NASA CONTRACTOR REPORT

NASA CR-2457

NASA CR-2457

MODERN SAMPLED-DATA CONTROL THEORY:

Design of the Large Space Telescope

by B. C. Kuo and G. Singh

Prepared by
SYSTEMS RESEARCH LABORATORY
Champaign, Ill. 61820
for George C. Marshall Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1974

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO. NASA CR- 2457	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE MODERN SAMPLED-DATA CONTROL THEORY: Design of the Large Space Telescope		5. REPORT DATE September 1974
		6. PERFORMING ORGANIZATION CODE M 132
7. AUTHOR(S) B. C. Kuo and G. Singh		8. PERFORMING ORGANIZATION REPORT #
9. PERFORMING ORGANIZATION NAME AND ADDRESS Systems Research Laboratory P. O. Box 2277, Station A Champaign, Illinois 61820		10. WORK UNIT NO.
		11. CONTRACT OR GRANT NO. NAS 8-29853
		13. TYPE OF REPORT & PERIOD COVERED CONTRACTOR Final
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D. C. 20546		14. SPONSORING AGENCY CODE
15. SUPPLEMENTARY NOTES		

16. ABSTRACT

This investigation was undertaken to determine the effect of varying the sampling period on the dynamic response of the sampled-data LST system.

A range of sampling periods is recommended based on the criterion that self-sustained oscillations are to be avoided in the LST system. The step responses of the LST system are then investigated when various sampling periods are used.

For small sampling periods, the dynamic behavior of the sampled-data system is very similar to that of the continuous-data system. When T is large (but less than 0.25 sec) the overshoot of the step response of the sampled-data system becomes greater. However, the dynamic behavior of the sampled-data system may be improved by redesigning the controller.

From this study it appears that a sampling period as high as 0.1 second is feasible for the LST system. However, it should be noted that the conclusions are obtained with the existing system model. Other practical considerations such as noise, coupling effects and quantization errors, may restrict the sampling period to a lower value.

17. KEY WORDS	18. DISTRIBUTION STATEMENT CAT. 10		
19. SECURITY CLASSIF. (of this report) Unclassified	20. SECURITY CLASSIF. (of this page) Unclassified	21. NO. OF PAGES 131	22. PRICE $4.75

TABLE OF CONTENTS

1. Selecting the Sampling Period of the LST System

The objective of this investigation is to determine the effect of varying the sampling period on the dynamic response of the sampled-data LST system.

A range of sampling periods is recommended based on the criterion that self-sustained oscillations are to be avoided in the LST system. The step responses of the LST system are then investigated when various sampling periods are used.

Detailed description of the LST system with the CMG nonlinearity is available in the Final Report, CONTINUOUS AND DISCRETE DESCRIBING FUNCTION ANALYSIS OF THE LST SYSTEM, January 1, 1974, prepared by the authors for NASA, Huntsville, under contract NAS8-29853. In that report describing function analyses are applied to the continuous-data and the sampled-data models of the LST system with the CMG nonlinearity. It is shown that the 9th-order LST system can be closely approximated by a 4th-order system.

Two sets of system parameters (System 1 and System 2) were considered in the Final Report. The study included in this report is concerned only with System 1.

It has been established that for System 1, and with $\gamma = 1.38 \times 10^5$ for the CMG nonlinearity, self-sustained oscillations will occur if the sampling period T exceeds 0.25 sec approximately.

In order to carry out the discrete describing function analysis for the sampled-data system, a sampler is inserted in the nonlinear loop, and thus a two-sampler system results. Computer simulation results show that the two-sampler model gives very good predictions on the occurance of self-sustained oscillations in the one-sampler system by the discrete describing function analysis.

Figures 1-1 and 1-2 show the zero-input responses of the LST system with two samplers; Figures 1-3 and 1-4 show the responses when there is only one sampler in the system. In both cases, the initial value of θ_V (vehicle position) is 5×10^{-8} rad while all other initial conditions are zero. The sampling period is 0.25 sec. With this large sampling period, the system actually settles into a self-sustained oscillation with a peak value of θ_V approximately equal to 10^{-10} rad, although this amplitude is not visible from the curves of Figures 1-1 through 1-4. As mentioned earlier, the sampling period of T = 0.25 sec can be considered as a boundary case between stability and instability.

It is of interest to investigate the step response of the LST system with and without sampling. A step input of amplitude $5 \times 10^{-8} K_0$ is applied which yields a final value of 5×10^{-8} rad for θ_V. Figures 1-5 and 1-6 illustrate the step responses of the continuous-data LST system. As expected, the continuous-data LST system with the designated controller parameter K_0 and K_1 has a fairly good step response, since it was demonstrated that the system has a relative damping ratio of 70% approximately. Figures 1-7 through 1-12 show the step responses of the sampled-data system with one sampler when T = 0.05, 0.1, and 0.25 sec, respectively. Figures 1-13 and 1-14 show the step responses of the two-sampler system with T = 0.25 sec.

When T = 0.25 sec, the step responses again have small oscillations in the steady state.

The step responses in Figures 1-5 through 1-14 show that the LST system with a step input behaves very similar (except for a shift in the reference of θ_V) to the system with zero input and nonzero initial value for θ_V.

The stability characteristics of the system with step input are also very similar to those of the system with zero input.

The conclusion is that with $\gamma = 1.38 \times 10^5$, the continuous-data system is always stable while the sampled-data system is stable for T less than 0.25 sec.

For small sampling periods, the dynamic behavior of the sampled-data system is very similar to that of the continuous-data system. When T is large (but less than 0.25 sec) the overshoot of the step response of the sampled-data system becomes greater. However, the dynamic behavior of the sampled-data system may be improved by redesigning the controller.

From this study it appears that a sampling period as high as 0.1 second is feasible for the LST system. However, it should be noted that the conclusions are obtained with the existing system model. Other practical considerations such as noise, coupling effects and quantization errors, may restrict the sampling period to a lower value.

System 1, γ = 1.38×10⁵ . T = 0.25 sec

Two Samplers

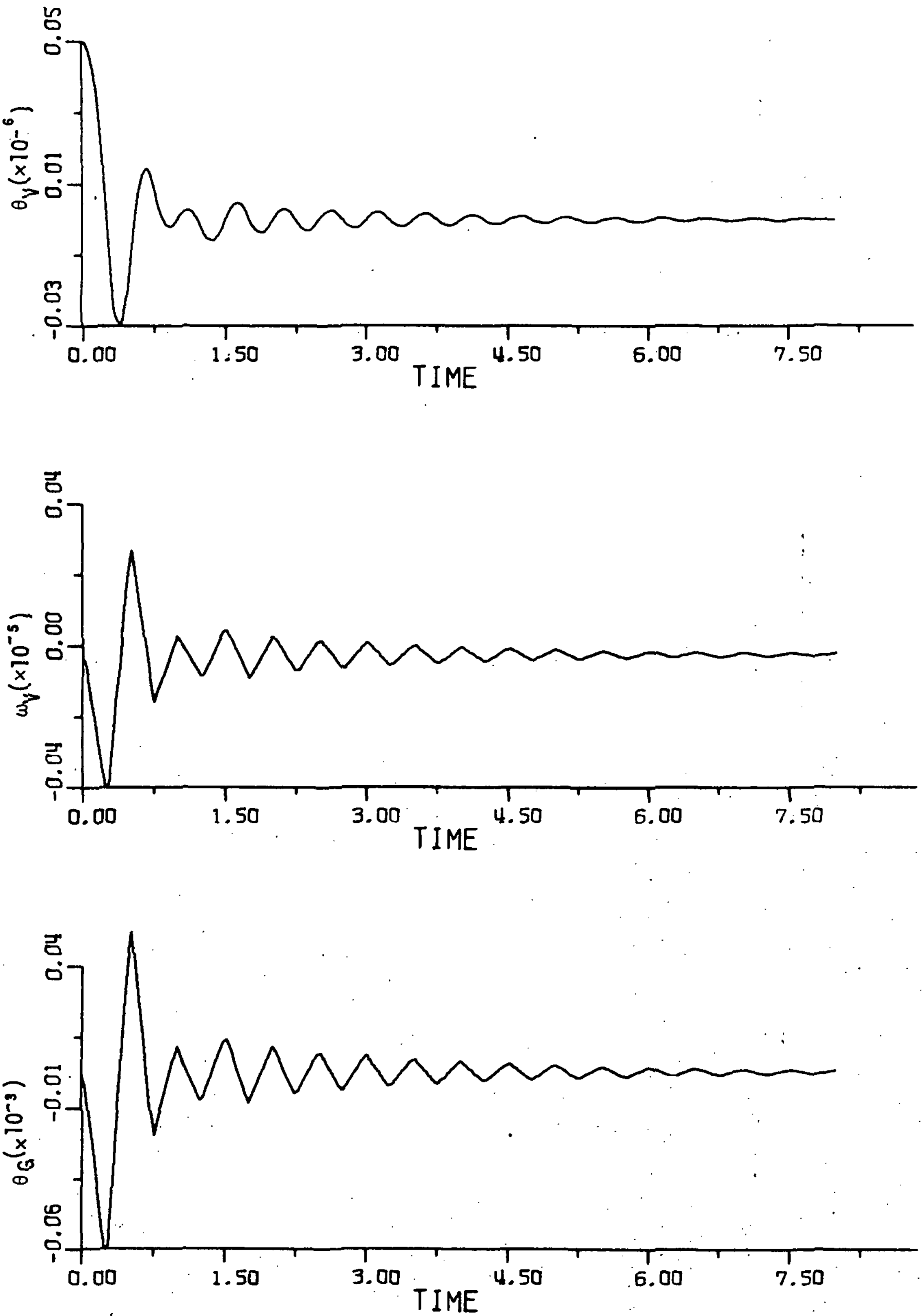


Figure 1-1

System 1, $\gamma = 1.38\times10^5$ T = 0.25 sec

Two Samplers

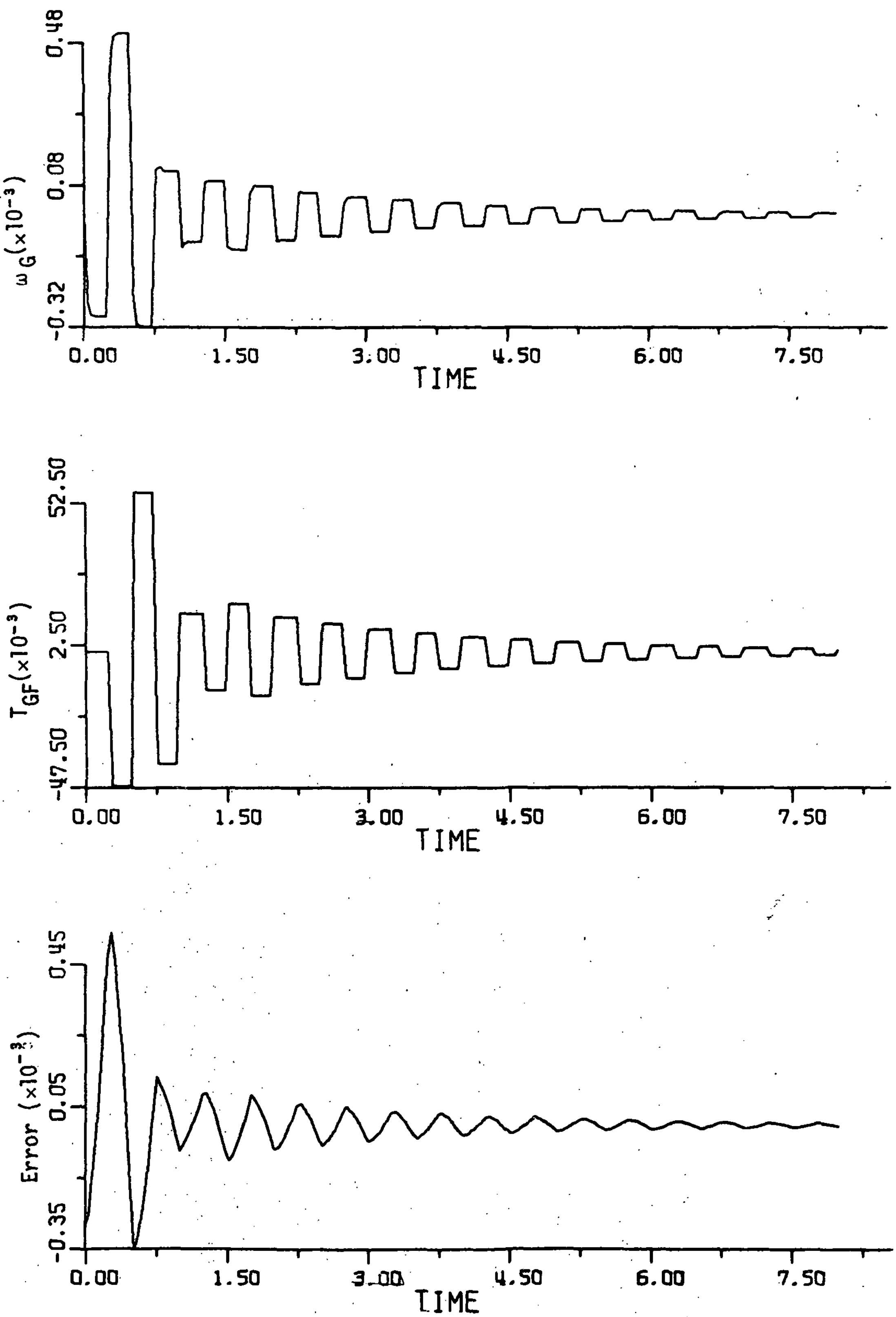


Figure 1-2

System 1, $\gamma = 1.38\times10^5$ T = 0.25 sec

One Sampler

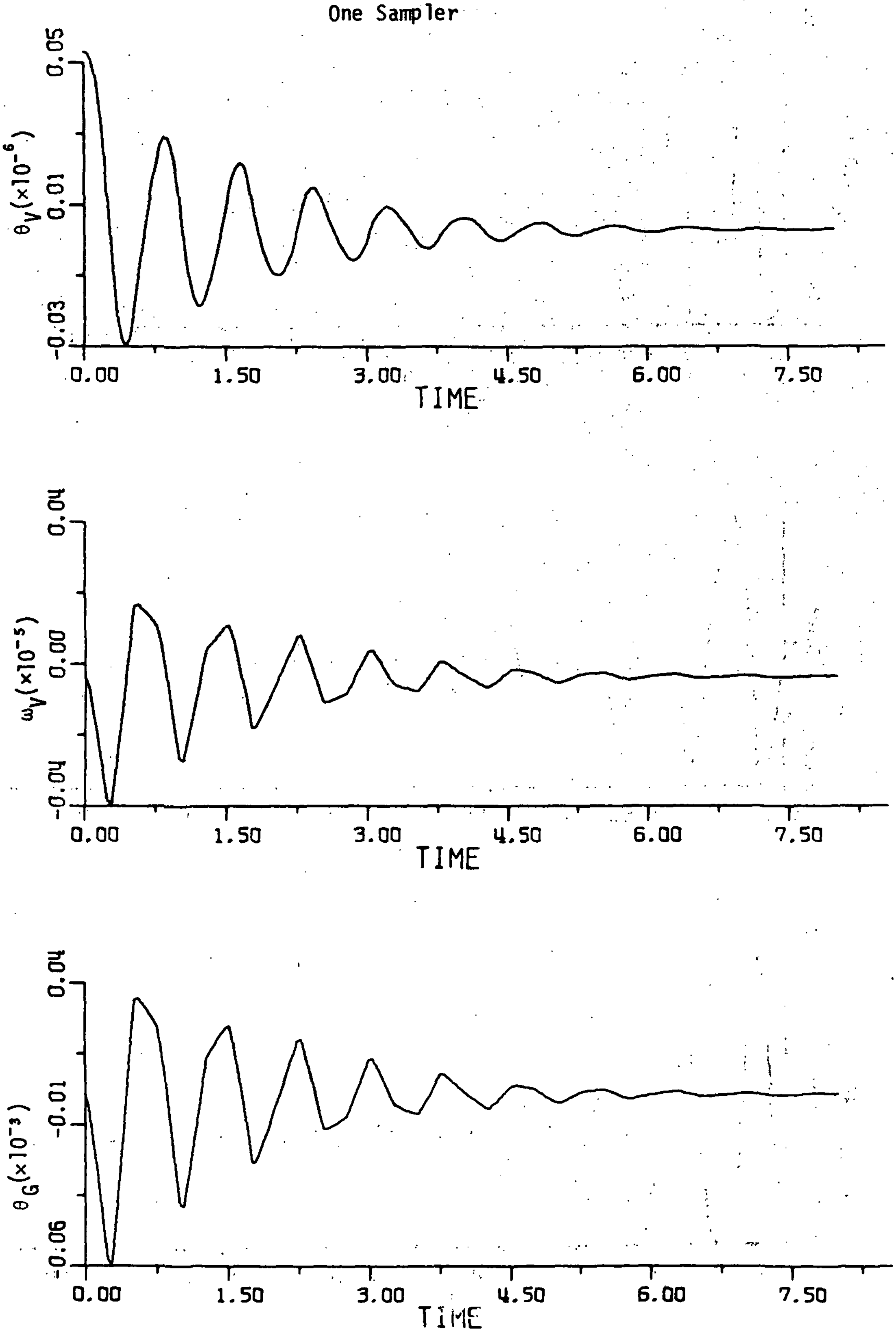


Figure 1-3

System 1, $\gamma = 1.38 \times 10^5$ T = 0.25 sec

One Sampler

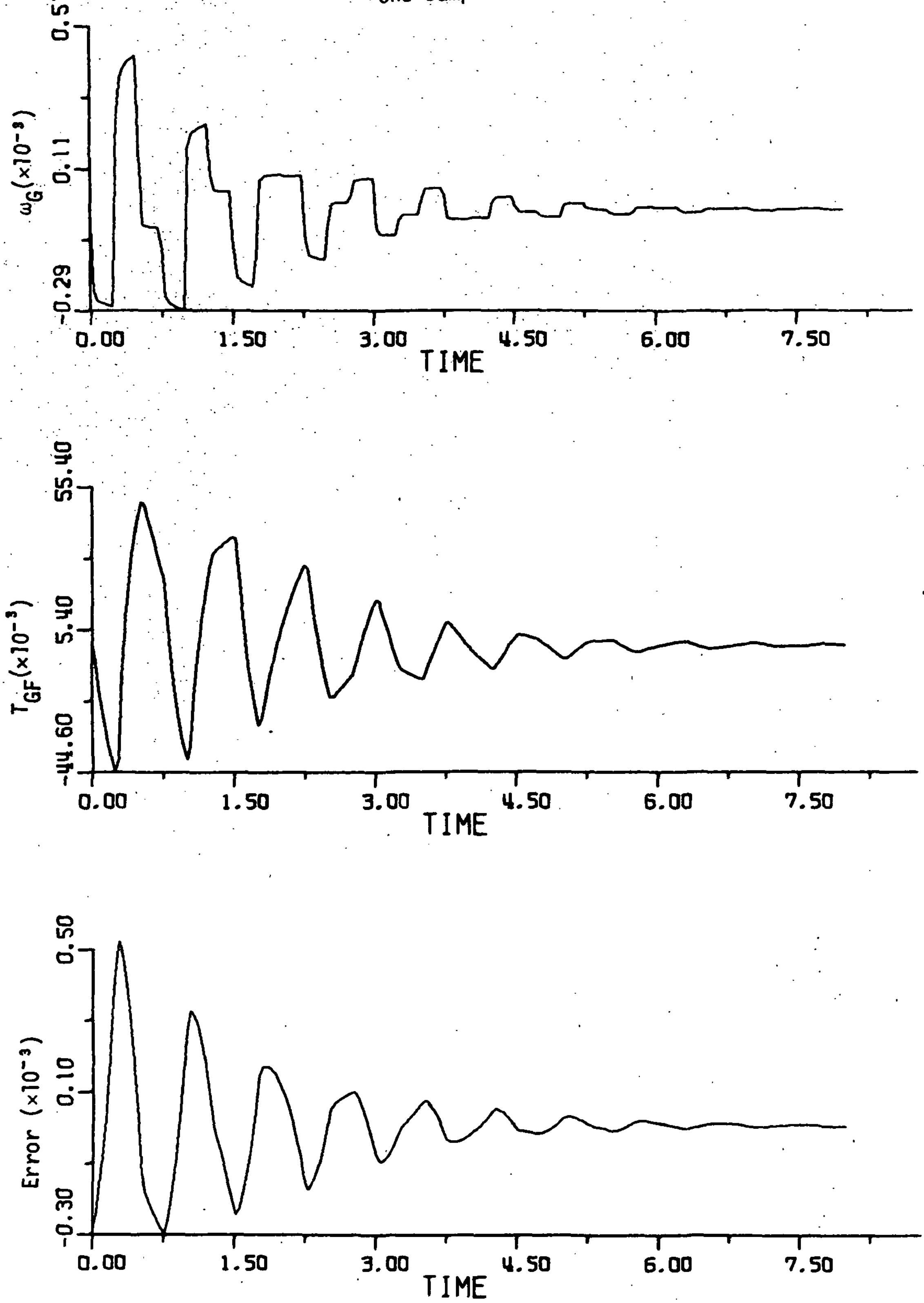


Figure 1-4

System 1, γ = 1.38×10^5 No Sampler

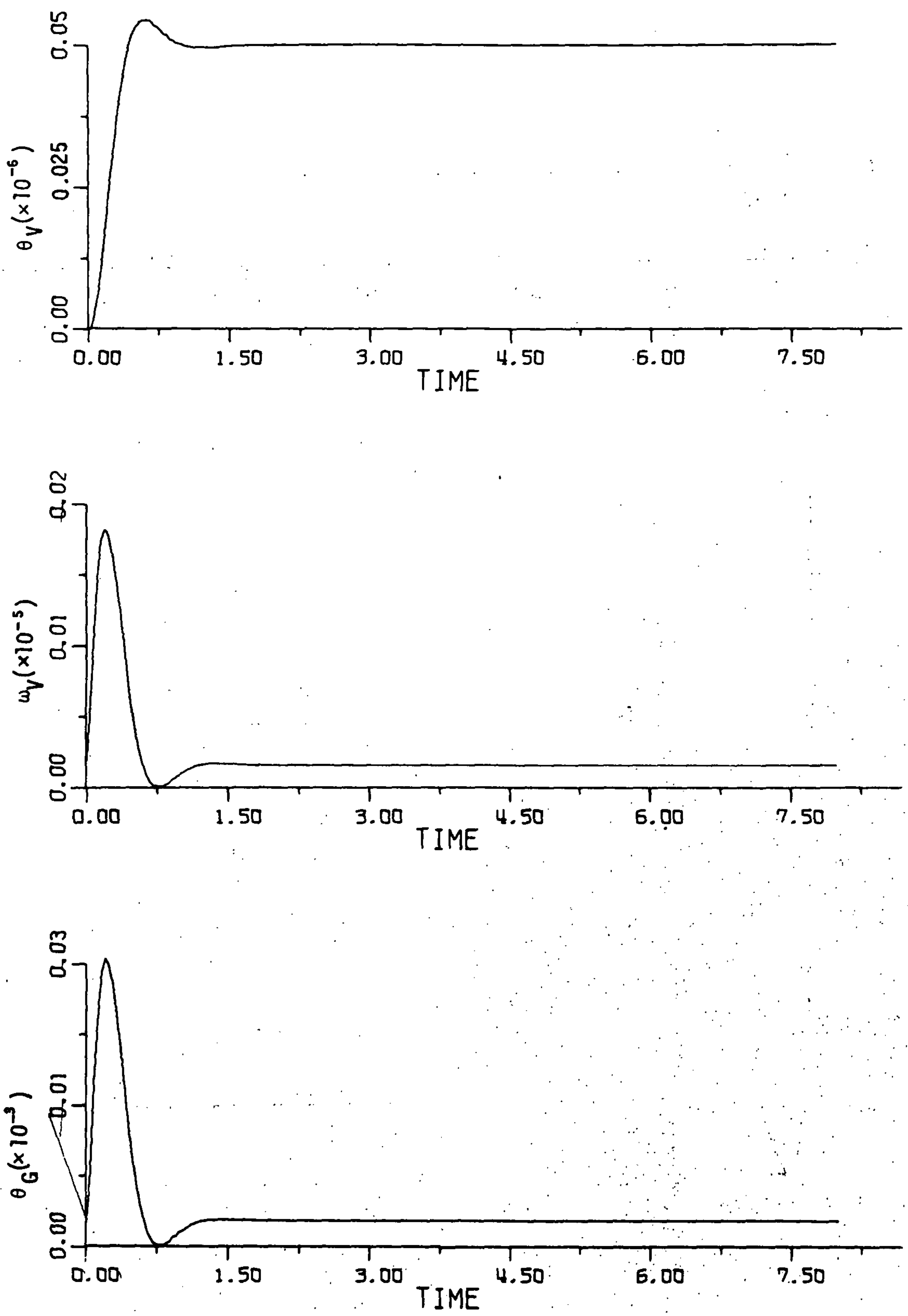


Figure 1-5

System 1, $\gamma = 1.38 \times 10^5$ No Sampler

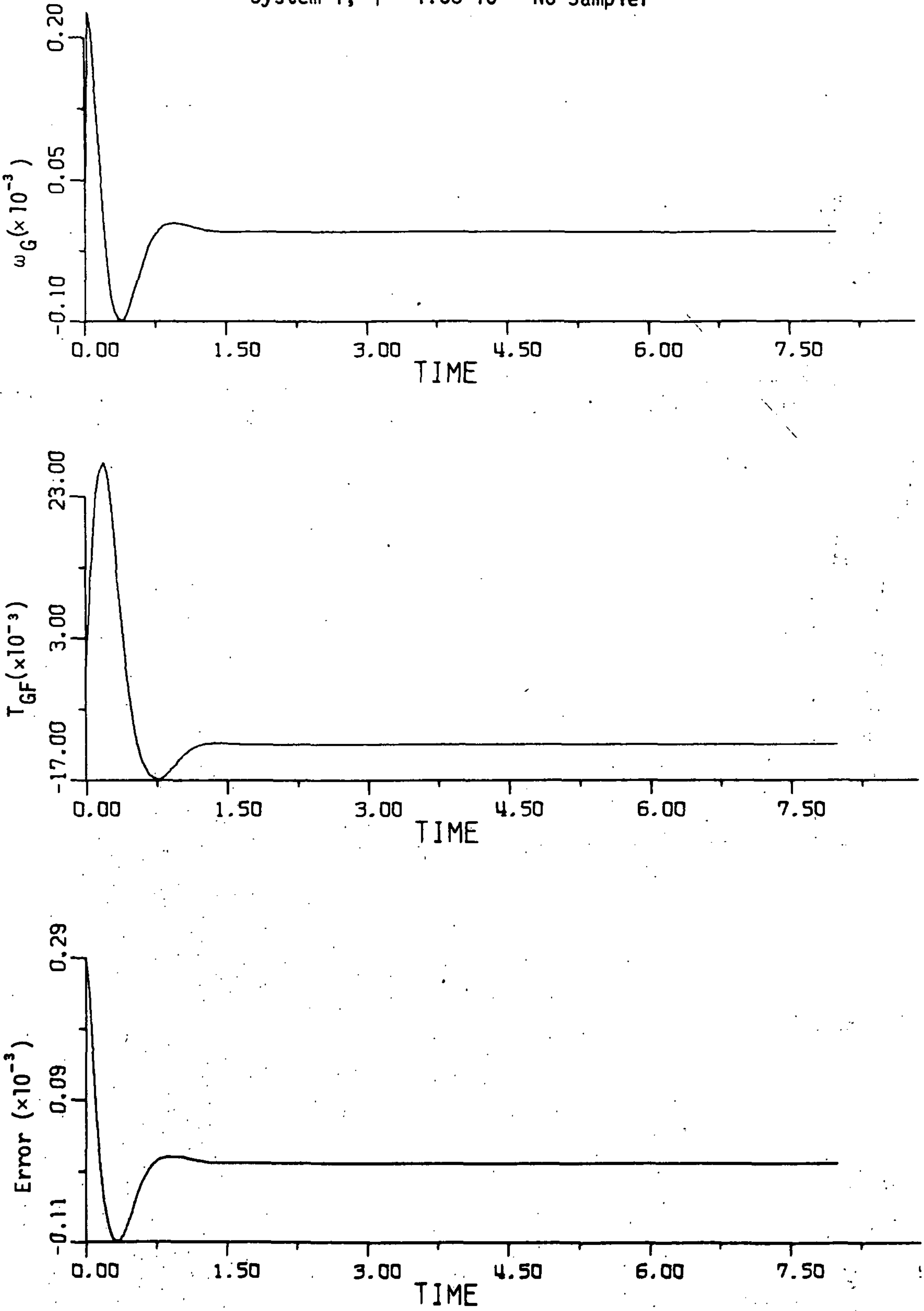


Figure 1-6

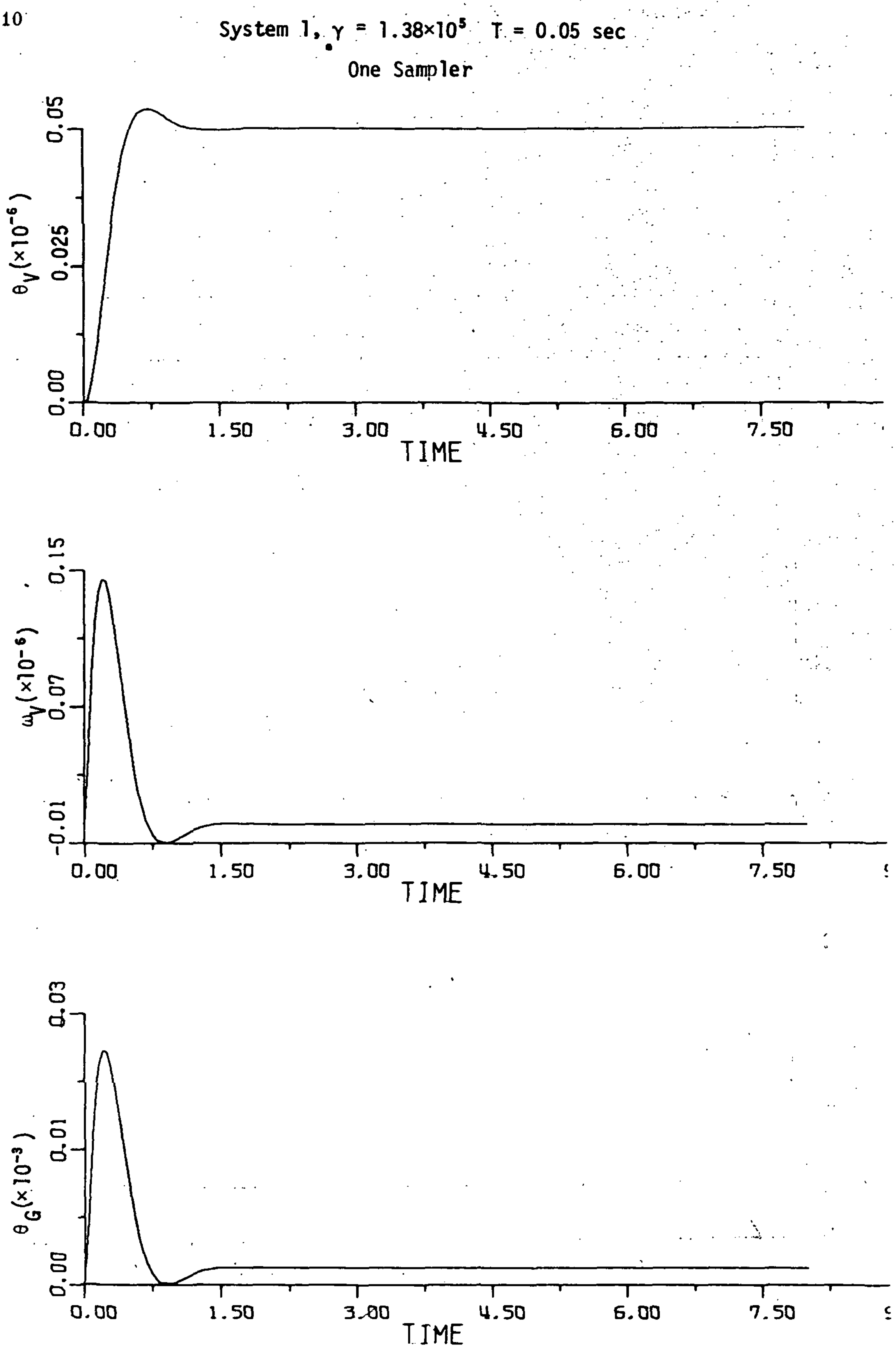

System 1, γ = 1.38×10⁵ T = 0.05 sec
One Sampler
θV(×10⁻⁶)
0.05
0.025
0.00
ωV(×10⁻⁶)
0.15
0.07
-0.01
θG(×10⁻³)
0.03
0.01
0.00
0.00
1.50
3.00
4.50
6.00
7.50
TIME

