Compressibility Effects in the Kemp-Sears Problem[1]

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The effects of including compressibility in the Kemp-Sears problem of aerodynamic interference between moving blade rows are examined. Methods of linearized, subsonic, plane, unsteady flow are adopted. The major new effect is that a resonance appears at certain combinations of flow Mach number, tip Mach number, and blade vane ratios. The resonance is at exactly the Tyler-Sofrin cutoff condition for rotor-stator interaction. At such conditions the unsteady lift on a blade row due to externally imposed nonstationary upwash vanishes. However, the resonance appears to be very sharp and seems to be more significant as an indication that around this condition the unsteady lift changes very rapidly.

This paper is concerned with unsteady blade forces developed on an axial compressor blade row due to unsteady periodic wave disturbances. The applications discussed pertain to unsteady disturbances produced on a blade row due to steady-state lifts of adjacent rows moving past the row of interest and due to viscous wakes shed by upstream rows.

The framework for the analytical approach to the solution of this complex problem is contained in two pioneering papers by Kemp and Sears (refs. 1, 2). Kemp and Sears adopt the representation of blade wheels as infinite cascades of two-dimensional airfoils. They consider a typical airfoil in the blade row on which unsteady forces are to be calculated as an isolated airfoil in linearized, unsteady, incompressible flow. This isolated airfoil is subject to unsteady disturbances from various sources. Reference 1 considers nonstationary upwash due to translation of steady-state design fields of an adjacent row. In this case the effect of a row upon

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an adjacent row downstream of it and the nonstationary upwash due to the "inviscid" wakes is included. Reference 2 considers nonstationary upwash due to viscous wakes shed by an upstream row. In this paper two refinements are considered. First, a typical airfoil of the row on which we wish to calculate the unsteady blade forces is regarded as a member of an isolated, infinite *cascade* of airfoils rather than as an isolated airfoil. Second, in calculating both the nonstationary upwash and the unsteady forces we include compressibility effects. Methods for linearized, subsonic, plane, compressible unsteady flow are adopted. For the present, we have not included nonstationary upwashes due to "inviscid" wakes. The motivation for incorporating these refinements is simply that in most present-day applications where unsteady blade forces in fans and compressors are of interest, the Mach numbers of the flow are too high for compressibility effects to be negligible.

Consider an isolated, infinite flat-plate cascade of identical airfoils spaced s apart whose chord lengths have been normalized to unity and with flow at Mach number M through the cascade. M is restricted to $M<1$ (subsonic flow through the blade passages). The problem is to calculate the unsteady lift on a typical blade of such a cascade due to a known nonstationary upwash on it where the time-dependence of the nonstationary upwash is of simple harmonic type. The nonstationary upwash will later be related to the translation of adjacent steady-state design fields and to viscous wakes shed by an upstream row. Due to the symmetry of the cascade the nonstationary upwash on the nth blade is essentially the same as on the zeroth blade, except for a phase difference factor of $\exp\,(jn\gamma)$[2], where we will later relate γ to the aerodynamic parameters.

The method of solution adopted is the method of distributed singularities and singular integral equations. An $x-y$ coordinate system is used as shown in figure 1. Let ξ denote the running coordinate on the zeroth blade that runs from $-\frac{1}{2}$ to $\frac{1}{2}$. Let the unsteady lift distribution on the zeroth chord be denoted by $F(\xi)e^{j\omega t}$. Let $K_v(x,\xi)$ denote a kernel function that gives the nonstationary upwash at the point $(x,0)$ on the zeroth blade due to an infinite row of equal oscillating forces of unit strength with phase shift γ located at ξ and its corresponding points; i.e., at

$$\xi+ns\sin\,(\alpha_s),\ ns\cos\,(\alpha_s)$$

where $n=0, \pm1, \pm2, \ldots$, etc. $K_v(x,\xi)$ is known since it is merely the sum of upwash contributions due to unit forces of known phase and location. Let $v_d(x)e^{j\omega t}$ be the known nonstationary upwash due to the adjacent rows. Then the boundary condition that the velocity normal to

[2] $j=\sqrt{-1}$

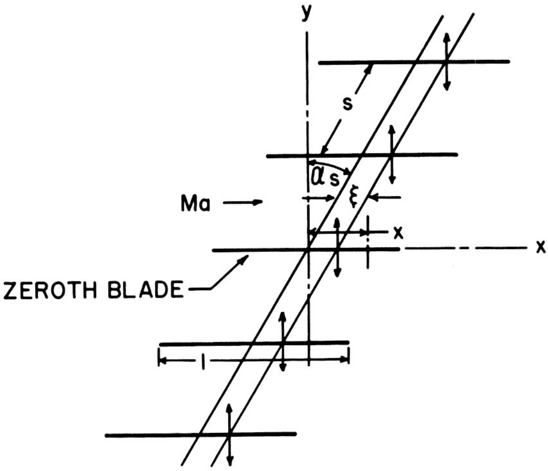

FIGURE 1.—*Coordinate system.*

the blade chords be zero leads to the integral equation:

$$\int_{-1/2}^{1/2} F(\xi)K_v(x,\xi)\, d\xi = -v_d(x) \ldots \tag{1}$$

where $K_v(x,\xi)$ and $v_d(x)$ are known. Only $F(\xi)$ is unknown, and, after solving equation (1), the total unsteady lift is found from

$$(\text{total lift}) = \int_{-1/2}^{1/2} F(\xi)\, d\xi \tag{2}$$

$K_v(x,\xi)$ has a singularity; i.e., a term as $1/(x-\xi)$; hence, equation (1) constitutes a singular integral equation for the force distribution. There are two other requirements on $F(\xi)$: first, that it vanish at the trailing edge $\xi=\frac{1}{2}$ (the Kutta-Joukowski condition), and, second, that it have a square-root singularity at the leading edge; i.e., the force distribution tends to ∞ as $\xi\rightarrow-\frac{1}{2}$ as $1/\sqrt{\xi+\frac{1}{2}}$.

CALCULATION OF KERNEL FUNCTION

The physical significance of the kernel function $K_v(x,\xi)$ is repeated: It is the upwash at point $(x,0)$ on the zeroth blade due to an infinite row of unit forces located at

$$\xi+ns\sin(\alpha_s),\ ns\cos(\alpha_s)$$

with $n=0, \pm1, \pm2, \ldots$, etc., where the phase of the oscillating force on the nth blade is related to that on the zeroth blade by a factor $\exp(jn\gamma)$.

