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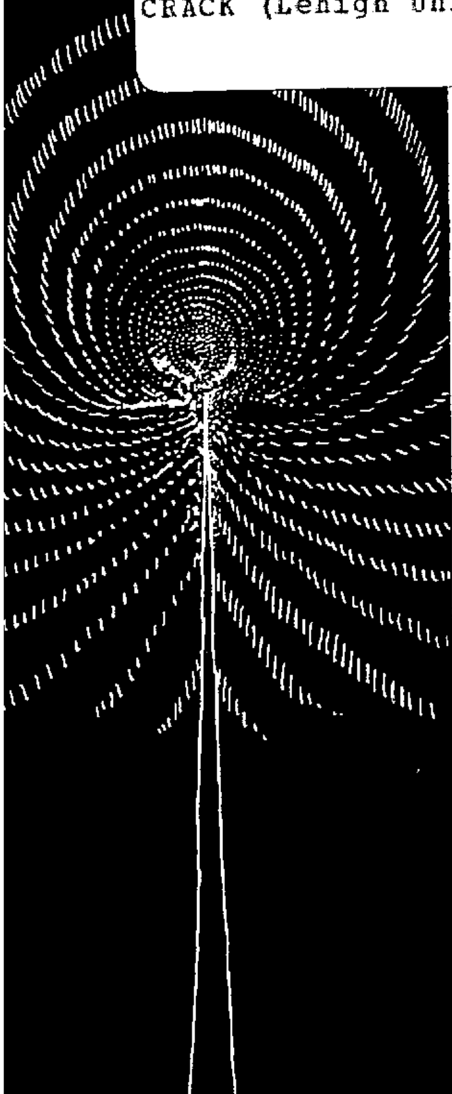

AN ORTHOTROPIC LAMINATE COMPOSITE CONTAINING A LAYER WITH A CRACK

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AN ORTHOTROPIC LAMINATE COMPOSITE CONTAINING A LAYER WITH A CRACK

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ABSTRACT

A laminate composite containing an orthotropic layer with a crack situated normal to the interfaces, and bonded to two orthotropic half-planes of dissimilar materials, is considered. The solutions for two different classes of orthotropic materials are presented separately. In each case, the problem is first reduced to a system of dual integral equations, then to a singular integral equation which is subsequently solved numerically for the stress intensity factors at the tip of the crack. The effect of the material properties on the stress intensity factor is investigated. Two cases, the generalized plane stress and the plane strain, are treated simultaneously.

1. INTRODUCTION

The increased use of composites can probably be attributed to the fact that they are a group of structural elements which optimize the requirements of safety and economy. As a result, this has prompted the solution to many problems involving various composite geometries [1] - [4], [10], [13] - [17]. One particular composite geometry of practical importance is the multi-layered isotropic medium consisting of many layers, including the presence of possible flaws. This problem has been treated by Hilton and Sih [1] by simplifying the geometry to the form of a single layer, with an internal crack normal to the interfaces, bonded to two dissimilar half-planes with averaged elastic properties. Through the application of an integral transform technique, a Fredholm integral equation is obtained and solved to obtain the crack tip stress field. Recently, the same problem has also been considered by Bogy [4]. Furthermore, the case of a crack touching the interfaces has been investigated by Ashbaugh [2] and Gupta [3], where the stress state around the crack tip has been found by the method of integral transforms, the singularity having been shown to be different than 1/2.

In this paper, the solution of the problem treated by Hilton and Sih will be extended to the case of orthotropic materials. Since the orthotropic materials can be classified in two groups, there are four possible material combinations. Two cases, where the layer and the matrix are both of the same material type, are formulated, and the corresponding solutions are presented. The solutions to the other two combinations can be obtained in an identical manner. In each case, the problem is formulated in terms of integral transforms, using the procedure described by Sneddon [5]. The dual integral equations resulting from the mixed boundary conditions are solved by reducing them to a singular integral equation, and then employing the numerical technique described in [7]. For certain choices of material parameters, the effect of crack size on the stress intensity factor is shown in Figures 2 - 6.

2. GENERAL FORMULAS

For an anisotropic elastic body the equilibrium equations, the stress-strain and the strain-displacement relations in the absence of body forces are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad (2.1)$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = [A_{ij}] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}, \qquad \begin{array}{l} i,j = 1,6 \\ A_{ij} = A_{ji} \end{array} \qquad (2.2)$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (2.3)$$

Substitution from (2.3) into (2.2) and (2.1) yields the equilibrium equations in terms of the displacements:

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + A_{55}\frac{\partial^2 u}{\partial z^2} + A_{16}\frac{\partial^2 v}{\partial x^2} + A_{26}\frac{\partial^2 v}{\partial y^2} + A_{45}\frac{\partial^2 v}{\partial z^2}$$
$$+ A_{15}\frac{\partial^2 w}{\partial x^2} + A_{46}\frac{\partial^2 w}{\partial y^2} + A_{35}\frac{\partial^2 w}{\partial z^2} + 2A_{16}\frac{\partial^2 u}{\partial x\partial y} + 2A_{15}\frac{\partial^2 u}{\partial x\partial z}$$
$$+ 2A_{56}\frac{\partial^2 u}{\partial y\partial z} + (A_{66}+A_{12})\frac{\partial^2 v}{\partial x\partial y} + (A_{56}+A_{14})\frac{\partial^2 v}{\partial x\partial z}$$
$$+ (A_{46}+A_{25})\frac{\partial^2 v}{\partial y\partial z} + (A_{14}+A_{56})\frac{\partial^2 w}{\partial x\partial y} + (A_{13}+A_{55})\frac{\partial^2 w}{\partial x\partial z}$$
$$+ (A_{36}+A_{45})\frac{\partial^2 w}{\partial y\partial z} = 0$$

