayesian risk

Let $\pi_t = P\{a = a_1 \mid y(s),\ 0 < s \leqq t\}$ be the posterior probability that H_1 is true based on observations y^t Define the function

$$K(\pi) = \text{Min } \{\tilde{a}\ \pi\ ,\ \tilde{b}\ (1-\pi)\}$$

Then it is shown in Appendix A that the risk function (4-64) can be expressed as:

$$R(\delta, \pi) \geq E_{\pi}[c\tau + K(\pi_{\tau})] \tag{4-66}$$

In other words, $K(\pi_{\tau})$ is the minimum risk from an immediate decision without further observations when the posterior probability of H_1 is π_{τ}. It will be assumed that whenever a terminal decision in D^0 is chosen after sampling has been terminated, that decision is an optimal decision against the posterior distribution of H_1. Hence in discussion of the sequential procedure, the optimal decision δ^* in essence is reduced to determination of optimal stopping time τ^*, after which the terminal decision d* can be easily defined by:

$$d^*(\omega) = \begin{cases} d_1 & \text{if } \tilde{a}\,\pi_{\tau^*} \geq \tilde{b}\,(1-\pi_{\tau^*}) \\ d_0 & \text{if } \tilde{a}\,\pi_{\tau^*} < \tilde{b}\,(1-\pi_{\tau^*}) \end{cases} \tag{4-67}$$

In this case the risk function (4-64) to be optimized can be written as:

$$R(\pi, \delta) = E_{\pi}[c\tau + K(\pi_{\tau})] \tag{4-68}$$

The risk function is thus reduced to the same form discussed in section 4.2. If only untruncated case is considered, the basic recurrent equation (4-29) for the optimal risk function in the discrete time case can be explicitly expressed with some notational changes as follows:

$$\rho(\pi_n) = \text{Min}\,[K(\pi_n), F(\pi_n)]$$

where the optimal terminal risk $K(\pi_n)$ is defined as

$$K(\pi_n) = \text{Min}\,[\tilde{a}\,\pi_n, \tilde{b}(1-\pi_n)] \tag{4-69}$$

and the risk for continuation $F(\pi_n)$ satisfies

$$F(\pi_n) = c + E'[F(\pi_{n+1}) \mid \pi_n] \tag{4-70}$$

The expectation operator is defined as,

$$E\,[F(\pi_{n+1})\mid\pi_n] = \int_{-\infty}^{\infty} F(\pi_{n+1}(\eta))\, p_y(\eta\mid\pi_n)\,d\eta \qquad (4\text{-}71)$$

where the posterior probability operator based on the observation y is expressed as:

$$\pi_{n+1}(y) = \frac{\pi_n \cdot p_1(y)}{(1-\pi_n)p_0(y) + \pi_n \cdot p_1(y)} \qquad (4\text{-}72)$$

and the probability density $p(y\mid\pi_n)$ against the posterior probability π_n can be written as:

$$p(y\mid\pi_n) = (1-\pi_n)\,p_0(y) + \pi_n\,p_1(y) \qquad (4\text{-}73)$$

where $p_0(y)$ and $p_1(y)$ are the probability density function of the observation y conditioned on H_0 and H_1 respectively.

The exact solution of the recurrent equations (4-70) in the discrete-time case is analytically difficult to solve. Whittle [86] discussed the general solution form for the problem of discriminating among m hypotheses with the function $K(\pi_n)$ assumed to be a linear function of π_n. However the explicit solution is not tractable. For the special case of discrimination between two simple hypotheses and with the approximation of neglecting overshoot of $F(\pi_n)$ at the final decision boundary, Whittle derived the general solution form as follows:

$$\begin{aligned} F(\pi_n) &= \psi_1 \cdot \pi_n + \psi_2 \cdot (1-\pi_n) \\ &\quad + \frac{c\,\pi_n}{\alpha_{10}}\,\ell n\left(\frac{1-\pi_n}{\pi_n}\right) + \frac{c(1-\pi_n)}{\alpha_{01}}\,\ell n\left(\frac{\pi_n}{1-\pi_n}\right) \end{aligned} \qquad (4\text{-}74)$$

where ψ_1 and ψ_2 are coefficients to be determined by the boundary conditions. The coefficients α_{ij} are known as measures of discriminatory information and defined as:

$$\alpha_{ij} = \int_{-\infty}^{\infty} \ell n \left(\frac{p_i(y)}{p_j(y)}\right) p_i(y)\, dy \qquad (4\text{-}75)$$

The solution (4-74) is identical with that which would be obtained by determining approximate values of the probabilities of error and expected sample size from Wald's identity in the usual manner (Wald[80]).

The solution of the recurrent equation in the continuous-time case will be derived in detail. The original discrete-time problem is approximated by a continuous-time diffusion process, in which one is essentially making use of the first and second derivatives of the risk function. Only the case of two hypotheses will be considered, the general solution form for the untruncated procedure in the continuous system can be simply derived by the solution of an ordinary differential equation.

It was shown in Chapter 3 that for the class of linear detection systems $\pi(t)$ satisfies the stochastic equation:

$$d\pi_t = -\frac{a_1^2}{\sigma^2}\, \pi_t^2 (1-\pi_t)\, dt + \frac{a_1}{\sigma^2}\, \pi_t (1-\pi_t)\, dy_t \qquad (4\text{-}76)$$

where a_0 is assumed to be zero. It was shown in (3-74) that a standard Wiener process exists such that:

$$dy_t = a\, dt + \sigma\, dw_t = a_1 \pi_t \cdot dt + \sigma\, dw_t \qquad (4\text{-}77)$$

Then the process $\pi(t)$ satisfies the stochastic differential equation:

$$d\pi_t = \frac{a_1}{\sigma}\, \pi_t (1-\pi_t)\, dw_t \qquad (4\text{-}78)$$

This easily yields

$$E[d\pi \mid \pi_t] = 0 \qquad (4\text{-}79)$$

$$E[(d\pi)^2 \mid \pi_t] = \frac{a_1^2}{\sigma^2} \pi_t^2 (1-\pi_t)^2 \, dt$$

The risk $F(\pi_t)$ for the untruncated case in the region D^1 of continuation of observations satisfies the relation

$$F(\pi_t) = c \cdot \Delta t + E_{\pi_t} \{ F(\pi_t + \Delta\pi) \} + 0(\Delta t) \qquad (4\text{-}80)$$

Assuming that the derivatives $\frac{\partial F}{\partial \pi}$ and $\frac{\partial^2 F}{\partial \pi^2}$ exist and expanding $F(\pi + \Delta\pi)$ into a Taylor series, it can be shown that

$$E_{\pi_t} \{F(\pi_t + \Delta\pi)\} = F(\pi_t) + \frac{\partial F}{\partial \pi_t}(\pi_t) E_{\pi_t}(\Delta\pi) + \frac{1}{2} \frac{\partial^2 F}{\partial \pi_t^2} (\pi_t) \cdot E_{\pi_t} (\Delta\pi)^2 + \ldots \qquad (4\text{-}81)$$

Using equation (4-79) produces

$$E_{\pi_t} \{ F(\pi_t + \Delta\pi) \} = F(\pi_t) + \frac{1}{2} \frac{a_1^2}{\sigma^2} \pi_t^2 (1-\pi_t)^2 \cdot \frac{\partial^2 F(\pi_t)}{\partial \pi_t^2} \cdot \Delta t + 0(\Delta t) \qquad (4\text{-}82)$$

Substituting this expression into equation (4-80) and passing to the limit as $\Delta t \to 0$ gives:

$$\frac{1}{2} \frac{a_1^2}{\sigma^2} \pi_t^2 (1-\pi_t)^2 \frac{d^2 F}{d\pi_t^2} + c = 0 \qquad (4\text{-}83)$$

The recurrent equation for the untruncated case is thus simplified to an ordinary differential equation. The general solution of (4-83) can be obtained as

$$F(\pi_t) = \frac{c}{(a_1^2/2\sigma^2)} (1-2\pi_t) \cdot \ell n(\frac{\pi_t}{1-\pi_t}) + \psi_3 \cdot \pi_t + \psi_4 \qquad (4-84)$$

A simple manipulation of (4-84) gives

$$F(\pi_t) = \frac{c}{(a_1^2/2\sigma^2)} (1-\pi_t) \ell n (\frac{\pi_t}{1-\pi_t}) + \frac{c}{(a_1^2/2\sigma^2)} \pi_t \ell n(\frac{1-\pi_t}{\pi_t})$$

$$+ \psi_1 \pi_t + \psi_2 \cdot (1-\pi_t) \qquad (4-85)$$

The density functions $p_1(y)$ and $p_0(y)$ in (4-75) are the normal distributions $N(a_1, \sigma)$ and $N(0, \sigma)$ respectively, since the observed processes are modeled as Gaussian. Then a simple computation of (4-75) shows the result:

$$\alpha_{01} = \alpha_{10} = (a_1^2/2\sigma^2) \qquad (4-86)$$

Thus a completely analogous form of the general solution $F(\pi_t)$ can be obtained in the discrete and continuous time cases.

It is instructive to study the solution form $F(\pi_t)$. The solution consists of two parts. Mathematically the two terms associated with the coefficient c are the particular solution, and the other two terms are the homogeneous solution. Denote the notation $J(\pi_t)$ as:

$$J(\pi_t) = \frac{c}{\alpha_{10}} \pi_t \cdot \ell n(\frac{1-\pi_t}{\pi_t}) + \frac{c}{\alpha_{01}} \cdot (1-\pi_t) \ell n(\frac{\pi_t}{1-\pi_t}) \qquad (4-87)$$

The term $-J(\pi_t)$ then represents the average of the current sampling cost of discrimination of the hypotheses. It is important to emphasize the $-J(\pi_t)$ represents the sampling cost used up to the current state π_t, and does not represent the sampling cost to complete the pro-

cess. If the state π_t is the termination state, then $-J(\pi_t)$ represents the total sampling cost for the whole process. The facts that $J(1/2) = 0$ implies that in Wald's formulation the sampling cost is referred to the datum $\pi_t = 1/2$. The state $\pi_t = 1/2$ reflects a state of no information about either hypothesis, and is the initial state to gain the information at the expense of the cost of sampling.

The coefficients ψ_1 and ψ_2 in (4-85) represent the non-negative weighting factor to be determined by the boundary conditions at the decision points. The results for determining the optimal decision boundary have been discussed by Chernoff,[13] Whittle[86] and others. The decision boundaries are to be chosen in such a way as to minimize the function $F(\pi_t)$ (Fig. 4.3). The main result states that if F possesses continuous first derivatives and the boundary γ is optimal, then

$$\frac{\partial F}{\partial \pi_i} = \frac{\partial K}{\partial \pi_i} \qquad (i=0,1) \tag{4-88}$$

$$\dot{F}(\pi_i) = K(\pi_i)$$

on the boundaries γ, at which $\partial K / \partial \pi_i$ exists. An heuristic proof of the result can be found in references[13], [86].

As mentioned before the optimal decision rule is characterized by determination of the optimal boundary, the derivation of the boundary thus gives the complete information about the structure of the optimal procedure. Let D^1 denote the region of π_t for continuing observations, and D^0 the region for the terminal decision, then the optimal risk function satisfies:

$$\begin{aligned} &\frac{1}{2}\frac{a_1^2}{\sigma^2}\pi_t^2(1-\pi_t)^2\frac{d^2F(\pi_t)}{d\pi_t^2} + c = 0 \qquad \pi_t \in D^1 \\ &K(\pi_t) = \text{Min}\,[\tilde{a}\,\pi_t,\ \tilde{b}(1-\pi_t)] \qquad \pi_t \in D^0 \end{aligned} \tag{4-89}$$

The solution to the equation (4-89) has been derived in (4-85).

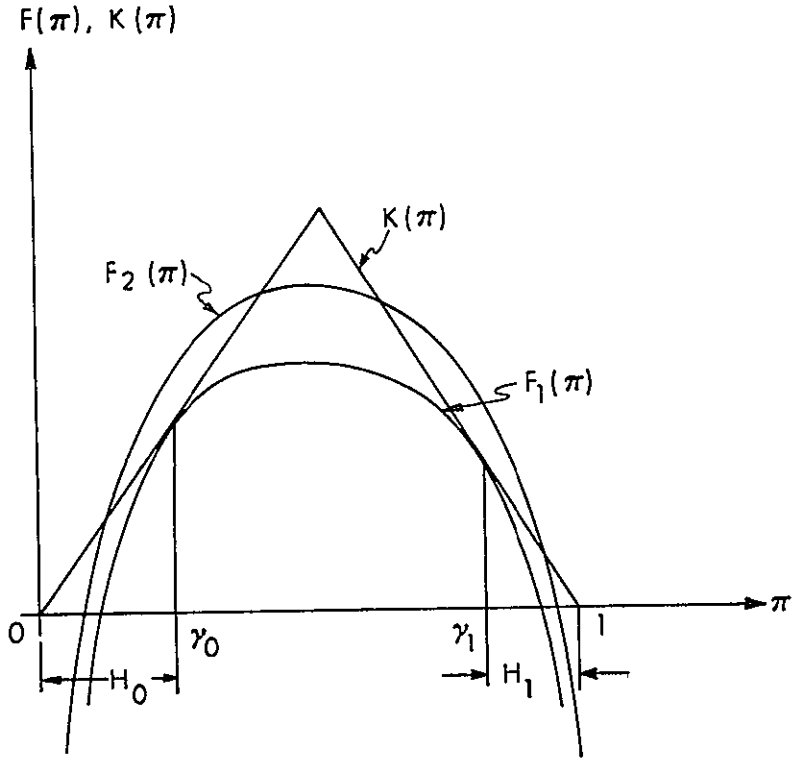


Fig. 4.3 Risk Functions of Sequential Decision Problem

The two straight lines together represent $K(\pi)$, the expected risk on terminal decision. The two curves represent conceivable forms of $F(\pi)$, the expected risk for continuation of observations.

$$F(\pi_t) = J(\pi_t) + G(\pi_t) \tag{4-90}$$

where $G(\pi_t) = \psi_1 \pi_t + \psi_2 \cdot (1-\pi_t)$, and $J(\pi_t)$ has been defined in (4-87). The minimum risk at the instant t is expressed as:

$$\rho(\pi_n) = \min_{\delta_n} \{K(\pi_n) , F(\pi_n)\}$$

The optimal boundary conditions give:

$$F(\gamma_i) = K(\gamma_i) \tag{4-91}$$

$$\left.\frac{dF}{d\pi}\right|_{\gamma_i} = \left.\frac{dK}{d\pi}\right|_{\gamma_i} \qquad i=0,1$$

where γ_i are the two optimal boundaries for accepting hypotheses H_i (i = 0, 1) respectively. In Wald's ordinary procedure, there are two unknown boundaries for H_0 and H_1, and two unknown constants ψ_1 and ψ_2. The four unknowns are determined by the four conditions at the optimal boundaries (4-91).

4.4.3 Formulation of a Stochastic Control Problem

In this section it will be shown that imbedded in the detection system there can be defined a combined optimal control and decision problem. In order to motivate the formulation the system of Wald's sequential analysis will first be discussed.

Consider the optimal risk function at stage n in the formulation of the sequential system:

$$\rho(\pi_n) = \min_{\delta_n} \{K(\pi_n), F(\pi_n)\} \tag{4-92}$$

where δ_n is the decision for continuation or termination of observations to be made at stage n. The solution form of the risk $F(\pi_n)$ for continuation of observations can be explicitly expressed as ((4-85) and (4-90)).

$$F(\pi_n) = G(\pi_n) + J(\pi_n) \tag{4-93}$$

where

$$G(\pi_n) = \psi_1 \cdot \pi_n + \psi_2 \cdot (1-\pi_n) \tag{4-94}$$

$$J(\pi_n) = -\frac{c}{\alpha_{10}} \pi \ln\left(\frac{\pi_n}{1-\pi_n}\right) - \frac{c}{\alpha_{01}} (1-\pi_n) \ln\left(\frac{1-\pi_n}{\pi_n}\right)$$

$$\triangleq \pi_n \cdot J_1(\pi_n) + (1-\pi_n) \cdot J_0(\pi_n) \tag{4-95}$$

It has been mentioned that the value $-J(\pi_n)$ represents a measure of the expected sampling cost used up to the current state π_n. The sampling cost of observations gives the information necessary to reach a greater degree of certainty such that the probability of making an incorrect terminal decision is decreased. Some insight can be obtained from $J(\pi_n)$ in terms of information measure. The measure of discriminatory information α_{ij} is defined in (4-75):

$$\alpha_{ij} = \int_{-\infty}^{\infty} \ln\left(\frac{p_i(y)}{p_j(y)}\right) p_1(y) \, dy$$

The coefficient α_{ij} represents the expected value of $\ln\left(\frac{p_i(y)}{p_j(y)}\right)$ if a single observation y is taken and H_i is true. The coefficient $-\left(\frac{c}{\alpha_{10}}\right)$ can be interpreted as a measure of the cost per unit of discrimination that H_1 is true. The term $-J_1(\pi_n)$ represents the sampling cost to generate the information $\ln\left(\frac{\pi_n}{1-\pi_n}\right)$ for discrimination in favor of H_1 against H_0. A similar interpretation can be given to $\left(\frac{c}{\alpha_{01}}\right)$ and $-J_0(\pi_n)$. $J(\pi_n)$ represents the expected value of the information measure over the current posterior probability π_n. For the case of sequential analysis where decisions on both H_0 and H_1 are to be made, $-J(\pi_n)$ is the necessary expected sampling cost of observations to give the information for discrimination of either hypothesis H_1 or H_0. It was shown by Wald and Wolfowitz[83] that in the class of all hypothesis tests of H_0 against H_1 subjected to the same false and miss alarm error probabilities, Wald's sequential analysis requires, on the average, the minimum number of

samples of all tests to come to a decision.

However, in the detection system one is only interested in minimizing the mean time delay in detecting a degradation. The information given by sampling which reduces the uncertainty for the decision of the mode H_1 is of important value, while the information given by sampling for discrimination in favor of the mode H_0 is of no importance, since no decision of the normal mode is to be made in the detection system. This analysis suggests that a further improvement in reducing the sampling cost can be obtained in the detection system. The improvement is to be obtained by utilization of uncertainty information feedback to formulate the detection system as a combined optimal control and decision problem.

The formulation is started by identifying the state variable, control (decision) variable, and the risk function. It is natural to define the posterior probability π_n as the state variable, since it represents the sufficient statistic of the detection process. The control variable is assumed to be an explicit function of the state variable π_n, i.e., $u_n = u(\pi_n)$. It is assumed that controls are to be executed only at the sampling instants n in the form of impulses with sizes denoted by u_n. With this mechanism the control can be simply implemented in the form of a jump in π at each sampling instant. Moreover, the nonlinear stochastic equation of the state $\pi(t)$, $t \in (n, n+1)$ during each sampling interval will not be affected. In this case the state variable $\pi(t)$ is in fact defined to be the posterior probability based on the observation history up to one sampling period, and at each sampling instant n the reset value π_n is treated as a new initial condition for updating the $\pi(t)$ - process. In general, the introduction of the control action into the detection system will certainly distort the evolution of the posterior probability, which is derived on the basis of updated observations, and the physical meaning of the state π_n as the posterior probability of system degradation is altered. However, an improvement of detection performance is made possible precisely due to the fact that the information value about the system in the normal mode is of no significance in the detection system, and distortion of the information for mode H_0 is allowable.

The admissible control region will now be defined. The uncertainty information π_n is investigated at each sampling instant. If the system is operated more likely in the normal mode, i.e., $\pi_n \in (0, 1/2)$, then one is not interested in utilizing the information in the operating system for taking any decision and is willing to throw away the previously accumulated information. In other words, the uncertainty information is only accumulated over a duration of one sample period, if the system appears to be in the normal mode. On the other hand, if the system is more likely operated in the degraded mode, then no distortion, i.e., neither "improvement" nor "degradation", of the information value of the system is allowed. In this case no control should be applied. The reduction of uncertainty information on mode H_1 must be derived from updated observations, and the problem is reduced to an optimal decision problem. With the above consideration the admissible control region U is defined as follows:

$$U = \begin{cases} u : \quad u(\pi) \in R \quad , \quad \pi \in (0, 1/2) \\ u : \quad u(\pi) \in \emptyset \quad , \quad \pi \in [1/2, 1) \end{cases} \tag{4-96}$$

where R is the entire real line, and $\emptyset$ is the empty set.

Before a precise optimal risk function of the detection system as a combined optimal problem is formulated it is useful to study the necessary modification of the optimal risk function form in the original decision process and the detection system. At state n the decision δ_n to be made for continuation or termination of observations for the detection system is defined by

$$\begin{aligned} &\min_{\delta_n} \{Q(\pi_n) \ , \ G(\pi_n)\} \\ &\quad = \min_{\delta_n} \{\tilde{K}(\pi_n) - J(\pi_n) \ , \ G(\pi_n)\} \end{aligned} \tag{4-97}$$

Two modifications of the formulation (4-92) have been made in (4-97). The form of the terminal risk for Wald's sequential analysis was defined in (4-69) as:

$$K(\pi_n) = \begin{cases} \tilde{a} \cdot \pi_n & \text{for decision of } H_0 \\ \tilde{b} \cdot (1-\pi_n) & \text{for decision of } H_1 \end{cases}$$

Since the decision of H_1 only is made in the detection system, the terminal risk is modified to be:

$$\tilde{K}(\pi_n) = \tilde{b} \cdot (1-\pi_n) \tag{4-98}$$

Physically the modification can be simply realized by implementation of a one-sided decision boundary.

The risk function for continuation in the detection system is formulated in the form $Q(\pi_n)$ rather than $F(\pi_n)$. The form $Q(\pi_n)$ represents the total risk for detection of mode H_1 (including the previous sampling cost $-J(\pi_n)$ up to the current state π_n), if the decision process is terminated at π_n. In comparison $F(\pi_n)$ represents the respective risk, depending on π_n defined in the region $R_0 = \{\pi_n : \pi_n \in (0, 1/2)\}$ or $R_1 = \{\pi_n : \pi_n \in (1/2, 1)\}$, to complete the process to acceptance of the mode H_0 or H_1. The optimal stopping boundary γ for the one-sided detection process can be determined by application of the optimal boundary conditions (4-91):

$$\left.\frac{dQ}{d\pi_n}\right|_{\gamma} = 0 \tag{4-99}$$

and

$$Q(\gamma) = G(\gamma)$$

The condition of the optimal decision process states that it is advantageous to continue the observation process until the rate of decrease of the terminal risk $\tilde{K}(\pi_n)$ and the rate of increase of the sampling cost $-J(\pi_n)$ are balanced. In this case a minimum total risk is obtained. It is noted that for a one-sided detection process, the form $G(\pi_n)$ in (4-94) to match the optimal decision boundary is simply a constant term.

The risk function of the detection system as a combined optimization problem will now be defined. Let the total risk function for a given a priori distribution π associated with a specified admissible control sequence $\{u\}$ and an admissible decision sequence $\{\delta\}$ be defined as:

$$Q_u(\pi) = E\left\{\sum_{n=0}^{\tau_u - 1} W(\pi_n) + K_u(\pi_{\tau_u})\right\} \tag{4-100}$$

Here τ_u is the first time (a random number) that the process reaches the stopping boundary π_{τ_u} defined by the sequence $\{\delta\}$. If an optimal decision sequence is applied, then the total risk can be reduced to $Q_u^0(\pi)$ with the optimal stopping time and boundary denoted as τ and π_τ respectively. In the specific problem considered, the risk $W(\pi_n)$ over each step is a constant c; the explicit representation $W(\pi_n)$ as a function π_n is to emphasize the fact that the effect of each step on the total risk depends on the step risk c as well as on the state π_n affected by the control variable u_n. The terminal risk $K_u(\pi_{\tau_u})$ represents a measure of the specified false alarm criterion. Since the state π_n no longer represents the probability that the system is in the degraded mode H_1 due to action of the control sequence $\{u\}$, it can not be justified to assume $K_u(\pi_{\tau_u})$ in the controlled process to be a linear form in π_{τ_u} as in Wald's sequential analysis. But it is reasonable to assume that $K_u(\pi_{\tau_u})$ is a monotonically decreasing function of the argument π_{τ_u}.

The detection system can now be formally formulated as a combined optimal control and decision problem with associated risk function defined as follows:

$$\begin{aligned}
&\underset{\{u,\delta\}}{\text{Min}} \{G(\pi), Q_u(\pi)\} \\
&\quad = \underset{\{u\}}{\text{Min}} \left\{ \underset{\{\delta\}}{\text{Min}} [G(\pi), Q_u(\pi)] \right\} \\
&\quad = \underset{\{u\}}{\text{Min}} \{ Q_u^0(\pi) \} = Q^*(\pi)
\end{aligned} \tag{4-101}$$

For any specified control sequence $\{u\}$, an optimal risk function $Q_u^0(\pi)$ can be derived by the optimal decision sequence $\{\delta^*\}$. An optimal control sequence $\{u^*\}$ is to be derived over all admissible control sequences $\{u\}$ such that

$$Q_{u^*}^{0*}(\pi) \triangleq Q^*(\pi) \leqq Q_u^0 \quad (\pi)$$

4.4.4 The Suboptimal Control System

In this section a suboptimal control scheme for the detection system is presented, and the suboptimal control sequence in the discrete-time case is derived. The difficulty of extension of results to the continuous-time system is discussed.

In the previous section the detection system was formally formulated as a combined control and decision optimization problem. However, it is exceedingly difficult to obtain the optimal solution to the combined problem formulation. Unlike Wald's sequential analysis where no transition of modes is assumed to occur during the whole observation process, the detection system is characterized by a transition from normal to degraded mode at a random instant. Moreover, it is difficult to formulate an explicit form of the terminal risk function in the controlled process due to the information distortion by the control action. In this section a suboptimal control scheme is presented.

The control philosophy of the suboptimal system is in a sense similar to the well-known open-loop-optimal feedback control (Dreyfus [20]). The operation can be described as follows. At the sampling instant k (k = 1, 2, ...), a current measurement π_k is taken. By definition of the admissible control region, a control u_k is allowed if $\pi_k \in (0, 1/2)$, otherwise no control is applied. The control u_k is computed to minimize the total risk function to complete the detection process:

$$Q(\pi_k, u_k) = E\left\{\sum_{i=0}^{\tau-1} c_i + K_u(\pi_\tau) \,\middle|\, \pi_k\right\} \qquad (4\text{-}102)$$

if a degradation will occur right after the current measurement, and the system remains in the degraded mode thereafter. Here, the cost c_i over each step is a constant c, and τ is a random number. It is assumed that only the current control u_k, but no future controls, is to be applied. This is justified by the definition of the admissible control region since the system is assumed to have degraded after the current time, and the process is reduced to an optimal decision process to terminate at an optimal stopping boundary. The current optimal control u_k^* is implemented. The operation is advanced to the next sample (k + 1). The procedure is repeated, and u_{k+1}^* is implemented.

The motivation of the design of the suboptimal system is to serve as a compromise between computation effort and an efficient control scheme. In the suboptimal system the problem of derivation of the optimal solution is reduced to a parameter optimization over the current control variable u_k, since no future controls are to be applied. Moreover, the control is designed on the basis of a specified state π_k. In comparison the control in the real system must be designed at every sample instant on the basis of the observed sample state which is a random value.

Since the control must be applied before the occurance of the degradation, and the degradation will occur at an unknown instant, the best information one knows whether the system has not degraded is the actually observed state at the current time. To design the control under the assumption that degradation will occur right after the present stage represents the condition that if a degradation does in fact occur, the control will give the minimum mean time of delay to detect the degradation under a specified false alarm criterion. A precise definition of false alarm criterion will be discussed later in this section. The performance (4-102) which the control u_k^* minimizes is not the detection performance of the actual suboptimal system. However, the suboptimal system implemented with the control sequence $\{u_k^*\}$ derived at each sampling instant k to minimize a form of the risk function (4-102) should give a good detection performance. A quantitative comparison has to be derived from the explicit control law, since the latter deter-

mines the optimal boundary of detection for a specified false alarm criterion. The effectiveness of the suboptimal system will later be verified in Section 4.5 by the analytical detection performance expression, and confirmed in Section 4.6 by the simulation results in comparison with the performance of the non-linear optimal system.

The optimal control u_k^* in minimizing the risk function (4-102) (it is the suboptimal control u_k^* for the detection system) must satisfy the basic recursion relation resulting from application of the principle of optimality:

$$Q^*(\pi_k) = \underset{u_k \in U}{\text{Min}}\, E\left\{ [c + E\{Q^*(\pi_{k+1}) \mid \pi_k, u_k\}] \mid \pi_k \right\} \tag{4-103}$$

By definition the control is assumed to be executed at the sampling instant k in the form of an impulse with size u_k. Since no control is applied during the sampling interval, the evaluation of the state π_t during the sampling interval Δ can be treated as the case without control. The state equations can then be represented as:

$$\begin{aligned} \pi_{k+1} &= \pi_k + u_k + \xi_k \\ &\triangleq \pi_k^+ + \xi_k \end{aligned} \tag{4-104}$$

$$\xi_k = \frac{a_1^2}{\sigma^2} \int_{k\Delta}^{(k+1)\Delta} \pi_t (1 - \pi_t)^2 \, dt + \frac{a_1}{\sigma} \int_{k\Delta}^{(k+1)\Delta} \pi_t (1 - \pi_t) \, dw_t \tag{4-105}$$

The equation (4-105) is derived under the assumption that the system has degraded. For any stage i, $i \in [k+1, \tau-1]$, the optimal risk function for continuation of observations must satisfy the recursion relation in the optimal decision process:

$$\begin{aligned} Q^*(\pi_1) &= c + E\left\{ Q^*(\pi_{i+1}) \mid \pi_i \right\} \\ &= c + E\left\{ Q^*(\pi_1 + \xi_1) \mid \pi_i \right\} \end{aligned} \tag{4-106}$$

where ζ_i is defined as in the equation (4-105). Substituting (4-104) and (4-106) into (4-103) produces the relation:

$$Q^*(\pi_k) = \min_{u_k \in U} E\{[c + E\{Q_1^*(\pi_k + u_k + \xi_k) \mid \pi_k^+\}] \mid \pi_k\} \qquad (4\text{-}107)$$

$$= \min_{u_k \in U} Q^*(\pi_k + u_k)$$

The solution u_k^* is still difficult to obtain in the above formulation. Even the randomness of the degradation instant has been removed, the solution of the problem requires the explicit form of the nonlinear terminal risk function which is difficult to define. However, the formulation provides a useful approach to the derivation of the optimal control. The optimal total risk $Q^*(\pi_k)$ in (4-107) is obtained by application of a sequence of an optimal control u_k^* and optimal decisions $(\delta_{k+1}^*, \ldots \delta_{\tau^*-1}^*)$ to continue until the optimal terminal time τ^* is reached. The risk $Q^*(\pi_k)$, though defined at the unknown terminal state π_{τ^*}, is by definition composed of two parts: a terminal risk function and an accumulated sampling cost function. It is certainly reasonable to assume the terminal risk $K_u(\pi_\tau)$ to be a decreasing function of the state π_τ even if its explicit form has not been formulated. Moreover, the portion of the sampling cost accumulated up to the current state π_k is of no information value for the detection system if the state π_k is defined in the region $R_o = \{\pi_k : \pi_k \in (0, 1/2)\}$. The analysis suggests that in this case it is advantageous to apply some control toward the boundary of the admissible control region. A useful information for derivation of the optimal control u_k^* is to prove monotonicity of the optimal risk function $Q^*(\pi_k^+)$ in its argument π_k^+, where $\pi_k^+ = \pi_k + u_k$.

The monotonicity property of the optimal risk is to be proved by application of the theory of the first passage time of diffusion processes. This will be discussed in detail in Section 4.5.2 on derivation of the detection performance. Only a brief introduction will be presented here. As discussed in Chapter 3 and Section 4.5, the state equation for Wald's sequential analysis can also be represented by the diffusion process $\lambda(t)$:

$$d\lambda(t) = \frac{a_1}{\sigma^2}\left(a - \frac{1}{2}a_1\right)dt + \frac{a_1}{\sigma}dw(t) \qquad (4\text{-}108)$$

$$\triangleq \mu(a)\,dt + b\,dw(t)$$

where the parameter $\mu(a)$ is defined as:

$$\mu(a_1) = \frac{1}{2}\frac{a_1^2}{\sigma^2} \qquad \text{for system defined in } H_1$$

$$\mu(a_o) = -\frac{1}{2}\frac{a_1^2}{\sigma^2} \qquad \text{for system defined in } H_o$$

Use of the diffusion process $\lambda(t)$ rather than $\pi(t)$ as the state variable is motivated by the linearity of the $\lambda(t)$ equation (4-108). Let $\tau(\lambda_u, B_u)$ be the mean time for the trajectory $\lambda(t)$ to exit through an upper boundary B_u under the assumption that the observed process is in the degraded mode H_1, and has the initial condition λ_u. The boundary B_u is derived under a specified false alarm criterion. It is convenient to use the mean time T between two false alarms, rather than the false alarm error probability, as the error criterion in the first passage time approach. It will be shown in Section 4.5.1 that there exists a deterministic relation between the two criteria and hence they are mathematically equivalent. It is clear from the definition of $Q^*(\pi_k)$ that the original problem to prove the monotonicity of $Q^*(\pi_k^+)$ in the argument π_k^+ can be reduced to an equivalent problem of proving the monotonicity of the mean time of delay $\tau(\lambda_u, B_u)$ in detection in the argument λ_u for a specified mean time $T(\lambda_u, B_u)$ between false alarms. Here λ_u is the reset state, equivalent to π_k^+ in the $\pi(t)$-process, by a control ζ_u, i.e., $\lambda_u = \lambda_k + \zeta_u$, where λ_k is the sampled value at k of the $\lambda(t)$-process.

The basic equation of interest for derivation of $\tau(\lambda_u, B_u)$ will be discussed. Let two constants $A \leqq 0$ and $B_u > 0$ be selected as the lower and upper boundaries. The first passage time $\tilde{t}(y, a)$ for the trajectory $\lambda(t)$ reaching the upper boundary B_u under the assumption that the initial condition $\lambda(0) = y$, $A \leqq y < B_u$, and that the value of the parameter a does not change during the observation interval, is shown

in Section 4.5.2 to satisfy

$$\frac{1}{2} b^2 \frac{d^2 \tilde{t}}{d\lambda^2} + \mu(a) \frac{d\tilde{t}}{d\lambda} = -1 \qquad (4\text{-}109)$$

The boundary conditions are derived by the definition of the control law. Let the parameter A be defined as follows. When the trajectory $\lambda(t)$ reaches the value A, a control is applied to reset the value to λ_u. If a degradation of the system does in fact occur, then no future controls will be applied. The $\lambda(t)$-process will eventually attain the upper boundary B_u, and the signal for detection of degradation is given. On the other hand, if the degradation of the system does not occur, a control of the same value is applied again to reset the value λ_u, whenever the trajectory $\lambda(t)$ defined in the normal mode H_o drifts back to the value A. It is assumed that the control is defined only as a function of the state, not of the time. The control operation is repeated until the trajectory exits through the upper boundary B_u to give a false alarm when the system is in fact defined in the normal mode H_o. Here one considers the same condition as in the optimal detection system described in Section 4.3, where the appearance of the degradation is preceded by a long process of observation defined in the normal mode H_o, in which a stationary regime is established, to be interrupted by the occurrence of infrequent false alarms. In this case, it is meaningful to define the false alarm criterion in the established stationary regime of observation in the normal mode H_o.

