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Matrix Methods for the Design of Cascades to Prescribed Surface Velocity Distributions and for Fully Compressible Flow

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This paper describes matrix methods that have been developed for calculating compressible flow on a blade-to-blade surface of revolution. The methods have been fully tested to date only for the design of plane cascades to prescribed blade surface distributions; the methods will be illustrated here for that problem only. Similar methods are presently being applied to both the direct and indirect problems and for flow on arbitrary surfaces of revolution in annular cascades with stream sheet thickness variations. It is believed that by such methods, both the direct and indirect calculations can be reduced to about 60 to 90 seconds of computing.

The trend in compressor and turbine design is toward fewer and more highly loaded stages. To do this and maintain high efficiency demands the ability to calculate in ever-increasing detail the gas flow through such a machine. So complex are the equations governing the flow and the geometries involved that practicable solutions can be found only after making simplifying assumptions. The degree of approximation is always a compromise between a realistic description of the physical processes and a mathematical model that can be solved within reasonable time and cost. This has led to design procedures which treat the flow in two stages—a two-dimensional through-flow calculation which neglects circumferential variations, followed by a two-dimensional blade-to-blade calculation in which the flow is assumed to take place on a surface of revolution. Although fully three-dimensional calculations are being attempted, these are slow and costly and have a long way to go before they become design tools.

It seems likely, therefore, that for a few years to come, two-dimensional approaches will remain the basis of most design work and, for this reason, it is worthwhile to make these calculations as realistic and fast as possible.

Methods will be described here for calculations on blade-to-blade surfaces of revolution. These methods are being applied both to the direct problem of calculating blade surface velocities when the blade geometries are prescribed and to the indirect problem of calculating the blade geometry when the blade surface velocities are prescribed. The methods will be illustrated by discussing the indirect problem for compressible flow in a plane cascade. This has been chosen because it is the only problem for which the methods have been fully tested to date and because the authors have seen no other fully compressible solution to this problem. It is believed that the methods described here extend easily to both the direct and indirect problems on surfaces of revolution with stream sheet thickness variations.

MATHEMATICAL ANALYSIS

Assumptions

The following assumptions have been made.

(1) The flow is steady, inviscid and irrotational.
(2) The fluid is a perfect gas.
(3) The total temperature is uniform across the entry to the cascade.
(4) The flow is plane two-dimensional flow and the normal component of velocity is zero on the blade surface.
(5) The cascade contains an infinite number of equally spaced blades of infinite length.

The assumption of irrotationality, together with the finite difference approximations to the differential equations and the boundary-value approach to the solution of the finite difference equations, tacitly assume that the flow is everywhere subsonic. However, the method will formally produce answers with supersonic patches and, where these are small and the peak Mach numbers only a little above sonic, these solutions are probably realistic.

Equations of Motion

In the analysis that follows, x and y are Cartesian coordinates with x measured in the "axial" direction and y in the "pitchwise" direction, as shown in figure 1. Velocities and density are normalized with respect to the stagnation sound speed and stagnation density, respectively.

The equations governing the flow are those of irrotationality and continuity which are, respectively

$$\frac{\partial V_x}{\partial y}-\frac{\partial V_y}{\partial x}=0 \tag{1}$$

$$\frac{\partial}{\partial x}(\rho V_x)+\frac{\partial}{\partial y}(\rho V_y)=0 \tag{2}$$

Density is related to velocity through Bernoulli's equation

$$\rho=\left\{1-\frac{\gamma-1}{2}(V_x^2+V_y^2)\right\}^{1/(\gamma-1)}$$

Equations (1) and (2) may be satisfied identically by a potential function ϕ and stream function ψ defined by

$$\frac{\partial\phi}{\partial x}=V_x$$

$$\frac{\partial\phi}{\partial y}=V_y$$

$$\frac{\partial\psi}{\partial y}=\rho V_x$$

$$\frac{\partial\psi}{\partial x}=-\rho V_y$$

It will be convenient also to work in terms of the net velocity V and flow direction θ, related to V_x and V_y by the equations

$$V_x=V\cos\theta$$

$$V_y=V\sin\theta$$

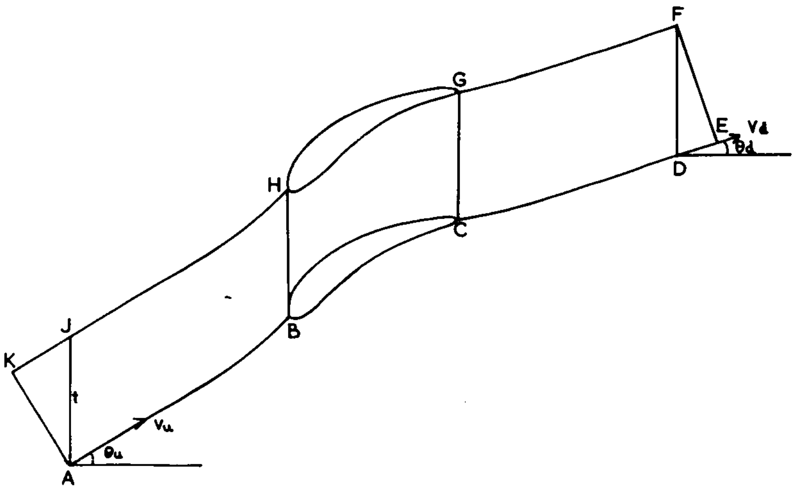

FIGURE 1.—*One strip of the cascade in the physical plane.*

If we now use ϕ and ψ as independent variables instead of x and y, equations (1) and (2) become

$$\rho V \frac{\partial V}{\partial \psi} - V^2 \frac{\partial \theta}{\partial \phi} = 0 \tag{3}$$

$$V \frac{\partial}{\partial \phi} (\rho V) + (\rho V)^2 \frac{\partial \theta}{\partial \psi} = 0 \tag{4}$$

and Bernoulli's equation is

$$\rho = \left[1 - \frac{\gamma - 1}{2} V^2\right]^{1/(\gamma-1)} \tag{5}$$

At this stage, Stanitz (ref. 1) linearized equations (3) and (4) by approximating equation (5) by

$$\rho = \frac{1}{\sqrt{1+V^2}}$$

At the equivalent stage in the direct problem, other workers have arranged the equations either in the form of a pseudo Poisson's equation, collecting the terms describing incompressible effects on the left in the form of a Laplacian and the terms describing compressible effects on the right in the form of a source term; or they have arranged the equations in the form of a general partial differential equation in which the coefficients contained derivatives of the density ρ. Finite difference and singularity methods have then been used to solve the equations in these forms iteratively by guessing the source term or coefficients, solving as though the equations were linear, and then re-estimating the terms that had been guessed. Iterative methods based on these forms of arrangements of the equations converge slowly at high Mach numbers because the guessed terms are by no means small perturbations and important contributions are left "trailing" one cycle behind in the iterations.