In the derivation of the kernel function we have relied heavily on a recent paper by Kaji and Okazaki, who treat an analogous problem (ref. 3). We are especially indebted to their repeated and ingenious use of the Poisson summation formula to switch from infinite series of Hankel functions to infinite series of exponential functions. In view of their paper, we omit many of the details of the derivation of the kernel function and follow Kaji's notation.

The first step is to note that the nonstationary upwash $V_n(x,0,t)$ at $(x,0)$ due to a force of complex strength $\exp(jn\gamma)$ located at $[\xi+ns\sin(\alpha_s), ns\cos(\alpha_s)]$ is as follows.

Let $k=\omega/a$, $\beta=\sqrt{1-M^2}$, ρ=density of uniformly flowing medium, a=speed of sound, $x_n=x-\xi-ns\sin(\alpha_s)$, $y_n=-ns\cos(\alpha_s)$, and $x_n'=x'-\xi-ns\cos(\alpha_s)$. Then

$$V_n(x,0,t)=e^{j\omega t}e^{jn\gamma}\left(\frac{k\beta}{4j\rho Ma}\exp\left(j\frac{kMx_n}{1-M^2}\right)\right.$$

$$\times\left\{-x_nH_1^{(2)}\frac{\left(\frac{\omega}{\beta^2a}\sqrt{x_n^2+\beta^2y_n^2}\right)}{\beta^2\sqrt{x_n^2+\beta^2y_n^2}}-\frac{j}{\beta^2M}H_0^{(2)}\right.$$

$$\left.\times\left[\frac{\omega}{\beta^2a}(x_n^2+\beta^2y_n^2)^{1/2}\right]\right\}-\frac{k^2\exp\left(-j\frac{k}{M}x_n\right)}{4jM^2\rho\beta Ma}$$

$$\left.\times\int_{-\infty}^{x_n}\exp\left[j\frac{kx_n'}{M(1-M^2)}\right]H_0^{(2)}\left(\frac{\omega}{\beta^2a}\sqrt{x_n'^2+\beta^2y_n^2}\right)dx\right) \qquad (3)^3$$

and

$$\left.K_v(x,\xi)=\sum_{n=-\infty}^{\infty}V_n(x,0,t)\right\} \qquad (4)$$

The above expression for V_n may be deduced from equations (8) and (11), section 14.3, of Y. C. Fung's "An Introduction to the Theory of Aeroelasticity." Thus,

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[3] $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions of the second kind of orders 0 and 1, respectively.

$$K_r(x,\xi) = \frac{jk}{4\pi\rho\beta^2 Ma} \exp\left\{j\left[\frac{kM(x-\xi)}{1-M^2} - P\delta\right]\right\}$$

$$\times\left[\frac{\ell(x-\xi)}{2w}\left(2\pi + s\bar{P}\sin(\alpha_s)I_1 - \frac{\ell s \sin(\alpha_s)}{2w} I_2\right)\right.$$

$$\left. + \frac{j(1-M^2)}{M^2}\left(\frac{\pi M I_3}{\sqrt{1-M^2}} + I_4\right)\right] \tag{5}$$

where

$$P = \frac{-2(x-\xi)\sin(\alpha_s)\pi}{s[1-M^2\cos^2(\alpha_s)]}$$

$$\delta = \frac{1}{2\pi}\left[\gamma - \frac{ksM\sin(\alpha_s)}{(1-M^2)}\right]$$

$$\ell = \frac{k}{\sqrt{1-M^2}}$$

$$\bar{P} = \frac{-2\pi\sin(\alpha_s)}{s[1-M^2\cos^2(\alpha_s)]}$$

and

$$w = \frac{ks\sqrt{1-M^2\cos^2(\alpha_s)}}{2\pi(1-M^2)}$$

$$\bar{\eta} = \frac{2\pi}{s}\,\frac{\sqrt{1-M^2}\cos(\alpha_s)}{[1-M^2\cos^2(\alpha_s)]}$$

and

$$\eta = \bar{\eta}\,|x-\xi|$$

and

$$I_1 = \frac{j}{\pi\omega\eta}\sum_{n=-\infty}^{\infty} H_1^{(2)}\frac{[w\sqrt{(2\pi n+P)^2+\eta^2}]}{[(2\pi n+P)^2+\eta^2]^{1/2}} \exp[j(2\pi n+P)\delta] \tag{5a}$$

$$I_2 = \sum_{-\infty}^{\infty}\frac{(2\pi n+P)H_1^{(2)}}{\sqrt{(2\pi n+P)^2+\eta^2}}[w\sqrt{(2\pi n+P)^2+\eta^2}] \exp[j(2\pi n+P)\delta] \tag{5b}$$

$$I_3 = \sum_{-\infty}^{\infty} H_0^{(2)}[w\sqrt{(2\pi n+P)^2+\eta^2}] \exp[j(2\pi n+P)\delta] \tag{5c}$$

I_4 represents the contribution from the integral term in equation (3); we give below the transformed form of it after application of the Poisson summation formula:

$$I_4=\ell\sum_{-\infty}^{\infty}\frac{\exp\left[jnP-\eta\sqrt{(\delta-n)^2-w^2}\right]}{\sqrt{(\delta-n)^2-w^2}\left[j(n-\delta)\bar{P}+j\dfrac{k}{M(1-M^2)}+\eta\sqrt{(\delta-n)^2-w^2}\right]}$$

for $(x-\xi)<0$. (5d)

$$I_4=\ell\sum_{-\infty}^{\infty}\frac{\exp\left[jnP-\eta\sqrt{(\delta-n)^2-w^2}\right]}{\sqrt{(\delta-n)^2-w^2}\left[j(n-\delta)\bar{P}+j\dfrac{k}{M(1-M^2)}-\eta\sqrt{(\delta-n)^2-w^2}\right]}$$

$$+\frac{2\eta\exp\left[jP\delta-j\dfrac{k(x-\xi)}{M(1-M^2)}\right]}{\left[(n-\delta)\bar{P}+\dfrac{k}{M(1-M^2)}\right]^2+\bar{\eta}^2\left[(\delta-n)^2-w^2\right]}$$

for $(x-\xi)>0$.