This description of the control action, in which a control is implemented whenever $\lambda(t)$ reaches a fixed value A, is used only for the computation of the mean time between false alarms. In the actual system $\lambda(t)$ is observed at periodic points t_k and control, if any, is implemented at those points. The values taken by the process at those times, λ_k, are random and this precludes calculation of the mean time between false alarms under the action of the control intended to be used. The alternate control action described is a reasonable approximation to the actual control action, and the approximation is especially good if the t_k are closely spaced so $\lambda(t)$ cannot drift far from λ_u between sample points. From the described control law, the boundary conditions are expressed as:

$$\tilde{t}(B_u) = 0$$

$$\tilde{t}(A) = \tilde{t}(\lambda_u) \tag{4-110}$$

The first passage time for the trajectory reaching the upper boundary B_u can be derived by solving the differential equation (4-109) with the boundary conditions (4-110). This can be shown to be:

$$\tilde{t}(y, a) = \frac{1}{\mu(a)} \left\{ B_u - y + (\lambda_u - A) \left(\frac{e^{-hy} - e^{-hB_u}}{e^{-h\lambda_u} - e^{-hA}} \right) \right\} \tag{4-111}$$

under the assumptions of the initial condition $\lambda(0) = y$, and of the parameter a. The parameter h is defined as:

$$h = \frac{2\mu(a)}{b^2} = \frac{2a - a_1}{a_1} \tag{4-112}$$

The mean time $\tau(\lambda_u, B_u)$ of delay in detection can be simply derived by use of (4-111) and (4-112):

$$\tau(\lambda_u, B_u) = \frac{1}{\mu(a_1)} \left\{ B_u - \lambda_u + (\lambda_u - A) \left(\frac{e^{-\lambda_u} - e^{-B_u}}{e^{-\lambda_u} - e^{-A}} \right) \right\} \tag{4-113}$$

where the system is defined without controls and is in fact Wald's system defined in the degraded mode with initial condition $y = \lambda_u$. Since no lower boundary is defined in the system for detection of degradation and one notices that:

$$\lim_{A \to \infty} \left(\frac{\lambda_u - A}{e^{-\lambda_u} - e^{-A}} \right) = 0 \tag{4-114}$$

then equation (4-113) can be simplified to:

$$\tau(\lambda_u, B_u) = \frac{1}{\mu(a_1)} (B_u - \lambda_u) \tag{4-114}$$

The remaining point of interest is then to derive an expression between the upper boundary B_u and the value λ_u for a specified time T between two false alarms through the relation $T(\lambda_u, B_u)$. The mean time is defined as $T = E(T_F)$, the expectation of the time interval T_F between two false alarms under a stationary regime of observation in the normal mode. The time T_F is defined as the total time elapsed until the trajectory defined in the mode H_o exits through B_u the first time, and a false alarm is signaled. With this definition, the relation $\tau(\lambda_u, B_u)$ can be derived by an application of (4-111) as:

$$T(\lambda_u, B_u) = \frac{1}{\mu(a_o)} \left\{ B_u + (\lambda_u - A) \left(\frac{1 - e^{B_u}}{e^{\lambda_u} - e^{A}} \right) \right\} \tag{4-115}$$

where the parameters $\mu(a)$ and h are defined to be $\mu(a_o)$ in (4-108) and -1. The initial condition is assumed to be zero, and the parameter A in this case is in fact the "generalized" sample value of the state λ_k according to the defined control law.

For a specified T, a relation between B_u and λ_u can be obtained by derivation of an expression $dB_u/d\lambda_u$. Using the condition $dT/d\lambda_u = 0$, one obtains:

$$\frac{dB_u}{d\lambda_u} = \frac{\left\{ \frac{(\lambda_u - A)\, e^{\lambda_u}}{(e^{\lambda_u} - e^{A})^2} - \frac{1}{(e^{\lambda_u} - e^{A})} \right\} (e^{B_u} - 1)}{\left\{ \frac{(\lambda_u - A)}{(e^{\lambda_u} - e^{A})} e^{B_u} - 1 \right\}} \tag{4-116}$$

The relation $dB_u/d\lambda_u \geqq 0$ can be verified by use of the inequalities:

$$\begin{aligned} x &\geq 1 - e^{-x} \qquad (\text{for } x \geq 0) \\ x &\leq e^{x} - 1 \end{aligned} \tag{4-117}$$

One notices that ($\lambda_u \leqq 0$, $B_u > 0$)

$$\frac{(\lambda_u - A)}{(e^{\lambda_u} - e^{A})} = e^{-\lambda_u}\left\{\frac{(\lambda_u - A)}{1 - e^{-(\lambda_u - A)}}\right\} > 1 \qquad (4\text{-}118)$$

by identifying $x \triangleq (\lambda_u - A) \geqq 0$, and $\lambda_u \leqq 0$. The assumption $\lambda_u > A$ is derived by the information provided by the optimal control problem formulation under equation (4-107). The relation (4-119) can also be established:

$$\frac{(\lambda_u - A)\, e^{\lambda_u}}{(e^{\lambda_u} - e^{A})^2} - \frac{1}{(e^{\lambda_u} - e^{A})}$$

$$= \left(\frac{1}{e^{\lambda_u} - e^{A}}\right)\left\{\frac{(\lambda_u - A)}{\left[1 - e^{-(\lambda_u - A)}\right]} - 1\right\} > 0 \qquad (4\text{-}119)$$

by use of the relation $(e^{\lambda_u} - e^{A}) > 0$, and the inequalities (4-117).

Moreover, an upper bound of $dB_u/d\lambda_u$ can be derived. Write

$$Y = \frac{(\lambda_u - A)\, e^{\lambda_u}}{(e^{\lambda_u} - e^{A})^2} - \frac{1}{(e^{\lambda_u} - e^{A})}$$

$$Z = \frac{(\lambda_u - A)}{(e^{\lambda_u} - e^{A})}$$

Then one can obtain

$$\frac{dB_u}{d\lambda_u} = \frac{Y(e^{B_u} - 1)}{(Z\, e^{B_u} - 1)}$$

$$= \frac{Y(e^{B_u} - 1)}{Z(e^{B_u} - 1) + (Z - 1)} < \frac{Y}{Z} \qquad (4\text{-}120)$$

by use of the relation (4-118). It can further be shown that

$$Y < Z \tag{4-121}$$

For verification of (4-121), it is sufficient to show

$$\frac{(\lambda_u - A)}{(1 - e^{-(\lambda_u - A)})} < (\lambda_u - A) + 1$$

since $(e^{\lambda_u} - e^{A}) > 0$. It suffices to show in turn that:

$$\left\{ \frac{(\lambda_u - A)\, e^{\lambda_u}}{(e^{\lambda_u} - e^{A})} - 1 \right\} < (\lambda_u - A). \tag{4-122}$$

Equation (4-122) can be simply proved by identifying

$$x \triangleq (\lambda_u - A) > 0$$

and using the relation (4-117)

$$x < e^{x} - 1 \qquad (x > 0)$$

Therefore, the following bounds can be established:

$$0 \leq \frac{dB_u}{d\lambda_u} \leq 1 \tag{4-123}$$

The monotonicity property of the mean time $\tau(\lambda_u, B_u)$ of delay in detection can now be derived. Let

$$\begin{aligned} \lambda_u &= A + \zeta_u \\ \lambda_v &= A + \zeta_v \end{aligned} \tag{4-124}$$

where ζ_u and ζ_v are two admissible controls defined by the admissible control region U_λ in the $\lambda(t)$-state space. A simple transforma-

tion of the admissible region U in the $\pi(t)$-state space will give the corresponding region U_λ as $U_\lambda = \{\lambda : \lambda \leqq 0\}$. By (4-114), the following relation can be obtained

$$\tau(\lambda_u, B_u, T) = \frac{1}{\mu(a_1)} (B_u - \lambda_u) \tag{4-125}$$

$$\tau(\lambda_v, B_v, T) = \frac{1}{\mu(a_1)} (B_v - \lambda_v) \tag{4-126}$$

where B_u and B_v are derived for a specified mean time T. The use of the explicit argument T in the notation τ is to emphasize its dependence on T.

Let

$$\Delta\tau = \tau(\lambda_v, B_v, T) - \tau(\lambda_u, B_u, T)$$

Then in order to prove $\tau(\lambda_u, B_u, T)$ to be a monotonic function in λ_u, it is equivalent to prove the condition $\Delta\tau \leqq 0$ for $\lambda_v \geqq \lambda_u$. By use of (4-123), (4-124), and the condition $\mu(a_1) > 0$, one obtains

$$\begin{aligned} \Delta\tau &= \frac{1}{\mu(a_1)} \{B_v - \lambda_v - (B_u - \lambda_u)\} \\ &= \frac{1}{\mu(a_1)} \{(\lambda_u - \lambda_v) - \int_{\lambda_v}^{\lambda_u} \frac{dB}{d\lambda} \cdot d\lambda\} \\ &\leq 0 \end{aligned} \tag{4-127}$$

for $\lambda_v \geqq \lambda_u$ since $\frac{dB}{d\lambda} < 1$. This proves that $\tau(\lambda_u, B_u, T)$ is a monotonically decreasing function in its argument λ_u.

It is noticed that the above conclusion can be extended to the case for any value $A (A \leqq 0)$, since the monotonicity is proved without reference to a particular value A. By identifying the parameter A as the generalized sample value of the state λ_k, the feedback control law of the $\lambda(t)$-process for the suboptimal system in the discrete-time case

can be defined as:

$$\zeta_k^*(\lambda_k) = \begin{cases} -\lambda_k & \lambda_k < 0 \\ 0 & \lambda_k \geq 0 \end{cases} \tag{4-128}$$

The suboptimal control ζ_k^*, corrects the current state λ_k to the boundary $\lambda = 0$ of the admissible control region U_λ, whenever the state λ_k is negative, and no control is applied if $\lambda_k \geq 0$. The equivalent control law of the $\pi(t)$-process can be defined as

$$u_k^*(\pi_u) = \begin{cases} (\frac{1}{2} - \pi_k) & \text{if } \pi_k \in (0,\ 1/2) \\ 0 & \text{if } \pi_k \in (1/2,\ 1) \end{cases} \tag{4-129}$$

The suboptimal control u_k^* corrects the current state π_k to the boundary $\pi = 1/2$ of the admissible control region U whenever the state $\pi_k \in (0, 1/2)$, and no control is applied otherwise.

The derived suboptimal control sequence $\{u^*\} = \{u_k^*, u_{k+1}^* \ldots\}$, where each u_k^* is derived according to the control law (4-128) or (4-129) is to be implemented in the suboptimal detection system. It is noticed that each control u_k^* is defined under the assumption that the degradation will occur at the instant k^+, while the real system will actually be degraded at an unknown instant. The detection performance of the suboptimal system will be derived under the implementation control sequence $\{u^*\}$ by averaging over all possible degradation states. This will be discussed in Section 4.5.3.

The suboptimal control law will give an intuitive explanation for the improvement of the detection performance. Consider the trajectory of the $\lambda(t)$-process. The control is equivalent to resetting the process $\lambda(t)$ to zero whenever it drifts to a negative value. Thus, no accumulation of negative likelihood ratio will exist. This implies that a transition from the normal mode to the degraded mode, if it happens, always occurs near the state $\lambda_k = 0$ in the suboptimal detection system. The resets carried out by controls initialize the occurrence of degradations in the detection process to an "equivalent" case with no transi-

tions. This gives an intuitive explanation of why the control sequence can provide an improved detection performance. However, the control sequence, on the other hand, increases the false alarm errors of the system, if the same decision boundary for detection is held. A detailed discussion of detection performance will be presented in Section 4.5.3.

The suboptimal control sequence in the discrete-time case is realized only at the sampling instants. The logical step toward improving the detection performance is to require that controls be carried out continuously. However, it is difficult to justify the extension of results in the discrete-time case to the continuous-time system. In the continuous system, a physically unrealizable continuous control $u(\pi_t)$ of infinite size will be required in order to compensate instantaneously the process π_t of a diffusion nature as the sampling time interval approaches zero. Moreover, in the derivation of the suboptimal control law in the discrete case, the state equation is used during the sampling interval as if no controls were applied. But no corresponding assumption can be justified in the continuous time case. The derivation of the state equation with continuous controls in a nonlinear stochastic system, and the verification of the conditions on existence of optimal stochastic controls (the continuous control in this case does not satisfy the uniform Lipschitz condition) are difficult tasks to be resolved.

The suboptimal continuous control problem will then be approximated by a control sequence in the sense that there exist control functions resulting in a risk function Q^{ϵ} arbitrarily close to the risk Q^{*} in the continuous suboptimal control system.

$$Q^{\epsilon}(u^{\epsilon}) - Q^{*}(u^{*}) < \epsilon \qquad \epsilon > 0 \tag{4-130}$$

It is trivial to find the solution u^{ϵ}. It is the corresponding control in the discrete time case with the sampling interval Δ selected arbitrarily small, yet finite. The concept applied here is similar to the one approximating a continuous optimal control by a so-called "ϵ-optimal"

control. The existence of the ϵ-optimal control has been discussed in the reference (Stratonovich[73]).

A block diagram of the discrete-time suboptimal detection system is illustrated in Fig. 4.4.

4.4.5 Identification of Polarity Information

In this section the problem of polarity identification of the mean bias will be discussed. A method to detect the degradation mode of unknown polarity by an averaging process will first be reviewed. It will be shown that the detection efficiency is attenuated by the averaging process. Moreover the test did not identify the polarity sign, which is useful information for the identification stage and for the efficient isolation of the degraded gyro as was discussed in Chapter 2. It will then be shown that the suboptimal system can be simply modified to detect the degradation with simultaneous identification of unknown polarity of mean bias, with little additional computation.

For testing the mean of unknown sign of a normal distribution with known variance σ, Wald[80] proposed the following procedure. Let a positive value b be specified to define the degradation mode H_1:

$$|\theta - \theta_0| \geq b$$

where θ_0 is the given mean under hypothesis H_0. The probability density of the sample $(x_1, \ldots, x_n)$ with independent and identical distribution under H_0 is given by

$$p_{0n} = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sigma^n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta_0)^2} \tag{4-131}$$

The density function p_{1n} under H_1 is defined as some weighted average of the probability densities corresponding to various values of θ in the definition of the degradation mode H_1. It was shown by Wald that an optimum weighted average is the simple average of two density functions corresponding to $\theta = \theta_0 \pm b$:

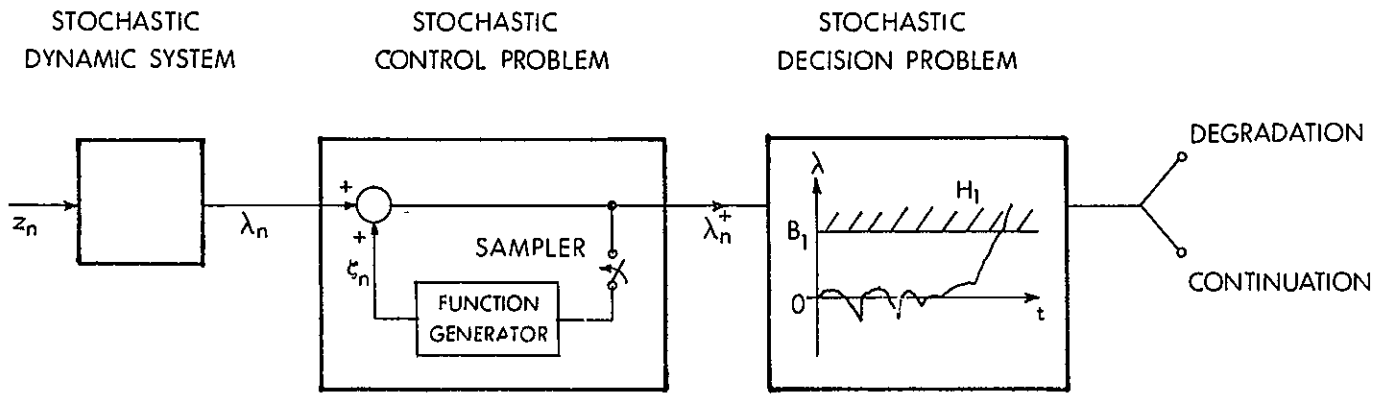


Fig. 4.4 Block Diagram of Sub-Optimal Detection System

$$p_{1n} = \frac{1}{2}\left[\frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0+b)^2}\right. \tag{4-132}$$

$$\left. + \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0-b)^2}\right]$$

The test is then carried out as follows: Continue taking observations as long as $A < (p_{1n}/p_{0n}) < B$. If $p_{1n}/p_{0n} \geqq B$, H_1 is accepted. If $p_{1n}/p_{0n} \leqq A$, H_0 is accepted. Here A and B are the specified lower and upper boundary respectively.

This is a reasonable approach if one has no more a priori information. However, the averaging process will significantly reduce the likelihood ratio sensitivity and thus attenuate the detection efficiency. This is illustrated in Fig. 4.5. For simplicity of representation θ_0 is assumed to be zero, and the probability density function is illustrated in a single observation. Suppose an observation x_1 has been taken, the likelihood ratio function $\frac{p_1(x_1)}{p_0(x_1)}$ of the averaging process will be attenuated by a ratio of $\frac{1}{2} \cdot \frac{[p_1(x_1 \mid H_1^+) + p_1(x_1 \mid H_1^-)]}{p_1(x_1 \mid H_1^+)}$

in comparison with the case of the identified polarities. Moreover the computation of the likelihood ratio function will be complicated by the presence of the addition term in (4-132).

It will be shown that in the suboptimal system a decision rule can be defined to detect the degradation mode with simultaneous identification of polarity. The modified detection system is the suboptimal system but conditioned on both polarities. Because of the uncertainty feedback the suboptimal detection system is defined practically by a single-side boundary for detection for degradation mode only. With the system conditioned on both polarities two boundaries are defined with each one specifying a given polarity. Thus a decision rule can be defined for identification of the unknown polarity. For simplifica-

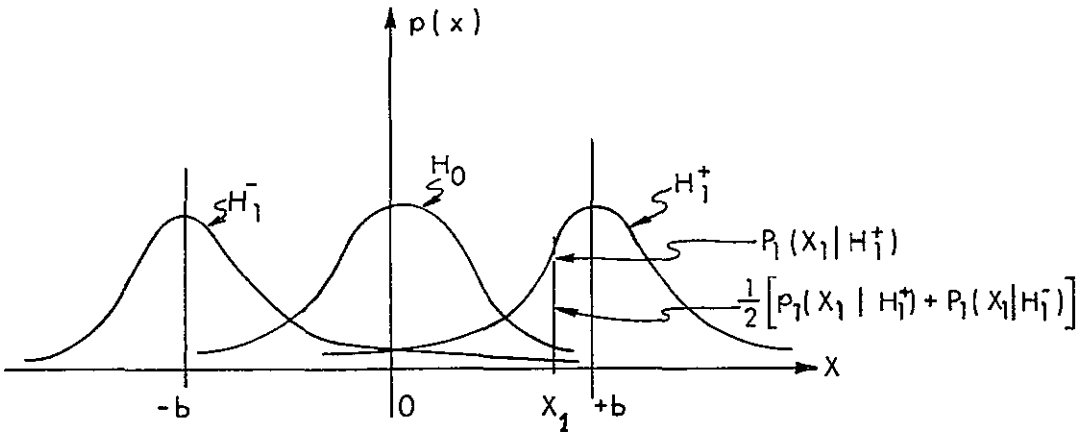


Fig. 4.5 Attenuation Effect of Averaging Process.

tion of presentation Wald's sequential test process will be used for illustration.

Let the normal distribution of the observations under consideration be defined as follows:

Normal mode H_0: $N(0, \sigma)$ (4-133)

Degradation mode with positive polarity H_1^+: $N(b, \sigma)$

Degradation mode with negative polarity H_1^-: $N(-b, \sigma)$

where b is a positive number, and σ is the given standard deviation.

Define the likelihood ratio function for degradation mode with positive polarity as

$$\ell i = \ell n \frac{p(x_i | H_1^+)}{p(x_i | H_0)} = \frac{b}{\sigma^2} \left(x_1 - \frac{b}{2}\right) \quad (4\text{-}134)$$

and for degradation mode with negative polarity as

$$m_i = \ell n \frac{p(x_i \mid H_1^-)}{p(x_i \mid H_0)} = -\frac{b}{\sigma^2}\left(x_i + \frac{b}{2}\right) \tag{4-135}$$

The test procedure for detecting mode H_1^+ is carried out as follows. Compute the likelihood ratio function of the joint probability density $p_n(\cdot)$:

$$\begin{aligned} \ell n \frac{p_n(\cdot \mid H_1^+)}{p_n(\cdot \mid H_0)} &\triangleq \sum_{i=1}^{n} \ell_i \\ &= \left[\sum_{i=1}^{n} x_i - \frac{n}{2} \cdot b\right] \cdot \frac{b}{\sigma^2} \\ &\triangleq \Lambda_n \cdot \frac{b}{\sigma^2} \end{aligned} \tag{4-136}$$

Observation is terminated with acceptance of mode H_1^+ if

$$\Lambda_n > \frac{\sigma^2}{b} \cdot B \tag{4-137}$$

and observation is terminated with acceptance of mode H_0 if

$$\Lambda_n < \frac{\sigma^2}{b} \cdot A \tag{4-138}$$

Otherwise continue the observation. Here B (> 0) and A (< 0) are the specified upper and lower boundary respectively. The test procedure for detecting mode H_1^- can be carried out similarly. Compute the likelihood ratio function:

$$\begin{aligned} \ell n \frac{p_n(\cdot \mid H_1^-)}{p_n(\cdot \mid H_0)} &\triangleq \sum_{i=1}^{n} m_i \\ &= -\left(\sum_{i=1}^{n} x_i + \frac{nb}{2}\right) \cdot \frac{b}{\sigma^2} \end{aligned} \tag{4-139}$$

(equation cont'd)

$$\triangleq -M_n \cdot \frac{b}{\sigma^2}$$

Accept the degradation mode H_1^- if

$$M_n < -\frac{\sigma^2}{b} \cdot B \tag{4-140}$$

and accept the normal mode H_0 if

$$M_n > -\frac{\sigma^2}{b} \cdot A \tag{4-141}$$

Otherwise continue the observation process.

Rewrite Λ_n and M_n in recursive forms:

$$\Lambda_n = \Lambda_{n-1} + (x_n - \frac{b}{2}) \tag{4-142}$$

$$M_n = M_{n-1} + (x_n + \frac{b}{2}) \tag{4-143}$$

Now suppose the real system has been degraded with a mean bias of positive polarity, then the value Λ_n on average must be positive. On the other hand, if $\Lambda_n \leqq 0$, then there are two cases: either the system is operated in the normal mode H_0 or in the degraded mode H_1^-. In the first case, one is not interested in verification of H_0, and in the second case Λ_n is not the correct decision function to apply. In both cases, one can apply the optimal uncertainty feedback control in the suboptimal system to reset $\Lambda_n = 0$.

Now suppose the real system has been degraded with a mean bias of negative polarity, then the value M_n must be negative. On the other hand, if $M_n \geqq 0$, then either the system is operated in the normal mode or in the degraded mode H_1^+. In both cases one can apply the optimal uncertainty feedback control in the suboptimal system to reset M_n to be zero. In summary, the following disjoint decision rules can be defined:

If $\Lambda_n > 0$, Λ_n is used as the decision function for the detection system until it crosses the upper boundary. Then one accepts the degraded mode H_1^+ and identifies the bias to be positive polarity.

If $M_n < 0$, M_n is used as the decision function for the detection system until it crosses the lower boundary. One accepts the degraded mode H_1^- and identifies the bias to be negative polarity.

If $\Lambda_n < 0$, (or $M_n > 0$), Λ_n (or M_n) is reset to zero.

4.5 Performance of the Detection System

4.5.1 Choice of Performance Criteria

In this section, the performance of the detection system is studied. It will be shown that for a sequential detection system with only a single-side boundary for degradation detection the meaningful performance criteria are the mean delay time in detection and the false alarm error probability. The definition of the false alarm error probability of the detection system is discussed, and a deterministic relation between the false alarm error probability and the mean time between two false alarms is mentioned.

It has been discussed in sections 4.2 and 4.3 that the optimal decision problem consists of observing the posterior probability and comparing it against threshold (boundaries) to determine the decision. The values of the thresholds are dependent upon the selection of specific performance criteria. The commonly used criteria for decision performance are the false alarm error probability α and the miss alarm error probability β. It is noted that the error probabilities α and β are defined only when appropriate decisions of H_0 and H_1 have been made, such that the corresponding thresholds can be determined. However, for the case of the detection systems discussed in section 4.4, where no explicit decision on the normal mode will be made, then the threshold for decision on the normal mode is not determined, and only a single threshold for decision on the degradation mode H_1 is defined. It will be shown that in this case only the false alarm error probability α is well-defined, while the miss alarm error probability β will be shown to be zero. The result is

to be proved by application of the theory of first passage times for diffusion processes. (Darling and Siegert[14]).

Consider the homogeneous diffusion process $\Lambda(t)$ with the transition coefficient μ and the diffusion coefficient b:

$$d\Lambda(t) = \mu\, dt + b\, dw \tag{4-144}$$

with $\Lambda(0) = \lambda_0$. It is required to derive the first passage time T of $\Lambda(t)$ to the boundary $\gamma > \lambda_0$. The first passage time T is defined by the events:

$$\Lambda(0) = \lambda_0,\ \Lambda(t) < \gamma\ \ (0 < t < T),\ \Lambda(T) = \gamma \tag{4-145}$$

It is convenient to study the first passage time by the use of an absorbing barrier at γ. Let the transition probability density be denoted by

$$p(\lambda_0, \lambda, t) \triangleq p_{\Lambda}\ (\lambda_0,\ t_0; \lambda, t) \tag{4-146}$$

The density function $p(\lambda_0;\ \lambda,\ t)$ represents the transition probability density that $\Lambda(0) = \lambda_0$ and $\Lambda(t) = \lambda$ and that the process $\Lambda(\cdot)$ does not reach the barrier γ in the time interval $(0, t)$. It is well known that the transition probability density satisfies the forward and backward Kolmogorov equations:

$$\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\ \{b^2 \cdot p(\lambda_0,\lambda,t)\} - \frac{\partial}{\partial\lambda}\ \{\mu \cdot p(\lambda_0,\lambda,t)\} = \frac{\partial p(\lambda_0;\lambda,t)}{\partial t} \tag{4-147}$$

and

$$\frac{1}{2} b^2 \frac{\partial^2}{\partial\lambda_0^2}\ \{p(\lambda_0,\lambda,t)\} + \mu \frac{\partial}{\partial\lambda_0}\ \{p(\lambda_0,\lambda,t)\} = -\frac{\partial p(\lambda_0,\lambda,t)}{\partial t_0}$$

$$= \frac{\partial p}{\partial t}(\lambda_0,\lambda,t) \tag{4-148}$$

Let

$$P(\lambda_0,\lambda,t) \triangleq \Pr\ \{\Lambda(\tau) < \gamma,\ \text{for } 0 < \tau < t,\ \Lambda(t) \leqq \lambda \mid \Lambda(0)=\lambda_0\}$$

$$= \int_{-\infty}^{\lambda} p(\lambda_0, x;t)dx \tag{4-149}$$

$$P(\lambda_0,\gamma;t) = \text{Prob. } (T \geqq t)$$

The distribution function $P(\lambda_0, \lambda, t)$ also satisfies the backward equation, as can be seen by integrating over the forward variable in the backward equation.

The backward equation is the appropriate one to use for passage times since it is required to determine the passage time distribution as a function of the initial state λ_0 for a fixed terminal state γ. Let $\tilde{g}(t \mid \lambda_0, \gamma)$ be the probability density function of T, then

$$\tilde{g}(t \mid \lambda_0,\gamma) = \frac{\partial}{\partial t} P(T \leqq t) = -\frac{\partial}{\partial t} P(\lambda_0,\gamma;t) \tag{4-150}$$

Substituting (4-150) into the backward equation for $P(\lambda_0, \lambda, t)$ produces:

$$\frac{1}{2} b^2 \frac{\partial^2}{\partial \lambda_0^2} \{\tilde{g}(t \mid \lambda_0, \gamma)\} + \mu \frac{\partial}{\partial \lambda_0} \{\tilde{g}(t \mid \lambda_0, \gamma)\} = \frac{\partial}{\partial t} \{\tilde{g}(t \mid \lambda_0, \gamma)\} \tag{4-151}$$

The partial differential equation (4-151) can be solved by Laplace transformation. Define the transform of $\tilde{g}(\cdot)$ as

$$\bar{g}(s \mid \lambda_0, \gamma) = \int_0^{\infty} e^{-st}\, \tilde{g}(t \mid \lambda_0, \gamma)\, dt \tag{4-152}$$

To ensure the existence of the limit of $\bar{g}(s \mid \lambda_0, \gamma)$ the condition $\mathrm{Re}(s) \geqq 0$ must be satisfied. Equation (4-152) is transformed into the ordinary differential equation:

$$\frac{1}{2} b^2 \frac{d^2\bar{g}}{d\lambda_0^2} + \mu \frac{d\bar{g}}{d\lambda_0} = s\bar{g} - \tilde{g}(t_0 \mid \lambda_0, \gamma) \tag{4-153}$$

But the term $\tilde{g}(t_0 | \lambda_0, \gamma) = \delta(\gamma - \lambda_0)$ is zero, if $\lambda_0 < \gamma$. Then the general solution of which is:

$$\bar{g}(\lambda_0) = A(s)\, e^{\lambda_0 \theta_1(s)} + B(s)\, e^{\lambda_0 \theta_2(s)} \qquad (4\text{-}154)$$

where $\bar{g}(\lambda_0) \triangleq \bar{g}(s | \lambda_0, \gamma)$

$$\theta_1(s),\ \theta_2(s) = \frac{-\mu \mp \sqrt{\mu^2 + 2s \cdot b^2}}{b^2} \qquad (4\text{-}155)$$

where $\theta_1(s) < 0 < \theta_2(s) \qquad (\mathrm{Re}(s) \geqq 0)$

$$\theta_1(0) = -2\mu / b^2,\ \theta_2(0) = 0 \qquad (\mu \geqq 0) \qquad (4\text{-}156)$$

$$\theta_1(0) = 0,\ \theta_2(0) = 2\,|\mu| / b^2 \qquad (\mu < 0)$$

To determine the coefficients A(s) and B(s), it is noted that due to boundedness $\bar{g}(\lambda_0)$ the coefficient A(s) must be zero, otherwise $\bar{g}(\lambda_0) \to \infty$, as $\lambda_0 \to -\infty$ $(\mathrm{Re}(s) \geqq 0)$. Furthermore for $\lambda_0 = \gamma$, then by definition, absorption occurs immediately and so $\bar{g}(\gamma) = 1$. This gives

$$\bar{g}(\lambda_0) = \bar{g}(s | \lambda_0, \gamma) = e^{(\lambda_0 - \gamma) \cdot \theta_2(s)} \qquad (\lambda_0 \leqq \gamma) \qquad (4\text{-}157)$$

Define $\pi(\lambda_0, \gamma)$ as the probability of ever reaching γ when starting from $\lambda_0 < \gamma$, then $\pi(\lambda_0, \gamma)$ can be simply evaluated from (4-157) by setting s=0,

$$\pi(\lambda_0, \gamma) = \int_0^\infty \tilde{g}(t | \lambda_0, \gamma)\, dt = \bar{g}(0 | \lambda_0, \gamma) \qquad (4\text{-}158)$$

Substituting (4-156) and (4-157) into (4-158) produces:

$$\pi(\lambda_0, \gamma) = \begin{cases} 1 & (\mu \geq 0) \\ e^{-2(\gamma - \lambda_0)|\mu|/b^2} & (\mu < 0) \end{cases} \tag{4-159}$$

This proved the following statement. For the diffusion process $\Lambda(t)$ with the transition coefficient μ and the initial condition λ_0, the probability of ultimate absorption of the process $\Lambda(t)$ in the boundary γ ($\gamma > \lambda_0$) is unity when there is a drift toward the boundary $\gamma(\mu > 0)$ or no drift ($\mu = 0$). When the drift μ is away from the boundary ($\mu < 0$) there is a non-zero probability of reaching the boundary γ.

The above result can be immediately applied to the case of the sequential system with a single threshold for detecting the degradation H_1. For simplicity of presentation the linear system of Wald SPRT will be considered. In this case the detection procedure consists in observing the likelihood ratio function $\lambda(t)$ and making decision H_1 if the process $\lambda(t)$ exits through the upper boundary γ. Consider the stochastic differential equation of $\lambda(t)$:

$$d\lambda(t) = -\frac{1}{2}\frac{a_1^2}{\sigma^2}dt + \frac{a_1}{\sigma^2}dy(t) \qquad \lambda(0) = \lambda_0 \tag{4-160}$$

and the observation process is:

$$\begin{aligned} dy(t) &= a_1\,dt + \sigma\,dw(t) && (y(t) \in H_1) \\ dy(t) &= \sigma\,dw(t) && (y(t) \in H_0) \end{aligned} \tag{4-161}$$

Substituting (4-161) into (4-160) yields:

$$\begin{aligned} d\lambda(t) &= \frac{1}{2}\frac{a_1^2}{\sigma^2}dt + \frac{a_1}{\sigma}dw(t) && (y(t) \in H_1) \\ d\lambda(t) &= -\frac{1}{2}\frac{a_1^2}{\sigma^2}dt + \frac{a_1}{\sigma}dw(t) && (y(t) \in H_0) \end{aligned} \tag{4-162}$$

with $\lambda(0) = \lambda_0$

On identifying (4-162) with (4-144), it can be seen that $\mu \triangleq \frac{1}{2}\frac{a_1^2}{\sigma^2} > 0$ and $\lambda_0 < \gamma$ for the system defined in the degraded mode H_1. Thus for the sequential procedure where the detection time is not fixed, the probability of ultimate detection is unity, i.e. $\beta = 0$. However, if the real system is defined in the normal mode such that there exists a negative drift rate, there is a non-zero probability of reaching the upper threshold γ to make the false decision on the degradation mode H_1. Moreover suppose the real system is defined as:

$$dy(t) = a\, dt + \sigma\, dw(t) \tag{4-163}$$

where a is an unknown parameter.

Substituting (4-163) into (4-160) produces

$$d\lambda(t) = \frac{a_1}{\sigma^2}\left(a - \frac{1}{2}a_1\right) dt + \frac{a_1}{\sigma} dw(t) \tag{4-164}$$

$$\lambda(0) = \lambda_0$$

Assuming that $a_1 > 0$, the relation

$$\mu = \frac{a_1}{\sigma^2}\left(a - \frac{1}{2}a_1\right) \geqq 0 \tag{4-165}$$

implies that $a \geqq \frac{1}{2}a_1$.

A simple application of the previous analysis will lead to the following statement: If the instrument is defined to be operating in the degradation mode on the condition that the real output data of the instrument show a mean bias magnitude greater than half of the "design" value a_1, then the miss alarm error probability will be zero in the single side sequential detection system. Thus the meaningful performance criteria in this case are the mean delay time in detection and the false alarm error probability.

The false alarms are presentation of degradation signals when the boundary of detection for degradation was reached, while the operating system is defined in the normal mode H_0. The false alarm error probability α of the detection system is defined as follows:

$$\begin{aligned} \alpha &= \Pr(\nu < \theta) \\ &= \int_0^\infty P(\nu < t \mid H_0) \cdot dP_\theta(t) \end{aligned} \tag{4-166}$$

In the equation ν is the time of decision that a degradation has occurred, while θ is the instant of occurrence of the true degradation with the a priori distribution:

$$P(\theta > t) = e^{-qt} \tag{4-167}$$

Two more criteria of characterizing false alarms are of interest:

N: the mathematical expectation of the number of false alarms before the time of the occurrence of a true degradation.

T: mean time between two false alarms.

For the assumed distribution (4-167), a simple deterministic relation N, T and α can be developed.