In order to introduce compressibility effects quickly into an iterative method, the authors consider it better to use Bernoulli's equation to express the derivatives of ρ in terms of those of the dependent variable and then to collect together all terms containing any particular derivative of that variable. The coefficients of these variables then do not contain derivatives of ρ which have to be guessed. For the indirect cascade problem considered here, the term $\partial(\rho V)/\partial\phi$ in equation (4) should not be expressed as $\rho(\partial V/\partial\phi) + V(\partial\rho/\partial\phi)$ with ρ and $\partial\rho/\partial\phi$ being guessed. Instead, equation (5) should be used to obtain

$$d(\rho V) = \rho \left(\frac{1 - [(\gamma+1)/2]V^2}{1 - [(\gamma-1)/2]V^2}\right) dV$$

so that equations (3) and (4) become

$$\frac{\rho}{V}\frac{\partial V}{\partial \psi}-\frac{\partial \theta}{\partial \phi}=0 \tag{6}$$

$$\frac{1}{\rho V}\left(\frac{1-[(\gamma+1)/2]V^2}{1-[(\gamma-1)/2]V^2}\right)\frac{\partial V}{\partial \phi}+\frac{\partial \theta}{\partial \psi}=0 \tag{7}$$

If one second-order equation was to be obtained by eliminating between (6) and (7), then, again, the derivatives of ρ introduced should be expressed in terms of those of V. For this problem there is, however, a neater approach. Define F and H by

$$dF=\frac{\rho}{V}\,dV \tag{8}$$

$$dH=\frac{1}{\rho V}\left(\frac{1-[(\gamma+1)/2]V^2}{1-[(\gamma-1)/2]V^2}\right)dV \tag{9}$$

so that equations (6) and (7) become

$$\frac{\partial H}{\partial \phi}+\frac{\partial \theta}{\partial \psi}=0 \tag{10}$$

$$\frac{\partial F}{\partial \psi}-\frac{\partial \theta}{\partial \phi}=0 \tag{11}$$

Using equation (5), equation (8) may be integrated directly for some values of γ. Taking $\gamma=\frac{4}{3}$ and writing $z=V^2/6$, we have

$$F(V)=\log V-\frac{3z}{2}+\frac{3z^2}{4}-\frac{z^3}{6} \tag{12}$$

Taking $\gamma=\frac{7}{5}$ and writing $z^2=1-(V^2/5)$, we have

$$F(V)=\log V+\frac{z^5}{5}+\frac{z^3}{3}+z-\log\left(\frac{1+z}{2}\right)-\frac{23}{15} \tag{13}$$

In each case, the constant of integration has been chosen such that $F(V)\to\log V$ as $V\to 0$.

The function F will now be taken as the dependent variable and equation (10) written in the form

$$\frac{dH}{dF}\frac{\partial F}{\partial \phi}+\frac{\partial \theta}{\partial \psi}=0 \tag{14}$$

where, from (8) and (9),

$$\frac{dH}{dF}=\frac{1}{\rho^2}\left(\frac{1-[(\gamma+1)/2]V^2}{1-[(\gamma-1)/2]V^2}\right)$$

The Potential and Deflection Conditions

In figure 1, which shows one strip of the cascade, AB, CE, GF and KH are dividing streamlines. At any point (x,y) on AB, the flow conditions are the same as at the point $(x,y+t)$ on JH and similarly for CD and GF. The lines KA and FE are far upstream and downstream of the cascade, where flow conditions are uniform. Figure 2 shows the same diagram mapped into the (ϕ,ψ)-plane with A, B, C, D, and E chosen as $\psi=0$. From the definitions of ϕ and ψ, it follows that

$$d\phi=V\,ds \tag{15a}$$

$$d\psi=\rho V\,dn \tag{15b}$$

Define

$$\Delta\phi_L=\phi_H-\phi_B$$

$$\Delta\phi_T=\phi_G-\phi_C$$

$$\Delta\phi_S=\phi_C-\phi_B$$

$$\Delta\phi_P=\phi_G-\phi_H$$

Clearly

$$\Delta\phi_L=\phi_J-\phi_K$$

$$\Delta\phi_T=\phi_E-\phi_D$$

From (15a), remembering that J, K, E and D are far from the cascade,

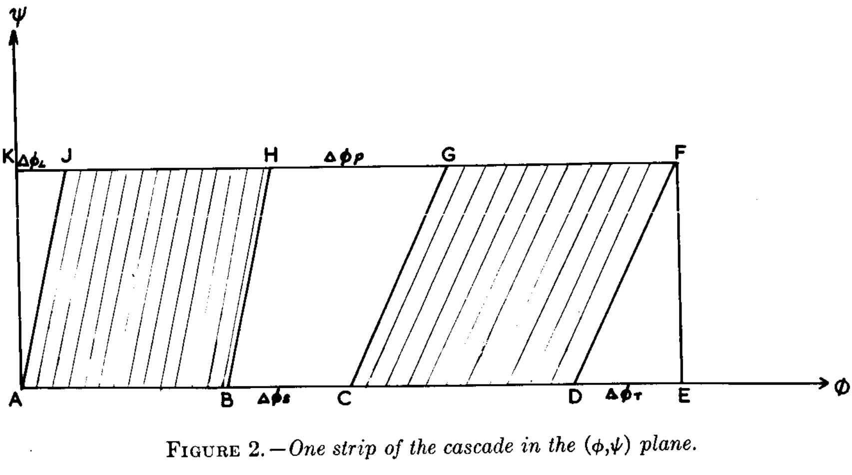

FIGURE 2.—*One strip of the cascade in the (ϕ,ψ) plane.*

$$\Delta\phi_L = V_u t \sin\theta_u$$

$$\Delta\phi_T = V_d t \sin\theta_d$$

It is also clear that

$$\Delta\phi_S - \Delta\phi_P = \Delta\phi_L - \Delta\phi_T$$

so that

$$\Delta\phi_S - \Delta\phi_P = t(V_u \sin\theta_u - V_d \sin\theta_d) \tag{16}$$

Equation (16) will be called the potential condition.

From equation (11), we have

$$\oint_{AEFK} F\,d\phi + \theta\,d\psi = 0$$

from which it follows that

$$\int_B^C F\,d\phi - \int_H^G F\,d\phi = \psi_0(\theta_u - \theta_d) + F_u\Delta\phi_L - F_d\Delta\phi_T \tag{17}$$

where ψ_0 is the value of ψ along KF. Equation (17) will be called the deflection condition.

