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Secondary Vorticity in Axial Compressor Blade Rows

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A theoretical investigation of secondary flow in compressor blade rows is presented. Formulas for calculating secondary flows in annular cascade blade passages are derived. The influence of the relative rotation vector on secondary velocity perturbations, using recent developments in shear-flow theory, is examined. A method of calculating the flow through successive blade rows is given and a comparison is made with experimental results.

The prediction of axial-flow compressor and turbine performance using mathematical modeling has long been a desired goal of turbomachinery analysts. Methods available at present for designing compressors and turbines are usually based on the assumption of inviscid flow. Several attempts have been made to predict the performance of axial compressor stages in which secondary-flow theory has been utilized (e.g., Horlock (ref. 1) and Dixon and Horlock (ref. 2)).

Horlock established that realistic estimates of the swirl angle distributions in the flow on and near the annulus walls may be made, provided that the entering vorticity was known and the secondary vorticities traced through successive blade rows. With the flow angle distributions known, the axial velocity profiles may then be calculated using three-dimensional inviscid analysis. Dixon and Horlock applied a simple secondary-flow theory to the calculation of the flow angles and velocity distributions through a compressor stage. Fairly close agreement with experimental values was obtained for the guide vanes. Comparison of calculated and experimental values for the heavily loaded rotor was rather poor and strongly influenced by "corner stall."

In the present paper, recent developments in the theory of shear flow by Hawthorne and Novak (ref. 3) are incorporated into a more accurate three-dimensional flow model to remove some of the approximations of references 1 and 2. It is now possible to calculate the secondary flow in an annular cascade of low hub/tip radius ratio and with comparatively few blades instead of using a two-dimensional approximation. In reference 1, secondary vorticity was calculated using the approximation derived by Squire and Winter (ref. 4) and in reference 2 an approximation to a formula of Smith (ref. 5) was employed. In this paper, secondary vorticity has been determined more accurately by means of Hawthorne and Novak's analysis (ref. 3). It is assumed that the flow is inviscid in all calculations and the fluid rotation is prescribed by the flow at entry to the guide vanes.

VARIATION OF VORTICITY ACROSS A BLADE ROW

An extensive literature has accumulated on the subject of secondary vorticity in cascades and blade rows. Review papers are available by Lakshminarayana and Horlock (ref. 6) and by Hawthorne (ref. 7). A discussion of the relative merits and differences between some of these theoretical treatments of secondary flow, although of great interest, is not possible in a short paper.

Recently, Hawthorne and Novak (ref. 3) have considered the transport of vortex filaments in a weakly sheared flow through a plane stationary blade cascade. In their treatment, vortex filaments were transported by a plane primary flow which was *irrotational.* They obtained an expression for the streamwise component of vorticity at exit, which is responsible for producing secondary flow, from the distortion and stretching of the vortex filaments by the primary flow.

A similar result is obtainable for the flow through an annular cascade for which the primary flow is irrotational. In this analysis, the stream surfaces of the primary flow are not necessarily at the same radius before and after the cascade (e.g., the annulus walls may be conical). Figure 1*a* shows the vorticity vectors lying on the development of a stream surface upstream and downstream of the cascade. For the assumed inviscid, incompressible flow, vorticity vector $\boldsymbol{\omega}_1$, at inlet is convected through the blade passage to become $\boldsymbol{\omega}_2$ at outlet. The change in orientation of the vector is caused by blade-passage-induced distortion of the primary flow, which can be determined approximately.

At exit, the streamwise component of vorticity is

$$\omega_{s2}=\frac{\overline{GD}\omega_{n2}}{p_2 \cos \beta_2}-\omega_{n2} \tan \beta_2 \tag{1}$$

where

$$\overline{GD}=\overline{AF}\frac{W_2}{W_1}+W_2 \int \frac{ds}{W_s}$$

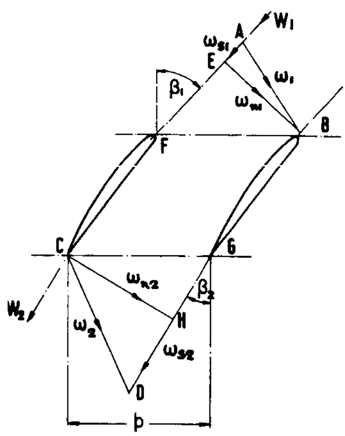

FIGURE 1*a*.—*Illustration of vorticity vector changes through a blade passage.*

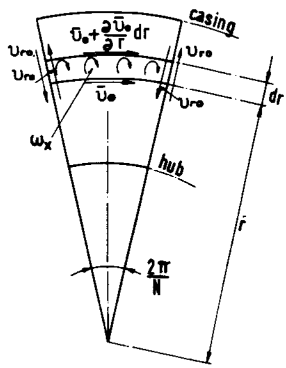

FIGURE 1*b*.—*Axial projection of blade passage showing passage and wake vorticity.*

$$\overline{AF}=p_1\left(\frac{\sin\beta_1+\cos\beta_1\omega_{s1}}{\omega_{n1}}\right)$$

The integral $\int ds/W_s$ is taken around the surface of an airfoil. It represents the difference in the transit time of a particle traveling from the leading to the trailing edge of the airfoil when passing along the suction surface and when passing along the pressure surface. The integral is not readily evaluated, but may be approximated by an expression given by Smith (ref. 5)

$$\int\frac{ds}{W_s}\doteq\frac{\Gamma}{W_\infty^2} \tag{2}$$

where W_∞ is the vector mean of the inlet and exit velocities and Γ is the blade circulation, i.e.

$$\Gamma = \frac{2\pi}{N}\,(r_1 V_{\theta 1} - r_2 V_{\theta 2}) = \text{constant}$$

Using Crocco's equation, grad $I = \mathbf{W} \times \boldsymbol{\omega}$, where I is relative stagnation pressure/density, a useful relation between the normal components of vorticity is obtained

$$W_1 \omega_{n1} = W_2 \omega_{n2} \tag{3}$$

Substituting the preceding expressions into equation (1), the outlet streamwise vorticity is found from

$$\frac{\omega_{s2}}{\omega_{n1}} \cos \beta_2 = \frac{r_1}{r_2}\left(\sin \beta_1 + \cos \beta_1 \frac{\omega_{s1}}{\omega_{n1}}\right) + \frac{W_1}{p_2} \int \frac{ds}{W_s} - \sin \beta_2 \frac{W_1}{W_2} \tag{4}$$

By a fairly trivial extension, changes in density can be included in the analysis so that it can apply to compressible flow.

For a flow at constant radius, $r_1 = r_2 = r$ and equation (4) reduces to

$$\omega_{s2} = \omega_{s1} \frac{\cos \beta_1}{\cos \beta_2} + \omega_{n1} \left(\frac{\sin \beta_1}{\cos \beta_2} - \frac{\sin \beta_2}{\cos \beta_1} + \frac{W_1}{p \cos \beta_2} \int \frac{ds}{W_s}\right) \tag{5}$$

This is essentially the result obtained by Hawthorne and Novak (ref. 3) for a plane flow in a stationary coordinate system. The streamwise component of vorticity at inlet ω_{s1} has been included in the above analysis from the outset. It is most important to realize that ω_{s1}, which is an axisymmetric vorticity, directly influences the magnitude of the passage (i.e., streamwise) vorticity ω_{s2} at outlet and therefore contributes to the secondary motion of the fluid.

The effect of the angular rotation vector $\boldsymbol{\Omega}$ on the production of secondary vorticity is considered in the following section.

ROTOR SECONDARY FLOW[1]

The calculation of the secondary flow in a rotor raises a fundamental point concerning the effect of rotor relative rotation on the vorticity used to determine the secondary velocities. Several writers (refs. 8, 9, and 10) have proposed flow models in which the angular rotation vector $\boldsymbol{\Omega}$ is subtracted from the absolute vorticity entering the rotor and have then used the resulting relative vorticity to determine the relative secondary

[1] The analysis of this section was carried out in collaboration with Professor Sir William Hawthorne.

flow. At exit from the rotor, the angular rotation vector is then added to the relative vorticity, which now includes the rotor secondary vorticity, to give the absolute vorticity at entry to the following stator row. It is shown in the following that the relative rotation vector plays no part in the calculation of secondary vorticity and only the absolute vorticity is relevant.

Hawthorne (ref. 11) demonstrated that in the case of a general rotational *steady* flow of an inviscid, incompressible fluid, the velocity of a fluid particle is represented by

$$\mathbf{V} = \text{grad}\ \phi - t\ \text{grad}\left(\frac{p_o}{\rho}\right)$$

in which ϕ is a potential function, t is the drift time of the particle, p_o is the stagnation pressure, and ρ is the density. In this flow, vortex filaments lie along the intersection of surfaces of constant t and constant stagnation pressure, the latter being also stream surfaces.

By means of an extension of this theory, it can be shown that in a rotating system of coordinates the *relative* velocity of the flow is

$$\mathbf{W} = \text{grad}\ \phi - t\nabla I - \mathbf{\Omega}\times\mathbf{r} \tag{6}$$

where I is relative stagnation pressure/density.

In the theory of shear flow, Hawthorne (ref. 11) used a *small shear* approximation to deal with the large disturbance type of flow such as the secondary flow in blade passages. The flow is assumed to be composed of a primary flow, fully described by a potential function ϕ_0 satisfying the boundary conditions on the walls and blade surfaces, and perturbations to take account of the rotationality of the flow. The velocities induced by rotationality are small, by hypothesis, compared with the primary flow so that the associated vortex filaments are convected by the primary flow.