By use of (4-166) and (4-167) the following relation can be established:

$$\begin{aligned} (1-\alpha) &= P\ (\nu \geqq \theta) \\ &= \int_0^\infty P_0\ (\nu \geqq t)\, d\, P_\theta(t) \\ &= q \int_0^\infty P_0\ (\nu \geqq t)\ e^{-qt}\, dt \end{aligned} \tag{4-168}$$

where $P(\nu \geqq t \mid H_0)$ is denoted as $P_0(\nu \geqq t)$. Expanding the exponential e^{-qt} produces

$$\left(\frac{1-\alpha}{q}\right) = \int_0^\infty P_0(\nu \geqq t)\, dt + 0\ (q) \tag{4-170}$$

Using definition T in (4-173) and integration by parts one can show

$$\int_0^{\infty} P_0(\nu \geq t)\, dt = \int_0^{\infty} t\, d\, P_0(\nu < t) \triangleq T \qquad (4\text{-}170)$$

Substituting (4-170) into (4-169) provides

$$\left(\frac{1-\alpha}{q}\right) = T + 0(q)$$

As $q \to 0$ with T fixed, i.e., the mean time of occurrence of a true degradation goes to infinity, then it is reasonable to assume that $\alpha \to 1$. In this case for any small $\epsilon > 0$ there exists a q_1 such that for $q < q_1$ the following relation holds true:

$$\left| \left(\frac{1-\alpha}{\alpha q}\right) - T \right| < \epsilon$$

Thus for $q < q_1$ the relation can be established:

$$\left(\frac{1-\alpha}{\alpha}\right) \cong q\,T \qquad (4\text{-}171)$$

A deterministic relation between N and α can also be developed.

The detection system is characterized by the multi-stage decision process. At each time an alarm signal occurs a verification is to be made by the identification system. The detection process terminates if a degradation is verified, and repeats if a false alarm is discovered. Because of the useful property of the exponential distribution:

$$P(\theta > s + \nu_i \mid \theta > \nu_i) = P(\theta > s)$$

The conditional distribution of the process θ, given that a false alarm has been discovered at an instant ν_i, coincides with the unconditional distribution, even if ν_i may be a random variable. Therefore until a true degradation has been verified, each detection stage can be regarded as identical and independent with the only difference being that one

more false alarm has been discovered.

Let A_1 be the event that i false alarms have been discovered before a true degradation has been verified. Then by definition

$$\begin{aligned} N &= \sum_{i=1}^{\infty} i \Pr \{A_1\} \\ &= \sum_{i=1}^{\infty} i \alpha^i (1-\alpha) \\ &= \alpha(1-\alpha) \sum_{i=1}^{\infty} i \alpha^{(i-1)} \\ &= \frac{\alpha}{1-\alpha} \end{aligned} \tag{4-172}$$

Since the case of most interest in practice is the one $q \to 0$, then the value of the error probability α of false alarms in detection system is usually incompatibly large in comparison with the specification of the false alarm error probability α in the Wald's sequential analysis process. In the Wald's case, the error probability α is always specified to be less than 1/2, and is usually specified to be an order of 1-5%. Thus for detection systems, the parameter T is usually the more meaningful criterion for the false alarm error specification. Since the parameters T, N and α are related by the deterministic relations, they are equivalent in mathematical formulation of the detection system.

4.5.2 Derivation of Performance of Optimal and Suboptimal Systems

Two approaches can be applied for derivation of the performance of the detection system. One is to formulate the detection system as a Bayesian problem and the performance is derived by solution of the Bayesian risk function (Stratonovich[73], and Shiryaev[64]). The other approach is to derive the performance by application of the theory of first passage times for diffusion processes. The latter approach will be used in this thesis, since it gives more physical insight and gives

a unique formulation for comparison of performance of the optimal and suboptimal system. The solution to the first passage problem for a diffusion process was given in an early paper by Darling and Siegert[14]. The result was extended to the solution of the performance (operating characteristics) in Wald's sequential analysis. The sequential analysis was originally formulated in the discrete form by Wald[80]. The exact solution of performance is analytically difficult. Wald solved the problem by developing a fundamental identity (Wald's identity) and by neglecting a possible excess of the likelihood ratio function over the decision boundary at the final absorption point. Darling and Siegert formulated the problem by approximating the discrete process in a continuous stochastic process, and derived the asymptotic solution by application of the Kolmogorov equations of a diffusion process. Aside from the difficulty of justifying the limiting operations, it turns out that this procedure leads to simple solutions. In this section the general formulation for the derivation of the detection system performance will first be presented. The main results of the optimal system performance by Shiryaev will then be mentioned. Finally, the derivation of the suboptimal system performance will be developed.

The problem of solution of detection performance is to derive the mean time delay in detecting the degradation under the constraint of a specified mean time between two false alarms. The problem is to be formulated as follows. Suppose the observation process y(t) defined in the normal mode is observed. With each sample function of the random process y(t), a random stopping time ν of detecting a "degradation" is observed. Associated with a whole ensemblance of observations, one can define the distribution F(u) of the stopping times ν.

Now consider the multi-stage observation method for the detection system defined in the normal mode H_0. The first stage of observation is defined in the usual manner. If after the first stopping time $\nu = \nu_1$, the observation is resumed according to the same rules as in the first stage of observation, and independently of results of the first stage, then the stopping time ν_2 does not depend on ν_1 and has the same

distribution F(u). A similar consideration applies to the duration ν_i of subsequent stages.

Let

$$\kappa_t = \max \{k: \nu_1 + \dots \nu_k < t\}$$

$$P_t(u) = P\{ [t - (\nu_1 + \dots \nu_{\kappa_t})] \leqq u\}$$

$$T = E_0(\nu) = \int_0^\infty u \, dF_\nu(u) \tag{4-173}$$

where $E_0(\cdot)$ means the expectation taken with respect to the observation process defined in the normal mode H_0. It is assumed that the appearance of a degradation is preceded by a long period of observations of the system in the normal mode, then in the case $T<\infty$, a stationary process of observations in the normal mode can be assumed to be established in the sense that

$$1 - P(u) = \lim_{t\to\infty} P \{ [t-(\nu_1 + \dots \nu_{\kappa_t})] > u\}$$

$$= \frac{1}{T} \int_u^\infty (1-F_\nu(x))dx \tag{4-174}$$

The equation (4-174) was proved in an early paper on renewal theory by Doob[19], in which $P(\cdot)$ is defined as the distribution function of the initial age, and $F(\cdot)$ the distribution function of lifetime.

Let

$$E_\delta(u) = E(\nu - u \mid \nu \geqq u) \tag{4-175}$$

where the conditional expectation is calculated under the assumption that the observation began at t=0, and is performed according to the decision rule δ, that a degradation appears at time t=u, and that $\nu \geqq u$ is the first stopping time. Then the quantity

$$\tau_t^{\delta}(T) = \int_0^t E_{\delta}(u) \; d\, P_t^{\delta}(u) \qquad (4\text{-}176)$$

characterizes the mean time delay in detecting a degradation which appears at the time θ=t, if the decision rule δ is used with a specified mean time T defined in (4-173). It is noted that the distribution $P_t^{\delta}(u)$ is calculated under the assumption of the system in the normal mode, and $E_{\delta}(u)$ is conditioned on the assumption that a degradation has appeared.

Define the quantity ($t \to \infty$)

$$\tau^{\delta}(T) = \int_0^{\infty} E_{\delta}(u)\, dP_{\delta}(u) \qquad (4\text{-}177)$$

as the mean delay time in detecting the degradation appearing against the background of an established stationary process under the normal mode. The problem is then to find the performance of the optimal method for which

$$\tau(T) = \inf_{\delta} \tau^{\delta}(T) \qquad (4\text{-}178)$$

for given T.

For clarity of presentation, the basic steps in performance derivation will be first discussed. It has already been shown that the optimal detection system consists of observing the posterior measure $\pi(\cdot)$ (or $\lambda(\cdot)$) and evaluating it against the threshold for detection. The diffusion process $\lambda(\cdot)$ will be used in performance derivation because of its simpler expression. Let $\Lambda_0(s)$ and $\Lambda_1(s)$ be the notation of the random process $\lambda(s)$ conditioned on the system defined in mode H_0 and H_1 respectively. The equation (4-177) will be explicitly written as:

$$\tau(T) = \int_0^{\infty} \tilde{t}(y, T)\, p(y)\, dy \qquad (4\text{-}179)$$

where p(y) is the one-dimensional stationary density function of the probability distribution established by the process $\Lambda_0(s)$. The quantity

$\tilde{t}(y, T)$ is the mathematical expectation of the first passage time at which the process $\Lambda_1(s)$ exits through the set decision boundary, under the assumption that $\Lambda_1(0) = y$ at the time of appearance of the degradation with a specified mean time T between false alarms. Since only the time delay after degradation is of interest, the degradation is assumed to occur at the instant s = 0.

Let the homogeneous diffusion process $\Lambda_0(s)$ be defined as

$$d\Lambda_0(s) = \mu_0(\Lambda_0(s))\, ds + \sigma_0(\Lambda_0(s))\, dw(s) \qquad (4\text{-}180)$$

It is noted that $\mu_0(\cdot)$ and $\sigma_0(\cdot)$ are not explicit functions of the time variable, as it has been shown for the process $\lambda(\cdot)$ in the untruncated case in section 4.2. Then the transition density function $p_{\Lambda_0}(\lambda_0, y, s)$ satisfies the Kolomogorov equations associated with the diffusion process $\Lambda_0(s)$. Since one is interested in the probability density of the forward variable y in (4-179), then it is appropriate to apply the forward equation. If the threshold for detection is denoted as γ, then it is equivalent to have an absorbing barrier at $y = \gamma$. Due to the definition of the transition probability of the forward variable, the appropriate boundary condition at γ is.

$$p_{\Lambda_0}(\gamma; \gamma, s) = 0 \quad \text{for } s>0 \qquad (4\text{-}181)$$

In the stationary case, the transition density function $p(\lambda_0; \gamma, s)$ will approach the stationary density p(y) with the appropriate boundary condition $p(\gamma) = 0$. This gives the approach for derivation of the density function p(y).

The distribution of the first passage time for the random process $\Lambda_1(s)$ is a rather complicated expression. However, if the first passage time is a proper random variable, that is, absorption (detection) is a certain event, then the moments of the passage time can be shown to be the solutions to simple differential equations by the following theorem for the homogeneous diffusion process (Darling and Siegert[14]).

Let $\Lambda_1(s)$ satisfy the homogeneous diffusion process:

$$d\Lambda_1(s) = \mu_1(\Lambda_1(s))ds + \sigma_1(\Lambda_1(s))\, dw(s) \qquad (4\text{-}182)$$

with $\Lambda_1(0) = y$, $B > y > A$, and A and B are the boundaries. Here $\mu_1(\cdot)$ and $\sigma_1(\cdot)$ are also not of explicit functions of time. Define the first passage time $T_{AB}(y)$ (crossing either the boundary A or B) as the random variable:

$$T_{AB}(y) = \sup \{\tilde{t} \mid A < \Lambda_1(s) < B,\ 0 \leqq s \leqq \tilde{t}\} \qquad (4\text{-}183)$$

Then if $T_{AB}(y)$ is a proper random variable whose moments of order $n \leqq n_0$ exist, $\tilde{t}^{(n)} = \tilde{t}^n_{AB}(y)$, the nth moment of $T_{AB}(y)$, satisfies the system:

$$\frac{1}{2}\sigma_1^2(y)\,\frac{d^2\tilde{t}^{(n)}}{dy^2} + \mu_1(y)\,\frac{d\tilde{t}^{(n)}}{dy} = -n\,\tilde{t}^{(n-1)}$$

$$\tilde{t}^{(0)} = 1 \qquad (4\text{-}184)$$

$$\tilde{t}^{(n)}_{AB}(A) = \tilde{t}^{(n)}_{AB}(B) = 0 \qquad n > 0$$

The theorem can be proved by substituting a series expansion for the moment generating function into the backward differential equation, and equaling the corresponding coefficients to zero. Since one is interested in the first passage time distribution to a fixed state of absorption as a function of the initial position $\Lambda_1(0) = y$, the backward equation provides the appropriate method. In particular if one is interested in the case of mean time for which n=1; then (4-184) is reduced to

$$\frac{1}{2}\sigma_1^2(y)\,\frac{d^2\tilde{t}}{dy^2} + \mu_1(\hat{y})\cdot\frac{d\tilde{t}}{dy} = -1$$

$$\tilde{t}(A) = \tilde{t}(B) = 0 \qquad (4\text{-}185)$$

The function $\tilde{t}(y, T)$ in (4-179) can be thus found as the solution to the equation (4-185).

With the general approach as the background, the performance of the detection system will be derived.

(1) Performance of the Optimal System

The performance of the optimal system has been derived by Shiryaev[64]. Here only the main results will be mentioned for reference, the details are left to the original paper.

It has been shown in Chapter 3 and section 4.3 that the optimal method for detecting the degradation is based on observing the stochastic equation:

$$d\lambda(t) = [q(e^{-\lambda(t)} + 1) - \frac{1}{2}\frac{a_1^2}{\sigma^2}]\,dt + \frac{a_1}{\sigma^2}\,dy(t) \qquad (4\text{-}186)$$

with the initial condition λ defined as

$$\lambda = \ell n\,(\frac{\pi}{1-\pi}) \qquad (4\text{-}187)$$

where π is the a priori probability that the system has been degraded. In Shiryaev's derivation it is assumed that $\pi \to 0$, then $\lambda \to -\infty$. The optimal boundary γ for detection can be derived from (4-47) in section 4.3, as

$$\gamma = \ell n\left\{\frac{1-\alpha\,(1-\pi)}{\alpha(1-\pi)}\right\} \tilde{=} \ell n(\frac{1-\alpha}{\alpha}) \qquad (4\text{-}188)$$

but it has been proved in (4-171)

$$\frac{1-\alpha}{\alpha} \cong q\,T \qquad (4\text{-}189)$$

Then
$$\gamma = \ell n(q\,T) \qquad (4\text{-}190)$$

Define

$$\rho(t) = \gamma - \lambda(t)$$

$$r \triangleq \frac{1}{2}\frac{a_1^2}{\sigma^2} \qquad (4\text{-}191a)$$

$$\frac{q}{r} \triangleq g \tag{4-191b}$$

$$\frac{qe^{-\lambda}}{r} = \frac{q \cdot \frac{1}{qT}}{r} = \frac{1}{rT} \triangleq c \tag{4-191c}$$

The equation (4-186) can be rewritten as:

$$d\rho(t) = r(1 - g - c\,e^{\rho(t)})dt - \frac{a_1}{\sigma^2}\,dy(t) \tag{4-192}$$

with the initial condition,

$$\rho(0) = \infty$$

and a degradation is signaled at the boundary γ and $\rho(\gamma(t)) = 0$.

The stochastic differential equation $\rho_0(t)$ for the system defined in the normal mode H_0 can be written as the diffusion process:

$$d\rho_0(t) = r(1 - g - c\,e^{\rho_0(t)})dt - \frac{a_1}{\sigma}\,dw(t) \tag{4-193}$$

since the observed process in the normal mode can be written as

$$dy(t) = \sigma\, dw(t)$$

The above equation can be further simplified by the transformation to a natural dimensionless scale s for time t:

$$s = rt \tag{4-194}$$

$$d\rho_0(s) = (1 - g - c\,e^{\rho_0(s)})\,ds - \sqrt{2}\,dw(s) \tag{4-195}$$

$$\rho_0(0) = \infty$$

and a false degradation will be detected at the boundary γ with $\rho_0(\gamma(s)) = 0$. The density p(y) can now be found by the forward equation:

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\{2\,p(y)\} - \frac{\partial}{\partial y}\{(1 - g - c e^{y}) \cdot p(y)\} = \frac{\partial p(y)}{\partial s} \qquad (4\text{-}196)$$

The equation for the stationary density function p(y) can be simplified to an ordinary differential equation with a variable coefficient:

$$\frac{d^2 p(y)}{dy^2} - \frac{d}{dy}\{(1 - g - c e^{y}) p(y)\} = 0 \qquad (4\text{-}197)$$

The unknown constants of the differential equation are removed by the condition (4-181) at the absorption boundary $\gamma = 0$,

$$p(\gamma) = p(0) = 0$$

and the normalizing condition,

$$\int_0^{\infty} p(y)\, dy = 1$$

The probability density p(y) can be solved as

$$p(y) = \frac{e^{f_0(y)} \int_0^y e^{-f_0(x)}\, dx}{\int_0^{\infty} e^{f_0(y)} [\int_0^y e^{-f_0(x)}\, dx]\, dy} \qquad (4\text{-}198)$$

where $$f_0(y) = \int_0^y (1 - g - ce^{z})\, dz \qquad (4\text{-}199)$$

To derive the function $\tilde{t}(y, T) \triangleq \tilde{t}(y)$, the stochastic differential equation for the system defined in the degradation mode H_1 must be used, the equation $\rho_1(s)$ can be easily derived from (4-192) to be

$$d\,\rho_1(s) = (-1 - g - c e^{\rho_1(s)})\, ds - \sqrt{2}\, dw(s) \qquad (4\text{-}200)$$

by using the relation

$$dy(t) = a_1\,dt + \sigma\,dw(t)$$

and by the time scale transformation (4-194).

The differential equation for $\tilde{t}(y)$ associated with the diffusion process (4-200) can be expressed as

$$\frac{d^2\tilde{t}(y)}{dy^2} + (-1 - g - c\,e^{y})\,\frac{d\tilde{t}(y)}{dy} = -1 \tag{4-201}$$

with the boundary conditions,

$$\tilde{t}(A) = \tilde{t}(B) = 0 \tag{4-202}$$

where $A = 0$, and $B \to \infty$.

Solving (4-201) produces

$$\tilde{t}(y) = \int_0^y \left[\int_z^\infty e^{f_1(x)}\,dx \right] e^{-f_1(z)} \cdot dz \tag{4-203}$$

where

$$\begin{aligned} f_1(x) &= \int_0^x (-1 - g - c\,e^{z})dz \\ &= -(1+g)\,x - c\,e^{x} + c \end{aligned} \tag{4-204}$$

Substituting (4-198) and (4-203) into (4-179) the detection performance in the time delay for the case $g \cong 0$ can be explicitly obtained by some manipulation:

$$\tau(c) = \frac{\int_c^\infty e^{-t} \left(\int_c^t \frac{e^{z}}{z^2}dz \right) \left\{ \int_c^t e^{z}\,dz \left(\int_z^\infty \frac{e^{-u}}{u^2}du \right) \right\} dt}{\int_c^\infty e^{-t} \left(\int_c^t \frac{e^{u}}{u^2}du \right) dt} \tag{4-205}$$

A further simplification has been worked out in details in the reference. The final result can be shown to be:

$$\tau(c) = e^{c}\,(-Ei(-c)) - 1 + c \int_{0}^{\infty} e^{-cv}\,\frac{\ell n(1+v)}{v}\,dv \qquad (4\text{-}206)$$

where the exponential integral is defined:

$$-Ei(-c) = \int_{c}^{\infty} \frac{e^{-u}}{u}\,du \qquad c>0 \qquad (4\text{-}207)$$

For the particular case of interest of sufficiently small values of c, i.e. T is large, (4-207) can be expanded into the asymptotic series:

$$-Ei(-c) = -\tilde{\gamma} - \ell n\, c + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{c^{n}}{n!\; n} \qquad (4\text{-}208)$$

where $\tilde{\gamma} = 0.57721$ is Euler's constant.

Using (4-208), the asymptotic solution can be simplified to be:

$$\tau\,(c) = -\,\ell n\, c - 1 - \tilde{\gamma} + 0(c) \qquad (4\text{-}209)$$

This is the optimal performance of mean delay time $\tau\,(c)$ in detecting the degradation as a function of the parameter c. Substituting (4-191) into (4-209) and transforming back from the natural scale to real time scale, $\tau\,(T)$ can be expressed as

$$\tau(T) = \frac{2\sigma^{2}}{a_{1}^{2}}\left\{ \ell n \left[\frac{a_{1}^{2}}{2\sigma^{2}} \cdot T\right] - 1.5772 + 0\left(\frac{2\sigma^{2}}{a_{1}^{2} T}\right)\right\} \qquad (4\text{-}210)$$

where T is the specified mean time between false alarms in the real time scale.

(2) Performance of the Suboptimal System

The derivation of the performance of the suboptimal sys-

tem is approached by solving the first passage time of an equivalent linear detection system with some modification of the boundary condition and application of a limiting operation. Consider the stochastic differential equation of Wald's linear detection system:

$$d\lambda(t) = \frac{a_1}{\sigma^2} \left[dy(t) - \frac{1}{2} a_1 \, dt \right] \tag{4-211}$$

The observed process is defined as

$$dy(t) = a \, dt + \sigma \, dw(t) \tag{4-212}$$

where a is the true unknown parameter defining the operation mode of the system, and the parameter a is specified as a_1 for mode H_1 and as zero for mode H_0.

Substituting (4-212) into (4-211) gives

$$\begin{aligned} d\lambda(t) &= \frac{a_1}{\sigma^2} \left(a - \frac{1}{2} a_1 \right) dt + \frac{a_1}{\sigma} dw(t) \\ &\triangleq \mu(a) \, dt + b \, dw(t) \end{aligned} \tag{4-213}$$

For Wald sequential analysis procedure, the decision rule is defined as follows: Two constants $A \leqq 0$ and $B \geqq 0$ are selected by the set of error probabilities. The decision rule is of single stage, the detection system terminates whenever $\lambda(t)$ reaches the lower boundary A or the upper boundary B. The first passage time for Wald's system, under the assumption that $\lambda(0) = y$, and that the value of the parameter a does not change during the whole observation interval, can be obtained by solving the following differential equation:

$$\frac{1}{2} b^2 \frac{d^2 \tilde{t}}{dy^2} + \mu(a) \frac{d\tilde{t}}{dy} = -1 \tag{4-214}$$

with the boundary conditions $\tilde{t}(A) = \tilde{t}(B) = 0$.

Notice that with assigned boundary conditions, there is no guarantee that absorption in the boundary B occurs before absorption in the bound-

ary A. To guarantee the absorption in the boundary B, the decision rule is modified as discussed in Section 4.4.1. when the process $\lambda(t)$ attains the value A, the process is reset to $\lambda=0$ and the observation is continued. When the process attains the value B the alarm signal for the detection of disorder is given. If the alarm is verified to be false, the process is renewed again from the point $\lambda=0$, and so on until disorder is detected.

From the described decision rule, the correct boundary conditions are expressed as

$$\begin{aligned} \tilde{t}(B) &= 0 \\ \tilde{t}(A) &= \tilde{t}(0) \end{aligned} \tag{4-215}$$

The first passage time for the trajectories reaching the upper boundary B for detecting a degradation can be derived by solving the differential equation (4-214) with the boundary conditions (4-215). This can be shown to be

$$\tilde{t}(y,a) = \frac{1}{\mu(a)}\left(B - y + A \cdot \frac{e^{-Bh} - e^{-yh}}{1 - e^{-Ah}}\right) \tag{4-216}$$

under the assumptions of the initial condition $\lambda(0) = y$ and of the parameter a. The parameter h is defined as

$$h = \frac{2\mu(a)}{b^2} = \frac{2a - a_1}{a_1} \tag{4-217}$$

The mean time T between two false alarms is defined as the mean time for the trajectories $\lambda(t)$ to exit through the upper boundary B under the assumptions that the observed process is of normal mode (a=0), and the initial condition $\lambda(0) = 0$. With these definitions, the mean time T can be derived from (4-216):

$$T = \tilde{t}(0,0) = \frac{1}{\mu(0)} \cdot \left(B + A\frac{e^{B} - 1}{1 - e^{A}}\right) \tag{4-218}$$

The detection performance of the suboptimal system can be derived as in (4-179) by:

$$\tau(a_1, T) = \int_A^B \tilde{t}(y, a_1, T)\, p(y)\, dy \tag{4-219}$$

Here p(y) is the probability density of the process $\lambda(t) = y$ at the instant of degradation. As in the optimal case, it is assumed that for this instant there exists a stationary condition, established in the observation process in the normal mode. The density p(y) is to be solved by the Kolmogorov forward equation. The final result can be shown to be:

$$p(y) = \begin{cases} \dfrac{(1-e^{B})(e^{A-y}-1)}{A(1-e^{B}) + B(e^{A}-1)} & A \leqq y < 0 \\ \dfrac{(e^{A}-1)(1-e^{B-y})}{A(1-e^{B}) + B(e^{A}-1)} & 0 < y \leqq B \end{cases} \tag{4-220}$$

The derivation is based on an application of the method of Green's function and will be presented in Appendix B.

The performance of the suboptimal system can now be derived by a simple limiting operation of the above general formulation. In the suboptimal system the linear stochastic system is designed with a feedback of uncertainty information defined by $\lambda(t)$. In the discrete-time system, the sampled value at each sampling instant is actually a random number, which is reset by the control to a value zero. It is difficult to derive the detection performance when the controls are applied at boundaries of random values. Instead, the performance is derived under the assumption that the controls are applied at a constant boundary A. The constant value A may be regarded as the expected value of the drift over one sample interval of the $\lambda(t)$-trajectory defined in the normal mode. The assumption of a constant value A can be further justified if one considers the case where the implement-

ed controls are u^{ϵ}-controls defined in Section 4.4.4, and are executed over very small sample intervals in approximating the continuous suboptimal control law. In the limiting case of the continuous-time system, the control will almost instantaneously reset $\lambda(t) = 0$ whenever $\lambda(t) < 0$. This is equivalent to observing the linear system with the lower boundary A set to zero. The detection performance is derived on the basis of this limiting case. In this case one obtains from (4-218),

$$\begin{aligned} T &= \lim_{A \to 0} \frac{1}{\mu(0)} \cdot \left(B + A \frac{e^B - 1}{1 - e^A}\right) \\ &= \frac{1}{\mu(0)} \cdot (B - e^B + 1) \end{aligned} \tag{4-221}$$

Let the upper boundary under the case $A = 0$ be denoted as B_1. The equation (4-216) can be rewritten as

$$\tilde{t}(y, a, T) = \frac{1}{\mu(a)} \cdot \left[B_1 - y + \frac{(e^{-B_1 h} - e^{-yh})}{h}\right] \tag{4-222}$$

Define $x = B_1 - y$, the following relations can be established by definitions:

$$h(a_1) = 1, \quad \mu(0) = -\frac{a_1^2}{2\sigma^2}, \quad \mu(a_1) = \frac{a_1^2}{2\sigma^2} \tag{4-223}$$

$$T = \frac{2\sigma^2}{a_1^2} (e^{B_1} - B_1 - 1) \tag{4-224}$$

$$\tilde{t}(x, a_1, T) = \frac{2\sigma^2}{a_1^2} \cdot \left[x + e^{-B_1}(1 - e^x)\right] \tag{4-225}$$

$$p(x) = \lim_{\substack{A\to 0 \\ B\to B_1}} \frac{(e^A-1)(1-e^{B-y})}{A(1-e^B)+B(e^A-1)} \qquad 0 \leqq y \leqq B$$

$$= \frac{(1-e^{B_1-y})}{1-e^{B_1}+B_1} \qquad 0 \leqq y \leqq B_1$$

$$= \frac{(e^x-1)}{e^{B_1}-B_1-1} \qquad 0 \leqq x \leqq B_1 \qquad (4\text{-}226)$$

The detection performance $\tau(t)$ can be derived by the simple integration of (4-219):

$$\tau(t) \triangleq \tau(a_1, T) = \int_0^{B_1} \tilde{t}(x, a_1, T) \cdot p(x)\, dx$$

$$= \left(\frac{2\sigma^2}{a_1^2}\right) \frac{1}{T} \left\{B_1 \left(e^{B_1} - \frac{1}{2}B_1 - e^{-B_1}\right)\right.$$

$$\left. -\frac{3}{2}(e^{B_1} - 2 + e^{-B_1})\right\} \qquad (4\text{-}227)$$

A simple transformation of (4-292) will give an interesting comparison between the detection performance of the optimal and suboptimal systems.

Consider the case that T is large which will be of main interest, an approximate value for the upper boundary B_1 can be derived

$$T = \frac{2\sigma^2}{a_1^2} \cdot (e^{B_1} - B_1 - 1)$$

$$\approx \frac{2\sigma^2}{a_1^2} \cdot e^{B_1}$$

or
$$B_1 \approx \ell n \left(\frac{a_1^2}{2\sigma^2} T\right) \tag{4-228}$$

The detection performance (4-227) of the suboptimal system under this condition takes then the form:

$$\begin{aligned} \tau(T) &\cong \left(\frac{2\sigma^2}{a_1^2}\right) \cdot \left(B_1 - \frac{3}{2}\right) \\ &= \left(\frac{2\sigma^2}{a_1^2}\right) \left[\ell n \left(\frac{a_1^2}{2\sigma^2} T\right) - \frac{3}{2}\right] \end{aligned} \tag{4-229}$$

The detection performance of the suboptimal system has thus been shown to be of a remarkably good approximation to that of the optimal system as expressed in (4-210):

$$\tau(T) = \frac{2\sigma^2}{a_1^2} \left\{ \ell n \left(\frac{a_1^2}{2\sigma^2} \cdot T\right) - 1.5772 + 0 \left(\frac{2\sigma^2}{a_1^2 T}\right) \right\}$$

An analysis of this result will be discussed in the next section.

4.5.3 Discussion of Detection Performance

In this section the performance comparison of the optimal and suboptimal detection system will be analyzed. In Section 4.4.4 the improvement of performance in the suboptimal detection system has been intuitively explained as a result of control actions that reset the system state π_n to the null-discrimination state whenever the system is operated in the normal mode. A more detailed analysis will be presented in the following discussion.

In deriving the performance of the detection system the problem was formulated as a first-passage time of the diffusion process in both optimal and suboptimal systems. This gives a better insight for performance comparison. Consider the basic equation of the mean time delay (4-219):

$$\tau(T) = \int_A^B \tilde{t}(a_1, y, T)\, p(y)\, dy \qquad (4\text{-}230)$$

There are two fundamental factors which enter into the basic equation: the behavior of the stochastic differential equation $\lambda(t)$ and the boundary values A and B. Each factor will be analyzed in both optimal and suboptimal systems. In the following the system stochastic equations will be expressed in the dimensionless natural scale s. The following transformation has been defined in (4-191):

$$r = \frac{1}{2}\frac{\dot{a}_1^2}{\sigma^2}$$

$$g = \frac{q}{r} \qquad (4\text{-}231)$$

$$s = rt$$

A. The Stochastic Differential Equation

In the optimal system the stochastic differential equation can be expressed as

$$d\lambda(s) = (-1 + g + g\, e^{-\lambda(s)})ds + (\frac{2a}{a_1})\, ds + \sqrt{2}\, dw(s) \qquad (4\text{-}232)$$

Let the a priori probability of degradation be denoted by π, then the initial condition of $\lambda(s)$ is defined as

$$\lambda = \ell n(\frac{\pi}{1-\pi})$$

The behavior of the process $\lambda(s)$ will be studied. It is assumed that g is negligible, i.e., the instrument has a long mean life time. However the term $g\, e^{-\lambda(s)}$ cannot be neglected, since $\lambda(s)$ can be a large negative number. The equation (4-232) can be rewritten as

$$d\,\lambda(s) = (g\, e^{-\lambda(s)} - 1)ds + (\frac{2a}{a_1})\, ds + \sqrt{2}\, dw(s) \qquad (4\text{-}232)$$

Denote $\lambda_e = \ell n\ g$. Consider the term in the brace of (4-233) around $\lambda(s) = \lambda_e$. There are three cases:

(1) $\lambda(s) = \lambda_e$, then the term

$$(ge^{-\lambda(s)} - 1) = (ge^{-\lambda_e} - 1) = (g \cdot \frac{1}{g} - 1) = 0$$

tends to zero, i.e., this term contributes nothing at the point $\lambda(s) = \lambda_e$.

(2) $\lambda(s) < \lambda_e$, let $\lambda(s) = \lambda_e - \delta\lambda$, $\delta\lambda > 0$. Then the term

$$(ge^{-\lambda(s)} - 1) = (e^{\delta\lambda} - 1) > 0$$

contributes on the average a positive trend for $\lambda(s)$ through the transition coefficient, and gives a compensation effect.

(3) $\lambda(s) > \lambda_e$, let $\lambda(s) = \lambda_e + \delta\lambda$, $\delta\lambda > 0$, the term

$$(ge^{-\lambda(s)} - 1) = (e^{-\delta\lambda} - 1) < 0$$

contributes a negative trend. In particular, if $\delta\lambda$ is large, then the term $e^{-\delta\lambda}$ tends to zero. This will be the situation when the system is operated in the degradation mode.

The fact that the process $\lambda(s)$ oscillates about the state $\lambda_e = \ell n\ g$ provides a convenient and interesting specification for the initial condition λ. In general for any practical system the initial state of degradation π should be a small value, this means that the initial state λ tends to be a negative infinite magnitude. From the above analysis in case (2), the trajectory $\lambda(s)$ will converge rapidly toward the value λ_e through the contribution of the large value of the term $(g\ e^{-\lambda} - 1)$. For the case of practical interest a degradation will occur only after a long time of normal operation, then in evaluating the detection performance it is unnecessary (and usually difficult) to get accurate information about the initial state. A convenient and reasonable specification for the initial state can be assigned to be

$$\lambda = \ell n\ g \qquad (4\text{-}234)$$

Next consider the corresponding stochastic equation of the suboptimal

system:

$$d\lambda(t) = (\zeta(t) - \frac{1}{2}\frac{a_1^2}{\sigma^2})\,dt + \frac{a_1}{\sigma^2}\,dy(t) \qquad (4\text{-}235)$$

Transforming (4-235) into the natural scale by (4-231) produces

$$d\lambda(s) = (\zeta(s) - 1)\,ds + (\frac{2a}{a_1})\,ds + \sqrt{2}\,dw(s) \qquad (4\text{-}236)$$

It has been shown in section 4.4.4 that the suboptimal control $\zeta(s)$ resets $\lambda(s)$ to zero if $\lambda(s) \leqq 0$, and no control will be applied if $\lambda(s) > 0$. The term $(\zeta(s) - 1)$ really contributes around the point $\lambda(s) = 0$ a similar effect as the term $(q\,e^{-\lambda(s)} - 1)$ on the behavior of the stochastic equation. If the system is operated in the normal mode, $\lambda(s)$ tends to be negative. The suboptimal resets the state $\lambda(s)$ to zero. This case is analogous to the condition (2) in the optimal system. If the system is operated in the degradation mode, $\lambda(s)$ tends to be positive, and no control is applied. This is analogous to the condition (3) in the optimal system where $\delta\lambda$ is large. This analysis of the trajectories of the stochastic differential equation is clearly confirmed by simulation results as illustrated in Fig. 4.6 and Fig. 4.10.

B. The Boundary Values A and B

The boundary values $[\lambda, \gamma]$ of the optimal detection system have been derived in (4-234) and (4-190).

$$\begin{aligned} \lambda &\simeq \ell n(g) = \ell n\,(\tfrac{q}{r}) \\ \gamma &= \ell n\left[\frac{1-\alpha(1-\pi)}{\alpha(1-\pi)}\right] \\ &\simeq \ell n\,(\tfrac{1-\alpha}{\alpha}) \simeq \ell n\,(qT) \\ &= \ell n\,(\tfrac{q}{r}) + \ell n\,(rT) \end{aligned} \qquad (4\text{-}237)$$

The corresponding boundary values [A, B_1] of the suboptimal system have also been derived in (4-228).

$$A = 0$$
$$B_1 = \ell n \left[\frac{a_1^2}{2\sigma^2} T\right] \triangleq \ell n [r\ T] \tag{4-238}$$

For a specified T, the boundary values of both systems differ only by a translation by $\ell n(q/r)$.