$$A_{16}\frac{\partial^2 u}{\partial x^2} + A_{26}\frac{\partial^2 u}{\partial y^2} + A_{45}\frac{\partial^2 u}{\partial z^2} + A_{66}\frac{\partial^2 v}{\partial x^2} + A_{22}\frac{\partial^2 v}{\partial y^2} + A_{44}\frac{\partial^2 v}{\partial z^2}$$
$$+ A_{56}\frac{\partial^2 w}{\partial x^2} + A_{24}\frac{\partial^2 w}{\partial y^2} + A_{34}\frac{\partial^2 w}{\partial z^2} + (A_{66}+A_{12})\frac{\partial^2 u}{\partial x\partial y}$$
$$+ (A_{56}+A_{14})\frac{\partial^2 u}{\partial x\partial z} + (A_{25}+A_{46})\frac{\partial^2 u}{\partial y\partial z} + 2A_{26}\frac{\partial^2 v}{\partial x\partial y}$$
$$+ 2A_{46}\frac{\partial^2 v}{\partial x\partial z} + 2A_{24}\frac{\partial^2 v}{\partial y\partial z} + (A_{25}+A_{46})\frac{\partial^2 w}{\partial x\partial y} + (A_{45}+A_{36})\frac{\partial^2 w}{\partial x\partial z}$$
$$+ (A_{23}+A_{44})\frac{\partial^2 w}{\partial y\partial z} = 0$$

$$A_{15}\frac{\partial^2 u}{\partial x^2} + A_{46}\frac{\partial^2 u}{\partial y^2} + A_{35}\frac{\partial^2 u}{\partial z^2} + A_{56}\frac{\partial^2 v}{\partial x^2} + A_{24}\frac{\partial^2 v}{\partial y^2} + A_{34}\frac{\partial^2 v}{\partial z^2}$$
$$+ A_{55}\frac{\partial^2 w}{\partial x^2} + A_{44}\frac{\partial^2 w}{\partial y^2} + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{14}+A_{56})\frac{\partial^2 u}{\partial x\partial y}$$
$$+ (A_{55}+A_{13})\frac{\partial^2 u}{\partial x\partial z} + (A_{36}+A_{45})\frac{\partial^2 u}{\partial y\partial z} + (A_{25}+A_{46})\frac{\partial^2 v}{\partial x\partial y}$$
$$+ (A_{36}+A_{45})\frac{\partial^2 v}{\partial x\partial z} + (A_{44}+A_{23})\frac{\partial^2 v}{\partial y\partial z} + 2A_{45}\frac{\partial^2 w}{\partial x\partial y} + 2A_{35}\frac{\partial^2 w}{\partial x\partial z}$$
$$+ 2A_{34}\frac{\partial^2 w}{\partial y\partial z} = 0 \qquad (2.4)$$

Defining the inverse of the matrix $[A_{ij}]$ by

$$[a] = [A]^{-1} \tag{2.5}$$

and noting that for orthotropic materials

$$[a] = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix}$$

$$E_x \nu_{yx} = E_y \nu_{xy}, \qquad E_y \nu_{zy} = E_z \nu_{yz}, \qquad E_z \nu_{xz} = E_x \nu_{zx}$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{13} & A_{23} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix} \tag{2.6}$$

Equilibrium equations (2.4) in the case of orthotropy are further simplified to give (see [6]):

$$A_{44}\frac{\partial^2 u}{\partial x^2} + A_{66}\frac{\partial^2 u}{\partial y^2} + A_{55}\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}\,[(A_{11}-A_{44})\frac{\partial u}{\partial x} + (A_{12}+A_{66})\frac{\partial v}{\partial y} + (A_{13}+A_{55})\frac{\partial w}{\partial z}] = 0$$

$$A_{66}\frac{\partial^2 v}{\partial x^2} + A_{55}\frac{\partial^2 v}{\partial y^2} + A_{44}\frac{\partial^2 v}{\partial z^2} + \frac{\partial}{\partial y}\,[(A_{12}+A_{66})\frac{\partial u}{\partial x} + (A_{22}-A_{55})\frac{\partial v}{\partial y} + (A_{23}+A_{44})\frac{\partial w}{\partial z}] = 0$$

$$A_{55}\frac{\partial^2 w}{\partial x^2} + A_{44}\frac{\partial^2 w}{\partial y^2} + A_{66}\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}\,[(A_{13}+A_{55})\frac{\partial u}{\partial x} + (A_{23}+A_{44})\frac{\partial v}{\partial y} + (A_{33}-A_{66})\frac{\partial w}{\partial z}] = 0 \qquad (2.7)$$

which reduce to the Navier's equations for isotropic materials.

Since the problem under consideration is one of the plane theory of elasticity, the formulas of the plane strain and the generalized plane stress will be given below.

2.1 Plane Strain

From the conditions of plane strain, i.e.,

$$w = 0\,, \qquad u = u(x,y)\,, \qquad v = v(x,y) \qquad (2.8)$$

and (2.3) we have

$$\varepsilon_x = \frac{\partial u}{\partial x}\,, \qquad \varepsilon_y = \frac{\partial v}{\partial y}\,, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0 \qquad (2.9)$$

Furthermore from (2.2) we conclude that for the bodies possessing at least one plane of elastic symmetry (the plane perpendicular to z-axis)

$$\tau_{xz} = \tau_{yz} = 0 \qquad (2.10)$$

and the first two equations of (2.4) can be solved to give the displacements u,v, the third being automatically satisfied.

In particular, for orthotropic bodies we have

$$A_{11} \frac{\partial^2 u}{\partial x^2} + A_{66} \frac{\partial^2 u}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$A_{66} \frac{\partial^2 v}{\partial x^2} + A_{22} \frac{\partial^2 v}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad (2.11)$$

$$\sigma_x = A_{11}\varepsilon_x + A_{12}\varepsilon_y$$

$$\sigma_y = A_{12}\varepsilon_x + A_{22}\varepsilon_y$$

$$\tau_{xy} = A_{66}\gamma_{xy}$$

$$\sigma_z = A_{13}\varepsilon_x + A_{23}\varepsilon_y$$

$$\tau_{yz} = \tau_{xz} = 0 \qquad (2.12)$$

2.2 Generalized Plane Stress

Assuming (see [12])

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad (2.13)$$

from (2.2), for the average stresses and strains we can write

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [\bar{a}] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad (2.14)$$

where

$$[\bar{a}] = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix}, \qquad [\bar{A}] = [\bar{a}]^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{12} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{13} & \bar{A}_{23} & \bar{A}_{33} \end{bmatrix} \qquad (2.15)$$