In the indirect problem, the velocity on the blade surfaces is prescribed as a function of fractional arc length S' measured from $S'=0$ at the leading edge stagnation point and $S'=1$ at the trailing edge stagnation point. Let these velocity distributions be $V_S(S')$ and $V_P(S')$ along the suction and pressure surfaces. If L_S and L_P represent the physical lengths of these surfaces, measured between stagnation points, the potential and deflection conditions may be written

$$L_S \int_0^1 V_S\,dS' - L_P \int_0^1 V_P\,dS' = t(V_u \sin\theta_u - V_d \sin\theta_d) \tag{18}$$

and

$$L_S \int_0^1 (VF)_S\,dS' - L_P \int_0^1 (VF)_P\,dS' = \psi_0(\theta_u - \theta_d) + F_u\Delta\phi_L - F_d\Delta\phi_T \tag{19}$$

From the prescribed velocity distributions and upstream and downstream conditions, the corresponding values of F may be found from (12) and (13) and L_S and L_P from (18) and (19). The lengths in the (ϕ,ψ)-plane, $\Delta\phi_S$ and $\Delta\phi_P$, may then be found from

$$\Delta\phi_S = L_S \int_0^1 V_S\,dS'$$

$$\Delta\phi_P = L_P \int_0^1 V_P\,dS'$$

and the diagram of the (ϕ,ψ)-plane constructed. Eliminating θ between equations (11) and (14) gives

$$\frac{\partial^2 F}{\partial \psi^2}+\frac{\partial}{\partial \phi}\left(\frac{dH}{dF}\frac{\partial F}{\partial \phi}\right)=0 \tag{20}$$

To determine the blade shape corresponding to the prescribed surface velocity distributions and far upstream and downstream conditions, we have to solve equation (20) inside and on the contour $ADFJ$, subject to the boundary conditions:

(1) F is prescribed on BC, HG, AJ and DF
(2) Along AB and JH

$$F(\phi,0)=F(\phi+\Delta\phi_L, \psi_0) \tag{21a}$$

$$\theta(\phi,0)=\theta(\phi+\Delta\phi_L, \psi_0) \tag{21b}$$

(3) Along CD and GF

$$F(\phi,0)=F(\phi+\Delta\phi_T, \psi_0) \tag{21c}$$

$$\theta(\phi,0)=\theta(\phi+\Delta\phi_T, \psi_0) \tag{21d}$$

Transformation of the (ϕ, ψ)-Plane

There are a number of possible approaches to a numerical solution of this boundary-value problem. The one given here involves an approximate transformation of the (ϕ,ψ)-plane and some tedious algebra. However, the error in the transformation can be controlled so that it is less than that involved in the numerical methods and leads to a boundary-value problem posed in a form for which this is a quick and elegant method of solution.

First, in order to get a good spacing of points on a finite difference grid and not to map part of the suction surface twice, it is convenient to invert the diagram in the (ϕ,ψ)-plane through a transformation $\psi \to \psi_0-\psi$. We can achieve this without altering the equations if we make the additional transformation $\theta \to -\theta$. In what follows, this transformation will be assumed to have been made. Define new variables ϕ' and ψ' through the equations

$$\psi=\frac{\psi_0}{\alpha}\psi' \tag{22a}$$

$$\phi=\frac{\Delta\phi_P}{2}+\frac{\Delta\phi_P}{\beta}\left[\phi'+\frac{\psi'}{\alpha}(a_1+a_2\tanh\phi')\right] \tag{22b}$$

where

$$a_1 = \frac{\beta}{\Delta\phi_P} \frac{\Delta\phi_T + \Delta\phi_L}{2}$$

$$a_2 = \frac{\beta}{\Delta\phi_P} \frac{\Delta\phi_T - \Delta\phi_L}{2}$$

The constant α is merely a scaling factor which can be chosen freely; β is a constant which, for values of $\phi' \geq \beta/2$, makes $\tanh \phi' \simeq 1$. This transformation approximately maps the contour $ADFJ$ of the (ϕ,ψ)-plane into three rectangular regions in the (ϕ',ψ')-plane as shown in figure 3. If we write

$$a' = \frac{\beta/\Delta\phi_P}{1 + (\psi/\psi_0) a_2 \operatorname{sech}^2 \phi'}$$

$$b' = \frac{-(a_1 + a_2 \tanh \phi')}{\psi_0[1 + (\psi/\psi_0) a_2 \operatorname{sech}^2 \phi']}$$

then we have

$$\left(\frac{\partial}{\partial\phi}\right)_\psi = a' \left(\frac{\partial}{\partial\phi'}\right)_{\psi'}$$

$$\left(\frac{\partial}{\partial\psi}\right)_\phi = b' \left(\frac{\partial}{\partial\phi'}\right)_{\psi'} + \frac{\alpha}{\psi_0} \left(\frac{\partial}{\partial\psi'}\right)_{\phi'}$$

Writing $\dot{H}$ for dH/dF, equation (20) becomes

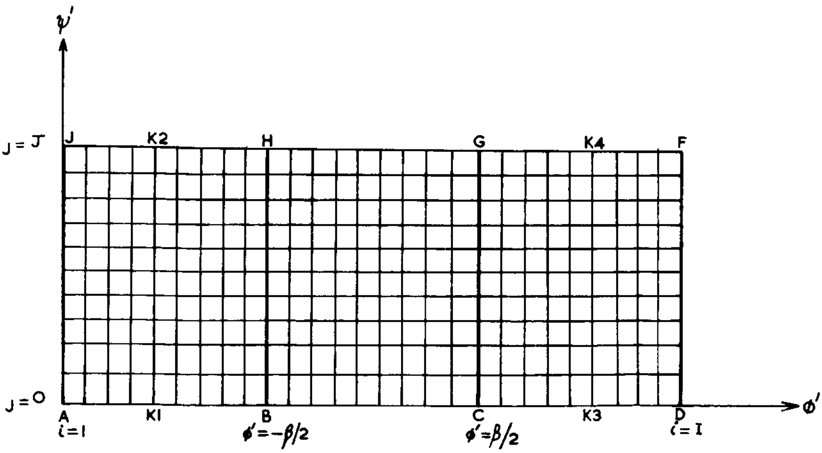

FIGURE 3.—*One strip of the cascade in the (ϕ',ψ') plane.*

$$(a'^2\dot{H}+b'^2)\frac{\partial^2 F}{\partial \phi'^2}+\frac{\partial F}{\partial \phi'}\left[a'\frac{\partial}{\partial \phi'}(a'\dot{H})+b'\frac{\partial b'}{\partial \phi'}+\frac{\alpha}{\psi_0}\frac{\partial b'}{\partial \psi'}\right]$$

$$+\frac{2b'\alpha}{\psi_0}\frac{\partial^2 F}{\partial \phi' \partial \psi'}+\frac{\alpha^2}{\psi_0^2}\frac{\partial^2 F}{\partial \psi'^2}=0 \qquad (23)$$

Although this equation looks more complicated than (20), the boundary conditions (21a) and (21b) are simplified to

$$F(\phi',0)=F(\phi',\alpha)$$

$$\theta(\phi',0)=\theta(\phi',\alpha)$$

along AB and JH and similarly along CD and FG.