A comparison of the system detection performance of the suboptimal system with the optimal system can now be analyzed as follows. Consider the basic equation (4-230) of the detection performance for the optimal system (denoted by subscript "0"):

$$\tau_0(t) = \int_{\lambda}^{\gamma} \tilde{t}_0(y, a_1, T) p_0(y)\, dy \qquad y \in [\lambda, \gamma]$$
$$= \int_0^{\gamma-\lambda} p_0(z)\, \tilde{t}_0(z, a_1, T)\, dz \qquad z \in [0, \gamma-\lambda] \tag{4-239}$$

where the transformed boundary can be derived from (4-237).

$$(\gamma-\lambda) \simeq \ell n \left[\frac{a_1^2}{2\sigma^2} \cdot T\right]$$

The detection performance of the suboptimal system (denoted by subscript "s") can be similarly expressed as.

$$\tau_s(T) = \int_0^{B_1} \tilde{t}_s(y, a_1, T)\, p_s(y)\, dy \qquad y \in [0, B_1] \tag{4-240}$$

where $B_1 \tilde{=} \ell n \left[\frac{a_1^2}{2\sigma^2} \cdot T\right]$

The time delay in detecting degradation $\tilde{t}(y, a_1, T)$ is evaluated

on the conditions that the system is operated in the degraded mode H_1, that the initial state is y, and that the mean time between false alarms is specified to be T. The stochastic equation $\lambda(s)$ for the optimal system defined in mode H_1 can be expressed as

$$d\lambda(s) = [g(1 + e^{-\lambda(s)}) + 1]\, ds + \sqrt{2}\, dw(s) \quad \lambda(0) = \lambda \qquad (4\text{-}241)$$

Define $\tilde{\lambda}(s) = \lambda(s) - \lambda$

Then (4-241) is transformed into

$$d\tilde{\lambda}(s) = (1 + g + e^{-\tilde{\lambda}(s)})\, ds + \sqrt{2}\, dw(s) \qquad \tilde{\lambda}(0) = 0 \qquad (4\text{-}242)$$

with $\tilde{\lambda}(s)$ defined in the interval $[0,\ \ell n(\frac{a_1^2}{2\sigma^2} \cdot T)]$

The stochastic equation $\lambda(s)$ for the suboptimal system can be shown to be

$$d\lambda(s) = [\xi(s) + 1]\, ds + \sqrt{2}\, dw(s) \qquad (4\text{-}243)$$

Since the process $\lambda(s)$ can be assumed on the average to be positive under the condition of the degraded mode H_1, then $\xi(s) = 0$, and (4-243) can be rewritten as

$$d\lambda(s) = ds + \sqrt{2}\, dw(s) \qquad \lambda(0) = 0 \qquad (4\text{-}244)$$

with $\lambda(s)$ defined in the interval $[\ 0,\ \ell n\ (\frac{a_1^2}{2\sigma^2} \cdot T)]$. A larger transition coefficient is present in the stochastic equation (4-242) for the optimal system. Hence by application of the comparison theorem the optimal system will in general show a shorter time delay to exit through the boundary $\ell n(\frac{a_1^2}{2\sigma^2} \cdot T)$ as expressed by the relation

$$\tilde{t}_0(y, a_1, T) \leqq \tilde{t}_s(y, a_1, T) \qquad (4\text{-}245)$$

However the difference will be insignificant if the parameter g is assumed to be small. The term $e^{-\tilde{\lambda}(s)}$ will initially contribute a large

effect, but will show a minor effect when the system is operated in the degraded mode with a large value of $\tilde{\lambda}(s)$. A detailed comparison of the probability density functions $p_0(y)$ and $p_s(y)$ would be quite involved, but there is evidence to believe that they will be approximately of the same distribution from the results of the analysis of the behavior of the stochastic equations of the optimal and suboptimal systems conditioned on the normal mode H_0. Finally, the boundaries of both systems have been shown to be the same, therefore the result that the detection performance of the suboptimal system shows a remarkably good approximation to that of the optimal system is not surprising.

Different detection systems of interest have been discussed in the literature. They include the detection system of the fixed-size Neyman-Pearson method (Shiryaev[65]) and a sequential test developed for use by radars detecting an emerging target (Gagliandi and Reed[30]). These however will not be reviewed here. In the following a detection system with the so-called degenerated Wald's sequential analysis method (Shiryaev[65]) will be briefly discussed. The system has been mentioned before in Section 4.4.1. When applied to the given problem, the method consists of observing the process $\lambda(t)$, $\lambda(0) = 0$, and making a decision at the instants when the trajectory $\lambda(t)$ first exits one of the boundaries. If $\lambda(\nu) = B > 0$ where B is an "upper" bound, then one accepts the hypothesis that a degradation is present and one verifies this decision. If a degradation has been verified, the observation terminates. In the case when it is a false alarm, and also for $\lambda(\nu) = A \leqq 0$, A being the "lower" boundary, the observation process restarts with λ reset to zero.

Let T and τ be the mean time between two false alarms and the corresponding mean time delay in detecting a degradation. Then it can be shown that both T and τ are functions of the boundaries A and B by application of the theory of first passage time problem. For a given T the solution for the optimal detection system is reduced to finding an optimal parameter A to minimize τ (A, T). Shiryaev has managed to prove by a rather cumbersome procedure the following

statement: there exists such a $\tilde{T} < \infty$ such that for all $T \geqq \tilde{T}$ the optimal choice of the lower boundary A is $A = 0$ in the class of Wald's sequential analysis (i.e. with the constraint $A \leqq 0$, $B > 0$).

Two remarks have been emphasized by Shiryaev in his approach. First Shiryaev claimed that there are no doubts that in reality for all T the choice A=0 is the optimal one, but the method of proof enables him to establish the result only for a certain bound $\tilde{T}$. This conclusion has been discussed later in a paper by Vorob'ev[79] Secondly, Shiryaev realized that for the condition A=0 there corresponds the error probabilities $\alpha = 0$, $\beta = 1$, and that from the point of view of the testing of statistical hypotheses, it is completely inadmissible. However, he argued that the probabilities α and β, by themselves, without taking account of the duration of the separate stages of observation, are not reasonable characteristics for the observation system, and that a probability of miss alarm β close to unity in no way discredits the observation system if the duration of the separate stages is sufficiently small.

The above result can be explained as follows: By definition Wald's sequential analysis is a single stage decision procedure. The process terminates whenever the trajectory exits through the upper or the lower boundary. For the normal system the trajectory will exit through the lower boundary with a probability of $(1-\alpha)$, and for the degraded system the trajectory will exit through the lower boundary with a probability of β. Since the system is operated a priori in the normal mode, then with the condition $\lambda(0)$ to be zero the trajectory will immediately exit through the lower boundary A of zero, and by definition it is a stage. Since there will be only a finite number of times for trajectories to exit through the upper boundary, while there are an infinite number of times for trajectories to exit through the lower boundary, this explains the reason that the false alarm error probability α is defined as zero. The miss alarm error probability β will be similarly defined as one. The drawback remarked by Shiryaev has been completely removed in the developed suboptimal detection system formulation.

4.6 Simulation Results

To demonstrate the actual numerical performance of the detection system described in this chapter, a redundant sensor system was simulated. The simulation results consist of three sets. The first set is to demonstrate the detection technique of the suboptimal system. The second set illustrates the comparison of detection performance of the optimal, the suboptimal and the linear detection system. The third set shows the isolation of the degraded instrument.

The first and second set of simulation results are based on a single gyro system. The dynamic equations for the gyro drift rate model in the discrete case are given in equation (2-4) by

$$D_G(k+1) = F\, D_G(k) + q_G(k)$$

The measurement equations are defined as

$$D(k) = D_G(k) + r_G(k) \qquad \text{for mode } H_0$$

$$D(k) = D_G(k) + m + r_G(k) \qquad \text{for mode } H_1$$

The statistics of the noises are given as

$$E\,[q_G(k)] = 0\ , \quad E\,[q_G(k)\, g_G(\ell)] = Q_G \cdot \delta_{k\ell}$$

$$E\,[r_G(k)] = 0 \qquad E\,[r_G(k)\, r_G(\ell)] = R_G \cdot \delta_{k\ell}$$

The measurement noise $r_G(k)$ represents the quantization error. If Q_g is the quantum size of the measurement of integrated rate and the increments of angle are calculated by differencing, the variance of the error in the increment angle is

$$R_{\Delta\theta} = \frac{1}{6} Q_g^2$$

The variance R_G in the angular velocity over the sampling period Δ is then derived as

$$R_G = \frac{1}{6} \cdot \frac{Q_g^2}{\Delta^2}$$

A set of real gyro test data is used as reference for the gyro model in simulation:

correlation time = 0.5 hour

sampling period = 6 minutes

quantization size = 6 arc seconds

standard deviation over long measurement interval = 2.5 meru

A simple computation gives the following set of data:

$$Q_G = 2 \text{ meru}^2, \quad R_G = 0.2 \text{ meru}^2, \quad F = 0.8$$

The system is normalized so that the following parametric values are used:

$$Q_G = 1, \quad R_G = 0.1, \quad m = 4, \quad F = 0.8, \quad \Delta = 0.1$$

As discussed in Chapter 3, the transformed measurement residual $\tilde{r}_G(k)$ for the single gyro system in the discrete case for degraded mode H_1 can be shown to have the normal distribution $N(a_1, \tilde{\sigma})$ with the asymptotic statistics as defined in (3-35) by similar manipulations:

$$a_1 = (1 + F V_x) \cdot m$$

$$\tilde{\sigma}^2 = R_G / (1 - K)$$

The values of V_x and the steady-state Kalman filter gain K can be obtained by either analytical or simulated computation:

$$V_x = -0.98146, \quad K = 0.91368$$

Thus the detection system is designed with the equivalent specifications to detect a degradation with magnitude $a_1 \cong 0.86$ in the background of noise with variance $\tilde{\sigma}^2 \cong 1.16$.

A discussion of the transient value of the transformed measurement residual $\tilde{r}_G(k)$ for degradation detection is required. Suppose a degradation of a constant bias m occurred at the (k+1)th sample, then the measurement residual $\tilde{r}_G(k+1)$ of the implemented system at that instant will reflect a bias of full magnitude, i.e., $a_1(k+1) = m$. The magnitudes a_1 at the next two samples can be respectively shown to be:

$$a_1(k+2) = (1 - F K)\, m$$

and

$$a_1(k+3) = (1 - F K (1 - F(1-K)))\, m$$

and the magnitude a_1 will be finally attenuated to its asymptotic value. The transient amplitudes of the measurement residuals will give a large signal-noise ratio for effective detection of a degradation.

The optimal boundaries of the stochastic process $\lambda(t)$ are determined by the following specifications:

(1) For Wald's linear detection system, the false alarm and miss alarm error probabilities are specified to be 1%. The upper and lower boundaries B and A are computed as

$$B = \ell n \left(\frac{1-\beta}{\alpha}\right) = 4.595$$

$$A = \ell n \left(\frac{\beta}{1-\alpha}\right) = -4.595$$

(2) For the suboptimal system, the upper boundary B, for detecting degradation is computed under the assumption of the same specified mean time T between two false alarms as Wald's linear system. The boundary B_1 can be solved by equations (4-218) and (4-221):

$$-\left(B + A \frac{e^B - 1}{1 - e^A}\right) = e^{B_1} - B_1 - 1$$

$$B_1 = 6.12$$

By (4-224), the mean time T can be computed as·

$$\frac{T}{\Delta} = \frac{2\tilde{\sigma}^2}{a_1^2} \cdot (e^{B_1} - B_1 - 1)$$

$$\cong 1200$$

(3) For the optimal system, the lower boundary λ and the upper boundary γ for detecting degradation are determined by (4-237):

$$\lambda = \ell n \left(\frac{q}{r}\right)$$

$$\gamma = \ell n \left(\frac{q}{r}\right) + \ell n (r T)$$

where the parameter r (4-191) in the discrete-time case is defined as

$$r = \frac{a_1^2}{2\tilde{\sigma}^2 \cdot \Delta}$$

The parametric value q of the failure rate is specified to be 0.001, this gives the mean life time T_m:

$$T_m = \frac{1}{q} = 1000$$

The implemented equation $\lambda(t)$ for the optimal detection system is the discrete form of equation (4-55), with some notational changes for the single gyro system using an integration algorithm of Euler's method:

$$\Lambda_n = \Lambda_{n-1} + \frac{a_1}{\tilde{\sigma}^2}\left(\tilde{r}_n - \frac{1}{2}a_1\right) + q\Delta\left(e^{-\Lambda_{n-1}} + 1\right),$$

where $\lambda(n)$ is denoted by Λ_n, and $\tilde{r}_n \triangleq \tilde{r}(n)$ is the measurement residual of the discrete time case discussed in Section 3.3.

The simulated equations $\lambda(t)$ for the suboptimal detection system in the discrete form are equations (4-142) and (4-143) implemented with reset control logic as discussed in Section 4.4.5.

(1) The recursion equation of the detection system for bias with positive polarity is represented by:

$$\Lambda_n = \Lambda_{n-1} + (\tilde{r}_n - \frac{1}{2}a_1)$$

and reset $\Lambda_n = 0$ in next iteration if $\Lambda_n < 0$.

(2) The recursion equation of the detection system for bias with negative polarity is represented by:

$$M_n = M_{n-1} + (\tilde{r}_n + \frac{1}{2}a_1)$$

and reset $M_n = 0$ in next iteration if $M_n > 0$. It is to be noted that the definition of the likelihood function Λ_n (and M_n) of the suboptimal system has been modified by a factor of $(\tilde{\sigma}^2/a_1)$ [See definitions in (4-136) and (4-139)]

Figures 4.6 - 4.9 demonstrate the detection trajectories of the suboptimal system. In Figs. 4.6 - 4.7, a degradation of bias value m = +4 occurs at sample time n = 258. The system detects the degradation at n = 292 in Fig. 4.6 with correct identification of a positive polarity. This illustrates a case in which the instrument happens to send out "good" data even after it became degraded. This is reflected by the fact that the system requires more observations to detect the degradation. Fig. 4.7 illustrates the detection trajectory on the basis of likelihood ratio function for a degradation of bias value with negative polarity. No degradation is indicated, since this trajectory is not the correct decision function to apply for a degradation with positive polarity. In Figs. 4.8 - 4.9 a degradation of bias value m = -4 occurs at the sample time n = 44. No degradation is indicated in Fig. 4.8, since the trajectory is not the correct decision function to apply for a degradation of bias value with negative polarity. The system detects the degradation at n = 48 in Fig. 4.9 with correct identification

of negative polarity. This illustrates a case where the instrument sends out bad data right after degradation. One should not claim a better detection performance for the case in Fig. 4.9, rather the difference in sample sizes required for detection reflects the different situation of the actual performance of the real system in two cases. The above fact illustrates the advantage of the sequential detection system, in which the sample size for detection reflects the quality of the actual instrument output data, over the detection system based on the predetermined number of observations. For the latter case the detection system will lose time in the case illustrated in Fig. 4.8 by having to wait for a decision at the prespecified sample size. In the case illustrated in Fig. 4.6 the system will likely miss the degradation at the test of a prespecified sample size, and will wait for the next test to detect the degradation.

The second set illustrates a comparison of the performance of the detection systems. Figures 4.6, 4.10, and 4.11 show the comparison of performance of the detection systems, with information of the bias polarity given for Wald's linear system and the optimal system. A set of identical random numbers is used in the simulation of the linear, optimal and suboptimal systems. The detection performance of the optimal and suboptimal system can be shown to be practically the same as illustrated in Fig. 4.10 and 4.6. A comparison of the detection trajectories of the optimal and suboptimal systems in Fig. 4.10 and 4.6 verifies the analysis discussed in Section 4.5.3. It is to be noted that in the suboptimal system the boundary value for degradation detection is calculated to be $B_1 = 6.12$, the same as the boundary value $(\gamma - \lambda)$ used in the optimal system in Fig. 4.10. The threshold value shown in Fig. 4.6 is modified by a factor of $(\tilde{\sigma}^2/a_1)$ to be consistent with the definition of the likelihood ratio function Λ_n which was modified for computational convenience. The extra time delay in detection by Wald's linear system is clearly illustrated in Fig. 4.11. Figs. 4.12 - 4.14 show the performance of detecting a degradation with an unknown bias polarity. The deterioration of detection performance of the Wald system modified by the averaging process described in section 4.4.5 is clearly shown in

Fig. 4.14, when one compares with that of the suboptimal system as illustrated in Fig. 4.12.

Figures 4.15 - 4.26 show the isolation of the degraded instrument. The set of parity equations used in the simulation is given in Table 2.4:

$$(\Omega_A - \Omega_B) \cdot c + (\Omega_C + \Omega_D) \cdot s = z_1$$

$$-(\Omega_A + \Omega_C) \cdot c + (\Omega_B + \Omega_F) \cdot s = z_2$$

$$(\Omega_E + \Omega_F) \cdot c - (\Omega_A + \Omega_B) \cdot s = z_3$$

$$(\Omega_A - \Omega_E) \cdot c + (\Omega_D - \Omega_F) \cdot s = z_4$$

$$(\Omega_C + \Omega_E) \cdot c - (\Omega_B + \Omega_D) \cdot s = z_5$$

$$(\Omega_C - \Omega_D) \cdot c + (\Omega_E - \Omega_F) \cdot s = z_6$$

where $c = \cos\phi = 0.851$

$s = \sin\phi = 0.526$

For all gyros defined in the normal mode, the dynamic equations for each parity equation residual (denoted as z) are defined in (3-32) and (3-33):

$$x(k+1) = F\, x(k) + q_1(k)$$

$$z(k+1) = x(k+1) + r_1(k+1)$$

The statistics of the noises are given by

$$E[q_1(k)] = 0,\ E[q_1(k)\, q_1(\ell)] = Q \cdot \delta_{k\ell} = 2Q_G \cdot \delta_{k\ell}$$

$$E[r_1(k)] = 0,\ E[r_1(k)\, r_1(\ell)] = R \cdot \delta_{k\ell} = 2R_G \cdot \delta_{k\ell}$$

The parametric values are then defined as

$$Q = 2,\ R = 0.2,\ F = 0.8,\ \Delta = 0.1$$

Suppose that instrument A has been degraded with a jump in bias with value of +10. Figures 4.15 - 4.22 demonstrate the detection of degradation by the suboptimal system using parity equation residuals z_1 - z_4 with simultaneous identification of respective degradation polarities, while Figs. 4.23 - 4.26 correctly demonstrate that the parity the parity equation residuals z_5 and z_6 are operated in the normal mode. In particular Figs. 4.19 - 4.20 illustrate the deteriorated performance for detecting the degradation of the instrument A by the parity equation residual z_3. In comparison with residuals z_1, z_2 and z_4 the residual z_3 suffers from an attenuation of the signal-noise ratio for detection by a factor $(\frac{\sin\phi}{\cos\phi})^2 = \frac{1}{2.5}$. The number of samples required for detection for the residual z_3 is 93 compared to that of 5 or 6 required for residuals z_1, z_2 or z_4. With the additional information of identification of the polarity the degraded instrument A can be reliably isolated by an efficient decision logic. The decision logic can be defined by identifying the parity equation residuals $z_1 = +1$ (or $z_2 = -1$) and $z_5 = 0$ (or $z_6 = 0$), and by ignoring the parity equation residual z_3. Similarly for isolation of the instrument B, the decision logic can be defined by identifying the parity equation residuals $z_1 = -1$ and $z_4 = 0$ (or $z_6 = 0$), and by ignoring the three parity equation residuals z_2, z_3, and z_5, which magnitudes are all attenuated by the factor $\sin\phi$. Here the state of the normal mode H_0 is denoted as "0", and of the degraded mode H_1 is denoted as "1" with the proper sign identifying the polarity.

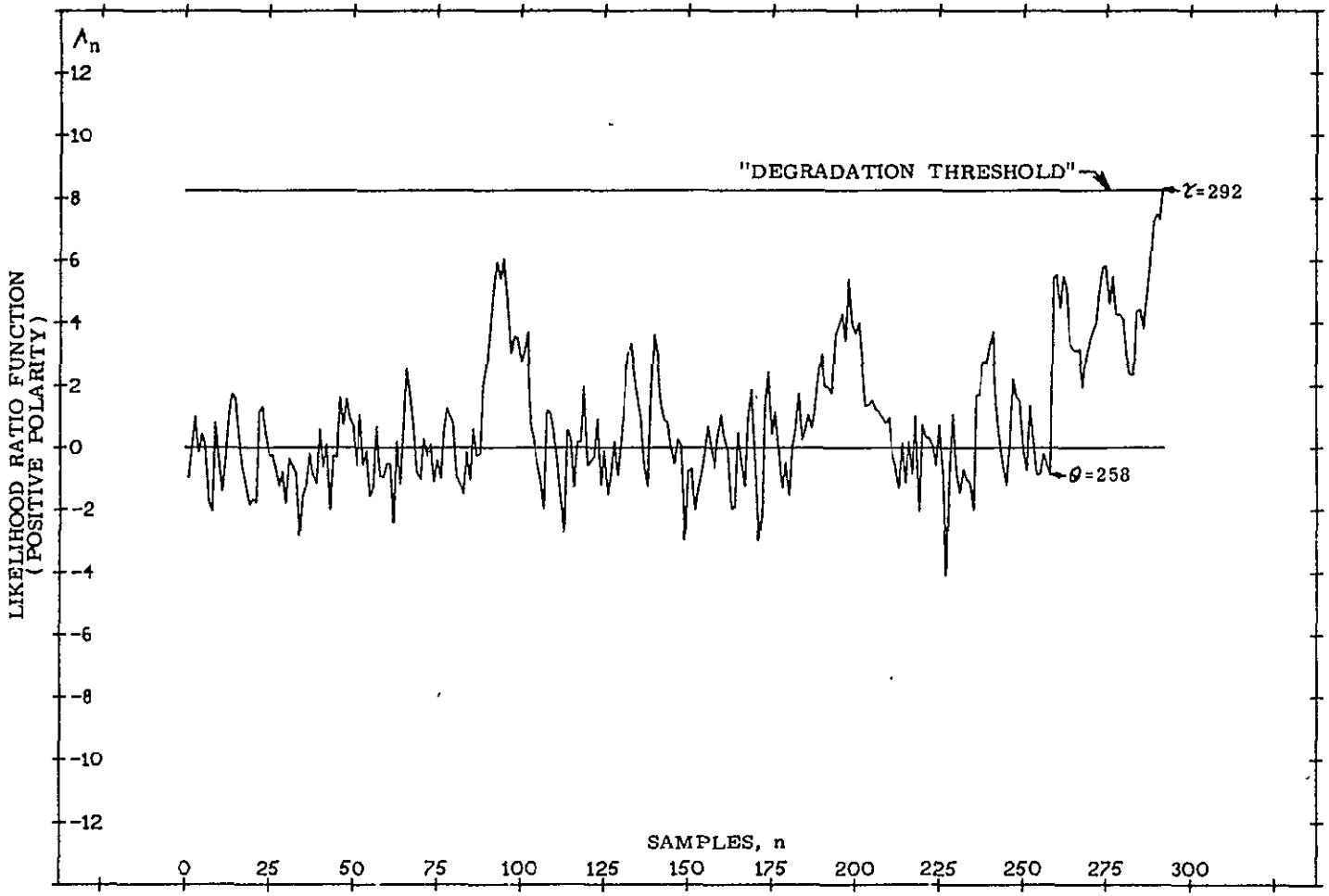


FIG. 4.6 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 1)

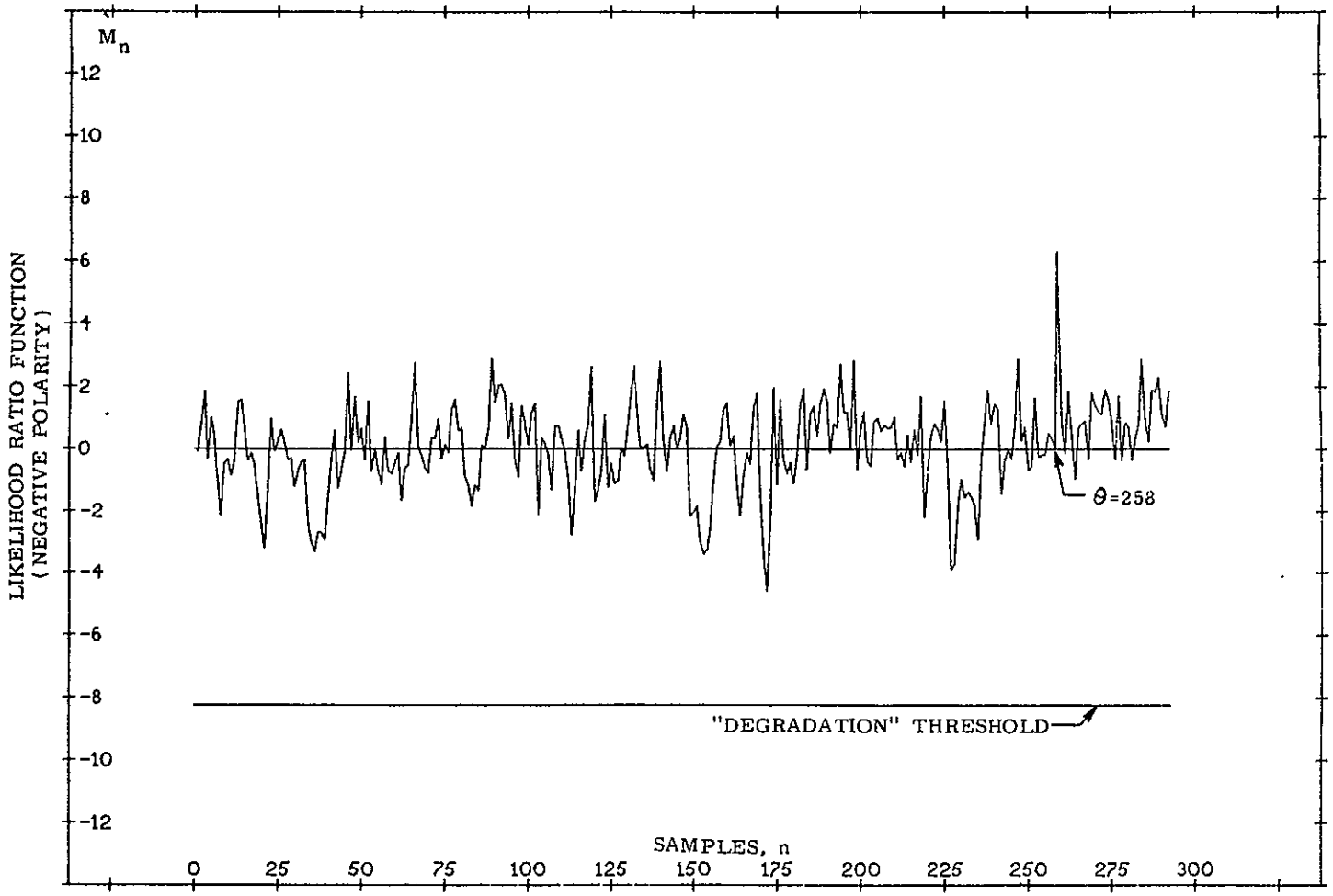


FIG. 4.7 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 1)

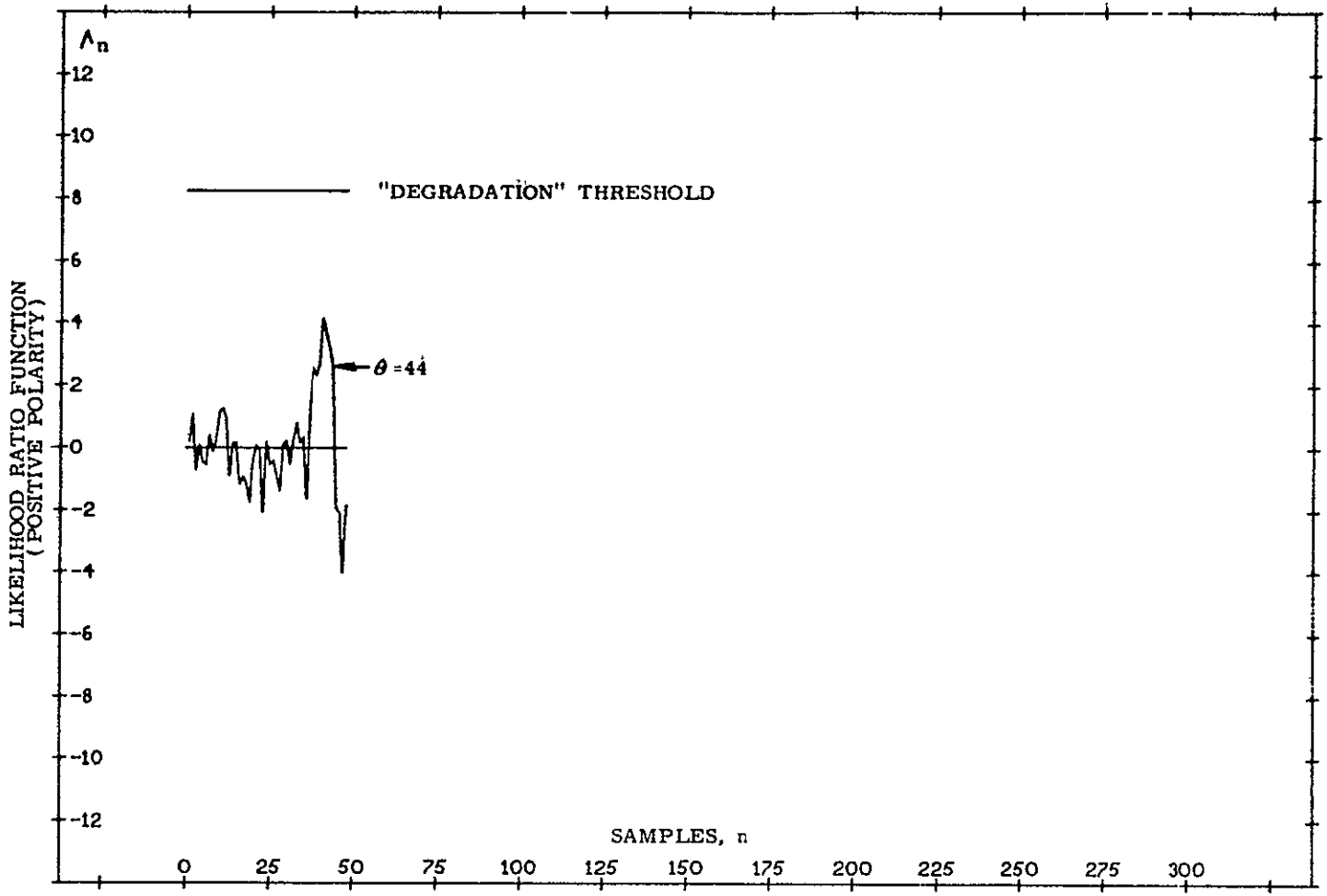


FIG. 4.8 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 2)

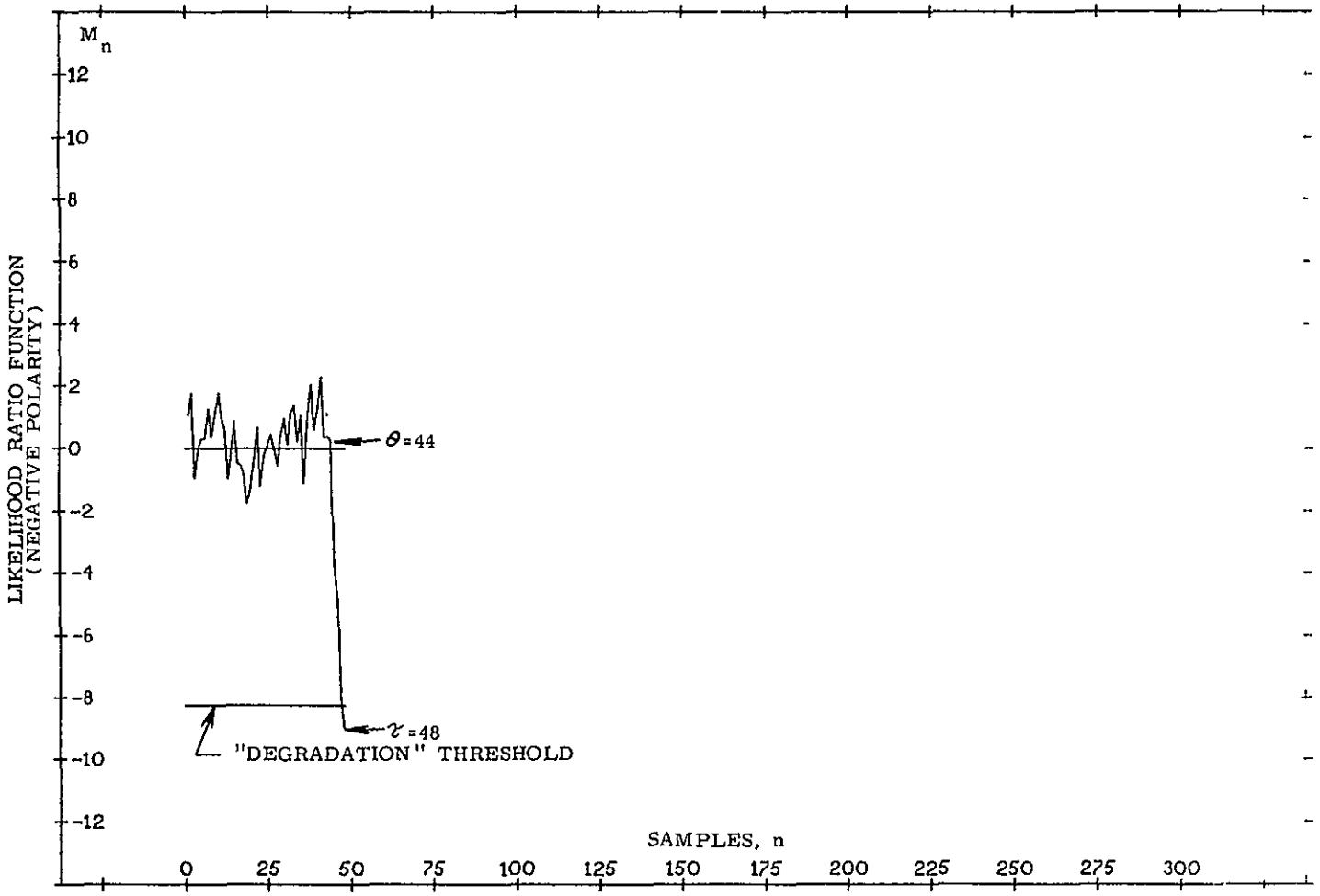


FIG. 4.9 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 2)

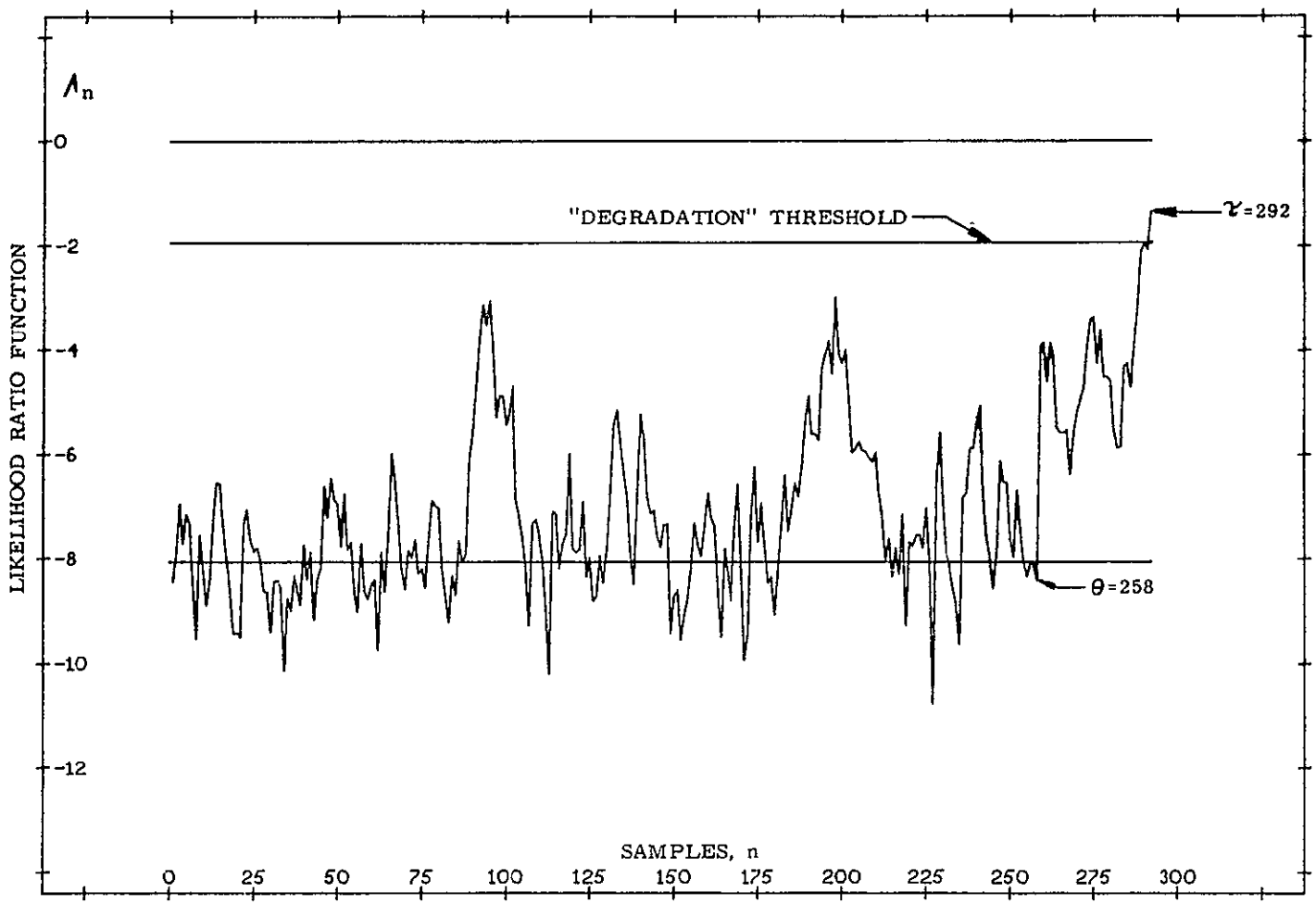


FIG. 4.10 DETECTION TRAJECTORY OF OPTIMAL SYSTEM (GIVEN BIAS POLARITY)