Writing,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots$$

$$t = t_0 + t_1 + t_2 + \cdots$$

where ϕ_1/ϕ_0, t_1/t_0 are of first order of smallness and ϕ_2/ϕ_0, t_2/t_0 are of second order of smallness, then, using equation (6), the relative velocity is

$$\mathbf{W} = (\nabla\phi_0 - \mathbf{\Omega}\times\mathbf{r}) + (\nabla\phi_1 - t_0\nabla I) + (\nabla\phi_2 - t_1\nabla I) + \cdots \tag{7}$$

Bracketed terms in equation (7) are in descending orders of magnitude from the left. The primary flow relative velocity is

$$\mathbf{W}_p = \nabla\phi_0 - \mathbf{\Omega}\times\mathbf{r} \tag{8}$$

and the velocity of the secondary flow, which is convected by the primary flow, is

$$\mathbf{W}_s = \nabla\phi_1 - t_0 \nabla I \tag{9}$$

It should be noted that t_0 is the drift time of the primary flow. The primary flow has zero absolute vorticity so that, to the first order, the vorticity is

$$\boldsymbol{\omega} = \text{curl } \mathbf{W}_s = \nabla I \times \nabla t_0 \tag{10}$$

using equation (9). Thus, from equation (9), the fundamental point is established that the secondary velocities within a rotor should be obtained from the absolute vorticity resolved in the relative flow direction. $\boldsymbol{\Omega}$, the relative rotation vector, does not enter into the calculation, except insofar as it appears implicitly in ∇I.

SECONDARY VELOCITIES IN BLADE PASSAGES

Formulas for calculating the two-dimensional solution of the secondary flow in cascade blade passages have been given by Hawthorne (ref. 12). More recently, the more difficult problem of secondary flow in a stationary annular cascade was investigated by Hawthorne and Novak (ref. 3) but the final solution was not derived. This annular cascade analysis is summarized below and is followed by a solution which can be adapted easily to the computation of secondary velocities.

In the case of a weakly sheared flow, the primary flow may be assumed to lie on cylindrical surfaces of constant radius and there is, therefore, no radial component of vorticity. At outlet from the blades, only ω_{s2}, the streamwise vorticity, contributes to the secondary flow, the effects of ω_{n2} being found from axisymmetric flow analysis. Referring to figure 2, the velocity perturbations induced by ω_{s2} have components v_r, v_θ, and v_x and the vorticity components are

$$\omega_{\theta 2} = \omega_{s2} \sin \alpha_2 = -\frac{dv_x}{dr}$$

$$\omega_{x2} = \omega_{s2} \cos \alpha_2 = \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial \theta}\right]$$

Noting that $v_x = v_x(r)$ only, and using the continuity condition, $\text{div}(\mathbf{v}) = 0$, a Stokes' stream function can be defined,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v_\theta = v_x \tan \alpha_2 - \frac{\partial \psi}{\partial r} \tag{11}$$

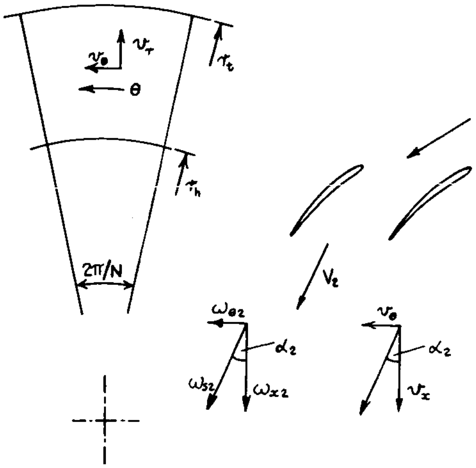

FIGURE 2.—*Vorticity and velocity perturbations in an axial projection of a blade passage.*

After substituting v_θ, v_r and $\omega_{\theta 2}$ into ω_{x2}, the following differential equation is derived,

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=\frac{v_x}{r}\frac{d}{dr}(r\tan\alpha_2)-\omega_{s2}\sec\alpha_2=F(r) \qquad (12)$$

Using equation (11), a mean value of v_θ can be derived and the mean flow angle perturbation obtained,

$$\Delta\alpha_{2s}=-\frac{N\cos^2\alpha_2}{2\pi V_x}\int_0^{2\pi/N}\frac{\partial\psi}{\partial r}\,d\theta \qquad (13)$$

where V_x is the primary flow axial velocity component, which can be replaced with small error by $\bar{V}_x$ (i.e., mean V_x), and N the number of blades.

A solution of equation (12) must satisfy the boundary values $\psi=0$ at $r=r_h$, r_t and also at $\theta=0$, $2\pi/N$, $4\pi/N$, etc. The Kutta condition at the trailing edge is then also satisfied, as $v_\theta=v_x\tan\alpha_2$ at $\theta=0$, $2\pi/N$, etc.

Hawthorne and Novak's equations, as given in the preceding paragraphs, are now solved. Writing

$$F(r)=F(r)\sum_{n\text{ odd}}^{\infty}\frac{4}{n\pi}\sin k_n\theta$$

and

$$\psi=\sum_{n\text{ odd}}^{\infty}\psi_n\sin k_n\theta \qquad (14)$$

equation (12) is reduced to

$$\frac{d^2\psi_n}{dr^2}+\frac{1}{r}\frac{d\psi_n}{dr}-\frac{k_n^2}{r^2}\psi_n=\frac{4}{n\pi}F(r) \tag{15}$$

where $k_n=nN/2$. The boundary conditions at $\theta=0$, $2\pi/N$, etc., have now been satisfied through equation (14).

Nondimensionalizing throughout (using r_t and $\bar{V}_x$) and solving for the mean flow angle perturbation $\Delta\alpha_{2s}$, using equation (14), gives

$$\Delta\alpha_{2s}=-\frac{2}{\pi}\cos^2\alpha_2\sum_{n\text{ odd}}^{\infty}\frac{1}{n}y_n' \tag{16}$$

and equation (15) becomes

$$y_n''+\frac{1}{\rho}y_n'-\frac{k_n^2}{\rho^2}=\frac{4}{n\pi}\frac{r_t}{\bar{V}_x}F(r)=\mathcal{R}(\rho) \tag{17}$$

where

$$\rho=r/r_t,\; y_n=\psi_n/(r_t\bar{V}_x),\; y_n'=dy_n/d\rho, \text{ etc.}$$

The solution of equation (17) must satisfy the boundary conditions $v_r=0$ at $\rho=\rho_h$ and $\rho=1$. Solving equation (15) by variation of parameters, the complete solutions for y_n and y_n' are

$$2k_ny_n=\frac{(\rho^{k_n}-\rho^{-k_n})}{(1-\rho_h^{2k_n})}\int_{\rho_h}^{1}\rho\mathcal{R}\left[\left(\frac{\rho}{\rho_h}\right)^{k_n}-\left(\frac{\rho}{\rho_h}\right)^{-k_n}\right]\rho_h^{k_n}\,d\rho$$

$$+\rho^{-k_n}\int_{\rho}^{1}\mathcal{R}\rho^{1+k_n}\,d\rho-\rho^{k_n}\int_{\rho}^{1}\mathcal{R}\rho^{1-k_n}\,d\rho \tag{18}$$

$$2\rho y_n'=\frac{(\rho^{k_n}+\rho^{-k_n})}{(1-\rho_h^{2k_n})}\int_{\rho_h}^{1}\rho\mathcal{R}\left[\left(\frac{\rho}{\rho_h}\right)^{k_n}-\left(\frac{\rho}{\rho_h}\right)^{-k_n}\right]\rho_h^{k_n}\,d\rho$$

$$-\rho^{-k_n}\int_{\rho}^{1}\mathcal{R}\rho^{1+k_n}\,d\rho-\rho^{k_n}\int_{\rho}^{1}\mathcal{R}\rho^{1-k_n}\,d\rho \tag{19}$$

For a typical blade row in which, for example, $N\geq 30$ and $\rho_h\leq 0.9$, such that $\rho_h^{2k_n}\ll 1$, a slightly more compact form of these expressions can be obtained. By combining equations (16) and (19), the average flow angle perturbation across the blade passage may be computed as a function of radius. Another useful result is the radial velocity perturbation v_{r_0} at the boundary $\theta=0$, $2\pi/N$, etc. From equation (14)

$$v_{r_0}=\frac{\bar{V}_x}{\rho}\sum_{n\text{ odd}}^{\infty}k_ny_n \tag{20}$$

where k_ny_n is obtained using equation (18).

In the third section, it was demonstrated that in a rotating coordinate framework, the secondary velocities relate to *absolute* vorticity, not relative vorticity. Thus, the solutions may also be applied to rotor rows, replacing α_2 by the rotor exit flow angle β_2 where required.

DERIVATION OF AXISYMMETRIC FLOW CONDITIONS DOWNSTREAM OF THE BLADE ROW

The calculation of the axial velocity distribution downstream of a blade row really presents a difficult problem unless the flow can be reasonably assumed to be axisymmetric. In the shear flow theory, the vorticity is assumed to be weak and the secondary velocities are then small in comparison with the primary flow velocities. Thus, under these conditions it would seem justifiable to assume that at entry to the following blade row, the flow is steady, or nearly so, the vorticity being distributed circumferentially.