For $x\neq\xi$, the expressions for I_1, I_2, and I_3 may be transformed by Poisson's summation formula as under

$$I_1=\frac{j}{\pi w\eta}\sum_{-\infty}^{\infty}\exp\left[jnP-\eta\sqrt{(\delta-n)^2-w^2}\right] \tag{5a}$$

$$I_2=\frac{1}{\pi w}\sum_{-\infty}^{\infty}\frac{(n-\delta)}{\sqrt{(\delta-n)^2-w^2}}\exp\left[jnP-\eta\sqrt{(\delta-n)^2-w^2}\right] \tag{5b}$$

$$I_3=\frac{j}{\pi}\sum_{-\infty}^{\infty}\frac{1}{\sqrt{(\delta-n)^2-w^2}}\exp\left[jnP-\eta\sqrt{(\delta-n)^2-w^2}\right] \tag{5c}$$

The expressions for I_1, I_2, and I_3 diverge as $x\to\xi$ in equations (5a), (5b), and (5c), or as P, $\eta\to 0$, which is merely the indication of the $1/|x-\xi|$ and $\log|x-\xi|$ types of singularities in $K_v(x,\xi)$. To locate these singularities, we examine the small value expansions of the Hankel functions as follows.

$$H_0^{(2)}(x)\sim-\frac{2j}{\pi}\log(x)+\left[1-\frac{2j}{\pi}(\gamma_0-\log 2)\right]$$

where $\gamma_0=0.5772=$ Euler's constant and $H_0^{(2)}(x)\sim 2j/\pi x$. With the aid of the above it is easily shown that

$$K_v(x,\xi)-\frac{jk}{4\pi\rho\beta^2 Ma}\left[\frac{2j\beta^3}{k(x-\xi)}+\frac{2\sqrt{1-M^2}}{M}\log\left(|x-\xi|\right)\right]=K_r(x,\xi) \quad (6)$$

is regular as $x\rightarrow\xi$. This motivates writing

$$K_v(x,\xi)=K_r(x,\xi)-\frac{\beta}{2\pi\rho Ma(x-\xi)}+j\frac{k\log\left(|x-\xi|\right)}{2\pi\rho\beta M^2 a} \quad (7)$$

Finally, let

$$f(x)=\frac{-2\rho Ma}{\beta}v_d(x) \quad (8)$$

Equation (1) now reads as

$$\int_{-1/2}^{1/2}\frac{F(\xi)}{\pi}\left[\frac{2\rho M\dot{a}\pi}{\beta}K_r(x,\xi)+\frac{jk\log\left(|x-\xi|\right)}{M(1-M^2)}-\frac{1}{(x-\xi)}\right]d\xi=f(x) \quad (9)$$

CALCULATION OF UPWASH DUE TO TRANSLATION OF ADJACENT STEADY-STATE FIELDS

Potential Interaction

The problem of potential interaction (see fig. 2) is to calculate the nonstationary upwash on line AB due to steady-state design lifts on a blade row on the right-hand side, which translates downward at a certain speed corresponding to a tip Mach number M_t. In a linearized treatment, clearly the stagger (α_r) and Mach number M_r of flow through the adjacent row are related to M_t, M, and α_s by

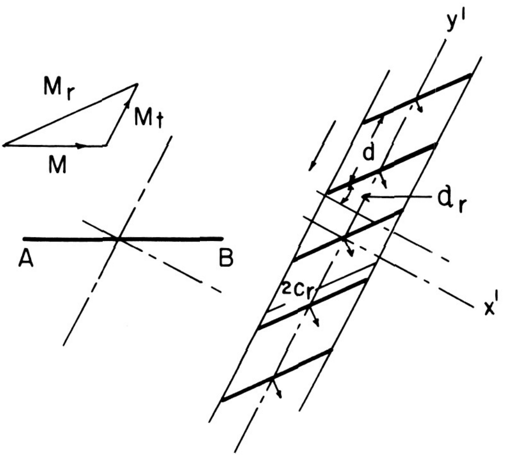

FIGURE 2.—*Geometry of adjacent row.*

$$\alpha_r = \tan^{-1}\left[\frac{M_t + M \sin(\alpha_s)}{M \cos(\alpha_s)}\right]$$

and

$$M_r = \sqrt{M_t^2 + M^2 + 2M_t M \sin(\alpha_s)}$$

Let $W_r = aM_r$. We first find the solution for velocity components u' and v', parallel to the x'-y' coordinate system of figure 2, due to equally spaced concentrated unit forces at the origin and its corresponding points as shown in figure 2. (Note that the blade exerts a force on the fluid equal and opposite to the force exerted by the fluid on the blade.) We use a frame of reference fixed with respect to the translating blade row so that we have a steady-state problem. We have to consider the effect of a sum of forces:

$$\mathbf{1} \sum_{n=-\infty}^{\infty} \delta(x')\delta\left(y' - 2n\frac{d}{2}\right)$$

where **1** denotes a unit force vector and δ stands for the Dirac delta function. By using a result on page 68 of reference 4 concerning the sum of an infinite row of equally spaced delta functions, we find that the above is clearly equal to

$$\frac{\mathbf{1}}{d}\delta(x')\left[1 + 2\sum_{n=1}^{\infty} \exp\left(j\frac{2\pi n y'}{d}\right)\right]$$

since our use of complex forms always implies that real parts are to be taken. The linearized equations of motion and continuity are

$$\rho\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + W_r\left[\cos(\alpha_r)\frac{\partial \rho'}{\partial x'} + \sin(\alpha_r)\frac{\partial \rho'}{\partial y'}\right] = 0 \tag{10a}$$

$$\cos(\alpha_r)\frac{\partial u'}{\partial x'} + \sin(\alpha_r)\frac{\partial u'}{\partial y'}$$

$$= -\frac{1}{\rho W_r}\frac{\partial p'}{\partial x'} + \frac{\sin(\alpha_r)\,\delta(x')}{\rho d W_r}\left[1 + 2\sum_{1}^{\infty} \exp\left(j\frac{2\pi n y'}{d}\right)\right] \tag{10b}$$

and

$$\cos(\alpha_r)\frac{\partial v'}{\partial x'} + \sin(\alpha_r)\frac{\partial v'}{\partial y'}$$

$$= -\frac{1}{\rho w_r}\frac{\partial p'}{\partial y'} \frac{-\cos(\alpha_r)\delta(x')}{\rho d W_r}\left[1 + 2\sum_{1}^{\infty} \exp\left(j\frac{2\pi n y'}{d}\right)\right] \tag{10c}$$