Figure 1-7

System 1, $\gamma = 1.38\times10^5$ T = 0.05 sec

One Sampler

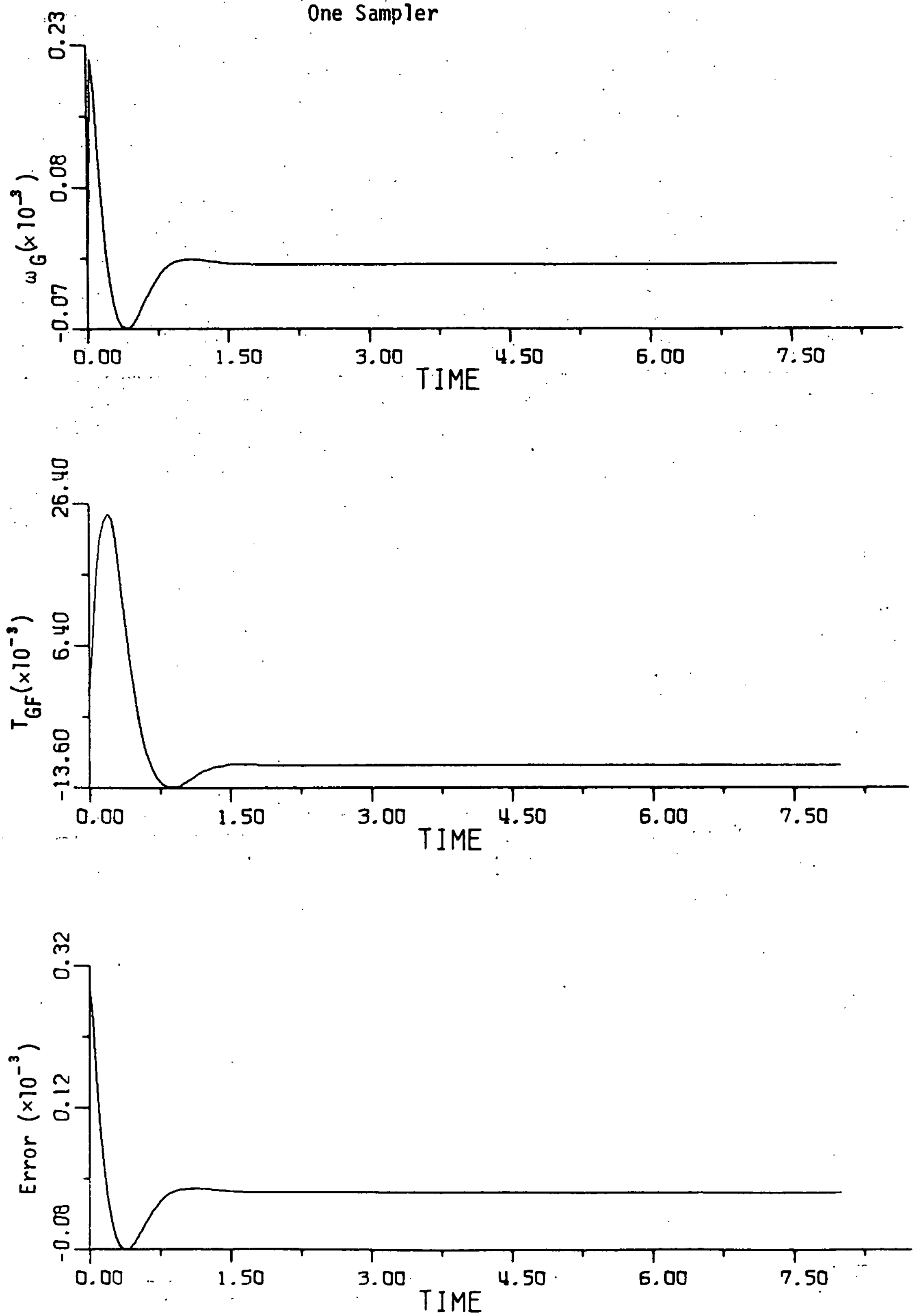


Figure 1-8

System 1, $\gamma = 1.38\times10^5$ T = 0.1 sec

One Sampler

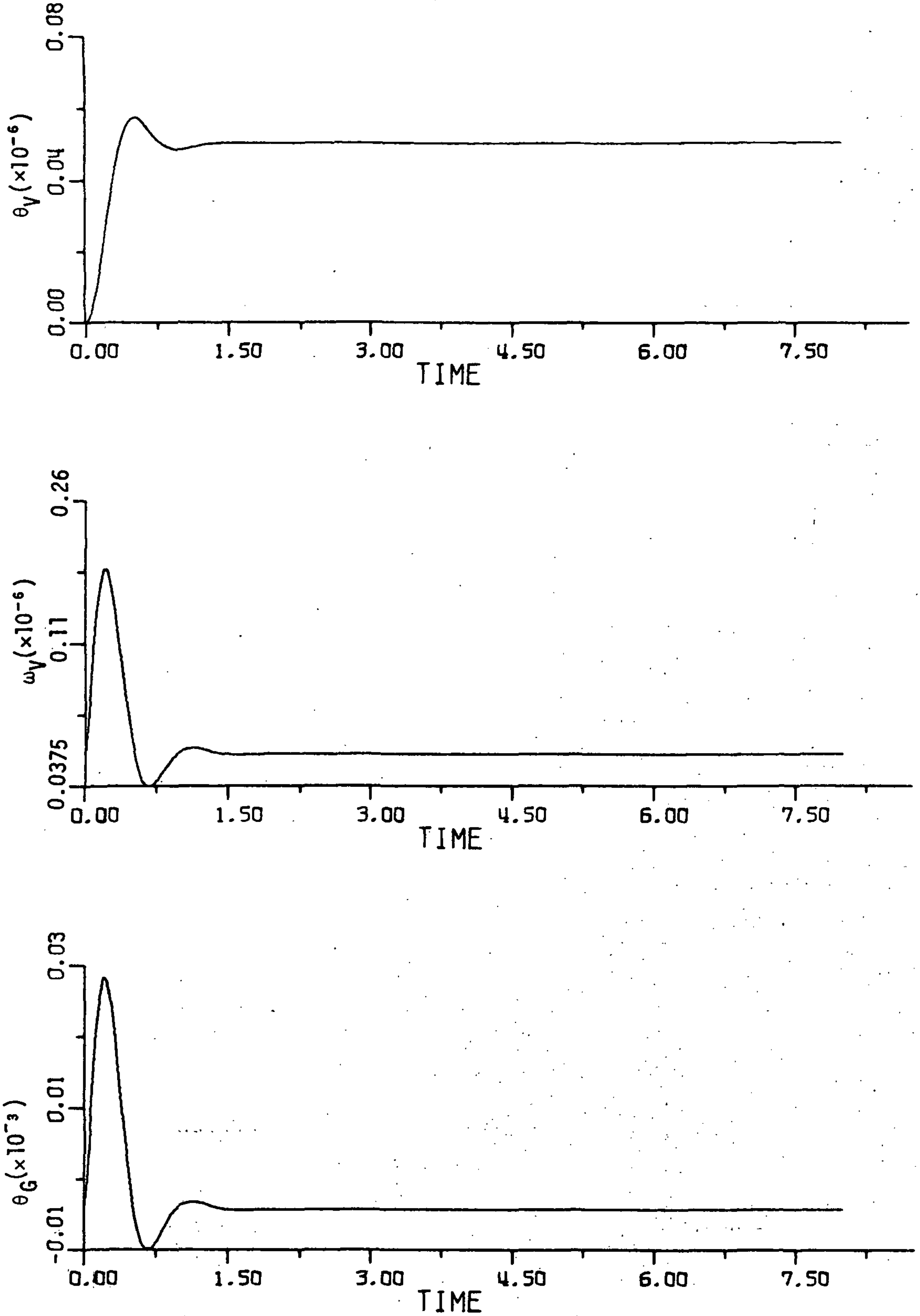


Figure 1-9

System 1, $\gamma = 1.38\times10^5$ $T = 0.1$ sec

One Sampler

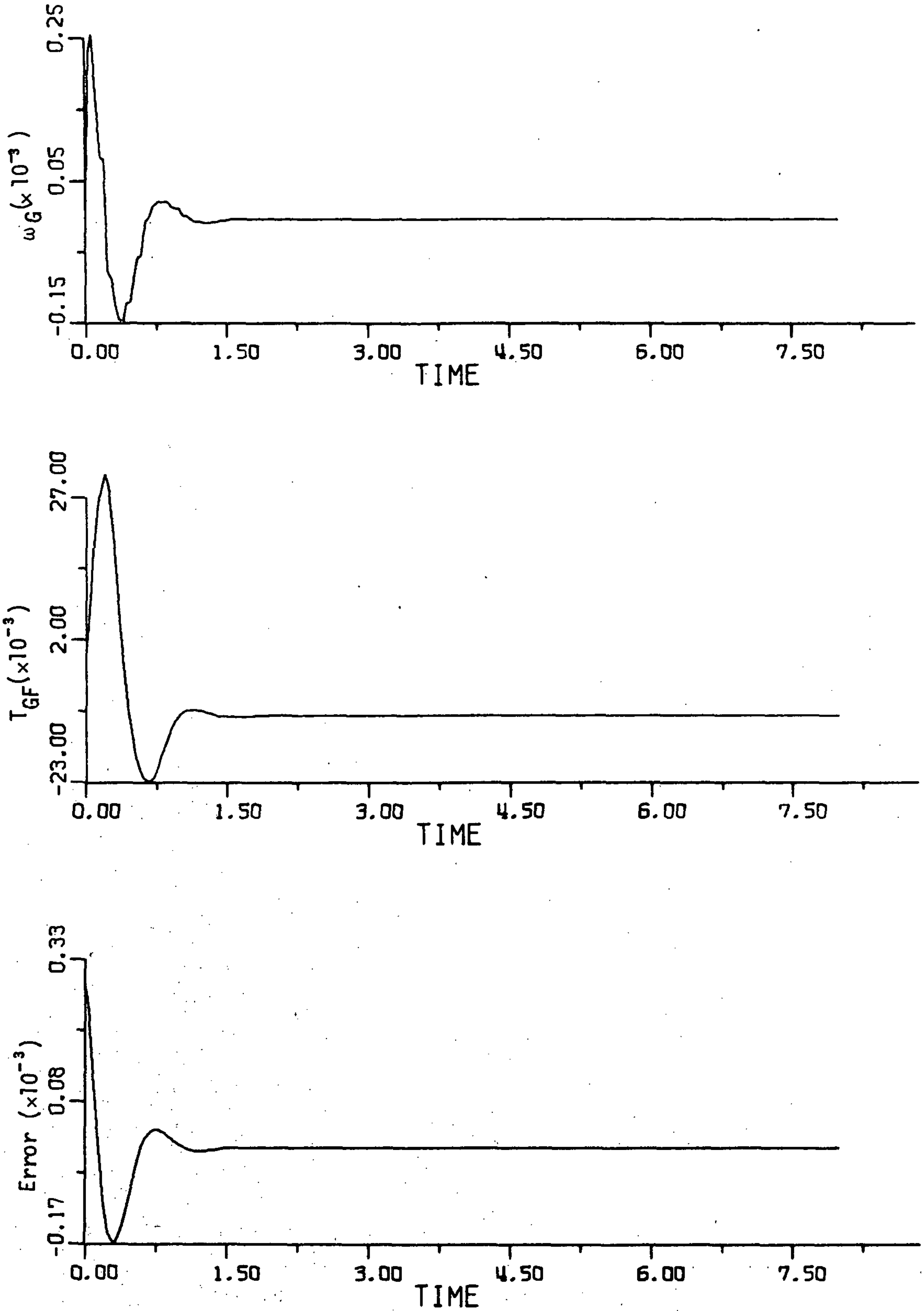


Figure 1-10

System 1, $\gamma = 1.38\times10^5$ $T = 0.25$ sec

One Sampler

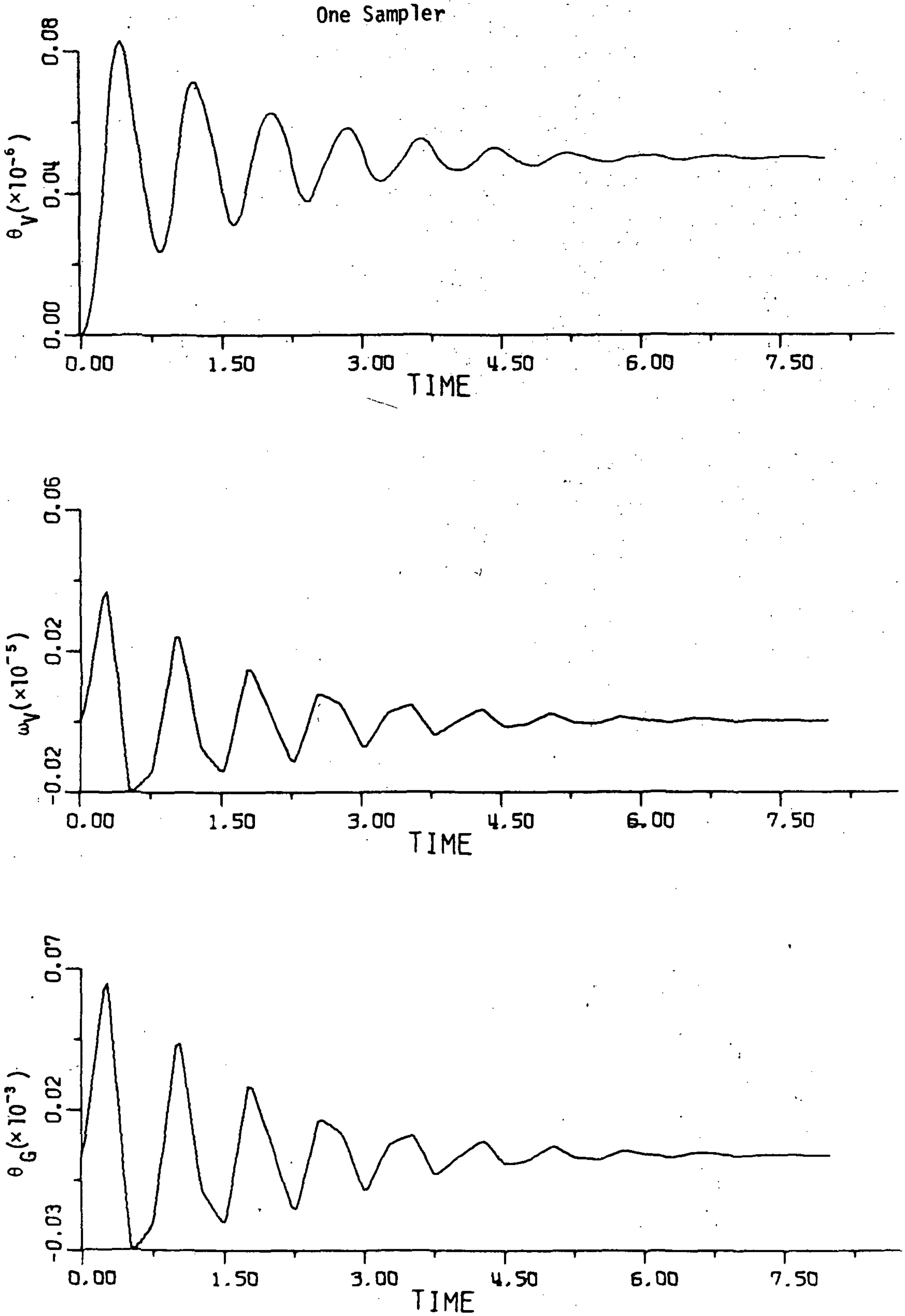


Figure 1-11

System 1, $\gamma = 1.38\times10^5$ T = 0.25 sec

One Sampler

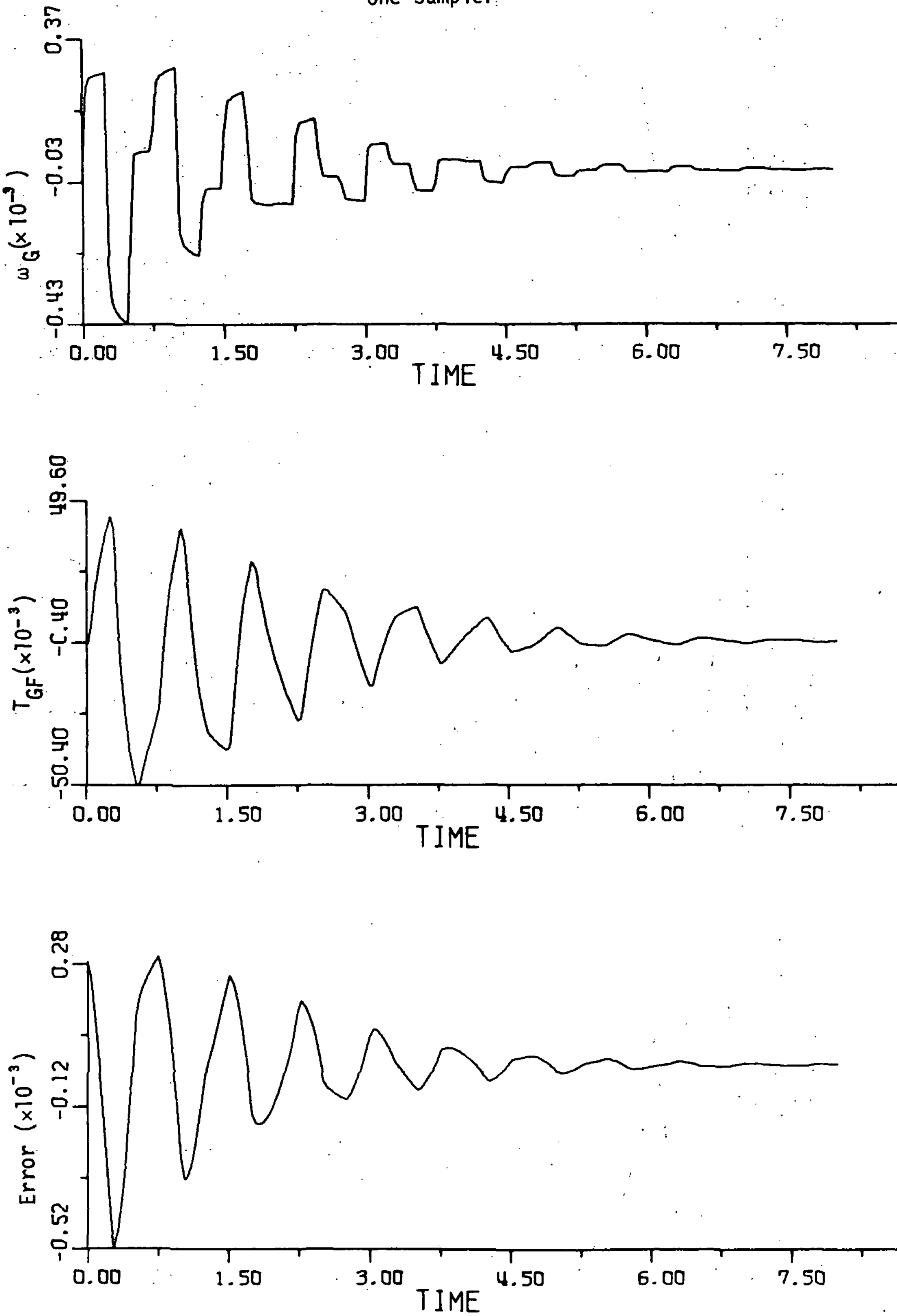


Figure 1-12

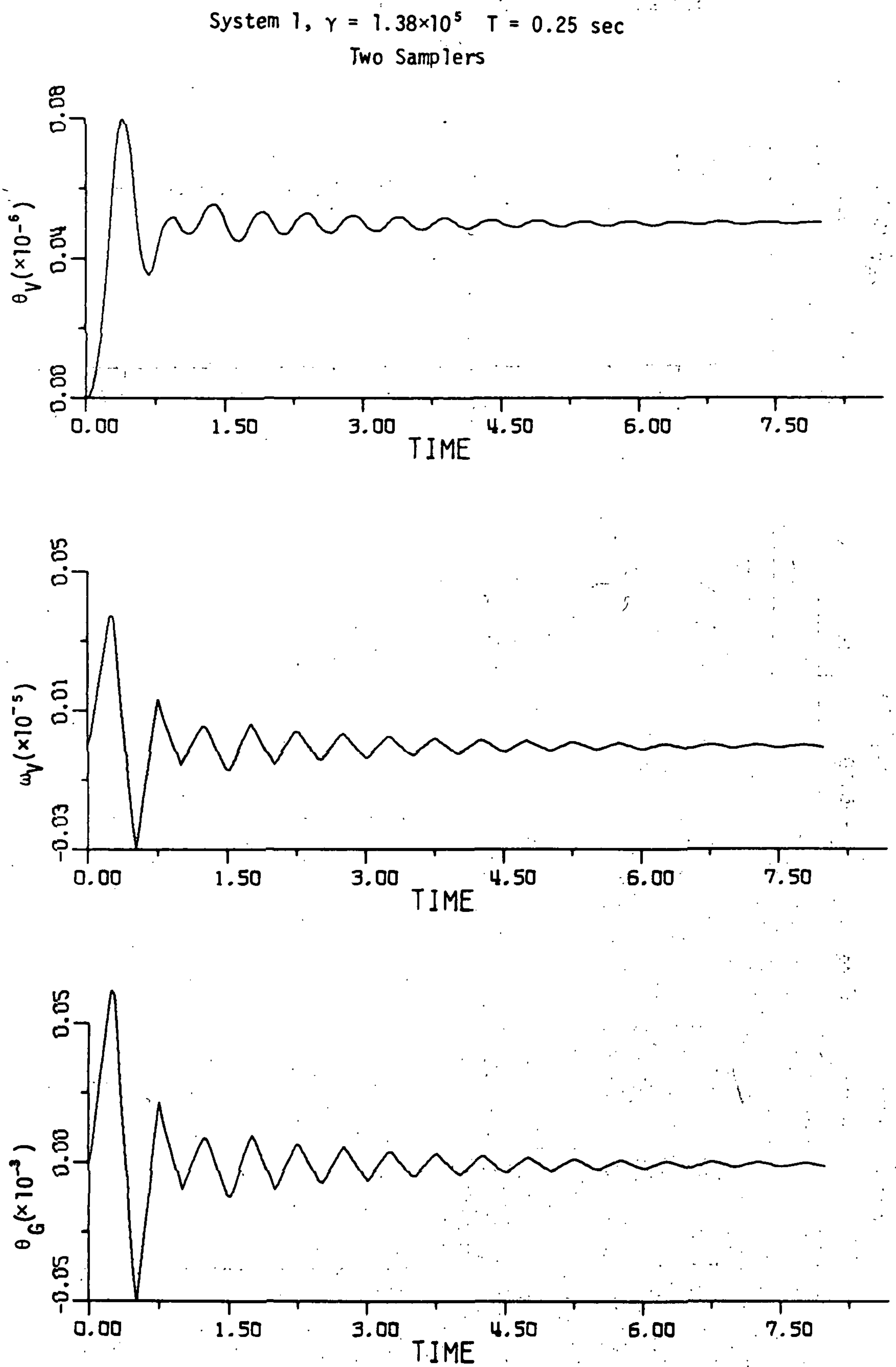
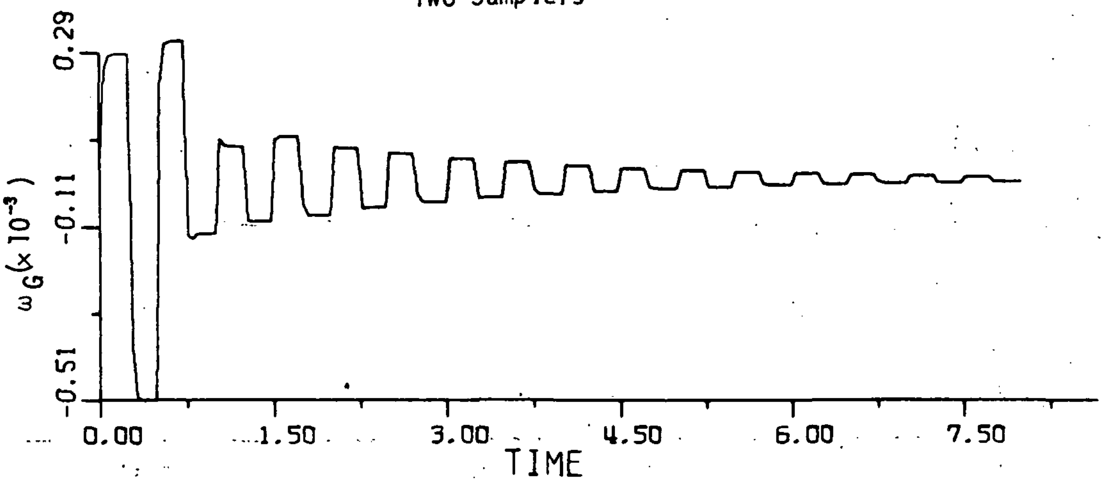
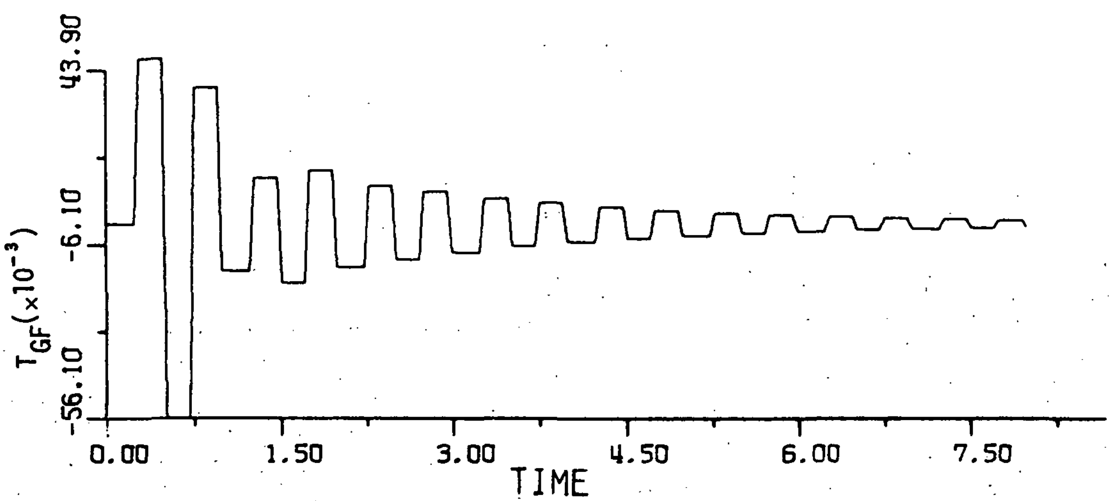

System 1, γ = 1.38×10⁵ T = 0.25 sec
Two Samplers
θ_V(×10⁻⁶)
0.08
0.04
0.00
0.00
1.50
3.00
4.50
6.00
7.50
TIME
ω_V(×10⁻⁵)
0.05
0.01
-0.03
0.00
1.50
3.00
4.50
6.00
7.50
TIME
θ_G(×10⁻³)
0.05
0.00
-0.05
0.00
1.50
3.00
4.50
6.00
7.50
TIME

Figure 1-13

System 1, $\gamma = 1.38\times10^5$ T = 0.25 sec

Two Samplers





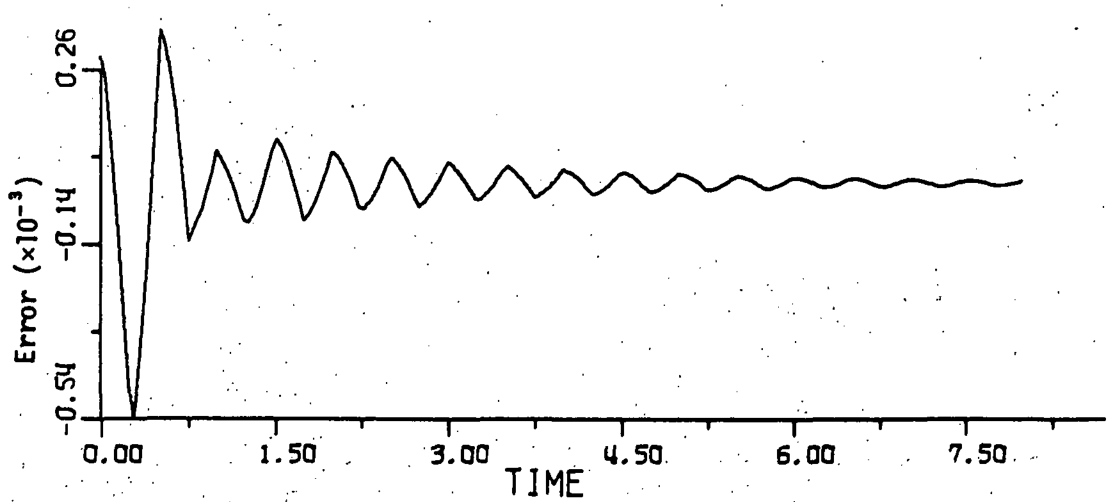


Figure 1-14

2. Design of the Continuous-Data LST System

The purpose of this section is to carry out an optimal design of the continuous-data LST system. The strategy is that the continuous-data controller may be used as a basis for a digital redesign, and at the same time, a digital control can be designed using a completely independent approach.

The system model of the LST was adopted from reference [1], and was later simplified from a 9th-order system to a 4th-order system in reference [2]. This simplification was justified from the standpoint of the system parameters, with no resulting loss of reality.

The block diagram of the 4th-order LST system is shown in Figure 2-1. Since K_0 and K_1 represent the parameters of the controller which are reported in reference [1], it is of interest to consider a complete redesign of the system. It was pointed out in [2] that with K_0 = 5758.35 and K_1 = 1371.02, the dominant CMG and vehicle modes are all with a damping ratio of approximately 0.707. However, in this report an attempt is made to arrive at a different control using the optimal control technique.

2-1. Decomposition of the LST System

Figure 2-2 shows a state diagram of the system of Figure 2-1. It is important to note that the nonlinear loop of the CMG dynamics is valid only from a symbolic viewpoint. In other words, the diagram of Figure 2-2 is obtained by treating N as a linear gain. For design purposes, the nonlinear loop is deleted, and for computer simulation, the system diagram of

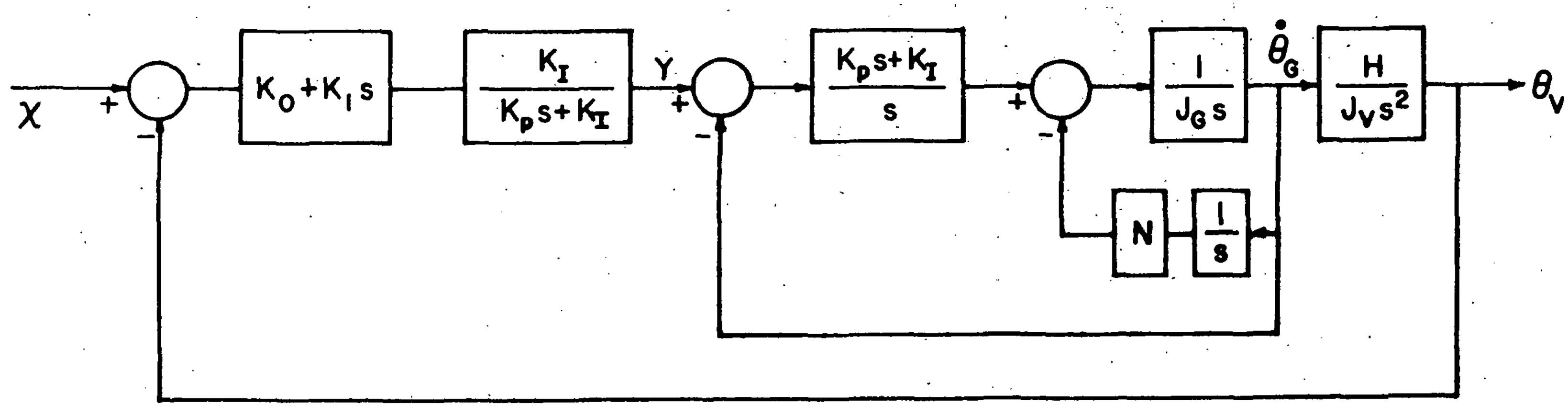


Figure 2-1. A Simplified LST system diagram.

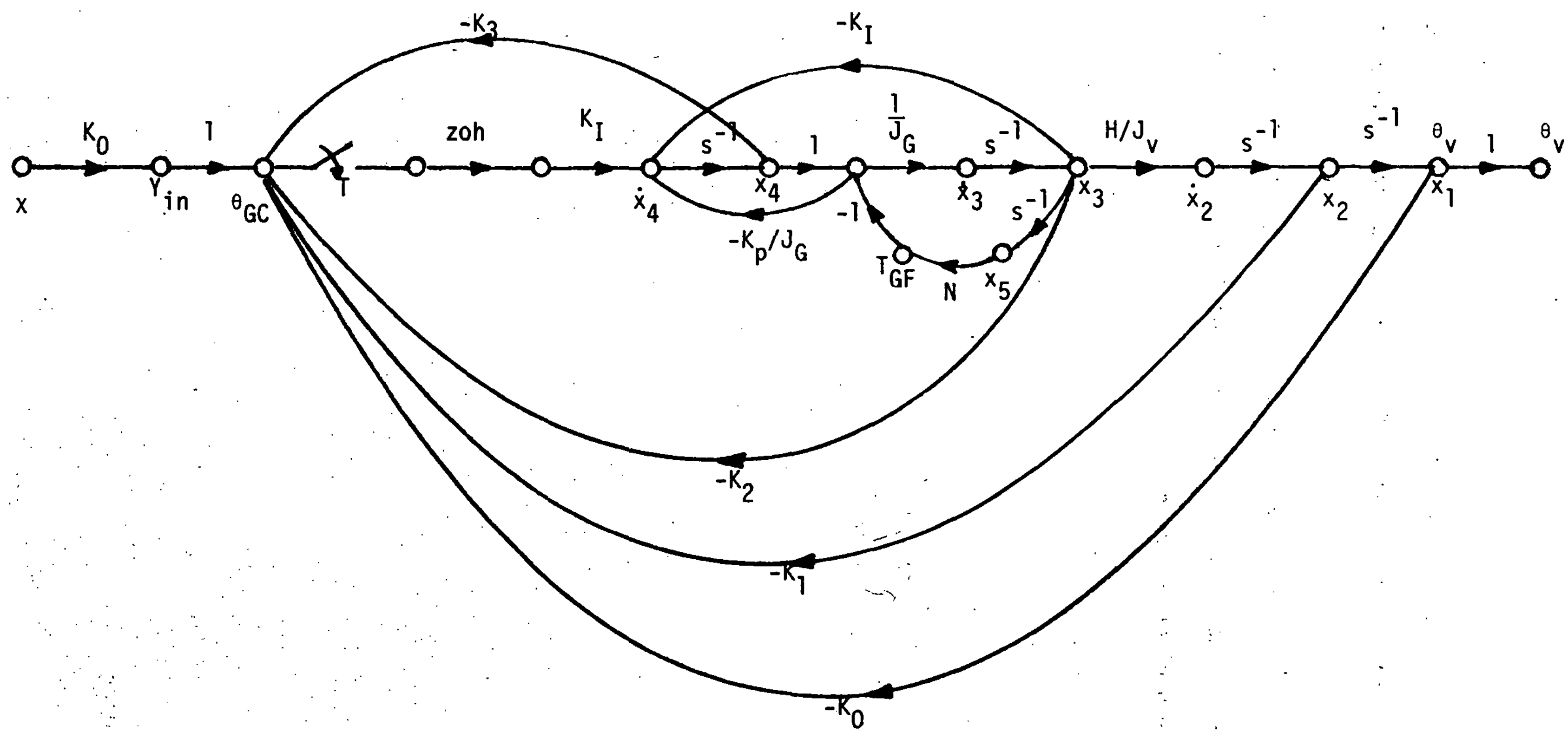


Fig. 2-2. A state diagram of the simplified LST system.

Figure 2-1 should be used.

Figure 2-2 also indicates that the control technique of state feedback is used. In reality, the system of Figure 2-1 feeds back two states in θ_V and $\dot{\theta}_V$ only.

The purpose of constructing the state diagram is so that we can represent the system in state variable form. The state equations of the system in Figure 2-2 with the nonlinear loop open are

$$\underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t) \tag{2-1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{H}{J_V} & 0 \\ 0 & 0 & 0 & \frac{1}{J_G} \\ 0 & 0 & -K_I & \frac{-K_p}{J_G} \end{bmatrix} \tag{2-2}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_I \end{bmatrix} \tag{2-3}$$

The control is given by

$$u(t) = K_o\chi(t) - K_o x_1(t) - K_1 x_2(t) - K_2 x_3(t) - K_3 x_4(t) \tag{2-4}$$

2-2. Linear Regulator Design

One of the advantages of using the state variable feedback is that the system can be designed in the sense of an optimal linear regulator. The performance index used for the optimization is

$$J = \int_0^\infty [\underline{x}'(t)Q\underline{x}(t) + u'(t)Ru(t)]dt \tag{2-5}$$

where Q is a symmetric semi-positive definite matrix, and R is symmetric and positive definite. The design objective is to determine the optimal control u(t) so that J in Equation (2-5) is a minimum, subject to the equality constraint of Equation (2-1).

It is well known that the solution to this optimal control problem is

$$u(t) = -R^{-1}B'K\underline{x}(t) \tag{2-6}$$

where K is the solution of the algebraic Riccati equation.

$$-KA - A'K + KBR^{-1}B'K - Q = 0 \tag{2-7}$$

The solutions of the Riccati equation and the optimal control have been programmed on a digital computer. Table 2-1 gives the solutions of K_0, K_1, K_2, and K_3, and the corresponding eigenvalues of the closed-loop system when various weighting matrices Q are used, where

$$Q = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{pmatrix} \tag{2-8}$$

TABLE 2-1

Linear Regulator Design of the Continuous-Data LST System ($q_3 = q_4 = 1$)

q_1	q_2	Eigenvalues			Feedback Gains			
					K_0	K_1	K_2	K_3
1	1	-0.673	-9700	-0.046 ± j0.046	1.0	23.17	0.6074	0.99
100	10	-0.676	-9700	-0.142 ± j0.149	10	82	10.2	0.99
10^3	10^3	-0.688	-9700	-0.247 ± j0.266	31.6	164.8	1.48	0.99
10^6	10^6	-0.973	-9700	-1.16 ± j1.262	1000	1818	5.9	0.99
10^7	10^6	-1.961	-9700	-1.206 ± j1.775	3162	3269	8.2	0.99
2×10^7	9000	-2.37	-9700	-1.185 ± j1.997	4472	3854	9.0	0.99
10^8	10^4	-3.08	-9700	-1.545 ± j2.626	10000	6570	12	0.99
3×10^7	5000	-2.53	-9700	-1.266 ± j2.14	5477	4407	9.6	0.99
5×10^7	5000	-2.75	-9700	-1.375 ± j2.33	7071	5220	10.6	0.99

Since the states x_1 and x_2 are of primary interest, q_3 and q_4 are kept constant at 1.0 while q_1 and q_2 are varied. Also, R = 1.

Several facts become clear from the results of Table 2-1.

(1) With large values of q_1 and q_2, that is, more weights on the states x_1 and x_2, the feedback gains K_2 and K_3 become negligible.

(2) The eigenvalue at -9700 is relatively insensitive to the various weighting matrices.

2-3. Design by Eigenvalue Assignment and the Inverse Problem

The development in the last section shows that it is difficult to have complete control of the eigenvalues of the closed-loop system by changing the elements of the weighting matrix Q. Since the original system from [1] with K_0 = 5758.35 and K_1 = 1371.02 resulted in a rather good step response, it is interesting to find out if it corresponds to an optimal linear regulator solution. This question is known as the inverse regulator problem [3].

The state equation of Equation (2-1) should first by transformed into the phase-variable canonical form. Substituting the system parameters into Equation (2-2) yields

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 6 \times 10^{-3} & 0 \\ 0 & 0 & 0 & 0.4762 \\ 0 & 0 & -9700 & -102.86 \end{bmatrix} \qquad (2\text{-}9)$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 9700 \end{pmatrix} \tag{2-10}$$

The transformation which transforms Equation (2-1) into the phase-variable canonical form is

$$\underline{y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.006 & 0 \\ 0 & 0 & 0 & 0.00286 \end{pmatrix} \underline{x} \tag{2-11}$$

$$v = 27.742u \tag{2-12}$$

The transformed state equation becomes

$$\dot{\underline{y}}(t) = A_1\underline{y}(t) + B_1v(t) \tag{2-13}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4623.7 & -102.86 \end{pmatrix} \tag{2-14}$$

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2-15}$$

The state feedback of the original system is described by

$$u = -G\underline{x} = -[K_o \quad K_1 \quad K_2 \quad K_3]\underline{x} \tag{2-16}$$

For the transformed system,

$$v = -H\underline{y} = -[H_o \quad H_1 \quad H_2 \quad H_3]\underline{y} \tag{2-17}$$

Thus, using Equations (2-11), (2-16) and (2-17), G and H are related through

$$G = \frac{H}{27.742}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.006 & 0 \\ 0 & 0 & 0 & 0.00286 \end{bmatrix}$$

$$= H\begin{bmatrix} 0.036 & 0 & 0 & 0 \\ 0 & 0.036 & 0 & 0 \\ 0 & 0 & 0.000216 & 0 \\ 0 & 0 & 0 & 0.000103 \end{bmatrix} \tag{2-18}$$

Thus,

$$\begin{aligned} K_o &= 0.036H_o \\ K_1 &= 0.036H_1 \\ K_2 &= 0.000216H_2 \\ K_3 &= 0.000103H_3 \end{aligned} \tag{2-19}$$

The inverse problem is that given the matrices A and B, and the feedback matrix G, is there a positive definite R and nonnegative Q such that Equation (2-16) is the optimal control for the system of Equation

(2-1) with the performance index in Equation (2-5)?