Hawthorne (ref. 13) has shown for the nonuniform flow through a cascade that, in the streamwise direction, there are three components of vorticity downstream of the trailing edge plane. The first is the distributed passage vorticity ω_{s2} already considered in the second section; the second and third are the trailing shed vorticity and trailing filament vorticity, both of which lie along the wake. Now, trailing shed circulation is caused by a gradient in circulation along the blade length and it is easily demonstrated that the contributions of both the primary and perturbation flows are already included in the analysis.

Trailing filament circulation arises in the "wakes" from the cellular motion induced in the blade passage by the secondary streamwise vorticity. The contribution made to the net vorticity by the trailing filament vorticity was shown by Smith (ref. 14) to be small for boundary layers which are thin compared with the blade spacing but appreciable when the boundary-layer thickness/blade spacing ratio is of order unity.

To show how the trailing filament modifies the distributed passage vorticity, consider first the axial component of passage vorticity,

$$\omega_x = \omega_{s2} \cos \alpha_2 = \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta}$$

Referring to figure 1*b* and applying Stokes' theorem to the fluid element of area $(2\pi r/N)\, dr$, excluding the blade wake, gives

$$\omega_x = \frac{1}{r} \frac{d}{dr} (r \bar{v}_\theta) + \frac{v_{r_0} N}{\pi r} \tag{21}$$

where $\bar{v}_\theta$ is the averaged perturbation in tangential velocity and v_{r_0} is the radial velocity perturbation at $\theta = 0$, $2\pi/N$, etc. Again, applying Stokes' theorem to the fluid element, but this time including a blade wake, gives

$$\omega_{xA} = \frac{1}{r} \frac{d}{dr} (r\bar{v}_\theta) \tag{22}$$

which is the axial component of the combined secondary and trailing filament vorticities. Combining equations (21) and (22), ω_{xA} can be found

$$\omega_{xA} = \omega_{s2} \cos \alpha_2 - v_{r0} \frac{N}{\pi r} \tag{23}$$

v_{r0} being computed from equations (18) and (20).

Now,

$$\omega_\theta = \omega_x \tan \alpha_2 = -\frac{dv_x}{dr}$$

and it is deduced that

$$\omega_{\theta A} = \omega_{xA} \tan \alpha_2$$

if the primary flow direction is not changed by the secondary flow. That this is so can be deduced from Crocco's equation,

$$1/\rho \text{ grad } p_o = \mathbf{V} \times \boldsymbol{\omega}$$

i.e., ω_{n2} is of fixed magnitude and the head of the *resultant* vorticity vector ω_{2A} must lie along the line AB in figure 3. A change in the axial velocity perturbation v_x must occur, consistent with the reduction of ω_θ to $\omega_{\theta A}$. This change in v_x is assumed to be completed far downstream of the trailing edge plane (i.e., as in actuator disc theory).

With the resultant axisymmetric vorticity known, the secondary flow in the following blade row can now be determined using the components of this vorticity resolved parallel and normal to the relative primary flow of that row.

For computing axial velocity profiles, the flow angle of the primary flow is added to the perturbation flow angle $\Delta\alpha_{2s}$ to give the flow angle in the trailing edge plane. All variations in v_x are assumed to occur downstream of this plane.

PERFORMANCE PREDICTION OF A COMPRESSOR STAGE

A revised theoretical model for predicting the performance of an axial-flow compressor based on inviscid secondary flow is now available. It is assumed that the velocity profile upstream of the inlet guide vanes is known and the primary flow (i.e., no secondary flow) efflux angles can be found for each blade row from cascade data.

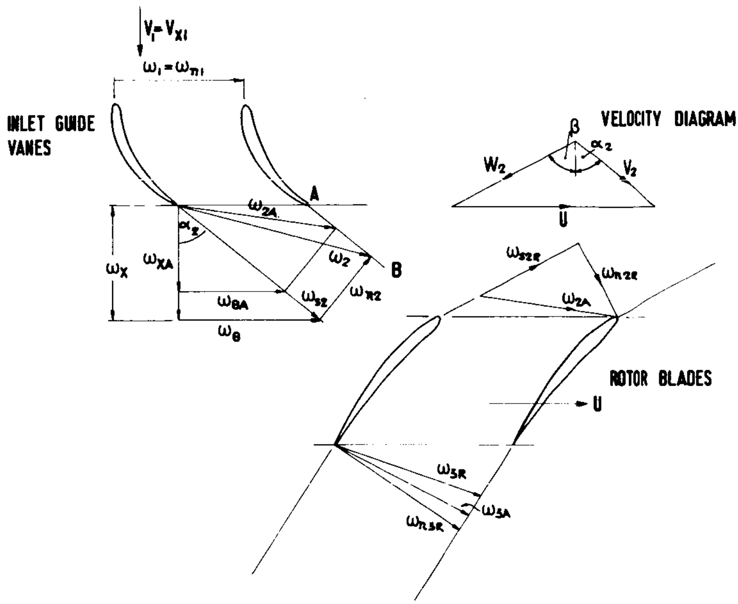

FIGURE 3.—*Vorticity vectors traced through two blade rows of a compressor.*

Referring to figure 3, the steps in the calculation of the flow are summarized as follows.

(1) Using equation (5), the passage vorticity ω_{s2} at exit from the inlet guide vanes is determined, ω_{s1} and α_1 being generally zero for this flow.

(2) With ω_{s2} known, the mean flow angle perturbation $\Delta\alpha_2(r)$ is calculated using equations (16) and (19). This distribution is added to the primary flow angle $\alpha_2(p)$, giving the total exit angle $\alpha_2(r)$ to be used in the axisymmetric flow calculation.

(3) The axial velocity distribution downstream of the guide vanes $V_{x2}(r)$ is calculated using $V_{x1}(r)$ and $\alpha_2(r)$ in an axisymmetric flow equation; e.g., equation (8) in Horlock (ref. 1).

(4) A modified axial component of the streamwise vorticity is calculated using equation (23); in this calculation v_{r_0} is obtained from equations (18) and (20). The resultant vorticity ω_{2A} can now be found, noting that ω_{n2}, the normal component of vorticity, must remain constant. The vorticity ω_{2A} could also be obtained from the downstream solution for axial velocity but this is less direct and is intrinsically less accurate, as one step in the computation involves differentiation.

(5) Resolving ω_{2A} into components parallel (ω_{s2R}) and normal (ω_{n2R}) to the *relative* flow at rotor entry, equation (5) is employed to determine the *absolute* streamwise vorticity (ω_{s3R}) at rotor exit. The streamwise

vorticity at entry will strongly influence the magnitude and direction of ω_{3R}.

(6) The flow angle perturbation $\Delta\beta_{3s}$ at rotor outlet is determined from ω_{s3R} and added to the primary flow angle $\beta_{3(p)}$ to give the rotor exit flow angle for determining the axial velocity distribution far downstream.

(7) Repeat the sequence, from step (3).

It will be noticed that at rotor (and stator) entry the streamwise vorticity relative to the blades is, in general, nonzero. This was pointed out by Horlock (ref. 1) who observed that the "conventional" direction of secondary rotation may be reversed because of the streamwise vorticity at entry. In inlet guide vanes, for which $\omega_{s1}=0$, the secondary rotation may produce *overturning* of the flow at the blade ends. For rotors and stators, secondary rotation produces *underturning* of the flow at the blade ends because of the inlet streamwise vorticity. Experimental results and theoretical calculations both show that this is a normal feature of the flow through rotors and stators. It is worth observing that the net effect of secondary vorticity and trailing filament vorticity is to produce a resultant vorticity which remains close to the tangential direction. This feature strongly influences the turning direction at the blade ends.

COMPARISON WITH EXPERIMENTAL RESULTS

Calculations based on the method summarized in the preceding section are still rather limited in scope. So far, only the flow through a set of inlet guide vanes has been determined but the results show a closer fit of experimental data than the earlier attempt described by Dixon and Horlock (ref. 2). The method of determining the secondary flow in reference 2 was of a more approximate nature and the theory was very much simplified.

The test results relate to a low-speed experimental compressor with a hub/tip ratio of 0.8 and with 60 inlet guide vanes having a blade outlet angle of 60°. The space/chord ratio (s/l) was 0.943 at the mean radius and the blade chord was constant (0.7 in.), so that s/l varied from 0.84 at the root to 1.05 at the tip. From this information, the primary flow angle at outlet from the vanes was estimated, using the deviation angle rule

$$\delta=0.17\theta\frac{s}{l}, \quad \text{deg}$$

where θ is the camber angle of the vanes.