Substitution from (2.3), (2.13) and (2.14) into (2.1) yields

$$\bar{A}_{11} \frac{\partial^2 u}{\partial x^2} + \bar{A}_{33} \frac{\partial^2 u}{\partial y^2} + 2\bar{A}_{13} \frac{\partial^2 u}{\partial x \partial y} + \bar{A}_{13} \frac{\partial^2 v}{\partial x^2} + \bar{A}_{23} \frac{\partial^2 v}{\partial y^2}$$

$$+ (\bar{A}_{33} + \bar{A}_{12}) \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\bar{A}_{13} \frac{\partial^2 u}{\partial x^2} + (\bar{A}_{12} + \bar{A}_{33}) \frac{\partial^2 u}{\partial x \partial y} + \bar{A}_{23} \frac{\partial^2 u}{\partial y^2} + \bar{A}_{33} \frac{\partial^2 v}{\partial x^2} + 2\bar{A}_{23} \frac{\partial^2 v}{\partial x \partial y}$$

$$+ \bar{A}_{22} \frac{\partial^2 v}{\partial y^2} = 0 \qquad (2.16)$$

For orthotropic bodies we have

$$[\bar{a}] = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & 1/G_{xy} \end{bmatrix}, \quad [\bar{A}] = \begin{bmatrix} \frac{1}{E_y \cdot \Delta} & \frac{\nu_{yx}}{E_y \cdot \Delta} & 0 \\ \frac{\nu_{xy}}{E_x \cdot \Delta} & \frac{1}{E_x \cdot \Delta} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix}$$

$$\Delta = \frac{1}{E_x \cdot E_y} (1 - \nu_{xy} \nu_{yx}) \qquad (2.17)$$

and (2.16) reduces to

$$\bar{A}_{11} \frac{\partial^2 u}{\partial x^2} + \bar{A}_{33} \frac{\partial^2 u}{\partial y^2} + (\bar{A}_{33} + \bar{A}_{12}) \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\bar{A}_{33} \frac{\partial^2 v}{\partial x^2} + \bar{A}_{22} \frac{\partial^2 v}{\partial y^2} + (\bar{A}_{33} + \bar{A}_{12}) \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad (2.18)$$

Hence for the orthotropic materials equations (2.11) and (2.18) can be expressed as

$$\beta_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta_3 \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial y^2} + \beta_3 \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad (2.19)$$

where

$$\beta_1 = \frac{A_{11}}{A_{66}}, \quad \beta_2 = \frac{A_{22}}{A_{66}}, \quad \beta_3 = 1 + \frac{A_{12}}{A_{66}} \quad \text{for plane strain}$$

$$\beta_1 = \frac{\bar{A}_{11}}{\bar{A}_{33}}, \quad \beta_2 = \frac{\bar{A}_{22}}{\bar{A}_{33}}, \quad \beta_3 = 1 + \frac{\bar{A}_{12}}{\bar{A}_{33}} \quad \text{for generalized plane stress} \tag{2.20}$$

3. FORMULATION OF THE PROBLEM

A single layer of width 2h is assumed to be bonded to two half-planes of dissimilar materials. Both the layer and the matrix are considered orthotropic. A crack of length 2a is symmetrically located being normal to the interfaces (see Figure 1). The case $a < h$ will be studied. The case $a = h$ is given in [10]. The crack surface loading consists of normal stresses distributed symmetrically. The solution of the skew-symmetric case can be obtained similarly. A superscript * for the elastic constants will refer to the matrix.

The approach described in [5] will be used. First, we will obtain the solutions of (2.19) satisfying certain boundary conditions of the layer and the matrix. The combination of these solutions will be forced to satisfy the rest of the boundary conditions.

3.1 Solutions $u_o(x,y)$, $v_o(x,y)$

Assume

$$u_o(x,y) = \frac{2}{\pi} \int_0^\infty f(t,x) \cos ty \, dt$$

$$v_o(x,y) = \frac{2}{\pi}\int_0^\infty f_o(t,x)\ \sin ty\, dt \tag{3.1}$$

Substitution from (3.1) into (2.19) gives

$$\beta_1 \frac{\partial^2 f}{\partial x^2} - t^2 f + \beta_3 t \frac{\partial f_o}{\partial x} = 0$$

$$\frac{\partial^2 f_o}{\partial x^2} - \beta_2 f_o t^2 - \beta_3 t \frac{\partial f}{\partial x} = 0 \tag{3.2}$$

which can be solved to give

$$f(t,x) = A(t)e^{s_1 tx} + B(t)e^{-s_1 tx} + C(t)e^{s_2 tx} + D(t)e^{-s_2 tx}$$

$$f_o(t,x) = \beta_7[A(t)e^{s_1 tx} - B(t)e^{-s_1 tx}]$$
$$+ \beta_8[C(t)e^{s_2 tx} - D(t)e^{-s_2 tx}] \tag{3.3}$$

where

$$\beta_4 = \frac{\beta_3^{\ 2} - \beta_1\beta_2 - 1}{\beta_1}, \qquad \beta_5 = \frac{\beta_2}{\beta_1}, \qquad \beta_6 = \sqrt{\beta_4^{\ 2} - 4\beta_5}$$

and s_1 and s_2 are the roots of

$$s^4 + \beta_4 s^2 + \beta_5 = 0$$

Hence

$$s_1 = \omega_1 + i\omega_2 = \sqrt{(-\beta_4 + \beta_6)/2}$$

$$s_2 = \omega_3 + i\omega_4 = \sqrt{(-\beta_4 - \beta_6)/2}$$

$$\beta_7 = \frac{1 - \beta_1 s_1^{\ 2}}{\beta_3 s_1}, \qquad \beta_8 = \frac{1 - \beta_1 s_2^{\ 2}}{\beta_3 s_2} \tag{3.4}$$

Hence $u_o(x,y)$ and $v_o(x,y)$ are determined in terms of unknown functions A(t), B(t), C(t), D(t). Note that β's here correspond to the layer.

