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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-713

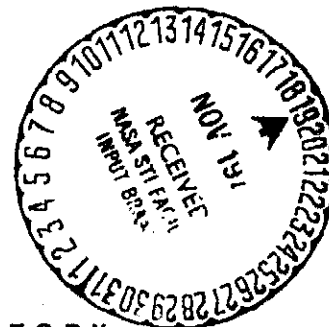
*Symbolic Integration of a Class
of Algebraic Functions*

Edward W. Ng

(NASA-CR-140707) SYMBOLIC INTEGRATION OF
A CLASS OF ALGEBRAIC FUNCTIONS (Jet
Propulsion Lab.) 21 p HC \$3.25 CSCL 12A

N75-10730

Unclas
G3/64 53163



JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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PREFACE

The work described in this report was performed by the Office of Computing and Information Systems of the Jet Propulsion Laboratory.

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ABSTRACT

An algorithm, based on a series of analyses by Carlson, is presented for the symbolic integration of a class of algebraic functions. This class consists of functions made up of rational expressions of an integration variable x and square roots of polynomials, trigonometric and hyperbolic functions of x . The algorithm is constituted of four major components, viz., (i) reduction of input integrand to canonical form, (ii) intermediate internal representation of integral, (iii) classification of output, and (iv) reduction and simplification of output to well-known functions. In the oral presentation, the algorithmic outline as well as some simple examples will be described.

1. INTRODUCTION

Symbolic integration is concerned with the computer automation of analytic or closed form integration. Recently Moses (Ref. 6) presented an excellent survey of the last decade of efforts in this area, from which he discerned three main streams of interests, viz., from artificial intelligence, from algebraic manipulation and from mathematics. For our discussion here we shall characterize these interests by two approaches, viz., the heuristic and the algorithmic. The former approach involves basically the application, by educated guess, of a class of methods each applicable to a specialized class of problems. The latter approach involves in the study of a general class of functions (e.g., rational functions) and attempts to determine if a given member of this class can be integrated in terms of a member of the same class or a well-defined extended class (e.g., rational functions extended by logarithms). On the theoretical side, Risch's result (Ref. 9) stands out as commanding paramount importance, both in its generality and rigor. On the practical side, Moses' implementation stands out as the most successful to date. Therefore we shall describe our proposed approach in relation to the work of both Risch and Moses, emphasizing the parts which our approach will yield that are not contained in that of Risch and Moses. To this end we first highlight the contributions of these two investigators.

Risch's algorithm is based on his reformulation of Liouville's theorem. It involves in first examining the hierarchical structure of the integrand, i.e., the finite number of elementary extensions that build it up from a ground field of rational functions. Once such hierarchy has been determined, the integration problem reduces to a systematic comparison of coefficients through differentiation. For our purpose here it is convenient to refer to

the integrand and the resultant integral as belonging to some input and output classes of functions. Using such terminology we should make clear that Risch's algorithm deals with only input and output that are elementary functions, that is, any of the following four classes, viz., (i) algebraic; (ii) exponential, trigonometric and hyperbolic (as rational functions of exponentials); (iii) logarithmic; (iv) inverse trigonometric and inverse hyperbolic (as logarithms of algebraic functions). Moreover, the current state of Risch's algorithm handles only inputs that are rational expressions involving cases (ii) and (iii). This point is made very clear in his most recent publication (Ref. 10). It is well known that many simple combinations of elementary functions do not possess elementary integrals, e.g., e^{x^2} , e^x/x , etc. In such cases it is necessary to extend Risch's algorithm so as to enlarge either the input or the output class, or both.

Another important point to note is that Risch's algorithm is designed mainly for the problem of indefinite integration. It is well known that many expressions are only integrable for a specialized interval, e.g., $[0, 2\pi]$, $[0, \infty]$, etc. Such definite integrations that possess no indefinite counterparts fall outside the realm of Risch's algorithm. Wang's (Ref. 11) investigations represent pioneering efforts in such definite symbolic integrations.

Turning to the practical side of symbolic integration, Moses' program SIN is certainly the most general and sophisticated to date and is in our opinion the only viable one. SIN is, roughly speaking, constituted of three components: one made up of a collection of specialized heuristic methods, one consisting of the capability for rational function integration, and one implementing Risch's results.