Figure 4 shows the axial velocity distribution at entry to the compressor which was used in all the calculations. The exact form of the velocity profile was not known and the equilibrium velocity profile

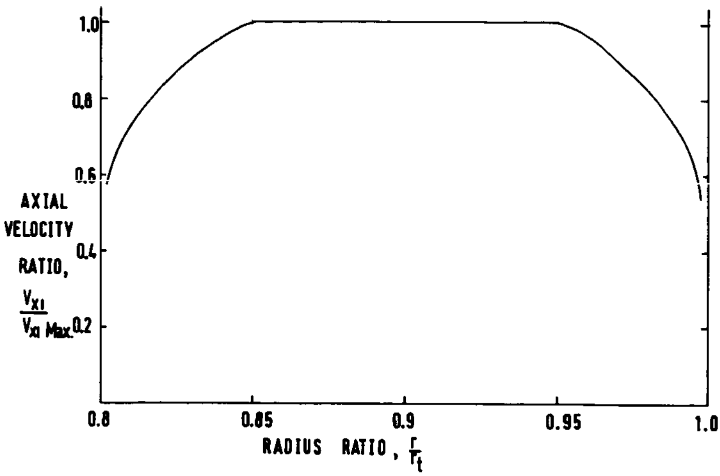

FIGURE 4.—*Axial velocity distribution at entry to inlet guide vanes.*

described by Coles (ref. 15) was used. This profile gives a good match with earlier experimental results by Horlock obtained on a similar compressor entry. The velocity distribution V in the annulus wall boundary layers as a fraction of the mainstream velocity V_1 is

$$\frac{V}{V_1}=\frac{V_{\tau 1}}{KV_1}\left[\ln\left(\frac{\delta_1 V_{\tau 1}}{\nu}\right)+\ln\eta+2+\Pi_x(1-\cos\pi\eta)\right]$$

where K and Π_x are constants having values of 0.4 and 0.55, respectively; $V_{\tau 1}$ is a "friction velocity"; δ_1 is the boundary-layer thickness; ν is the kinematic viscosity; and η is the distance from the wall as a fraction of δ_1.

The calculated distribution of α_2 based on the theory is shown in figure 5, together with the experimental values. Agreement between calculated and measured values appears to be very good. The corresponding variation in radial velocity perturbation along the blade wake (i.e., at $\theta=0$) is given in figure 6 as a fraction of the mean axial velocity, which indicates the rather high velocity perturbation caused by secondary flow. The radial velocity perturbation is of most use in determining the resultant vorticity of the axisymmetric flow. Figure 7 shows the axial component of the streamwise vorticity at outlet ω_x, together with the radial velocity correction $v_{r_0}N/\pi r$, both in a nondimensional form. The difference between the two curves results in the net axial component of streamwise vorticity. It is of interest to note that the radial velocity contribution to vorticity predominates over the secondary vorticity toward the boundary-layer edge.

The axial velocity distribution downstream of the guide vanes was calculated using the flow angle distribution of figure 5 and the inlet

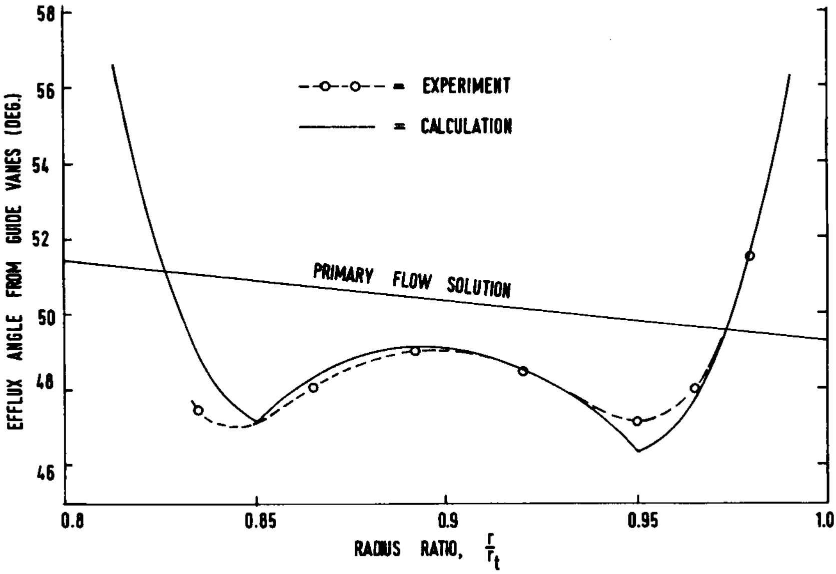

FIGURE 5.—*Flow angles at exit from guide vanes.*

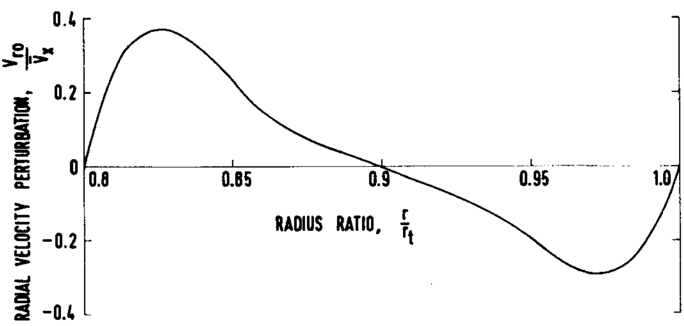

FIGURE 6.—*Variation of radial velocity at exit from guide vanes for $\theta = 0$.*

velocity profile of figure 4. As the hub/tip ratio was high and the blade aspect ratio (the ratio of blade height to distance between centerlines of adjacent blade rows) was small (less than 2), the axisymmetric flow calculations were based on simple radial equilibrium between blade rows. The blade row was replaced by an actuator disc located at the midchord position and interference effects from other blade rows were neglected. Figure 8 shows the calculated axial velocity profile which can be compared with an experimentally derived profile. Agreement between the curves is very good except toward the hub where, as it turns out, the flow angle prediction also differs from the experimental values.

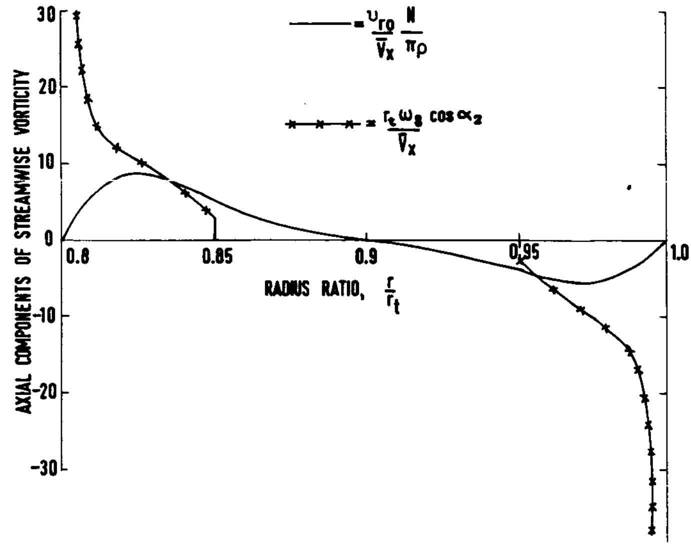

FIGURE 7.—*Axial components of streamwise vorticity at exit from guide vanes.*

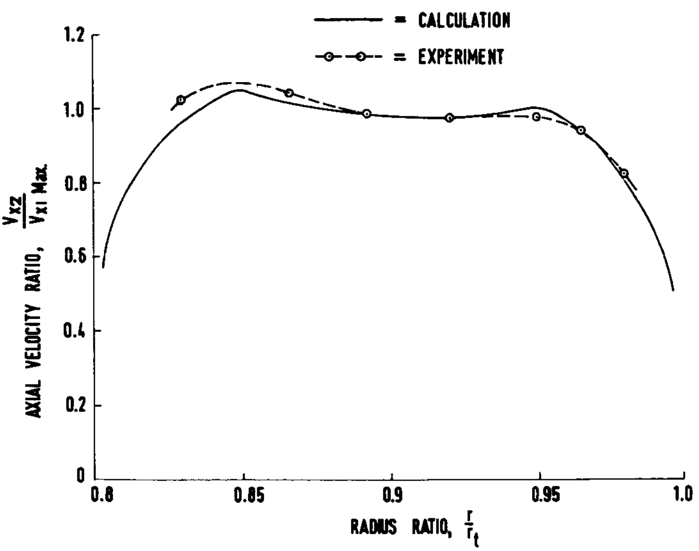

FIGURE 8.—*Axial velocity distribution downstream from guide vanes.*

The calculation of the flow in the following rotor row using the procedure given in this paper has not yet been attempted. However, flow calculations of the rotor have been made by means of a similar but less advanced theory. A fairly crude approximation was used to determine the secondary vorticity and secondary velocities were obtained from a simple two-dimensional model. Nevertheless, valuable conclusions can be drawn from these results, which are presented here.

The rotor blades had the same space/chord values as the guide vanes and comprised constant section blades (10C5/20C50 profiles) set at 50° stagger. From experimental data available, it was noticed that at the particular test conditions being considered the blades operated close to the peak pressure rise. A primary flow angle at rotor exit was estimated based on the maximum unstalled deviation of the blades. Figure 9 shows the calculated distribution of the flow angle, β_3, at rotor exit, together with experimental values. The results can be seen to be qualitatively similar, the discrepancies between the two curves being due probably to the approximate nature of the theory and the assumption of constant boundary-layer thickness.