3.2 Solutions $u_2(x,y)$, $v_2(x,y)$

Assume

$$u_2(x,y) = \frac{2}{\pi}\int_0^\infty f_1(t,x)\cos ty\,dt$$

$$v_2(x,y) = \frac{2}{\pi}\int_0^\infty f_2(t,x)\sin ty\,dt \qquad (3.5)$$

Then the solution will be identical to $u_o(x,y)$, $v_o(x,y)$ except β's will be replaced by β^*'s to refer to the matrix. Hence,

$$f_1(t,x) = K(t)e^{s_1^* tx} + L(t)e^{-s_1^* tx} + M(t)e^{s_2^* tx} + N(t)e^{-s_2^* tx}$$

$$f_2(t,x) = \beta_7^*[K(t)e^{s_1^* tx} - L(t)e^{-s_1^* tx}]$$

$$+ \beta_8^*[M(t)e^{s_2^* tx} - N(t)e^{-s_2^* tx}] \qquad (3.6)$$

3.3 Solutions $u_3(x,y)$, $v_3(x,y)$

Assume

$$u_3(x,y) = \frac{2}{\pi}\int_0^\infty f_3(t,y)\sin tx\,dt$$

$$v_3(x,y) = \frac{2}{\pi}\int_0^\infty f_4(t,y)\cos tx\,dt \qquad (3.7)$$

Substitution from (3.7) into (2.19) yields

$$\frac{\partial^2 f_3}{\partial y^2} - \beta_3 t\frac{\partial f_4}{\partial y} - \beta_1 t^2 f_3 = 0$$

$$\beta_2\frac{\partial^2 f_4}{\partial y^2} + \beta_3 t\frac{\partial f_3}{\partial y} - f_4 t^2 = 0 \qquad (3.8)$$

Solving (3.8) we obtain

$$f_3(t,y) = E(t)e^{s_1 ty/\sqrt{\beta_5}} + F(t)e^{-s_1 ty/\sqrt{\beta_5}} + G(t)e^{s_2 ty/\sqrt{\beta_5}} + H(t)e^{-s_2 ty/\sqrt{\beta_5}}$$

$$f_4(t,y) = \beta_9[E(t)e^{s_1 ty/\sqrt{\beta_5}} - F(t)e^{-s_1 ty/\sqrt{\beta_5}}] + \beta_{10}[G(t)e^{s_2 ty/\sqrt{\beta_5}} - H(t)e^{-s_2 ty/\sqrt{\beta_5}}] \quad (3.9)$$

where

$$\beta_9 = \frac{1}{\beta_3}\left[-\frac{\beta_1\sqrt{\beta_5}}{s_1} + \frac{s_1}{\sqrt{\beta_5}}\right]$$

$$\beta_{10} = \frac{1}{\beta_3}\left[-\frac{\beta_1\sqrt{\beta_5}}{s_2} + \frac{s_2}{\sqrt{\beta_5}}\right] \quad (3.10)$$

and the other constants are as in $u_o(x,y)$ and $v_o(x,y)$.

3.4 Stress and the displacement fields of the laminate composite

The displacements of the layer and the matrix will be expressed as

$u_1(x,y) = u_o(x,y) + u_3(x,y)$

$v_1(x,y) = v_o(x,y) + v_3(x,y)$ for the layer, and

$u_2(x,y)$, $v_2(x,y)$ for the matrix (3.11)

From the symmetry conditions we have

$B(t) = -A(t)$, $D(t) = -C(t)$

Furthermore from the condition that $u_3(x,y)$ and $v_3(x,y)$ will tend to zero as $y \to \infty$, the unknowns G(t) and F(t) are eliminated. Also, the condition that $u_2(x,y)$ and $v_2(x,y)$ go to zero as $x \to \infty$ will be used to eliminate K(t), M(t).

The stress field is obtained from (2.14) and (2.17) for the case of generalized plane stress. Stress field for the plane strain may simply be obtained by substituting A_{12}/A_{11}, A_{12}/A_{22}, $1/A_{11}$ and $1/A_{22}$ in place of ν_{yx}, ν_{xy}, $(E_y \cdot \Delta)$ and $(E_x \cdot \Delta)$ in the stress expressions.

In Appendix A, it is shown that s_1 and s_2 are either real or complex conjugates. Accordingly, the expressions below are given for two different types of orthotropic materials. Hence, after intermediate manipulations we have,

<u>For Material Type I</u>

$$u_1(x,y) = u_o + u_3$$

$$= \frac{4}{\pi} \int_0^\infty [A(t)\sinh(\omega_1 tx) + C(t)\sinh(\omega_3 tx)]\cos ty\, dt$$

$$+ \frac{2}{\pi} \int_0^\infty [E(t)e^{-|\omega_1| ty/\sqrt{\beta_5}} + H(t)e^{-|\omega_3| ty/\sqrt{\beta_5}}] \sin tx\, dt$$

$$v_1(x,y) = v_o + v_3$$

$$= \frac{4}{\pi} \int_0^\infty [A(t)\beta_7 \cosh(\omega_1 tx) + C(t)\beta_8 \cosh(\omega_3 tx)]\sin ty\, dt$$

$$- \frac{2}{\pi} \int_0^\infty [\mathrm{sign}(\omega_1)\beta_9 E(t)e^{-|\omega_1| ty/\sqrt{\beta_5}}$$

$$+ \mathrm{sign}(\omega_3)\beta_{10} H(t)e^{-|\omega_3| ty/\sqrt{\beta_5}}] \cos tx\, dt$$

$$u_2(x,y) = \frac{2}{\pi} \int_0^\infty [L(t)e^{-|\omega_1^*| tx} + N(t)e^{-|\omega_3^*| tx}] \cos ty\, dt$$

$$v_2(x,y) = -\frac{2}{\pi}\int_0^\infty [\text{sign}(\omega_1^*)\beta_7^* L(t)e^{-|\omega_1^*|tx} + \text{sign}(\omega_3^*)\beta_8^* N(t)e^{-|\omega_3^*|tx}] \sin ty \, dt \qquad (3.12)$$

and the stresses

$$\frac{\pi(1 - \nu_{xy}\nu_{yx})}{2E_x}\sigma_{x1}(x,y) =$$

$$\int_0^\infty [\gamma_1 E(t)e^{-|\omega_1|ty/\sqrt{\beta_5}} + \gamma_2 H(t)e^{-|\omega_3|ty/\sqrt{\beta_5}}] \cos tx \, tdt$$

$$+ \int_0^\infty [2\gamma_3 A(t)\cosh(\omega_1 tx) + 2\gamma_4 C(t)\cosh(\omega_3 tx)] \cos ty \, tdt$$