With the foregoing preliminaries, we are ready to describe an algorithmic approach to handle a class of algebraic integrands. We shall label this approach Carlson's algorithm because it is based on a series of analyses rendered by Carlson and his associates in the last ten years (Refs. 2, 3, 4, 8, and 12). The class of integrands is of the form $r(x,y)$, where y^2 is a polynomial in x , and r a rational function in x and y . This is the type of integrand that classically led to the study of elliptic integrals. At first glance this is a rather restricted class of algebraic functions. But in fact many trigonometric and hyperbolic integrands reduce to this form. The richness of this class of integrands is exemplified by a recently published handbook of 3000 integral formulas (Ref. 1). Our proposed approach will cover fifty to seventy percent of the items in the handbook. Furthermore the non-classical approach we shall describe holds great promise of developing to the case where definite integrals can be evaluated in terms of a host of other well-known functions (e.g., Bessel and Legendre). It is important to make clear what we aspire to achieve that is not possible with Risch's algorithm. For this purpose let us divide our approach into four cases: (i) indefinite integration - elementary output; (ii) indefinite integration - non-elementary output; (iii) definite integration - elementary output; (iv) definite integration - non-elementary output, in all cases the input being algebraic functions. By definite integrations we mean those that possess no indefinite counterparts. It is clear from the previous discussions that at its current stage, Risch's algorithm cannot handle any of these cases, and even in its prospective completion it will only handle our case (i) here. On the other hand Risch's algorithm is far more general

in scope than our proposed one. Therefore the latter should be considered an adjunction or supplement, albeit an important one, to the general capability in symbolic integration.

It is appropriate here to give some examples of the types of integrals that fall in the class under investigation:

$$\int t^{-\frac{1}{2}}(1+t^3)^{-\frac{1}{2}} dt \quad (1)$$

$$\int t^2(t^4+at^2+b)^{-3/2} dt \quad (2)$$

$$\int (A+B \cos \theta + C \sin \theta + D \cos^2 \theta + E \sin^2 \theta + F \sin \theta \cos \theta)^{-\frac{1}{2}} d\theta \quad (3)$$

$$\int \frac{(A \sinh^2 \theta + B \sinh \theta + C) d\theta}{(E \sinh \theta + F)^2 (G \sinh \theta + H \cosh \theta)^{\frac{1}{2}}} \quad (4)$$

It is also appropriate here to delimit the scope of our output. We are interested in adding to the output only three new functions to the elementary functions. These functions play the same role as the logarithm in the integration of rational functions. These functions can be chosen as Legendre's well known canonical integrals F, E, and π or Carlson's standard functions (cf. Ref. 12).

2. CARLSON'S ALGORITHM

The class of integrands, viz., $r(x,y)$, where $y^2 = P_n(x)$, has been the subject of intense study for almost two centuries. It is the simplest class beyond rational functions, but even such a class frequently leads to non-elementary integrals. For $n = 0, 1$, or 2 , the integral is elementary; for $n = 3$ or 4 , the integral is elementary or elliptic in which case up to three new functions need be introduced; and for $n \geq 5$, the integral is called

hyperelliptic, in which case more than three new functions are required. Two main classical approaches to handle this class were due to Legendre and Jacobi, respectively (Ref's. 1, 7), and a more recent one due to Carlson and his associates. We believe that Carlson's approach is superior to the other two for the purpose of symbolic integration. Such belief is based on the observation that Carlson's representation possesses certain symmetry that offers significant simplification in formal manipulation and economy in storage of formulae. Here we give a few examples to illustrate this remark, reserving for a future report a detailed comparison of the three approaches. First, many integrals involving different Jacobian elliptic functions, say, $\text{cn } u$, $\text{nc } u$, $\text{dn } u$, $\text{nd } u$, $\text{cd } u$, $\text{dc } u$ appear in Carlson's representation as various permutations of arguments of the same integral. Thus in such a situation one reduction formula is required for Carlson's case as compared to six in Jacobi's case, (see remark and example in Ref. 8, p. 227). Second, in the case of reduction of Legendre's integrals, different substitutions are needed according to various types of roots of $P_n(x)$. Here one either heuristically chooses the substitution (which is not always successful, see Ref. 7, Pp. 600-602), or stores rather substantial tables of transformation formulae (see Ref. 13, Pp. 78-79, 84-91), in addition to the regular reduction formulae. With Carlson's approach this procedure is not necessary because of the possibility of rendering the integrand into a canonical form. Third, a number of linear and quadratic transformation formulae are needed in Legendre's and Jacobi's approaches (see Ref. 1, Pp. 12-14, 38-41). In the present context these formulae would be required for algebraic simplification, which is crucial to the success of any program in symbolic

computation. With Carlson's approach the linear transformations are rendered trivial by permutation symmetry while the quadratic transformations take on a canonical form (cf. Ref. 12, Pp. 203-208).