In what follows we omit the y' independent term in the force since it gives a stationary upwash (cf. p. 591 of ref. 1). In equations (12a), (12b), and (12c), ρ' and p' stand for small perturbations of the density and pressure. Eliminating p' from equations (10b) and (10c) we derive

$$\left[\cos(\alpha_r)\frac{\partial}{\partial x'}+\sin(\alpha_r)\frac{\partial}{\partial y'}\right]\left[\frac{\partial u'}{\partial y'}-\frac{\partial v'}{\partial x'}-\frac{2\delta(x')}{\rho W_r d}\sum_1^\infty \exp\left(j\frac{2\pi n y'}{d}\right)\right]=0$$

Since u', v', and the delta function term vanish far from the blade row,

$$\frac{\partial u'}{\partial y'}-\frac{\partial v'}{\partial x'}=\frac{2\delta(x')}{\rho W_r d}\sum_1^\infty \exp\left[j\frac{2\pi n y'}{d}\right] \tag{11}$$

(Kutta-Joukowski Law).

Next we eliminate the force terms in equations (12b) and (12c); assuming an isentropic relation between p' and ρ', we obtain:

$$\frac{\partial u'}{\partial x'}[1-M_r^2\cos^2(\alpha_r)]+\frac{\partial v'}{\partial y'}[1-M_r^2\sin^2(\alpha_r)]$$

$$=M_r^2\sin(\alpha_r)\cos(\alpha_r)\left[\frac{\partial u'}{\partial y'}+\frac{\partial v'}{\partial x'}\right] \tag{12}$$

(modified continuity equation).

Using equations (11) and (12), single equations for u' and v' may be obtained that may be solved by requiring that u' and v' vanish as $x'\to\pm\infty$. We omit the details and give the result.

$$u' \text{ for } x'\gtrless 0=-\frac{1}{\rho d W_r}\frac{[j\sqrt{1-M_r^2}\pm M_r^2\sin(\alpha_r)\cos(\alpha_r)]}{1-M_r^2\cos^2(\alpha_r)}$$

$$\times\sum_1^\infty \exp\left(j\frac{2\pi n y'}{d}\right)$$

$$\times\exp\left\{\frac{2\pi n x'[jM_r^2\sin(\alpha_r)\cos(\alpha_r)\mp\sqrt{1-M_r^2}]}{d(1-M_r^2)\cos^2(\alpha_r)}\right\} \tag{13a}$$

Similarly:

$$v' \text{ for } x'\gtrless 0=\frac{\mp 1}{\rho d W_r}\sum_{n=1}^\infty \exp\left(j\frac{2\pi n y'}{d}\right)\exp\left\{\frac{2\pi n x'}{d[1-M_r^2\cos^2(\alpha_r)]}\right.$$

$$\left.\times[jM_r^2\cos(\alpha_r)\sin(\alpha_r)\mp\sqrt{1-M_r^2}]\right\} \tag{13b}$$

The upwash normal to line AB (fig. 2) is

$$v_d = v' \cos(\alpha_s) - u' \sin(\alpha_s) \tag{13c}$$

Thus v_d for $x' \gtrless 0$ is

$$\frac{\mp 1}{\rho d W_r}\left[\cos(\alpha_s) - \frac{M_r^2 \sin(\alpha_r)\cos(\alpha_r)\sin(\alpha_s)}{1-M_r^2\cos^2(\alpha_r)} \mp \frac{j\sin(\alpha_s)\sqrt{1-M_r^2}}{1-M_r^2\cos^2(\alpha_r)}\right]$$

$$\times \sum_1^\infty \exp\left(j\,\frac{2\pi n y'}{d}\right)\exp\left[\frac{2\pi n x'}{d}\,\frac{jM_r^2\sin(\alpha_r)\cos(\alpha_r)\mp\sqrt{1-M_r^2}}{1-M_r^2\cos^2(\alpha_r)}\right] \tag{13d}$$

The effect of distributed loading on a finite chord may be estimated by integrating the results of equation (13d) over the finite chord. We again omit the details since the derivation is very similar to that outlined on pages 589 through 590 of reference 1. Noting that these upwash fields translate with respect to the adjacent rows, one readily obtains the nonstationary upwash on the adjacent rows.

Viscous Wake Interaction

The form of the nonstationary upwash contributed by the viscous wake intraction mechanism was assumed to be the same as in reference-2. The pertinent equation giving the upwash is equation (28) of reference 2.

FINAL FORMULATION OF INTEGRAL EQUATION (9) AND METHOD OF SOLUTION

From figure 3, clearly the frequency ω of unsteady lift is $(2\pi n/d)M_t\, a$ and thus $k=\omega/a=(2\pi n/d)M_t$. In what follows we consider each harmonic n separately. The phase lag γ is discussed on page 592 of reference 1. It is easily shown that

$$\gamma = 2\pi n\left(\frac{s}{d}\right)$$

Assume for $F(\xi)$ the form

$$A_0 \cot\left(\frac{\phi}{2}\right) + \sum_1^N A_n \sin(n\phi)$$

where $\xi = -\frac{1}{2}\cos\phi$, $\phi=0$ at the leading edge, and $\phi=\pi$ at the trailing edge.