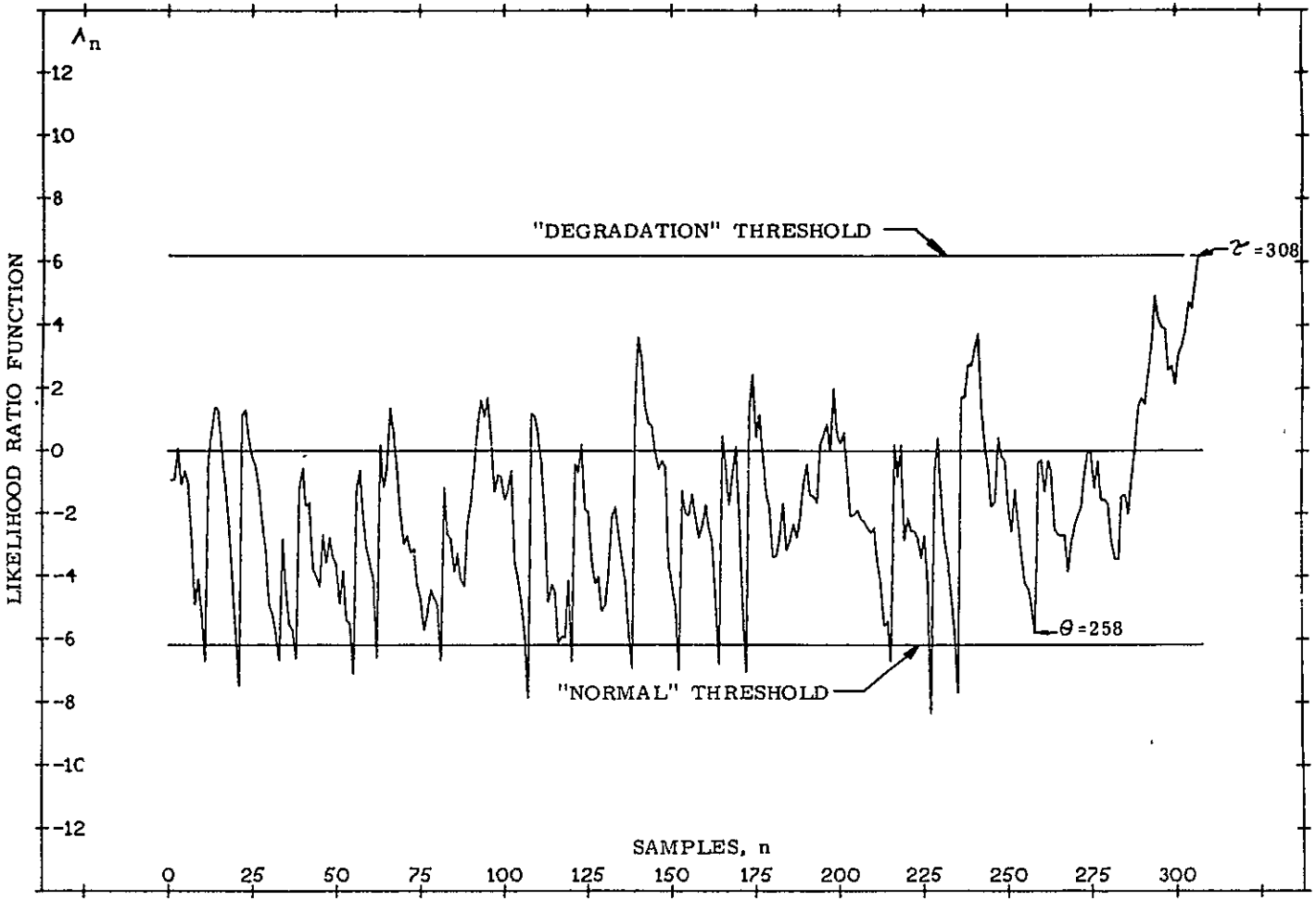


FIG. 4.11 DETECTION TRAJECTORY OF WALD'S LINEAR SYSTEM (GIVEN BIAS POLARITY)

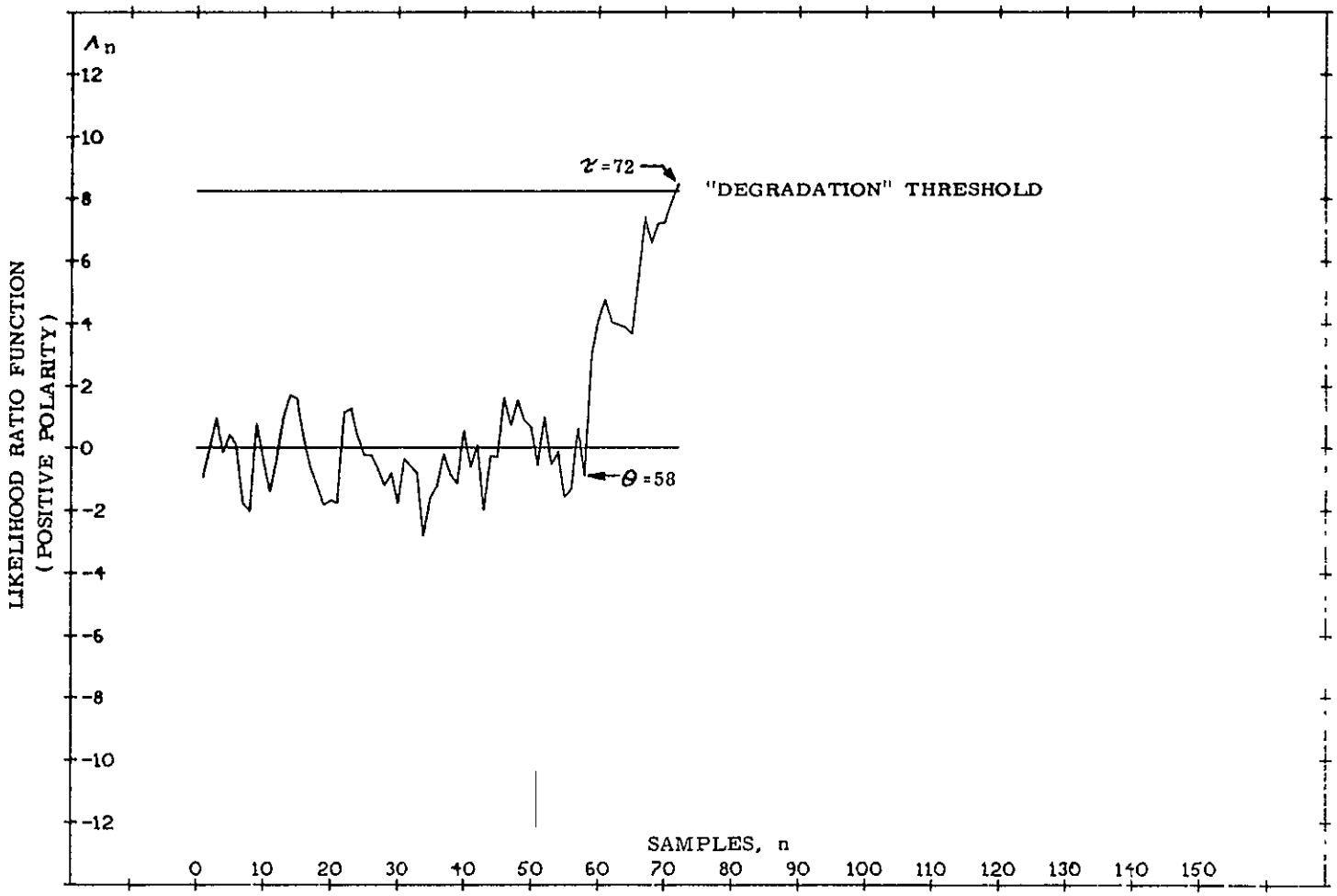


FIG. 4.12 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 3)

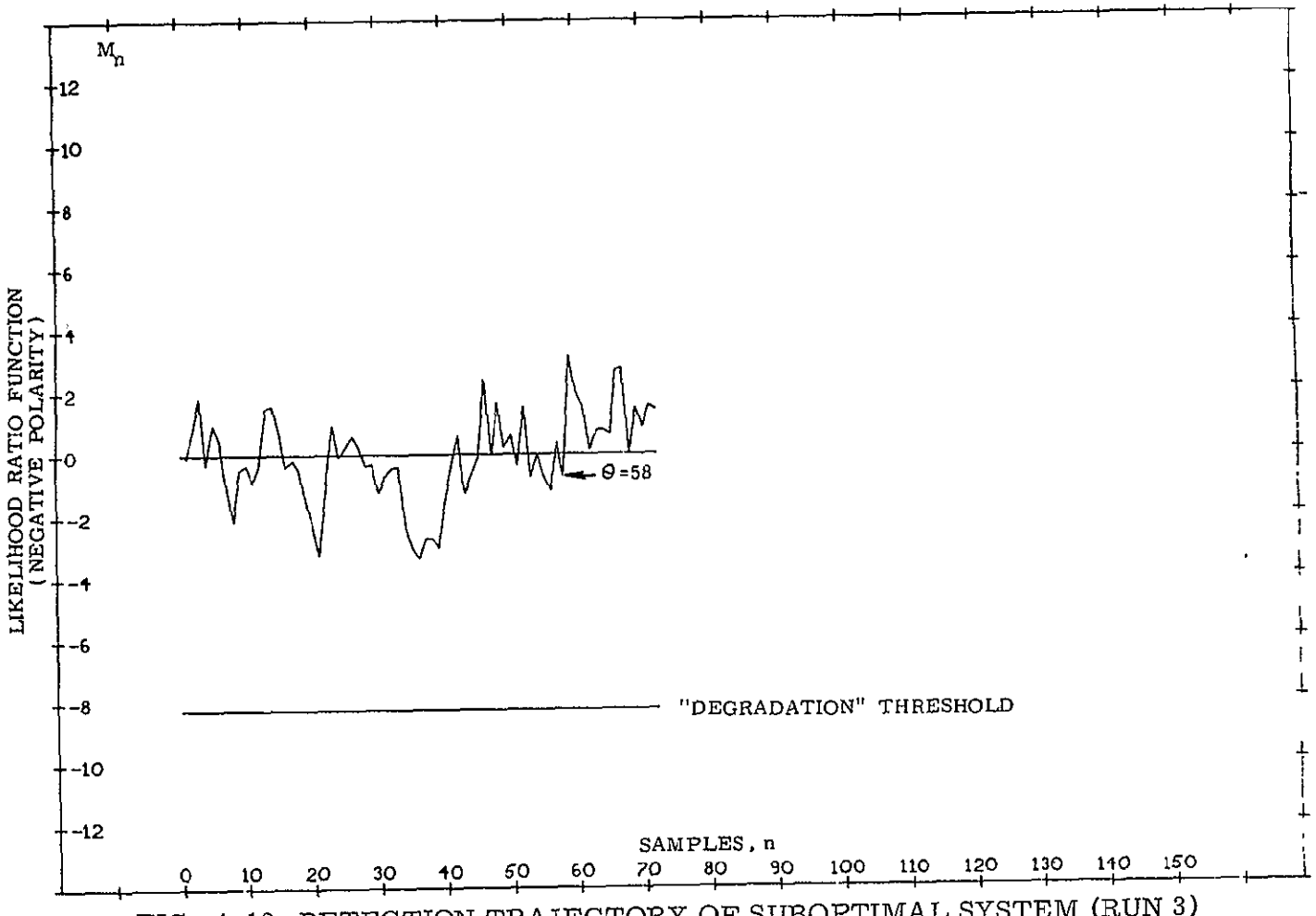


FIG. 4.13 DETECTION TRAJECTORY OF SUBOPTIMAL SYSTEM (RUN 3)

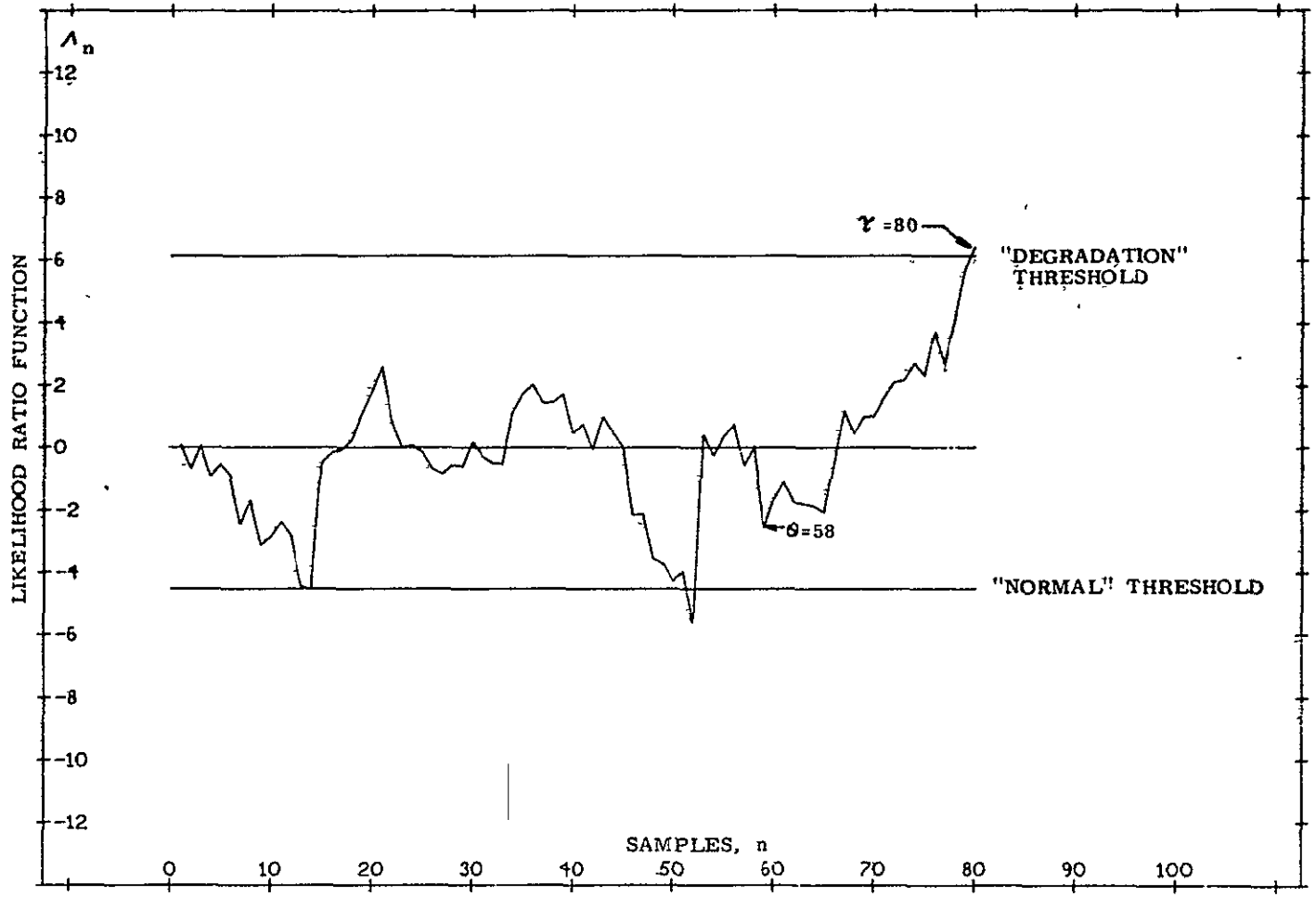


FIG. 4.14 DETECTION TRAJECTORY OF WALD'S LINEAR SYSTEM (UNKNOWN POLARITY)