In this section we shall translate Carlson's results into an algorithmic scheme. This scheme can be summarized by the following steps: (1) Reduce the input function to a canonical form; (2) represent the output integral formally as a certain Carlson's R-function (which is just an intermediate representation); (3) examine the parameters of the R-function to characterize the output, for example, whether it is elementary or elliptic; (4) reduce the output to a combination of elementary functions and well-known elliptic integrals.

Here we are mainly interested in the "algorithmic outline" and therefore to avoid substantial digressions into mathematical details, we reserve the description of Carlson's R-function to Appendix I. It suffices here to specify the notation $R(a; b_1, \dots, b_k; z_1, \dots, z_k)$ to indicate that in general this function is characterized by k variables and $k+1$ parameters. For the major bulk of our work here k is between 2 and 4. For brevity we shall refer to this function as R and indicate whatever change in parameter or variable by specifying that parameter only, e.g., $R(b_j+1) \equiv R(a; b_1, \dots, b_j+1, \dots, b_k; z_1, \dots, z_k)$.

Procedure I: Canonical Reduction of Input

- I.1 Given input expression $r(x,y)$; if it is already in the form of linear factors $r(x,y) = \prod_{i=1}^k (g_i x + w_i)$, go directly to Procedure II, else proceed.

- I.2 Reduce to linear fraction $r(x,y) \rightarrow [P_1(x)+yP_2(x)]/[P_3(x)+yP_4(x)]$, where the $P_i(x)$ are polynomials in x ; this reduction is possible because in any polynomial in x and y , the even powers of y can be expressed as straight polynomials of x and the odd powers as polynomials of x multiplied by y .
- I.3 Multiply both the numerator and denominator by $[P_3(x)-yP_4(x)]$ and then by y to transform it to the form $r_1(x)+r_2(x)/y$, where r_1 and r_2 are rational functions of x .
- I.4 Integrate $r_1(x)$ by standard rational function integrator.
- I.5 Apply square-free and partial fraction decomposition to $r_2(x)$, resulting in $r_2(x) = \frac{A_1}{S_1} + \dots + \frac{A_l}{S_l}$ (see, e.g., Ref. 6, p. 551), where the S_l 's have only simple roots and are relatively prime to each other.
- I.6 Factorize each S_l into linear form $\prod_{j=1}^k (x+z_j)$, where z_j may be real or complex. Now the integration of $r_2(x)/y$ reduces to that of $A_l/(S_l^l y)$ which in turn reduces to that of $x^m y^{-1} \prod_{j=1}^k (x+z_j)^{-l}$. There is of course the practical problem of factorization. But the problem here is no worse than the case for rational function integration. Furthermore, for most cases of practical interest the S_l 's are polynomials of very low degrees, as evidenced by the several thousand formulae in Ref. 1.

Procedure II: Intermediate Representation of Integral

- II.1 For the integration of algebraic functions, extreme care must be exerted to avoid the indeterminacy of integration across a branch

point. For example, in the Legendre formulation, most of the above-mentioned substitutions are required to avoid such ambiguity. In Carlson's formulation the situation is simpler. Here we need only consider the indefinite integration variable as either always to the left of the smallest root of y on the real axis or to the right of the largest real root. For definite integration we need to allow the degenerate cases of one or both integration limits being branch points. For the sake of such distinction we need first to factorize

$$y \rightarrow \prod_{i=1}^k (x+z_i).$$

II.2 For indefinite integration, a user of this algorithm is asked to indicate the appropriate integration range, i.e., whether it is to the left or the right. We then proceed to give either one of the following representations:

$$a \int_x^\infty \prod_{i=1}^k (t+z_i)^{-b_i} dt = R(a; \vec{b}; \overrightarrow{x+z}), \quad (5)$$

$$\int_0^x \prod_{i=1}^k (t+z_i)^{-b_i} dt = x \prod_{i=1}^k (x+z_i)^{-b_i} R(1; a, \vec{b}; 1, 1 + \frac{x}{z}), \quad (6)$$

where $a = b_1 + \dots + b_k - 1$, in (5); $a = b_1 + \dots + b_k - 2$, in (6); $\vec{b} = (b_1, \dots, b_k)$, $\overrightarrow{x+z} = (x+z_1, x+z_2, \dots, x+z_k)$, etc.