We form the characterisitic equation of the closed-loop system in the phase-variable canonical form.

$$p(s) = |sI - A_1 + B_1H| = s^4 + (102.86 + H_3)s^3 + (4623.7 + H_2)s^2 + H_1s + H_0 \tag{2-20}$$

The characteristic equation of the open-loop system is

$$\psi(s) = |sI - A_1| = s^4 + 102.86s^3 + 4623.7s^2 \tag{2-21}$$

It is shown in [3] that the feedback matrix H is indeed optimal for the choice of

$$Q = D'D \tag{2-22}$$

and R = 1, where

$$D = [d_1 \quad d_2 \quad d_3 \quad d_4] \tag{2-23}$$

The elements of D are the coefficients of the polynomial

$$m(s) = d_4s^3 + d_3s^2 + d_2s + d_1 \tag{2-24}$$

where m(s) satisfies

$$p(s)p(-s) = \psi(s)\psi(-s) + m(s)m(-s) \tag{2-25}$$

Substitution of Equations (2-20), (2-21), and (2-24) into Equation (2-25), we have the following relationships after simplification:

$$-1332.78 - d_4^2 = 2(H_2 + 4623.7) - (102.86 + H_3)^2 \tag{2-26}$$

$$(4623.7)^2 + d_3^2 - 2d_2d_4 = 2H_o - 2H_1(102.86 + H_3) + (H_2 + 4623.7)^2 \tag{2-27}$$

$$2d_1d_3 - d_2^2 = 2H_o(H_2 + 4623.7) - H_1^2 \tag{2-28}$$

$$d_1^2 = H_o^2 \tag{2-29}$$

For

$K_o = 5758.35,$

$K_1 = 1371.02,$

$K_2 = K_3 = 0,$

we have

$H_o = 159954,$

$H_1 = 38083.9,$

$H_2 = H_3 = 0.$

Thus, the last four equations lead to

$$d_1 = H_o = 159954 \tag{2-30}$$

$$d_4 = 0 \tag{2-31}$$

$$d_3^2 = 2H_o - 205.72H_1 = -7.51 \times 10^6 \tag{2-32}$$

Since d_3^2 is negative, we do not have a real solution for d_3. Thus, the system with the prescribed feedback gains does not correspond to an optimal linear regulator solution. Equation (2-32) shows that to have a linear regulator solution, the following conditions must be satisfied (with $K_2 = K_3 = 0$):

$$H_0 \geq 102.86H_1 \tag{2-33}$$

or

$$K_0 \geq 102.86K_1 \tag{2-34}$$

<u>Eigenvalue Assignment</u>

An alternative to the linear state regulator method for designing linear feedback systems is the method of pole placement or eigenvalue assignment. In this method, the approach is to place the eigenvalues of the closed-loop system at certain desired locations by appropriate choice of the feedback gains. If the system is in the phase-variable canonical form, this method is directly applicable [3].

Consider the linear system represented by Equations (2-14), (2-15), and (2-17), the characteristic polynomial of this closed-loop system is as in Equation (2-20),

$$\begin{aligned} |sI - A_1 + B_1H| = s^4 + (102.86 + H_3)s^3 + (4623.7 + H_2)s^2 \\ + H_1s + H_0 \end{aligned} \tag{2-35}$$

Let the desired location of the closed-loop eigenvalues be $-\alpha_1$, $-\alpha_2$, $-\alpha_3$, $-\alpha_4$. The characteristic polynomial which yields these eigenvalues is

$$(s + \alpha_1)(s + \alpha_2)(s + \alpha_3)(s + \alpha_4) = s^4 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)s^3$$
$$+ (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4)s^2$$
$$+ (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4)s + \alpha_1\alpha_2\alpha_3\alpha_4 \qquad (2\text{-}36)$$

The desired feedback gains are obtained by equating the coefficients of the polynomials of Equations (2-35) and (2-36). Thus

$$H_o = \alpha_1\alpha_2\alpha_3\alpha_4$$
$$H_1 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4$$
$$H_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 - 4623.7$$
$$H_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 102.86 \qquad (2\text{-}37)$$

In addition, it is desired that all elements of H be positive so that negative feedback is maintained. Once H is determined the feedback matrix G can be obtained from Equation (2-18).

Table 2-2 shows the feedback gains G and H, obtained for several choices of eigenvalues.

TABLE 2-2

Eigenvalue Assignment Design for the Continuous-Data LST System

				H				G			
α_1	α_2	α_3	α_4	H_o	H_1	H_2	H_3	K_o	K_1	K_2	K_3
100	10	4 + j4	4 - j4	128000	36480	533	45.25	4618	1316	0.115	0.005
250	10	4 + j4	4 - j4	320000	89280	7733	195.2	11546	3221.4	1.67	0.02
500	10	4 + j4	4 - j4	640000	177280	19733	445.25	23092	6397	4.27	0.05
300	40	4 + j4	4 - j4	96000	33920	893	215.25	3463	1224	0.19	0.02
500	40	4 + j4	4 - j4	160000	56320	4493	415.25	5773	2032	0.97	0.04
1000	40	4 + j4	4 - j4	320000	112320	13491	915.25	11546	4052	2.91	0.09

3. Computer Simulation of the Simplified LST System with the Linear State Regulator and Eigenvalue Assignment Designs

A computer simulation is presented here to show the response of the continuous-data LST system with the feedback gains designed by the linear state regulator method and the eigenvalue assignment method. Although the design was performed in Section 2 without the nonlinearity, the simulation is of the nonlinear system. As mentioned earlier, the system of Figure 2-1 is used for this purpose.

The numerical data for the system and nonlinearity are

$J_V = 10^5$

$J_G = 2.1$

$K_p = 216$

$K_I = 9700$

$H = 600$

$\gamma = 1.38 \times 10^5$

$T_{GFO} = 0.1$

For the computer simulation, the input to the LST system χ is set to $5 \times 10^{-8} K_o$ so that the final value of $\theta_V = 5 \times 10^{-8}$. All initial conditions are zero. The following quantities are plotted

θ_V = vehicle position (radians)

ω_V = vehicle velocity (radians/second)

θ_G = Gimbal position (radians)

ω_G = Gimbal velocity (radians/second)

T_{GF} = Torque output of the nonlinearity (ft.lb.)

Error = Error input to the CMG.

$= \chi - K_0\theta_V - K_1\omega_V$

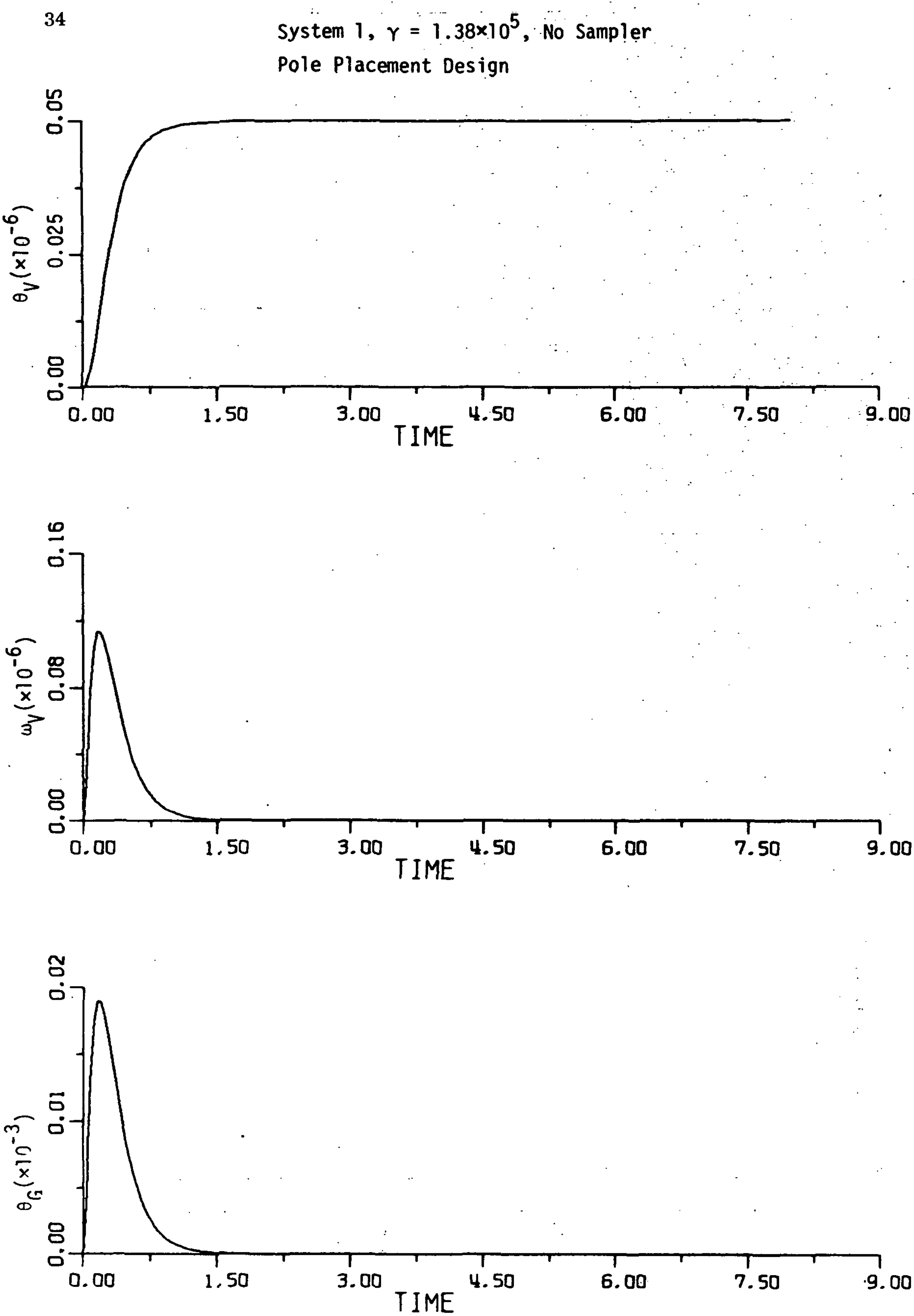
In all the simulations, only the states θ_V and ω_V are fed back. Although the designs yielded feedback from all four states, the gains K_2, K_3 are ignored as they are very small and are also not accessible in the system which is simulated.

Figures 3-1 and 3-2 show the results with a design using the eigenvalue assignment method. The feedback gains are K_0 = 5773, K_1 = 2032. The corresponding eigenvalue locations are -500, -40, -4 ± j4.

Figures 3-3 and 3-4 show the results with a state regulator design. The feedback gains are K_0 = 10000, K_1 = 6579.5. The system eigenvalues are -9700, -3.08, -1.545 ± j2.626. The weighting matrices used for this design are Q = diagonal $[10^8 \quad 10^4 \quad 1 \quad 1]$, R = 1.

Figures 3-5 and 3-6 show the results with another state regulator design. Here K_0 = 7071, K_1 = 5220, the eigenvalues are -9700, -2.75, -1.375 ± j2.33 and the weighting matrices are Q = diagonal $[5 \times 10^7 \quad 5000 \quad 1 \quad 1]$, R = 1.

The results of Figures 3-1 through 3-6 show that the linear design methods of Section 2 can yield closed-loop systems with very acceptable response characteristics. In comparison with the system used in reference [2], the new designs result with no overshoot in the response of θ_V. The state regulator designs are, however, slower in comparison to the eigenvalue assignment design and the system in reference [2].



System 1, γ = 1.38×10^5, No Sampler
Pole Placement Design
θV(×10^-6)
0.00
0.025
0.05
ωV(×10^-6)
0.00
0.08
0.16
θG(×10^-3)
0.00
0.01
0.02
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 3-1

System 1, $\gamma = 1.38\times10^5$, No Sampler
Pole Placement Design

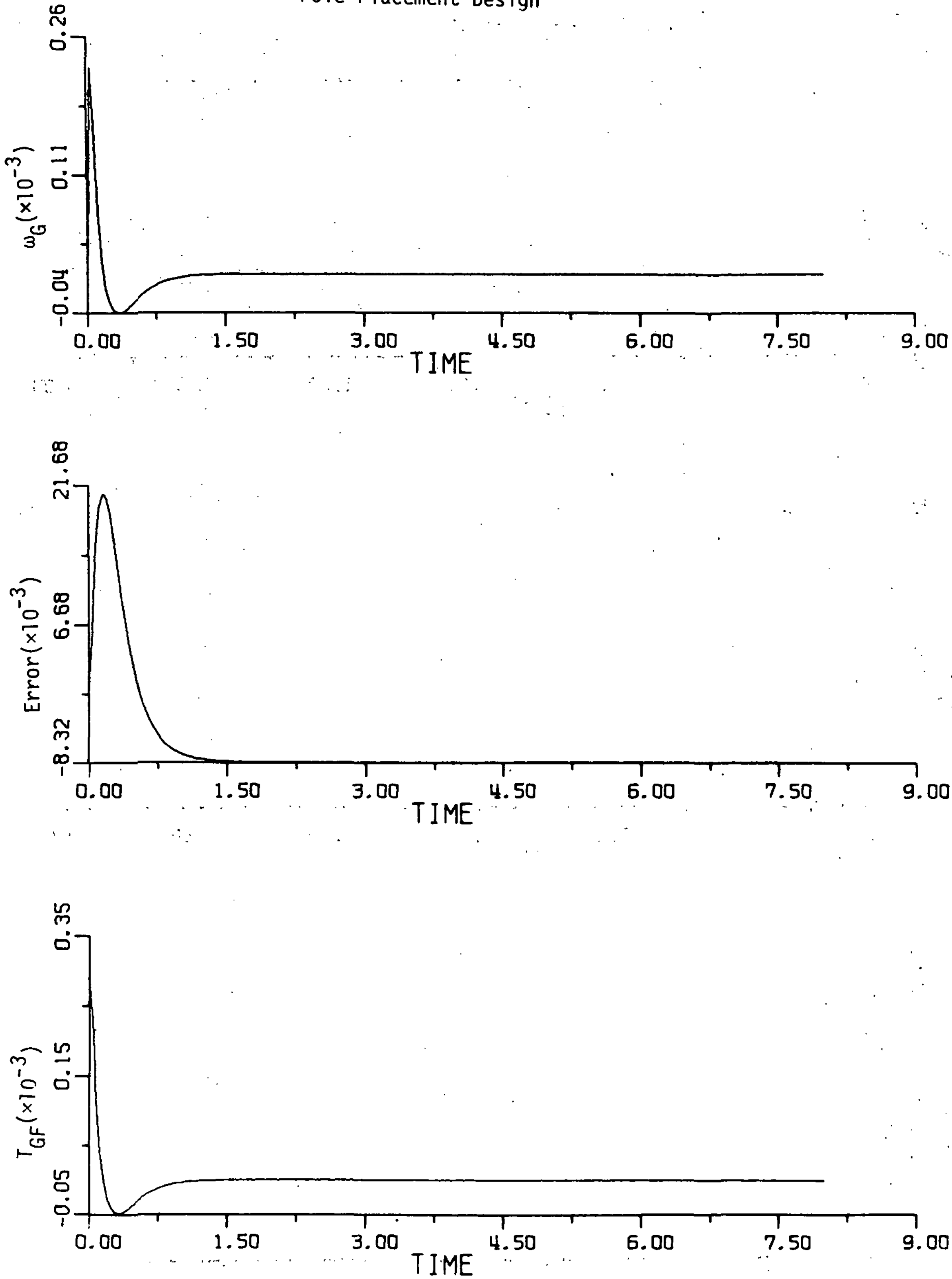


Figure 3-2

System 1, $\gamma = 1.38\times10^5$, No Sampler

State Regulator Design

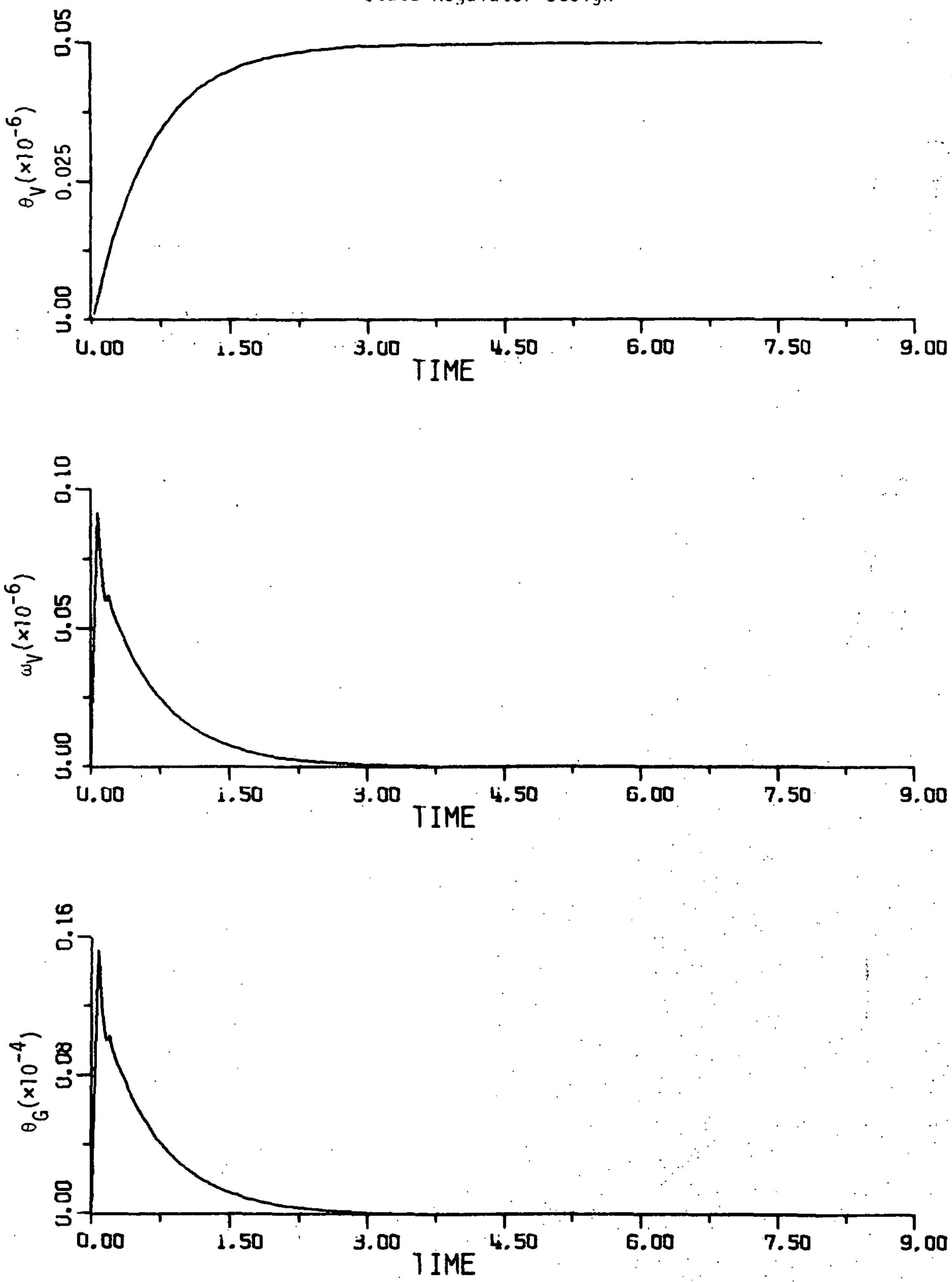


Figure 3-3

System 1, $\gamma = 1.38\times10^5$, No Sampler

State Regulator Design

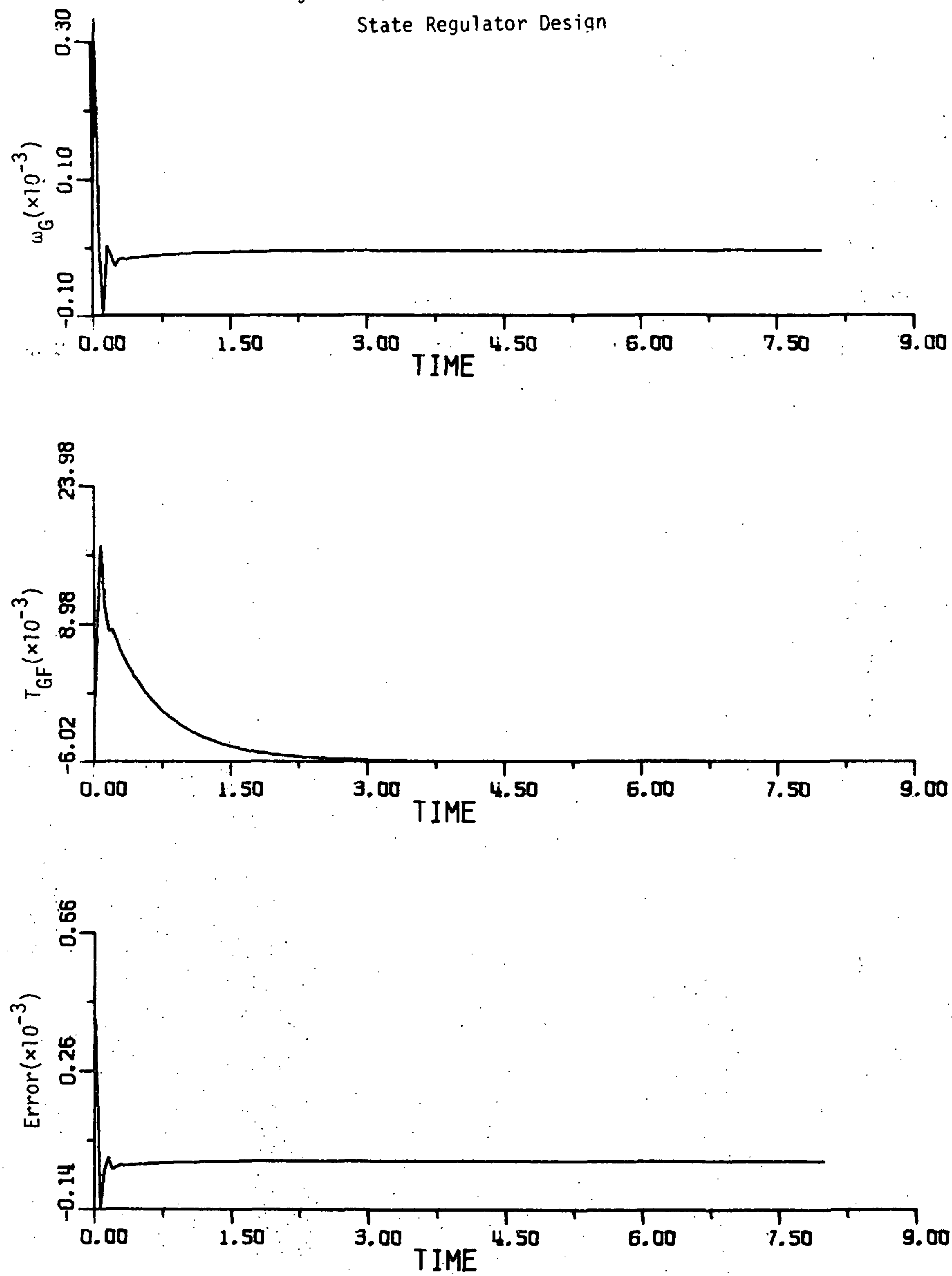


Figure 3-4

System 1, $\gamma = 1.38\times10^5$, No Sampler

State Regulator Design

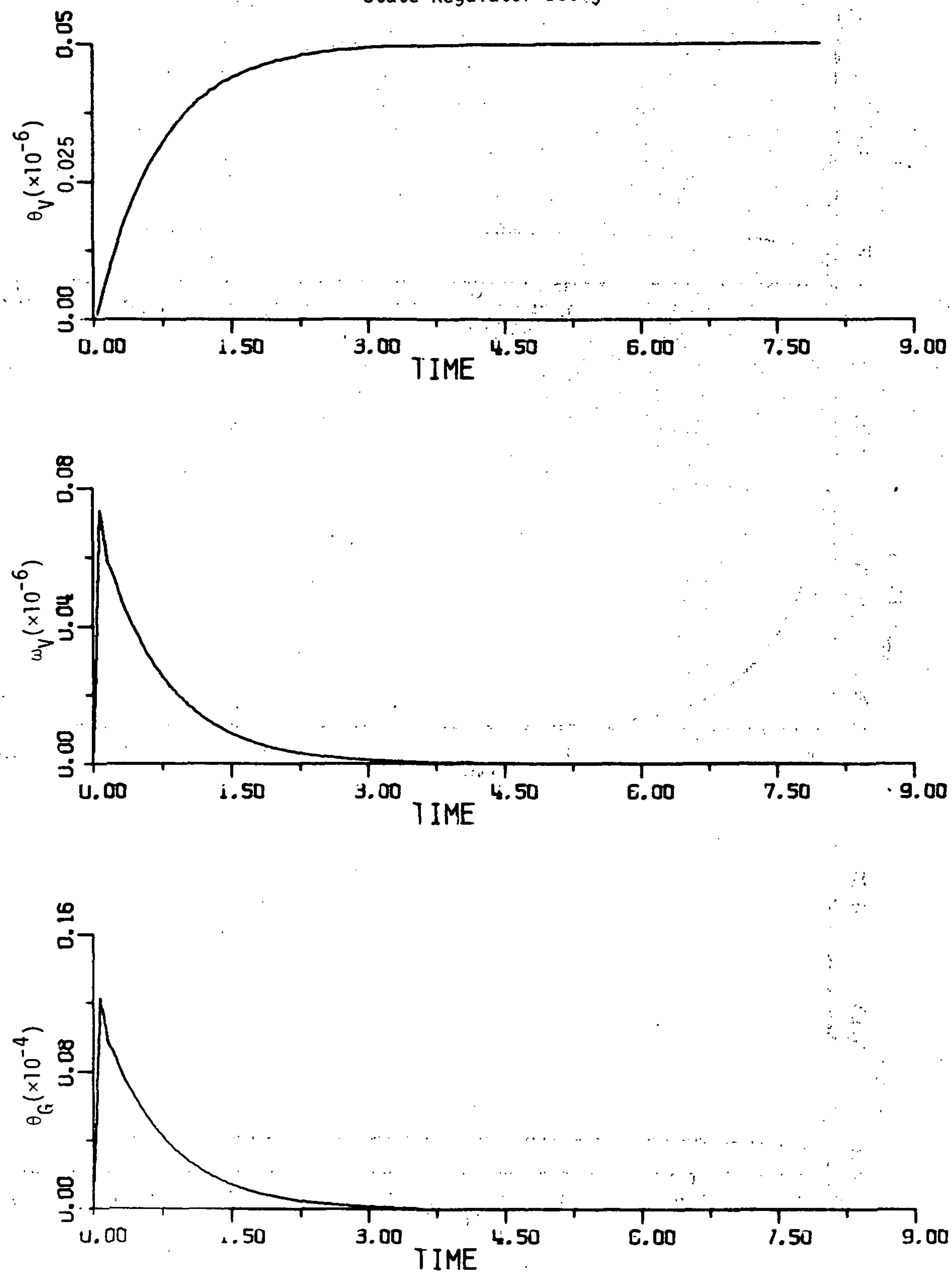


Figure 3-5

System 1, $\gamma = 1.38\times10^{5}$, No Sampler

State Regulator Design

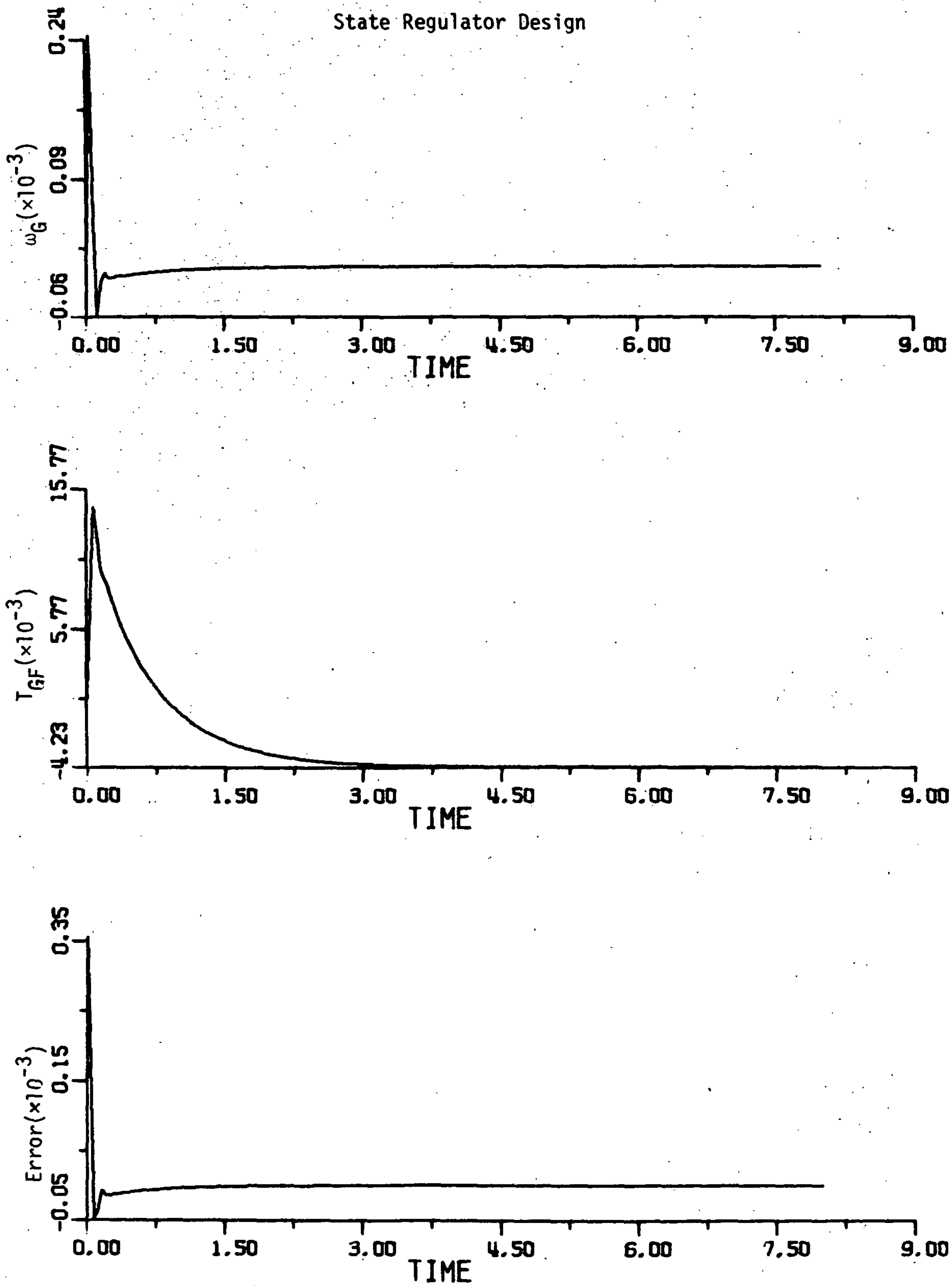


Figure 3-6

4. Digital Redesign of the Large Space Telescope (LST) System

In this chapter, the simplified digital LST system is designed by use of the point-by-point method of digital redesign [6]. Three different continuous-data control laws are considered.

Control Law A

This is the original control law of the LST system, obtained by classical techniques for a damping ratio of 0.707. The gain matrices for this case are

$$G(0) = [5758 \quad 1371 \quad 0 \quad 0]$$

$$E(0) = [5758]$$

This control has been used extensively in analyzing the stability of the LST system with the nonlinear CMG gimbal friction [2].

Control Law B

This control law is obtained by the eigenvalue assignment method [7].

$$G(0) = [5773 \quad 2032 \quad 0.97 \quad 0.04]$$

$$E(0) = [5773]$$

Control Law C

This control law is obtained by the linear regulator design method [7].

$$G(0) = [7071 \quad 5220 \quad 10.6 \quad 0.99]$$

$$E(0) = [7071]$$

The simplified model of the continuous-data LST system is used in its decomposed form [7].

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) \tag{4-1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{H}{J_V} & 0 \\ 0 & 0 & 0 & \frac{1}{J_G} \\ 0 & 0 & -K_I & \frac{-K_p}{J_G} \end{bmatrix} \tag{4-2}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ K_I \end{bmatrix} \tag{4-3}$$

The control is given by (with zero input)

$$u(t) = -G(0)\underline{x}(t)$$

The numerical values used are

$H = 600$

$J_V = 1 \times 10^5$

$J_G = 2.1$

$K_p = 216$

$K_I = 9700$

It should be noted that the system model of Equations (4-1), (4-2) and (4-3) is obtained by neglecting the nonlinearity of the CMG. This is necessary in order to obtain a state variable representation of the system. In reality, however, only the first two states of the system, θ_v and $\dot{\theta}_v$, will be fed back. Thus, the last two gains of control laws B and C are for redesign purposes only; they will not be used in the simulations.

The closed-loop eigenvalues of the LST system, with the three continuous-data control laws are:

Control law A:
$$s_1 = -46.86 + j39.04$$
$$s_2 = -46.86 - j39.04$$
$$s_3 = -4.57 + j4.70$$
$$s_4 = -4.57 - j4.70$$

Control law B:
$$s_1 = -471$$
$$s_2 = -11.1$$
$$s_3 = -3.97 + j3.85$$
$$s_4 = -3.97 - j3.85$$

Control law C:
$$s_1 = -9700$$
$$s_2 = -2.77$$
$$s_3 = -1.37 + j2.32$$
$$s_4 = -1.37 - j2.32$$

The point-by-point method of state matching is used to digitally redesign the LST system for each of the continuous-data control laws.

With each control law, a weighting matrix H is chosen and the redesign is performed for a range of sampling periods T. With the feedback gain matrix $G_W(T)$, the digital system is represented by

$$\underline{x}(k + 1) = \phi(T)\underline{x}(k) + \theta(T)u(k) \tag{4-4}$$

$$u(k) = -G_W(T)\underline{x}(k) \tag{4-5}$$

where

$$\phi(T) = e^{AT} \tag{4-6}$$

$$\theta(T) = \int_0^T e^{A\tau}d\tau B \tag{4-7}$$

The z-plane characteristic roots of the sampled-data system are given by the eigenvalues of $(\phi(T) - \theta(T)G_W(T))$. These roots are calculated for each value of T with some very interesting results.

The results with control law A are shown in Table 4-1 for H = [1 1 0 0], Table 4-2 for H = [1 1 1 1], and Table 4-3 for H = [1 0 0 0]. The results with control law B for H = [1 1 0 0] are shown in Table 4-4, and the results with control law C for H = [1 1 0 0] and H = [1 0 0 0] are shown in Tables 4-5 and 4-6, respectively.

The z-plane characteristic roots of the redesigned systems in Tables 4-1 through 4-6 are shown in Figures 4-1 through 4-6, respectively.

These results show that a large variety of sampled-data systems are available to control the digital LST. It is apparent that the choice of H plays a very dominant role in the resulting gain $G_W(T)$. In fact, with control law A, while one choice of H (Table 4-1 and Figure 4-1) provides a stable sampled-data system for a wide range of sampling periods T, another choice of H (Table 4-2 and Figure 4-2) yields a system which is unstable at higher sampling periods (T greater than 0.5 sec). Still a third choice of H (Table 4-3 and Figure 4-3) yields a system which is stable only at very low sampling periods (T = 0.06 sec or less). With control law C, and H = [1 1 0 0] or H = [1 0 0 0], the redesigned system is unstable for very small sampling periods (T less than 0.018 sec and 0.02 sec, respectively).

In general, it appears that control laws A and B are more effective when digitally redesigned.

Some computer simulations are now presented to show the effects of digital redesign with the various control laws. As before, only the first two feedback gains are utilized, and the simplified LST system is simulated on a digital computer with the CMG nonlinearity included. The parameters of the nonlinearity are $\gamma = 1.38 \times 10^5$, $T_{GFO} = 0.1$. In all the simulations, the input $\chi = 0$ and the initial value of $\theta_V(0) = 1 \times 10^{-8}$, with all other initial states equal to zero.

The following simulations have been performed:

Figure No.	Control Law	H	Sampling Period T (sec)
4-7, 4-8	A	-	Continuous-data System
4-9, 4-10	A	[1 1 0 0]	0.02
4-11, 4-12	A	[1 1 0 0]	0.1
4-13, 4-14	B	-	Continuous-data System
4-15, 4-16	B	[1 1 0 0]	0.02
4-17, 4-18	B	[1 1 0 0]	0.1

Figure No.	Control Law	H	Sampling Period T (sec)
4-19, 4-20	C	-	Continuous-data System
4-21, 2-22	C	[1 1 0 0]	0.02
4-23, 4-24	C	[1 1 0 0]	0.1

In each simulation, the following quantities are plotted:

θ_V = vehicle position (radians)

ω_V = vehicle velocity (radians/sec)

θ_G = Gimbal position (radians)

ω_G = Gimbal velocity (radians/sec)

T_{GF} = Nonlinearity Torque (ft-lb)

Error = $\chi - K_0\theta_V - K_1\omega_V$

= Error input to CMG controller.

The simulation results show that adequate digital control schemes can be obtained for wide ranges of sampling periods by appropriate redesign of the feedback gains. Again, it appears that the method of redesign is more effective in the case of control laws A and B.