Figures 10 and 11 show a series of axial velocity distributions calculated systematically for the rotor with prescribed conditions. In figure 10, both sets of calculations, *A* and *B*, employed the previously calculated rotor exit angles shown in figure 9. However, for curve *B* the measured total pressure losses in the rotor were included in the axisymmetric flow calculation, whereas for curve *A* they were ignored. Comparing these calculated results with experiment, a very marked improvement in the accuracy of curve *B* is evident. In figure 11, the measured rotor exit flow angle was

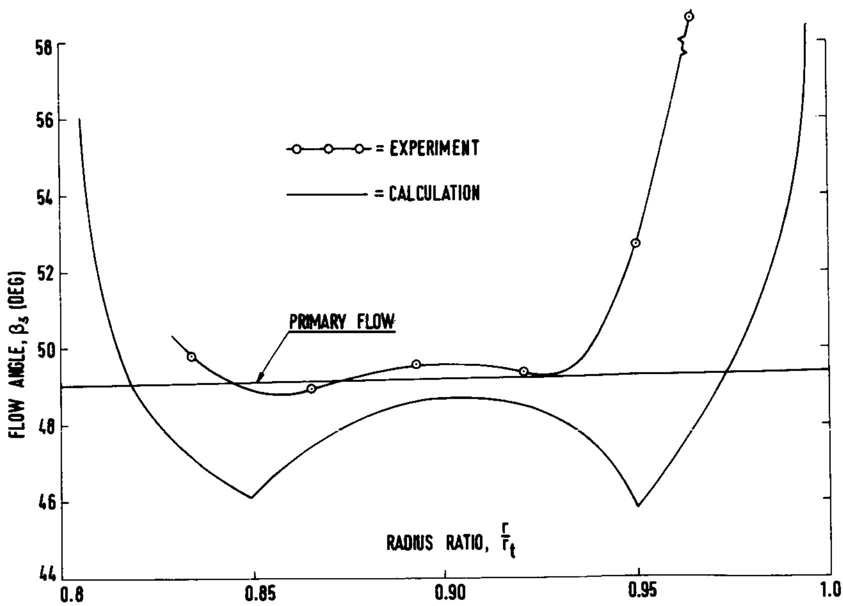

FIGURE 9.—*Flow angle distribution at exit from rotor blades.*

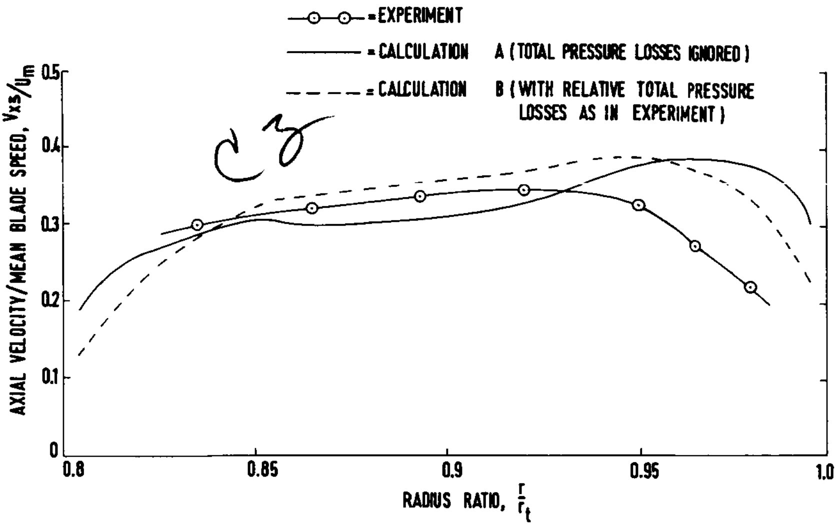

FIGURE 10.—*Axial velocity distributions downstream of rotor row using calculated flow angle distribution.*

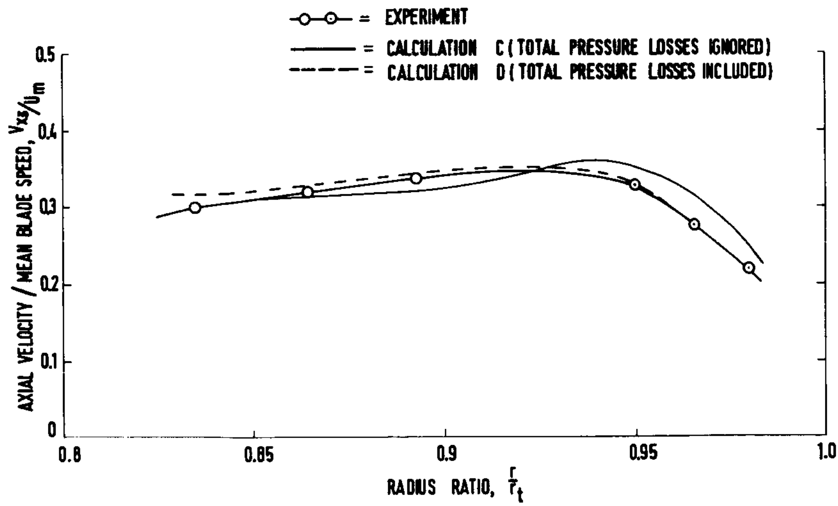

FIGURE 11.—*Axial velocity distributions downstream of rotor row with experimental flow angle distribution.*

used in both calculations, *C* and *D*. For the former, the relative total pressure losses were ignored and for the latter, they were included. It is clear from curves *C* and *A* together that having the correctly calculated flow angle distribution is vital for accurate performance prediction in compressors. However, it is evident from curve D that account must be taken also of the "losses," at least in diffusing blade rows.

CONCLUSIONS

A comprehensive theoretical analysis is presented for calculating the secondary flow through the successive rows of axial-flow turbomachines, based on a known inlet velocity distribution. The controversial relative rotation vector is shown in the analysis to be irrelevant to the calculation of secondary vorticity. Application of the analysis to the inlet guide vanes of an axial-flow compressor predicts an exit flow angle distribution which is very close to the measured distribution. Subsequent calculation of the downstream axial velocity distribution using the predicted flow angles gives close agreement with the measured axial velocities. Thus, it is concluded that for guide vanes, at least, the exit flow is fairly accurately predicted by assuming inviscid loss-free flow with no growth in boundary-layer thickness across the row.

The prediction of flow through the rotor, which employed a simplified, more approximate treatment than that given in the paper, showed only a moderate agreement with experiment. The rotor was heavily loaded and relative total pressure losses were significant. These calculations emphasize the importance of the accurate prediction of both outlet angle and pressure loss distributions in the calculation of axial velocity distributions. It should now be possible for a more accurate assessment of rotor outlet angles to be made with the theory given in this paper.

ACKNOWLEDGMENT

The author is grateful for the help and interest of Professor Sir William Hawthorne, Cambridge University. The experimental data used here were supplied by A. B. McKenzie of Rolls-Royce Ltd., Derby, and originally used in reference 2.

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DISCUSSION

W. R. HAWTHORNE (Cambridge University): The author's solution of equation (12) given in equations (18) and (19), although correct, is somewhat difficult to evaluate. It is easier to reduce equation (12) to a rectangular coordinate system by writing

$$r/r_h = \exp\left(\frac{2\pi z}{N}\right)$$

and

$$\theta = \frac{2\pi y}{N}$$

Then the author's equation (12) becomes

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial y^2} = \left(\frac{2\pi r}{N}\right)^2 F(r) = G(z)$$

with boundary condition $\psi = 0$ at $y = 0$ and $y = 1$ and at $z = 0$ and $l = (N/2\pi) \log_e (r_t/r_h)$.

Writing

$$G(z) = G(z) \sum_{1,3,5}^{\infty} \frac{4}{n\pi} \sin n\pi y$$

and

$$\psi = \sum_{1,3}^{\infty} \psi_n \sin n\pi y$$

we obtain

$$\frac{d^2\psi_n}{dz} - n^2\pi^2\psi_n = \frac{4}{n\pi} G(z)$$

By the method of variation of parameters, we obtain

$$\psi_n(z) = -\frac{4/n^2\pi^2}{\sinh \pi n l}\left[\sinh n\pi z \int_z^l G(t) \sinh n\pi(l-t)\, dt \right.$$

$$\left. + \sinh n\pi(l-z) \int_0^z G(t) \sinh n\pi t \, dt\right]$$

and

$$\frac{d\psi_n}{dz}=\frac{4/n\pi}{\sinh n\pi l}\left[\cosh n\pi(l-z)\int_0^z G(t)\sinh n\pi t\,dt\right.$$

$$\left.-\cosh n\pi z\int_z^l G(t)\sinh n\pi(l-t)\,dt\right]$$

The author refers to an extension of my representation of velocity in the steady rotational flow of an inviscid, incompressible fluid, namely

$$\mathbf{V}=\nabla\phi-\frac{t\nabla p_o}{\rho}$$

to a rotating coordinate system. To complete the record it is desirable to give a derivation of the author's equation (6), since it has not been published elsewhere. If $\mathbf{W}$ is the velocity relative to a coordinate system rotating with angular velocity $\mathbf{\Omega}$, then Euler's equation for steady relative flow is

$$(\mathbf{W}\cdot\nabla)\mathbf{W}+2\mathbf{\Omega}\times\mathbf{W}+\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})=\frac{1}{\rho}\nabla p$$

where $\mathbf{r}$ is a position vector and p and ρ are pressure and density, respectively. It may be transformed to

$$\mathbf{W}\times(\nabla\times\mathbf{W}+2\mathbf{\Omega})=\nabla\left[\frac{p}{\rho}+\tfrac{1}{2}(W^2-\Omega^2r^2)\right]$$

where r is the radius from the rotating axis. The term

$$p_R=p+\tfrac{1}{2}\rho(W^2-\Omega^2r^2)=\rho I$$

is better called the relative Bernoulli pressure to avoid confusion, since stagnation pressure could be written $p+\frac{1}{2}\rho W^2$. The term

$$\nabla\times\mathbf{W}+2\mathbf{\Omega}=\boldsymbol{\omega}$$

is the absolute vorticity. Now Clebsch (ref. D–1) has shown that it is possible to represent the absolute velocity in a three-dimensional flow as

$$\mathbf{V}=\nabla\phi-\tau\nabla\sigma$$

where ϕ, σ, and τ are scalars which may be chosen arbitrarily. The relative velocity may then be written

$$\mathbf{W}=\nabla\phi-\tau\nabla\sigma-\mathbf{\Omega}\times\mathbf{r}$$

and the absolute vorticity

$$\boldsymbol{\omega}=\nabla\sigma\times\nabla\tau$$

Now we choose $\sigma = I$, so that

$$\nabla\sigma = \nabla I = \mathbf{W} \times \boldsymbol{\omega}$$

Hence

$$\boldsymbol{\omega} = (\mathbf{W} \times \boldsymbol{\omega}) \times \nabla\tau$$
$$= (\mathbf{W} \cdot \nabla\tau)\,\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla\tau)\,\mathbf{W}$$