Numerical Analysis

Equation (23) may now be solved numerically by finite differences on a rectangular grid in the (ϕ',ψ')-plane. The method will be described for a grid with spacing $\delta\phi'$ and $\delta\psi'$ constant in the ϕ' and ψ' directions, respectively. In practice, it is better to use an unequally spaced grid but, to avoid unnecessary complication in the description, a discussion of unequal grid spacing will be left until later. The grid described here is shown in figure 3. Write equation (23) in the form:

$$A(\delta\phi')^2\frac{\partial^2 F}{\partial \phi'^2}+2B\,\delta\phi'\frac{\partial F}{\partial \phi'}+4C\,\delta\phi'\,\delta\psi'\frac{\partial^2 F}{\partial \phi'\,\partial \psi'}+D(\delta\psi')^2\frac{\partial^2 F}{\partial \psi'^2}=0 \qquad (24)$$

The method of solution will be to estimate the coefficients A, B, C and D, solve (24) as a linear equation, and re-estimate these coefficients. The process is continued until converged, which usually requires about three or four cycles. Equation (24) may be approximated by finite differences in the form:

$$A_j^i(F_j^{i+1}-2F_j^i+F_j^{i-1})+B_j^i(F_j^{i+1}-F_j^{i-1})+C_j^i(F_{j+1}^{i+1}-F_{j+1}^{i-1}-F_{j-1}^{i+1}+F_{j-1}^{i-1})$$

$$+D_j^i(F_{j+1}^i-2F_j^i+F_{j-1}^i)=0 \qquad (25)$$

for $1\leq i\leq I-1$; $\quad 1\leq j\leq J-1$.

The boundary conditions are (1) that F_0^i and F_J^i are given on BC and HG, together with F_j^0 and F_j^I for $j=0\ldots J$, and (2) that $F_0^i=F_J^i$ and $\theta_0^i=\theta_J^i$ along AB and JH and along CD and GF (eqs. 21(a)–21(d)). The method of solving these finite difference equations is a slight modification of a method suggested to the authors by Stocker (ref. 2). Rewrite equation (25), grouping terms according to superscripts $i+1$, i, and $i-1$.

$$\begin{aligned}[-C_j{}^iF_{j-1}^{i+1}+(A_j{}^i+B_j{}^i)F_j{}^{i+1}+C_j{}^iF_{j+1}^{i+1}]\\+[D_j{}^iF_{j-1}{}^i-2(A_j{}^i+D_j{}^i)F_j{}^i+D_j{}^iF_{j+1}{}^i]\\+[C_j{}^iF_{j-1}^{i-1}+(A_j{}^i-B_j{}^i)F_j{}^{i-1}-C_j{}^iF_{j+1}^{i-1}]=0\end{aligned} \tag{26}$$

Inside the rectangle $BCGH$, augment equation (26) with the equations

$$F_0{}^i=F_0{}^i$$

$$F_J{}^i=F_J{}^i$$

remembering that both $F_0{}^i$ and $F_J{}^i$ are known. In the rectangles $ABHJ$ and $CDFG$, augment equation (26) with

$$F_0{}^i=F_J{}^i$$

$$\theta_0{}^i=\theta_J{}^i$$

The last relation must be expressed in terms of F. This could be done using equation (11), which implies that

$$\left(\frac{\partial F}{\partial\psi'}\right)_0^i=\left(\frac{\partial F}{\partial\psi'}\right)_j^i$$

and approximating this relation by finite differences. This was tried, but it led to small but unacceptable errors. Instead, therefore, equation (11) was integrated along $\phi'=$constant and the boundary conditions $\theta_0{}^i=\theta_J{}^i$ inserted into the integral. The integral was then approximated by finite differences using Simpson's rule, giving

$$\sum_{j=0}^{J}\left(\dot{H}_j{}^i+\frac{b'^2}{a'^2}\right)K_j(F_j{}^{i+1}-F_j{}^{i-1})=0 \tag{27}$$

where $K_j=1, 4, 2 \ldots 2, 4, 1$.
Therefore, inside the rectangles $ABHJ$ and $CDFG$, equation (26) is augmented by (27) and $F_0{}^i=F_J{}^i$.

Defining $\mathbf{F}^i$ to be the column vector $(F_0{}^i, F_1{}^i, \ldots, F_J{}^i)$, equation (26), together with the augmenting equations, may be written in the form

$$M^i\mathbf{F}^{i+1}+N^i\mathbf{F}^i+P^i\mathbf{F}^{i-1}=\mathbf{Q}^i \tag{28}$$

where M^i, N^i and P^i are square matrices and $\mathbf{Q}^i$ is a column vector which contains only zeros inside the rectangles $ABHJ$ and $CDFG$ and is of the form $(F_0{}^i, 0, 0, 0, \ldots 0, F_J{}^i)$ inside the rectangle $BCGH$. To solve equation (28), we begin by estimating $F_j{}^i$ at every mesh point other than those along $i=0$ and $i=I$ where F is prescribed. From these estimates, the coefficients A, B, C, and D of (24) may be calculated at each point and hence the matrices M^i, N^i and P^i of (28) may be determined. We then look for a solution of (28) of the form

$$\mathbf{F}^i = R^i\mathbf{F}^{i+1} + \mathbf{t}^i \tag{29}$$

where the R^i are square matrices and the $\mathbf{t}^i$ are column vectors. To determine R^i and $\mathbf{t}^i$, we substitute (29) into (28) and, after some rearrangement, obtain

$$\mathbf{F}^i = -(N^i + P^iR^{i-1})^{-1}M^i\mathbf{F}^{i+1} + (N^i + P^iR^{i-1})^{-1}(\mathbf{Q}^i - P^i\mathbf{t}^{i-1}) \tag{30}$$

Comparing (29) and (30), we obtain by inspection

$$R^i = -(N^i + P^iR^{i-1})^{-1}M^i \tag{31}$$

$$\mathbf{t}^i = -(N^i + P^iR^{i-1})^{-1}(P^i\mathbf{t}^{i-1} - \mathbf{Q}^i) \tag{32}$$

Equations (31) and (32) may be solved recursively for R^i and $\mathbf{t}^i$, for $1 \leq i \leq I-1$, once R^0 and $\mathbf{t}^0$ are known. These are obtained from the prescribed value of $\mathbf{F}^0$, for

$$\mathbf{F}^0 = R^0\mathbf{F}^1 + \mathbf{t}^0 \tag{33}$$

If (33) is to be satisfied, whatever the value of $\mathbf{F}^1$, we must have

$$R^0 = 0$$

$$\mathbf{t}^0 = \mathbf{F}^0$$

Having determined R^i and $\mathbf{t}^i$, $0 \leq i \leq I-1$, we can now solve for F everywhere, using (29) and commencing from

$$\mathbf{F}^{I-1} = R^{I-1}\mathbf{F}^I + \mathbf{t}^{I-1}$$

where $\mathbf{F}^I$ is the prescribed boundary condition on $i = I$. Having determined $F_j{}^i$ everywhere, the coefficients A, B, C, and D of (24) may be re-estimated and the process repeated until successive estimates of F everywhere converge to within some tolerance. In practical cases, two to four iterations are usually required, depending on the level of Mach number.

There is a further point in the calculation of F which requires discussion; namely, the treatment of the stagnation points, the points B, C, G, and H in figure 3. Near stagnation points, $V \to 0$ and $F \to -\infty$. If, when prescribing the velocities along BC and HG, zero velocities are prescribed at the stagnation points, then it is clear that the methods described so far cannot be applied.