All the above terms are zero at the trailing edge (Kutta condition), and the series has the usual square-root singularity at the leading edge. $A_0, A_1, A_2 \ldots A_n$ are, of course, unknown. We let $x=-\frac{1}{2}\cos\theta$ so that $\theta=0$ corresponds to $x=-\frac{1}{2}$ and $\theta=\pi$ corresponds to $x=\frac{1}{2}$ and denote by

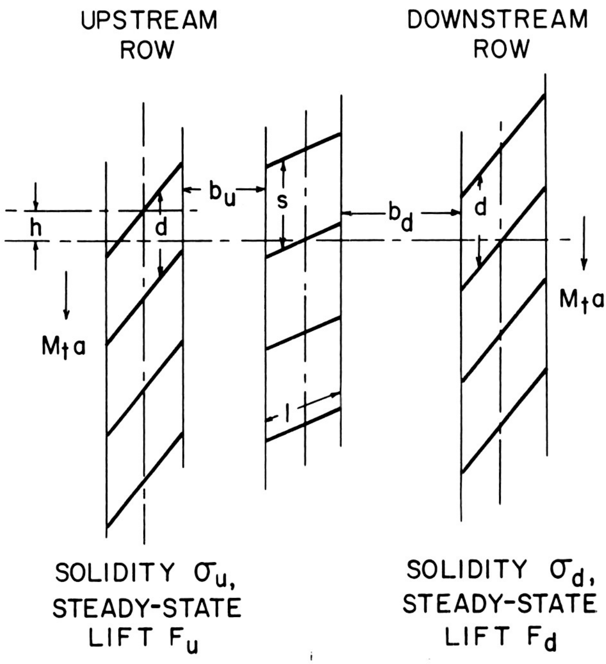

FIGURE 3.—*Sketch for potential interaction due to two adjacent rows.*

$$G(n,\theta)=\int_0^\pi \cos\,(n\phi)K_r(x,\xi)\;d\phi$$

where $n=0,\,1,\,\ldots,\,N+1$.

The $G(n,\theta)$ will be evaluated numerically by Simpson's rule and will present no difficulties since $K_r(x,\xi)$ is bounded. Then it is easily shown by using the results of equations (16), (17), and (18) of the appendix that equation (9) may be written as

$$A_0\left\{\tfrac{1}{2}[G(0,\theta)+G(1,\theta)]\frac{-\beta}{2\rho Ma}-\frac{jk(\log 2+\frac{1}{2}\cos\theta)}{2\rho\beta M^2a}\right\}$$

$$+A_1\left\{\tfrac{1}{4}[G(0,\theta)-G(2,\theta)]+\frac{\beta\cos\theta}{2\rho Ma}-\frac{jk}{4\rho\beta M^2a}\left[\log 2-\frac{\cos\,(2\theta)}{4}\right]\right\}$$

$$+\sum_{n=2}^{N}A_n\left\{\tfrac{1}{4}[G(n-1,\theta)-G(n+1,\theta)]+\frac{\beta\cos\,(n\theta)}{2\rho Ma}\right.$$

$$\left.-\frac{jk}{8\rho\beta M^2a}\left[\frac{\cos\,(n-1)\;\theta}{n-1}-\frac{\cos\,(n+1)\;\theta}{n+1}\right]\right\}=f(x)$$

The above equation holds for $0 \leq \theta \leq \pi$, and the method of collocation of points involves satisfying the above equation exactly for $(N+1)$ equally spaced values of θ between 0 and π and thus deriving $(N+1)$ simultaneous equations for $A_0, A_1, \ldots, A_N$, which may be solved by matrix inversion.

The net lift is

$$\int_{-1/2}^{1/2} F(\xi)\, d\xi = \frac{\pi}{2}\left(A_0 + \frac{A_1}{2}\right)$$

and the magnitude of unsteady lift is the amplitude of $\pi/2[A_0 + (A_1/2)]$.

NUMERICAL DETAILS AND TYPICAL RESULTS

The unsteady lift distribution was assumed to have the form of an $(M+1)$ term series. The check on whether the chosen M is satisfactory is whether the terms $A_0, A_1, \ldots, A_M$ resulting from the solution of the simultaneous equations (14) converge rapidly enough. Judging from the calculations performed in this paper, the value of M to be used increases with the Mach number of the flow through the blade passages. Up to a Mach number of about 0.5, $M=7$ suffices. Between Mach numbers of 0.5 and 0.8, $M=11$ suffices. Beyond a Mach number of 0.8 it seems necessary to use $M=15$ to get good convergence. The use of the present analysis for Mach numbers close to unity is not very valid anyway because for such high Mach numbers the convected wave equation (from which eq. (3) is derived) is not a valid linearized equation for describing the nonsteady flow.

In figure 4 we have plotted results for potential interaction[4] on a row with flow at a Mach number of 0.1 due to a row downstream. This case should be analogous (owing to the low Mach numbers) to a case calculated in figure 5 of reference 1. The Kemp-Sears results and results of this paper compare reasonably well.

A similar check with the Kemp-Sears results is obviously desirable for viscous wake interaction. In reference 2, in the interest of obtaining a closed-form solution, the upwash used to calculate the unsteady lift is taken at selected points on the airfoil. Two sets of results pertaining to a stator rotor sequence as sketched in figure 5, one corresponding to the upwash at quarter chord from the leading edge and another corresponding to the upwash at quarter chord from the trailing edge, are presented in table 1 of reference 2. The methods used in this paper make such an

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[4] For all potential interaction calculations reported in this paper (Figs. 4, 6*a*, 7*b*) the steady lift distribution is assumed to be of the flat-plate type (see p. 594 of ref. 1).

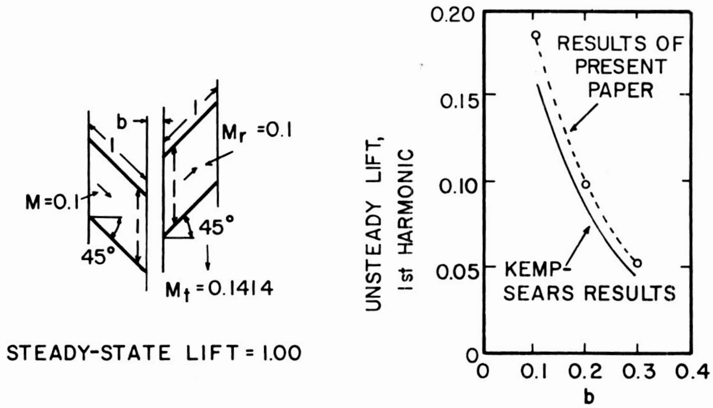

FIGURE 4.—*Potential interaction due to downstream row. Comparison with Kemp-Sears (reference 1).*

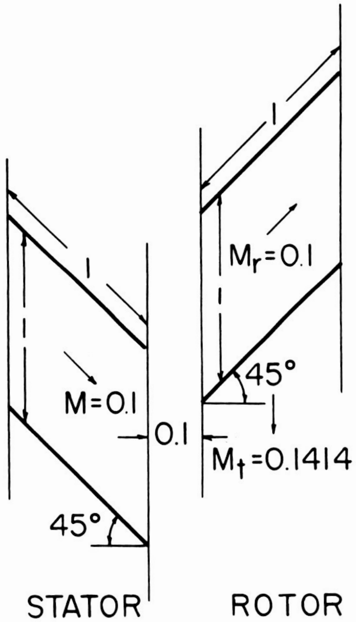

FIGURE 5.—*Stator rotor sequence used to compare results for viscous wake interaction with Kemp-Sears (reference 2).*

approximation unnecessary; our results are compared with the Kemp-Sears results in table I below. The calculations in this paper are again for a row with flow at a Mach number of 0.1.