II.3 For definite integration, check for degeneracy; if none apply 2 above, else use the following:

$$\int_{x_0}^{x_1} (t-x_0)^{a-1} (x_1-t)^{a'-1} \prod_{i=2}^k (t+z_i)^{-b_i} dt \quad (7)$$

$$= (x_1-x_0)^{a+a'-1} \frac{\Gamma(a)\Gamma(a')}{\Gamma(a+a')} \prod_{i=2}^k (x_0+z_i)^{-b_i} R\left(a; \vec{b}; 1, \frac{\overrightarrow{z+x_1}}{z+x_0}\right),$$

where $a+a' = b_1 + \dots + b_k$.

II.4 The R-function so obtained has a very special form, viz., the b_i 's (and therefore a) are integers or half-integers. Since y is the square root of an n^{th} degree polynomial, there must be n parameters among a and b_i 's that are half-integers, where $h = n$ or $n+1$ for n even or odd, respectively. This is the so-called R-function of index h , which can always be expressed in terms of R-functions of h variables in which $a, b_1, b_2, \dots, b_{h-1}$ are half-integers and b_h is an integer. The next step in the algorithm is to actually achieve that via recurrences. First rearrange the R-function to the form such that $b_1 \dots b_{h-1}$ are integers in increasing order and $b_k = -m$; this can always be done due to permutation symmetry; in most cases of practical interest, $b_h \dots b_k$ should be small integers, say ± 1 and ± 2 .

II.5 Recall from Step I.6 that we now have integrals of a polynomial divided by $S_{\ell}^{\ell} y$, i.e., $\alpha_m \int x^m / (S_{\ell}^{\ell} y) dx + \alpha_{m-1} \int x^{m-1} / (S_{\ell}^{\ell} y) dx + \dots$, where each $m, m-1, \dots$ appear as the index b_k in the corresponding R-function. Recursive operation n -times with the identity

$$cR = a'R(b_k+1) + a \hat{z}_k R(a+1, b_k+1) \quad (8)$$

will combine these m terms and reduce each b_k to zero. Here \hat{z}_k

is the k^{th} variable of the R-function and the identity is readily derived from the general recurrence relation, eq. A(11).

II.6 Now $b_{h+1} \dots b_{k-1}$ are all exactly 1, which makes it efficient for the following recursive operation:

$$(a-1)(\hat{z}_{k-1} - \hat{z}_{k-2})R = (c-1)[R(a-1, b_{k-1}-1) - R(a-1, b_{k-2}-1)]. \quad (9)$$

Here $\hat{z}_{k-1} \neq \hat{z}_{k-2}$ because S_ℓ has only simple roots.

II.7 The above two steps reduce the integral $\int A_\ell / (S'_\ell y) dx$ to an R-function of index n , i.e.,

$$R(a; b_1, \dots, b_h, 0, \dots, 0; \hat{z}_1, \dots, \hat{z}_k) \rightarrow R(a; b_1, \dots, b_h; \hat{z}_1, \dots, \hat{z}_h). \quad (10)$$

Procedure III: Classification of Output

The classification depends on the parameter $n = 2, 4, 6 \dots$

III.1 If $n=0$, that means the input is a rational function of x only and we have only come this far by mistake. Here an error return with such a message would be appropriate.

III.2 If $n=2$, the integral is elementary and each R-function is expressible in terms of two "basis" functions, viz.,

$$R(\frac{1}{2}; \frac{1}{2}; \hat{z}_1) = 1/\sqrt{\hat{z}_1}, \text{ and} \quad (11)$$

$$R(\frac{1}{2}; \frac{1}{2}, 1; \hat{z}_1, \hat{z}_2) = (\hat{z}_2 - \hat{z}_1)^{-\frac{1}{2}} \cos^{-1}(\hat{z}_1/\hat{z}_2)^{\frac{1}{2}}. \quad (12)$$

This case would have been handled more easily by rational substitutions, but here we also want to "catch" integrals that have a misleading non-elementary appearance, and so this capability is retained here. Now set flag $P = \text{true}$ and apply Procedure IV.2. (Note: The flag is to distinguish between the pseudoelliptic and elliptic cases.)

III.3 The case $n=4$ constitutes the major bulk of our effort here. For example at least two thousand formulae in Ref. 1 belong in this category.