A simple and approximate method of overcoming this difficulty, which is equivalent to removing the stagnation points by cusping the blade, is as follows. At the start of each compressibility iteration, a nonzero velocity is assigned to the points B and H and another nonzero velocity to the points C and G. With these values, together with the other prescribed boundary conditions, we can now solve for F everywhere by the methods already described and this solution will satisfy all the prescribed boundary conditions. However, for arbitrary choices of velocity at the points B and H and C and G, the function F is not constant at upstream and downstream infinity; that is, although $\partial F/\partial\psi'$ is zero there, $\partial F/\partial\phi'$ is not zero. Furthermore, for given boundary conditions, the value of $\partial F/\partial\phi'$ at upstream infinity is primarily controlled by the velocity assigned to B and H, and $\partial F/\partial\phi'$ at downstream infinity by the velocity assigned to C and G. Therefore, at the start of each iteration, as well as recalculating the matrices M^i, N^i, and P^i, new estimates are made of the velocities at B and H and at C and G to make $\partial F/\partial\phi'$ zero at points far upstream and downstream. This additional change does not seriously affect the convergence of the main iteration.

Although this is a rather crude treatment of the stagnation points, it does lead to accurate answers in the following sense. When θ is calculated from F, equation (11) is integrated along a streamline starting from far downstream where θ is prescribed. The closeness of agreement of the calculated and prescribed values of θ far upstream is one measure of the accuracy of the calculation. This agreement is best (about 0.2 percent for 100° of turning) when the adjustments described have converged. Methods such as those of Woods (ref. 3) for dealing with singular points were tried but did not appear to increase the accuracy of the calculation, possibly because the computing grid was coarse compared with the small region over which the velocity is close to zero.

From the converged solution for F, the blade coordinates may be calculated. This is done by first integrating equation (11) along $\psi'=\alpha/2$ to give θ along the center of the blade passage and then integrating equation (10) away from this mean line to give θ on the blade surface. Having found θ, the blade coordinates are found by integrating the equations

$$dx=\frac{d\phi}{V}\cos\theta-\frac{d\psi}{\rho V}\sin\theta$$

$$dy=\frac{d\phi}{V}\sin\theta+\frac{d\psi}{\rho V}\cos\theta$$

The integration is performed in the (ϕ',ψ')-plane and commences from arbitrary values of x and y in the middle of the blade passage, out along the line $\phi'=\text{constant}$ to the blade surface and then along the blade surfaces, $\psi'=0$ and $\psi'=\alpha$. This path of integration avoids the necessity of crossing the stagnation point region. The blade shapes obtained show the cusps over the first and last two points on each surface and the leading and trailing edges are generally rounded by eye.

SIZE AND SPEED OF COMPUTER PROGRAM

The methods described have been programed on an IBM 360/65 computer. Using 40 points of each blade surface, 50 upstream and 50 downstream points, and 11 points across the blade passage, the program size is 162K bytes. For a fully converged solution, three to five cycles are required at an average of 24 seconds per cycle. For a fully compressible calculation on such a large grid, the method is therefore very fast. To obtain this speed of computation, an unequally spaced grid has been used, with the grid becoming more widely spaced far upstream and downstream. The only change required in the methods described is to modify the finite difference approximations to derivatives in the obvious way.

Sample Calculation

The program has been tested on a number of examples, one of which, a NASA blade taken from reference 4, is described here. In figure 4, the circles and triangles represent the measured velocity distribution while the full line is the velocity used in the calculation. The measured outlet angle was changed by about 0.7° to −67.7° because the calculation cannot take into account viscous effects. The true and calculated blade shapes are shown in figure 5, where it will be seen that the agreement is generally good. Agreement is worst near the leading and trailing edges. The shape of the leading edge depends critically on the velocity distribution and this is

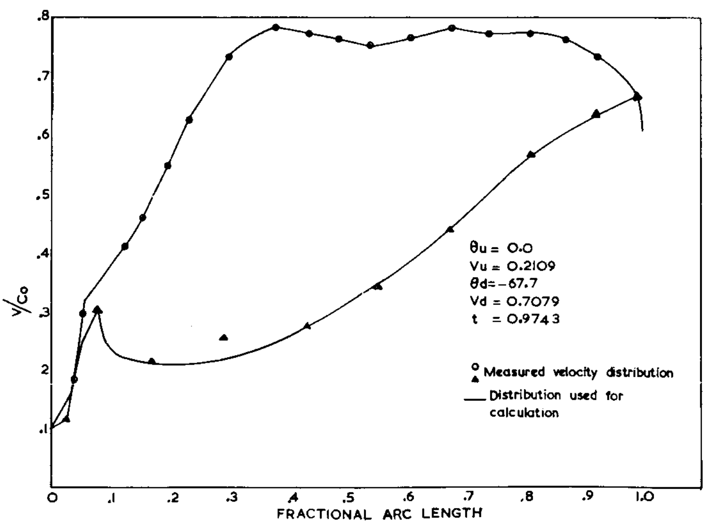

FIGURE 4.—*Velocity distribution of blade of reference 4.*

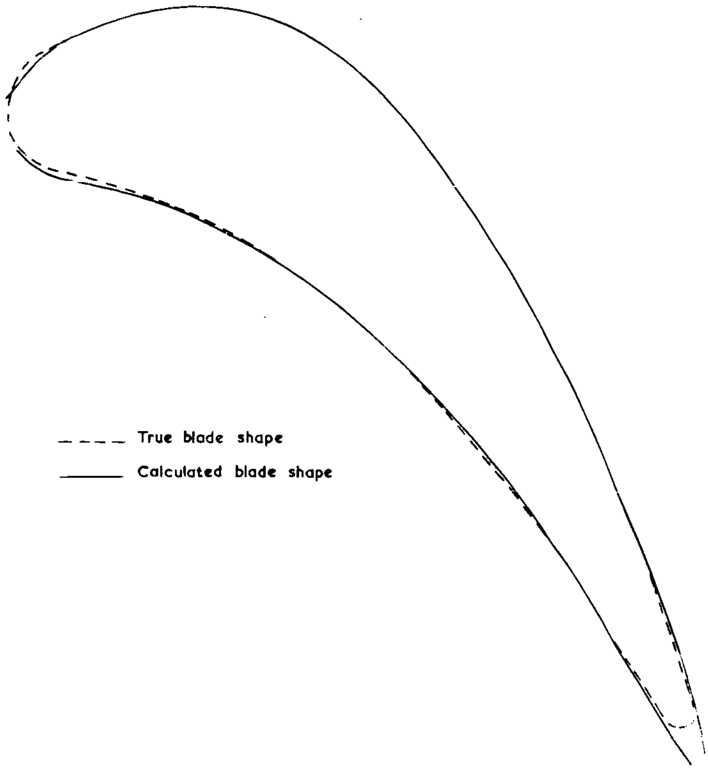

FIGURE 5.—*Comparison of the true and calculated blade shapes.*

impossible to measure at points sufficiently close together to give accurate definition. Also, one of the measured points on the pressure surface has been ignored, for it was found that a smooth velocity distribution through that point did not reproduce the correct blade shape. The velocity distribution used in this region is merely guessed to give a reasonably good blade shape.