The second term

$$\boldsymbol{\omega} \cdot \nabla\tau = \nabla\sigma \times \nabla\tau \cdot \nabla\tau = 0$$

hence

$$\mathbf{W} \cdot \nabla\tau = 1$$

or

$$\tau = \int \frac{ds}{W}$$

where the integral is taken along a streamline of the relative flow and s is the distance along the streamline. If, at any instant, one of the surfaces $\tau =$ constant is identified, then, at a time t later, the fluid particles will have drifted a distance downstream such that

$$t = \int \frac{ds}{W}$$

Hence, we may write

$$\mathbf{W} = \nabla\phi - t\nabla I - \boldsymbol{\Omega} \times \mathbf{r}$$

and

$$\boldsymbol{\omega} = \nabla I \times \nabla t$$

The author has adopted the approximation in which the stagnation pressure gradients are small but the disturbance from the upstream flow is large.

There is the possibility of some confusion in the use of the terms "primary flow" and "secondary flow." In all the methods given in the literature on the secondary-flow approximation, the stagnation pressure gradient, $\nabla p_o/\rho$ (or its equivalent, ∇I, in a rotating coordinate system) is assumed to be small of $O(\epsilon)$. We determine the components of vorticities in the flow by considering their convection by a flow of $O(1)$ for which $\nabla p_o/\rho = 0$ (or $\nabla I = 0$) and which satisfies the boundary conditions. This flow of $O(1)$ has frequently been described as the primary flow.

Writing the total velocity as

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}$$

where $\mathbf{V}_0$ is the primary flow and $\mathbf{v}$ is $O(\epsilon)$

$$\nabla \times \mathbf{V} = \nabla \times \mathbf{V}_0 + \nabla \times \mathbf{v}$$

Now

$$\mathbf{V}_0 \times (\nabla \times \mathbf{V}_0) = 0$$

because the primary flow has no gradient of p_o. Hence, the primary flow must either be a potential flow, $\nabla \times \mathbf{V}_0 = 0$, or a Beltrami flow with

$$\boldsymbol{\omega}_0 = \nabla \times \mathbf{V}_0 = \lambda \mathbf{V}_0$$

where λ is a scalar which is constant along a streamline of the primary flow. The only component of the vorticity $\boldsymbol{\omega}_0$ is ω_{0s} in the direction of $\mathbf{V}_0$, such that $\omega_{0s}/V_0 = \lambda =$ constant along a streamline. Beltrami flows are found downstream of rows of twisted blades around which the circulation varies along the span when the fluid is ideal and the flow upstream of the blades is irrotational. For such a flow, we may define $\boldsymbol{\omega}_0$ as the primary vorticity and note that it may be $O(1)$. Then $\nabla \times \mathbf{v}$ of $O(\epsilon)$ is the secondary vorticity. If the Beltrami flow is weak so that λ is $O(\epsilon)$, then we may choose a potential flow as the primary flow and incorporate the velocities induced by the vorticity ω_{0s} in the velocity $\mathbf{v}$.

This result may be extended to flows in rotating coordinate systems by noting that in the primary flow

$$\mathbf{W}_0 \times \boldsymbol{\omega}_0 = 0$$

so that

$$\boldsymbol{\omega}_0 = \lambda \mathbf{W}_0$$

where λ is now constant along a streamline of the relative flow. When λ is $O(\epsilon)$, it is possible to define a primary flow

$$\mathbf{W}_0 = \nabla \phi_0 - \boldsymbol{\Omega} \times \mathbf{r}$$

for which the absolute velocity is irrotational.

Hitherto, I have been discussing the definition of the primary flow used in calculating the vorticity convection. Some confusion has, perhaps, arisen because when considering the computation of the velocity components through cascades, it has been convenient to split the flow into two parts—namely, a flow which can be computed by axisymmetric methods (L. H. Smith calls this the primary flow) and a flow component which requires blade-to-blade analysis and in which the effects of the passage vorticity are obtained from equations such as the author's equations (16) and (19). In this latter calculation and in the axisymmetric calculations, we assume that the Kutta condition is adequately satisfied, in the latter case by using a blade element or potential flow theory.

For flows in which $\omega_{s1} = 0$ there appears to be agreement, supported by experimental data, that the actuator disc or axisymmetric approach combined with a passage vortex computation is satisfactory (Hawthorne and Novak, ref. 3).

When $\omega_{s1} \neq 0$ or in the extreme case when the approaching flow is a Beltrami flow, some question arises as to the adequacy of the method assumed for satisfying the Kutta condition in the axisymmetric or actuator-disc computations. It is not obvious that the blade element theory can adequately predict the outlet angle for such flows. Work is proceeding on this point, which is of considerable importance in step (6) of the sixth section of the paper. At the moment, all we can say is that step (6) contains the best assumptions that are available.

The most rigorous attempt to satisfy the Kutta condition for the secondary flows has been given by M. Gomi (ref. D–2).

J. H. HORLOCK (Cambridge University): With Dr. Dixon, I have for some years followed, and participated in, discussions between Dr. L. H. Smith and Sir William Hawthorne on the problem of secondary flow in stationary and rotating rows of blades. This introduction to the discussion is an attempt to summarize the position, to draw attention to differences in two approaches to the problem, and to pose some questions to which I still do not know the answers. These differences and questions are illustrated by some simple flows, concentrating first on the secondary flows in stators and later on the secondary flows in rotors. Much of the work presented here is not my own, but is drawn from correspondence between the four of us and notes of the discussions that have taken place.

A critical difference between the approaches of L. H. Smith on one hand and of Hawthorne and Dixon on the other lies in their definitions of secondary flow. Smith defines the secondary (passage) vorticity as the difference between the *actual* streamwise vorticity leaving the blade row (ω_s) and the "primary" vorticity that would exist downstream of the blade row if there were an infinite number of blades (ω_{s_0}). Hawthorne and Dixon define the secondary vorticity as the actual streamwise vorticity in the channel (ω_s). In my opinion, neither of these approaches is incorrect for stationary coordinates—it is simply a question of definition.

Let us first summarize the various equations that have been derived (table D–I) for secondary flow of an incompressible fluid, expressed in stationary coordinates. We shall use the notation of Dixon's figure 3 for the flow through the stationary row, working in absolute velocities and vorticities in the first instance, but using z instead of r (implying that the radius of the machine is very large, and we can work in Cartesian coordinates). Subscripts s and n mean parallel and normal to the absolute velocity, respectively.

We shall consider the use of these equations in two flows through stationary blade rows. In the first case there is no secondary vorticity at entry ($\omega_{s_1}=0$, $\omega_{n_1}\neq 0$) and in the second case there is no normal vorticity at entry ($\omega_{n_1}=0$, $\omega_{s_1}\neq 0$; a Beltrami flow). We compare the Hawthorne equation (D–3) with the Smith equation (D–5) in each case.

TABLE D–I.—*Equations for Secondary Flow (Stationary Coordinates)*

Equation	Reference and notes
$\bar{V}\cdot\nabla\left(\frac{\omega_s}{V}\right)=-\frac{1}{V^4}\left[2\bar{V}\times\nabla\left(\frac{p_0}{\rho}\right)\right]\cdot(\bar{V}\cdot\nabla)\bar{V}$ (D–1)	Hawthorne (ref. 13)—general equation
$\omega_{s_2}-\omega_{s_1}=-2\epsilon\omega_{n_1}$ (D–2)	Squire and Winter (ref. 4)—small deflection
$\delta\Gamma_s=\omega_{s_2}\sigma\cos\alpha_2$ $=\frac{dV_1}{dz}\left[-V_1\int\frac{ds}{V}+\frac{\sigma\cos\alpha_2}{2}\times\left(\frac{\sin\alpha_2}{\cos\alpha_1}-\frac{\sin\alpha_1}{\cos\alpha_2}\right)\right]$ (D–3)	Hawthorne (ref. 13)—$\omega_{s_1}=0$
$\omega_{s_2}=\left(\frac{\nabla p_0}{\rho}\right)\times(\nabla t_0)$ (D–4)	Hawthorne and Novak (ref. 3) —$\omega_{s_1}=0$
$\delta\Gamma_s=(\omega_s-\omega_{s0})_2\cos\alpha_2$ $=\left(\omega_{n_1}\frac{V_1\Gamma_{VA}}{V_\infty^2}+\frac{d\Gamma_V}{dn_1}\right)\frac{dn_1}{dn_2}$ (D–5)	L. H. Smith (ref. 5)—general
$\delta\Gamma_s=-V_1\frac{dV_1}{dz}\frac{\Gamma_{VA}}{V_\infty^2}+\frac{d\Gamma_V}{dz}$ (D–6)	L. H. Smith (ref. 5)—special case with $dn_1=dn_2$ $\omega_{n_1}=-\frac{dV_1}{dz}$
Γ_{VA} = actual (primary + secondary) circulation Γ_V = primary circulation n = distance normal to axisymmetric streamline	