Table 4-1

Digital Redesign of the LST System, Continuous-Data Control Law A

$G(0) = [5758 \quad 1371 \quad 0 \quad 0]$, $E(0) = [5758]$

$H = [1 \quad 1 \quad 0 \quad 0]$

Sampling Period T (sec)	G_W: G_{W11}	G_{W12}	G_{W13}	G_{W14}	E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
0.01	5757	1386	0.02	1.8×10^{-5}	5757	$0.58 \pm j0.23$, $0.95 \pm j0.045$
0.02	5745	1400	0.04	6.6×10^{-5}	5745	$0.29 \pm j0.25$, $0.90 \pm j0.088$
0.03	5714	1411	0.07	1.35×10^{-4}	5714	$0.12 \pm j0.19$, $0.86 \pm j0.13$
0.04	5658	1417	0.09	2.16×10^{-4}	5658	$0.81 \pm j0.16$, $0.044 \pm j0.11$
0.05	5573	1419	0.1	3.0×10^{-4}	5573	$0.016 \pm j0.044$, $0.76 \pm j0.19$
0.06	5458	1414	0.12	3.8×10^{-4}	5458	-0.026, 0.051, $0.71 \pm j0.22$
0.07	5315	1403	0.13	4.5×10^{-4}	5315	-0.027, 0.061, $0.7 \pm j0.25$
0.08	5149	1385	0.14	5.1×10^{-4}	5149	-0.018, 0.062, $0.6 \pm j0.27$
0.09	4964	1364	0.15	5.6×10^{-4}	4964	-0.01, 0.058, $0.58 \pm j0.29$
0.1	4769	1338	0.15	6.0×10^{-4}	4769	-0.005, 0.052, $0.54 \pm j0.30$
0.25	2211	899	0.12	5.3×10^{-4}	2211	-0.004, 0.0, $0.16 \pm j0.34$
0.5	509	467	0.06	2.9×10^{-4}	509	-0.003, 0.0, $0.14 \pm j0.048$
1.0	103.8	228	0.03	1.4×10^{-4}	103.8	-0.003, 0.0, 0.43, -0.096
2.0	42	127	0.017	7.8×10^{-5}	42	-1.8×10^{-5}, 0.0, 0.064, -0.086

Table 4-2

Digital Redesign of the LST System, Continuous-Data Control Law A

$G(0) = [5758 \quad 1371 \quad 0 \quad 0]$, $E(0) = [5758]$

$H = [1 \quad 1 \quad 1 \quad 1]$

Sampling Period T (sec)	G_W				E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
	G_{W11}	G_{W12}	G_{W13}	G_{W14}		
0.02	5658	1437	0.12	2.8×10^{-4}	5658	$0.26 \pm j0.28$, $0.91 \pm j0.083$
0.04	4333	1397	0.32	1.4×10^{-3}	4333	$0.86 \pm j0.12$, $-0.12 \pm j0.122$
0.06	-8678	-128	0.72	5.2×10^{-3}	-8678	1.37, -0.82, 0.71, -0.033
0.08	2.5×10^5	3.7×10^4	1.39	-0.017	2.5×10^5	-15.1, 0.56, -0.38, -0.024
0.1	4.6×10^5	7.6×10^4	10.7	0.03	4.6×10^5	-53.2, 0.47, $0.025 \pm j0.012$

Table 4-3

Digital Redesign of the LST System, Continuous-Data Control Law A

$G(0) = [5758 \quad 1371 \quad 0 \quad 0]$, $E(0) = [5758]$

$H = [1 \quad 0 \quad 0 \quad 0]$

Sampling Period T (sec)	G_W				E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
	G_{W11}	G_{W12}	G_{W13}	G_{W14}		
0.02	5751	1394	0.03	4.4×10^{-5}	5751	$0.29 \pm j0.24$, $0.91 \pm j0.089$
0.04	5704	1411	0.07	1.4×10^{-4}	5704	$0.81 \pm j0.17$, $0.058 \pm j0.098$
0.06	5597	1416	0.096	2.6×10^{-4}	5597	$0.7 \pm j0.21$, 0.1, -0.0325
0.08	5429	1407	0.18	3.7×10^{-4}	5429	$0.6 \pm j0.29$, 0.13, -0.017
0.10	5213	1387	0.13	4.6×10^{-4}	5213	$0.5 \pm j0.33$, 0.13, -0.0036
0.25	3338	1101	0.13	5.8×10^{-4}	3338	$-0.11 \pm j0.32$, 0.06, 0.0
0.50	1433	701	0.09	4.1×10^{-4}	1433	-1.01, -0.074, 0.006, 0.0
1.0	353	350	0.046	2.1×10^{-4}	353	-1.1, -0.0055, -0.0024, 0.0
2.0	85.2	17	0.02	1.0×10^{-4}	85.2	-1.04, -2.6×10^{-4}, 2.4×10^{-5}, 0.0

Table 4-4

Digital Redesign of the LST System, Continuous-Data Control Law B

$G(0) = [5773 \quad 2032 \quad 0.97 \quad 0.04]$, $E(0) = [5773]$

$H = [1 \quad 1 \quad 0 \quad 0]$

Sampling Period T (sec)	G_W				E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
	G_{W11}	G_{W12}	G_{W13}	G_{W14}		
0.02	2231	799	-0.2	0.0086	2231	$-0.65, 0.81, 0.92 \pm j0.073$
0.04	1822	666	-0.33	0.0037	1822	$-0.31, 0.69, 0.84 \pm j0.14$
0.06	1780	664	-0.31	0.002	1780	$-0.08, 0.64, 0.76 \pm j0.22$
0.08	1823	693	-0.26	0.0012	1823	$-0.0038, 0.61, 0.69 \pm j0.29$
0.10	1872	730	-0.2	9×10^{-4}	1872	$0.007, 0.57, 0.63 \pm j0.35$
0.14	1899	777	-0.098	6.3×10^{-4}	1898	$8 \times 10^{-4}, 0.47, 0.5 \pm j0.41$
0.20	1771	784	4×10^{-4}	4.9×10^{-4}	1771	$-5 \times 10^{-5}, 0.34, 0.33 \pm j0.44$

Table 4-5

Digital Redesign of the LST System, Continuous-Data Control Law C

$G(0) = [7071 \quad 5220 \quad 10.6 \quad 0.99]$, $E(0) = [7071]$

$H = [1 \quad 1 \quad 0 \quad 0]$

Sampling Period T (sec)	G_W				E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
	G_{W11}	G_{W12}	G_{W13}	G_{W14}		
0.002	1035	769	0.71	0.14	1035	-1.58, 0.99, 0.991 ± j0.005
0.006	407	301	-0.33	0.05	407	-1.51, 0.98, 0.99 ± j0.014
0.01	273	202	-0.55	0.027	273	-1.32, 0.97, 0.986 ± j0.023
0.02	176	131	-0.7	0.012	176	-0.88, 0.95, 0.97 ± j0.045
0.04	142	107	-0.76	0.005	142	-0.33, 0.9, 0.95 ± j0.09
0.06	146	111	-0.75	0.003	146	-0.08, 0.87, 0.93 ± j0.14
0.08	161	123	-0.71	0.002	161	-0.004, 0.84, 0.91 ± j0.18
0.10	178	138	-0.68	0.001	178	0.007, 0.82, 0.9 ± j0.22
0.14	213	168	-0.6	8.8×10^{-4}	213	8×10^{-4}, 0.76, 0.86 ± j0.29
0.20	254	206	-0.5	6×10^{-4}	254	-5×10^{-5}, 0.69, 0.78 ± j0.39

Table 4-6

Digital Redesign of the LST System, Continuous-Data Control Law C

$$G(0) = [7071 \quad 5220 \quad 10.6 \quad 0.99],\ E(0) = [7071]$$
$$H = [1 \quad 0 \quad 0 \quad 0]$$

Sampling Period T (sec)	G_W				E_W	z-Plane Characteristic Roots of the Sampled-Data System z_1, z_2, z_3, z_4
	G_{W11}	G_{W12}	G_{W13}	G_{W14}		
0.004	730	539	0.2	0.09	730	-2.3, 0.99, 0.99 ± j0.009
0.012	298	221	-0.5	0.03	298	-1.75, 0.97, 0.98 ± j0.027
0.02	211	157	-0.65	0.02	211	-1.24, 0.95, 0.97 ± j0.045
0.04	155	116	-0.74	0.008	155	-0.43, 0.9, 0.95 ± j0.09
0.06	146	110	-0.75	0.005	146	-0.09, 0.87, 0.93 ± j0.14
0.08	151	114	-0.74	0.003	151	-0.001, 0.85, 0.92 ± j0.18
0.10	159	122	-0.72	0.003	159	0.009, 0.83, 0.91 ± j0.22

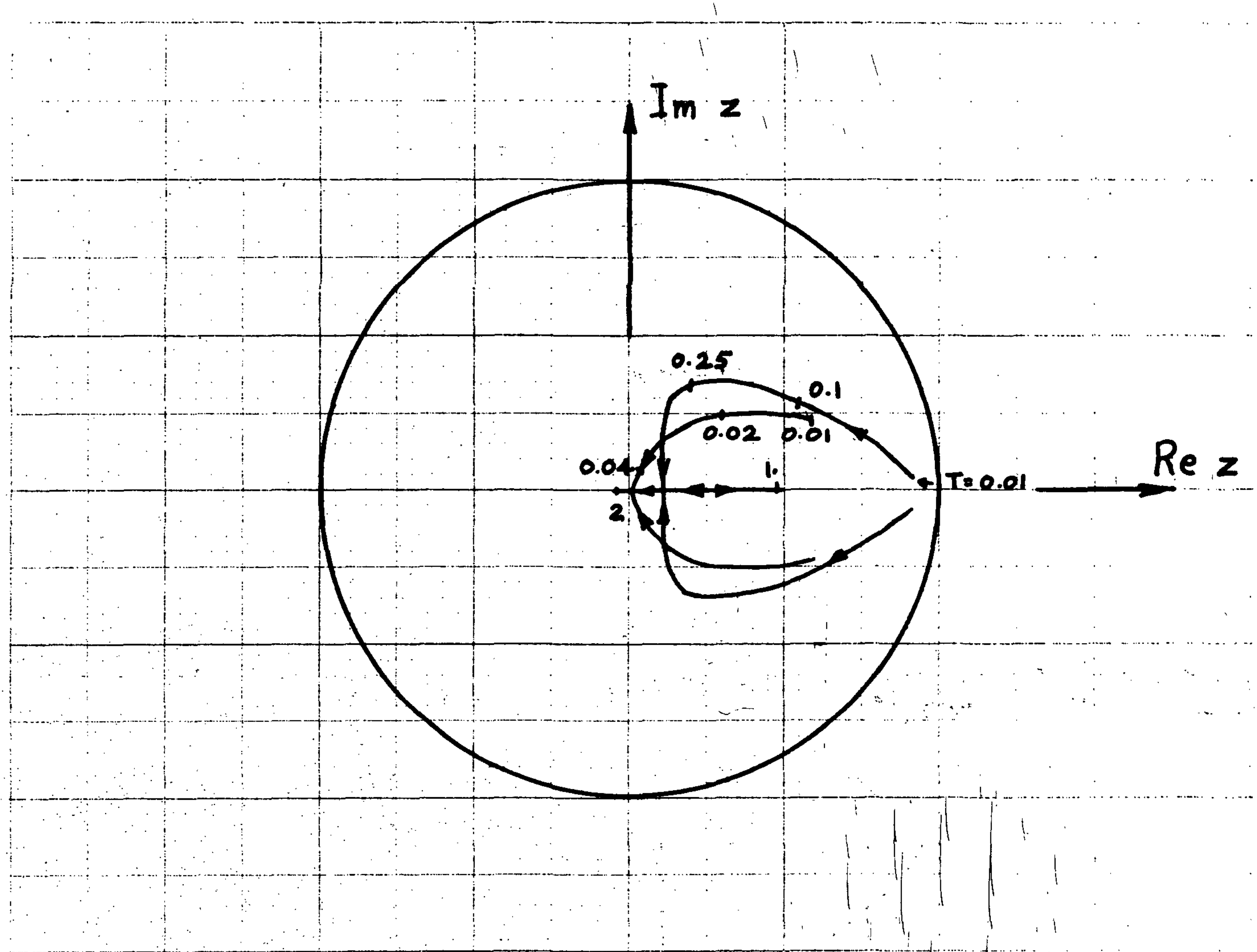


Figure 4-1. z-plane characteristic roots of the simplified LST system. Control Law A, H = [1 1 0 0]

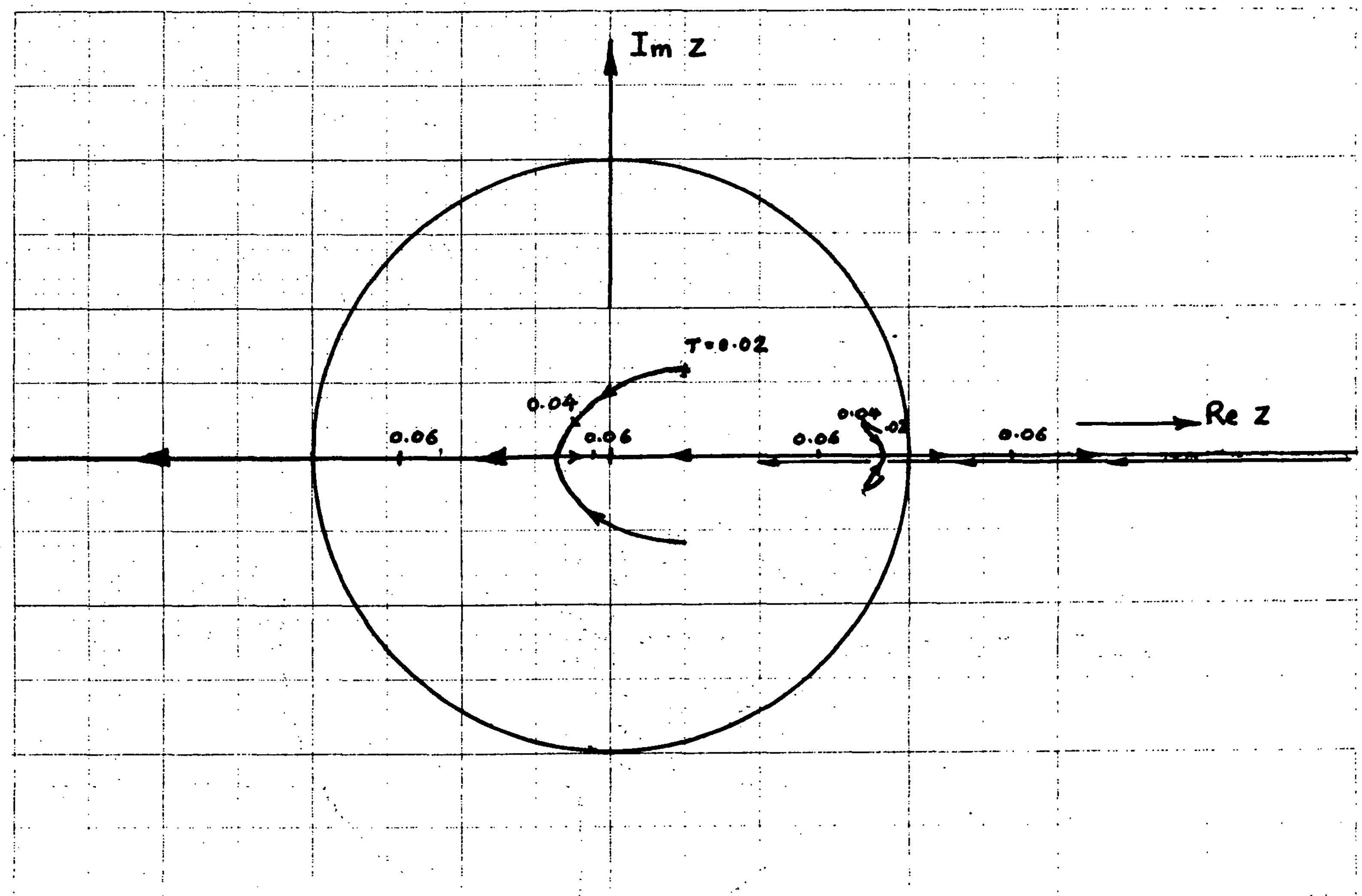


Figure 4-2. z-plane characteristic roots of the simplified LST system.
Control Law A, H = [1 1 1 1]

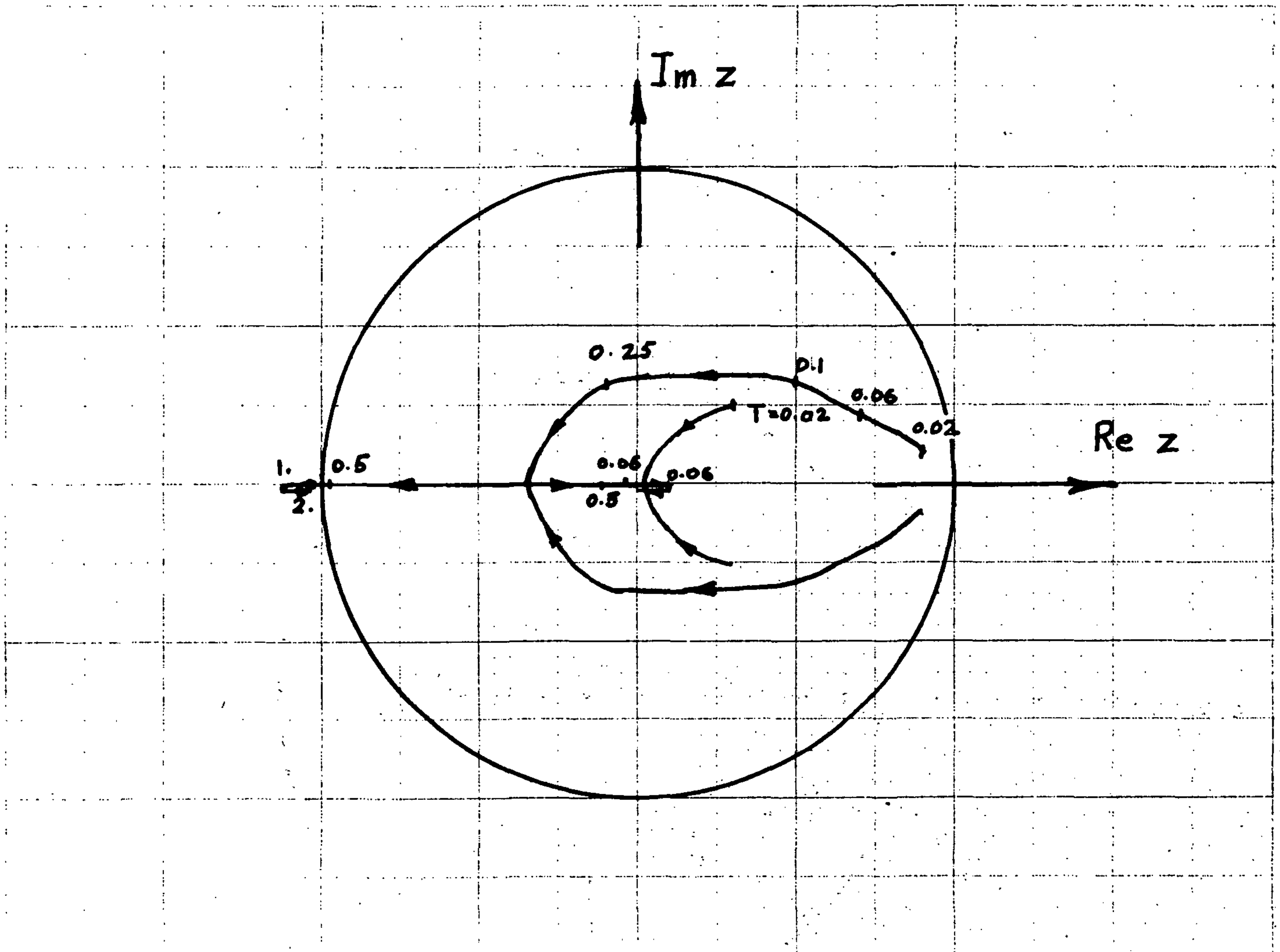


Figure 4-3. z-plane characteristic roots of the simplified LST system. Control Law A, H = [1 0 0 0]

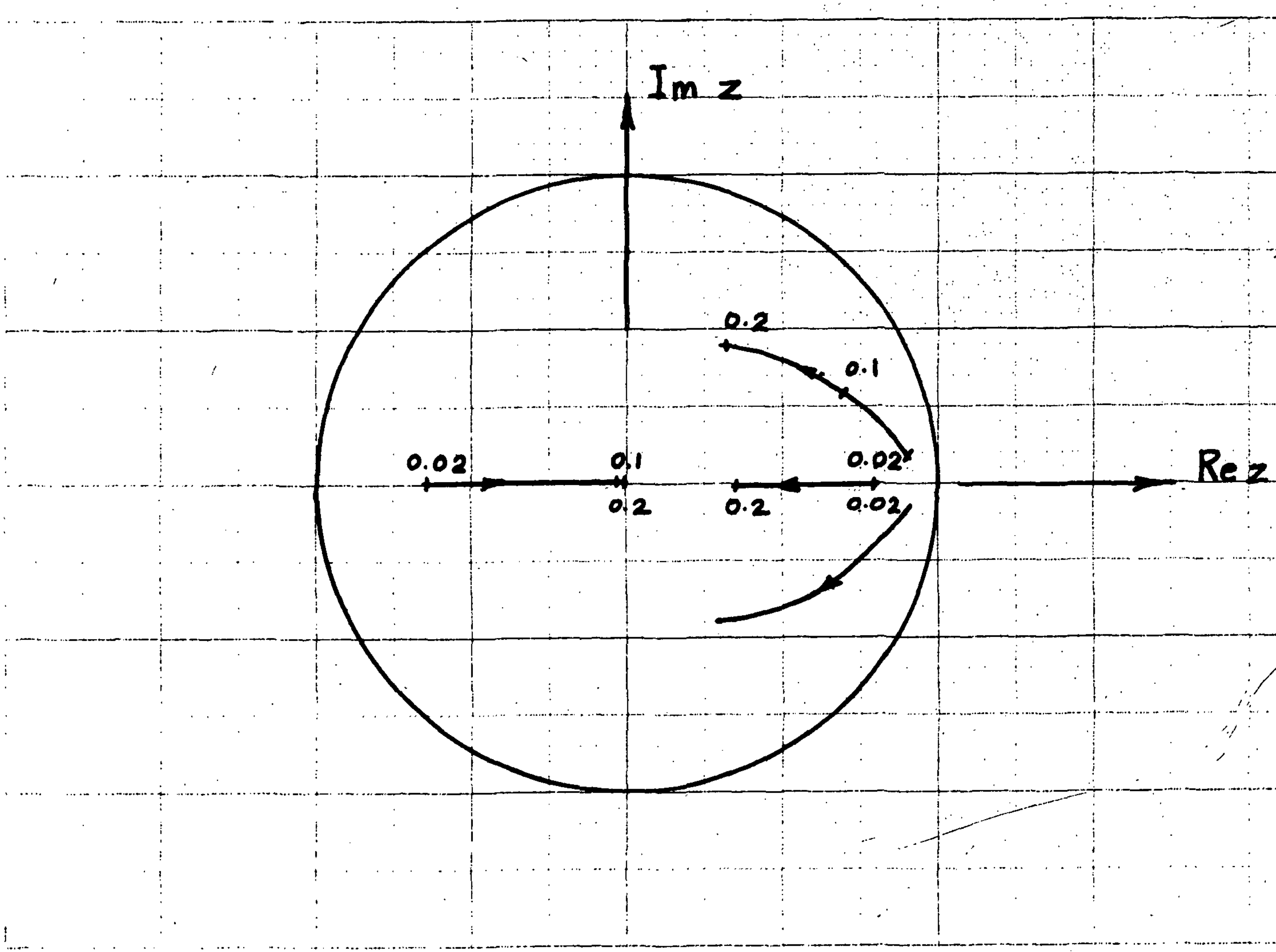


Figure 4-4. z-plane characteristic roots of the simplified LST system. Control Law B, H = [1 1 0 0]

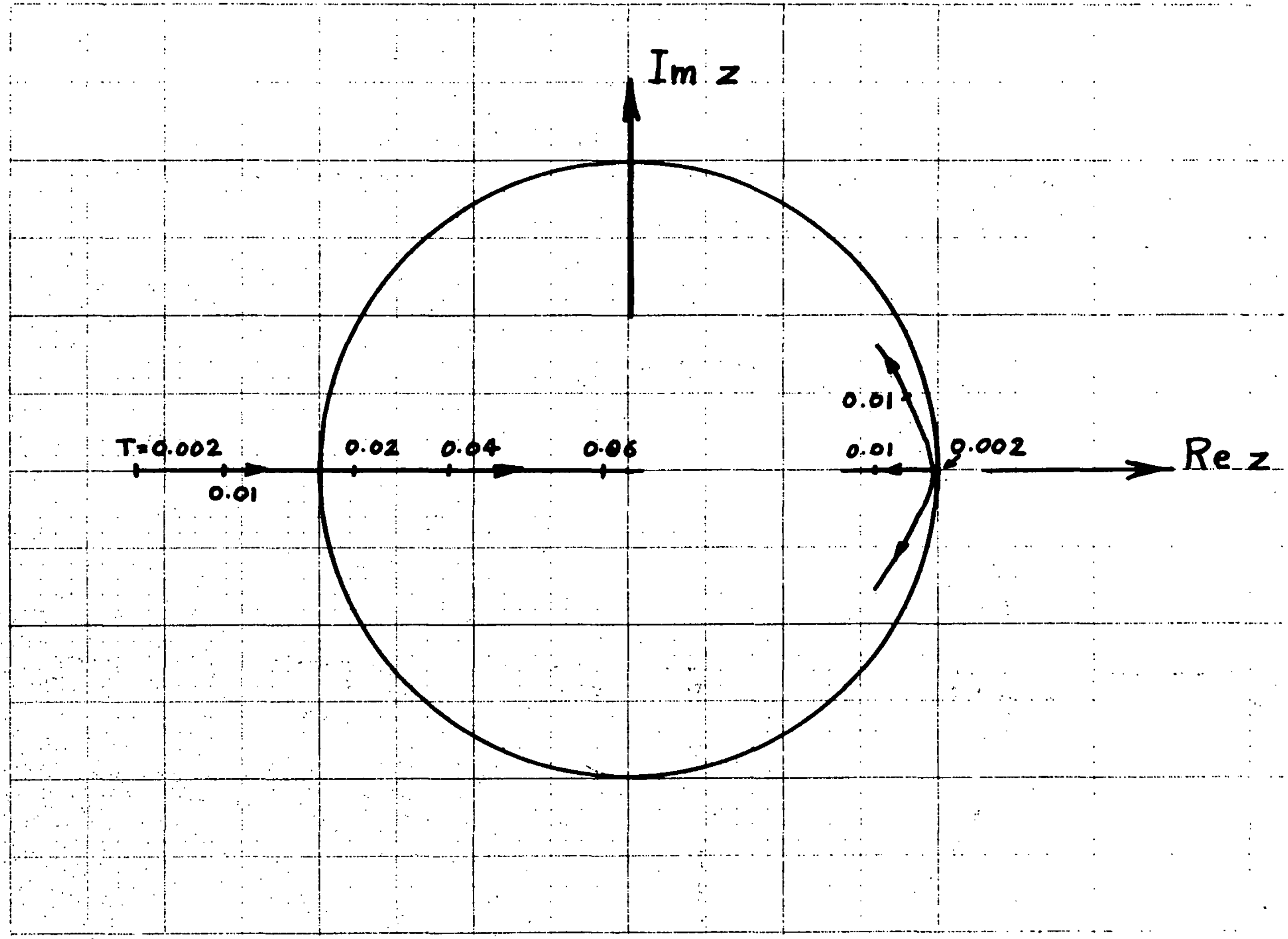


Figure 4-5. z-plane characteristic roots of the simplified LST system. Control Law C, H = [1 1 0 0]

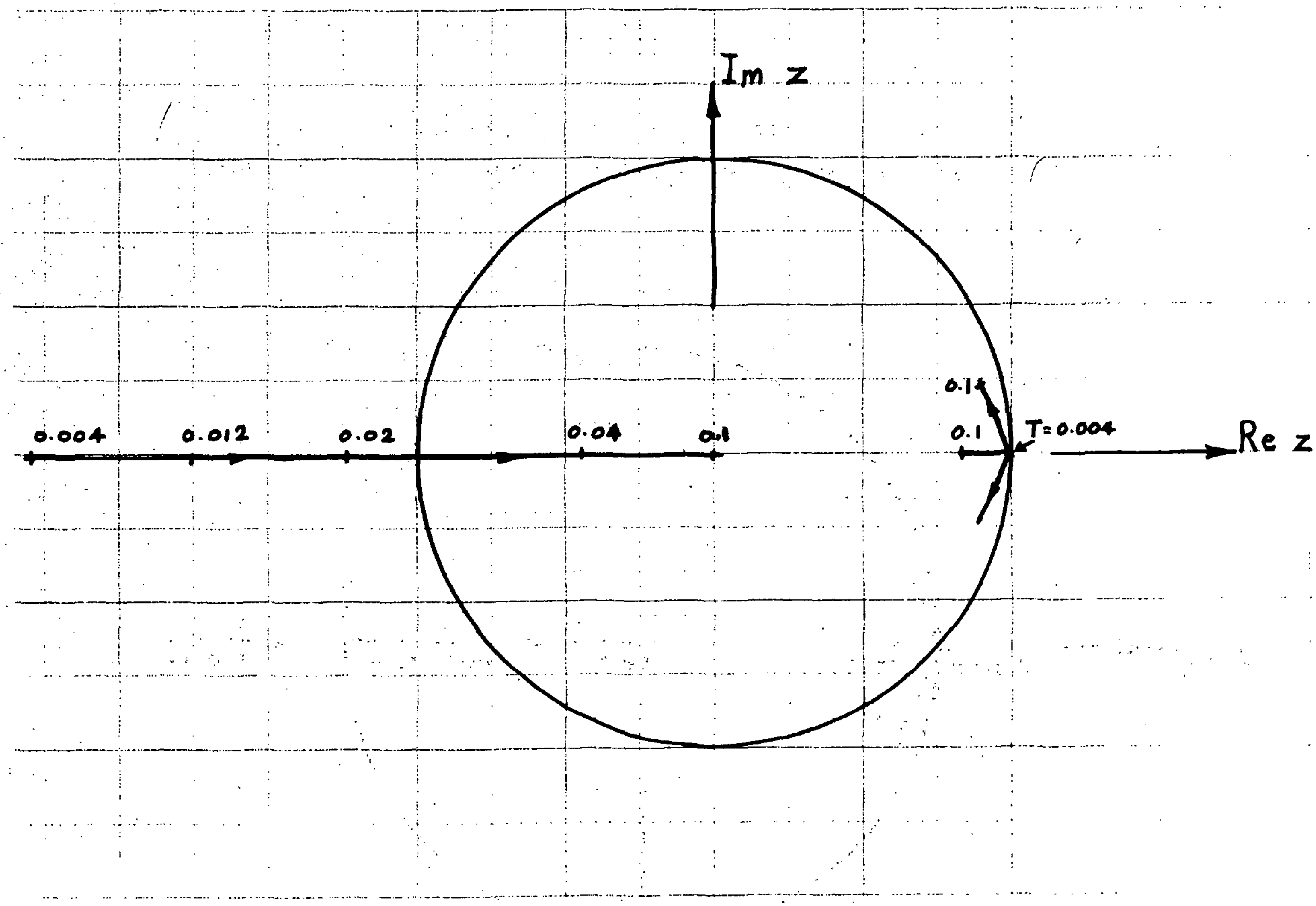


Figure 4-6. z-plane characteristic roots of the simplified LST system.
Control Law C, H = [1 0 0 0]

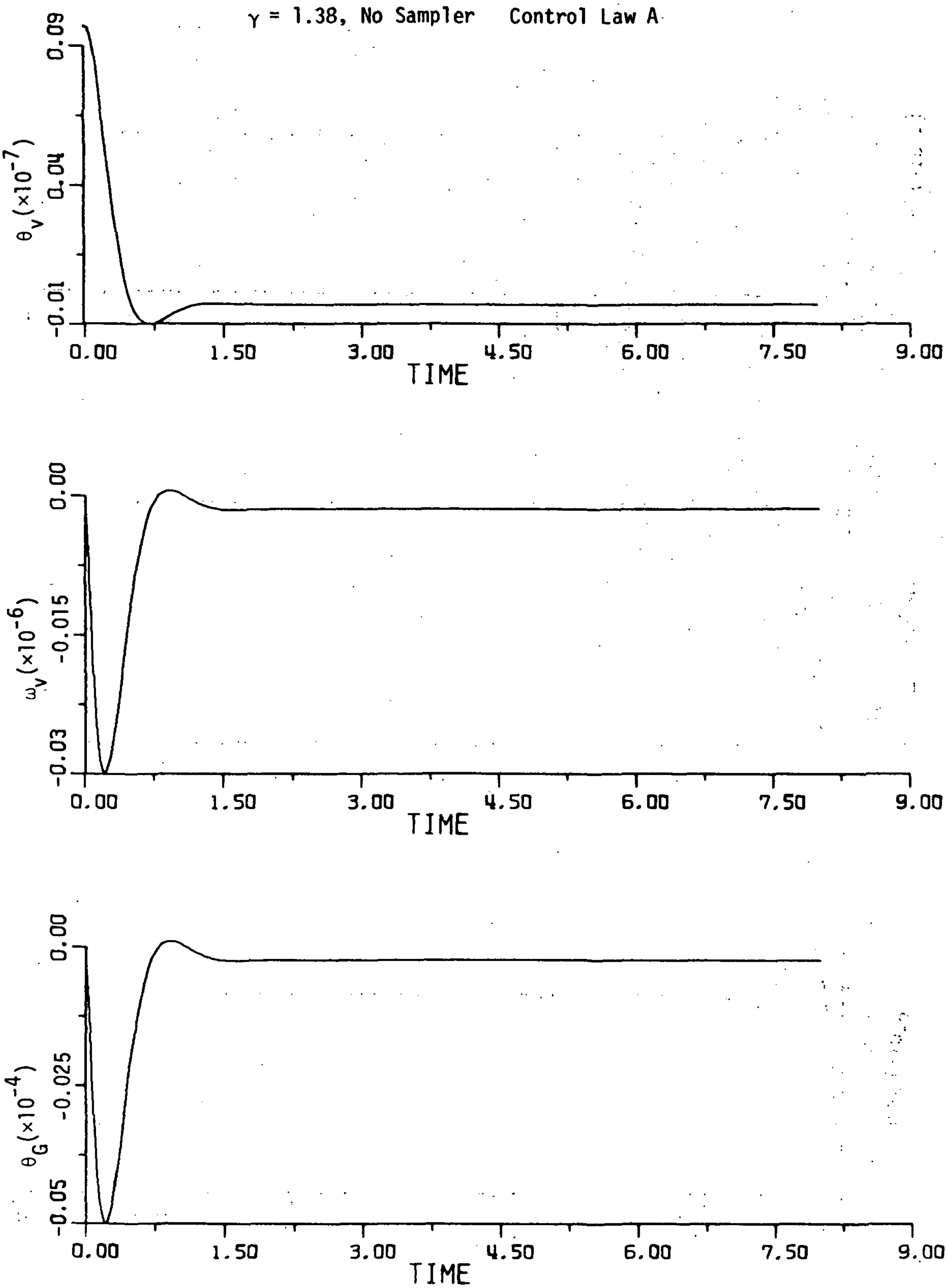

γ = 1.38, No Sampler   Control Law A
θv(×10^-7)
0.09
0.04
-0.01
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
ωv(×10^-6)
0.00
-0.015
-0.03
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
θG(×10^-4)
0.00
-0.025
-0.05
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-7