LIST OF SYMBOLS

F A function of velocity
H A function of velocity
n Distance normal to a streamline
S Distance along a streamline
S' Fractional length along a blade surface measured between stagnation points
t Pitch

V Velocity, normalized with respect to the stagnation sound speed
V_x The x-component of V
V_y The y-component of V
x Cartesian coordinate measured in the axial direction
y Cartesian coordinate measured in the pitchwise direction
γ Ratio of specific heats
θ Flow direction measured counterclockwise from the positive x direction
ρ Density, normalized with respect to the stagnation density
ϕ Potential function
ϕ' Transformed potential function
ψ Stream function
ψ' Transformed stream function

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Subscripts and Superscripts

d Far downstream
i Index referring to the value of ϕ'
j Index referring to the value of ψ'
L Leading edge
T Trailing edge
u Far upstream

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DISCUSSION

D. PAYNE (Rolls-Royce): The authors are to be congratulated on applying a highly efficient matrix method to the solution of the boundary-value problem which cascade design presents in the compressible flow function plane.

The technique used to solve each iterate of equation (24) (as yet unpublished by Professor Stocker) transmits boundary-value information just once to the right through the vectors **t** and just once to the left through the **F** vectors themselves. This elegant technique thrives on highly rectangular grids, such as the one established here in the (ϕ',ψ')-plane, although the slightly approximate transformation (22b) into this plane could possibly be avoided by the use of a variable skew mesh in the (ϕ,ψ)-plane.

The starting approximation to the coefficients in equation (28), although not explicitly stated, presumably results from taking $H=F$, and this assumption is, in itself, quite accurate for Mach numbers less than about 0.8 (ref. D–1).

The desirability of basing the design of gas turbine blading on a prescribed distribution of surface velocity can be justified by consideration of the mechanical, aerodynamic, and mathematical aspects of the overall design problem (ref. D–2). For the past eight years, all turbine blades designed at the Bristol Engine Division of Rolls Royce Ltd. (previously Bristol Siddeley Engines) have been produced on this basis, using the Bristol Design Transformation (ref. D–3) to generate the necessary cascade geometry. Until now, it has been a rather unfortunate handicap that, while a complete velocity distribution theory (parametric description of velocity distribution, optimization under geometric constraints, wake models, etc.) could be established for an arbitrary density-speed relation (ref. D–1), the actual transformation from the (ϕ,ψ)-plane to the (x,y)-plane was only practical, on a routine basis, for a simplified form of the $\rho(V)$ function (linearized compressible flow, or Chaplygin gas). Although the linearized transformation can be shown to agree closely with plane-flow experiments for Mach numbers up to about 0.85 (ref. D–1), it is to be expected that the methods of Mr. Silvester and Miss Fitch will produce a routine design transformation able to cope with near-sonic and, perhaps, slightly supersonic flow, as well as allowing incorporation of blade-to-blade variations of radial aerodynamic influences, as such variations become better understood.

J. P. GOSTELOW (Cambridge University): The authors introduce their promising new matrix techniques as being suitable for both the direct and the indirect problems of cascade flow prediction. Since it is well known that some considerable effort at their company (ref. D–4) has been invested in iterative solutions of the direct problem, using the Martensen method (ref. D–5) as a basis, it would be of interest to know whether the iterative approach has failed and has therefore been written off. The difficulty with the iterative schemes is that the source distribution contains first derivatives of ρ and, therefore, second derivatives of ψ. It would not be surprising, therefore, if convergence difficulties were experienced at high subsonic Mach numbers where the desired result is masked by rounding-off errors. This question does not concern simply the direct problem since, as Murugesan and Railly (ref. D–6) have shown, the Martensen method can become a successful design tool in solving the indirect problem.

It is interesting to observe that Silvester and Fitch deliberately rearrange the equations so that density-change information is transmitted immediately into the flow solution. It is more conventional for the density calculation to lag the stream function calculation by one iteration, again deliberately to improve stability. This latter approach is employed in Smith's excellent paper (ref. D–7) and in most streamline curvature solutions to the axisymmetric problem. It was clear from Smith's presentation that the trailing density approach is justified for cases where the local Mach number does not exceed 0.85, but even linearized flow models can cope with such examples. It would be interesting to know whether Smith can retain numerical stability, with density lagging by one iteration, when sonic conditions are reached and exceeded on the blade surface.

The kernel of the question is whether one ought to follow Silvester and Fitch in rearranging the equations when local sonic conditions are approached.

L. MEYERHOFF (Eastern Research Group): I have three questions. The first is about trailing edge conditions. I'm curious to know what the author believes would happen if, in our reiteration, the stagnation point of the trailing edge was set right at the trailing edge to zero velocity and the iteration continued with that fact reinserted in each iteration. The other questions are (1) What is meant by the term "cycle" for fully converged solution in your report? Is the word "cycle" meant to be "iteration number"? and (2) What is the total number of mesh points allowed by the program at present?

H. YEH (University of Pennsylvania): You refer to the need for an estimate by the computer for the velocities near the inlet and the trailing edge in order to have the prerequisite velocity at plus and minus infinity. Now, isn't this due to the fact that they really cannot completely describe the velocities anyway because you must have prerequisite separation for the whole profile to fulfill your inlet and exit conditions at infinity. Furthermore, there is a condition for which you get a closed profile. In other words, the profile may not be closed; this is a so-called conditioned closure. Now, if you had to make use of these conditions beforehand, and if you had considerable freedom in adjusting the velocity distribution and did so, it seems to me that you would not really need a computer to make further adjustments.

J. W. DZIALLAS (General Electric Co.): Here are a few questions which should be of general interest.

(1) If the flow is assumed everywhere subsonic, how can the field contain "supersonic patches"? If there are these patches, where are they located? Doesn't the authors' selection of the function $F(V)$ near the stagnation points strongly affect these supersonic patches?

(2) Is the Kutta condition satisfied?

(3) How close to the sonic velocity can the authors' method go on the profile surface? Does the solution become unstable?

(4) What useful information can the authors extract from their hodograph?

(5) Recalling the comparison with the experimental velocity distribution presented in a slide, I ask: How valid is this comparison since, through smoothing of the data, adjustments on the function $F(V)$, and variable grid size it seems possible to arrive at predetermined results. How many trials are necessary to recover the profile?

(6) It would be interesting to see a comparison with an exact direct-method airfoil computation.

P. N. R. SHEKHAR (University of Liverpool, England): At Liverpool University, we have been concerned with the problem of designing airfoils in two-dimensional cascades. Hence, we would like to raise the following points:

(1) Equation (17) is valid only for special cases of γ. According to reference D–8, the deflection condition for any γ is (fig. 2)

$$\int (\log q \; d\phi + \theta/\rho \; d\psi) - \int_{\psi=0}^{\pi/2} \int_{\phi=-\infty}^{+\infty} \theta[\partial(1/\rho)/\partial\phi] d\phi \, d\psi = 0$$

This equation can only be solved iteratively. The final solution is consistent with the Price-Martensen theory (ref. D–9).

(2) In the main paper, the problem is considered as well posed in the (ϕ',ψ')-plane. However, the problem can be well posed in the (ϕ,ψ)-plane itself by Green's function of the second kind as demonstrated in reference D–8. Hence, one wonders if it is not advantageous to work in the (ϕ,ψ)-plane itself?