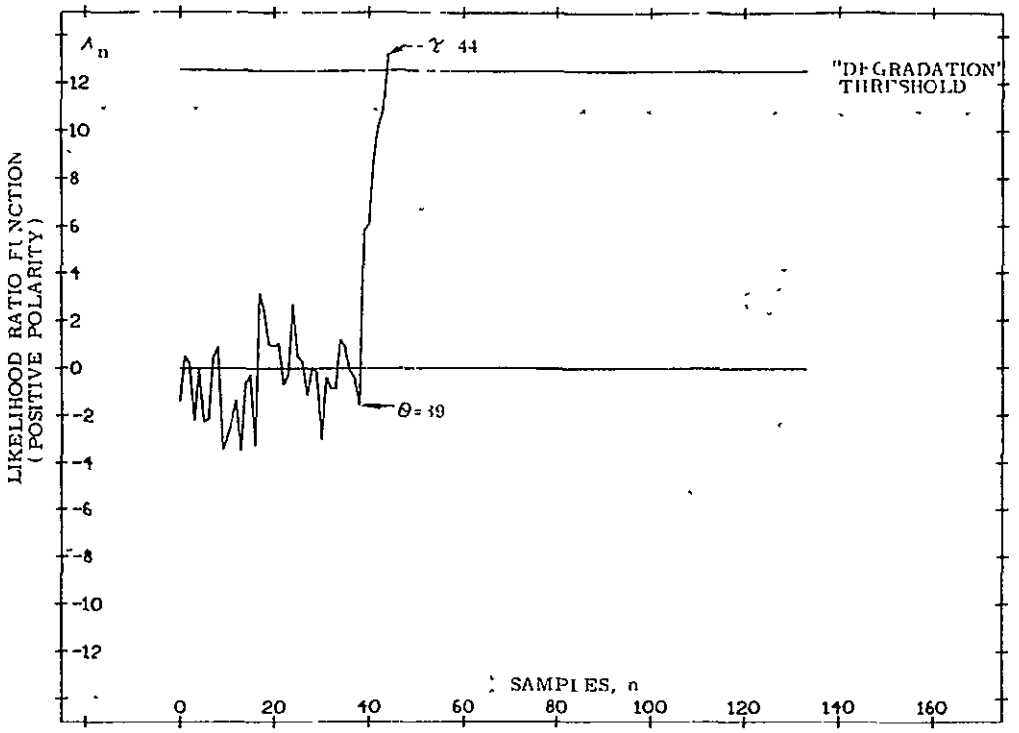


FIG. 4.15 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_1

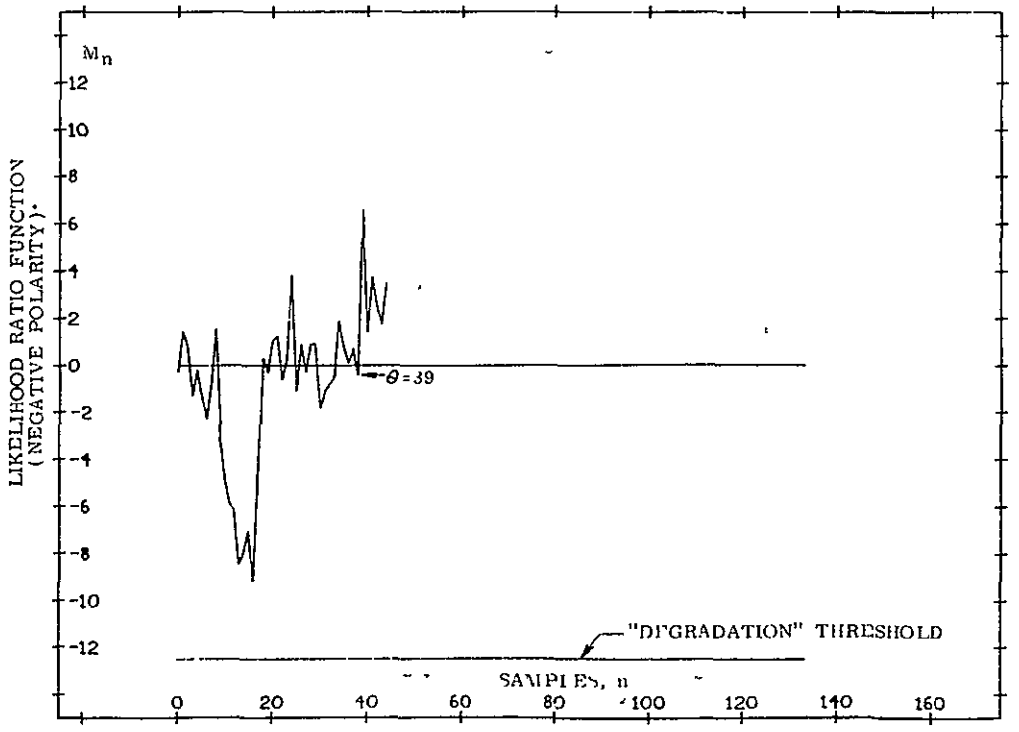


FIG. 4.16 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_1

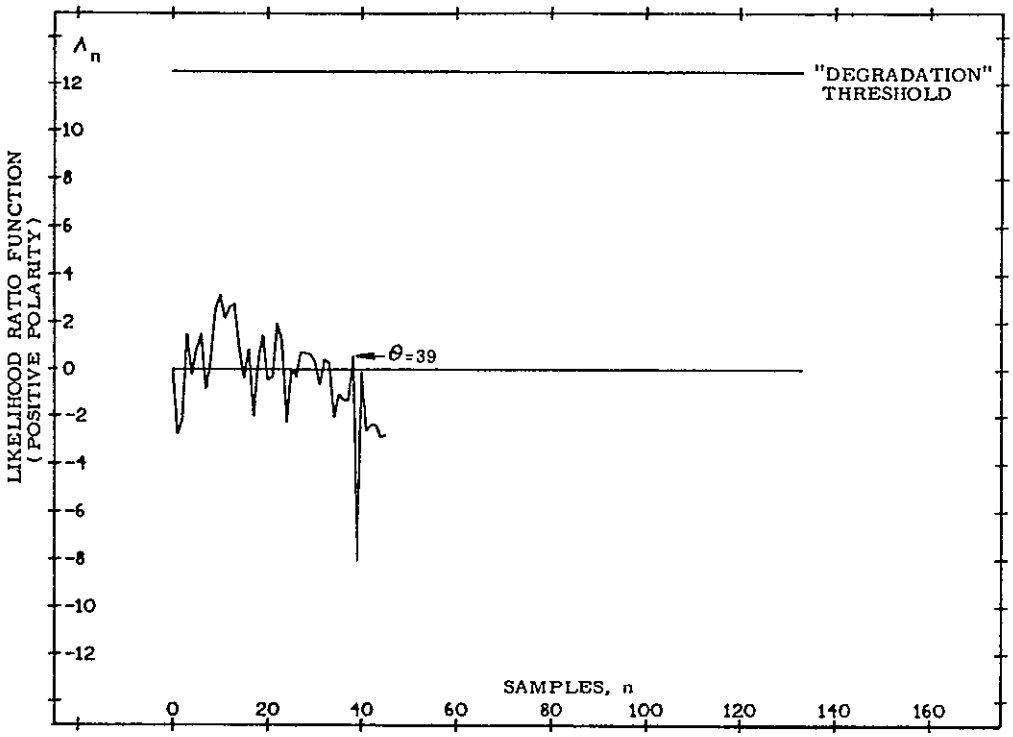


FIG. 4.17 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_2

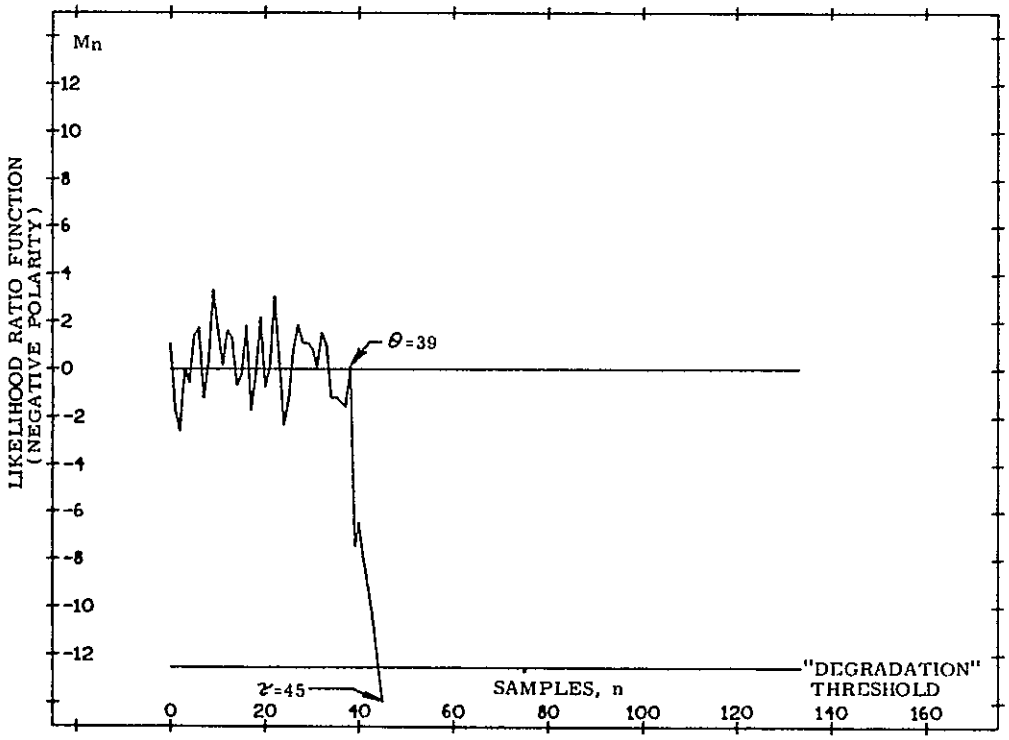


FIG. 4.18 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_2

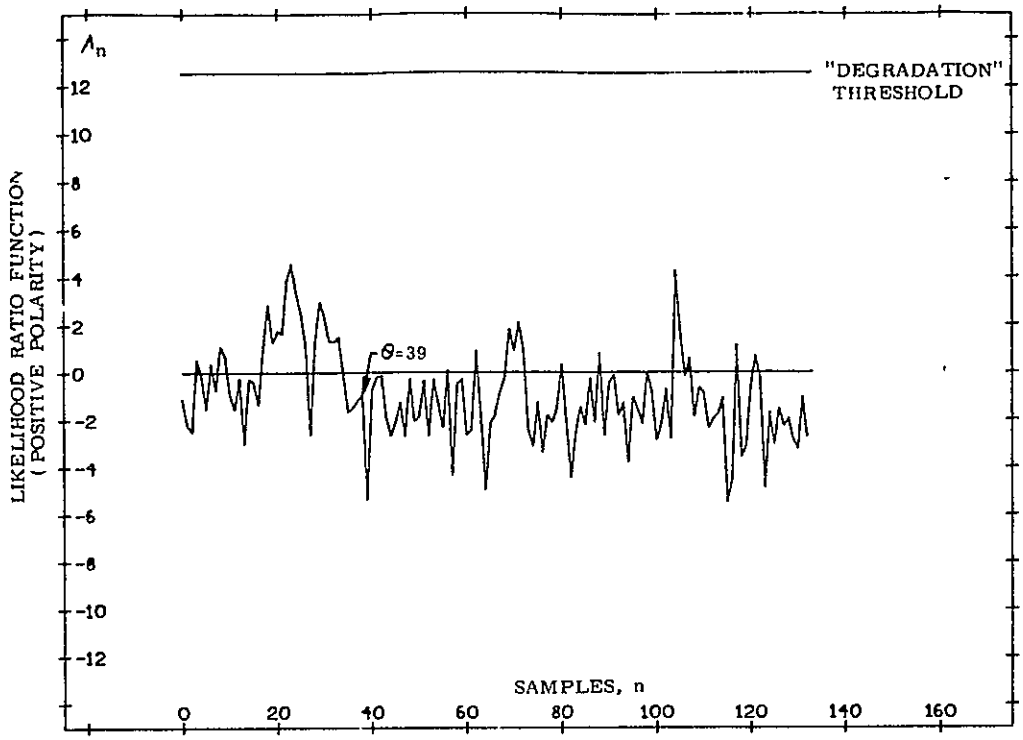


FIG. 4.19 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_3

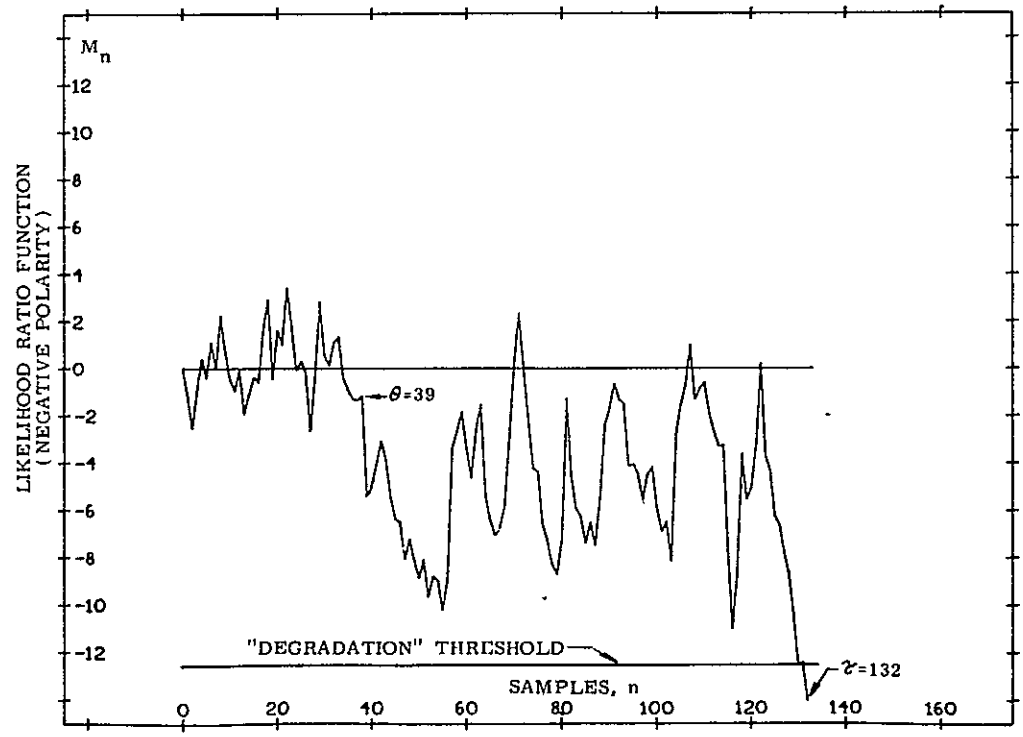


FIG. 4.20 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_3

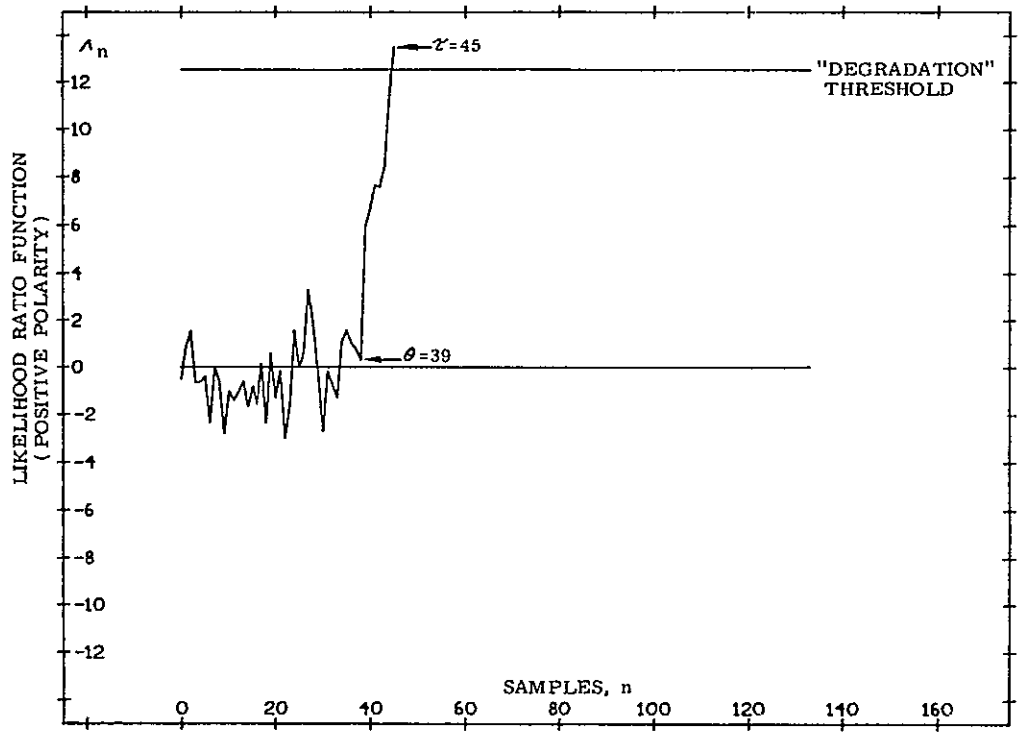


FIG. 4.21 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_4

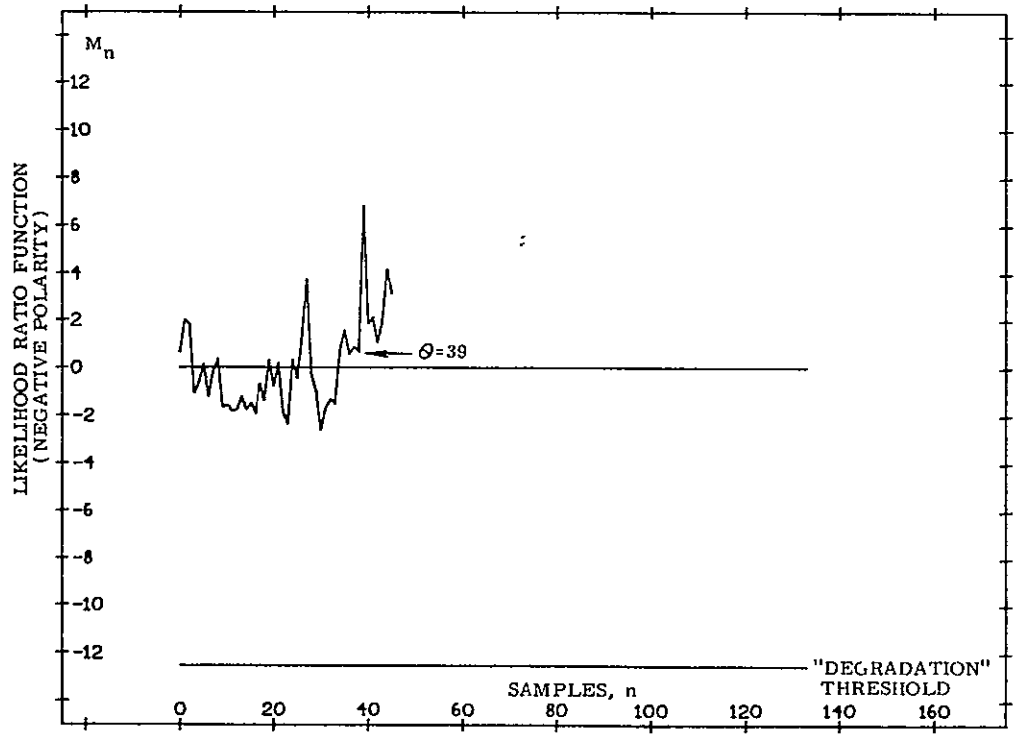


FIG. 4.22 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_4

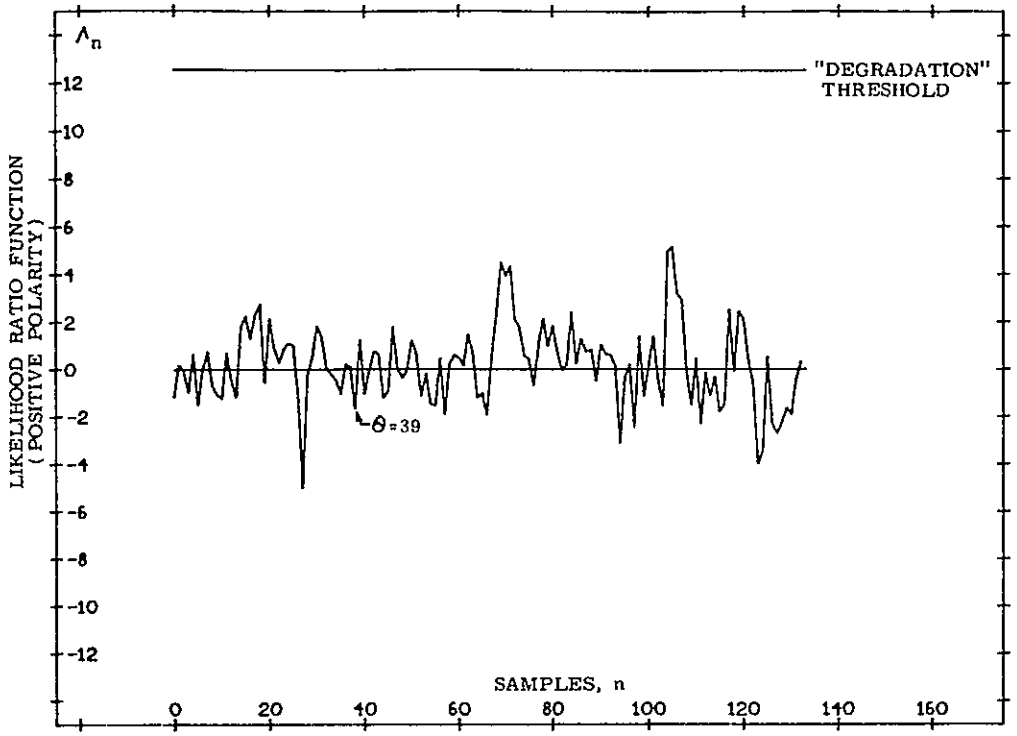


FIG. 4.23 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_5

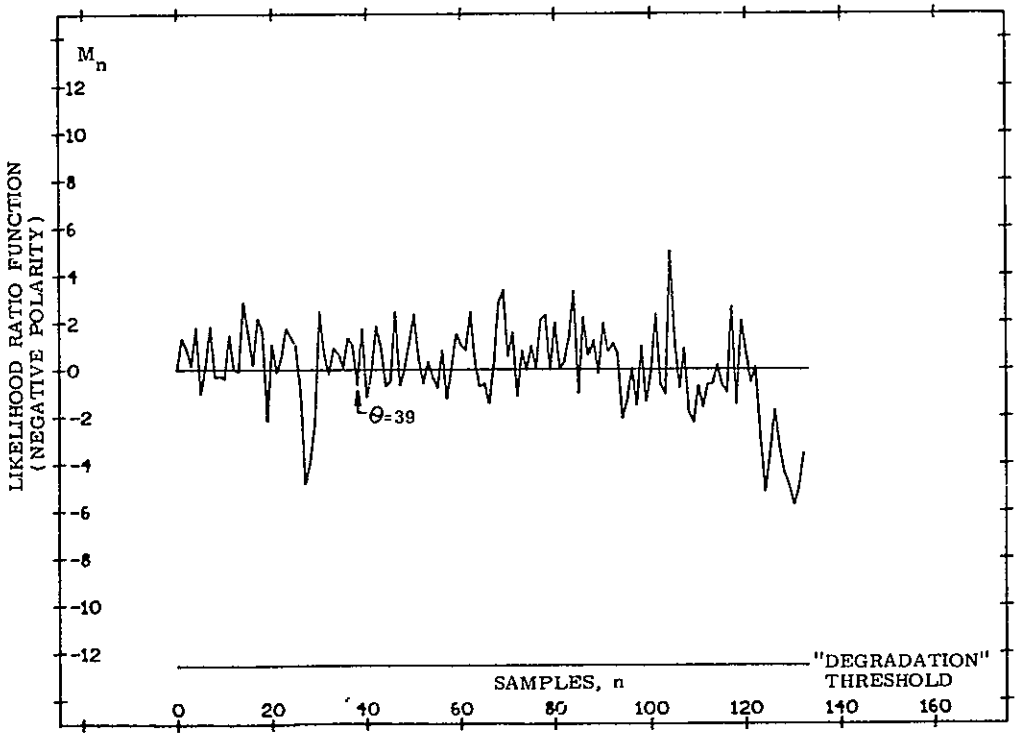


FIG. 4.24 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_5

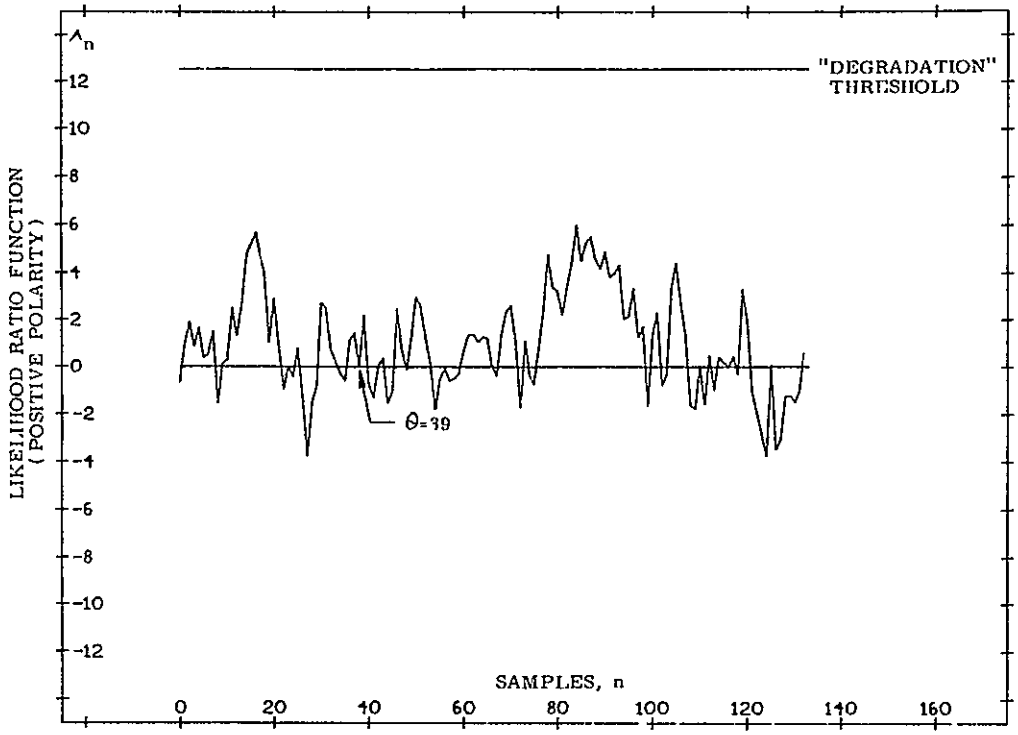


FIG. 4.25 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_6

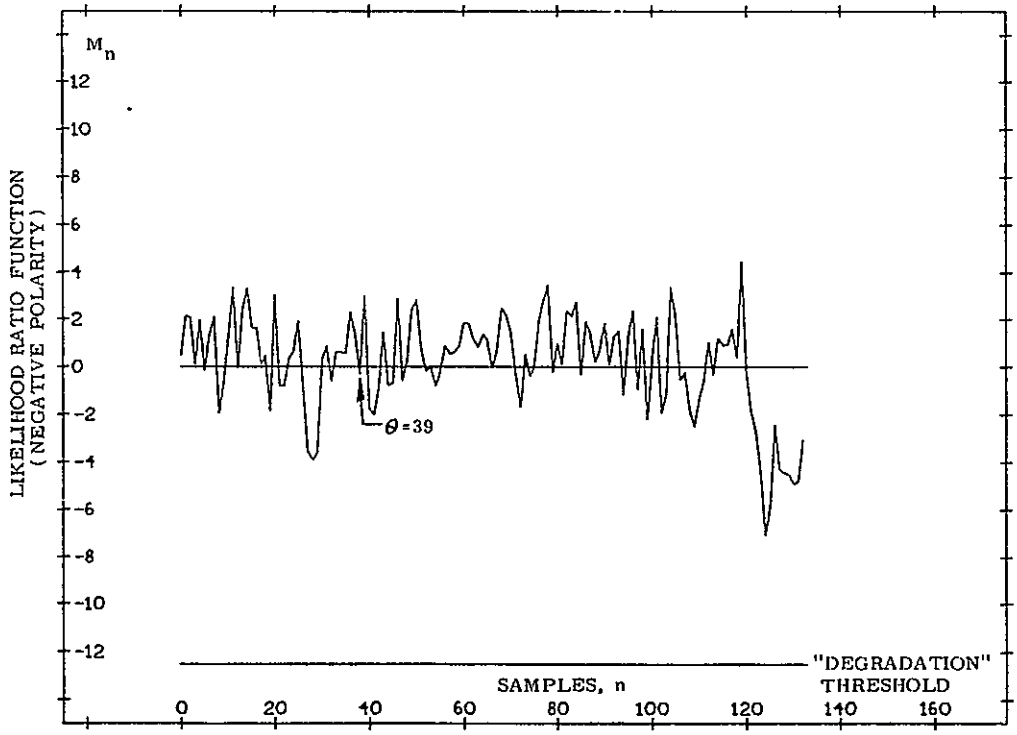


FIG. 4.26 DETECTION TRAJECTORY OF PARITY EQUATION RESIDUAL Z_6

CHAPTER 5

Identification System

5 1 General Discussion

The identification system is the decision stage to determine the nature of the degradation modes of the operating system Verification of the normal mode and classification of degradation modes are the decisions to be made in the identification system Verification is necessary because of the probable false alarms made in the detection stage or the probable requalification of a "degraded" instrument Classification identifies the information about the nature of degradation in the form of a constant bias or ramp bias change in the drift rate

The data processing of the identification system is implemented as follows After a "degraded" instrument has been isolated, the identification system will process a specific parity equation residuals from the parity equation set C (Table 2-4). The residual is chosen to give the highest sensitivity of the degradation bias signal with respect to the isolated "degraded" instrument The basic technique employed for the identification system formulation is Wald's sequential probability ratio test procedure An invariant transformation is developed to transform the ambiguous three-class identification problem into a set of pair wise disjoint two-class identification problems.

5 2 The Problem Formulation

The identification system is characterized by the following features First, the "degraded" instrument is operating and remains operating in one of the specified modes, no transition of modes is assumed to occur In the identification stage of a relatively short duration with respect to the mean life time of instruments this will be a reasonable assumption

Second, no a priori information about the instrument operating condition can be assumed The system must consider all alternate modes and the two-sided decision process must be used Identification is a single stage decision process The process terminates whenever acceptance of either class is decided Third, the reliability of the decision is of main concern in the identification system The false alarm and miss alarm error probabilities are chosen as the performance criteria Since in the identification system both boundaries of alternate modes must be specified, therefore both error probabilities are well-defined

The above analysis suggests that Wald's sequential probability ratio test (SPRT) is a natural formulation for the identification system The original discrete-time case of the SPRT is well-known (Wald[80]), the continuous-time version of the SPRT will be discussed In Chapter 3 the stochastic differential equation $\lambda(t)$ for the case of an unobservable process in the form of an unknown yet unchanged constant was derived,

$$d\lambda(t) = \frac{(a_1 - a_0)}{\sigma^2} \left(dy - \frac{1}{2}(a_1 + a_0)\, dt\right) \qquad (5\text{-}1)$$

where

$$\lambda(t) = \ell n \left[\frac{\pi(t)}{1-\pi(t)}\right] \quad , \lambda(0) = 0$$

The observable process is defined as

$$dy = a\,dt + \sigma\, d\omega(t)$$

where $a = a_1$ for the degradation mode, or $a = a_0$ for the normal mode.

A study of the definition of the posterior probability $\pi(t)$ will show the fact that the quantity $\lambda(t)$ is the same as the logarithm of the likelihood ratio function defined in Wald's SPRT process The initial condition $\lambda(0)$ with a value of zero reflects a state of no a priori information of the system being in either normal or degradation mode

The performance criteria α and β will now be derived by application of the theory of first passage problem. Let A and B be the lower and

upper decision boundaries for identification of the normal and degraded modes Let $\alpha(x)$ and $\beta(x)$ denote the error probabilities with respect to the process $\lambda(t)$ given $\lambda(0) = x$, and $A < x < B$ Then by definition the false alarm error probability $\alpha(x)$ denotes the probability of the trajectories of the process $\lambda(t)$ which is defined in the normal mode and which first exits through the upper boundary B Similarly the error probability $\beta(x)$ denotes the probability of the trajectories of the process $\lambda(t)$ which is defined in the degradation mode and which first exits through the lower boundary A

For evaluation of the error probabilities $\alpha(x)$ and $\beta(x)$ defined for the specified initial condition x, the backward equation is the appropriate equation to apply Let $p_\lambda(x;y,t)$ be the transition probability of the homogeneous diffusion process

$$d\lambda = \mu(\lambda)\, dt + \sigma(\lambda)\, dw(t) \tag{5-2}$$

then $p_\lambda(x;y,t)$ satisfies the backward equation:

$$\frac{\partial p}{\partial t} = \mu(x) \frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2(x) \frac{\partial^2 p}{\partial x^2} \tag{5-3}$$

Let $\phi(x)$ be the ultimate probability of absorption of trajectories $\lambda(t)$ in one of the boundaries A or B, conditioned on $\lambda(0) = x$. By Laplace transformation of (5-3) and manipulations as in Section 4 5.1, it can be shown that $\phi(x)$ is the solution to the equation[14]:

$$\frac{1}{2}\sigma^2(x) \frac{d^2\phi}{dx^2} + \mu(x) \frac{d\phi}{dx} = 0 \tag{5-4}$$

The appropriate boundary conditions must be used For example, if $\phi_B(x)$ is the probability that absorption in the boundary B occurs before A, then the appropriate boundary conditions are $\phi_B(B) = 1$, and $\phi_B(A) = 0$

The above technique will be applied to evaluate $\alpha(x)$ and $\beta(x)$, since in sequential procedures the error probability by definition represents the ultimate probability of absorption of the trajectory $\lambda(t)$ in the boundary The stochastic process $\lambda_0(t)$ for evaluation of the error probability $\alpha(x)$ is defined in the normal mode H_0, and is represented by

$$d\lambda_0(t) = -\frac{1}{2}\frac{a_1^2}{\sigma^2}dt + \frac{a_1}{\sigma}dw(t) \triangleq \mu_0\,dt + b\,dw(t) \tag{5-5}$$

Then the error probability $\alpha(x)$ is the solution to the equation

$$\frac{1}{2}b^2\frac{d^2\psi}{dx^2} + \mu_0\frac{d\psi}{dx} = 0 \tag{5-6}$$

with the boundary conditions $\psi(A) = 0$, and $\psi(B) = 1$ The boundary conditions imply that the absorption in the upper boundary B occurs before the lower boundary A. Substituting the definitions of b and μ_0 into (5-6) produces

$$\frac{d^2\psi}{dx^2} - \frac{d\psi}{dx} = 0 \tag{5-7}$$

then $\alpha(x)$ can be solved to be

$$\alpha(x) = \frac{e^x - e^A}{e^B - e^A} \tag{5-8}$$

Similarly the error probability $\beta(x)$ is the solution to the differential equation

$$\frac{d^2\psi}{dx^2} + \frac{d\psi}{dx} = 0 \tag{5-9}$$

with the boundary conditions: $\psi(B) = 0$, $\psi(A) = 1$. The solution $\beta(x)$ is shown to be

$$\beta(x) = \frac{e^A(e^{B-x} - 1)}{e^B - e^A} \tag{5-10}$$

It is interesting to compare these results with those in Wald's original

formulation. By definition,

$$\alpha \triangleq \alpha(0) = \frac{1-e^{A}}{e^{B}-e^{A}}$$

$$\beta \triangleq \beta(0) = \frac{e^{A}(e^{B}-1)}{e^{B}-e^{A}} \tag{5-11}$$

Solving the simultaneous equations one obtains Wald's well-known formula connecting the boundaries A, B with the error probabilities α and β

$$A = \ell n\left(\frac{\beta}{1-\alpha}\right)$$

$$B = \ell n\left(\frac{1-\beta}{\alpha}\right) \tag{5-12}$$

These results have been mentioned in (4-58) of Section 4.4.1. In the discrete-time case, Wald has managed to derive the same results by the approximation of neglecting the excess of the likelihood function over the boundaries In the continuous-time case the formula gives the exact solution and the derivation is much simpler

5 3 An Invariant Transformation

It was shown in Chapter 3 that the following approach is used in the design of the detection system

(1) The unknown bias parameters of the degradation modes are not estimated, instead the detection process is constructed on the basis of a "design" value, specified by the mission performance requirement

(2) The degradation mode in the form of a ramp change of the mean drift rate is not distinguished from that in the form of a constant bias jump Instead only one detection system is designed for the degradation mode in the form of the constant bias jump.

The approach is justified by the application of a comparison theorem for diffusion processes. However the above approach cannot be applied

in the identification system In the identification system the basic information used for identification of different degradation modes is the likelihood ratio function of the joint conditional probability density defined as

$$\Lambda_n = \ell n \frac{p(y_1, \ldots, y_n \mid H_i, b_i)}{p(y_1, \ldots, y_n \mid H_j, b_j)} \tag{5-13}$$

where H_i is the ith-degradation mode, b_i the associated unknown parameter, and $\{y_n\}$ is the sequence of the observation data The determination of the joint likelihood ratio function Λ_n in the identification process requires not only the specification of a given functional form for the conditional probability density of the degradation modes but also the unknown values of the associated parameters Thus the parameters must be estimated for calculating Λ_n Since the observation data come from unknown classes of degradation modes, the parameter b must be estimated by weighing over all possible classes:

$$p(b \mid y_1, \cdots y_n)$$
$$= \frac{p(y_n \mid b, y_1 \cdots y_{n-1}, H_i) \cdot P(H_i) + p(y_n \mid b, y_1, \cdots y_{n-1}, H_j) \cdot P(H_j)}{p(y_n \mid y_1, \cdots y_{n-1}, H_i) \cdot P(H_i) + p(y_n \mid y_1, \cdots y_{n-1}, H_j) \cdot P(H_j)} \cdot p(b \mid y_1, \cdots y_{n-1}) \tag{5-14}$$

where b represents the unknown parameters in both class, b_i and b_j Since samples are unclassified, (5-14) cannot be carried out seperately for each class in general, and the computation is very difficult (Ho et al [34]) To circumvent the difficult computation procedure, an invariant transformation is developed to eliminate the effect of unknown parameters on the identification process

The transformation can be stated as follows Let $\{y_i\}$ be a sequence of identically distributed (i d) and independent random variables of normal distribution $N(a, \sigma)$, where a is an unknown mean and σ is given Let $\{\tilde{z}_i\}$ be a sequence defined by the transformation

$$\tilde{z}_1 = \frac{1}{\sqrt{2}} y_1 - \frac{1}{\sqrt{2}} y_2$$

$$\cdots\cdots$$

$$\tilde{z}_{n-1} = \frac{1}{\sqrt{n(n-1)}}(y_1 + \cdots y_{n-1}) - \frac{n-1}{\sqrt{n(n-1)}} y_n \tag{5-15}$$

Then $\{\tilde{z}_j\}$ is an equivalent sequence of i. d and independent variables having the normal distribution $N(0, \sigma)$ with the same variance and zero mean

Moreover the following relation holds:

$$\sum_{j=1}^{n-1} \tilde{z}_j^2 = \sum_{i=1}^{n} (y_i - \bar{y}_n)^2 \tag{5-16}$$

where $$\bar{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i$$

The proof of the results will be presented in Appendix C.

The transformation reduces the given data $\{y\}$ to the transformed data $\{\tilde{z}\}$, and replaces in effect the original family of joint distribution $P_n(y_n, a)$ with the new family of distribution $P_n(\tilde{z}_n, \tilde{\lambda})$ Here $\tilde{\lambda} = \tilde{\lambda}(a)$ denotes the parameter corresponding to the original unknown parameter a induced by the transformation The transformation leaves the identification problem invariant in the sense that the decision procedure at any stage about the original process with an unknown parameter can be transformed into an invariant decision procedure based on the transformed process

This transformation is useful only for the identification stage, where no transition of states exists, i e the observations are random variables of identical distribution For the detection stage, where a transition of states is possible, the mean cannot be eliminated by this transformation

The identity (5-16) gives the intuition that the mean is eliminated by utilization of the best available information of the updated sample mean. For finite storage the transformation must be implemented in the following form

$$\tilde{z}_{n-1} = \frac{1}{\sqrt{n(n-1)}} \sum_{i=1}^{n-1} y_i - \frac{n-1}{\sqrt{n(n-1)}} y_n$$

In this case only the cumulated sum of past data need be stored

In Chapter 3 it was shown that if the measurement process z(t) is defined to be in the degraded mode with an unknown mean bias, the transformed measurement residuals (3-39) will be asymptotically of i d. and independent random variables of normal distribution with a corresponding unknown mean bias and a given variance In the case of the ramp bias the transformed measurement residual (3-40) will be asymptotically of independent random variables of normal distribution with a corresponding unknown mean of approximate ramp bias and a given variance An application of the invariant transformation to the measurement residuals will then transform the original process with unknown constant or ramp bias into the corresponding process with zero or an attenuated ramp bias.

Original Class	Transformed Class
H_1: $N(a, \sigma)$	$N(0, \sigma)$
H_2: $N(bk, \sigma)$	$N(-\frac{1}{2}bk, \sigma)$

where a and b are unknown constant parameters, k is the sample index

The attenuated ramp rate $-\frac{1}{2}bk$ can be derived as follows Let the original process {y} be defined in the degradation mode with a mean ramp rate b Then from (5-15):

$$
\begin{aligned}
E[\tilde{z}_{n-1}] &= \frac{1}{\sqrt{n(n-1)}} \cdot [b(\sum_{i=1}^{n-1} i) - (n-1) \cdot n \cdot b] \\
&= -\frac{1}{2}\sqrt{n(n-1)} \cdot b \\
&\cong -\frac{1}{2}(n-1) \cdot b \qquad (\text{if } n \text{ is large})
\end{aligned}
\tag{5-17}
$$

It is noted that a positive ramp bias in the original process {y} is transformed into half the ramp bias with a negative polarity The presence of a ramp rate in the transformed process will give an effective information for identification of the degradation mode H_2, even if the parameter of the

ramp rate in the original process is of an insignificant magnitude

To eliminate the effect of the constant bias in the original process $\{y\}$ a simple transformation defined in (5-18) can be designed

$$\tilde{z}_{n-1} = y_n - y_{n-1} \; ; \quad n \geq 2 \tag{5-18}$$

In this case a ramp bias with rate b of one sample interval in the original process is transformed into a constant bias with magnitude b In comparison, the proposed transformation (5-15) will give more effective information for identification after a sample size of three observations of the original process Since the parameter b of the ramp rate in the practical system is usually of an insignificant magnitude, the identification capability of a small magnitude b by the transformation (5-18) in a comparatively strong background noise level σ^2 is questionable

It has thus been shown that the ambiguous multi-class (i.e , three classes) identification process can now be transformed into a set of pair wise disjoint two-class identification processes The verification of the normal mode is done by processing the measurement residuals of the original process The classification of degradation modes into constant bias or ramp bias mode is done by processing the transformed measurement residuals. Both decisions are reduced to a simple testing procedure against the presence of a constant mean (with known polarity) A mean is considered to be present if its magnitude exceeds a specified value $B = \delta \cdot \sigma$ where σ is the given standard deviation and δ is a positive specified parameter.

In Chapter 4 it was shown that the suboptimal detection system can simultaneously identify the bias polarity in detecting the degradation. It is important to assign the correct polarity to the specified value of the decision threshold B for the transformed process It is noted from (5-25) that a positive (negative) ramp bias rate in the original process will be transformed into a negative (positive) ramp bias rate A specification of incorrect polarity will cause the sequential probability ratio test process to make a wrong identification decision This is illustrated in Fig 5 1

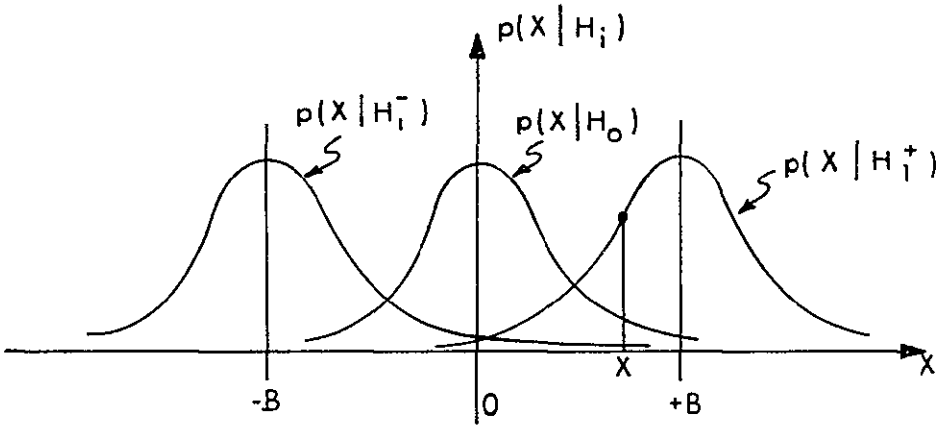


Fig. 5.1. Effectof Polarity on Identification System.

In this figure, the sample x will be correctly classified into the degradation mode if the right polarity is assigned for the specified value B of the decision threshold for identification. However the sample x will be incorrectly classified into the normal mode if the wrong polarity is assigned.

The block diagram of the identification system is given in Fig. 5.2.

5.4 Simulation Results

Figures 5.3-5.6 demonstrate the classification of the degradation mode into the class of constant bias or ramp bias by application of the technique of the invariant transformation (5-18). The classification is done by processing the transformed measurement residuals. The discussion is based on Wald's sequential testing process against the presence of a specified constant mean. In the simulation results the degradation mode of a constant bias of +4 against a ramp bias of rate value of -0.11 is to be identified. The threshold value of the constant mean is specified to be $\delta \cdot \sigma$ with an appropriate polarity. In simulation the threshold value δ is specified to be a value of 0.5 and specified with a positive polarity due to the negative ramp bias. The parameter σ of the single gyro system was defined in section 4.6. Figure 5.5 illustrates the identification of the degradation mode of the ramp

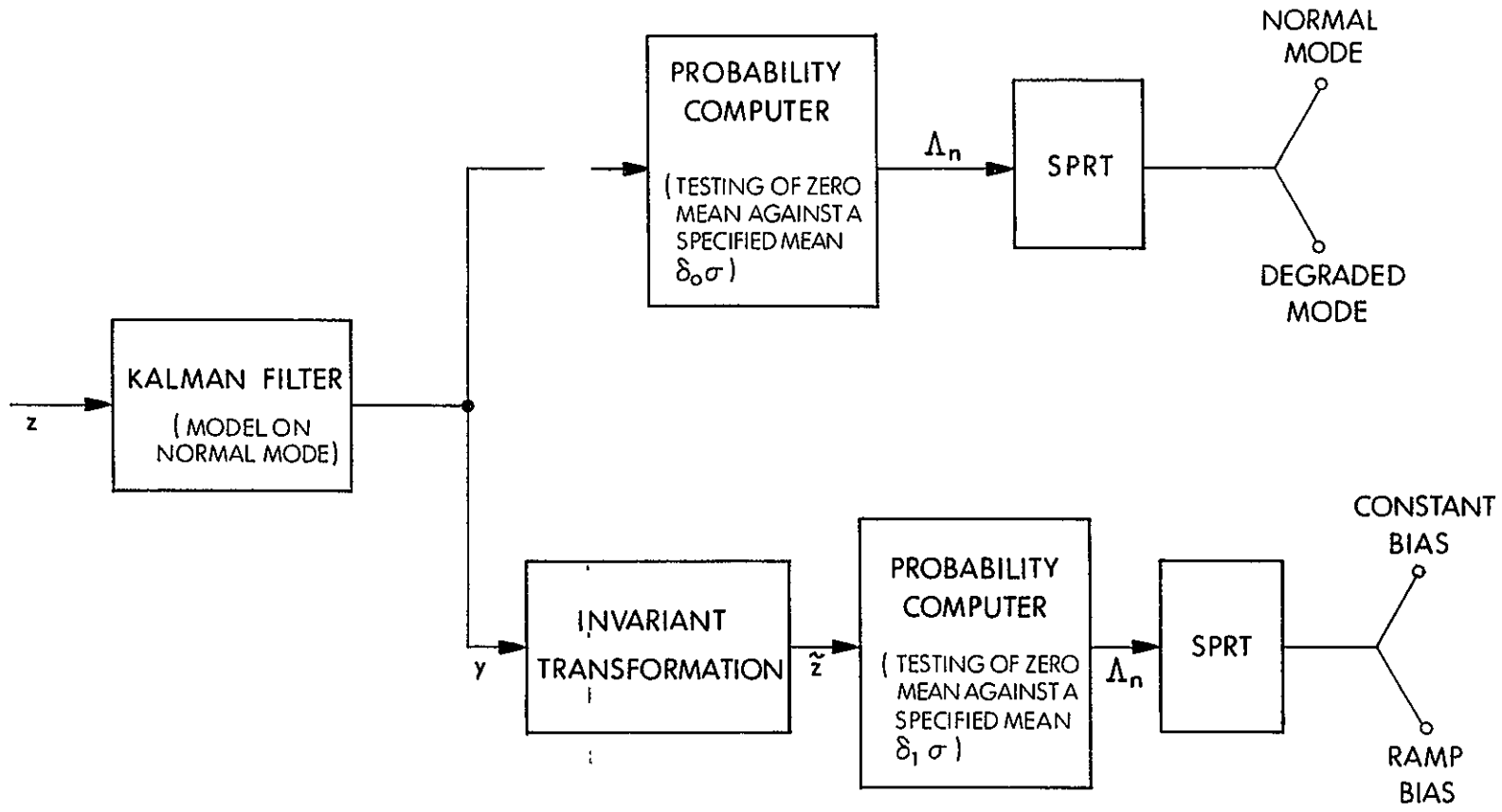


Fig. 5.2 Block Diagram of Identification System

bias. The likelihood ratio function shows that during the early stage of identification the decision moves toward the class of the constant bias due to the relatively small parameteric value of the ramp rate. As the number of samples increases, the weighting factor of the current data in the transformed residuals become more significant to indicate the presence of the ramp bias. The likelihood function reflects the information and identifies correctly the class of the ramp bias The identification of the class of the constant bias is illustrated in Fig 5. 3 Figure 5 4 and Figure 5 6 demonstrate the trajectories of the transformed measurement residuals. It can be seen, though not very clearly, that the trajectory in Fig 5 4 shows a zero mean, and the trajectory in Fig 5 6 shows the trend of a positive ramp rate This illustrates the justification of the invariant transformation discussed in section 5. 3.