γ = 1.38, No Sampler   Control Law A

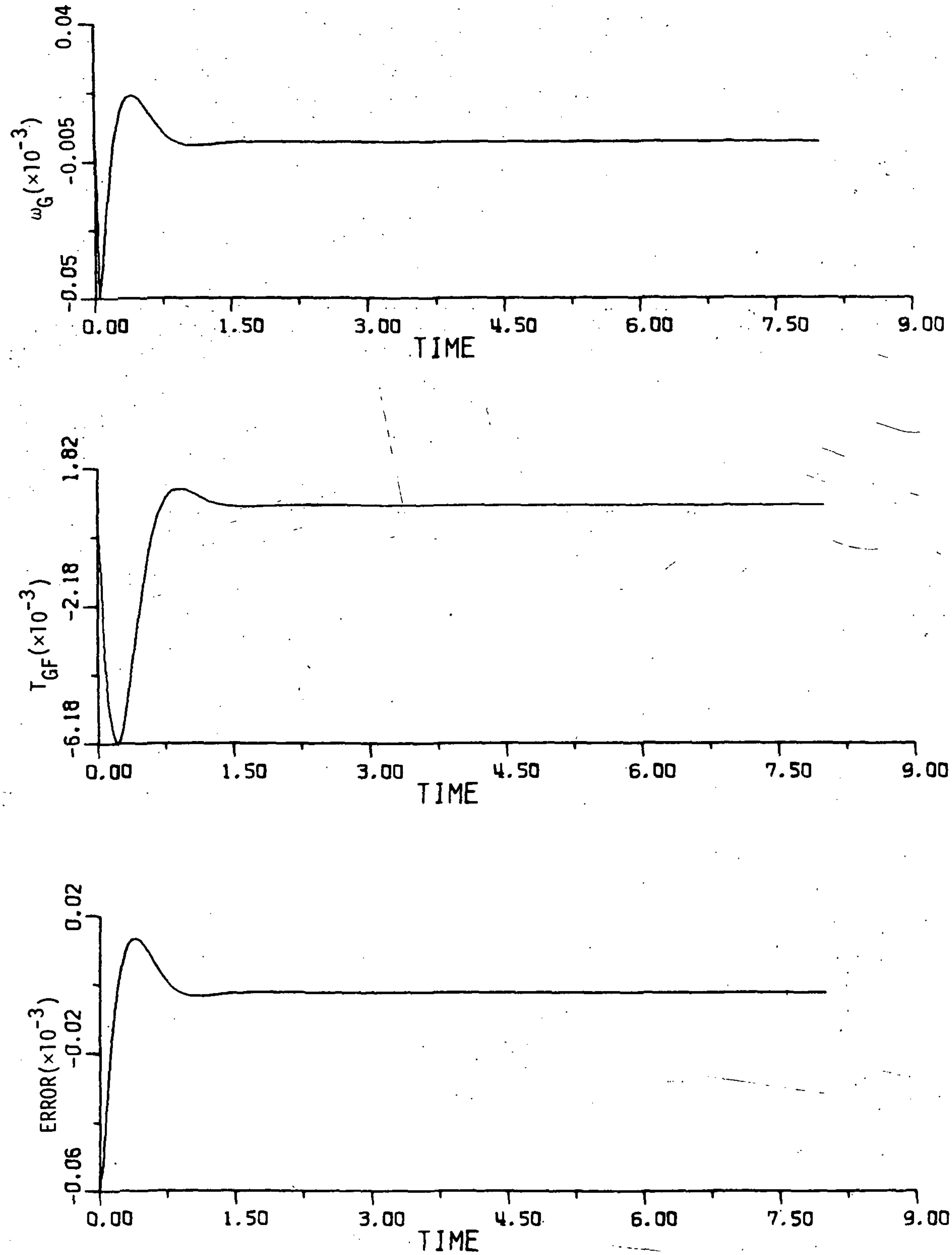


Figure 4-8

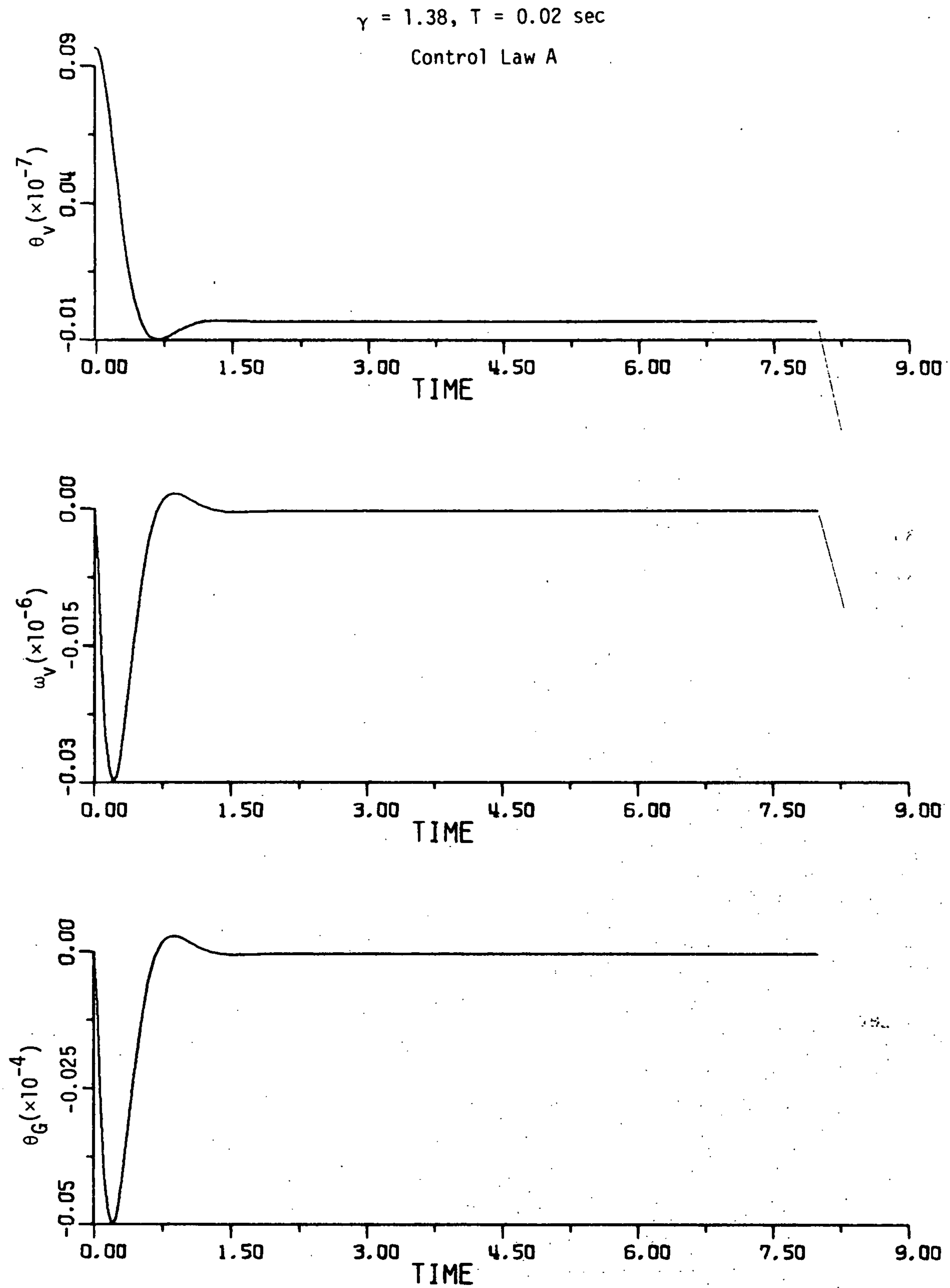

γ = 1.38, T = 0.02 sec
Control Law A
0.09
0.04
-0.01
θv(×10^-7)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.00
-0.015
-0.03
ωv(×10^-6)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.00
-0.025
-0.05
θG(×10^-4)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-9

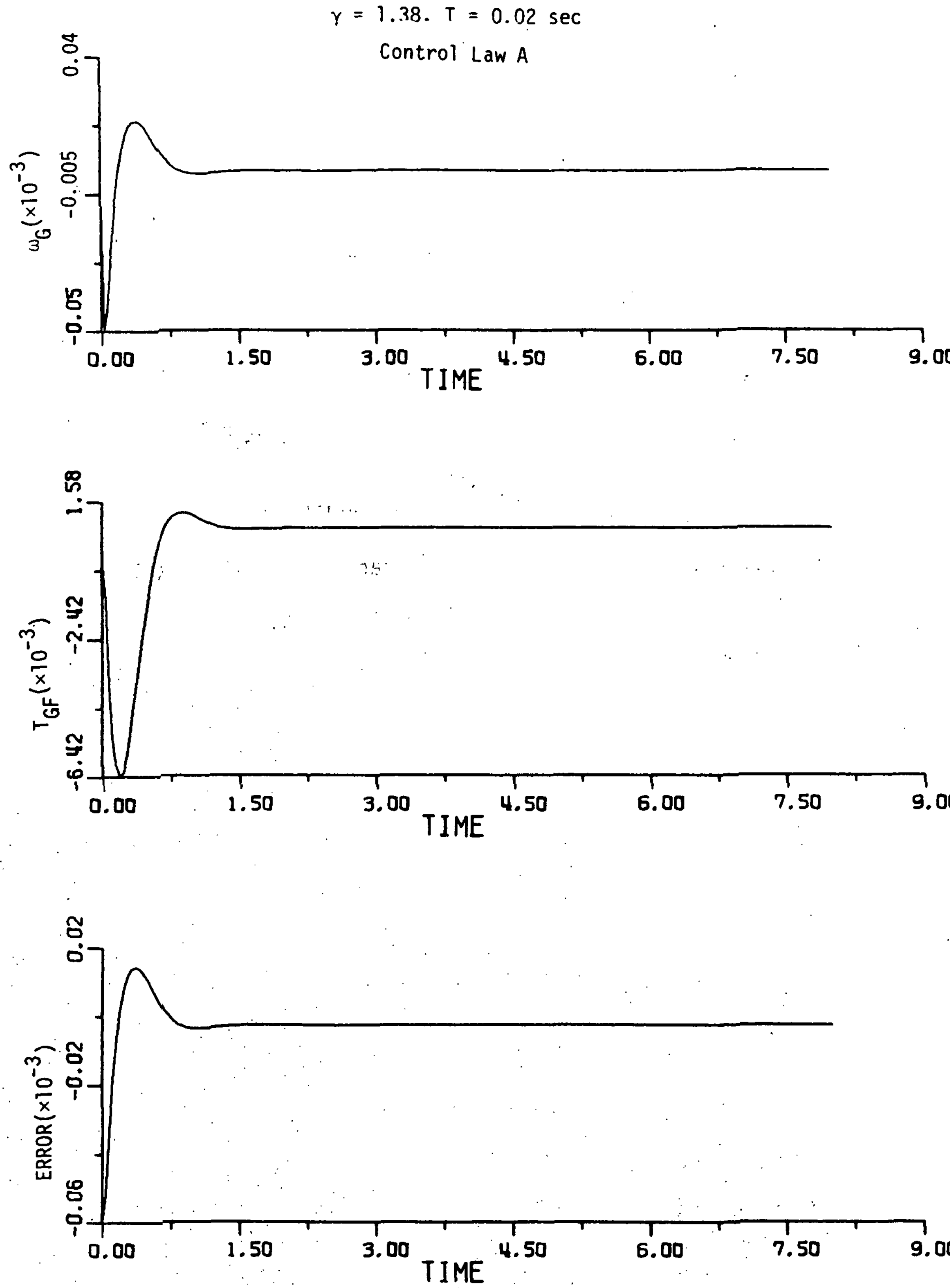

γ = 1.38. T = 0.02 sec
Control Law A
0.04
-0.005
-0.05
ω_G(×10^-3)
1.58
-2.42
-6.42
T_GF(×10^-3)
0.02
-0.02
-0.06
ERROR(×10^-3)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-10

γ = 1.38, T = 0.1 sec

Control Law A

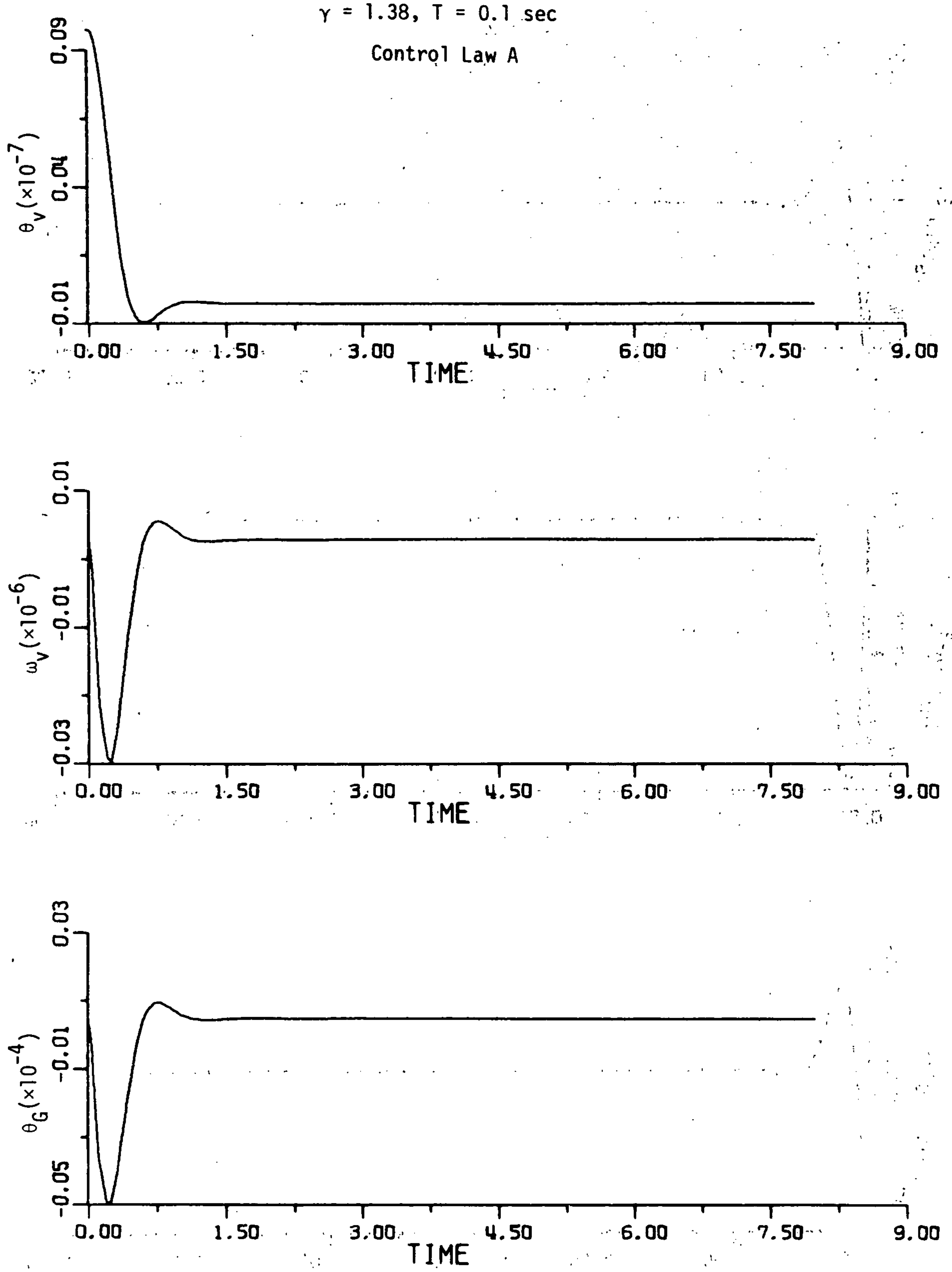


Figure 4-11

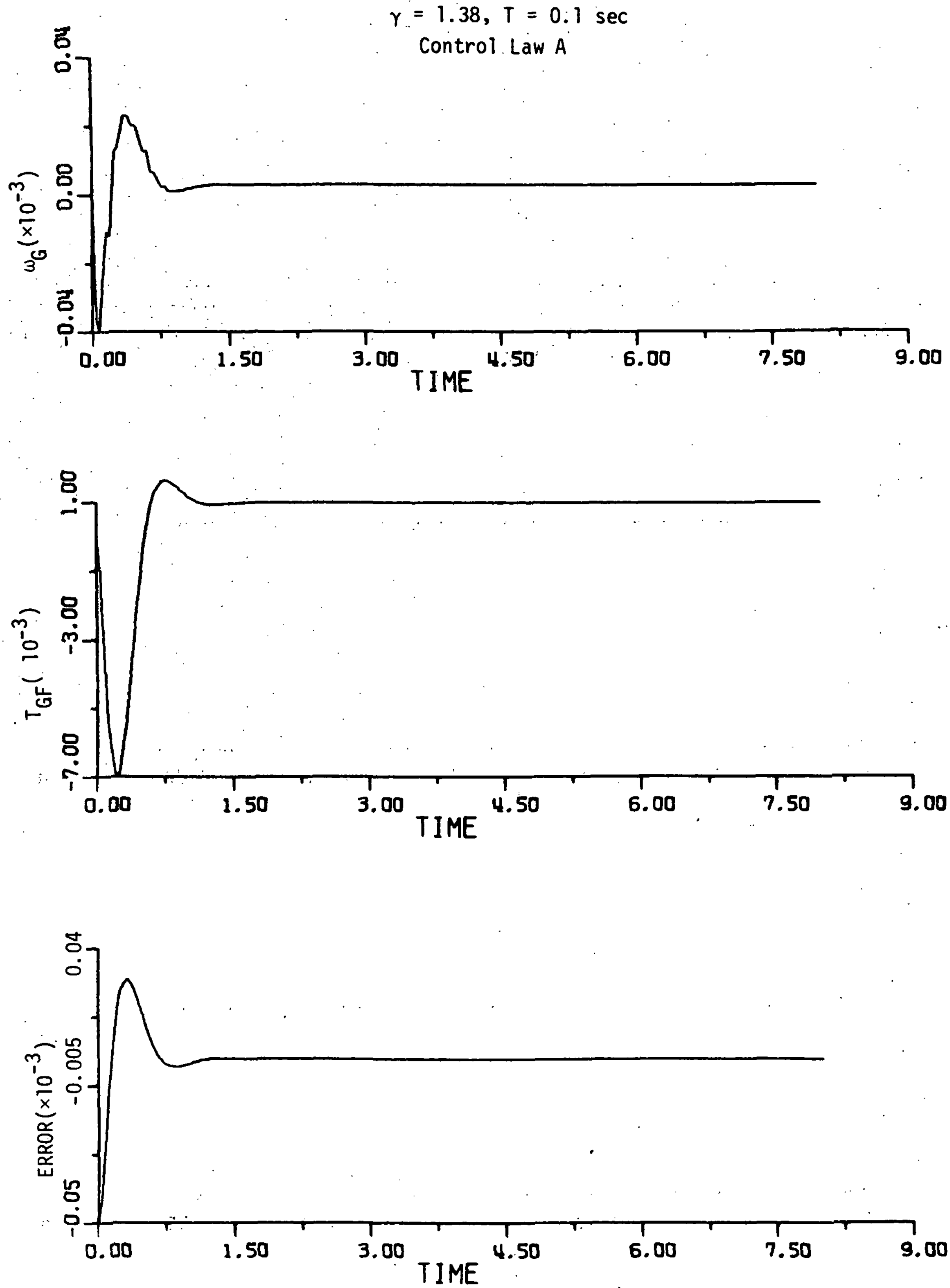

γ = 1.38, T = 0.1 sec
Control Law A
$\omega_G(\times 10^{-3})$
0.04
0.00
-0.04
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$T_{GF}(10^{-3})$
1.00
-3.00
-7.00
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$ERROR(\times 10^{-3})$
0.04
-0.005
-0.05
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-12

γ = 1.38, No Sampler

Control Law B

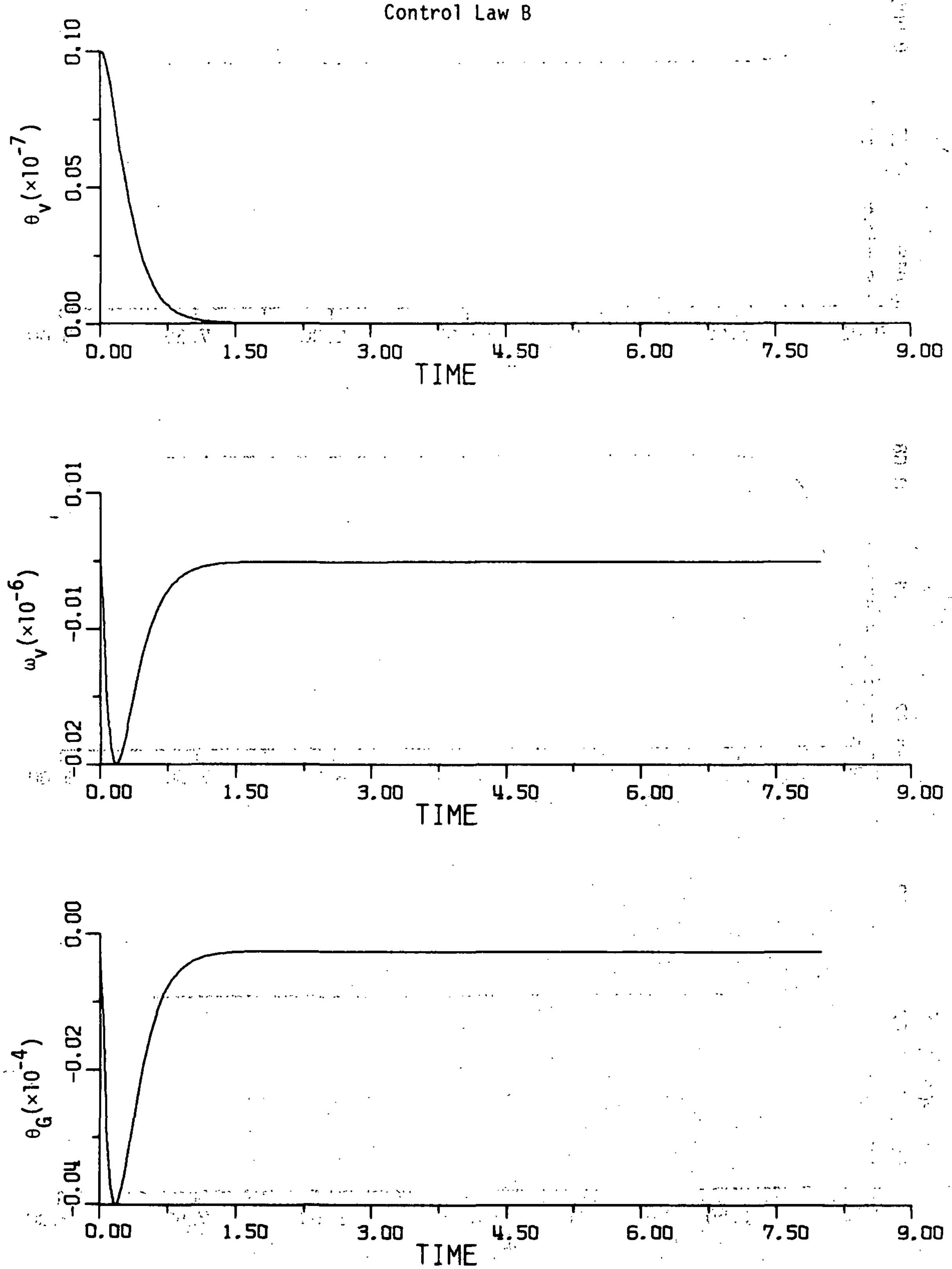


Figure 4-13

γ = 1.38, No Sampler

Control Law B

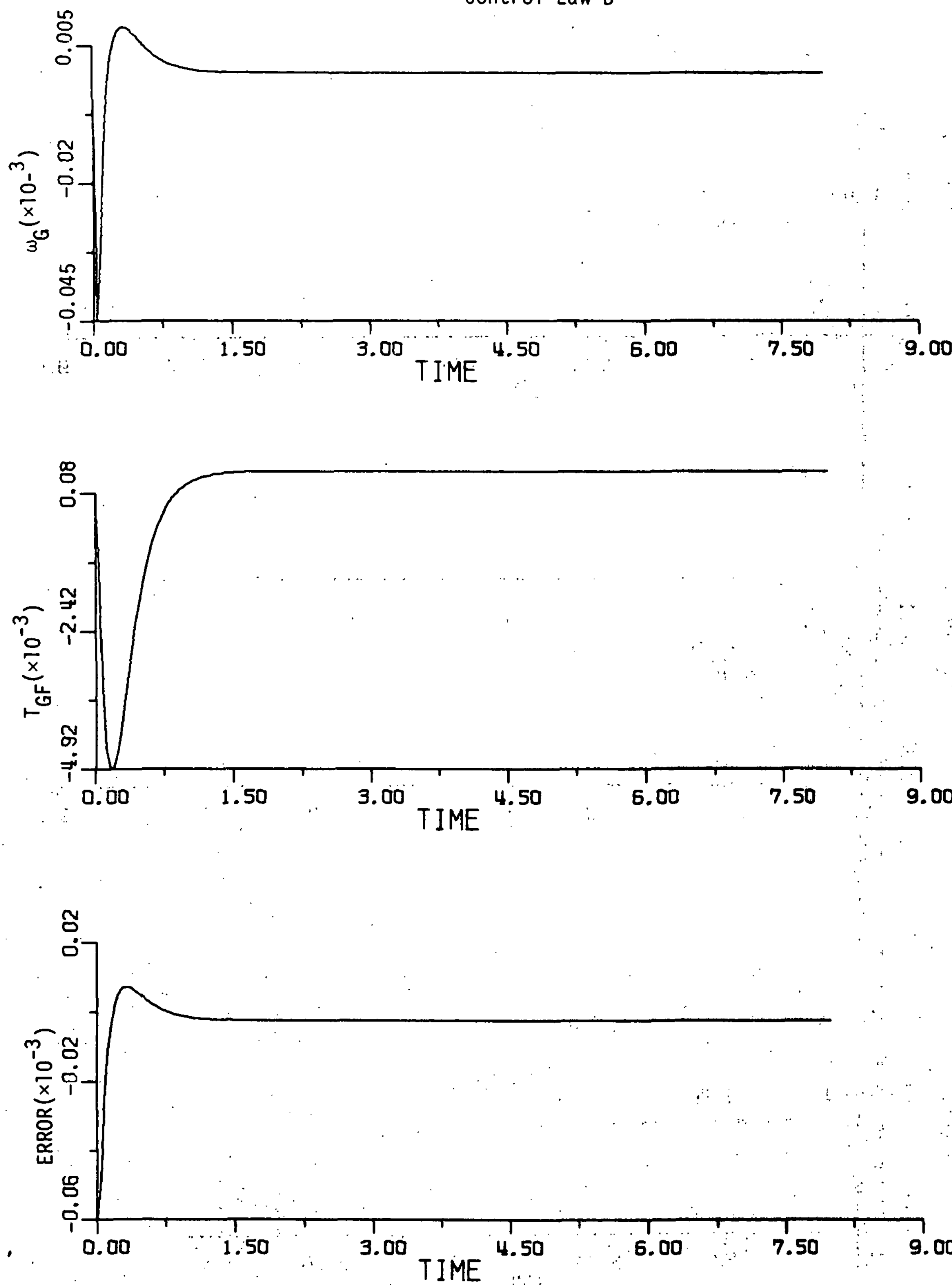


Figure 4-14

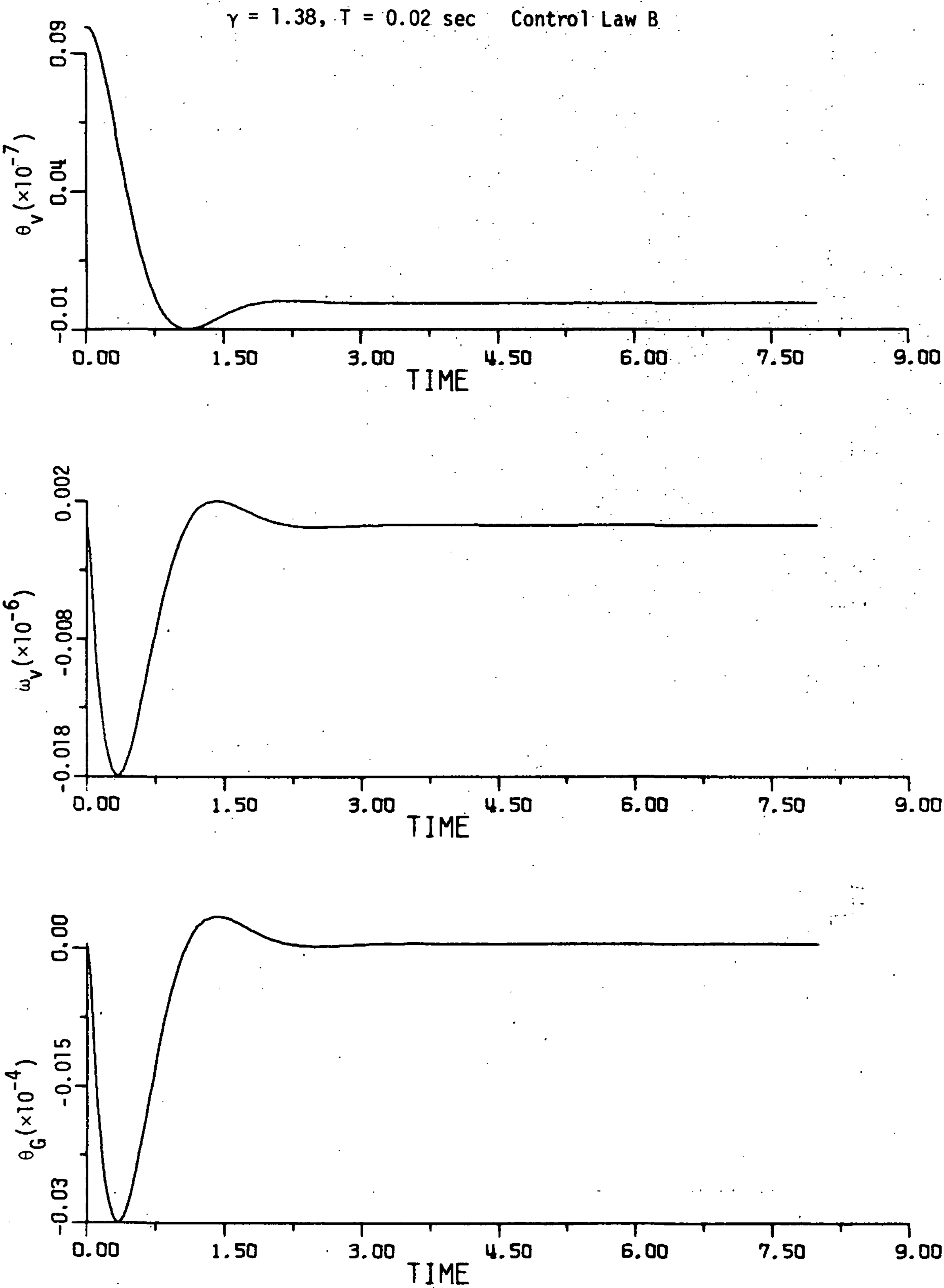

γ = 1.38, T = 0.02 sec Control Law B
θv(×10^-7)
0.09
0.04
-0.01
ωv(×10^-6)
0.002
-0.008
-0.018
θG(×10^-4)
0.00
-0.015
-0.03
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-15

γ = 1.38, T = 0.02 sec   Control Law B

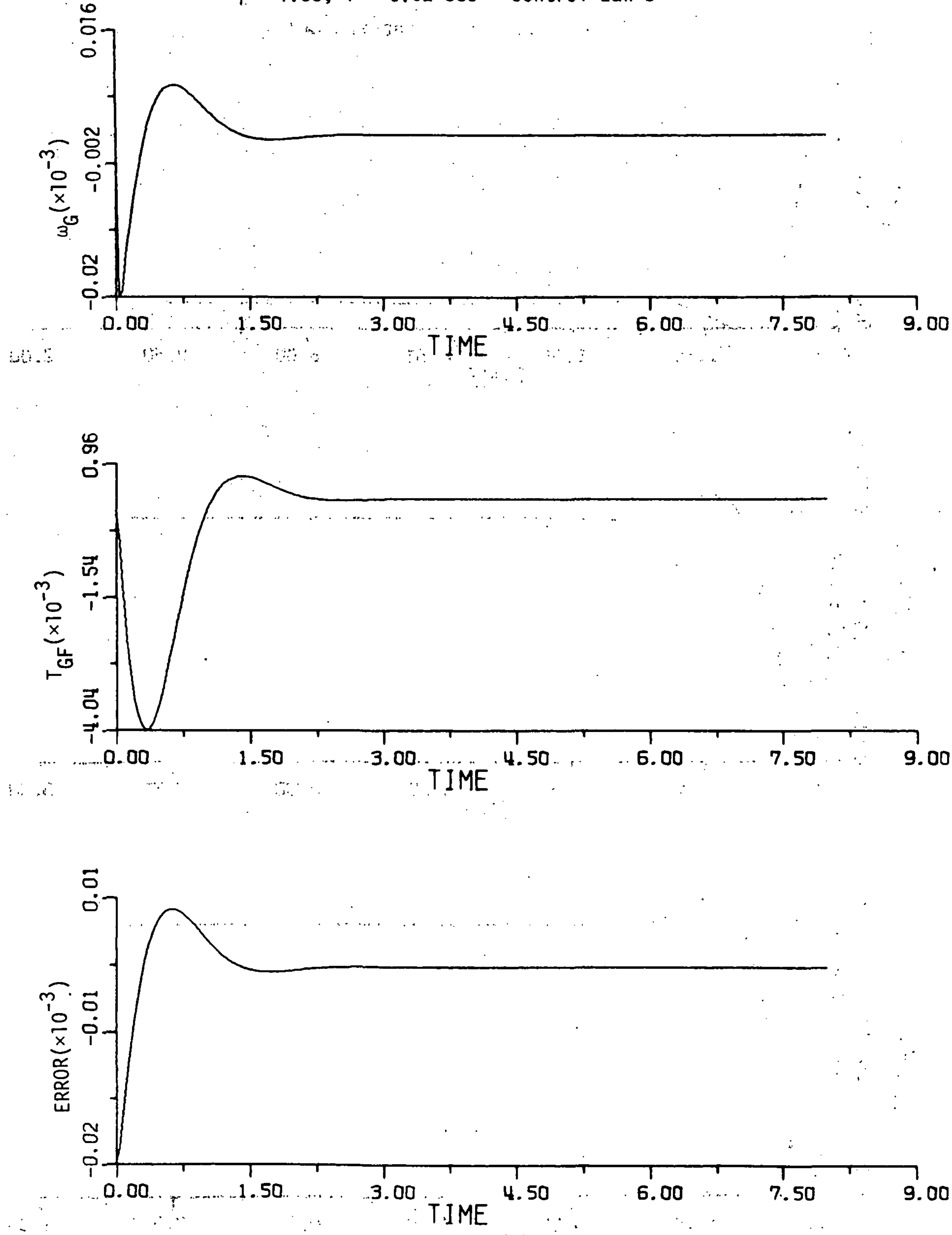


Figure 4-16

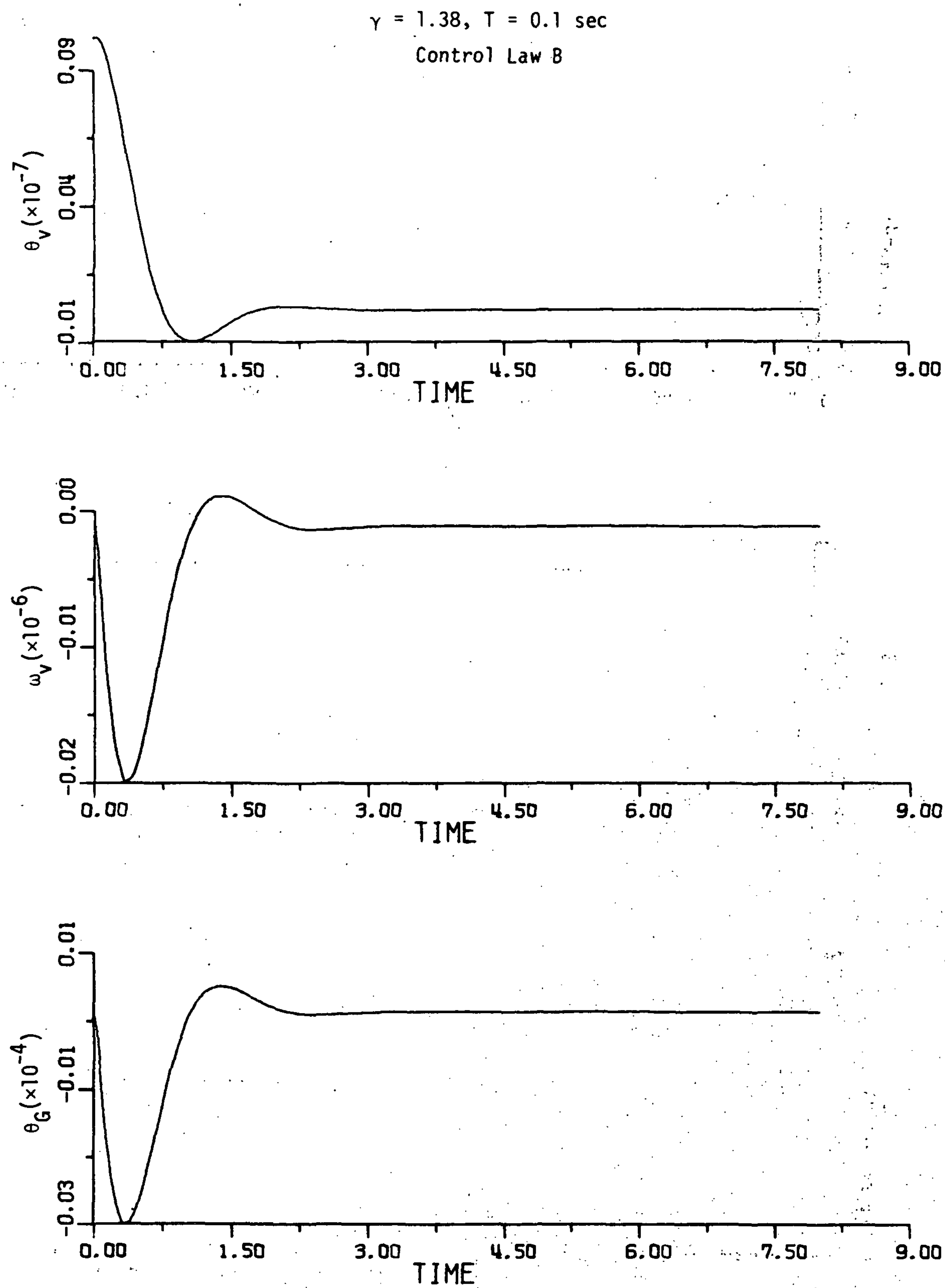

γ = 1.38, T = 0.1 sec
Control Law B
$\theta_v(\times 10^{-7})$
0.09
0.04
-0.01
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$\omega_v(\times 10^{-6})$
0.00
-0.01
-0.02
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
$\theta_G(\times 10^{-4})$
0.01
-0.01
-0.03
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-17