(3) Once the boundary conditions are formulated, the problem can be treated as a typical boundary-value problem. The methods available include (i) Green's function, (ii) finite difference scheme, and (iii) variational finite element. Even Stocker's method (ref. 2) could be used advantageously. However, in reference D–8, it is used to get the following matrix:

$$[A][q]=[f]$$

where A is a codiagonal block matrix with submatrices that are also codiagonal. No equation contains more than five nonzero elements and only the nonzero elements are stored; hence, the storage requirement is minimized. This method is attractive compared to the marching procedure for two reasons, at least.

(a) The distribution of $\log q$ is found at all the interior points of the rectangle in one go.

(b) The boundary conditions are consistent with the interior solution, whereas in marching procedures this is not so. In our opinion, once the boundary conditions are known, it really does not matter which method is adopted for determining $\log q$ inside the rectangle. We are sure Stocker's method could also be adopted very effectively.

(4) At stagnation points, $\log q$ has logarithmic infinity and θ is multivalued. According to L. C. Woods, the movement of the front stagnation points by $\frac{1}{1000}$ of chord distance affects the velocity peak by more than 10 percent for isolated airfoils (let alone cascades). What is really important is not so much the presence of the stagnation points as the effect it might have on the rest of the solution. It is probably true that Woods' method needs a very refined mesh. However, Payne has proposed a very attractive method for determining the effect of stagnation points on the rest of the solution by integral equation techniques. A detailed analysis is available in references D–8 and D–1. We find it very difficult to accept the concept that the solution achieved by ignoring four stagnation points is satisfactory. One could even say that the classic channel model proposed by Stanitz is satisfactory for cascades. Stanitz has produced some very realistic profiles in NACA 1116.

(5) Last, we would like to examine the following two problems:

(a) Inconsistency with the Price-Martensen theory

(b) The simplicity of Green's function solution.

The Price-Martensen theory has been used extensively and, to a great extent, satisfactorily. However, we understand from Silvester-Fitch that Smith's solution is consistent with the design problem. Presumably this means that the stagnation points, shape of the stagnation, and other streamlines tie up completely. Hence, it should be pointed out that with the help of design and Smith problem, and treated on an iterative basis, it should be possible to produce a one-to-one correspondence and a closed profile. The simplicity of Green's function solution can be illustrated very simply by taking the incompressible or linearized flows. The profile shape depends completely on the boundary conditions and it is immaterial what is happening inside the boundary. In the method reported, it is necessary to know what is happening not only on the boundary but inside the boundary as well. The difference between the incompressible and compressible flow lies in the presence of a double integral term and some minor points.

SILVESTER AND FITCH (authors): The authors agree with Dr. Payne about the desirability of being able to design blades to prescribed surface velocity distributions. We believe that both the direct and indirect approaches can be useful to the blade designer and it was for this reason alone that we developed an indirect method alongside our existing direct method based on the work of Martensen and Price.

We have found that the Martensen-Price method converges well for subsonic flow and that convergence can be obtained, although somewhat more slowly, for flows containing supersonic patches, provided these are not too large. We have also been looking at matrix methods for the direct problem because we believe that they can be made faster than singularity methods. We also believe, but have not shown, that the immediate introduction of density change terms into the equations will improve the stability and rate of convergence.

Concerning details of the calculation, we agree with Dr. Payne that we could have worked on a skew mesh in the (ϕ,ψ)-plane. With the method adopted, the algebra is more tedious, but this is compensated for by the fact that the approximation of partial derivatives by finite differences with small truncation error and using only the lines, $i-1$, i, $i+1$ is easier in the (ϕ',ψ')-plane.

We do not commence the calculation by assuming $H=F$. Referring to figure 3, we assume velocities of V_u on AB and JH, V_d on CD and GF, and the prescribed velocities on BC and HG. The velocity elsewhere is assumed to vary linearly with ψ' at constant ϕ'.

In response to the comments of Meyerhoff and Yeh, it is certainly true that for given values of V and θ, far upstream and downstream, not every velocity distribution that the designer may prescribe will give a closed, nonintersecting curve for his blade profile, but only those velocity distributions which satisfy the so-called closure conditions. It is also true that although we have tacitly assumed a closed profile when deriving the equations, we have not placed any restrictions on the velocity distributions that may be prescribed and so may not obtain sensible blade shapes for every velocity distribution. When we use this program, we assume that the designer is able to specify a velocity distribution which nearly satisfies the closure condition and which will require only slight modification within the program. In practice, we use the program in conjunction with a direct method, so this is usually true. We justify this approach with two arguments. First, when designing a cooled turbine blade, there are factors other than aerodynamic (stressing and cooling considerations) which place restrictions on an acceptable blade geometry. In order to satisfy all these conditions, it is likely that the designer will require three or more runs of the program to achieve a satisfactory blade, modifying his velocity distribution with each successive run. Since the program prints out the modifications to the velocity distributions that it makes internally and these can be fed into the next run, after the first one or two runs, little if any internal modification is required. Second, we know of no method of determining velocity distributions for compressible flow satisfying the closure condition which would involve the designer in any less work than the method we use.

The choice of nonzero velocities at the stagnation points is a necessity since it is impossible to evaluate F or H for zero velocity.

If arbitrary, nonzero values are used and reinserted each cycle, a converged solution may or may not be found. If convergence is obtained, then it will be found that, although the velocity attains the values V_u and V_d far upstream and downstream, $\partial V/\partial\phi$ is not zero there. In addition, integration of $\partial\theta/\partial\phi$ between far upstream and far downstream will not result in the prescribed turning, $\theta_d-\theta_u$. The authors regard the finding of velocities at the stagnation points as a process of finding the shape and direction of cusps which must be added to the rounded profile in order to support the prescribed velocity on the remainder of the profile.

In the paper, the words "cycle" and "iteration" have both been used to denote the process of solving numerically equation (24) for fixed estimates of the coefficients A, B, C, and D at every point and of the velocities of the stagnation points.

The program allows up to 50 points upstream and downstream of the blade, 40 points on each surface of the blade, and 11 points across the blade passage. Best results have been obtained by using a mesh of variable spacing in both the ϕ' and ψ' directions, having points closer together near the blade surfaces and particularly so near the leading and trailing edges.

As to the remarks of Dziallas, it must be made clear that the indirect method described here is intended for use as a design tool in which blade shapes are determined from velocity distributions prescribed by the designer. Although we have tested the program by using it as a direct method (that is, by trying to recover blade shapes from measured velocity distributions) this is not the mode in which it was intended the program should be used. The method described here is not intended as an alternative to the direct methods but as an additional aid to the blade designer. With this intended use of the program in mind, such questions as, "How many trials are necessary to recover a profile?" are not strictly applicable.