From table I it is seen that the first harmonic results compare well if Kemp-Sears results corresponding to an upwash at a quarter chord from the leading edge are used.

In figure 6*a* we have plotted results for unsteady lift assuming the velocity triangles of figure 4 to be linearly scaled up in Mach number. This could be representative of changes in unsteady lift as one runs up a fan or compressor on a constant loadline.

The forms for I_2, I_3, and I_4 obtained by the use of the Poisson summation formula (eqs. 5*b*, 5*c*, and 5*d*) indicate that if, for any integer m,

$$(\delta - m)^2 - \omega^2 = 0 \ldots \tag{14}$$

then I_2, I_3, and $I_4 \rightarrow \infty$. Since $K_v(x,\xi)$ in equation (1) involves I_2, I_3, and I_4, this means that if equation (14) is satisfied then $K_v(x,\xi) \rightarrow \infty$. The only way in which one can obtain a bounded $v_d(x)$ under the condition that $K_v(x,\xi) \rightarrow \infty$ in equation (1) is to have $F(\xi) \rightarrow 0$. Thus the resonance condition denoted by equation (14) is one for which the unsteady blade forces vanish. Physically this condition arises when purely transverse waves are produced in the blade passages (i.e., waves traveling only in the tangential direction). At this resonance condition, waves emitted from one surface, say the upper surface of a blade, travel transversely and arrive at the lower surface of an adjacent blade with the time of travel being such that the incident wave phase is exactly antiphase (i.e., with a phase difference of some odd integer multiple of 180°) with the phase of

TABLE I.—*Viscous Wake Interaction*[1]

Kemp-Sears results				Results of present paper		
Harmonic number	C_D	Upwash evaluated at	C_L	Harmonic number	C_D	C_L
1	0.01	Q.C. from L.E.	0.029	1	0.01	0.029
		Q.C. from T.E.	0.023			
	0.02	Q.C. from L.E.	0.057		0.02	0.05757
		Q.C. from T.E.	0.045			

Q.C. = quarter chord, C_D = profile drag coefficient, L.E. = leading edge, T.E. = trailing edge, C_L = coefficient of unsteady lift.

[1] Comparison with Kemp-Sears results for case sketched in figure 5.

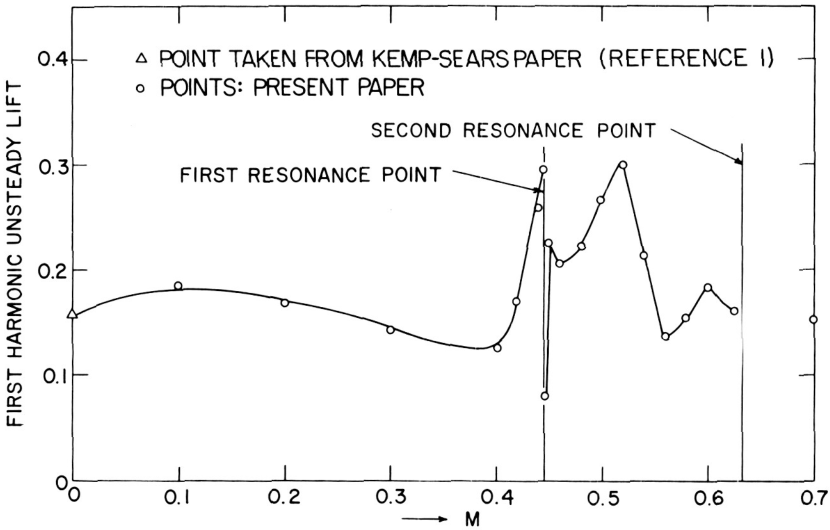

FIGURE 6a.—*Potential interaction for dynamically similar velocity triangles. Effect of Mach number.*

the sources on the lower surface of this adjacent blade. This phase cancellation prevents development of any unsteady lift.

The resonance condition of equation (14) may be rewritten in terms of tip Mach numbers and flow Mach numbers as

$$M_t=\left(1-\frac{md}{s}\right)[\pm\sqrt{1-M^2\cos^2(\alpha_s)}-M\sin(\alpha_s)] \tag{15}$$

For velocity triangles of the type shown in figure 4, $M_t=M\sqrt{2}$, $\alpha_s=-45°$, and the solution of (15) yields resonant Mach numbers of

$$M=\frac{(m-1)}{\sqrt{m^2+1}}$$

where m is a positive integer. The first significant resonant Mach number M is thus

$$M=\frac{1}{\sqrt{5}}=0.447$$

and the second is

$$M=\frac{2}{\sqrt{10}}=0.632$$

These two ordinate lines (corresponding to $M=0.447$ and 0.632) are shown in figure 6*a* as first and second resonance points. As can be noted, the resonance is extremely sharp in that the approach of the unsteady lift to zero as $M \rightarrow 1/\sqrt{5}$ is extremely sharp, being represented by an almost vertical drop in figure 6*a*.

The results of figure 6*a* suggest that the resonance, while undoubtedly denoting a point of zero unsteady lift, is much too sharp to have practical significance as a condition of low unsteady lift. However, the resonance points do have considerable significance (as may be observed from fig. 6*a*) as delineating rather different families of variations of unsteady lift with Mach number.