The first case is as follows:

$$\omega_{s_1}=0$$

$$\omega_{n_1}=-\frac{dV_1}{dz}$$

Hawthorne's equation (D–3) may be written as

$$\delta\Gamma_s=-\frac{dV_1}{dz}\left(V_1\int\frac{ds}{V}\right)-\frac{dV_1}{dz}(\sigma\sin\alpha_1)+\frac{dV_1}{dz}\left(\sigma\frac{V_1}{V_2}\sin\alpha_2\right)\quad\text{(D–3a)}$$

Smith's equation (D–6) may be written as

$$\delta\Gamma_s = -\frac{dV_1}{dz} V_1 \frac{\Gamma_{VA}}{V_\infty^2} - \frac{d\Gamma_V}{dz} \tag{D–6a}$$

The first two terms are clearly equivalent. Smith also shows that in the primary flow

$$-\frac{d\Gamma_V}{dz} = -\sigma \frac{d}{dz}(V_{\theta_1} - V_{\theta_2})$$

$$= -\sigma\left(\sin\alpha_1 \frac{dV_1}{dz} - \sin\alpha_2 \frac{dV_2}{dz}\right)$$

and since

$$\frac{dp_{o1}}{dz} = \frac{dp_{o2}}{dz}$$

$$\frac{dV_2}{dz} = \frac{V_1}{V_2}\frac{dV_1}{dz} = \frac{\cos\alpha_2}{\cos\alpha_1}\frac{dV_1}{dz}$$

so that

$$\frac{d\Gamma_V}{dz} = \sigma\left(\sin\alpha_1 \frac{dV_1}{dz} - \sin\alpha_2 \frac{V_1}{V_2}\frac{dV_1}{dz}\right) \tag{D–7}$$

which establishes the identity of the other terms in the equations.

It is important to note here that in Smith's primary flow for this example there is no streamwise vorticity, so that there is no conflict in the definition of the secondary flow in this case.

Let us now examine the second case

$$\omega_{n_1} = 0 \qquad \omega_{s_1} \text{ finite}$$

Consider this Beltrami flow moving through a cascade of twisted flat plates that receive the flow at zero incidence and do not deflect it at all.

We cannot use Hawthorne's equation (D–3), which was derived for $\omega_{s_1}=0$, but the general equation (D–1) shows that with no change in velocity, $(\bar{V}\cdot\nabla)\bar{V}=0$ and

$$\omega_{s_2} = \omega_{s_1}$$

Thus, on Hawthorne's definition, the secondary vorticity at exit is equal to that at entry. It is from this total secondary vorticity that we may calculate the total flow velocity perpendicular to the vorticity vector.

From Smith's equation, since $\Gamma_{VA}=\Gamma_V=0$, $\delta\Gamma_s=0$; so, on Smith's definition, there is no secondary flow $(\omega_s-\omega_{s_0})_2=0$. However, primary vorticity $(\omega_{s_0})_2=(\omega_{s_0})_1$ exists, so that there is vorticity $\omega_{s_2}=(\omega_{s_0})_1$ along the streamline in the total flow, as in Hawthorne's calculation.

This example illustrates the important difference in definition of secondary flow.

It appears then, from these examples, that the expressions given for secondary flow through stators are entirely consistent, provided the difference in definition is appreciated.

We now try to make a similar comparison for a rotating blade row and again consider a number of examples to which the various approaches must provide a solution. Equations now available are presented in table D–II, with $\bar{W}$ now the relative velocity.

We may at this stage note that A. G. Smith has simplified his equation to the form

$$\left(\frac{\omega_W}{W}\right)_2-\left(\frac{\omega_W}{W}\right)_1=\int_1^2 2\frac{\nabla I}{\rho}\frac{\sin\gamma}{W^2}\,d\epsilon_R$$

$$+\int_1^2\frac{2\Omega}{W^3}\frac{\nabla I}{\rho}\cos\delta\,ds \qquad \text{(D–12)}$$

where δ is the angle between $\nabla I/\rho$ and Ω and is usually nearly $\pi/2$, so that the second term may be considered second-order for the purposes of this discussion; γ is the angle between $(\nabla I/\rho)\times\bar{W}$ and the direction of curvature of the relative streamline, which is also approximately $\pi/2$; and ϵ_R is the relative deflection. Thus

$$\left(\frac{\omega_W}{W}\right)_2-\left(\frac{\omega_W}{W}\right)_1=\int_0^{\epsilon_R} 2\frac{\nabla I}{\rho}\frac{d\epsilon_R}{W^2} \qquad \text{(D–13)}$$

But $\nabla I/\rho=\bar{W}\times\bar{\omega}$; thus, if the *absolute* vorticity perpendicular to the relative stream line is $\omega_{R_{p1}}$ and the velocity changes little in a small relative deflection of the flow, then

$$\omega_{W_2}-\omega_{W_1}=-2\epsilon_R\omega_{R_{p1}} \qquad \text{(D–14)}$$

The present paper argues that the perturbations in the relative flow arise as a result of the absolute vorticity $\bar{\omega}$, but L. H. Smith again argues that the vorticity that should be used is the difference between the total absolute vorticity and the absolute vorticity in the primary flow $(\bar{\omega}-\bar{\omega}_0)$. This argument may be illustrated by a statement of the various velocities and their curl.

	Velocity	Curl
Absolute primary flow	$\bar{V}_0$	$\nabla\times V_0=\bar{\omega}_0$
Disturbed total absolute flow	$\bar{V}=\bar{V}_0+\bar{v}'$	$\bar{\omega}=\nabla\times V=\bar{\omega}_0+\bar{\omega}'$
Disturbance flow (absolute)	$\bar{V}-\bar{V}_0=\bar{v}'$	$\bar{\omega}'$
Relative primary flow	$\bar{W}_0=\bar{V}_0-\bar{\Omega}\times\bar{r}$	$\nabla\times\bar{W}_0=\bar{\omega}_0-2\bar{\Omega}$
Disturbed total relative flow	$\bar{W}=\bar{V}_0+\bar{v}'-\bar{\Omega}\times\bar{r}$	$\nabla\times\bar{W}=\bar{\omega}_0-2\bar{\Omega}+\bar{\omega}'$
Disturbance flow (relative)	$\bar{W}-\bar{W}_0=v'$	$\nabla\times(\bar{W}-\bar{W}_0)=\bar{\omega}'$

TABLE D–II. *Equations for Secondary Flow (Rotating Coordinates)*

Equation	Reference and notes
$$(\bar{W}\cdot\nabla)\frac{\omega_W}{W}=\frac{1}{W^4}\left[2\left(\frac{\nabla I}{\rho}\right)\times\bar{W}\right]\cdot(\bar{W}\cdot\nabla)\bar{W}+\frac{2}{W^2}\left[\bar{\Omega}\cdot\left(\frac{\nabla I}{\rho}\right)\right] \qquad \text{(D–8)}$$ where subscript w indicates resolution along the lines of the relative velocity, but ω is still the absolute vorticity.	A. G. Smith (ref. 8)
$$\omega_W=\nabla I\times\nabla t_0 \qquad \text{(D–9)}$$ where, from Hawthorne's discussion of the present paper, $$t_0=\int\frac{ds_R}{W} \qquad \text{(D–10)}$$	Dixon (present paper)—presumably for $\omega_1=0$ only (potential flow at entry)
$$(\omega_W-\omega_{W_0})=\left(W_1\omega_{R_{p1}}\frac{\Gamma_{VA}}{W_\infty^2}+\frac{d\Gamma}{dn_1}\right)\frac{dn_1}{dn_2} \qquad \text{(D–11)}$$ where $\omega_{R_{p1}}$ is the absolute vorticity, resolved normal to the relative streamlines at entry.	L. H. Smith (ref. 5)

In his derivation, Smith employs the Helmholz laws to determine how the absolute vorticity changes as the flow passes through a blade row, which may be rotating or stationary. He defines the secondary vorticity as the difference between the actual absolute vorticity and the absolute vorticity of the primary flow. His reason for using the difference between two absolute vorticities to obtain flow perturbations in the rotating coordinate system is illustrated in the paragraphs following Equation (D–14) of this discussion.

Thus, the disturbance flow is calculated from $\bar{\omega}'=\bar{\omega}-\bar{\omega}_0$, the difference between the total absolute vorticity and the absolute vorticity in the primary flow.

In Dixon's example in the paper, the primary flow is potential, so that $\bar{\omega}_0=$ zero, and both Dixon and L. H. Smith give the same answer.

We again consider a number of examples of flows through rotors in order to establish whether the various approaches give the same result.

We shall first consider A. G. Smith's example of uniform flow at the entry to a rotor.

If the entry flow is axial and uniform ($V_1=$constant, $\omega_1=0$), then

$$\nabla\left(\frac{I_1}{\rho}\right)=\frac{\nabla p}{\rho}+\nabla\left(\frac{V_1^2+U^2}{2}\right)-\nabla\left(\frac{U^2}{2}\right)=0$$

It is then evident from A. G. Smith's equation that no *absolute* vorticity ω_W (resolved in the relative direction) can be developed along the streamline, even if the relative flow is deflected. The Hawthorne/Dixon equations imply that no absolute secondary vorticity (ω) can be generated since $\nabla I \times \nabla t_0$ is zero, and the primary flow is potential.