Taking the last two questions first, the program has been tested by recovering blade shapes both from measured velocity distributions and from distributions calculated by the singularity method of Martensen and Price, the matrix method of D. J. Smith, and the author's own matrix method, which is still under development. (The method of Katsanis was not available to the authors.) It was found that, for a given blade, the three direct methods gave velocity distributions which differed by small but significant amounts, so that each produced a slightly different blade with the indirect program. Best agreement was with the authors' own direct method. Because of the inconclusive nature of these tests and because it is a more valid test of a program to a designer, most testing of the program has been with measured velocity distributions. In those cases where the tests were two-dimensional and shock-free, blade shapes could be recovered well with velocity distributions close to those measured. There is some freedom in choosing F near the leading and trailing edges because pressure tappings are rarely close enough to give an adequate picture there. Such a test of the program is a useful one, provided one remains close to the measured results (which may include small experimental errors). It simply is not true that one can arrive at predetermined results. Blade shapes are independent of the mesh size, provided it is fine enough.

Although the numerical methods used can be justified only for elliptic equations and hence subsonic flows, even so we can and sometimes do prescribe velocity distributions with supersonic regions. If converged answers are obtained, these must necessarily contain supersonic patches adjacent to the regions of the blade when supersonic velocities have been specified. It can be assumed that in these supersonic regions, small errors due to round-off increase the more distant a mesh point is from the boundary where the velocity is specified. If the patches are small, the errors may not have a chance to grow too large, so it seems possible that sensible answers may be obtained. There is, however, no provision in the program for discontinuous solutions as would be caused by shocks.

The program effectively selects F near stagnation points. We still have some further work to do on this, but we have found that it is more difficult to converge on velocities at the stagnation points when supersonic patches have been prescribed.

As for the Kutta condition, remember that we produce cusped blades. The velocities on the cusps (that is, the velocities which are chosen to satisfy upstream and downstream boundary conditions) are chosen to be equal on both pressure and suction surfaces. In addition, we usually prescribe velocity distributions which become equal on both surfaces close to the leading and trailing edges. This treatment is something like a Kutta condition, although we do not talk specifically in terms of zero velocity.

We have not looked seriously at solutions in the hodograph plane.

In reply to Mr. Shekhar, if the function F is defined by equation (8), then equation (17) is true for all values of γ and for all Mach numbers. Closed analytic forms for F can be found for rational values of γ, and the values of $\frac{7}{5}$ and $\frac{4}{3}$ given by the authors should be quite sufficient for all practical purposes. The error involved in using $\gamma = 4/3$ for the design of a hot turbine blade will be negligible compared with the errors introduced by other assumptions in the mathematical model such as, for example, isentropic inviscid flow. If this is accepted, then it is certainly more convenient and almost certainly more accurate to solve the potential and deflection conditions explicitly by simple numerical integration of the prescribed velocity distribution than iteratively by methods requiring the numerical evaluation of double integrals over the whole flow field with each compressibility cycle. Also, it is worth mentioning that the deflection condition, as we have formulated it, depends only on the boundary conditions. This is useful in two ways. First, it enables the surface lengths to be calculated without any knowledge of the flow field elsewhere and so allows the possibility of abandoning the program before any major computation has been performed if the prescribed velocity leads to unrealistic surface lengths. Second, it allows a check of the accuracy of the solution of equation (24) for fixed boundary conditions, because when the coordinates (x,y) of the blade are eventually found from the solution of (24), the lengths can be calculated and compared against those calculated from the potential and deflection conditions. We have found that for fully converged answers, the lengths calculated by the two methods agree to within less than 0.2 percent.

The relative advantages of the (ϕ,ψ) and (ϕ',ψ') planes have been given in the reply to Dr. Payne.

With reference to question (3), it seems as though Mr. Shekhar believes that the method used by the authors is a marching procedure and one in which, in some way, the solution is inconsistent with the boundary conditions. We do not use a marching procedure; the solution obtained depends at every point upon all the boundary conditions and is completely consistent with them. Moreover, the solution is obtained "all in one go" just as much as in his own method, for (using Mr. Shekhar's own notation), Stocker's method is simply a method of solving the matrix equation

$$[A][q]=[f]$$

Concerning question (4), again, it is not true to say that the authors have neglected the stagnation points. There is a striking similarity between the method used by the authors and the treatment described by Payne as a relaxed treatment. Referring to figure 3, Payne's method consists of the following steps:

(1) Choose nonzero velocity at the stagnation points.

(2) Set the velocity at V_u along AK_1 and JK_2 and V_d on K_4F and K_3D.

(3) Apply the cyclic or repeat conditions to the segments K_2H, K_1B, GK_4 and CK_3.

(4) Set up a variable, but one-parameter, mesh spacing.

(5) Solve as though there were no singularities.

Payne points out that if a solution is now obtained, ignoring the singularities and with an arbitrary mesh parameter, the points B and H, for example, will not be one blade pitch apart in the physical plane—but for a particular choice of the mesh parameter, this can be achieved. Notice that by making the assumptions (2), Payne is, in fact, forcing $\partial V/\partial\phi = 0$ far upstream and downstream but, at the same time, relaxing the repeat conditions on θ over the segments on which V is prescribed. Relaxing these conditions permits the streamlines JH and AB to be of different shape when they should be identical, but the error is reduced by forcing B and H to be nearly one pitch apart.

This approach is very similar to that of the authors. We apply the repeat conditions on both V and θ over the whole length of the dividing streamlines, so that the streamlines corresponding to AB and JH, for example, are identical in shape. It then follows that because A and J are one pitch apart in the physical plane, B and H must be. If we were to follow Payne and choose fixed velocities at the stagnation points, then we would have to choose a mesh spacing to make $\partial V/\partial\phi = 0$ far upstream and downstream. Instead, we keep the mesh spacing fixed and vary the velocity at the stagnation points. The mesh spacing and the chosen velocity are to some extent interchangeable, for both affect the calculated values of derivatives of V near the leading and trailing edges. Payne also justifies the use of such approximate methods of dealing with stagnation points.

As already stated in reply to Mr. Dzillias, the authors do not obtain complete agreement with any direct method, just as none of the direct methods is in complete agreement with any other. The differences are small, but some further work is needed.

Finally, I would agree that where the equations of motion can be reduced to Laplace's equation (that is, for two-dimensional incompressible flows or flows of sufficiently low Mach number that one could reasonably assume $H = F$) an integral method is probably to be preferred to a differential equation approach. Most practical cases, however, cannot be described adequately by Laplace's equation, either because the Mach number level is too high or because it is necessary to take into account effects such as stream-tube thickness variation or the fact that a turbine blade row does not form a linear two-dimensional cascade. (These effects have yet to be incorporated in the authors' program.) Therefore, in most practical cases, it is necessary to compute the fluid velocity everywhere (not only on the boundaries), whichever method is used. In such cases, it is debatable whether integral methods are to be preferred to differential methods. It should be pointed out that the double integral to which Mr. Shekhar refers is not simply a minor term in integral methods; apart from increasing the amount of computation to be done (compared with incompressible flow) it does express the difference between incompressible and compressible flow and this can be quite marked when Mach number levels are high. Further, if the other effects referred to were included, the double integral term would express the difference between plane incompressible flow and compressible flow with stream-tube thickness variation and in an annular cascade.

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