$$\frac{\pi(1 - \nu_{xy}\nu_{yx})}{2E_y}\sigma_{y1}(x,y) =$$

$$\int_0^\infty [\gamma_5 E(t)e^{-|\omega_1|ty/\sqrt{\beta_5}} + \gamma_6 H(t)e^{-|\omega_3|ty/\sqrt{\beta_5}}] \cos tx \, tdt$$

$$+ \int_0^\infty [2\gamma_7 A(t)\cosh(\omega_1 tx) + 2\gamma_8 C(t)\cosh(\omega_3 tx)] \cos ty \, tdt$$

$$\frac{\pi}{2G_{xy}}\tau_{xy1}(x,y) =$$

$$\int_0^\infty [2\gamma_9 A(t)\sinh(\omega_1 tx) + 2\gamma_{10} C(t)\sinh(\omega_3 tx)] \sin ty \, tdt$$

$$+ \int_0^\infty [\gamma_{11} E(t)e^{-|\omega_1|ty/\sqrt{\beta_5}} + \gamma_{12} H(t)e^{-|\omega_3|ty/\sqrt{\beta_5}}] \sin tx \, tdt$$

$$\frac{\pi(1 - \nu_{xy}^*\nu_{yx}^*)}{2E_x^*}\sigma_{x2}(x,y) =$$

$$\int_0^\infty [\gamma_{13}^* L(t)e^{-|\omega_1^*|tx} + \gamma_{14}^* N(t)e^{-|\omega_3^*|tx}] \cos ty \, tdt$$

$$\frac{\pi(1 - \nu_{xy}^{*}\nu_{yx}^{*})}{2E_{y}^{*}} \sigma_{y2}(x,y) =$$

$$\int_{0}^{\infty} [\gamma_{15}^{*}L(t)e^{-|\omega_1^{*}|tx} + \gamma_{16}^{*}N(t)e^{-|\omega_3^{*}|tx}] \cos ty\, tdt$$

$$\frac{\pi}{2G_{xy}^{*}} \tau_{xy2}(x,y) =$$

$$\int_{0}^{\infty} [\gamma_{9}^{*}L(t)e^{-|\omega_1^{*}|tx} + \gamma_{10}^{*}N(t)e^{-|\omega_3^{*}|tx}] \sin ty\, tdt \qquad (3.13)$$

<u>For Material Type II</u>

$$u_1(x,y) = u_o + u_3$$

$$= \frac{4}{\pi} \int_{0}^{\infty} [A(t)\cos(\omega_2 tx)\sinh(\omega_1 tx)$$

$$+ C(t)\sin(\omega_2 tx)\cosh(\omega_1 tx)]\cos ty\, dt$$

$$+ \frac{2}{\pi} \int_{0}^{\infty} [E(t)\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + H(t)\sin\frac{\omega_2 ty}{\sqrt{\beta_5}}] e^{-|\omega_1| ty/\sqrt{\beta_5}} \sin tx\, dt$$

$$v_1(x,y) = v_o + v_3$$

$$= \frac{4}{\pi} \int_{0}^{\infty} \{A(t)[\beta_7'\cos(\omega_2 tx)\cosh(\omega_1 tx) - \beta_7''\sin(\omega_2 tx)\sinh(\omega_1 tx)]$$

$$+ C(t)[\beta_7''\cos(\omega_2 tx)\cosh(\omega_1 tx)$$

$$+ \beta_7'\sin(\omega_2 tx)\sinh(\omega_1 tx)]\} \sin ty\, dt$$

$$+ \frac{2}{\pi} \int_{0}^{\infty} \{E(t)[- \beta_9'\mathrm{sign}(\omega_1)\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} - \beta_9''\sin\frac{\omega_2 ty}{\sqrt{\beta_5}}]$$

$$+ H(t)[\beta_9''\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} - \beta_9'\mathrm{sign}(\omega_1)\sin\frac{\omega_2 ty}{\sqrt{\beta_5}}]\} e^{-|\omega_1| ty/\sqrt{\beta_5}} \cos tx dt$$

$$u_2(x,y) = \frac{2}{\pi}\int_0^\infty [K(t)\cos(\omega_2^* tx) + M(t)\sin(\omega_2^* tx)]e^{-|\omega_1^*|tx}\cos ty\,dt$$

$$v_2(x,y) = \frac{2}{\pi}\int_0^\infty \{K(t)[-\beta_7^{*'}\operatorname{sign}(\omega_1^*)\cos(\omega_2^* tx) - \beta_7^{*''}\sin(\omega_2^* tx)]$$

$$+ M(t)[\beta_7^{*''}\cos(\omega_2^* tx)$$

$$- \beta_7^{*'}\operatorname{sign}(\omega_1^*)\sin(\omega_2^* tx)]\}e^{-|\omega_1^*|tx}\sin ty\,dt \qquad (3.14)$$

and the stresses

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_x}\sigma_{x1}(x,y) =$$

$$\int_0^\infty \{2A(t)[-\Delta_2\sin(\omega_2 tx)\sinh(\omega_1 tx) + \Delta_1\cos(\omega_2 tx)\cosh(\omega_1 tx)]$$

$$+ 2C(t)[\Delta_1\sin(\omega_2 tx)\sinh(\omega_1 tx)$$

$$+ \Delta_2\cos(\omega_2 tx)\cosh(\omega_1 tx)]\}\cos ty\,t\,dt$$

$$+ \int_0^\infty [E(t)(\Delta_3\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_4\sin\frac{\omega_2 ty}{\sqrt{\beta_5}})$$

$$+ H(t)(-\Delta_4\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_3\sin\frac{\omega_2 ty}{\sqrt{\beta_5}})]e^{-|\omega_1|ty/\sqrt{\beta_5}}\cos tx\,t\,dt$$