γ = 1.38, T = 0.1 sec

Control Law B

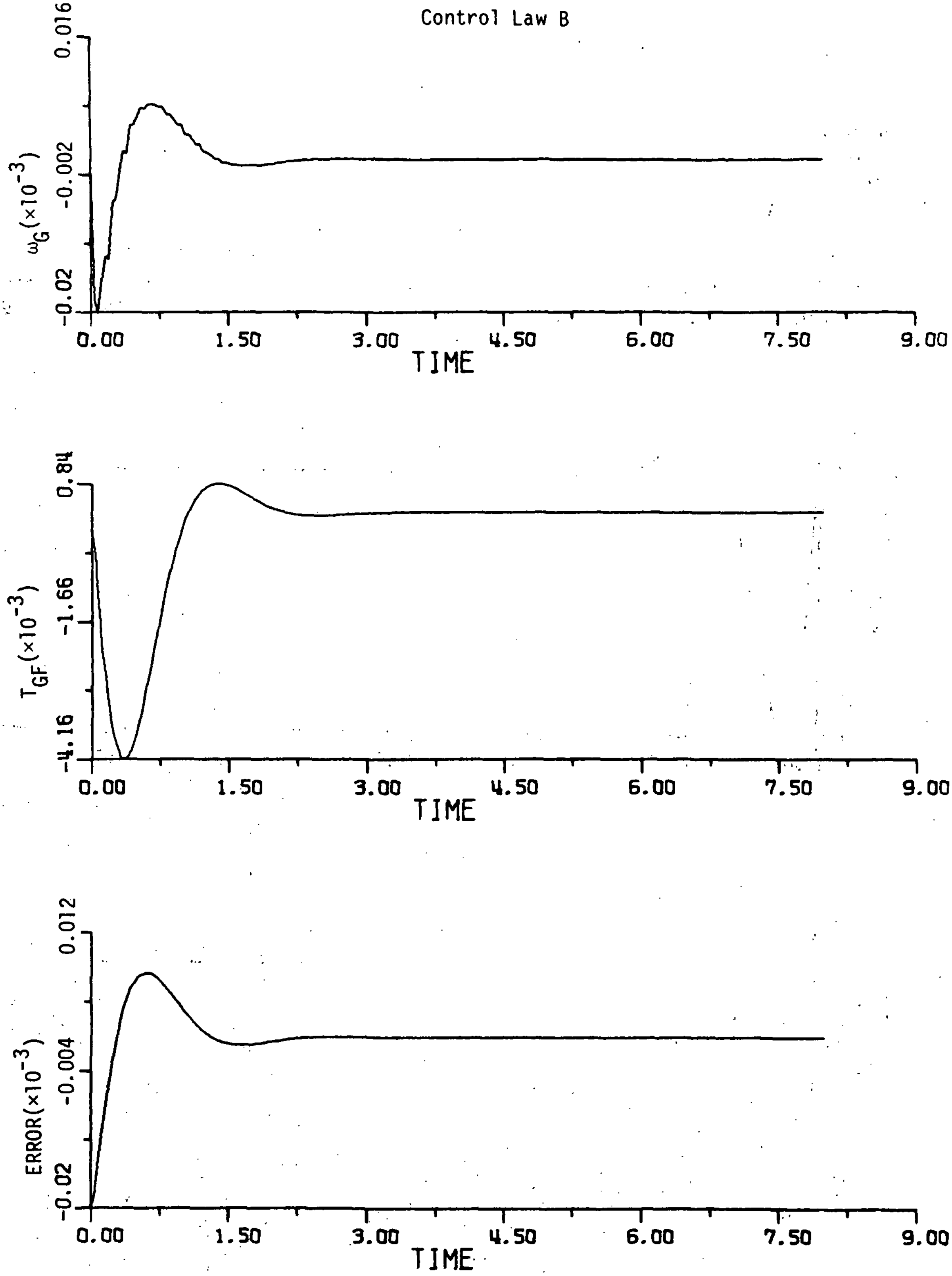


Figure 4-18

γ = 1.38, No Sampler

Control Law C

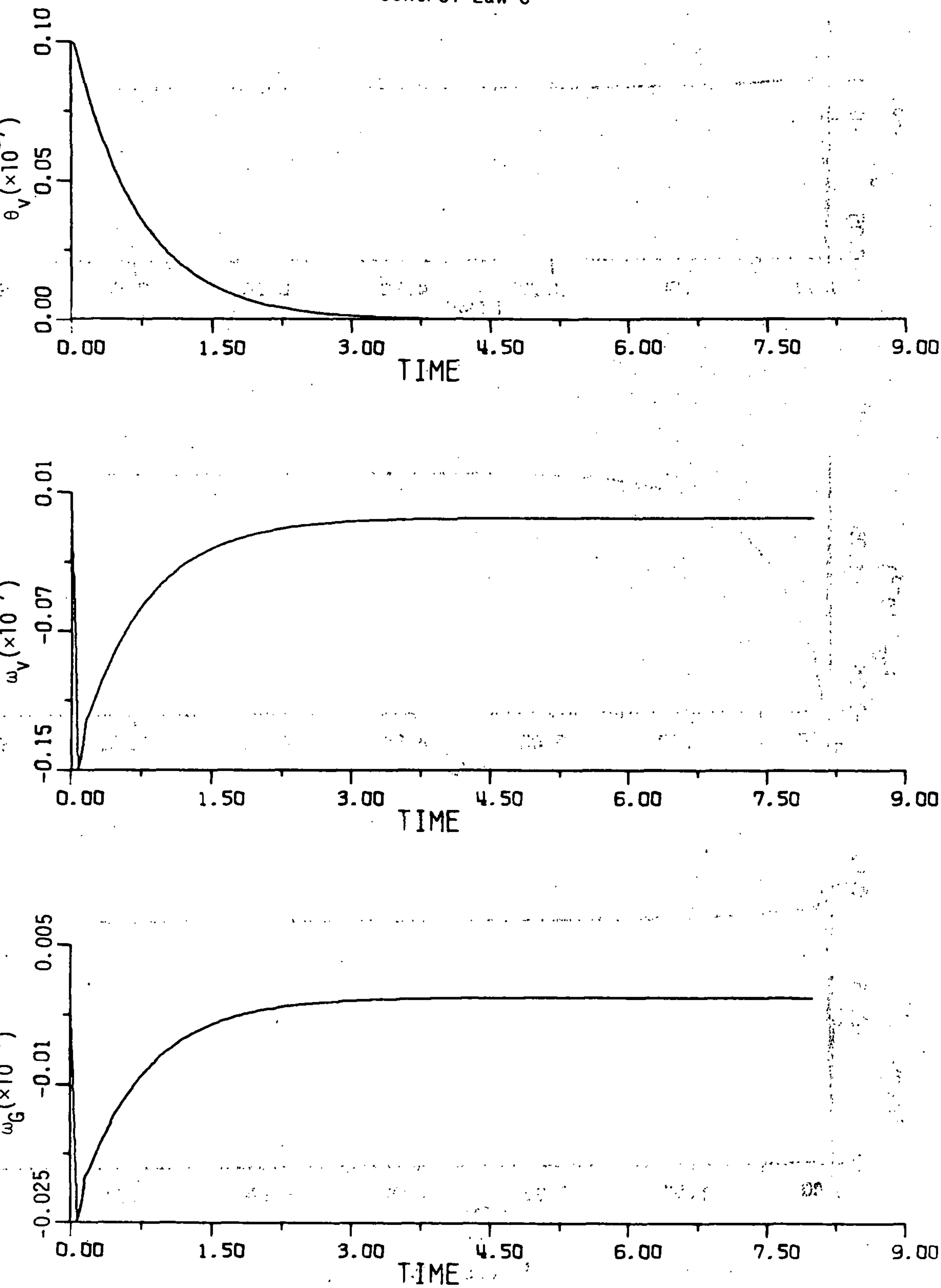


Figure 4-19

γ = 1.38, No Sampler

Control Law C

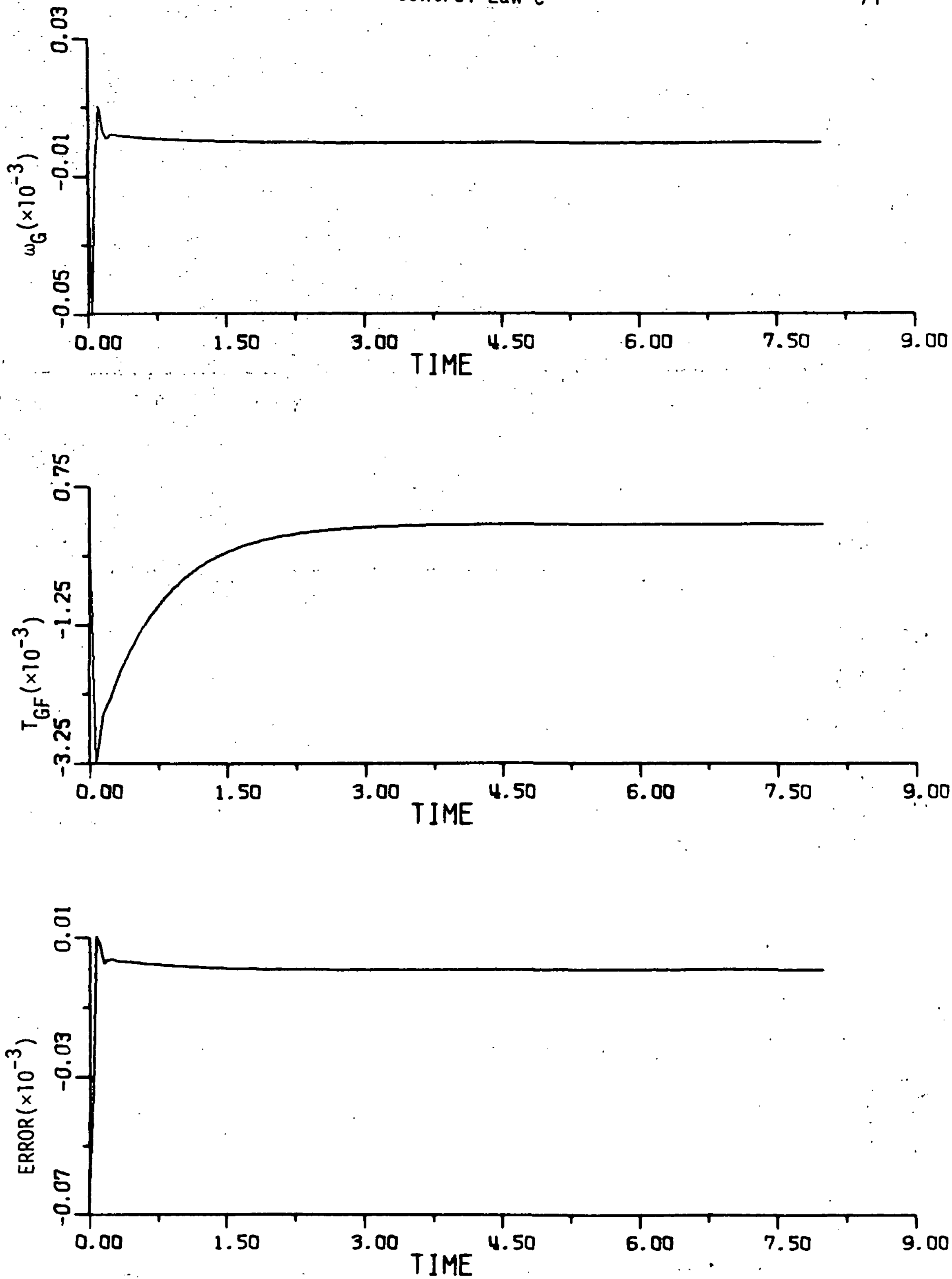


Figure 4-20

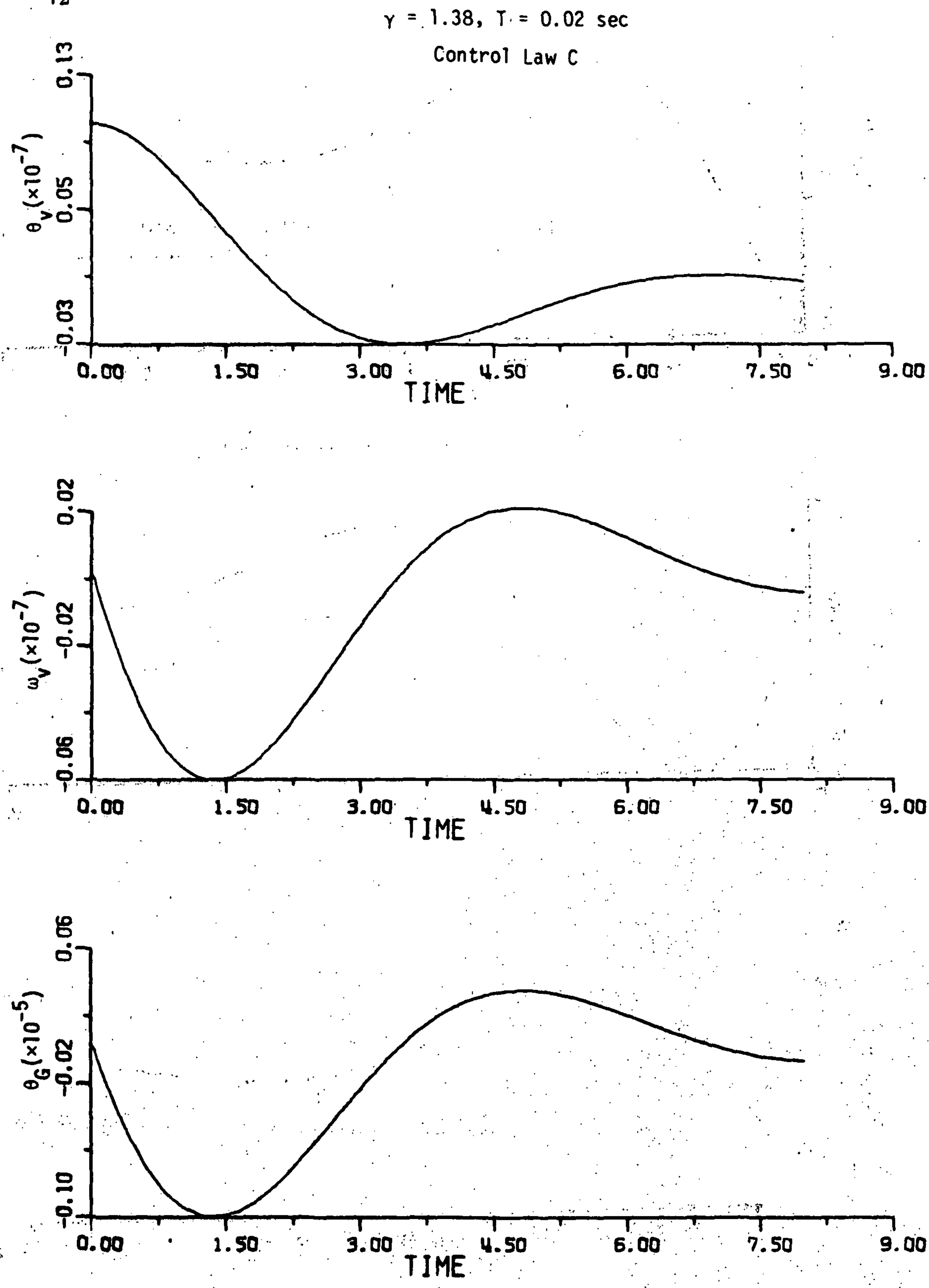

γ = 1.38, T = 0.02 sec
Control Law C
0.13
0.05
-0.03
$\theta_v(\times 10^{-7})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.02
-0.02
-0.06
$\omega_v(\times 10^{-7})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.06
-0.02
-0.10
$\theta_G(\times 10^{-5})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-21

γ = 1.38, T = 0.02 sec

Control Law C

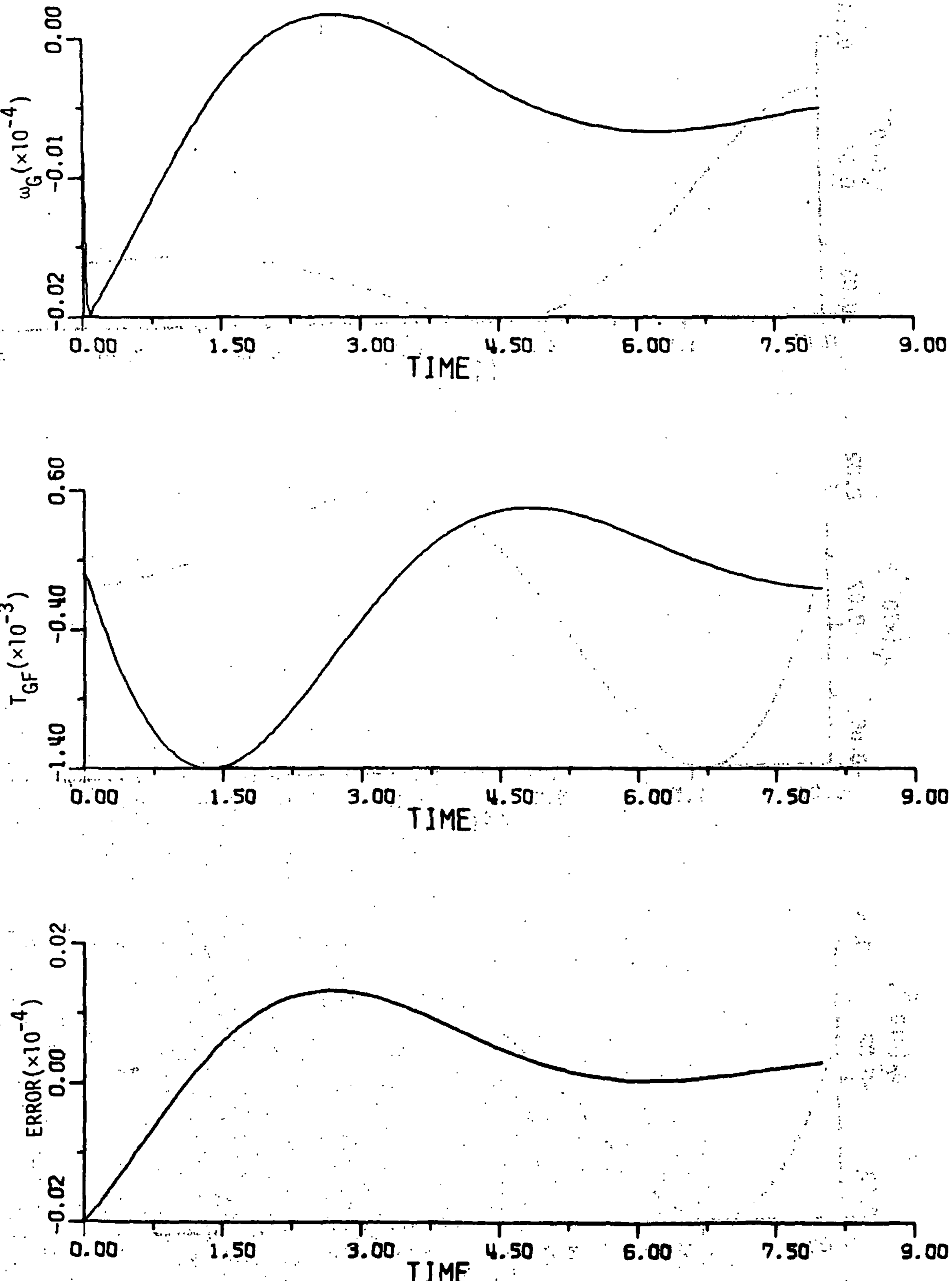


Figure 4-22

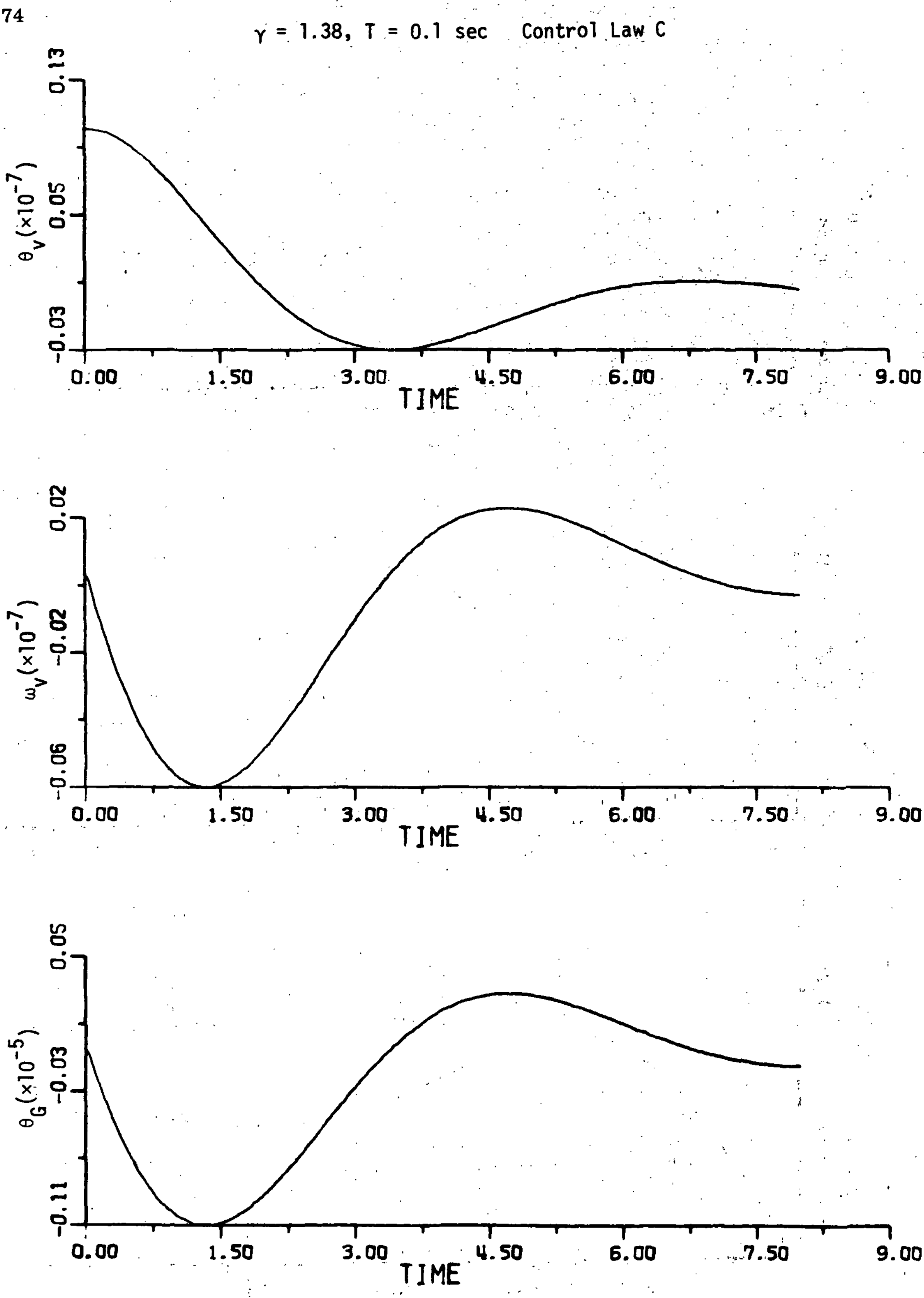

γ = 1.38, T = 0.1 sec Control Law C
0.13
0.05
-0.03
$\theta_v(\times 10^{-7})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.02
-0.02
-0.06
$\omega_v(\times 10^{-7})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.05
-0.03
-0.11
$\theta_G(\times 10^{-5})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-23

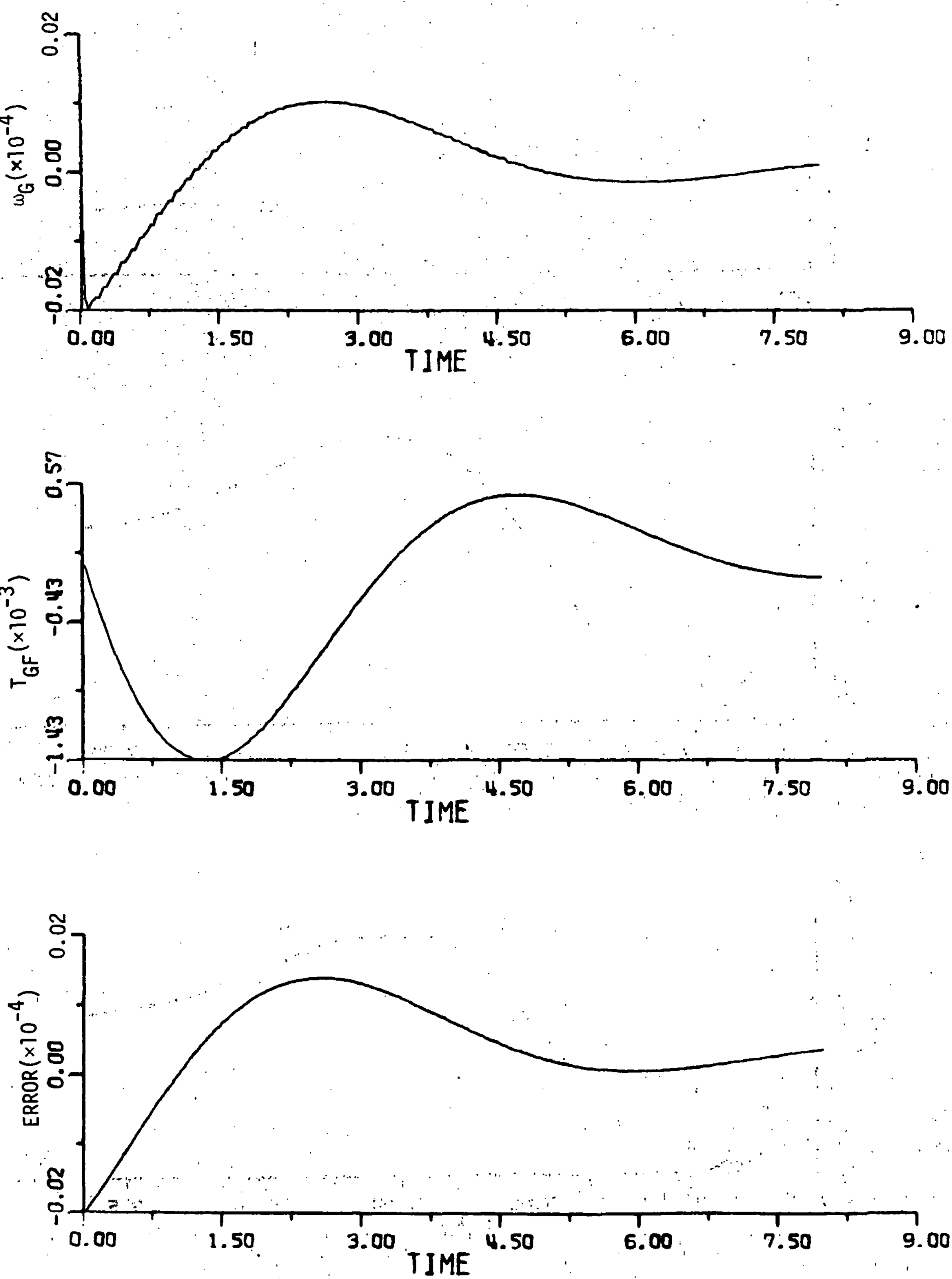

0.02
0.00
-0.02
$\omega_G(\times 10^{-4})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.57
-0.43
-1.43
$T_{GF}(\times 10^{-3})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME
0.02
0.00
-0.02
ERROR$(\times 10^{-4})$
0.00
1.50
3.00
4.50
6.00
7.50
9.00
TIME

Figure 4-24

5. Stability Considerations and Constraints on the Selection of the Weighting Matrix of the Digital Redesign Technique

It has been reported [6] that given a continuous-data system

$$\dot{\underline{x}}(t) = [A - BG(0)]\underline{x}(t) \tag{5-1}$$

the solution of the feedback matrix G(T) of an equivalent digital system, designed with the point-by-point state comparison method, must satisfy the following equation:

$$\theta(T)G(T) = e^{AT} - e^{[A-BG(0)]T} \tag{5-2}$$

Since $\theta(T)$ is usually not square, we cannot solve for G(T) directly from the last equation. One remedy to the problem is to introduce a weighting matrix H, such that the inverse of $H\theta(T)$ exists. Then,

$$G_W(T) = [H\theta(T)]^{-1}H[e^{AT} - e^{[A-BG(0)]T}] \tag{5-3}$$

However, the weighting matrix H cannot be chosen arbitrarily. The solution in Eq. (5-3) is significant only if the digitally redesigned system is stable.

In chapter 4, it has been demonstrated in the digital redesign of the LST system that for some sampling period T and some H, the resultant $G_W(T)$ gives rise to an unstable closed-loop digital system. This means that given the continuous-data control system, the weighting matrix H cannot by chosen arbitrarily. The conclusion is that if the closed-loop digital system is unstable, the solution to $G_W(T)$, corresponding to the selected H, will be meaningless.

The problem now is to find the condition under which an H can be found such that the digital system is stable.

Stability of the Closed-Loop Digital System

The state equations of the digital system are written

$$\underline{x}[(k+1)T] = \phi(T)\underline{x}(kT) + \theta(T)u(kT) \tag{5-4}$$

where

$$\phi(T) = e^{AT} \tag{5-5}$$

$$\theta(T) = \int_0^T e^{A\lambda} d\lambda B \tag{5-6}$$

The feedback control is

$$u(kT) = -G(T)\underline{x}(kT) \tag{5-7}$$

Then, Eq. (5-4) becomes

$$\underline{x}[(k+1)T] = [\phi(T) - \theta(T)G(T)]\underline{x}(kT) \tag{5-8}$$

The digital system is stable if all the eigenvalues of $[\phi(T) - \theta(T)G(T)]$ are located inside the unit circle $|z| = 1$. Since $\phi(T)$ and $\theta(T)$ are known once the sampling period T is specified, the conditions on G(T) for stability can be established using well-established techniques.

Let

$$e^{AT} - e^{[A-BG(0)]T} = D(T) \tag{5-9}$$

and premultiplying both sides of Eq. (5-2) by the $1 \times n$ matrix H, we have

$$H\theta(T)G_W(T) = HD(T) \tag{5-10}$$

where G(T) has been replaced by $G_W(T)$ to indicate the weighed matching of states.

Taking the matrix transpose on both sides of the last equation, we get

$$G_W'(T)\theta'(T)H' = D'(T)H' \tag{5-11}$$

Rearranging, Eq. (5-11) becomes

$$[G_W'(T)\theta'(T) - D'(T)]H' = \underline{0} \tag{5-12}$$

This equation represents a set of n linear homogeneous equations which have nontrival solutions if and only if the following condition is satisfied:

$$|G_W'(T)\theta'(T) - D'(T)| = 0 \tag{5-13}$$

which is also equivalent to

$$|\theta(T)G_W(T) - D(T)| = 0 \tag{5-14}$$

Thus, if Eq. (5-14) is satisfied, there is always a nonzero H which will satisfy

$$G_W(T) = [H\theta(T)]^{-1}HD(T) \tag{5-15}$$

<u>Illustrative Example</u>

Consider the continuous-data system

$$\underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t) \tag{5-16}$$

$$u(t) = -G(0)\underline{x}(t) \tag{5-17}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$G(0) = [2 \quad 3]$$

It is desired to design a digital system which will match the response of the continuous-data system at the sampling instants. The sampling period is 1 second.

The following matrices are computed:

$$\phi(T) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\theta(T) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$D(T) = \begin{bmatrix} 0.767 & 1.233 \\ 0.465 & 1.097 \end{bmatrix}$$

The characteristic equation of the closed-loop system is

$$F(z) = |zI - \phi(T) + \theta G_W(T)| = z^2 + [-2 + 0.5G_1(T) + G_2(T)]z + [1 - 0.5G_1(T) - G_2(T) + TG_1(T)] = 0 \tag{5-18}$$

where $G_1(T)$ and $G_2(T)$ are the elements of $G_W(T)$; that is,

$$G_W(T) = [G_1(T) \quad G_2(T)] \tag{5-19}$$

Using the Schur-Cohen stability criterion, the roots of Eq. (5-18) are all inside the unit circle if

$$F(0) = 1 + 0.5G_1(T) - G_2(T) < 1 \tag{5-20}$$

$$F(1) = G_1(T) > 0 \tag{5-21}$$

$$F(-1) = 4 - 2G_2(T) > 0 \tag{5-22}$$

These conditions on $G_1(T)$ and $G_2(T)$ are plotted in the parameter plane of $G_2(T)$ versus $G_1(T)$, as shown in Fig. 5-1.

Having established the conditions on the elements of $G_W(T)$ for the stability of the digital system, we turn to the condition under which an H exists which also satisfies Eq. (5-15).

Equation (5-14) leads to

$$|\theta(T)G_W(T) - D(T)| = \begin{vmatrix} 0.5G_1(T) - 0.767 & 0.5G_2(T) - 1.233 \\ G_1(T) - 0.465 & G_2(T) - 1.097 \end{vmatrix} = 0 \tag{5-23}$$

or

$$-0.535G_2(T) + 0.684G_1(T) + 0.268 = 0 \tag{5-24}$$

Equation (5-24) represents a straight line in the $G_2(T)$ versus $G_1(T)$ parameter plane. The intersect between the line represent by Eq. (5-24) and the stable region gives the stable trajectory for $G_1(T)$ and $G_2(T)$, as shown in Fig. 5-1.

If the intersect between Eq. (5-24) and the stability region of $G_1(T)$ and $G_2(T)$ is convex in general, the vertices of the intersect can be used to find the bounds on the weighting matrix H.

In the present case, the vertices of $G_1(T)$ and $G_2(T)$ are at (0, 0.5023) and (1.171, 2).

Substituting the vertices of $G_1(T)$ and $G_2(T)$ in Eq. (5-12), we have the two boundary equations for the elements of $H = [h_1 \quad h_2]$.

$$G_1(T) = 0, \qquad h_1 = -0.606h_2 \tag{5-25}$$

$$G_1(T) = 1.171 \qquad h_1 = 3.875h_2 \tag{5-26}$$

Figure 5-2 shows the region in which h_1 and h_2 should lie so that Eq. (5-3) will always yield a state feedback control such that the digital system is stable.

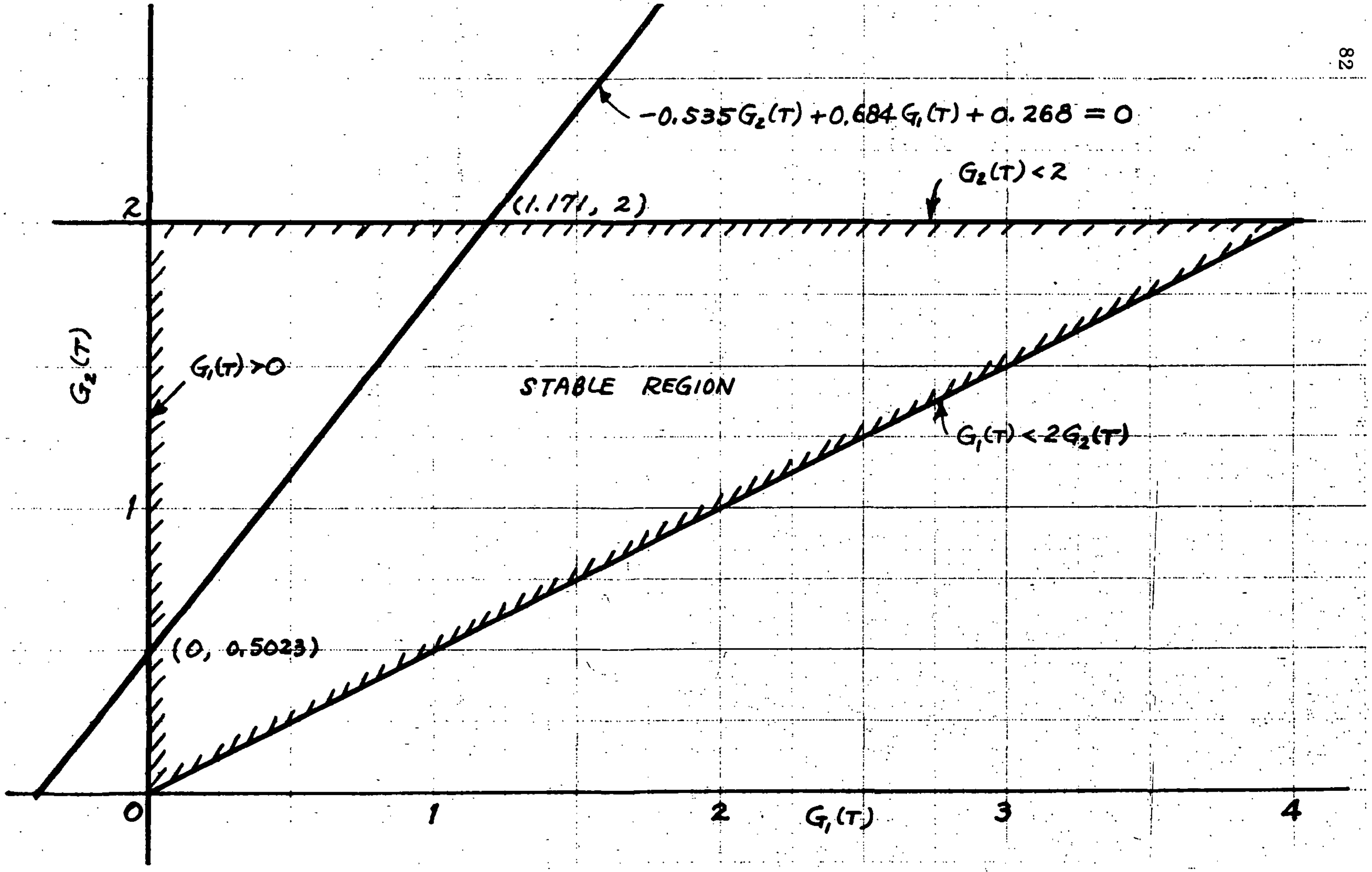


Figure 5-1. Stable region and locus for nontrival solution of H.