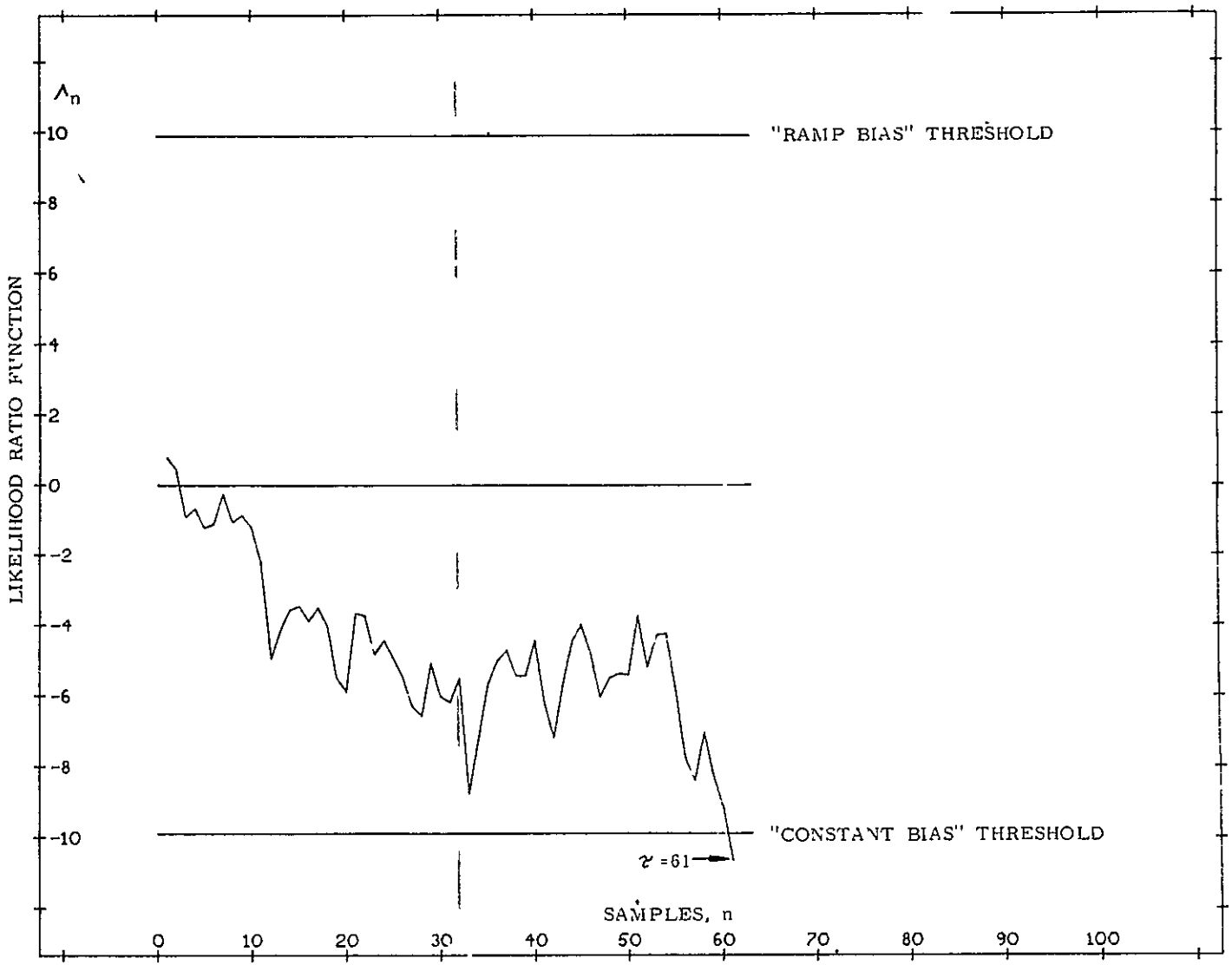


FIG 5 3 IDENTIFICATION TRAJECTORY OF CONSTANT BIAS MODE

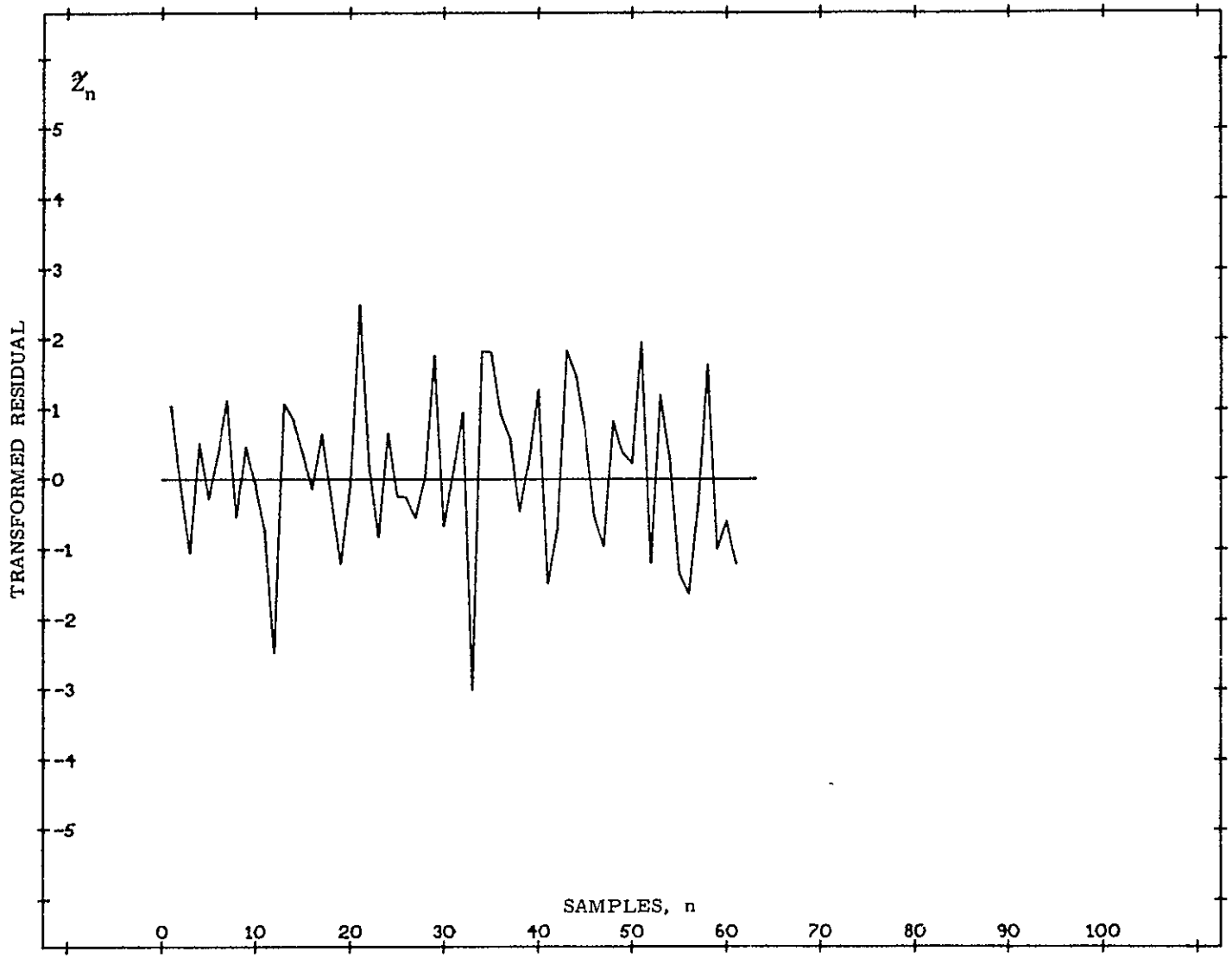


FIG. 5.4 TRAJECTORY OF TRANSFORMED RESIDUAL (CONSTANT BIAS MODE)

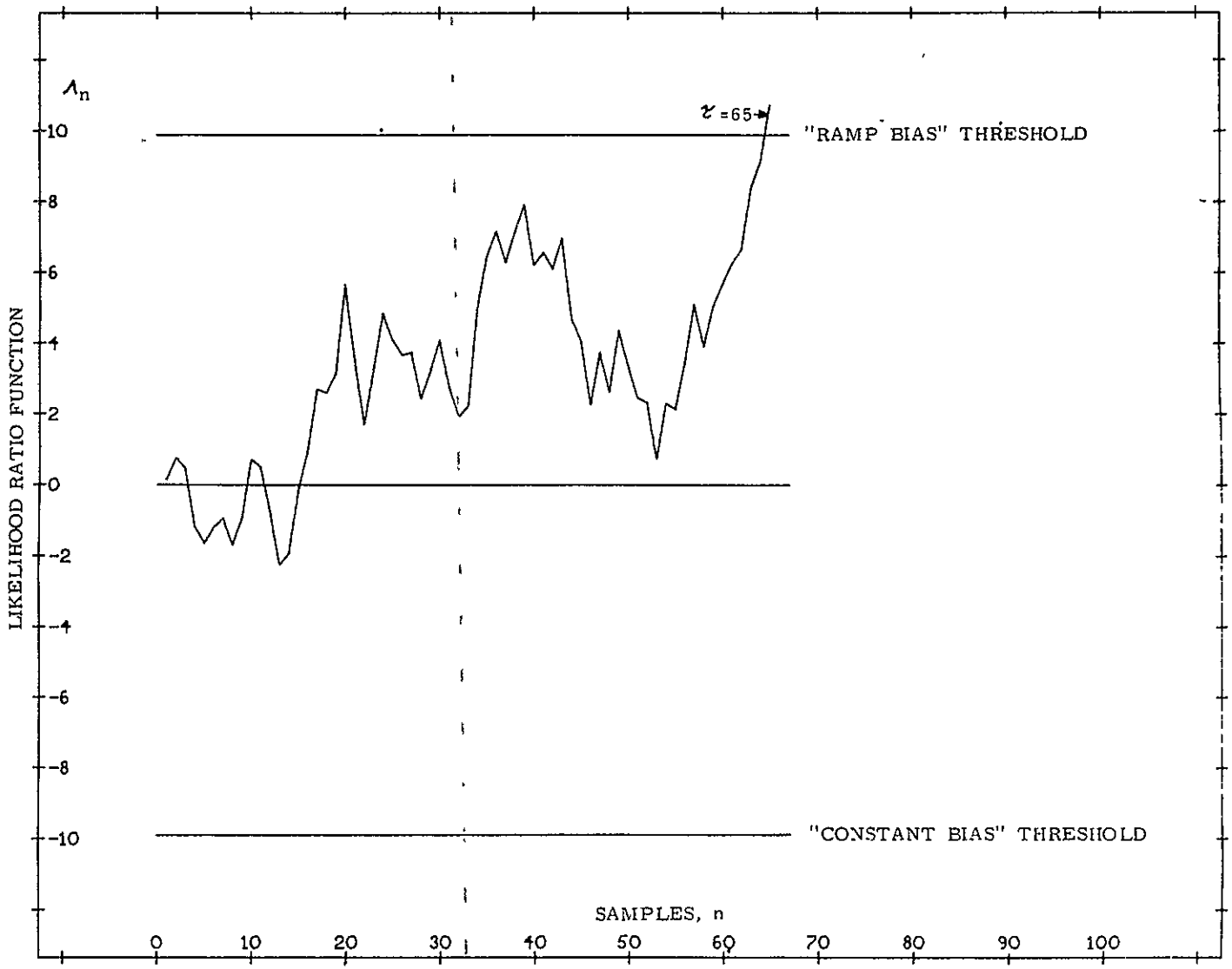


FIG. 5 5 IDENTIFICATION TRAJECTORY OF RAMP BIAS MODE

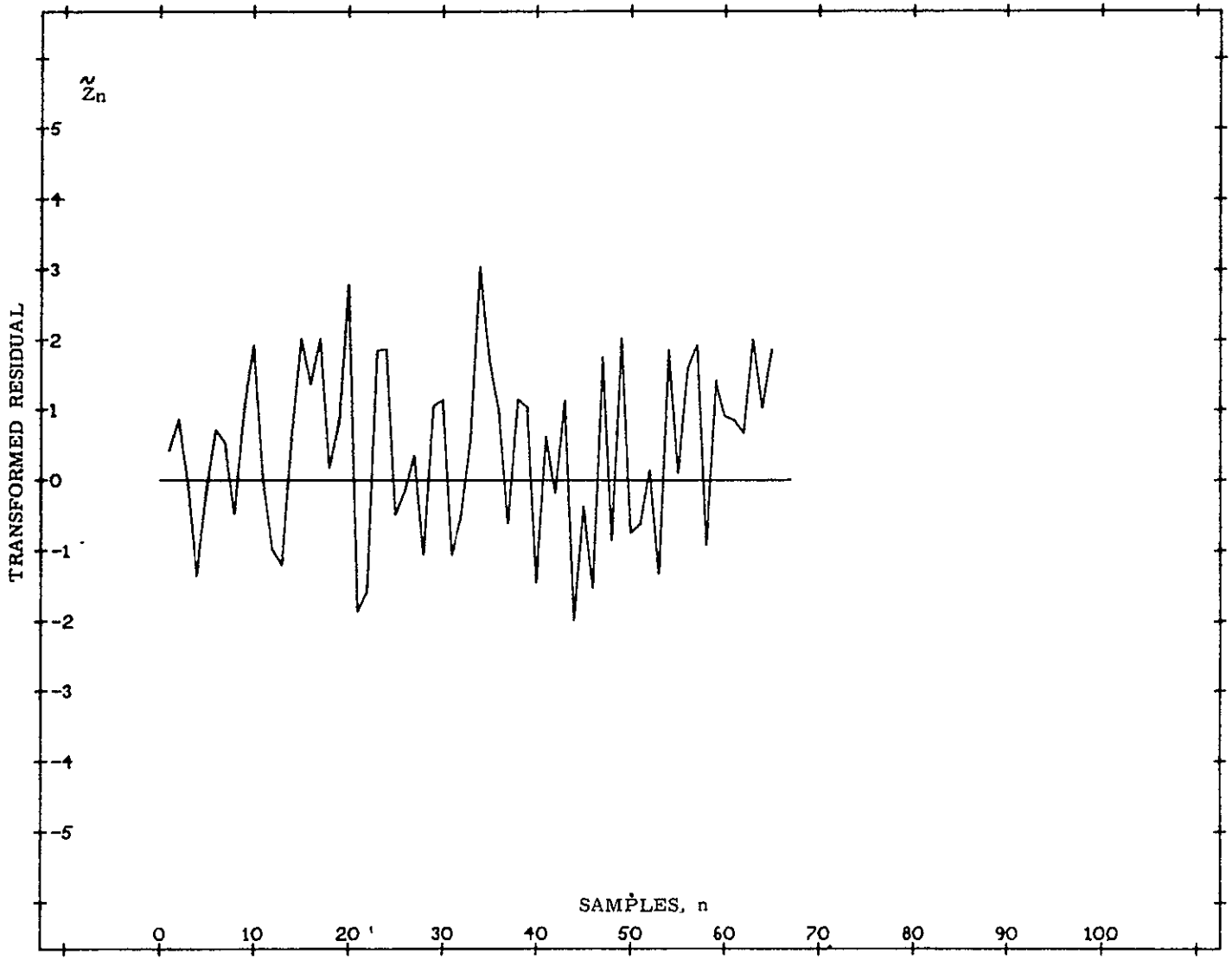


FIG. 5.6 TRAJECTORY OF TRANSFORMED RESIDUAL (RAMP BIAS MODE)

CHAPTER 6

Compensation System

6 1 Problem Statement

After the degradation mode has been identified, a compensation system is designed to recover the degraded instrument performance, if possible. Since an accurate knowledge of the model of the degraded system is now available, the design of the compensator is simply reduced to that of a statistical estimator to estimate the unknown bias of the identified mode (H_1 or H_2)

The problem of estimating a constant but unknown bias in recursive filtering has been extensively studied In this chapter, a technique of decoupled estimator is applied to estimate the unknown bias The convergence rate of the bias estimation will be studied Finally, some discussion of the simulation results will be presented

6 2 A Decoupled Bias Estimator

B Friedland[29] has developed a technique of estimating an unknown bias by a decoupled estimator. In this technique the bias estimate can be computed in terms of residuals in the bias-free estimate, and thus effectively is decoupled from estimation of the state A block diagram of the estimator is shown in Figure 6.1.

The motivation of this technique is to reduce the dimension of the estimate of the state variable in the filtering process, thus avoiding the computational difficulties, speed as well as numerical inaccuracies, associated with large matrices This will be the case when the number of bias terms is comparable to the number of state variables of the original problem The new state vector to be estimated will be substantially increased in dimension by the augmented additional bias terms However, the motivation to

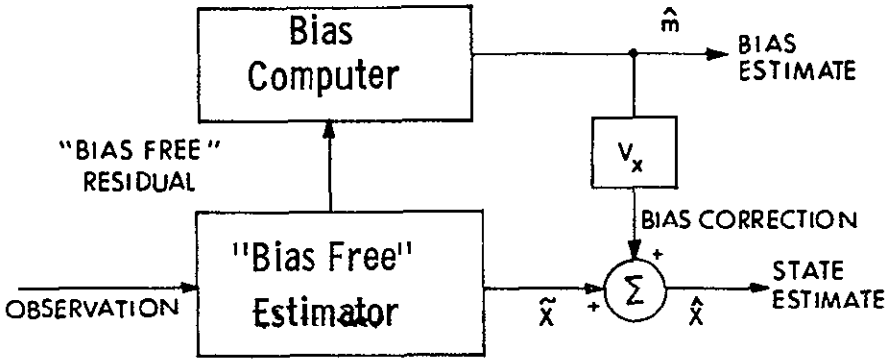


Fig. 6.1. Block Diagram of Bias Estimator

use this technique for the compensation system design in this thesis results from a different consideration. During most of the mission period the system will be operated in the normal mode In this case a recursive filter is designed in the bias-free condition to generate the input information for the detection and identification systems. Once an instrument performance degradation has been detected and identified, the filter should be modified to estimate the associated bias value This must be done with as simple modification of the original filter as possible so that the detection and identification processes can proceed as before without any complicated reorganization. This motivates the use of a decoupled bias estimator

(1) Computation of Decoupled Bias Estimate

The results of the decoupled estimator are based on expressing the solution of the variance equation of the problem with bias present in terms of the solution of the variance equation for bias-free estimation and other matrices which depend only on the bias-free computations The following presentation is essentially a simplified restatement of Friedland's results

The notations used in the derivation are consistent with those of Chapters 3 and 4

Let the dynamic equations be expressed as follows:

$$\dot{x}(t) = -\omega x(t) + g(t) \tag{6-1}$$

$$\dot{m} = 0 \tag{6-2}$$

and the observation equation is expressed as

$$z(t) = x(t) + c(t) \cdot m + v(t) \tag{6-3}$$

The following notations are used

x = original state (scalar)

m = bias term (scalar)

z = observation (scalar)

g = process noise, with given statistics of zero mean and variance $E[g(t)g(\tau)] = W \cdot \delta(t-\tau)$

v = observation noise, with given statistics of zero mean and variance $E[v(t)v(\tau)] = V \cdot \delta(t-\tau)$

The factor $c(t)$ determines how the bias m enters into the observations For the designed system, $c = 1$ for the degraded mode H_1 and $c = t$ for the mode H_2

Define a new augmented state vector

$$\tilde{y} = \begin{bmatrix} x \\ m \end{bmatrix}$$

The dynamic and observation equations can be written as

$$\dot{\tilde{y}} = A\,\tilde{y} + G\,g \tag{6-4}$$

$$z = L\,\tilde{y} + v \tag{6-5}$$

where

$$A = \begin{bmatrix} -\omega & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L = [1\ c]$$

Application of the Kalman-Bucy filtering theory results in the following equations for optimum estimate $\hat{\tilde{y}}$ of the state $\tilde{y}$

$$\dot{\hat{y}} = A\hat{y} + P L^T V^{-1} (z - L\hat{y}) \tag{6-6}$$

where the a posteriori covariance matrix P is the solution to the covariance equation.

$$\dot{P} = AP + P A^T - P L^T V^{-1} LP + GW G^T \tag{6-7}$$

The covariance matrix P is partitioned as follows

$$P = \begin{bmatrix} P_x & P_{xm} \\ P_{xm} & P_m \end{bmatrix} \tag{6-8}$$

The variance equations then take the form of the following scalar equations

$$\dot{P}_x = -z\omega\ P_x - \frac{(P_x + c\,P_{xm})^2}{V} + W \tag{6-9}$$

$$\dot{P}_{xm} = \left[-\omega - \frac{(P_x + c \cdot P_{xm})}{V}\right] P_{xm} - \frac{(P_x + c \cdot P_{xm}) \cdot c}{V} \cdot P_m \tag{6-10}$$

$$\dot{P}_m = - \frac{(P_{xm} + c\,P_m)^2}{V} \tag{6-11}$$

It is noted that equations (6-10) and (6-11) together are homogeneous in P_{xm} and P_m. Hence, if

$$P_{xm}(0) = 0$$

$$P_m(0) = 0$$

then for all $t > 0$,

$$P_{xm}(t) = 0$$

$$P_m(t) = 0 \tag{6-12}$$

and in this case P_x satisfies

$$\dot{P}_x = -2\omega P_x - \frac{P_x^2}{V} + W \tag{6-13}$$

The interpretation of the equation (6-12) and (6-13) is that if the bias m is perfectly known ($P_m(0) = 0$) at $t = 0$, then by virtue of (6-2), it is perfectly known thereafter and the estimation problem reduces to that in which there is no bias The variance equation (6-13) is the same as would result if m were known to be zero (with probability 1). Let the covariance matrix in this case be denoted by

$$\tilde{P} = \begin{bmatrix} \tilde{P}_x & 0 \\ 0 & 0 \end{bmatrix} \tag{6-14}$$

where $\tilde{P}$ is the solution to Eq. (6-7) for the initial condition,

$$\tilde{P}(0) = \begin{bmatrix} \tilde{P}_x(0) & 0 \\ 0 & 0 \end{bmatrix} \tag{6-15}$$

and $\tilde{P}_x$ is the solution to Eq. (6-13) with $\tilde{P}_x(0)$ given.

If the bias is not perfectly known, then (6-15) is not the correct initial condition for the problem to be solved Let

$$P(0) = \begin{bmatrix} \tilde{P}_x(0) & P_{xm}(0) \\ P_{xm}(0) & P_m(0) \end{bmatrix}$$

where $P_m(0)$ must be non-zero

It is noted that in (6-9) - (6-11) the equations for P_x, P_{xm}, and P_m are all coupled and hence must be solved together A transformation will now be derived to decouple the solution of the variance equations The following result is used that if $\tilde{P}$ is a solution to (6-7) then any other solution can be expressed as follows

$$P = \tilde{P} + \tilde{V} M \tilde{V}^T \tag{6-16}$$

where

$$\dot{\tilde{V}} = (A - \tilde{P} L^T V^{-1} L)\tilde{V} \tag{6-17}$$

$$\dot{M} = -M \tilde{V}^T L^T V^{-1} L \tilde{V} M \tag{6-18}$$

where M is a scalar in the designed system case and $\tilde{V}$ can be partitioned as

$$\tilde{V} = \begin{bmatrix} V_x \\ V_m \end{bmatrix}$$

Then Eq. (6-17) and (6-18) can be expressed as three scalar equations

$$\dot{V}_x = \left(-\omega - \frac{\tilde{P}_x}{V} \right) V_x - \frac{\tilde{P}_x \cdot c}{V} V_m \tag{6-19}$$

$$\dot{V}_m = 0 \text{ i.e. } V_m = \text{constant}$$

and

$$\dot{M} = - \frac{(V_x + V_m \cdot c)^2 \quad M^2}{V} \tag{6-21}$$

The variance equations have been decoupled It is possible to solve for $\tilde{P}_x$ independent of P_{xm} and P_m, then solve V_x, V_m, and then M

The actual values of the desired variances are evaluated as

$$P_x = \tilde{P}_x + V_x^2 \cdot M$$

$$P_{xm} = V_x M V_m \tag{6-22}$$

$$P_m = V_m^2 \quad M$$

The initial conditions are not unique In the important special case in which $P_{xm}(0) = 0$, i e , there is not a priori correlation between the state and the bias, a convenient choice of initial conditions is·

$$M(0) = P_m(0) \neq 0$$

$$V_x(0) = 0 \tag{6-23}$$

$$V_m = 1$$

In this case

$$P_{xm} = V_x M$$

$$P_m = M \tag{6-24}$$

The results of this analysis can now be applied to obtain a form of the decoupled bias estimate from the equation (6-6)

$$\dot{\hat{x}} = -\omega \hat{x} + \frac{(P_x + c P_{xm})}{V} (z - \hat{x} - c \hat{m}) \tag{6-25}$$

$$\dot{\hat{m}} = \frac{(P_{xm} + P_m \quad c)}{V} (z - \hat{x} - c \hat{m}) \tag{6-26}$$

The equations (6-25) and (6-26) can be shown to be expressed in the decoupled form by the transformation:

$$\hat{x} = \tilde{x} + V_x \hat{m} \tag{6-27}$$

Then the final result for the bias estimate can be expressed as

$$\dot{\hat{m}} = -\frac{M(V_x + c)^2}{V} \hat{m} + \frac{(V_x + c)M}{V} (z - \tilde{x}) \tag{6-28}$$

where $\tilde{x}$ is the estimate of x which would be obtained in the absence of bias, i e $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = -\omega \tilde{x} + \frac{\tilde{P}_x}{V} (z - \tilde{x}) \tag{6-29}$$

The asymptotic solutions of (6-27) and (6-28) are of interest It can

easily be shown by substituting (6-24), and the equations (6-50) and (6-54), (6-51) and (6-59) in the next subsection into (6-28) that the asymptotic solution of $\hat{m}$ can be represented as

$$\hat{m} = m \tag{6-30}$$

and therefore the asymptotic solution of $\hat{x}$ will be

$$\hat{x} = \tilde{x} + V_x \quad m \tag{6-31}$$

These results have been utilized in the discussion of the transformation of the measurement processes (3-12) and (3-25) in Section 3 3

A schematic representation of the computation of estimates in the presence of bias is shown in Fig 6 2.

The corresponding results for the discrete-time case can be summarized as follows

The dynamic equations are defined as

$$\begin{aligned} x(k) &= F\, x(k-1) + q_1\,(k-1) \\ m(k) &= m(k-1) \end{aligned} \tag{6-32}$$

The observation equation is

$$z(k) = x(k) + c(k)\, m(k) + r_1(k) \tag{6-33}$$

where $c(k) = 1$ in degraded mode H_1, and $c(k) = k$ in mode H_2.

The process noise and observation noise are given with statistics of zero mean and variances: $E[q_1(k)\, q_1(\ell)] = Q\, \delta_{k\ell}$ and $E[r_1(k) \cdot r_1(\ell)] = R\, \delta_{k\ell}$ respectively.

The derivation for the discrete-time case is more complicated, but is based on a transformation of the discrete variance equation as in the form of equation (6-16) for the continuous-time case

$$P'(k) = \tilde{P}'(k) + U(k)\, M(k)\, U^T\,(k)$$

where $P'(k)$ is the a priori covariance matrix of the estimate of the augmented state at time k conditioned on the measurement history

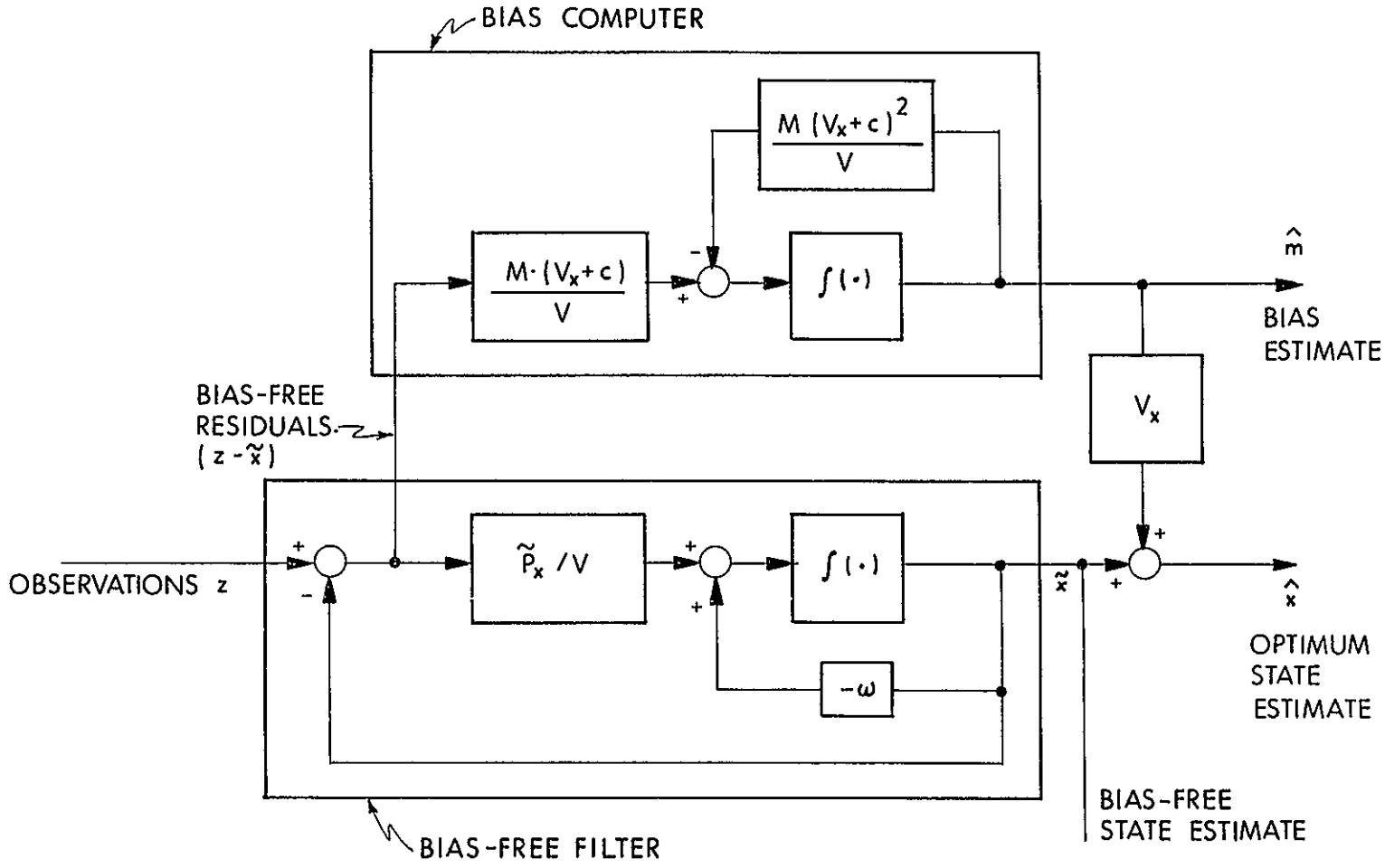


Fig. 6.2 Scheme of Computation of Estimates in Presence of Bias

$\{z(1) \ldots z(k-1)\}$. $\tilde{P}'(k)$ is any solution to the covariance equation of $P'(k)$ with $\tilde{P}'(0) \neq P'(0)$. $U(k)$ and $M(k)$ are auxiliary matrices The derivation will then be similarly carried out by partitioning the augmented matrix $P'(k)$ as in the continuous-time case

Let $P'_x(k)$ be the a priori variance of the estimate of $x(k)$ conditioned on the measurement history $\{z(1) . \quad z(k-1)\}$, $P_x(k)$ be the a posteriori variance of estimate $x(k)$ conditioned on $\{z(1) .. \quad z(k)\}$. The a priori variance equation and the posterior variance equation are transformed into the following forms as in the continuous-time case,

$$\begin{aligned} P'_x(k) &= \tilde{P}'_x(k) + U_x^2(k)\, M(k) \\ P'_{xm}(k) &= U_x(k)\, M(k) \\ P'_m(k) &= M(k) \end{aligned} \tag{6-34}$$

and

$$\begin{aligned} P_x(k) &= \tilde{P}_x(k) + V_x^2(k)\, M(k+1) \\ P_{xm}(k) &= V_x(k)\, M(k+1) \\ P_m(k) &= M(k+1) \end{aligned} \tag{6-35}$$

where

$$U(k) = \begin{bmatrix} U_x(k) \\ U_m(k) \end{bmatrix}, \qquad V(k) = \begin{bmatrix} V_x(k) \\ V_m(k) \end{bmatrix}$$

and $M(k)$ is a scalar quantity computed as

$$M(k+1) = M(k) - \frac{M(k)\, S^2(k)}{\tilde{P}'_x(k) + R + S^2(k)\, M(k)} \quad M(k) \tag{6-36}$$

The auxiliary factors $U_x(k)$, $V_x(k)$ and $S(k)$ are recursively computed as follows·

$$V_x(k) = U_x(k) - \tilde{K}_x(k)\, S(k) \tag{6-37}$$

where $$S(k) = U_x(k) + c(k)$$

$$U_x(k+1) = F\, V_x(k)$$

The following initial conditions are used:

$$U_x(0) = 0\,, \quad P_{xm}(0) = 0\,, \quad U_m(0) = 1$$

The variance $\tilde{P}'_x$, $\tilde{P}_x$ and the bias-free gain $\tilde{K}_x$ for the bias-free estimator are computed in the normal manner

$$\tilde{K}_x(k) = \tilde{P}'_x(k) \,/\, (\tilde{P}'_x(k) + R) \tag{6-38}$$

$$\tilde{P}_x(k) = \left(1 - \tilde{K}_x(k)\right) \cdot \tilde{P}'_x(k) \tag{6-39}$$

$$\tilde{P}'_x(k+1) = F^2\, \tilde{P}_x(k) + Q \tag{6-40}$$

The corresponding equations for estimates $\hat{x}$ and $\hat{m}$ can be written in a form similar to the continuous-time case

$$\hat{x}(k) = \tilde{x}(k) + V_x(k)\, \hat{m}(k) \tag{6-41}$$

$$\hat{m}(k) = (1 - K_m(k)S(k))\, \hat{m}(k-1) + K_m(k)\, \tilde{r}(k) \tag{6-42}$$

where

$$\tilde{x}(k) = F\, \tilde{x}(k-1) + \tilde{K}_x(k)\, [z(k) - F\, \tilde{x}(k-1)]$$

and

$$\tilde{r}(k) = z(k) - F\, \tilde{x}(k-1)$$

are the bias-free estimate and the residual of bias-free estimation respectively. The gain $K_m(k)$ of the bias-estimator is given by

$$K_m(k) = \frac{M(k+1)\, [V_x(k) + c(k)]}{R} \tag{6-45}$$

(2) Convergence Rate of Bias Estimate:

For the on-line adaptive system the convergence rate of the bias estimate determines the rate of compensation for performance degradation, and will thus have a significant influence on over-all system reliability and performance In Chapter 2 a preliminary discussion of convergence rate of the bias estimate was presented. In this section the condition for the optimal convergence rate will be analyzed and the optimal rate of the bias estimate will be evaluated

Since the bias estimate will converge to its true value, as the mean square error of the estimate converges to zero, the rate of convergence for an estimator is often expressed in terms of mean square error of the estimate In the structure of the linear estimator the problem of obtaining the optimal convergence rate is reduced to design of an optimal gain function such that the mean square error at each and every time is minimized. By definition this is the optimal gain function derived in the minimum variance Kalman estimator.

Thus the optimal convergence rate can be evaluated by the variance equations derived for the decoupled estimator The variance $P_m(k)$ for the discrete-time case must be computed by a set of recursive formulas It is difficult to get a closed form of the convergence rate of $P_m(k)$ as a function of the time index k in the discrete-time case. A simpler development and a more readily interpretable result can be obtained for the continuous-time case. The variance equation for $P_m(t)$ in the continuous time case is represented in (6-24), (6-21) and (6-23),

$$P_m(t) = M(t) \tag{6-24}$$

$$\dot{M}(t) = -\frac{M^2(t) \cdot (V_x(t) + c(t))^2}{V} \tag{6-46}$$

with the initial condition $P_M(0) = M(0)$. Thus the problem is reduced to solution of an ordinary differential equation (6-46) Substituting (6-24) into (6-46) produces

$$\dot{P}_m(t) = -\frac{P_m^2(t)\ (V_x(t) + c(t))^2}{V}, \quad P_m(0) = P_0 \tag{6-47}$$

where $c(t) = 1$ for mode H_1, and $c(t) = t$ for mode H_2.

The quantity $V_x(t)$ satisfies the equation,

$$\dot{V}_x(t) = -\left(\omega + \frac{\tilde{P}_x(t)}{V}\right) V_x(t) - \frac{\tilde{P}_x(t)}{V} c(t), \quad V_x(0) = 0 \tag{6-48}$$

This is an ordinary differential equation with a time-variant coefficient If only the steady-state Kalman filter is implemented, then (6-48) can be simply solved Define the quantity

$$\begin{aligned} \tilde{P}_\infty &= \text{the steady state Kalman filter error variance} \\ \tilde{\beta} &= \frac{\tilde{P}_\infty}{V} > 0 \\ \tilde{\gamma} &= \left(\omega + \frac{\tilde{P}_\infty}{V}\right) > 0 \end{aligned} \tag{6-49}$$

The quantity $V_x(t)$ can be determined with the result

$$V_x(t) = -\frac{\tilde{\beta}}{\tilde{\gamma}}\left(1 - e^{-\tilde{\gamma}t}\right) \tag{6-50}$$

for degradation mode H_1, and

$$V_x(t) = -\frac{\tilde{\beta}}{\tilde{\gamma}}\left(\tilde{\gamma}t - 1 + e^{-\tilde{\gamma}t}\right) \tag{6-51}$$

for degradation mode H_2.

Define

$$W_m(t) = \frac{1}{P_m(t)} \tag{6-52}$$

Substituting the solution $V_x(t)$ (6-50) and the definition $W_m(t)$ into (6-47) produces the expression of $W_m(t)$ for degradation mode H_1

$$W_m(t) = W_m(0) + \frac{\left(1 - \frac{\tilde{\beta}}{\tilde{\gamma}}\right)^2}{V} \cdot t + \frac{2\frac{\tilde{\beta}}{\tilde{\gamma}}\left(1 - \frac{\tilde{\beta}}{\tilde{\gamma}}\right)}{\tilde{\gamma} \cdot V}\left(1 - e^{-\tilde{\gamma}t}\right)$$

$$+ \frac{\tilde{\beta}^2}{2\,V\,\tilde{\gamma}^3}\left(1 - e^{-2\tilde{\gamma}t}\right) \tag{6-53}$$

The convergence rate $P_m(t)$ for mode H_1 for sufficiently large t can be expressed as

$$P_m(t) \cong \frac{1}{k_1 + k_2 t} \tag{6-54}$$

where the quantities k_1 and k_2 are defined

$$k_1 = \frac{1}{P_m(0)} + \frac{2\left(\frac{\tilde{\beta}}{\tilde{\gamma}}\right)\left(1-\frac{\tilde{\beta}}{\tilde{\gamma}}\right)}{\tilde{\gamma} \cdot V} + \frac{\tilde{\beta}^2}{2V\tilde{\gamma}^3} \tag{6-55}$$

$$k_2 = \frac{\left(1-\frac{\tilde{\beta}}{\tilde{\gamma}}\right)^2}{V} \tag{6-56}$$

The expression of $W_m(t)$ for degradation mode H_2 can be similarly derived by use of the solution $V_x(t)$ in (6-51) and the definition (6-52).

$$W_m(t) = W_m(0) + \frac{1}{V}\int_0^t \left\{\left(1-\frac{\tilde{\beta}}{\tilde{\gamma}}\right)\sigma + \frac{\tilde{\beta}}{\tilde{\gamma}^2}\left(1-e^{-\tilde{\gamma}\sigma}\right)\right\}^2 d\sigma \tag{6-57}$$

The result is somewhat tedious, therefore only the solution $W_m(t)$ for large t will be considered.

$$W_m(t) \cong W_m(0) + \frac{1}{V} \cdot \left\{\frac{1}{3}\left(1-\frac{\tilde{\beta}}{\tilde{\gamma}}\right)^2 t^3 + \left(1+\frac{\tilde{\beta}}{\tilde{\gamma}}\right)\frac{\tilde{\beta}}{\tilde{\gamma}^2} \cdot t^2 + \frac{\tilde{\beta}^2}{\tilde{\gamma}^4} \cdot t\right\}$$

$$\cong W_m(0) + \frac{1}{3}\,\frac{\left(1-\frac{\tilde{\beta}}{\tilde{\gamma}}\right)^2}{V} \cdot t^3 \tag{6-58}$$

$$\triangleq \frac{1}{P_m(0)} + \frac{1}{3} \cdot k_2 \cdot t^3$$

where k_2 is defined in (6-56).