A similar result is shown by Kaji and Okazaki in reference 3. They consider in reference 3 the effect of a flat-plate cascade with flow on an incident sound wave as shown in figure 6*b*. An unsteady force distribution on the blades to cancel the velocities induced by the incident sound wave is sought as the solution to an integral equation of the same type as equation (1). Later the effect of the unsteady force distributions is integrated to obtain far-field pressure (sound) waves in the transmitted and reflected regions. The cascade in figure 6*b* behaves as a diffraction grating with respect to the incident sound wave. One propagating reflected and one propagating transmitted wave are always produced by the interaction of the incident sound wave and the blade row. The reflected wave corresponds to a specular reflection of the incident wave by the blade row,

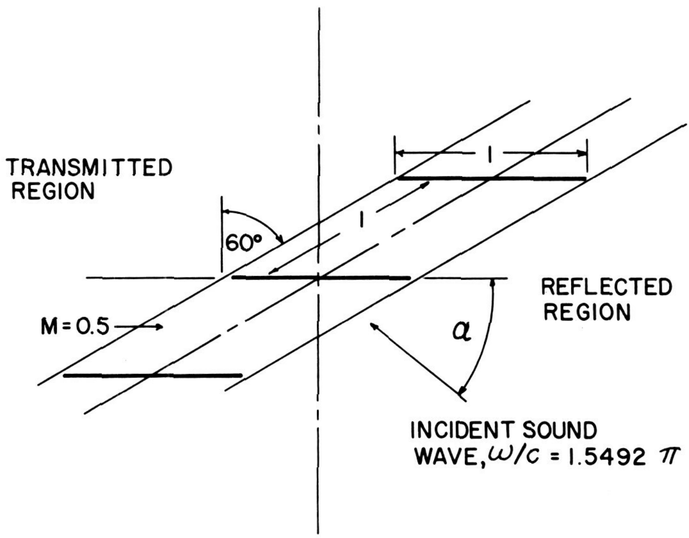

FIGURE 6b.—*Configuration of cascade whose results are shown in figure 6c of reference 3.*

and the transmitted wave has the same orientation as the incident wave. The basic transmitted wave, produced under all circumstances, is denoted by the authors of reference 3 as a $\nu=0$ mode[5] in figure 5 of their paper (the upper half of which is reproduced as fig. 6c in this paper). Higher order modes are also produced if, for the interaction of the incident wave and the blade row, more than one resulting mode is above "cutoff." In figure 6c, for α less than about 40°, a basic transmitted mode (labeled $\nu=0$) and a higher order mode (labeled $\nu=-1$) are produced. For α greater than 40°, the basic transmitted mode and a higher order mode (labeled $\nu=+1$) are produced. The pressure transmission coefficient is the ratio of amplitude of transmitted wave to the amplitude of incident wave. The ordinate around $\alpha=40°$ in figure 6c represents a resonant condition of the type of equation (14). In the example of figure 6c, the orientation of the higher order transmitted wave undergoes an abrupt change as one passes across the resonant incidence angle. Note how the pressure transmission

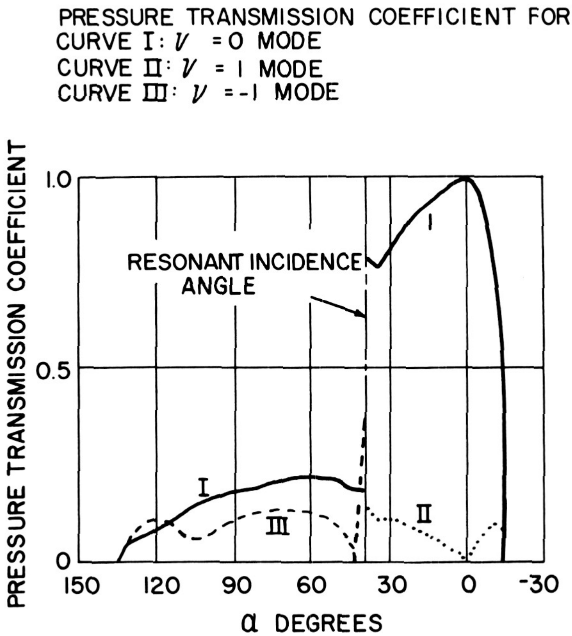

FIGURE 6c.—*Curve from upper half of figure 5 of reference 3.*

[5] Corresponds to $m-1=0$ in this paper.

coefficient associated with the basic transmitted wave ($\nu=0$ mode) undergoes an abrupt change as one crosses over the resonant incidence angle.

This type of resonance effect was apparently first observed by Runyan, Woolston, and Rainey (ref. 5). Their concern was with the effect of wind-tunnel walls on the lift forces developed by an oscillating wing in two-dimensional subsonic compressible flow. The resonance phenomenon was experimentally confirmed by these authors.

In view of the very rapid change of events just around resonance, it might appear worthwhile to attempt an analytical solution of equation (1) close to resonance. However, McCune, in a different context, has examined the merits of replacing cylindrical wave functions (which should be employed in a proper three-dimensional analysis) by two-dimensional approximation in such problems (ref. 6). He shows clearly that such an approximation breaks down precisely at these resonance frequencies. Thus, it seems of dubious advantage to pursue the cascade plane analysis any further near resonance. Finally, one may easily show that the resonance conditions are precisely the conditions at which successive rotor-stator interaction modes of the classical Tyler-Sofrin analysis (ref. 6) are cut off. Thus the cutoff frequencies introduced by Tyler and Sofrin as delineating regimes of acoustic propagation or decay of successive interaction modes are seen to play an important role in the estimation of the unsteady blade force problem.

From equation (13*d*), one notices that the exponential decay rate of the potential flow field of an adjacent row is altered from its incompressible value by the factor

$$\frac{\sqrt{1-M_r^2}}{1-M_r^2 \cos^2 (\alpha_r)}$$

To show the effect of this factor, we have plotted the decay of unsteady lift with spacing for $\alpha_r=0$, 45°, and 60° in figure 7.