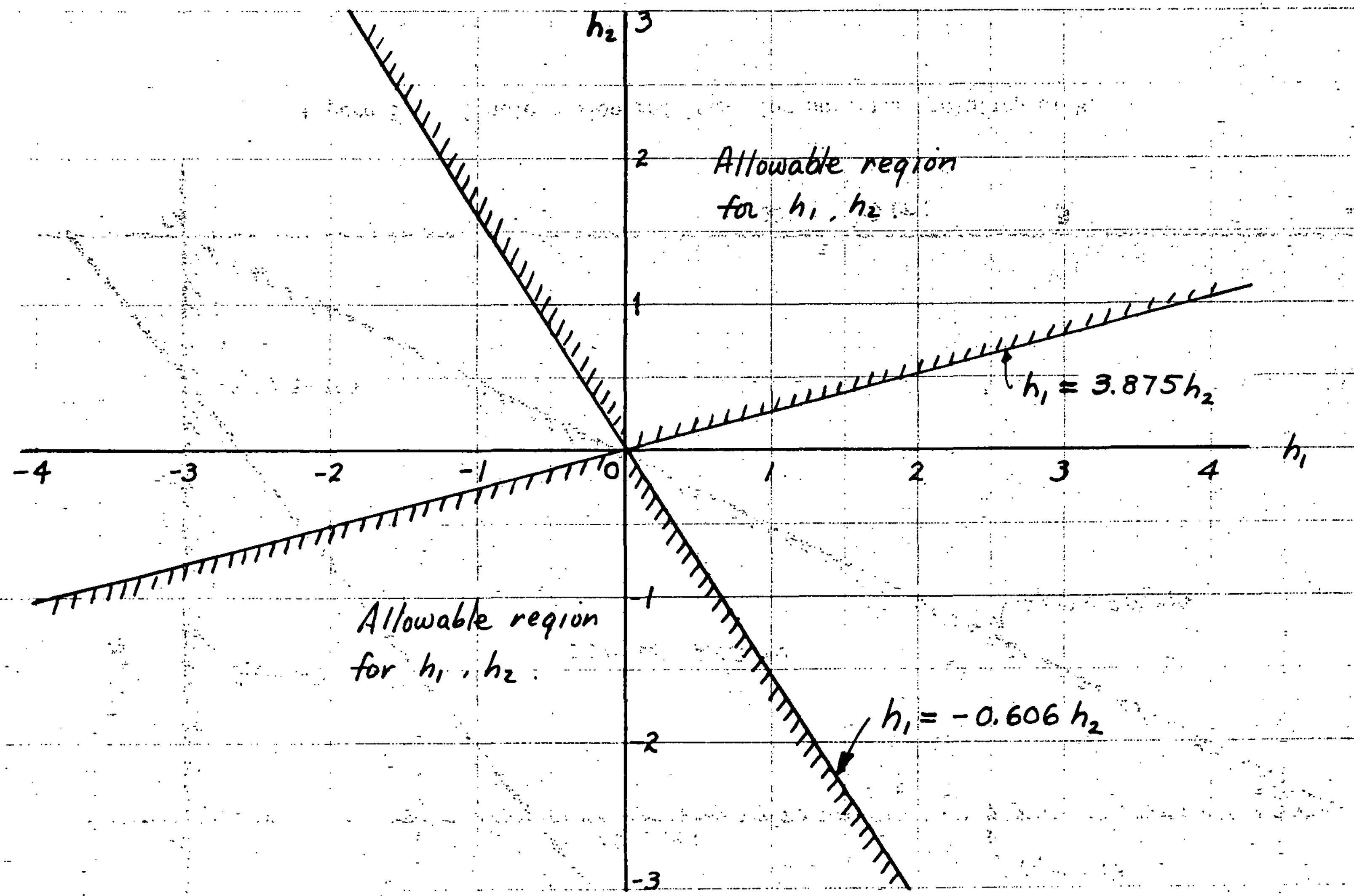


Figure 5-2. Allowable regions of h_1 and h_2 for stable digital system.

6. Realization of State Feedback by Dynamic Controllers

One of the unique characteristics of modern control theory is that optimal control is often realized by state feedback. For instance, it is well known that if a system is completely controllable, its eigenvalues can be arbitrarily assigned through state feedback, and the optimal linear regulator design always leads to a state feedback solution. Unfortunately, in practice, not all the state variables of a physical system are accessible. Considerable amount of results have been reported in the past on the design of optimal systems with partial state feedback.

The basis of the classical control system design is that the configuration of the controller is selected a priori. The controller used in practical systems usually assume the form of cascade or feedback controllers, or a combination of these. In these cases, only the outputs of the system are fed back. One advantage of the classical controllers is that they can be implemented often by passive filters or electronic circuits.

In this chapter we shall present a method whereby a system with state feedback is approximated by a system with a cascade controller. How the state feedback is determined is immaterial for the present analysis; it can be obtained from the pole-location solution or the Riccati equation solution, or some other optimal control design methods.

Continuous-Data Systems

Consider the system

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \tag{6-1}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \tag{6-2}$$

where

$\underline{x}(t) = n \times 1$ state vector

$\underline{u}(t) = r \times 1$ input vector

$\underline{y}(t) = m \times 1$ output vector

A, B, C, and D are coefficient matrices of appropriate dimensions.

Assume that state feedback is given such that

$$\underline{u}(t) = -G\underline{x}(t) \tag{6-3}$$

where G is an $r \times n$ feedback gain matrix.

The design objective is to approximate the system of Fig. 6-1a which is described by Eqs. (6-1), (6-2), and (6-3), by the system of Fig. 6-1b which has a feedback controller with feedback from the output variables. Let the transfer relation of the controller be represented by

$$\underline{U}(s) = -H(s)\underline{Y}(s) \tag{6-4}$$

where H(s) is the controller transfer function matrix:

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1m}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2m}(s) \\ \vdots & \vdots & & \vdots \\ H_{r1}(s) & H_{r2}(s) & \cdots & H_{rm}(s) \end{bmatrix} \tag{6-5}$$

Let $H_{ij}(s)$ ($i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, m$) be a pth order transfer function,

$$H_{ij}(s) = \frac{K_{ij}(1 + \alpha_{ij1}s + \alpha_{ij2}s^2 + \ldots + \alpha_{ijp}s^p)}{(1 + \beta_{ij1}s + \beta_{ij2}s^2 + \ldots + \beta_{ijp}s^p)} \tag{6-6}$$

The transfer function $H_{ij}(s)$ is expanded into a Taylor series about $s = 0$,

$$H_{ij}(s) = K_{ij} \sum_{k=0}^{\infty} d_{ijk}s^k \tag{6-7}$$

where

$$d_{ijk} = \left. \frac{\partial^k H_{ij}(s)}{\partial s^k} \right|_{s = 0} \tag{6-8}$$

Evaluating the coefficients of the Taylor series, we have

$$d_{ijo} = 1$$

$$d_{ij1} = \alpha_{ij1} - \beta_{ij1}$$

and for $k > 1$,

$$d_{ijk} = \alpha_{ijk} - \beta_{ijk} - \sum_{v=1}^{k-1} \beta_{ij(k-v)} d_{ijv} \tag{6-9}$$

Since the state feedback represents the feedback of the system output and its higher-order derivatives, a truncated series expansion of $H_{ij}(s)$ may be used as a dynamic implementation of state feedback by feeding back only the output variables.

If the infinite series of (6-7) converges, we may approximate it truncating it after p terms, where p is not yet specified. Let us introduce the notation, $H_{ijp}(s)$, for the truncated version of $H_{ij}(s)$; then

$$H_{ijp}(s) = K_{ij} \sum_{k=0}^{p-1} d_{ijk} s^k \tag{6-10}$$

where d_{ijk} is as defined in Eqs. (6-9).

Substituting Eq. (6-10) in Eq. (6-4) for the elements of H(s), we have

$$\underline{U}(s) = -\begin{bmatrix} K_{11}[d_{110}\ d_{111}\ \cdots\ d_{11(p-1)}] & K_{12}[d_{120}\ d_{121}\ \cdots\ d_{12(p-1)}] & \cdots & K_{1m}[d_{1m0}\ d_{1m1}\ \cdots\ d_{1m(p-1)}] \\ K_{21}[d_{210}\ d_{211}\ \cdots\ d_{21(p-1)}] & K_{22}[d_{220}\ d_{221}\ \cdots\ d_{22(p-1)}] & \cdots & K_{2m}[d_{2m0}\ d_{2m1}\ \cdots\ d_{2m(p-1)}] \\ \vdots & \vdots & & \vdots \\ K_{r1}[d_{r10}\ d_{r11}\ \cdots\ d_{r1(p-1)}] & K_{r2}[d_{r20}\ d_{r21}\ \cdots\ d_{r2(p-1)}] & \cdots & K_{rm}[d_{rm0}\ d_{rm1}\ \cdots\ d_{rm(p-1)}] \end{bmatrix} \begin{bmatrix} Y_1(s) \\ sY_1(s) \\ \vdots \\ s^{p-1}Y_1(s) \\ \hline Y_2(s) \\ sY_2(s) \\ \vdots \\ s^{p-1}Y_2(s) \\ \hline \vdots \\ \hline Y_m(s) \\ sY_m(s) \\ \vdots \\ s^{p-1}Y_m(s) \end{bmatrix} \tag{6-11}$$

The elements of the last equation are rearranged to give

$$\underline{U}(s) \cong -\begin{bmatrix} [K_{11}d_{110} & K_{12}d_{120} & \cdots & K_{1m}d_{1m0}][K_{11}d_{111} & K_{12}d_{121} & \cdots & K_{1m}d_{1m1}] & \cdots & [K_{11}d_{11(p-1)} & K_{12}d_{12(p-1)} & \cdots & K_{1m}d_{1m(p-1)}] \\ [K_{21}d_{210} & K_{22}d_{220} & \cdots & K_{2m}d_{2m0}][K_{21}d_{211} & K_{22}d_{221} & \cdots & K_{2m}d_{2m1}] & \cdots & [K_{21}d_{21(p-1)} & K_{22}d_{22(p-1)} & \cdots & K_{2m}d_{2m(p-1)}] \\ \vdots & & & \vdots & & \vdots \\ [K_{r1}d_{r10} & K_{r2}d_{r20} & \cdots & K_{rm}d_{rm0}][K_{r1}d_{r11} & K_{r2}d_{r21} & \cdots & K_{rm}d_{rm1}] & \cdots & [K_{r1}d_{r1(p-1)} & K_{r2}d_{r2(p-1)} & \cdots & K_{rm}d_{rm(p-1)}] \end{bmatrix} \begin{Bmatrix} \underline{Y}(s) \\ s\underline{Y}(s) \\ s^2\underline{Y}(s) \\ \vdots \\ s^{p-1}\underline{Y}(s) \end{Bmatrix} \tag{6-12}$$

The time-domain equivalence of the last equation is

$$\underline{u}(t) \cong -F \begin{Bmatrix} \underline{y}(t) \\ \dot{\underline{y}}(t) \\ \ddot{\underline{y}}(t) \\ \vdots \\ \underline{y}^{(p-1)}(t) \end{Bmatrix} \tag{6-13}$$

where F denotes the $r \times mp$ coefficient matrix in Eq. (6-12).

From Eq. (6-2),

$$\begin{aligned} \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \\ &= (C - DG)\underline{x}(t) \end{aligned} \tag{6-14}$$

Then,

$$\dot{\underline{y}}(t) = (C - DG)(A - BG)\underline{x}(t) \tag{6-15}$$

$$\vdots$$

$$\underline{y}^{(p-1)}(t) = (C - DG)(A - BG)^{p-1}\underline{x}(t) \tag{6-16}$$

Substituting the last three equations in Eq. (6-13), we have

$$\underline{u}(t) \cong -F \begin{bmatrix} (C - DG) \\ (C - DG)(A - BG) \\ \cdot \\ \cdot \\ \cdot \\ (C - DG)(A - BG)^{p-1} \end{bmatrix} \underline{x}(t) \tag{6-17}$$

Comparing Eq. (6-17) with Eq. (6-2), we have

$$\underset{(r \times mp)}{F} \underset{(mp \times n)}{\begin{bmatrix} (C - DG) \\ (C - DG)(A - BG) \\ \cdot \\ \cdot \\ \cdot \\ (C - DG)(A - BG)^{p-1} \end{bmatrix}} = \underset{(r \times n)}{G} \tag{6-18}$$

In order to solve for F from the last equation, we write

$$F = G \begin{bmatrix} (C - DG) \\ (C - DG)(A - BG) \\ \vdots \\ (C - DG)(A - BG)^{p-1} \end{bmatrix}^{-1} \tag{6-19}$$

if $mp = n$, or $p = n/m$. This means that if n/m is an integer, we may truncate the Taylor series expansions of $G_{ij}(s)$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, m$, at $p = n/m$ terms. If n/m is not an integer, we may choose p to be an integer which satisfies

$$\frac{n}{m} < p < \frac{n}{m} + 1 \tag{6-20}$$

Since F is $r \times mp$, there will be $(r)(m)(p)$ unknowns. However, there are only rn equations in Eq. (6-18). Thus, $r(mp - n)$ of the elements of F may be assigned arbitrarily.

The solution of F from Eq. (6-19) also depends on the existence of the inverse in the equation.

It should be noted that solution of the elements of F gives only the values of the coefficients in Eq. (6-7). The coefficients of the transfer function of (6-6) still have to be determined using Eq. (6-9). In general, there are more unknowns than equations in Eq. (6-9). This simply means that in the ideal situation we simply set

$$d_{ijo} = 1$$
$$d_{ijk} = \alpha_{ijk}$$

and all $\beta_{ijk} = 0$, for $k = 1, 2, \ldots, m$. However, for a physically realizable transfer function, $H_{ijp}(s)$ must not have more zeros than poles. Therefore, the values of β_{ijk} should be assigned such that the dynamic behavior of the overall system is not appreciably affected by the presence of β_{ijk}, $k = 1, 2, \ldots, m$. This is similar to the classical design practice of designing the zeros of $H_{ijp}(s)$ to control the dynamic behavior of the system, while placing the poles of $H_{ijp}(s)$ so that they do not have appreciable effects on the system performance.

<u>Single-Variable Continuous-Data Systems</u>

When the control $u(t)$ is a scalar, $u(t) = -G\underline{x}(t)$, where

$$G = [g_1 \quad g_2 \quad \cdots \quad g_n] \tag{6-21}$$

Eq. (6-6) becomes

$$H(s) = \frac{K(1 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_n s^n)}{(1 + \beta_1 s + \beta_2 s^2 + \ldots + \beta_n s^n)} \tag{6-22}$$

Then,

$$H(s) \cong K(1 + d_1 s + d_2 s^2 + \ldots + d_{n-1} s^{n-1}) \tag{6-23}$$

where

$$d_k = \alpha_k - \beta_k - \sum_{v=1}^{k-1} \beta_{k-v} d_v \tag{6-24}$$

for $k = 1, 2, \ldots, n-1$.

Equation (6-19) becomes

$$F = K[1 \quad d_1 \quad d_2 \quad \cdots \quad d_{n-1}] = G \begin{bmatrix} C - DG \\ (C - DG)(A - BG) \\ \vdots \\ (C - DG)(A - BG)^{n-1} \end{bmatrix}^{-1} \qquad (6\text{-}25)$$

Single-Variable Continuous-Data Systems in Phase-Variable Canonical Form

If the system to be controlled is in the phase-variable canonical form, then

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \qquad (6\text{-}26)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad (6\text{-}27)$$

and the output equation if characterized by $D = \underline{0}$, and

$$C = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \qquad (6\text{-}28)$$

the formulation given in the preceding section is further simplified. Since $\underline{D} = \underline{0}$, and $CBG = \underline{0}$,

$$\begin{bmatrix} C - DG \\ (C - DG)(A - BG) \\ \cdot \\ \cdot \\ \cdot \\ (C - DG)(A - BG)^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} = I \text{ (identity matrix)} \qquad (6\text{-}29)$$

Then, Eq. (6-25) becomes

$$\begin{aligned} F &= K[1 \quad d_1 \quad d_2 \quad \cdots \quad d_{n-1}] = G \\ &= [g_1 \quad g_2 \quad \cdots \quad g_n] \end{aligned} \qquad (6\text{-}30)$$

and

$$K = g_1$$

$$\alpha_1 = \frac{g_1\beta_1 + g_2}{g_1}$$

$$\vdots \qquad (6\text{-}31)$$

$$\alpha_k = \frac{g_{k+1}}{g_1} + \beta_k + \sum_{v=1}^{k-1} \beta_{k-v} g_{v+1}$$

$k = 1, 2, \ldots, n.$

Equivalent Cascade Controller

The development carried out in the preceding sections is based on a controller being placed in the feedback path of the system as shown in Fig. 5-1b. When the reference input $r(t)$ is zero, that is, when the system is a regulator, it does not matter whether the controller $H(s)$ is in the forward path or the feedback path. However, when the input is not zero, it may be desirable to determine an equivalent system which has the controller in the forward path of the system as shown in Fig. 5-2. In the following, single-variable notation is used for simplicity. The problem is to find the transfer function of the cascade controller $G_C(s)$ so that the closed-loop transfer functions of the two systems with feedback controller and the forward-path controller are identical. The solution of $G_C(s)$ is

$$G_C(s) = \frac{1}{1 + G(s)[H(s) - 1]} \tag{6-32}$$

where

$$G(s) = C(sI - A)^{-1}B \qquad D = 0 \tag{6-33}$$

In general, given G(s), and having determined H(s), the order of $G_C(s)$ will usually be higher than that of H(s).

The following example will illustrate the design method outlined in the preceding sections.

<u>Example 6-1</u>

Consider that the dynamic equations of a linear time-invariant system are given by

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + Bu(t) \\ y(t) &= C\underline{x}(t) \end{aligned} \tag{6-34}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$C = [1 \quad 0 \quad 0]$

Since the system is completely controllable, we may assign the eigenvalues of the system arbitrarily. Furthermore, the state equations are already in phase-variable canonical form. With state feedback, $u = -G\underline{x}$, the closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (g_3 + 1)s^2 + g_2 s + g_1} \tag{6-35}$$

The characteristic equation is

$$s^3 + (g_3 + 1)s^2 + g_2 s + g_1 = 0 \tag{6-36}$$

Let us assume that we wish to place the eigenvalues of the closed-loop system at $s = -10$, $-1 \pm j1$, and $-1 - j1$. Then, Eq. (5-36) gives

$$g_1 = 20, \quad g_2 = 22, \quad g_3 = 11$$

or

$$G = [20 \quad 22 \quad 11] \tag{6-37}$$

Now consider that the states x_2 and x_3 are not directly accessible, and it is desired to approximate the state-feedback solution by a feedback controller and output feedback. Since the system is of the third order, $n = 3$, the dynamic controller may be of the second order; that is,

$$H(s) = K \frac{1 + \alpha_1 s + \alpha_2 s^2}{1 + \beta_1 s + \beta_2 s^2} \tag{6-38}$$

In the present case, the results of Eq. (6-30) may be used. Thus,

$$K = g_1 = 20$$

$$\alpha_1 = \frac{g_1\beta_1 + g_2}{g_1} = 1.1 + \beta_1$$

$$\alpha_2 = \frac{g_3}{g_1} + \beta_2 + \beta_1 \frac{g_2}{g_1} = 0.55 + 1.1\beta_1 + \beta_2$$

Assuming that physical circuit elements allow the selection of β_1 and β_2 to be relatively small as compared with the resulting values of α_1 and α_2, we let $\beta_1 = 0.15$ and $\beta_2 = 0.005$. Then,

$$\alpha_1 = 1.25$$

$$\alpha_2 = 0.72$$

The transfer function of the feedback controller is

$$H(s) = 2880 \frac{s^2 + 1.736s + 1.389}{s^2 + 30s + 200} \tag{6-39}$$

The closed-loop transfer function of the system with the cascade controller is

$$\frac{Y(s)}{R(s)} = \frac{2880(s^2 + 1.736s + 1.389)}{s^5 + 31s^4 + 230s^3 + 3080s^2 + 5000s + 4000} \tag{6-40}$$

A comparison of the step responses of the system with state feedback and the system with the feedback controller is shown in Fig. 6-3.

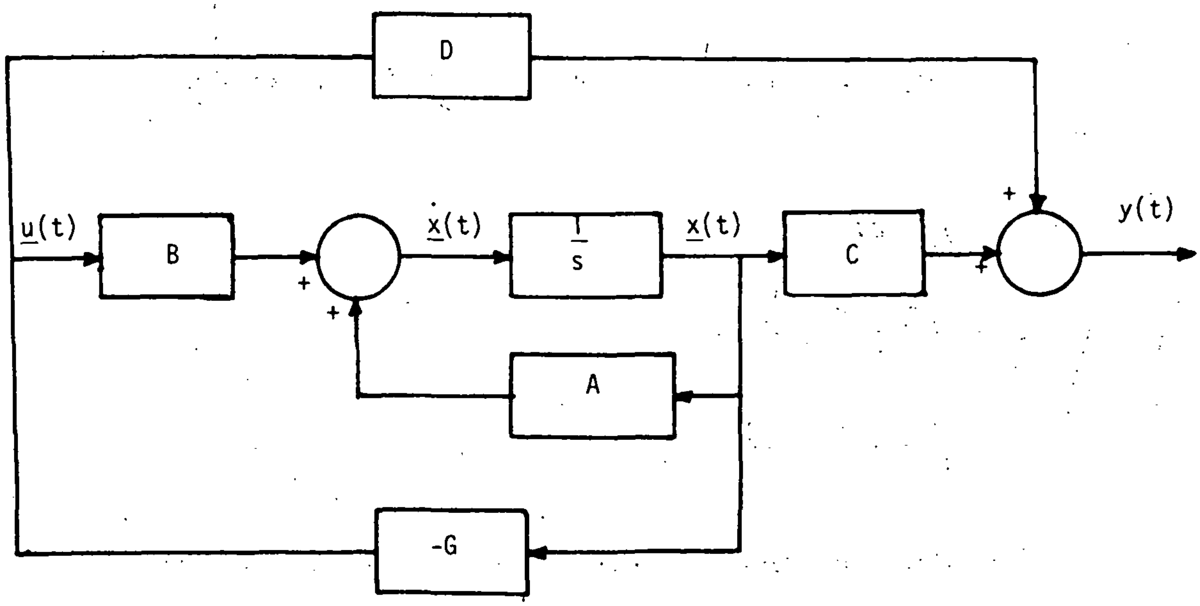


Figure 6-1a. Block diagram of system with state feedback.

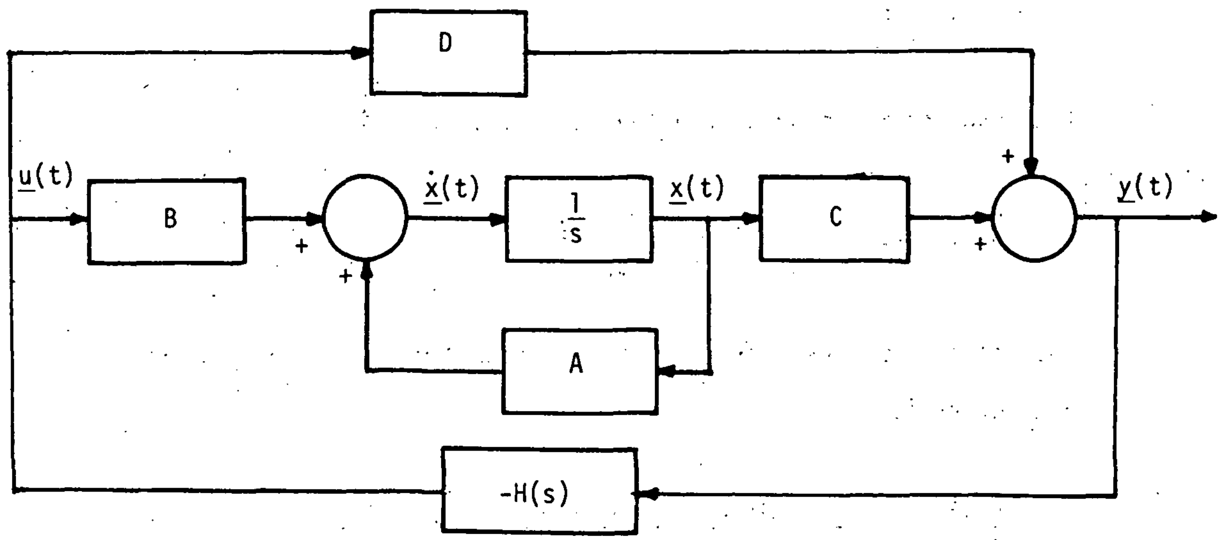
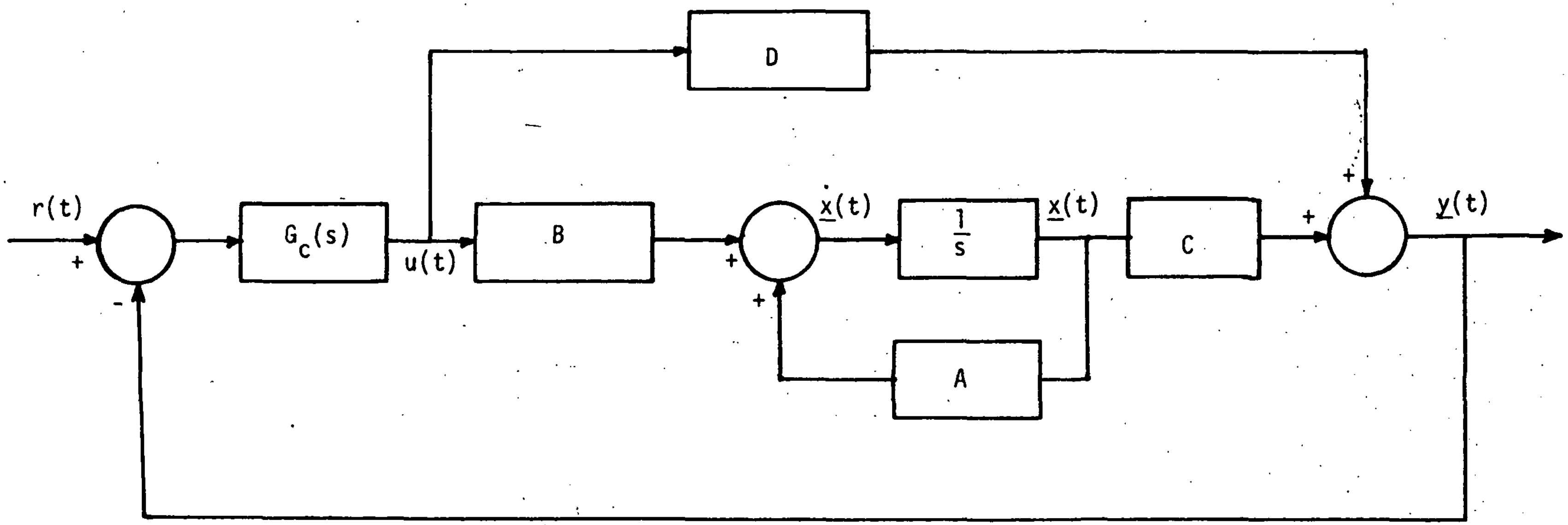


Figure 6-1b. Block diagram of system with cascade controller from output feedback.



r(t)
+
-
$G_c(s)$
u(t)
B
D
+
+
$\dot{\underline{x}}(t)$
$\frac{1}{s}$
$\underline{x}(t)$
A
C
+
+
$\underline{y}(t)$

Figure 6-2.

Discrete-Data Systems

The dynamic controller design technique described in the last section can be applied to discrete-data systems. Consider the dynamic equations,

$$\underline{x}(k + 1) = A\underline{x}(k) + B\underline{u}(k) \tag{6-41}$$

$$\underline{y}(k) = C\underline{x}(k) + D\underline{u}(k) \tag{6-42}$$

where

$\underline{x}(k) = n \times 1$ state vector

$\underline{u}(k) = r \times 1$ input vector

$\underline{y}(k) = m \times 1$ output vector

A, B, C, and D are coefficient matrices of appropriate dimensions.

Assume that the state feedback is used such that

$$\underline{u}(k) = -G\underline{x}(k) \tag{6-43}$$

where G is an $r \times n$ feedback gain matrix.

Let the controller be modeled as a feedback controller with the transfer function relation,

$$\underline{U}(z) = -H(z)\underline{Y}(z) \tag{6-44}$$

where H(z) is given by

$$H(z) = \begin{bmatrix} H_{11}(z) & H_{12}(z) & \cdots & H_{1m}(z) \\ H_{21}(z) & H_{22}(z) & \cdots & H_{2m}(z) \\ \vdots & & & \\ H_{r1}(z) & H_{rr}(z) & \cdots & H_{rm}(z) \end{bmatrix} \tag{6-45}$$

Let $H_{ij}(z)$ be a pth-order transfer function,

$$H_{ij}(z) = \frac{K_{ij}(z^p + \alpha_{ij1}z^{p-1} + \alpha_{ij2}z^{p-2} + \ldots + \alpha_{ijp})}{z^p + \beta_{ij1}z^{p-1} + \beta_{ij2}z^{p-2} + \ldots + \beta_{ijp}} \tag{6-46}$$

Let us expand $H_{ij}(z)$ into a Laurent's series about $z = 0$,

$$H_{ij}(z) = K_{ij} \sum_{k=0}^{\infty} d_{ijk} z^{-k} \tag{6-47}$$

where

$$\begin{aligned} d_{ijo} &= 1 \\ d_{ij1} &= \alpha_{ij1} - \beta_{ij1} \\ d_{ijk} &= \alpha_{ijk} - \beta_{ijk} - \sum_{v=1}^{k-1} \beta_{k-v} d_{ijv} \end{aligned} \tag{6-48}$$

Truncating $H_{ij}(z)$ at p terms, we have

$$H_{ij}(z) \cong K_{ij} \sum_{k=0}^{p-1} d_{ijk} z^{-k} \tag{6-49}$$

Similar to the development in Eqs. (6-11) and (6-12), the time-domain correspondence of Eq. (6-44) is

$$\underline{u}(k) \cong -F \begin{bmatrix} \underline{y}(k) \\ \underline{y}(k-1) \\ \vdots \\ \underline{y}(k-p+1) \end{bmatrix} \tag{6-50}$$

where F is identical to the $r \times mp$ coefficient matrix defined in Eqs. (6-12) and (6-13), except with its elements correspond to the coefficients of Eq. (6-46).

From Eqs. (6-42) and (6-43),

$$\underline{y}(k) = (C - DG)\underline{x}(k) \tag{6-51}$$

Thus,

$$\underline{y}(k - 1) = (C - DG)\underline{x}(k - 1) \tag{6-52}$$

Also,

$$\begin{aligned} A\underline{x}(k - 1) &= \underline{x}(k) - B\underline{u}(k - 1) \\ &= \underline{x}(k) - BG\underline{x}(k - 1) \end{aligned} \tag{6-53}$$

Therefore,

$$\underline{x}(k - 1) = (A - BG)^{-1}\underline{x}(k) \tag{6-54}$$

Substitution of Eq. (6-54) in Eq. (6-52), we have

$$\underline{y}(k - 1) = (C - DG)(A - BG)^{-1}\underline{x}(k) \tag{6-55}$$

Similarly,

$$\underline{y}(k - 2) = (C - DG)(A - BG)^{-2}\underline{x}(k) \tag{6-56}$$

$$\vdots$$

$$\underline{y}(k - p + 1) = (C - DG)(A - BG)^{-p+1}\underline{x}(k) \tag{6-57}$$

Thus, Eq. (6-50) becomes

$$\underline{u}(k) \cong -F \begin{bmatrix} C - DG \\ (C - DG)(A - BG)^{-1} \\ \cdot \\ \cdot \\ \cdot \\ (C - DG)(A - BG)^{-p+1} \end{bmatrix} \underline{x}(k) \tag{6-58}$$

Comparing Eq. (6-58) with Eq. (6-42), we have

$$F = G \begin{bmatrix} C - DG \\ (C - DG)(A - BG)^{-1} \\ \cdot \\ \cdot \\ \cdot \\ (C - DG)(A - BG)^{-p+1} \end{bmatrix}^{-1} \tag{6-59}$$

if $mp = n$, or $p = n/m$, and the indicated inverse exists.

For a single-input, single-output system, $r = m = 1$. Furthermore, if $D = 0$, Eq. (6-59) is simplified to

$$F = K[d_o \quad d_1 \quad d_2 \quad \ldots \quad d_{n-1}] = G \begin{bmatrix} C \\ CA^{-1} \\ CA^{-2} \\ \cdot \\ \cdot \\ \cdot \\ CA^{-n+1} \end{bmatrix}^{-1} \tag{6-60}$$

<u>Example 6-2</u>

Consider the sampled-data process shown in Fig. 6-4. The z-transfer function of the process is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368} \tag{6-61}$$

The state equations of the system can be written in the form of Eq. (6-41) with

$$A = \begin{bmatrix} 0 & 1 \\ -0.368 & 1.368 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The output equation, Eq. (6-42), is

$$y(k) = C\underline{x}(k) = [0.264 \quad 0.368]\underline{x}(k) \tag{6-62}$$

Let the state feedback be denoted by

$$u(k) = -G\underline{x}(k) = -[g_1 \quad g_2]\underline{x}(k) \tag{6-63}$$

The characteristic equation of the closed-loop system is written

$$|(zI - A + BG)| = z^2 + (g_2 - 1.368)z + (g_1 + 0.368)$$

Let us select the feedback gains as $g_1 = 0.132$ and $g_2 = 0.368$ so that the eigenvalues are at

$$\lambda_1 = 0.5 + j0.5, \quad \lambda_2 = 0.5 - j0.5$$

To obtain an equivalent cascade controller to replace the state feedback, we let

$$H(z) = K\frac{z + \alpha_1}{z + \beta_1} \cong K(1 + d_1 z^{-1}) \tag{6-64}$$

Thus, Eq. (6-60) gives

$$F = K[1 \quad d_1] = G \begin{pmatrix} C \\ CA^{-1} \end{pmatrix}^{-1}$$

$$= [0.132 \quad 0.368] \begin{pmatrix} 0.264 & 0.368 \\ 1.35 & -0.717 \end{pmatrix}^{-1}$$

$$= 0.86[1 \quad -0.0708] \tag{6-65}$$

Then, $K = 0.86$, $d_1 = -0.0708$. From Eq. (6-48),

$$d_1 = \alpha_1 - \beta_1$$

Selecting $\beta_1 = 0.0005$, we have $\alpha_1 = -0.0703$. The transfer function of the feedback controller is

$$H(z) = 0.86 \frac{z - 0.0703}{z - 0.0005} \tag{6-66}$$

The overall system is shown in Fig. 6-5.

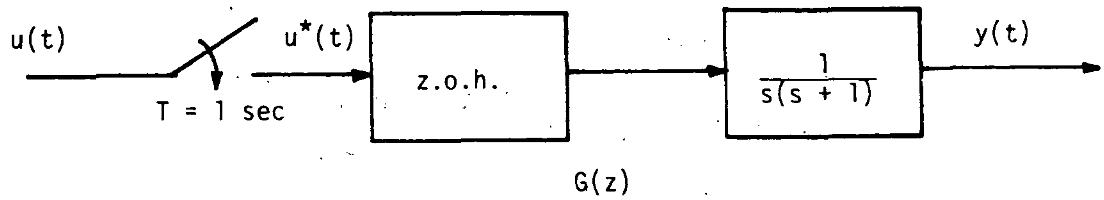


Figure 6-4. A sampled-data process.

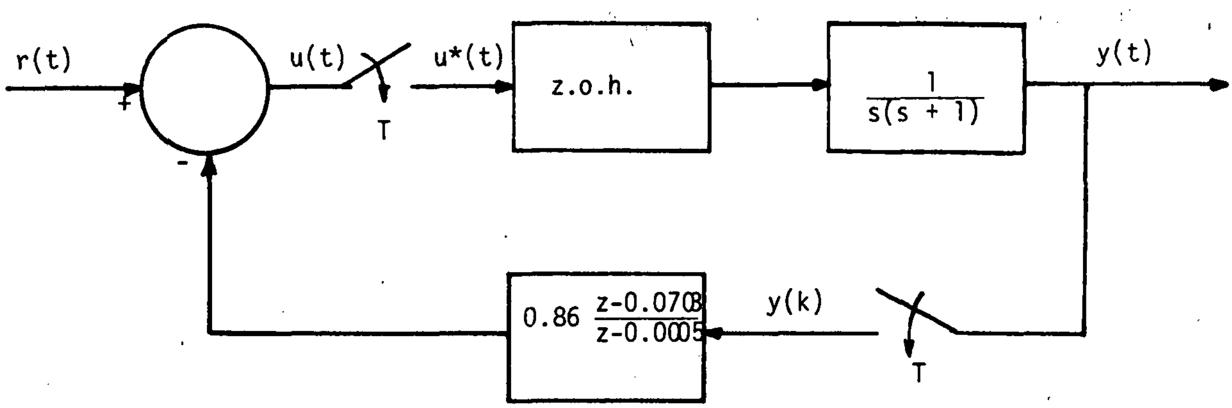


Figure 6-5. Closed-loop sampled-data system.

7. A Numerical Technique for Predicting Self-Sustained Oscillations in the Nonlinear LST System With the Continuous and Discrete Describing Function Methods

7.1 Introduction

It has been demonstrated [2] that the continuous and discrete describing function methods are useful tools for predicting the existence of self-sustained oscillations due to the nonlinear CMG friction in a single-axis model of the LST. The corresponding stability equations which have to be solved are

$$1 + N(A)G(j\omega) = 0 \qquad \text{(continuous case)} \tag{7-1}$$

and

$$1 + N(A, n)G(T, n) = 0 \qquad \text{(discrete case)} \tag{7-2}$$

In Reference [2], these equations were solved graphically, and the conditions for existence of self-sustained oscillations were established. In each case the intersection of the -1/N curves with the G curves of the system was used as a criterion for the solution of the stability equations, (7-1) and (7-2).

Although this approach is convenient in the case of a single nonlinearity, it becomes very cumbersome, if not impossible to use, when more than one nonlinearity exists, except in a few special cases [8]. For example, with two coupled LST axes, the stability equations in the continuous case may be of the form,

$$1 + G_1(j\omega)N(A) + G_2(j\omega)N^2(A) = 0 \tag{7-3}$$

Clearly, a graphical solution for Eq. (7-3) is impractical, and the situation very quickly deteriorates if more axes are added, or if the amplitudes of oscillations in the different axes are not the same.

In view of this situation, it appears worthwhile to consider an alternate method for solving the stability equations of the type of (7-1) through (7-3). In this chapter the results obtained by solving Eqs. (7-1) and (7-2) for the single-axis LST system by means of an iterative numerical method are presented. This method has been successful and provides numerical solutions to Eqs. (7-1) and (7-2), these solutions being identical to those obtained by the earlier graphical method. The method is promising and can be directly extended to the more complicated cases, as in Eq. (7-3).

7-2 The Numerical Method

Consider the set of two nonlinear equations

$$f_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) = 0 \tag{7-4}$$

or in more compact notation

$$\underline{F}(\underline{x}) = \underline{0} \tag{7-5}$$

where

$$\underline{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7-6}$$

An algorithm for an iterative solution of Eqs. (7-4) or (7-5) is

$$\underline{x}^{k+1} = \underline{x}^k - [\underline{F}_{\underline{x}}(\underline{x}^k)]^{-1}\underline{F}(\underline{x}^k) \tag{7-7}$$

where $\underline{x}^k$ is the value of $\underline{x}$ at the kth step, and $\underline{F}_{\underline{x}}$ is the Jacobian

$$\underline{F}_{\underline{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \tag{7-8}$$

The algorithm in Eq. (7-7) is the well-known Newton's method for the multivariable case, and can be shown to locally converge to the solution $\underline{x}^*$ of Eq. (7-5) if

i) $\underline{F}(\underline{x}^*) = 0$

and

ii) $\underline{F}_{\underline{x}}(\underline{x}^*)$ exists.