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y}\sigma_{y1}(x,y) =$$

$$\int_0^\infty \{2A(t)[-\Delta_6\sin(\omega_2 tx)\sinh(\omega_1 tx) + \Delta_5\cos(\omega_2 tx)\cosh(\omega_1 tx)]$$

$$+ 2C(t)[\Delta_5\sin(\omega_2 tx)\sinh(\omega_1 tx)$$

$$+ \Delta_6\cos(\omega_2 tx)\cosh(\omega_1 tx)]\}\cos ty\,t\,dt$$

$$+ \int_0^\infty [E(t)(\Delta_7\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_8\sin\frac{\omega_2 ty}{\sqrt{\beta_5}})$$

$$+ H(t)(-\Delta_8\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_7\sin\frac{\omega_2 ty}{\sqrt{\beta_5}})]e^{-|\omega_1|ty/\sqrt{\beta_5}}\cos tx\,t\,dt$$

$$\frac{\pi}{2G_{xy}} \tau_{xy1} =$$

$$\int_0^\infty \{2A(t)[\Delta_9 \cos(\omega_2 tx)\sinh(\omega_1 tx) - \Delta_{10}\sin(\omega_2 tx)\cosh(\omega_1 tx)]$$

$$+ 2C(t)[\Delta_9 \sin(\omega_2 tx)\cosh(\omega_1 tx)$$

$$+ \Delta_{10}\cos(\omega_2 tx)\sinh(\omega_1 tx)]\} \sin ty\, tdt$$

$$+ \int_0^\infty [E(t)(-\Delta_{11}\sin\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_{12}\cos\frac{\omega_2 ty}{\sqrt{\beta_5}})$$

$$+ H(t)(\Delta_{11}\cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_{12}\sin\frac{\omega_2 ty}{\sqrt{\beta_5}})]e^{-|\omega_1| ty/\sqrt{\beta_5}} \sin tx\, tdt$$

$$\frac{\pi(1 - \nu^*_{xy}\nu^*_{yx})}{2E^*_x} \sigma_{x2}(x,y) =$$

$$\int_0^\infty \{-K(t)[\Delta^*_2 \sin(\omega^*_2 tx) + \Delta^*_{13}\cos(\omega^*_2 tx)]$$

$$+ M(t)[\Delta^*_2 \cos(\omega^*_2 tx) - \Delta^*_{13}\sin(\omega^*_2 tx)]\}e^{-|\omega^*_1| tx} \cos ty\, tdt$$

$$\frac{\pi(1 - \nu^*_{xy}\nu^*_{yx})}{2E^*_y} \sigma_{y2}(x,y) =$$

$$\int_0^\infty \{-K(t)[\Delta^*_6 \sin(\omega^*_2 tx) + \Delta^*_{14}\cos(\omega^*_2 tx)]$$

$$+ M(t)[\Delta^*_6 \cos(\omega^*_2 tx) - \Delta^*_{14}\sin(\omega^*_2 tx)]\}e^{-|\omega^*_1| tx} \cos ty\, tdt$$

$$\frac{\pi}{2G^*_{xy}} \tau_{xy2}(x,y) =$$

$$\int_0^\infty \{K(t)[\Delta^*_9 \cos(\omega^*_2 tx) + \Delta^*_{15}\sin(\omega^*_2 tx)]$$

$$+ M(t)[\Delta^*_9 \sin(\omega^*_2 tx) - \Delta^*_{15}\cos(\omega^*_2 tx)]\}e^{-|\omega^*_1| tx} \sin ty\, tdt \qquad (3.15)$$

where A(t), C(t), E(t), H(t), K(t), M(t) stand for the combinations of the previous unknowns.

The bielastic constants Δ_i and γ_i are given in Appendix B.

Hence, there are four possible material combinations for the laminate composite. However, we will obtain the solutions for two cases where the matrix and the layer both are either of material type I or II. The other two combinations can be derived in an identical manner.

The boundary conditions of the problem may be summarized as follows.

At $x = h$

$$u_1(h,y) = u_2(h,y) \; ; \qquad v_1(h,y) = v_2(h,y)$$

$$\sigma_{x1}(h,y) = \sigma_{x2}(h,y) \; ; \qquad \tau_{xy1}(h,y) = \tau_{xy2}(h,y) \tag{3.16}$$

At $y = 0$

$$\tau_{xy1}(x,0) = 0 \; , \quad |x| < h; \qquad \tau_{xy2}(x,0) = 0 \; , \quad |x| > h$$

$$v_2(x,0) = 0 \; , \quad |x| > h \tag{3.17}$$

$$\sigma_{y1}(x,0) = -p(x) \; , \quad |x| < a; \qquad v_1(x,0) = 0 \; , \quad a < |x| < h \tag{3.18}$$

At $x = 0$

$$u_1(0,y) = \tau_{xy1}(0,y) = 0 \tag{3.19}$$

4. THE DERIVATION OF THE INTEGRAL EQUATION AND THE SOLUTION

4.1 For Material Type I

The stress and the displacement fields given by (3.12) and (3.13) automatically satisfy the boundary conditions (3.17) if we choose

$$H(t) = -\frac{\gamma_{11}}{\gamma_{12}} E(t) \tag{4.1}$$

On the other hand the mixed conditions (3.18) give

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y}\sigma_{y1}(x,0) =$$

$$\lim_{y\to 0}\int_0^\infty E(t)\left(\gamma_5 e^{-|\omega_1|ty/\sqrt{\beta_5}} - \gamma_6\frac{\gamma_{11}}{\gamma_{12}}e^{-|\omega_3|ty/\sqrt{\beta_5}}\right)\cos tx\,tdt$$

$$+\int_0^\infty [2\gamma_7 A(t)\cosh(\omega_1 tx) + 2\gamma_8 C(t)\cosh(\omega_3 tx)]tdt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y}p(x)\ , \qquad |x| < a$$

$$v_1(x,0) = -\frac{2}{\pi}\gamma_{19}\int_0^\infty E(t)\cos tx\,dt = 0\ , \qquad a < |x| < h \tag{4.2}$$