The convergence rate $P_m(t)$ for mode H_2 for sufficiently large t can

then be expressed as

$$P_m(t) \cong \frac{1}{\frac{1}{P_m(0)} + \frac{1}{3}k_2 \cdot t^3} \tag{6-59}$$

The following conclusions can be derived from (6-54) and (6-59):

(A) The convergence rate $P_m(t)$ is also determined by the quantity (denoted as $-V_x$ in (3-21) of Section 3 3)·

$$\frac{\tilde{\beta}}{\tilde{\gamma}} = \frac{(\tilde{P}_\infty / V)}{(\tilde{P}_\infty / V + \omega)} \tag{6-60}$$

which is dependent on the correlation parameter ω of the dynamic system. For the highly correlated dynamic system with a small parametric value ω, the quantity

$$\frac{\tilde{\beta}}{\tilde{\gamma}} \to 1 \tag{6-61}$$

and

$$(1 - \frac{\tilde{\beta}}{\tilde{\gamma}}) \ll 1$$

in this case a slow convergence rate will be expected

(B) The convergence rate $P_m(t)$ for degradation mode H_2 will be much faster than the convergence rate for degradation mode H_1.

These conclusions are confirmed by the simulation results

6 3 Simulation Results

Figures 6 3 - 6 4 show the estimates of the bias by the technique of the decoupled estimator Figure 6 3 illustrates the estimate of a constant bias of a value +4. As discussed in section 6 2, the convergence rate is shown to be extremely slow For an estimate within 95% of the true value a size of more than 800 samples are required. For comparison the typical sample size required for detection is about 30 samples as illustrated in simulated results in section 4. 6 The saving in time required for detection compared to that for estimation will thus be of the order of 25. Figure 6 4 illustrates the estimate of a ramp bias with a rate value of +0.1. The con-

vergence rate in the ramp bias case is much faster as is expected from the analytical expression (6-59).

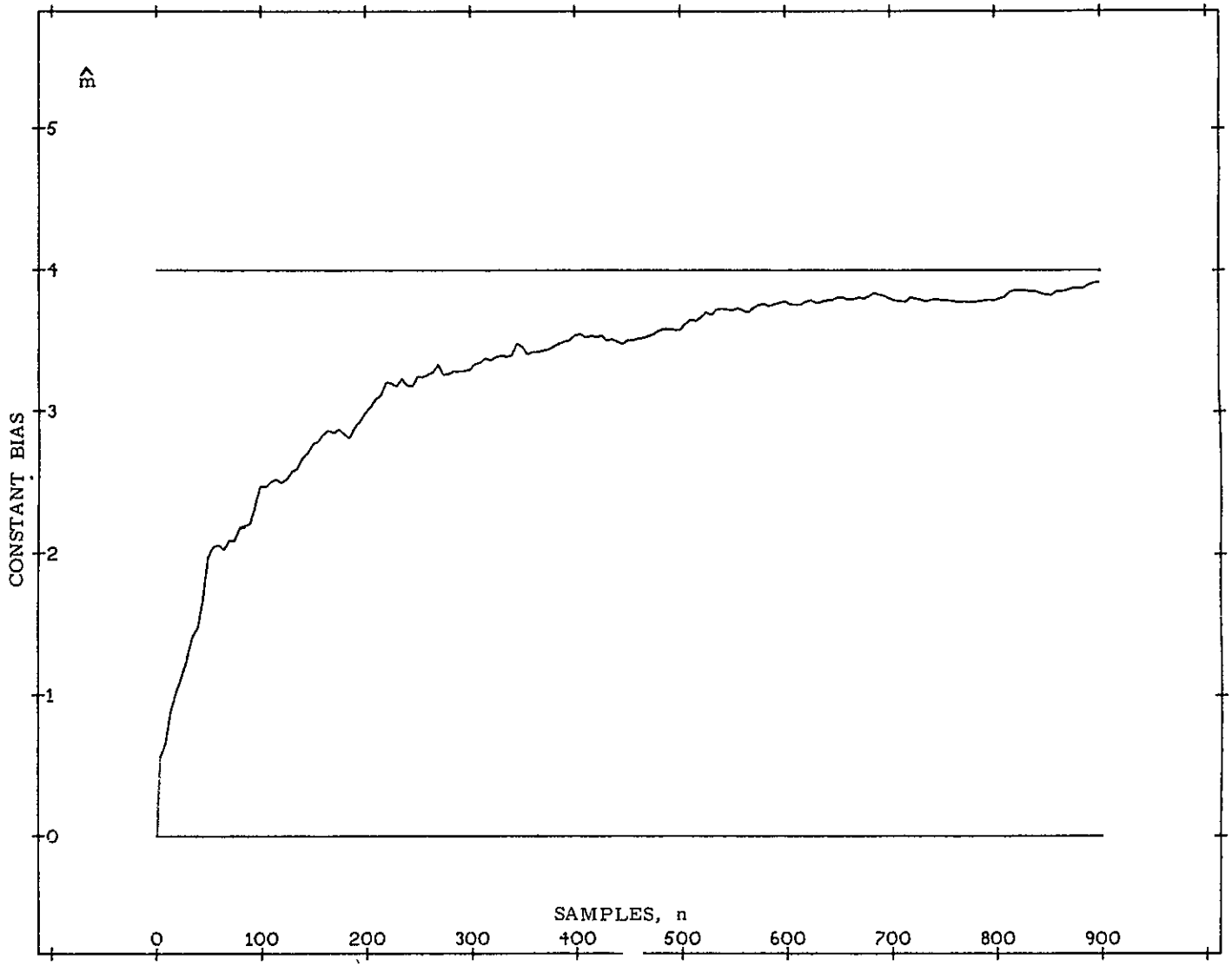


FIG. 6.3 ESTIMATE OF CONSTANT BIAS

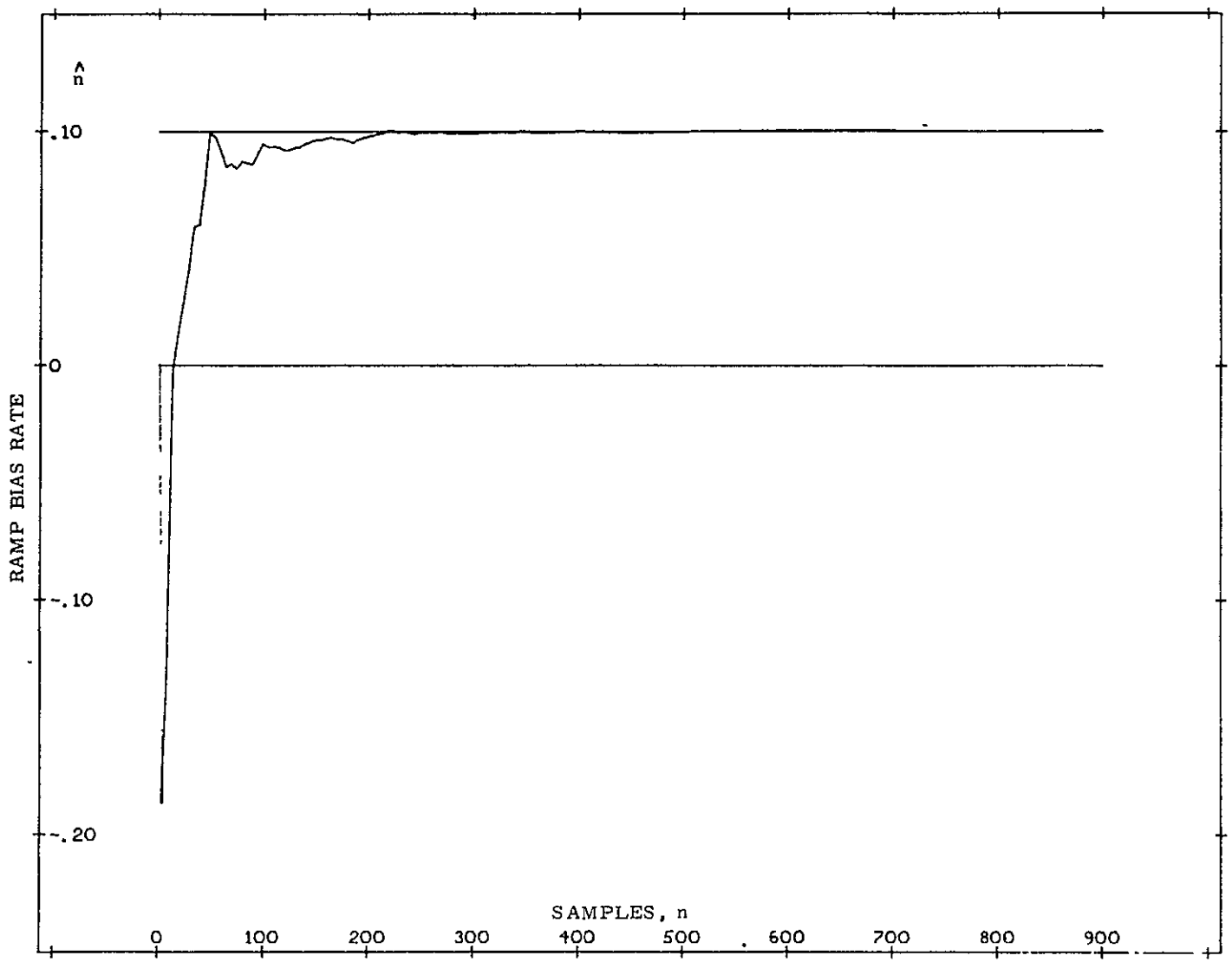


FIG 6 4 ESTIMATE OF RAMP BIAS RATE

CHAPTER 7

Conclusions and Recommendations

7 1 Conclusions and Contributions

An on-line adaptive technique is developed to provide a self-contained redundant-sensor navigation system with a capability to utilize its full potentiality in reliability and performance The gyro drift rate is modeled as a Gauss-Markov stochastic process The degradation modes are defined by changes in characteristics specified by parameters associated with the model. The dominant modes resulting in navigation performance degradation are shown to be of the form of a jump of a constant and a ramp bias of the gyro drift rate

The adaptive system is formulated as a multi-stage stochastic decision process a detection system to detect the existence of a degradation and to isolate the degraded sensor, an identification system to verify a degradation and to determine its nature, and a compensation system to recover the degraded sensor performance The inputs to be processed by the adaptive system are the aggregated scalar states of residuals generated by a set of parity equations of outputs of redundant sensors

The detection system is characterized as a process with partial information, in which the state describing the degradation is inaccessible for direct observation The problem of solving the partially observable process is approached by deriving the stochastic differential equation for the posterior probability of the unobservable state conditioned on the a priori information and the observable measurement history It is shown that for the optimal detection system formulated as a Bayesian problem with additive risk and conditionally independent observations the posterior probability is the sufficient statistic.

A detection system in a class of the linear system of Wald's sequential probability ratio test is developed. The feature of the system design is utilization of feedback of the uncertainty information indicated by the posterior probability The design of the detection system is formulated as a combined stochastic control and decision problem The admissible control region is defined with reference to the value of the information as an indicator of system degradation. A suboptimal control scheme is designed to avoid the complicated computation for obtaining the solution to the combined optimization problem formulation The suboptimal control law is derived by utilization of the fact about monotinicity property of the risk function of detection The suboptimal control for the discrete-time case is shown to have the characteristic that the control corrects the posterior probability to a constant threshold number, if it is less than the threshold. If the posterior probability is larger than the threshold, no feedback is used. The difficulty in extending the results of the suboptimal control law in the discrete case to the continuous-time case is discussed, and a control law based on the concept of ϵ-optimal control is suggested.

The performance of the detection system is derived by application of the theory of first passage times for diffusion processes It is shown for a sequential system with one-sided boundary for degradation detection the meaningful performance criteria are the mean delay time in detection and the false alarm error probability.

The developed system is very simple for on-line implementation The system shows a remarkably close detection performance to that of the optimal non-linear detection system It is shown that the developed system can be simply modified to detect the degradation with simultaneous identification of its unknown polarity, which is shown to be an important piece of information for efficient isolation of the degraded gyro

It is shown that only one detection system designed for the degradation mode in the form of a jump of a constant bias is sufficient for all degradation modes characterized by a systematic mean change Moreover, the detection process can be constructed on the basis of a "design" value, specified by the mission requirement, of the unknown parameter in the real system.

In the design of the identification system an invariant transformation is derived to eliminate the effect of nuisance parameters. An application of the technique to the measurement residuals will transform the original process with unknown constant and ramp rate into a corresponding process with a zero mean and with an attenuated ramp rate The transformation gives a sensitive identification capability even for a degradation with a small ramp rate It is shown that the ambiguous three-class identification process can be transformed into a set of pairwise disjoint two-class identification process

A technique of decoupled estimator is applied in the compensation system to estimate the unknown bias such that the adaptive system can be operated without any complicated reorganization. The long time interval required for processing information in the estimation process in comparison with that required in the decision process illustrates the justification of the multi-stage formulation of the thesis

7 2 Recommendations

In this section suggestions for further study will be briefly discussed

The treatment of the degradation mode having an increase in variance of the gyro drift rate will be considered The detection of a change in variance can be treated by a simple extension to the theory of sequential analysis with correlated measurements to test the hypothesis H_0 of the normal variance against the hypothesis H_1 of a specified "degraded" variance. The test procedure consists of observing the likelihood function of the joint conditional probability density against two thresholds related to the set error bounds. The likelihood function can be recursively generated on-line by the two sequences of measurement residuals derived from two Kalman fitlers modeled on hypothesis H_0 and H_1 respectively This has been discussed by Newbold and Ho [56] However, the problem of the adaptive system design in this case must be handled with some modifications

First the sequence of measurement residuals is an asymptotically stationary but not an independent sequence if the Kalman filter model is not based on the true variance One simple way to ensure independence is to sample the data at a sufficiently long interval. Since it was discussed in Chapter 2 that the increase in variance during degradation of the instrument

is gradual and small, and one notices that the sequence in this case will be only weakly correlated, then the effect of a moderate increase in the sampling interval will not likely cause a serious degradation of the navigation system performance Secondly, since the variance increase during degradation is gradual, there is no justification to assume that a degradation mode of a change in mean bias will not occur during the same interval. This complicates the design of the identification system If independence of the measurement residuals can be ensured, the invariant transformation developed in this thesis can be applied to remove the constant bias mean In this case by processing the transformed measurement residuals, the degradation modes of variance increase and of a constant bias jump can be classified. However, there will be an ambiguity in classification between the degradation modes of the variance increase and the ramp bias, and some other technique or information for their classification is required Finally, the problems of the design of optimal and suboptimal systems for the on-line estimation of the "degraded" noise variance, for example, by using analysis of measurement residuals, have been discussed by Mehra [52] and Abramson [1] It is noticed that the performance of a degraded instrument due to an increase in variance cannot be requalified, but the updated variance may give the information for optimal weighing of the measurement data among different instruments in the navigation system, or give an indication that the instrument will have an impending hard failure

The technique developed in this thesis can be extended to compensate the drift rate non-stationary component modeled as a random walk process The random walk component of the gyro drift rate is proposed to be treated as an unknown mean of arbitrary form. No modification is required in the detection system to augment this component into the gyro drift rate However, in compensation for performance degradation, one would have to be content to match the unknown complicated function to only one of the two given classes, a constant or a ramp mean bias. The compensation will be based on a model with either the constant or ramp bias degradation mode according to the classification result by the identification system.

An important area is the study of problems involved in choosing useful information in adaptive system designs. One direction of research is

to introduce into the information theory the element of value (Brillouin [10], Howard [36], Stratonovich [73]). In the present theory information is treated as an absolute quantity which has the same numerical value for any user. On the other hand, the value of the information would necessarily be a relative quantity, and would have different values for different users. This intuition leads one to seek a notion of the value of information

The design of the detection system in this thesis illustrates an approach by relating the concept of the information value to the theory of optimal solutions This is equivalent to selecting the information according to a certain figure of merit The question of interest is then to derive the most effective information with respect to the cost functional. One may conceive of a situation where more information may lower the effectiveness of choosing the optimal solution The design of the detection system illustrates a case in which not all information is useful, and it is possible to carry out further minimization of the risk function of detection by choice of only useful relevant information

The above formulation introduces an application of the concept of the value of the information In many adaptive systems, the question of interest is how measurements should be made so that one can extract most effectively all the relevant information which will be utilized in designing the adaptive system This problem remains an open question

APPENDIX A

DERIVATION OF WALD'S RISK FUNCTION

Let $\delta(\omega) = \{\tau(\omega), d(\omega)\}$ be some decision function from the results of observations on the process $\{y(s), s \leq \tau\}$. Here $\tau(\omega)$ is the random stopping time for terminating the observation, $d(\omega)$ is the terminal decision rule. The terminal decision space D^0 consists of two points d_0 and d_1. The decision d_i will be interpreted as the decision that the hypotheses H_i is accepted. Let $\alpha(\delta) = P\{d(\omega) = d_1 | H_0\}$ and $\beta(\delta) = P\{d(\omega) = d_0 | H_1\}$ be the false and miss alarm error probabilities at the terminal decision for the decision function $\delta(\omega)$. Then the risk function associated with $\delta(\omega)$ is defined as

$$R(\delta, \pi) = \pi \{c E_1(\tau) + \tilde{a} \cdot \beta(\delta)\} + (1-\pi) \{c E_0(\tau) + \tilde{b} \cdot \alpha(\delta)\} \tag{A-1}$$

Notice that $R(\delta, \pi)$ denotes the total expected cost starting from the initial state π to the terminal decision based on the decision function δ. For the case of specified α and β, $\tilde{a}$ and $\tilde{b}$ are the Lagrange multipliers. The coefficient c is the sampling cost per unit time.

Let $\pi_t = P\{a = a_1 | y(s), s \leq t\}$ be the posterior probability that H_1 is true based on observations $\{y(s), s \leq t\}$. Define the operator

$$E_\pi(\cdot) = \pi \cdot E_1(\cdot) + (1-\pi) \cdot E_0(\cdot)$$

Then (A-1) can be written as

$$R(\delta, \pi) = \pi\{c E_1(\tau) + \tilde{a} \cdot \beta(\delta)\} + (1-\pi)\{c E_0(\tau) + \tilde{b} \cdot \alpha(\delta)\} \tag{A-1}$$

$$R(\delta,\pi) = c\,E_{\pi}(\tau) + \pi\,\tilde{a}\cdot P\{d(\omega) = d_0 \mid H_1\}$$
$$+ (1-\pi)\tilde{b}\cdot P\{d(\omega) = d_1 \mid H_0\}$$
$$= c\,E_{\pi}(\tau) + \pi\,\tilde{a}\int_{\Omega} P\{d(\omega) = d_0 \mid y(s),\ s \leqq \tau\}\, dP_1(\omega)$$
$$+ (1-\pi)\tilde{b}\cdot \int_{\Omega} P\{d(\omega) = d_1 \mid y(s),\ s \leqq \tau\}\, dP_0(\omega) \qquad (A\text{-}2)$$

Now for any function $S(\pi_\tau, \tau)$ of states τ, $\pi_\tau \triangleq \pi(\tau)$ at the final decision

$$E_1\left\{\frac{\pi_i}{\pi_1(\tau)}\cdot S(\pi_\tau, \tau)\right\} = E_{\pi}\{S(\pi_\tau, \tau)\} \qquad (A\text{-}3)$$

where $\pi_1(\tau)$ is the posterior probability that H_i is true. The relation (A-3) can be verified as follows

$$E_1\left\{\frac{\pi_i}{\pi_i(\tau)}\cdot S(\pi_\tau, \tau)\right\}$$
$$= \sum_{\tau}\int_{\Omega_\tau} S(\pi_\tau, \tau)\,\frac{\pi_1}{\pi_1(\tau)}\cdot dP_i(\omega) \qquad (A\text{-}4)$$

where Ω_τ is the set of observations which lead to decision of termination at the instant τ. $P_i(\omega)$ denotes the conditional joint distribution of observations $y_1, \ldots, y_\tau$ generated by the random process $\{y(s),\ s \leqq \tau\}$. Writing explicitly $P_i(\omega)$ gives

$$P_1(\omega) = \mu_i(y_1)\cdots\mu_1(y_\tau).$$

As discussed under equation (4-36), the "generalized" notation $\{y_\tau\}$ is used, while y_τ denotes the observation $\Delta y_\tau \triangleq (y_\tau - y_{\tau-1})$ in the real system. It can be seen that the observations are independent random variables and moreover of identical distribution denoted as $\mu_1(\cdot)$. The latter comes from the statement of the Wald's problem that the state a is an unknown, yet constant value, and there is no transition of states in the obser-

vation process. In this case the posterior probability $\pi_1(\tau)$ can be expressed as

$$\pi_i(\tau) = \frac{\pi_i \cdot d\mu_1(y_1) \ldots d\mu_1(y_\tau)}{\sum_{k=0}^{1} \pi_k \, d\mu_k(y_1) \ldots d\mu_k(y_\tau)} \quad i = 0, \text{ or } 1$$

Then one can obtain:

$$\frac{\pi_1}{\pi_1(\tau)} \cdot dP_i(\omega) = \sum_{k=0}^{1} \pi_k \cdot d\mu_k(y_1) \ldots d\mu_k(y_\tau) \qquad \text{(A-5)}$$

The relation (A-3) can be proved by substitution of (A-5) into (A-4).

On using the relation (A-3) the following relation can be established

$$\int_\Omega \pi \, P\{d(\omega) = d_0 \mid y(s), \; s \leqq \tau\} \cdot dP_1(\omega)$$
$$= \int_\Omega \pi_\tau(\omega) \cdot P\{d(\omega) = d_0 \mid y(s), \; s \leqq \tau\} \, dP_\pi(\omega)$$

by identifying $S(\cdot) = \pi(\tau) \cdot P\{d(\omega) = d_0 \mid y(s), \; s \leq \tau\}$ and $E_i = E_1$.

Similarly

$$\int_\Omega (1 - \pi) P\{d(\omega) = d_1 \mid y(s), \; s \leqq \tau\} \, dP_0(\omega)$$
$$= \int_\Omega (1 - \pi_\tau(\omega)) P\{d(\omega) = d_0 \mid y(s), \; s \leqq \tau\} \, dP_\pi(\omega)$$

by identifying $S(\cdot) = (1 - \pi(\tau)) \cdot P\{d(\omega) = d_1 \mid y(s), \; s \leq \tau\}$ and $E_i = E_0$.

Then (A-2) can be rewritten as

$$R(\delta, \pi) = c \, E_\pi(\tau) + \int_\Omega \big[\tilde{a} \pi_\tau(\omega) \cdot P\{d(\omega) = d_0 \mid y(s), \; s \leqq \tau\}$$
$$+ \tilde{b} \cdot (1 - \pi_\tau(\omega)) P\{d(\omega) = d_1 \mid y(s), \; s \leqq \tau\}\big] dP_\pi(\omega)$$
$$\geqq c \, E_\pi(\tau) + \int_\Omega \min\big[\tilde{a} \pi_\tau(\omega), \; \tilde{b}(1 - \pi_\tau(\omega))\big] \, dP_\pi(\omega)$$

Define

$$K(\pi) = \text{Min}\left[\tilde{a}\,\pi\,,\ \tilde{b}(1-\pi)\right]$$

Then

$$R(\delta\,,\,\pi) \geq E_{\pi}\ \left[\,c\tau + K(\pi_{\tau})\right] \qquad \text{(A-6)}$$

APPENDIX B

DERIVATION OF STATIONARY PROBABILITY DENSITY FUNCTION

It has been discussed in section 4. 5. 2 that at the instant of degradation a stationary condition, established in the observation process in the normal mode, can be assumed to exist The stationary probability density function p(y) generated under this condition is to be derived in this appendix. The density p(y) is solved by the Kolmogorov forward equation The forward equation for $p_\lambda(y, t)$ can be shown to be

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\{b^2\, p(y,t)\} - \frac{\partial}{\partial y}\{\mu(0) \cdot p(y,t)\} = \frac{\partial p(y,t)}{\partial t} \qquad \text{(B-1)}$$

where the observation process $\lambda(t)$ defined in the normal mode is represented by

$$d\lambda(t) = \mu(0)dt + b\, dw(t) \qquad \text{(B-2)}$$

with parameters

$$\mu(0) = \frac{a_1^2}{2\sigma^2}$$

$$b = \frac{a_1}{\sigma}$$

The equation (B-l) is subjected to the homogeneous boundary conditions at the absorbing boundaries A and B,

$$p_\lambda(A, t) = p_\lambda(B, t) = 0 \qquad t > 0 \qquad \text{(B-3)}$$

and to the initial condition $\lambda(0) = 0$,

$$p_\lambda(y, 0) = \delta(y) \tag{B-4}$$

The partial differential equation (B-1) can be solved by application of the Laplace transformation

$$\bar{p}(y, s) = \int_0^\infty \exp(-st) \cdot p(y, t)\, dt \tag{B-5}$$

Then (B-1) is transformed into

$$\frac{1}{2}\frac{a_1^2}{\sigma^2}\left(\frac{\partial^2 \bar{p}}{\partial y^2} + \frac{\partial \bar{p}}{\partial y}\right) = s\,\bar{p} - p(y, 0) \tag{B-6}$$

Substituting (B-4) into (B-6) yields

$$\frac{1}{2}\frac{a_1^2}{\sigma^2}\left(\frac{\partial^2 \bar{p}}{\partial y^2} + \frac{\partial \bar{p}}{\partial y}\right) - s\,\bar{p} = -\delta(y) \tag{B-7}$$

subject to the corresponding boundary conditions in (B-3).

The equation (B-7) can be solved by the method of Green's function $g(y, 0, s)$. The function $g(y, 0, s)$ satisfies

$$\frac{1}{2}\frac{a_1^2}{\sigma^2}\left(\frac{\partial^2 \bar{p}}{\partial y^2} + \frac{\partial \bar{p}}{\partial y}\right) - s\,\bar{p} = 0 \tag{B-8}$$

everywhere except at $y = 0$. The equation (B-8) can be solved as an ordinary differential equation with s considered as constant Since the stationary condition of the probability density is the only piece of information desired, the final value theorem will be applied to save the effort of taking the inverse transformation The final value theorem is stated as

$$\lim_{t \to \infty} p(y, t) = \lim_{s \to 0} s\,\bar{p}(y, s) \tag{B-9}$$

Multiplying (B-8) by the constant s, and using the relation (B-9) the equation (B-8) can be simplified to

$$\frac{d^2p}{dy^2} + \frac{dp}{dy} = 0 \tag{B-10}$$

where $p = p(y) \triangleq p(y, \infty)$

The Green function $g(y, 0)$ satisfies the equation (B-10) everywhere except at $y=0$, and is subjected to the conditions in (B-3). Let $g^-(y)$ and $g^+(y)$ be the solutions to the equation (B-10) in the region $A \leqq y < 0$ and $0 < y \leqq B$ respectively with boundary conditions

$$g^-(A) = g^+(B) = 0 \tag{B-11}$$

then $p^-(y)$ and $p^+(y)$ of the Green function $g(y, 0)$ take the form

$$\begin{aligned} p^-(y) &= c \cdot g^+(0) \cdot g^-(y) \qquad A \leqq y < 0 \\ p^+(y) &= c \cdot g^-(0) \cdot g^+(y) \qquad B \geqq y > 0 \end{aligned} \tag{B-12}$$

where the associated functions $g^-(y)$ and $g^+(y)$ can be easily solved to a constant factor as

$$\begin{aligned} g^-(y) &= (e^{A-y} - 1) \\ g^+(y) &= (1 - e^{B-y}) \end{aligned} \tag{B-13}$$

Substituting (B-13) into (B-12) and normalizing $p(y)$ to a density function produces

$$p(y) = \begin{cases} \dfrac{(1-e^B)\ (e^{A-y}-1)}{A(1-e^B) + B(e^A-1)} & A \leqq y \leqq 0 \\ \dfrac{(e^A-1)\ (1-e^{B-y})}{A(1-e^B) + B(e^A-1)} & 0 < y \leqq B \end{cases} \tag{B-14}$$

APPENDIX C

PROOF OF PROPERTIES OF INVARIANT TRANSFORMATION

Let $\{\tilde{z}_i\}$ be a sequence defined by the transformation

$$\tilde{z}_{n-1} = \frac{1}{\sqrt{n(n-1)}}(y_1 + \ldots + y_{n-1}) - \frac{n-1}{\sqrt{n(n-1)}} y_n, \quad (n \geq 2) \tag{C-1}$$

where $\{y_i\}$ is a sequence of independent and identically distributed (i. d) random variables of normal distribution with an unknown mean a and a given variance σ. Then $\{\tilde{z}_j\}$ is a sequence of i d. and independent random variables having the normal distribution with zero mean and the same variance Moreover the following relation holds

$$\sum_{j=1}^{n-1} \tilde{z}_j^2 = \sum_{i=1}^{n} (y_i - \bar{y}_n)^2 ; \; \bar{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i \tag{C-2}$$

This result can be proved as follows Since the $\{\tilde{z}_j\}$ are the result of a linear transformation of y_1's and $\{y_i\}$ are of normal distribution, so $\{\tilde{z}_j\}$ are of normal distribution. The statistics of $\{\tilde{z}_j\}$ can be derived as follows (for $\eta = n-1$):

$$E[\tilde{z}_{n-1}] = \frac{1}{\sqrt{n(n-1)}} \{E(\sum_{i=1}^{n-1} y_i) - (n-1)E(y_n)\} = 0 \tag{C-3}$$

$$E[\tilde{z}_{n-1}^2] = \frac{1}{n(n-1)} E\{\sum_{i=1}^{n-1} y_i - (n-1) y_n\}^2$$

$$= \frac{1}{n(n-1)} E\{(\sum_{\imath=1}^{n-1} y_\imath)^2 - 2(n-1)y_n (\sum_{\imath=1}^{n-1} y_\imath)$$

(equation continued)

$$+ (n-1)^2 y_n^2 \}$$

$$= \frac{1}{n(n-1)} \{(n-1)E(y^2) + (n-1)(n-2) \cdot [E(y)]^2$$

$$-2(n-1)^2 \cdot [E(y)]^2 + (n-1)^2 E(y^2) \}$$

$$= E(y^2) - [E(y)]^2 = \sigma^2 \qquad \text{(C-4)}$$

Here the i. d. and independent properties of $\{y_i\}$ distribution have been used in the derivation. Hence it has been shown that $\{\tilde{z}_i\}$ are i. d random variables of normal distribution $N(0, \sigma)$. To prove the independence of random variables of normal distribution, it is sufficient to show that the random variables are uncorrelated:

$E[\tilde{z}_k \tilde{z}_l] = E[\tilde{z}_k] \; E[\tilde{z}_j] = 0 \quad k \neq j$

$$E[\tilde{z}_j \tilde{z}_k] = E\left[\frac{1}{\sqrt{j(j+1)}} \left\{ \sum_{i=1}^{j} y_i - j y_{j+1} \right\} \cdot \frac{1}{\sqrt{k(k+1)}} \right. \qquad \text{(C-5)}$$

$$\left. \left\{ \sum_{i=1}^{k} y_i - k\, y_{k+1} \right\} \right]$$

$$= \frac{1}{\sqrt{j(j+1)}} \cdot \frac{1}{\sqrt{k(k+1)}} \cdot E\left[\sum_{i=1}^{j} (y_i - y_{j+1}) \cdot \left\{ \sum_{i=1}^{k} y_i - k\, y_{k+1} \right\} \right]$$

$$= \frac{1}{\sqrt{j(j+1)}} \cdot \frac{1}{\sqrt{k(k+1)}} \left[\sum_{i=1}^{j} E\left\{ (y_i - y_{j+1}) \left[\sum_{\ell=1}^{k} y_\ell - k\, y_{k+1} \right] \right\} \right]$$

It is easy to show that $(k > j)$

$$E\left\{ (y_i - y_{j+1}) \left[\sum_{\ell=1}^{k} y_\ell - k\, y_{k+1} \right] \right\}$$

$$= E\left\{ (y_i - y_{j+1}) y_i + (y_i - y_{j+1}) y_{j+1} \right] = 0$$

by using the fact that the random variables $\{y\}$ are i d and independent,

$$E(y_i - y_j) = 0$$

$$E(y_i y_j) = E(y_i)\,E(y_j) \qquad i \neq j$$

$$E\,[y_i^2 - y_j^2] = 0$$

The symmetry of $E[\tilde{z}_j\ \tilde{z}_k]$ with respect to its indices j and k implies the same result for case k<j. This concludes the proof of the first part.

To prove the identity

$$\sum_{j=1}^{n-1} \tilde{z}_j^2 = \sum_{i=1}^{n} (y_i - \bar{y}_n)^2 \tag{C-6}$$

where

$$\bar{y}_n = \frac{1}{n} \sum_{i=1}^{n} y_i$$

Consider the equivalent identity

$$\sum_{i=1}^{n} (y_i - \bar{y}_n)^2 - \sum_{i=1}^{n-1} (y_i - \bar{y}_{n-1})^2$$

$$= \left[\frac{1}{\sqrt{n(n-1)}} \left(\sum_{i=1}^{n-1} y_i - (n-1)y_n \right)\right]^2 \tag{C-7}$$

This can be proved by some manipulation

$$\sum_{i=1}^{n} (y_i - \bar{y}_n)^2 - \sum_{i=1}^{n-1} (y_i - \bar{y}_{n-1})^2$$

$$= \sum_{i=1}^{n-1} \{ (y_i - \bar{y}_n)^2 - (y_i - \bar{y}_{n-1})^2 \} + (y_n - \bar{y}_n)^2$$

$$= (n-1)\ (\bar{y}_n - \bar{y}_{n-1})^2 + (y_n - \bar{y}_n)^2$$

But $(y_n - \bar{y}_n) = (n-1)(\bar{y}_n - \bar{y}_{n-1})$

Thus the relation (C-7) can be shown:

$$\sum_{i=1}^{n} (y_i - \bar{y}_n)^2 - \sum_{i=1}^{n-1} (y_i - \bar{y}_{n-1})^2$$

$$= \frac{(y_n - \bar{y}_n)^2}{n-1} + (y_n - \bar{y}_n)^2$$

$$= \left(\frac{n}{n-1}\right) (y_n - \bar{y}_n)^2$$

$$= \left\{\frac{1}{\sqrt{n(n-1)}} \left[\sum_{i=1}^{n} y_i - n\, y_n \right] \right\}^2$$

$$= \left[\frac{1}{\sqrt{n(n-1)}} \left[\sum_{i=1}^{n-1} y_i - (n-1) y_n \right] \right]^2$$

or

$$\sum_{i=1}^{n} (y_i - \bar{y}_n)^2 - \sum_{i=1}^{n-1} (y_i - \bar{y}_{n-1})^2 = \tilde{z}_{n-1}^2 \qquad \text{(C-8)}$$

By repeated applications of the equation (C-8) and by a summation of both sides of the formed equations the following identity can be proved.

$$\sum_{i=1}^{n} (y_i - \bar{y}_n)^2 = \sum_{j=1}^{n-1} \tilde{z}_j^2 \qquad \text{(C-9)}$$

This concludes the proof of the statements about the transformation

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