(i) In general a function of this class is of the form $R(a; b_1, b_2, b_3, b_4; \hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ and is reducible to 4 basis functions, viz.,

$$(\hat{z}_1 \hat{z}_2 \hat{z}_3)^{-\frac{1}{2}} \equiv R\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \hat{z}_1, \hat{z}_2, \hat{z}_3\right), \quad (13)$$

$$R_F(\hat{z}_1, \hat{z}_2, \hat{z}_3) \equiv R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \hat{z}_1, \hat{z}_2, \hat{z}_3\right), \quad (14)$$

$$R_G(\hat{z}_1, \hat{z}_2, \hat{z}_3) \equiv R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \hat{z}_1, \hat{z}_2, \hat{z}_3\right), \quad (15)$$

$$R_H(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4) \equiv R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1; \hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4\right). \quad (16)$$

The last three functions are analogous to Legendre's canonical integrals F, E and π (Ref. 1, p. 8). All these functions are symmetric in the variables \hat{z}_1 , \hat{z}_2 , and \hat{z}_3 .

(ii) There are two important special cases caused by one or two of the following conditions: (a) any $\hat{z}_j = 0$, (b) $b_4 = 0$, and (c) any $\hat{z}_i = \hat{z}_j$. Here a flag $k = 4$ should be set and for any occurrence of the above conditions, decrease k by 1 and reduce the R-function by one of equations (25) - (27). Now go to the three cases according to $k = 2, 3, 4$.

(iii) Special case $k=2$: R has been reduced to $R(a; b_1, b_2; \hat{z}_1, \hat{z}_2)$.

(a) If b_1 or b_2 is integer, we have pseudoelliptic case. Now set $P = \text{true}$ and apply Procedure IV.2.

(b) Else the answer is expressible in the form $CR = C_K R_K + C_E R_E$, where the C's are polynomials in \hat{z}_1 and \hat{z}_2 , and

$$R_K \equiv R\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \hat{z}_1, \hat{z}_2\right), \quad R_E \equiv R\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \hat{z}_1, \hat{z}_2\right). \quad (17)$$

They are analogous to Legendre's complete integrals K and E.

Now set $P = \text{false}$ and apply Procedure IV.2 to obtain this answer.

- (iv) Special case $k=3$: R has been reduced to $R(a; b_1, b_2, b_3; \hat{z}_1, \hat{z}_2, \hat{z}_3)$.

The answer is expressible in the form

$$CR = C_F R_F + C_G R_G + C_A (\hat{z}_1 \hat{z}_2 \hat{z}_3)^{-\frac{1}{3}}, \quad (18)$$

where the C's are polynomials in \hat{z}_1, \hat{z}_2 and \hat{z}_3 . Apply Procedure IV.3 to obtain this answer.

- (v) General case $k=4$. The answer is expressible in the form

$$CR = C_F R_F + C_G R_G + C_H R_H + C_A (\hat{z}_1 \hat{z}_2 \hat{z}_3)^{-\frac{1}{3}}, \quad (19)$$

where the C's are polynomials in \hat{z}_1, \hat{z}_2 and \hat{z}_3 . Apply Procedure IV. to obtain this answer.

III.k The cases $h = 6, 8, \dots$ in general deal with hyperelliptic integrals.

In practice there are many such integrals that have hyperelliptic appearance but are actually reducible to the elliptic case (e.g., Ref. 1, Pp. 254-271). We are only interested in such degenerate cases. As before first set a flag $k=h$.

- (i) First check for obvious degeneracies (a) $b_h = 0$, (b) $\hat{z}_j = 0$, or (c) $\hat{z}_i = \hat{z}_j$. Each occurrence of any of these conditions reduces k by 1.

- (ii) For each equal pair (\hat{z}_i, \hat{z}_j) compute the new parameter $b_{ij} = b_i + b_j$. Collect all integer b_{ij} 's and now apply the same operations as in II.4 - II.7 to further reduce the R-function.

The reduction of each parameter is indexed by a corresponding decrease of k by 1.

- (iii) Now if $k < 5$, go to step III.3, else return the hyperelliptic R-function with no further reduction. A typical user may still

find this R-function to be useful because many properties are known about it (Ref. 2)

Procedure IV. Reduction of R-function to Basis Functions

This procedure consists of three sub-procedures ($k=2, 3, 4$) for the reduction of $R(a; b_1, b_2, \dots, b_k; z_1, \dots, z_k)$ to a set of basis functions. The basic technique in this Procedure involves the application of the recurrence formulae (20)-(23). For this preliminary report we shall only present the case $k=2$, leaving the more tedious ones for the final detailed report.