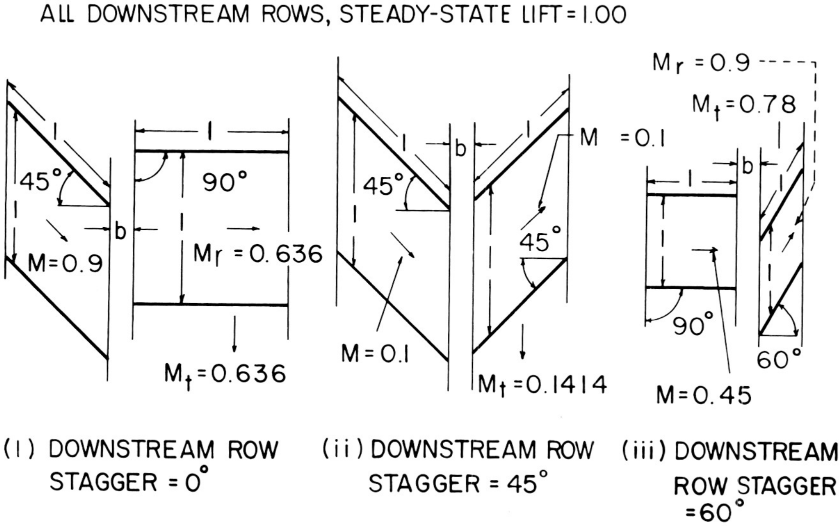

FIGURE 7a.—*Effects of stagger angle on decay of potential flow fields—configurations considered.*

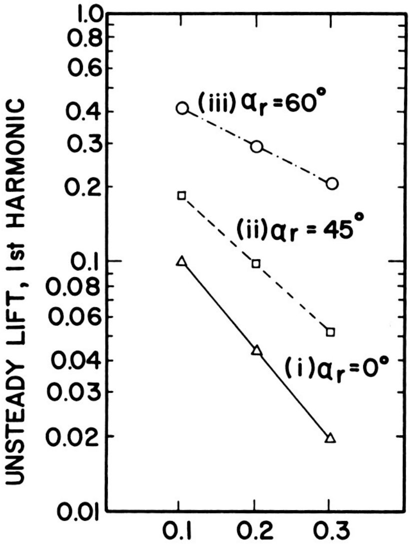

FIGURE 7b.—*Effects of stagger angle on decay of potential flow fields—results for unsteady lift for configurations of figure 7a.*

APPENDIX

Two well-known principal value integrals are

$$\frac{1}{\pi}\int_0^\pi \left[\frac{\cos(n\phi)\,d\phi}{\cos(\phi)-\cos(\theta)}\right]=\frac{\sin(n\theta)}{\sin\theta}\ldots \tag{16}$$

and

$$\frac{1}{\pi}\int_0^\pi \frac{\sin(n\phi)\sin(\phi)\,d\phi}{\cos(\theta)-\cos(\phi)}=\cos(n\theta)\ldots \tag{17}$$

An expansion of $jk(\log|x-\xi|)/2M(1-M^2)$ convergent for $x\neq\xi$ or $\theta\neq\phi$ is

$$\frac{-jk}{M(1-M^2)}\left[\log(2)+\sum_1^\infty\frac{\cos(n\theta)\cos(n\phi)}{n}\right]\ldots \tag{18}$$

REFERENCES

1. Kemp, N. H., and W. R. Sears, Aerodynamic Interference Between Moving Blade Rows. *J. Aeron. Sci.*, Vol. 20, No. 9, September 1953, p. 585.
2. Kemp, N. H., and W. R. Sears, The Unsteady Forces Due to Viscous Wakes in Turbomachines. *J. Aeron. Sci.*, Vol. 22, No. 7, July 1955.
3. Kaji, S., and T. Okazaki, *Study on Plane Sound Waves Passing Through a Compressor Blade Row.* Paper F-5-1, presented at Sixth International Congress on Acoustics (Tokyo), August 21–28, 1968.
4. Lighthill, M. J., *Fourier Analysis and Generalized Functions.* Cambridge U. Press, 1962.
5. Runyan, H. L., D. S. Woolston, and A. G. Rainey, *Theoretical and Experimental Investigation of the Effect of Tunnel Walls on the Forces on an Oscillating Airfoil in Two-Dimensional Subsonic Compressible Flow.* NACA Report 1262, 1955.
6. McCune, J. E., The Transonic Flow Field of an Axial Compressor Blade Row. *J. Aeron. Sci.*, Vol. 25, 1958, pp. 616–626.
7. Tyler, J. M., and T. G. Sofrin, Axial Compressor Noise Studies. *Trans. SAE*, Vol. 70, 1962, pp. 309–332.

DISCUSSION

A. ABDELHAMID (Carleton University): I would like to comment on the effect of the blade being in a row or being isolated and on whether it's a first- or a second-order effect. I believe it's a second-order effect because the steady vorticity on the blades of the same row as the blade that you are considering will not contribute to the contour it imposes.

J. E. FFOWCS WILLIAMS (Imperial College of Science): It seems to me that when you have this guaranteed repetitive system you must generate a resonance. You've got to pick them up somehow. My point of concern is the way in which the boundary conditions are put in. There are difficulties concerned in judgments, and these turn up when the resonant wave fronts are going at a parallel, one normally observes. Now if they're going parallel to the surface, the procedure to adopt is one to control the velocity on the surface. If one asks what pressure is required on that surface to bring about a control on the velocity, it turns out to be infinite by the way his blade is set, so a more realistic boundary condition for any practical system would be a pressure-release condition.

MANI (author): Two points have been raised in the discussion period. The first concerns the proper boundary condition to be used near the resonant (cutoff) frequency. The author agrees that if very large pressures result the assumption of perfectly rigid blades is not a suitable one and should be replaced perhaps by an impedance condition. It is worth reiterating, however, that from the point of view of the isolated two-dimensional cascade model's representativeness of the actual situation in the turbomachine, it would not be worthwhile to pursue matters much further in the cascade plane near cutoff. The effect of adjacent blade rows, open-end terminations, and three-dimensional effects become all-important at this condition.

The second pertains to the effect of adjacent blade rows. It should be noted that estimates of effect of adjacent rows as carried out in the original Kemp-Sears papers is not entirely sufficient. Kemp and Sears assumed the fluid to be incompressible and, with this assumption, all nonaxisymmetric flow patterns exhibit exponential axial decay. When the fluid is regarded as compressible, some of these flow patterns (those above cutoff) exhibit no axial decay and hence the treatment of the problem on the basis of isolated blade rows subject to given external upwash becomes questionable.