To obtain the proper singularity, we will follow the procedure described in [13] and [14], and define a new unknown function by

$$\frac{\partial v_1(x,0)}{\partial x} = \frac{2}{\pi}\gamma_{19}\int_0^\infty E(t)\sin tx\,tdt = \phi(x) \tag{4.3}$$

such that

$$\phi(x) = 0\ , \qquad |x| > a$$

and finally invert (4.3) to get

$$\gamma_{19}tE(t) = \int_0^a \phi(x)\sin tx\,dx \tag{4.4}$$

Substituting from (4.4) into the first equation of (4.2) one arrives at the singular integral equation. This will be done by making use of the integrals given in Appendix C. Hence,

$$\gamma_{20} \int_{-a}^{a} \frac{\phi(t)dt}{t-x} + \int_{0}^{\infty} [2\gamma_7 A(t)\cosh(\omega_1 tx) + 2\gamma_8 C(t)\cosh(\omega_3 tx)]tdt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y} p(x) , \qquad |x| < a \tag{4.5}$$

where the definition of $\phi(t)$ has been extended into the (-a,0) range as an odd function. The single-valuedness of the displacements can then be expressed as

$$\int_{-a}^{a} \phi(t)dt = 0 \tag{4.6}$$

The next step is to determine A(t) and C(t) from the equations (3.16). From (3.16), (3.12) and (3.13) after some manipulations and an inversion we obtain

$$-2A(t)\sinh(\omega_1 th) - 2C(t)\sinh(\omega_3 th)$$

$$+ L(t)e^{-|\omega_1^*|th} + N(t)e^{-|\omega_3^*|th}$$

$$= \frac{2}{\pi}\int_0^\infty \cos ty\,dt \int_0^\infty E(\xi)\,[e^{-|\omega_1|\xi y/\sqrt{\beta_5}}$$

$$- \frac{\gamma_{11}}{\gamma_{12}} e^{-|\omega_3|\xi y/\sqrt{\beta_5}}]\sin\xi h\,d\xi$$

$$= \frac{2}{\pi}\int_0^\infty E(\xi)\sin\xi h\,d\xi\,[\frac{|\omega_1|\xi/\sqrt{\beta_5}}{(\omega_1^2\xi^2/\beta_5) + t^2} - \frac{\gamma_{11}}{\gamma_{12}}\frac{|\omega_3|\xi/\sqrt{\beta_5}}{(\omega_3^2\xi^2/\beta_5) + t^2}]$$

$$= R_1(t)$$

$$2A(t)\beta_7\cosh(\omega_1 th) + 2C(t)\beta_8\cosh(\omega_3 th)$$

$$+ \operatorname{sign}(\omega_1^*)\beta_7^* L(t)e^{-|\omega_1^*|th} + \operatorname{sign}(\omega_3^*)\beta_8^* N(t)e^{-|\omega_3^*|th}$$

$$= \frac{2}{\pi}\int_0^\infty \sin ty\, dy \int_0^\infty E(\xi)[\operatorname{sign}(\omega_1)\beta_9 e^{-|\omega_1|\xi y/\sqrt{\beta_5}}$$

$$- \frac{\gamma_{11}}{\gamma_{12}} \operatorname{sign}(\omega_3)\beta_{10} e^{-|\omega_3|\xi y/\sqrt{\beta_5}}] \cos \xi h\, d\xi$$

$$= \frac{2}{\pi}\int_0^\infty E(\xi)\cos \xi h\, d\xi [\operatorname{sign}(\omega_1)\beta_9 \frac{t}{(\omega_1^2\xi^2/\beta_5) + t^2}$$

$$- \frac{\gamma_{11}}{\gamma_{12}} \operatorname{sign}(\omega_3)\beta_{10} \frac{t}{(\omega_3^2\xi^2/\beta_5) + t^2}] = R_2(t)$$

$$[-2\gamma_3 A(t)\cosh(\omega_1 th) - 2\gamma_4 C(t)\cosh(\omega_3 th)$$

$$+ \gamma_{17}\gamma_{13}^* L(t)e^{-|\omega_1^*|th} + \gamma_{17}\gamma_{14}^* N(t)e^{-|\omega_3^*|th}]\, t$$

$$= \frac{2}{\pi}\int_0^\infty \cos ty\, dy \int_0^\infty E(\xi)[\gamma_1 e^{-|\omega_1|\xi y/\sqrt{\beta_5}}$$

$$- \gamma_2 \frac{\gamma_{11}}{\gamma_{12}} e^{-|\omega_3|\xi y/\sqrt{\beta_5}}] \cos \xi h\, \xi d\xi$$

$$= \frac{2}{\pi}\int_0^\infty E(\xi)\cos \xi h\, \xi d\xi [\gamma_1 \frac{|\omega_1|\xi/\sqrt{\beta_5}}{(\omega_1^2\xi^2/\beta_5) + t^2}$$

$$- \gamma_2 \frac{\gamma_{11}}{\gamma_{12}} \frac{|\omega_3|\xi/\sqrt{\beta_5}}{(\omega_3^2\xi^2/\beta_5) + t^2}] = tR_3(t)$$

$$[-2\gamma_9 A(t)\sinh(\omega_1 th) - 2\gamma_{10}C(t)\sinh(\omega_3 th)$$

$$+ \gamma_{18}\gamma_9^* L(t)e^{-|\omega_1^*|th} + \gamma_{18}\gamma_{10}^* N(t)e^{-|\omega_3^*|th}]\,t$$

$$= \frac{2}{\pi}\int_0^\infty \sin ty\,dy \int_0^\infty E(\xi)\gamma_{11}[e^{-|\omega_1|\xi y/\sqrt{\beta_5}}$$

$$- e^{-|\omega_3|\xi y/\sqrt{\beta_5}}]\sin \xi h\,\xi d\xi$$

$$= \frac{2\gamma_{11}}{\pi}\int_0^\infty E(\xi)\sin \xi h\,\xi d\xi\left[\frac{t}{(\omega_1^2\xi^2/\beta_5) + t^2} - \frac{t}{(\omega_3^2\xi^2/\beta_5) + t^2}\right]$$

$$= tR_4(t) \tag{4.7}$$

Functions $R_i(t)$ can be obtained in terms of $\phi(t)$ by substituting from (4.4) into (4.7) and using the integrals given in Appendix C. Hence, after lengthy manipulations,

$$R_1(t) = \frac{1}{2\gamma_{19}t}\int_{-a}^{a}[e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_1|}$$

$$- \frac{\gamma_{11}}{\gamma_{12}} e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_3|}]\,\phi(\eta)d\eta$$

$$R_2(t) = \frac{1}{2\gamma_{19}t}\int_{-a}^{a}[\mathrm{sign}(\omega_1)\beta_9 e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_1|}$$

$$- \frac{\gamma_{11}}{\gamma_{12}}\mathrm{sign}(\omega_3)\beta_{10} e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_3|}]\,\phi(\eta)d\eta$$

$$R_3(t) = -\frac{\sqrt{\beta_5}}{2\gamma_{19}t}\int_{-a}^{a}\left[\frac{\gamma_1}{|\omega_1|}e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_1|}\right.$$

$$\left. - \frac{\gamma_{11}}{\gamma_{12}}\frac{\gamma_2}{|\omega_3|}e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_3|}\right]\phi(\eta)d\eta$$

$$R_4(t) = \frac{\gamma_{11}\sqrt{\beta_5}}{2\gamma_{19}t} \int_{-a}^{a} [\frac{1}{|\omega_1|} e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_1|}$$

$$- \frac{1}{|\omega_3|} e^{-(h-\eta)t\sqrt{\beta_5}/|\omega_3|}] \phi(\eta)d\eta \tag{4.8}$$