IV.2 Reduction of 2-parameter R-functions:

If the flag $P = \text{true}$, the two basis functions are elementary (11)-(12), else they are R_K and R_E (17)-(18). For this Procedure we need the following four formulas which may be derived from (20)-(23).

$$(c-1)R(b_i-1) = (a'-1)R + az_iR(a+1); \quad (20)$$

$$z_i(c-1)R(b_j-1) = z_j(c-1)R(b_i-1) - (z_j-z_i)(a'-1)R; \quad (21)$$

$$az_i z_j R(a+1) = a'R(a-1) - (b_i-a)z_i R - (b_j-a)z_j R \quad (22)$$

$$a'b_i(z_i-z_j)R(b_i+1) = c\{(b_i-a)(z_j-z_i)R + z_j(c-1)[R-R(b_i-1)]\}; \quad (23)$$

Now the algorithmic steps are as follows:

(i) Given an R-function $R(\bar{a}; \bar{b}_1, \bar{b}_2; z_1, z_2)$ and the basis functions R_1 and R_2 ;

(ii) apply eq. (22) repeatedly to R_1 and R_2 to obtain

$$R(\bar{a}; b_1, b_2; z_1, z_2) \text{ and } R(\bar{a}+1; b_1, b_2; z_1, z_2); \quad (24)$$

(iii) apply eq. (20) to compute $R(\bar{a}; b_1-1, b_2; z_1, z_2)$;

(iv) recur on b_1 , with eq. (23) to compute $R(\bar{a}; \bar{b}_1-1; b_2; z_1, z_2)$;

(v) apply eq. (21) to compute $R(\bar{a}; \bar{b}_1, b_2-1; z_1, z_2)$;

- (vi) recur on b_2 , with eq. (23) to obtain $R(\bar{a}; \bar{b}_1, \bar{b}_2; z_1, z_2)$;
- (vii) return result in the form $C_1 R_1 + C_2 R_2$.

Procedure V. Transformation of Output to Legendre's Notation

This Procedure is used only optionally for those who prefer Legendre's notations. The implementation of such transformation is basically clerical in nature, with the appropriate formula given by Carlson (Ref. 3). Therefore no algorithmic flow is indicated here.

ACKNOWLEDGEMENT

This author is indebted to Professor B. C. Carlson for numerous helpful discussions, and to Mr. V. Snyder for helping to design Procedure IV.2.

This research has been supported by a grant, #044-440, from the Office of Naval Research.

APPENDIX I Properties of R-functions

The R-function is essentially a generalized hypergeometric function of n -variables. It is symmetric with respect to simultaneous permutation of the b -parameters and z -variables. For detail properties see Carlson (Ref. 2). Here we summarize those properties useful in the text.

1. Special values of parameters and variables:

$$R(a; b_1, \dots, b_{n-1}, 0; z_1, \dots, z_n) = R(a; b_1, \dots, b_{n-1}; z_1, \dots, z_{n-1}) \quad (25)$$

$$R(a; b_1, \dots, b_n; z_1, \dots, z_{n-1}, 0) = \frac{B(a, a' - b_n)}{B(a, a')} R(a; b_1, \dots, b_{n-1}; z_1, \dots, z_{n-1}), \quad (26)$$

$$R(a; b_1, \dots, b_n; z_1, \dots, z_{n-1}, z_{n-1}) = R(a; b_1, \dots, b_{n-1} + b_n; z_1, \dots, z_{n-1}), \quad (27)$$

where

$$C = b_1 + \dots + b_n, \quad a' = C - a, \quad \text{and} \quad B(a, a') = \Gamma(a)\Gamma(a')/\Gamma(a+a'). \quad (28)$$

Owing to permutation symmetry, the zero or equal pair of variables in the last three formulas is not confined to the last position. Moreover, these identities can obviously be generalized to multiple zeros or multiple pairs, or any k -tuplets of equal variables.

2. Recurrence relations:

$$(c-1)R(b_i-1) = (a'-1)R + az_i R(a+1), \quad i=1, \dots, n \quad (29)$$

$$cR = \sum_{i=1}^n b_i R(b_i+1), \quad (30)$$

$$cR(a-1) = \sum_{i=1}^n b_i z_i R(b_i+1), \quad (31)$$

$$acR(a+1) = \sum_{i=1}^n b_i z_i^{-1} [cR - a'R(b_i+1)]. \quad (32)$$

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