Consider the rotor to be made up of helical plates which do not deflect the flow and have no lift or circulation.

Both A. G. Smith's equations and the Dixon/Hawthorne equations show that the absolute vorticity (ω_W) does not change along the relative streamline in this case. ω_W is zero at entry and at exit. The primary flow also has zero vorticity $\omega_{W_0}=0$ and the difference $\omega_W-\omega_{W_0}=0$.T here is no secondary flow.

L. H. Smith's equation shows that $\omega_W=\omega_{W_0}$ is zero since $\Gamma_{VA}=\Gamma_V=0$.

We now consider a forced vortex flow entering a rotor; the entry tangential velocity is everywhere equal to the blade speed, but the entry axial velocity is uniform. The entry vorticity is in the axial direction, so that

$$\frac{\nabla I_1}{\rho}=\bar{W}_1\times 2\bar{\Omega}=0$$

The relative velocity and absolute vorticity vectors are parallel—a kind of rotating Beltrami flow. The Dixon/Hawthorne equation cannot strictly be used in this case since the entry flow is not potential.

Consider next this flow moving through rotating flat plates, aligned in the axial direction, operating at zero incidence with zero circulation and zero deflection. A. G. Smith's equations give the absolute vorticity $\omega_W=2\Omega$ as unchanged. However, the primary flow also has vorticity $\omega_{0W}=2\Omega$, so the difference $\omega_W-\omega_{W_0}$ is zero, and there is no secondary flow. (Note that 2Ω is subtracted from ω_W here, not because it is the curl of the blade speed but because it is the vorticity in the primary flow.)

L. H. Smith's equation (15) also gives $\omega_W-\omega_{W_0}$ as zero although the primary flow has leaving vorticity 2Ω.

The discussor's tentative conclusions are

(1) The equations of Hawthorne and L. H. Smith are consistent for flow through stators, if account is taken of the different definitions of secondary flow.

(2) The Dixon/Hawthorne equation (10) refers only to perturbations of a primary potential flow.

(3) A. G. Smith's equation accurately describes the change in total absolute vorticity resolved along the relative streamline.

(4) To determine the secondary flow in rotors, either L. H. Smith's equation or A. G. Smith's equation may be used, as long as the primary vorticity is first subtracted from the absolute vorticity in the latter case. The secondary perturbation flow is calculated from $\bar{\omega}_W-\bar{\omega}_{W_0}$.

B. LAKSHMINARAYANA (The Pennsylvania State University): The secondary flow approximations (namely, small shear and large disturbance) are inadequate for application to annulus wall or hub wall boundary layers. The Bernoulli surface rotation and viscous effects tend to reduce the development of secondary flow. This was indicated by B. Lakshminarayana and J. H. Horlock in reference D–3 where they adequately demonstrated that Bernoulli surface rotation and boundary-layer growth through the blade row should be taken into account for accurate prediction of secondary velocities and outlet angles, especially near the wall. In view of this work, it is somewhat surprising that the author was able to predict the outlet angles accurately (fig. 5) ignoring these effects. The radial or spanwise velocities plotted in figure 6 indicate that the distortions of Bernoulli surfaces in the author's inlet guide vanes are not negligible. The author's conclusion that the outlet angle predictions are good is based on only two experimental points inside the annulus wall boundary layer (fig. 5). It would be useful if the author could provide a few more data points, especially near the wall, and indicate whether the predictions are good in this region.

With regard to the rotor secondary flow, the components ω_{s2R} and ω_{n2R} at the inlet to the rotor (fig. 3) will vary through the boundary layer due to change in relative flow direction. This effect has been neglected in this paper, thus leading to inaccurate estimation of the streamwise vorticity downstream and the resulting perturbations. Inclusion of these effects is essential if accurate prediction near the wall is sought. I have illustrated this effect, quantitatively, for an isolated rotor with axial entry and neglecting the contributions to the downstream streamwise vorticity of the wake vortex sheets.

Using A. G. Smith's equation (eq. D–13 in the discussion by Horlock), the streamwise vorticity in the relative flow direction can be written as

$$\frac{\omega_{W_2}}{\Omega} = \sin\beta_1 - 2\epsilon\cos\beta_1$$

where Ω is the inlet absolute vorticity for axial entry of the absolute flow.

If $\Delta\beta$ is the change in relative flow direction at any location inside the boundary layer and $\epsilon_o + \Delta\beta$ is the corresponding turning angle of the relative flow, the following expression can be derived on the assumption that $\cos\Delta\beta \simeq 1$ and $\sin\Delta\beta \simeq \Delta\beta$

$$\frac{\omega_{W_2}}{\Omega} = \sin\beta_{1_o} - 2\epsilon_o\cos\beta_{1_o} + \Delta\beta(r)\sin\beta_{1_o}[2\epsilon_o + 2\Delta\beta(r) - \phi]$$

where subscript o refers to free-stream values and $\phi = \cot\beta_o$. The last term in the equation represents the error in neglecting the change in β_1 through the boundary layer. $\Delta\beta(r)$ can be derived from the known absolute velocity distribution in the boundary layer. For example, for $\phi=0.5$, $\epsilon_0=25°$ and at the location where the absolute velocity is 60 percent of the free-stream velocity, the effect of neglecting this effect results in 20 percent error in the estimate of ω_{W_2}/Ω at this location. The magnitude of this error depends on the values of ϕ and ϵ; the error is largest for low-speed, large-turning or high-speed, low-turning blade rows. A similar correction can be incorporated in the author's general equation (5).

Thus the author's poor predictions for the rotor (fig. 9) may be due to

(1) Neglect of the variation in ω_{s2R} and ω_{n2R} through the annulus wall boundary layer

(2) Tip clearance effect, which has a tendency to underturn the relative flow (This effect has been neglected in this analysis.)

(3) Boundary-layer growth through the rotor

S. L. DIXON (author): In the discussions presented by Professors Hawthorne and Horlock, the main point at issue is the effect of the streamwise vorticity entering a blade row on the secondary motion generated at exit. The extreme case of a Beltrami flow passing through a blade row poses some presently unanswerable questions on the flow angle leaving the blades. In this paper, I have been concerned with weakly sheared flow in which vorticity is *convected* by a primary potential flow. This potential flow *could* convect a Beltrami flow provided that, in the notation used by Hawthorne,

$$\lambda=\frac{\omega_{0s}}{W}=O(\epsilon)$$

Horlock has considered a forced vortex flow entering a rotor with a tangential velocity equal to the blade speed and having uniform absolute velocity in the axial direction. For this "rotating Beltrami flow," the primary flow has vorticity $O(1)$ and there is *no* transportation of secondary vorticity—at least not in the sense of the paper. The vorticity is an *integral* part of the primary flow. We could replace this flow by a primary potential flow convecting another flow whose vorticity is 2Ω. However, the theory is not valid for flows in which the vorticity is $O(1)$ and should not be applied to such an extreme case.

More light may be thrown on the way Beltrami flows behave by an extension of some analysis due to Hawthorne (ref. 11). We note that for a Beltrami flow in rotating coordinates

$$\boldsymbol{\omega}=\lambda\mathbf{W} \tag{D–15}$$

where λ is a scalar. By taking the curl of both sides of equation (D–15) and noting that div $\mathbf{W}=$ curl $\mathbf{\Omega}=0$, we find

$$\nabla^2\mathbf{W}+\nabla\lambda\times\mathbf{W}=-\lambda\,\text{curl}\,\mathbf{W} \tag{D–16}$$

If we consider the special class of flows where $\lambda=\omega/W=$ constant, then $\nabla\lambda=0$ and

$$\nabla^2\mathbf{W}=-\lambda \text{ curl } \mathbf{W}=-\lambda(\boldsymbol{\omega}-2\boldsymbol{\Omega}) \tag{D-17}$$

Using equation (D–15) in equation (D–17), we get

$$(\nabla^2+\lambda^2)\mathbf{W}=2\Omega\lambda \tag{D-18}$$

For the special case in which $\boldsymbol{\omega}=2\boldsymbol{\Omega}$ (Horlock's "rotating Beltrami flow"), equation (D–17) gives us

$$\nabla^2\mathbf{W}=0$$

for which the only solution is $\mathbf{W}=$ constant (i.e., no secondary flow). When $\boldsymbol{\omega}\neq 2\Omega$, equation (D–18) may be solved provided sufficient boundary conditions are known.

Professor Lakshminarayana has rightly mentioned that boundary-layer growth effects should be included in the method. In addition to this, I would include the changes in outlet angle due to flow separation. However, at present no reliable analytical method for predicting these changes is known. Whenever such a method becomes available its inclusion should significantly improve the accuracy of the predicted axial-velocity profiles after diffusing blade rows.

It is no longer possible to obtain any further experimental data from the original source.

The reason why directional changes of the vorticity vector induced by the vorticity itself have been neglected in the analysis is bound up with the nature of the approximations made in the secondary-flow theory. Hawthorne (ref. 3) has indicated that for small vorticity the primary flow may be assumed to remain on cylindrical surfaces of constant radius. If there is a distortion of these surfaces of $O(\epsilon)$, the effect on the vorticity components is $O(\epsilon^2)$ and may be neglected. The analysis given in the paper is based on the assumption of small vorticity, so that it would be incorrect to attempt the higher-order approximations suggested by Professor Lakshminarayana.

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