Recently, a numerical method has been proposed by Brown [9, 10] which is similar but computationally more efficient than Newton's method and still posesses the same convergence properties. This method is now used to solve Eqs. (7-1) and (7-2).

7.3 The Continuous Case

The stability equation in the continuous case is as in Eq. (7-1),

$$1 + N(A)G(j\omega) = 0 \tag{7-1}$$

where

ω is the frequency in rads/sec

A is the amplitude of the input sinusoid to the nonlinearity

N is the describing function

and G is the system transfer function seen by the nonlinearity.

Since N and G are both complex quantities, they may be written as

$$N(A) = N_R(A) + jN_I(A)$$

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega) \tag{7-9}$$

where N_R, N_I, G_R, and G_I are all real quantities.

Substituting Eq. (7-9) into Eq. (7-1) yields

$$1 + (N_R + jN_I)(G_R + jG_I) = 0 \tag{7-10}$$

Collecting the real and imaginary terms in Eq. (7-10) and equating them to zero gives

$$1 + N_R(A)G_R(j\omega) - N_I(A)G_I(j\omega) = 0$$

$$N_R(A)G_I(j\omega) + N_I(A)G_R(j\omega) = 0 \tag{7-11}$$

Using the notation of Eq. (7-5), Eq. (7-11) becomes

$$\underline{F}(\underline{x}) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7-12}$$

where

$$f_1 = 1 + N_R G_R - N_I G_I$$

$$f_2 = N_R G_I + N_I G_R$$

$$x_1 = \omega$$

$$x_2 = A \tag{7-13}$$

Thus, Eq. (7-11) represents a system of two simultaneous nonlinear equations in two unknowns. This system is solved by the proposed numerical method with the following system parameters

$T_{GFO} = 0.1$ ft-lb

$H = 600$ ft-lb-sec

$J_G = 2.1$ ft-lb-sec^2

$K_o = 5758.35$

$K_1 = 1371.02$

$K_p = 216$ ft-lb/rad/sec

$K_I = 9700$ ft-lb/rad

$J_V = 10^5$ ft-lb-sec^2

With $\gamma = 1.38 \times 10^7$, it is known from the graphical results [2] that two solutions to Eq. (7-11) exist. Figures 7-1 and 7-2 show the numerical iterations for several initial solutions. The two solutions are

1. $\omega = 4.27$ rad/sec
   $A = 5.57 \times 10^{-6}$ rad

2. $\omega = 1.748$ rad/sec
   $A = 4.45 \times 10^{-7}$ rad

Although solution 2 is an unstable equilibrium and solution 1 is a stable equilibrium, the numerical method does not differentiate between them. It converges on the solution in whose domain of attraction the inital solution is chosen. Figure 7-3 shows a graphical interpretation of the domains of attraction of the two solutions. If the initial

LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION	FREQUENCY	AMPLITUDE
0	0.4000000000D 01	0.1000000000D-04
1	0.4153389340D 01	0.9505734718D-06
2	0.3572611372D 01	0.2092881357D-05
3	0.4052445838D 01	0.3844786143D-05
4	0.4215366015D 01	0.5098580073D-05
5	0.4268157550D 01	0.5538061301D-05
6	0.4272124884D 01	0.5573295442D-05
7	0.4272145403D 01	0.5573466150D-05

ND= 1 FREQUENCY= 4.27215D 00RAD/SEC AMPLITUDE= 5.57347D-06

LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION	FREQUENCY	AMPLITUDE
0	0.5000000000D 01	0.1000000000D-04
1	0.3954077761D 01	0.3288577583D-05
2	0.4170249852D 01	0.4754914169D-05
3	0.4260352606D 01	0.5468966540D-05
4	0.4271952990D 01	0.5571790199D-05
5	0.4272145453D 01	0.5573467104D-05
6	0.4272145393D 01	0.5573465998D-05

ND= 1 FREQUENCY= 4.27215D 00RAD/SEC AMPLITUDE= 5.57347D-06

LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION	FREQUENCY	AMPLITUDE
0	0.6000000000D 01	0.1000000000D-04
1	0.4464704973D 01	0.7074854552D-05
2	0.4233071867D 01	0.5230806768D-05
3	0.4270025323D 01	0.5554770302D-05
4	0.4272140286D 01	0.5573426828D-05
5	0.4272145395D 01	0.5573466033D-05

ND= 1 FREQUENCY= 4.27215D 00RAD/SEC AMPLITUDE= 5.57347D-06

LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION	FREQUENCY	AMPLITUDE
0	0.7000000000D 01	0.1000000000D-04
1	0.4978234956D 01	0.9948595213D-05
2	0.3950753609D 01	0.3265377534D-05
3	0.4167741217D 01	0.4737468019D-05
4	0.4259855835D 01	0.5464563603D-05
5	0.4271936111D 01	0.5571641673D-05
6	0.4272145453D 01	0.5573467159D-05
7	0.4272145393D 01	0.5573465998D-05

ND= 1 FREQUENCY= 4.27215D 00RAD/SEC AMPLITUDE= 5.57347D-06

Figure 7-1. Numerical iterations in the continuous case; $\gamma = 1.38 \times 10^7$.

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION     FREQUENCY              AMPLITUDE
    0      0.5000000000D 01      0.5000000000D-05
    1      0.4330824789D 01      0.5716070923D-05
    2      0.4272669199D 01      0.5575178057D-05
    3      0.4272145539D 01      0.5573465283D-05
    4      0.4272145393D 01      0.5573466000D-05

ND= 1    FREQUENCY= 4.27215D 00RAD/SEC    AMPLITUDE= 5.57347D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION     FREQUENCY              AMPLITUDE
    0      0.5000000000D 01      0.1000000000D-05
    1      0.3986281834D 01      0.2781931045D-05
    2      0.4081011589D 01      0.4234094904D-05
    3      0.4242333400D 01      0.5306434465D-05
    4      0.4270849057D 01      0.5562073786D-05
    5      0.4272143796D 01      0.5573455629D-05
    6      0.4272145393D 01      0.5573466007D-05

ND= 1    FREQUENCY= 4.27215D 00RAD/SEC    AMPLITUDE= 5.57347D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION     FREQUENCY              AMPLITUDE
    0      0.3000000000D 01      0.1000000000D-05
    1      0.4106761584D 01      0.3045086482D-05
    2      0.4110062736D 01      0.4414687420D-05
    3      0.4249871806D 01      0.5373028097D-05
    4      0.4271417631D 01      0.5567084315D-05
    5      0.4272145079D 01      0.5573465364D-05
    6      0.4272145393D 01      0.5573465999D-05

ND= 1    FREQUENCY= 4.27215D 00RAD/SEC    AMPLITUDE= 5.57347D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 07

ITERATION     FREQUENCY              AMPLITUDE
    0      0.2000000000D 01      0.5000000000D-06
    1      0.1633524697D 01      0.2812970363D-06
    2      0.1712930384D 01      0.4060843547D-06
    3      0.1745466022D 01      0.4418355628D-06
    4      0.1748094452D 01      0.4451098805D-06
    5      0.1748104365D 01      0.4451182416D-06

ND= 1    FREQUENCY= 1.74810D 00RAD/SEC    AMPLITUDE= 4.45118D-07
```

Figure 7-2. Numerical iterations in the continuous case; $\gamma = 1.38 \times 10^7$.

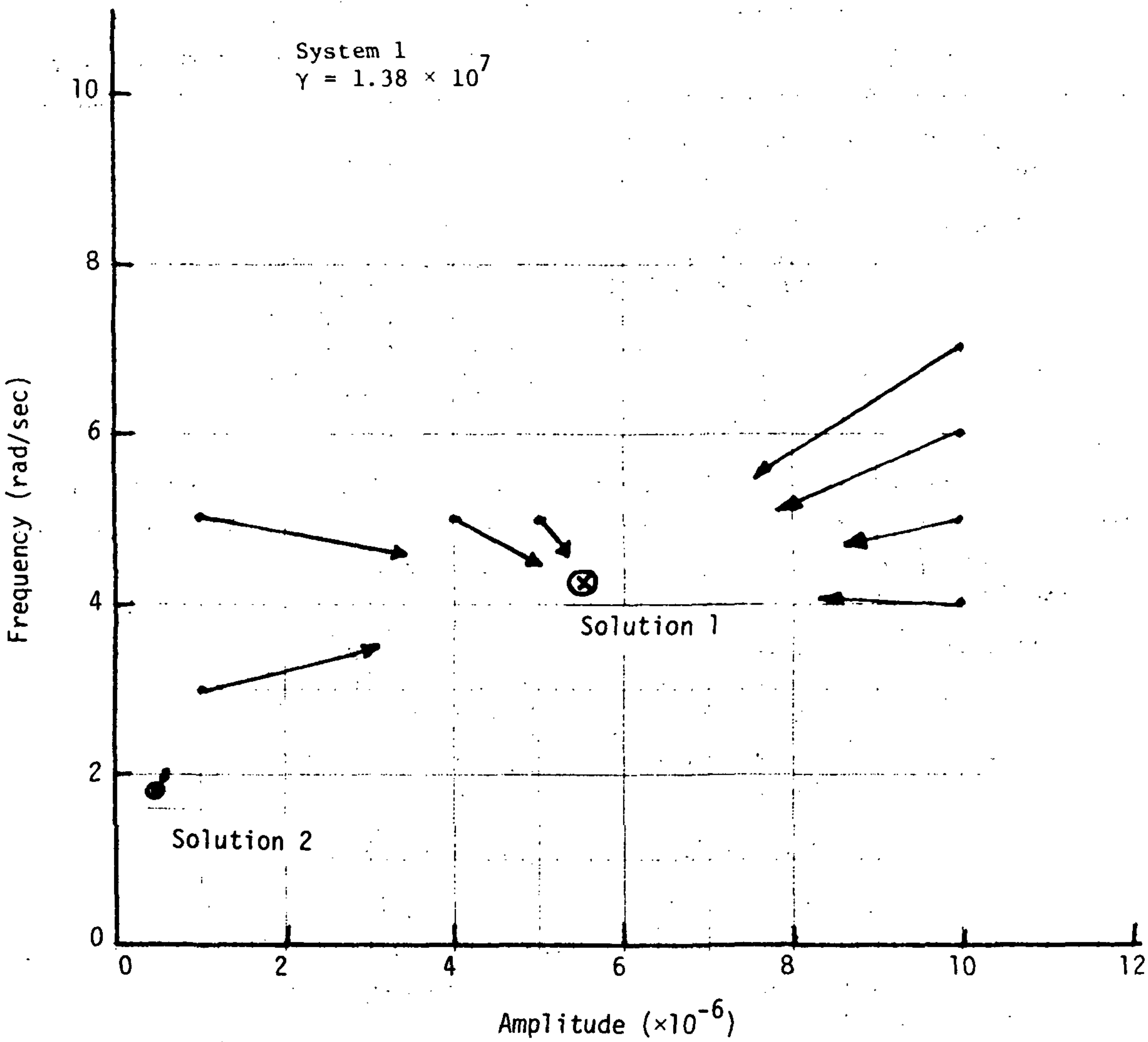


Figure 7-3. Domains of attraction of the two solutions with $\gamma = 1.38 \times 10^7$.

conditions are chosen outside of the domain of attraction, the method does not converge.

With changes in γ, the solution points shift, and Fig. 7-4 shows the stable solutions with $\gamma = 0.69 \times 10^8$ and 1.38×10^8 respectively.

7.4 The Discrete Case

In this case the input to the nonlinearity is assumed to be

$$e(t) = A\cos(\omega t + \phi) \tag{7-14}$$

and the stability equation is

$$1 + N(A, \phi, n)G(T, n, z) = 0 \tag{7-15}$$

where

T is the sampling period of the system

n is the order of oscillation, i.e., the period of oscillation is Tn.

A is the amplitude of the sinusoidal input to the nonlinearity

ϕ is the phase of this input relation to the sampling process

$z = \exp(j2\pi/nT)$

G is the z-domain transfer function seen by the nonlinearity

N is the discrete describing function of the nonlinearity.

To maintain consistency, Eq. (7-15) can have two and only two variables; thus, n and ϕ are assumed to be fixed parameters, and A and T are the two variables. With each value of n and ϕ, a solution of Eq. (7-15) is desired.

As in the continuous case, define

$$G(T) = G_R(T) + jG_I(T)$$

$$N(A) = N_R(A) + jN_I(A) \tag{7-16}$$

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 6.90000D 07

  ITERATION     FREQUENCY                AMPLITUDE
        0     0.7000000000D 01     0.1000000000D-04
        1     0.5323793060D 01     0.1064495497D-04
        2     0.5203573837D 01     0.8113476180D-05
        3     0.5220917925D 01     0.8556741622D-05
        4     0.5221777385D 01     0.8580953482D-05
        5     0.5221778880D 01     0.8580997113D-05

ND= 1    FREQUENCY= 5.22178D 00RAD/SEC    AMPLITUDE= 8.58100D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 6.90000D 07

  ITERATION     FREQUENCY                AMPLITUDE
        0     0.5000000000D 01     0.1000000000D-04
        1     0.5204808810D 01     0.8217060863D-05
        2     0.5221270619D 01     0.8566684031D-05
        3     0.5221778574D 01     0.8580987575D-05
        4     0.5221778879D 01     0.8580997080D-05

ND= 1    FREQUENCY= 5.22178D 00RAD/SEC    AMPLITUDE= 8.58100D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 08

  ITERATION     FREQUENCY                AMPLITUDE
        0     0.7000000000D 01     0.1000000000D-04
        1     0.5463694124D 01     0.1074650663D-04
        2     0.5412668486D 01     0.8894201414D-05
        3     0.5419316590D 01     0.9155958622D-05
        4     0.5419498774D 01     0.9163497173D-05
        5     0.5419498719D 01     0.9163495937D-05

ND= 1    FREQUENCY= 5.41950D 00RAD/SEC    AMPLITUDE= 9.16350D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF 1+N(A)G(W)=0.
GAMMA= 1.38000D 08

  ITERATION     FREQUENCY                AMPLITUDE
        0     0.5000000000D 01     0.1000000000D-04
        1     0.5396836346D 01     0.8992653820D-05
        2     0.5419441243D 01     0.9161503119D-05
        3     0.5419498763D 01     0.9163497435D-05
        4     0.5419498719D 01     0.9163495936D-05

ND= 1    FREQUENCY= 5.41950D 00RAD/SEC    AMPLITUDE= 9.16350D-06
```

Figure 7-4. Numerical iterations in the continuous case; $\gamma = 0.69 \times 10^8$ and 1.38×10^8.

Substituting Eq. (7-16) into Eq. (7-15) and separating the real and imaginary parts yields

$$\underline{F}(\underline{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7-17}$$

where

$$f_1 = 1 + N_R G_R - N_I G_I$$

$$f_2 = N_R G_I + N_I G_R$$

$$x_1 = T$$

$$x_2 = A \tag{7-18}$$

Equation (7-17) now represents two equations in two unknowns and can be solved by the proposed numerical method. With the same system parameters as in the continuous case, and $\gamma = 1.38 \times 10^7$, a solution for $n = 20$, $\phi = 0$ is obtained. The initial solution in this case is obtained from a knowledge of the continuous system solution. This solution, $n = 20$ and $\phi = 0$, is now used as an initial guess to determine the solution for $n = 18$, $\phi = 0$, which is then used as an initial solution to obtain the solution for $n = 16$, $\phi = 0$, and so on. Figures 7-5, 7-6 and 7-7 show the iterations for $n = 20$ through $n = 4$. Due to the different characteristics of the odd-n and even-n solutions, the decrement or increment of n is made in multiples of 2. This way, the solution for all n can be obtained.

Once a solution is available for a particular n at $\phi = 0$, the ϕ-space can be spanned by slowly varying ϕ and using the previous

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07   N=  20   PHI= 0.0
  ITERATION  SAMPLING PERIOD       AMPLITUDE
      0      0.6000000000D-01    0.1000000000D-04
      1      0.6635022584D-01    0.7168351284D-05
      2      0.6684849597D-01    0.7391591411D-05
      3      0.6684184809D-01    0.7402030884D-05
      4      0.6684185386D-01    0.7402032739D-05

ND= 1   SAMPLING PERIOD= 6.68419D-02SEC   AMPLITUDE= 7.40203D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07   N=  18   PHI= 0.0
  ITERATION  SAMPLING PERIOD       AMPLITUDE
      0      0.6684185385D-01    0.7402032738D-05
      1      0.7258981714D-01    0.7831800820D-05
      2      0.7332495463D-01    0.7741965186D-05
      3      0.7333309288D-01    0.7740553676D-05
      4      0.7333308321D-01    0.7740556323D-05

ND= 1   SAMPLING PERIOD= 7.33331D-02SEC   AMPLITUDE= 7.74056D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07   N=  16   PHI= 0.0
  ITERATION  SAMPLING PERIOD       AMPLITUDE
      0      0.7333308323D-01    0.7740556318D-05
      1      0.8035141927D-01    0.8308808703D-05
      2      0.8136810323D-01    0.8182471046D-05
      3      0.8138185695D-01    0.8180174756D-05
      4      0.8138184229D-01    0.8180178901D-05

ND= 1   SAMPLING PERIOD= 8.13818D-02SEC   AMPLITUDE= 8.18018D-06
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07   N=  14   PHI= 0.0
  ITERATION  SAMPLING PERIOD       AMPLITUDE
      0      0.8138184231D-01    0.8180178895D-05
      1      0.9022050932D-01    0.8961126589D-05
      2      0.9175128977D-01    0.8758961044D-05
      3      0.9177880263D-01    0.8754660548D-05
      4      0.9177877850D-01    0.8754667872D-05

ND= 1   SAMPLING PERIOD= 9.17788D-02SEC   AMPLITUDE= 8.75467D-06
```

Figure 7-5. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $\phi = 0$, variable n.

LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07 N= 12 PHI= 0.0

ITERATION	SAMPLING PERIOD	AMPLITUDE
0	0.9177877853D-01	0.8754667861D-05
1	0.1033226304D 00	0.9891276675D-05
2	0.1058887009D 00	0.9522086634D-05
3	0.1059569620D 00	0.9512652996D-05
4	0.1059569299D 00	0.9512665872D-05

ND= 1 SAMPLING PERIOD= 1.05957D-01SEC AMPLITUDE= 9.51267D-06

LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07 N= 10 PHI= 0.0

ITERATION	SAMPLING PERIOD	AMPLITUDE
0	0.1059569300D 00	0.9512665855D-05
1	0.1216012311D 00	0.1131708575D-04
2	0.1264908711D 00	0.1056350696D-04
3	0.1267133857D 00	0.1053796573D-04
4	0.1267135528D 00	0.1053796318D-04

ND= 1 SAMPLING PERIOD= 1.26714D-01SEC AMPLITUDE= 1.05380D-05

LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07 N= 8 PHI= 0.0

ITERATION	SAMPLING PERIOD	AMPLITUDE
0	0.1267135526D 00	0.1053796320D-04
1	0.1484753967D 00	0.1378146933D-04
2	0.1593391474D 00	0.1210637147D-04
3	0.1604043798D 00	0.1200135682D-04
4	0.1604109954D 00	0.1200077001D-04

ND= 1 SAMPLING PERIOD= 1.60411D-01SEC AMPLITUDE= 1.20008D-05

LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07 N= 6 PHI= 0.0

ITERATION	SAMPLING PERIOD	AMPLITUDE
0	0.1604109885D 00	0.1200077106D-04
1	0.1897914063D 00	0.1878650442D-04
2	0.2182241830D 00	0.1497205004D-04
3	0.2272750301D 00	0.1410272682D-04
4	0.2273371225D 00	0.1408970958D-04
5	0.2273765514D 00	0.1408969070D-04

ND= 1 SAMPLING PERIOD= 2.27738D-01SEC AMPLITUDE= 1.40897D-05

Figure 7-6. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $\phi = 0$, variable n.

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=    4    PHI= 0.0
  ITERATION   SAMPLING PERIOD        AMPLITUDE
         0     0.2277376508D 00     0.1408969073D-04
         1     0.2545144363D 00     0.2792562588D-04
         2     0.3119051493D 00     0.2584222751D-04
         3     0.4186570592D 00     0.1482765201D-04
         4     0.5071624020D 00     0.1557192198D-04
         5     0.5354877655D 00     0.1584931294D-04
         6     0.5371469161D 00     0.1587070593D-04
         7     0.5371503035D 00     0.1587076013D-04

ND= 1    SAMPLING PERIOD= 5.37150D-01SEC    AMPLITUDE= 1.58708D-05
```

Figure 7-7. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $\phi = 0$, variable n.

solution as the initial guess for the next solution. The range of ϕ is $2\pi/n$ for n even and π/n for n odd. Figures 7-8 through 7-11 show the iterations for n = 3 and variable ϕ. A plot of all solutions of T and A for n = 3 and various ϕ's is shown in Fig. 7-12. Here the plots for n = 4, 6, 8 and 10 are also shown. As n increases the range of ϕ decreases and the plots of T and A with ϕ as a parameter shrink to single points. Figure 7-13 shows the plot of T and A with $\phi = 0$ for n = 10 through n = 100.

The plots of Figs. 7-12 and 7-13 can be used to determine the frequency ($2\pi/nT$) and amplitude (A) of self-sustained oscillations when the sampling period T is given. They provide very concise information on the conditions under which self-sustained oscillations can occur in the discrete-data system. The corresponding solutions by the graphical method require overlapping the -1/N and G curves for each n and checking for intersection or containment.

As n increases beyond 100, the discrete solution asymptotes towards the continuous solution.

7.5 Conclusions and Extensions

The results of the previous sections have demonstrated the ease and effectiveness of the proposed numerical method. It provides the exact frequency and amplitude of oscillation in comparison to the approximate values obtained by the graphical technique. In the discrete case particularly, the numerical approach provides the information in a more convenient form. The frequency and amplitude combinations which can exist for each value of sampling period are available from a single curve.

The major limitation of the numerical method is the convergence characteristics of the solutions. Unless an adequate initial solution is available, no useful information can be obtained. The initial solution must be near to or within the domain of attraction of the exact solution. Once a solution is obtained, it is simple to slowly vary the parameters and obtain all the desired solutions.

The method is unrestricted to the form of the stability equation or the number of nonlinearities present. If adequate initial solutions can be selected, this approach can yield useful results with the more complicated multiple-axis models of the LST.

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 6.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0      0.1000000000D 01     0.2000000000D-04
       1      0.1206736268D 01     0.2180807968D-04
       2      0.1246507649D 01     0.2200966570D-04
       3      0.1247557574D 01     0.2201127396D-04
       4      0.1247557444D 01     0.2201127250D-04

ND= 1    SAMPLING PERIOD= 1.24756D 00SEC    AMPLITUDE= 2.20113D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 5.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0      0.1247557444D 01     0.2201127250D-04
       1      0.7576140716D 00     0.2210012199D-04
       2      0.8593652656D 00     0.2212015179D-04
       3      0.8710667172D 00     0.2219954845D-04
       4      0.8711804237D 00     0.2220037576D-04
       5      0.8711803498D 00     0.2220037467D-04

ND= 1    SAMPLING PERIOD= 8.71180D-01SEC    AMPLITUDE= 2.22004D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 4.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0      0.8711803498D 00     0.2220037468D-04
       1      0.6048096982D 00     0.2299103460D-04
       2      0.6500626463D 00     0.2334330100D-04
       3      0.6527787250D 00     0.2342663401D-04
       4      0.6527844416D 00     0.2342710075D-04

ND= 1    SAMPLING PERIOD= 6.52784D-01SEC    AMPLITUDE= 2.34271D-05
```

Figure 7-8. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 3$, variable ϕ.

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 3.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0     0.6527844384D 00     0.2342710022D-04
       1     0.4972491700D 00     0.2522586042D-04
       2     0.5185445957D 00     0.2594052161D-04
       3     0.5191962437D 00     0.2604025247D-04
       4     0.5191954834D 00     0.2604065065D-04

ND= 1    SAMPLING PERIOD= 5.19195D-01SEC    AMPLITUDE= 2.60407D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 2.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0     0.5191954849D 00     0.2604065029D-04
       1     0.4479887235D 00     0.2776194476D-04
       2     0.4542599663D 00     0.2831780647D-04
       3     0.4542548541D 00     0.2835057365D-04
       4     0.4542548401D 00     0.2835057996D-04

ND= 1    SAMPLING PERIOD= 4.54255D-01SEC    AMPLITUDE= 2.83506D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 1.00000D 01
  ITERATION  SAMPLING PERIOD          AMPLITUDE
       0     0.4542548402D 00     0.2835057996D-04
       1     0.6655150491D 00     0.1157357946D-04
       2     0.7303889506D 00     0.1642213821D-04
       3     0.6664793457D 00     0.1959748765D-04
       4     0.6559819499D 00     0.2045437953D-04
       5     0.6553780327D 00     0.2049946525D-04
       6     0.6553777502D 00     0.2049953882D-04

ND= 1    SAMPLING PERIOD= 6.55378D-01SEC    AMPLITUDE= 2.04995D-05
```

Figure 7-9. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 3$, variable ϕ.

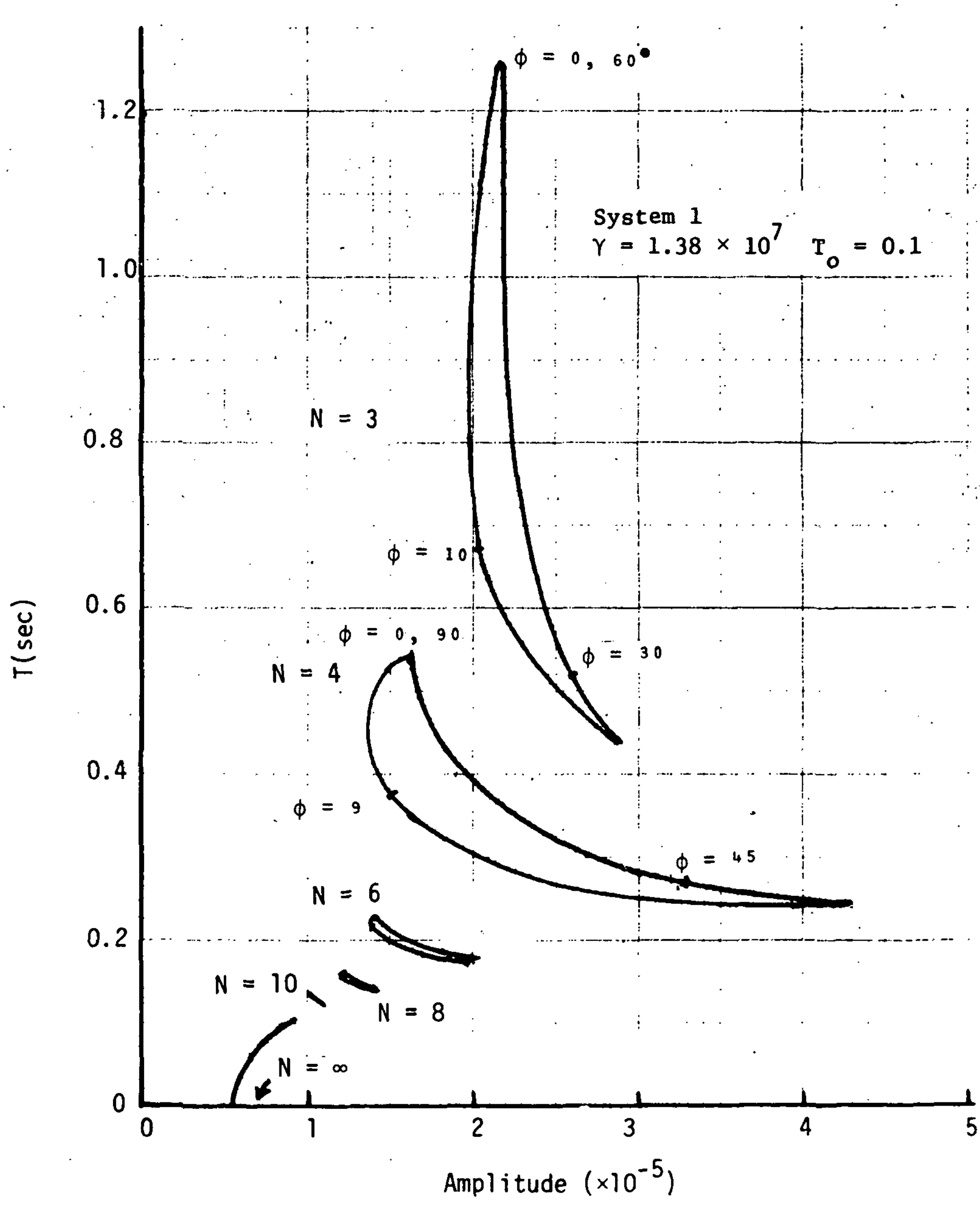


Figure 7-12. Amplitude and frequency ($2\pi/nT$) of self sustained oscillations for various sampling periods in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 3$ through $n = 10$.

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 4.00000D 00
  ITERATION  SAMPLING PERIOD        AMPLITUDE
      0      0.9443921729D 00     0.2000431869D-04
      1      0.1098499702D 01     0.2069887736D-04
      2      0.1120422983D 01     0.2075227791D-04
      3      0.1120755444D 01     0.2075252072D-04
      4      0.1120755263D 01     0.2075252039D-04

ND= 1    SAMPLING PERIOD= 1.12076D 00SEC    AMPLITUDE= 2.07525D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=   3    PHI= 2.00000D 00
  ITERATION  SAMPLING PERIOD        AMPLITUDE
      0      0.1120755263D 01     0.2075252039D-04
      1      0.1243920308D 01     0.2159514112D-04
      2      0.1255495749D 01     0.2162959060D-04
      3      0.1255574083D 01     0.2162960044D-04

ND= 1    SAMPLING PERIOD= 1.25557D 00SEC    AMPLITUDE= 2.16296D-05
```

Figure 7-11. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 3$, variable ϕ.

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=    3    PHI= 1.00000D 01
  ITERATION   SAMPLING PERIOD        AMPLITUDE
       0       0.6000000000D 00     0.2000000000D-04
       1       0.6530111015D 00     0.2038544865D-04
       2       0.6553859033D 00     0.2049873663D-04
       3       0.6553777320D 00     0.2049953953D-04

ND= 1    SAMPLING PERIOD= 6.55378D-01SEC    AMPLITUDE= 2.04995D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=    3    PHI= 8.00000D 00
  ITERATION   SAMPLING PERIOD        AMPLITUDE
       0       0.6553777518D 00     0.2049953873D-04
       1       0.7703562041D 00     0.1958935672D-04
       2       0.7841108478D 00     0.1982121919D-04
       3       0.7842075811D 00     0.1982549843D-04
       4       0.7842075975D 00     0.1982549437D-04

ND= 1    SAMPLING PERIOD= 7.84208D-01SEC    AMPLITUDE= 1.98255D-05
```

```
LST SYSTEM-NUMERICAL SOLUTION OF DISCRETE DESCRIBING FUNCTION
GAMMA= 1.38000D 07    N=    3    PHI= 6.00000D 00
  ITERATION   SAMPLING PERIOD        AMPLITUDE
       0       0.7842075974D 00     0.1982549438D-04
       1       0.9235810128D 00     0.1990482637D-04
       2       0.9440712611D 00     0.2000313242D-04
       3       0.9443923074D 00     0.2000432019D-04
       4       0.9443921728D 00     0.2000431868D-04

ND= 1    SAMPLING PERIOD= 9.44392D-01SEC    AMPLITUDE= 2.00043D-05
```

Figure 7-10. Numerical iterations in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 3$, variable ϕ.

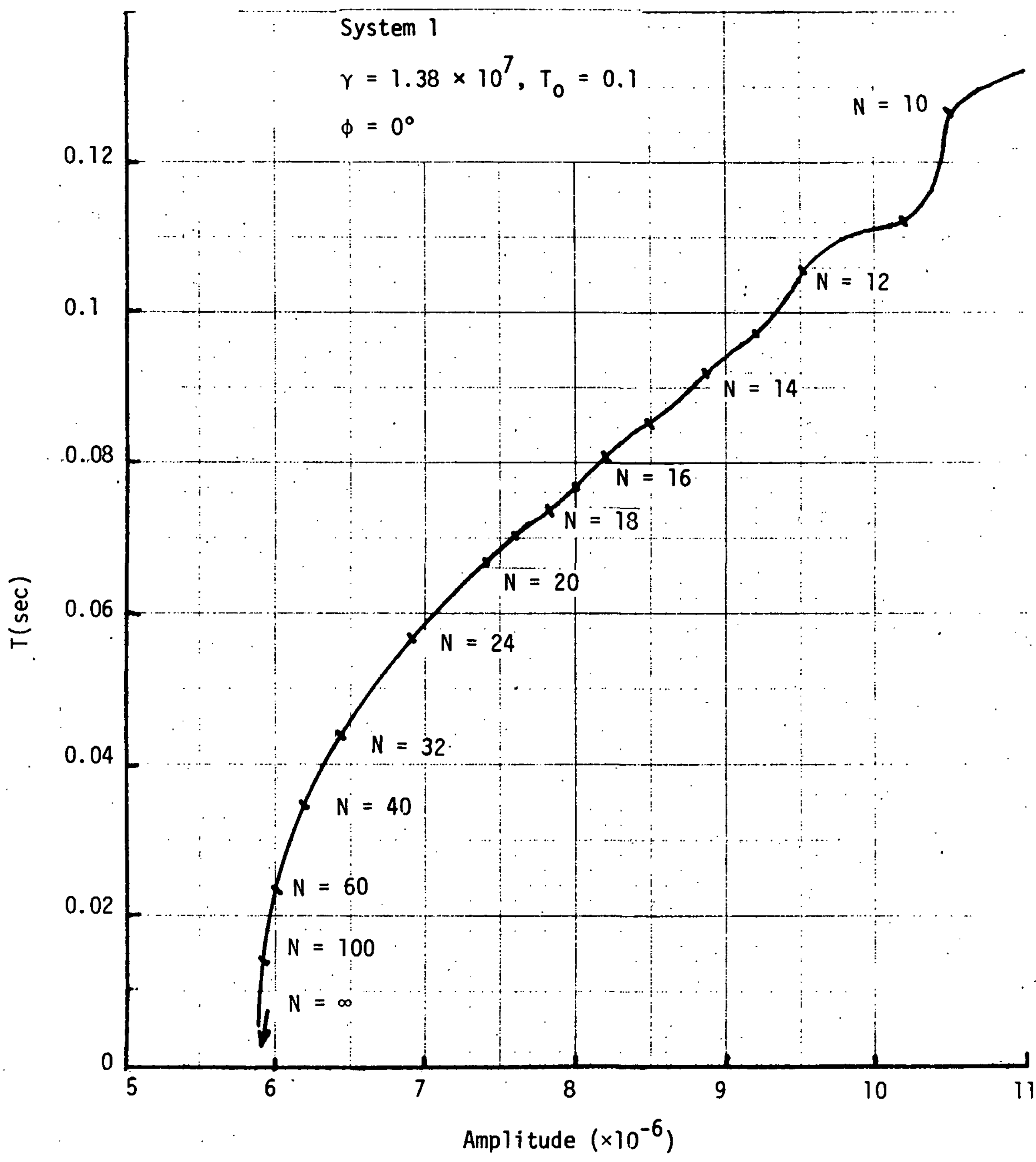


Figure 7-13. Amplitude and frequency ($2\pi/nT$) of self sustained oscillations for various sampling periods in the discrete case; $\gamma = 1.38 \times 10^7$, $n = 10$ through $n = 100$.

References

1. HEAO Supplemental Information: Gimbal Control System Analysis, Sperry Flight Systems Division, Pheonix, Arizona, May, 1972.

2. B. C. Kuo and G. Singh, Continuous and Discrete Describing Function Analysis of the LST System, Final Report, for NAS8-29853, Systems Research Laboratory, Champaign, Illinois, January 1, 1974.

3. B. D. O. Anderson and J. B. Moore, Linear Optimal Control, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971.

4. M. Healey, 'Study of Methods of Computing Transition Matrices', Proceedings IEE, Vol. 120, No. 8, August, 1973, pp. 905-912.

5. B. C. Kuo, G. Singh, R. A. Yackel, Research Study on Stabilization and Control-Modern Sampled-Data Control Theory, Final Report for NAS8-28584, Systems Research Laboratory, Champaign, Illinois, April, 1973.

6. B. C. Kuo, G. Singh, and R. A. Yackel, "Digital Approximation of Continuous-Data Systems by Point-by-Point State Comparison," Computer and Electrical Engineering, Vol. 1, pp. 155-170, 1973.

7. B. C. Kuo and G. Singh, Design of the Large Space Telescope Control System, Bimonthly Report I-74, for NAS8-29853, Systems Research Laboratory, Champaign, Illinois, March 1, 1974.

8. E. J. Davison and D. Constantinescu, "A Describing Function Technique for Multiple Nonlinearities in a Single-Loop Feedback System," IEEE Transactions on Automatic Control, Vol. AC-16, February 1971, pp. 56-60.

9. K. M. Brown and S. D. Conte, "The Solution of Simultaneous Nonlinear Equations," Proceedings ACM 22nd National Conference, pp. 111-114.

10. K. M. Brown, "Solution of Simultaneous Nonlinear Equations," Collected Algorithms from CACM. Algorithm 316.

☆ U.S. GOVERNMENT PRINTING OFFICE: 1974—635-044/32

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons. Also includes conference proceedings with either limited or unlimited distribution.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include final reports of major projects, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546