Solving (4.7) for A(t) and C(t) and substituting into (4.5) gives the following singular integral equation.

$$\frac{1}{\pi} \int_{-a}^{a} \frac{\phi(t)dt}{t-x} + \int_{-a}^{a} k(x,t)\phi(t)dt = - \frac{(1 - \nu_{xy}\nu_{yx})}{2\gamma_{20}E_y} p(x) \;, \qquad |x| < a \tag{4.9}$$

subject to

$$\int_{-a}^{a} \phi(t)dt = 0 \tag{4.10}$$

where

$$k(x,t) = \frac{1}{\pi\gamma_{20}} \int_{0}^{\infty} [k_1(x,\eta)e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_1|}$$

$$+ k_2(x,\eta)e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_3|}] d\eta$$

$$k_1(x,\eta) = \frac{1}{2\gamma_{19}f(\eta)} [\gamma_7 f_5(x,\eta)f_7(\eta) + \gamma_8 f_6(x,\eta)f_8(\eta)]$$

$$k_2(x,\eta) = \frac{1}{2\gamma_{19}f(\eta)} [\gamma_7 f_5(x,\eta)f_9(\eta) + \gamma_8 f_6(x,\eta)f_{10}(\eta)] \tag{4.11}$$

and

$$f(\eta) = f_3(\eta)f_1(\eta) - f_2(\eta)f_4(\eta)$$

$$f_1(\eta) = \gamma_{25} + \gamma_{24}\tanh(\omega_3\eta h)$$

$$f_2(\eta) = \gamma_{29} + \gamma_{28}\tanh(\omega_3\eta h)$$

$$f_3(\eta) = \gamma_{27} + \gamma_{26}\tanh(\omega_1\eta h)$$

$$f_4(\eta) = \gamma_{23} + \gamma_{24}\tanh(\omega_1\eta h)$$

$$f_5(\eta,x) = \cosh(\omega_1\eta x)/\cosh(\omega_1\eta h)$$

$$f_6(\eta,x) = \cosh(\omega_3\eta x)/\cosh(\omega_3\eta h)$$

$$f_7(\eta) = \gamma_{38} + \gamma_{39}\tanh(\omega_3\eta h)$$

$$f_8(\eta) = \gamma_{40} + \gamma_{41}\tanh(\omega_1\eta h)$$

$$f_9(\eta) = \gamma_{42} + \gamma_{43}\tanh(\omega_3\eta h)$$

$$f_{10}(\eta) = \gamma_{44} + \gamma_{45}\tanh(\omega_1\eta h) \tag{4.12}$$

Changing the variables

$$t = a\tau , \qquad x = a\chi \qquad \text{and}$$

$$\phi(t) = \phi_o(\tau) , \quad p(x) = p_o(\chi)$$

$$k_o(\chi,\tau) = ak(x,t)$$

$$g(\chi) = -\frac{(1 - \nu_{xy}\nu_{yx})}{2\gamma_{20}E_y} p_o(\chi) \tag{4.13}$$

we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_o(\tau)}{\tau-\chi}d\tau + \int_{-1}^{1}k_o(\chi,\tau)\phi_o(\tau)d\tau = g(\chi) , \qquad |\chi| < 1$$

$$\int_{-1}^{1}\phi_o(\tau)d\tau = 0 \tag{4.14}$$

The solution with the proper singularities will be sought in the form

$$\phi_o(\tau) = \frac{F(\tau)}{\sqrt{1-\tau^2}} \tag{4.15}$$

where $F(\tau)$ is Hölder continuous in the interval $-1 \leq \tau \leq 1$.

Hence following the procedure described in [7] we arrive at

$$\sum_{k=1}^{N} \frac{F(\tau_j)}{N} \left[\frac{1}{\tau_j - \chi_i} + \pi k_o(\chi_i, \tau_j) \right] = g(\chi_i) \ , \quad i = 1,\ldots,N-1$$

$$\sum_{k=1}^{N} \frac{\pi}{N} F(\tau_j) = 0 \tag{4.16}$$

where

$$\tau_j = \cos\frac{\pi}{2N}(2j-1) \ , \qquad j = 1,\ldots,N$$

$$\chi_i = \cos\frac{\pi i}{N} \ , \qquad i = 1,\ldots,N-1 \tag{4.17}$$

From (4.16) N unknowns $F(\tau_j)$, $j = 1,\ldots,N$ can be solved.

4.2 For Material Type II

Choosing

$$H(t) = -\frac{\Delta_{12}}{\Delta_{11}} E(t) \tag{4.18}$$

we see that the expressions (3.14) and (3.15) for the stress and the displacement fields automatically satisfy the boundary conditions (3.17).

The mixed conditions (3.18) give

$$\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y} \sigma_{y1}(x,0) =$$

$$\int_0^\infty \{2A(t)[-\Delta_6 \sin(\omega_2 tx)\sinh(\omega_1 tx) + \Delta_5 \cos(\omega_2 tx)\cosh(\omega_1 tx)]$$

$$+ 2C(t)[\Delta_5 \sin(\omega_2 tx)\sinh(\omega_1 tx) + \Delta_6 \cos(\omega_2 tx)\cosh(\omega_1 tx)]\} tdt$$

$$+ \lim_{y\to 0} \int_0^\infty E(t)[(\Delta_7 \cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_8 \sin\frac{\omega_2 ty}{\sqrt{\beta_5}})$$

$$- \frac{\Delta_{12}}{\Delta_{11}}(-\Delta_8 \cos\frac{\omega_2 ty}{\sqrt{\beta_5}} + \Delta_7 \sin\frac{\omega_2 ty}{\sqrt{\beta_5}})] e^{-|\omega_1| ty/\sqrt{\beta_5}} \cos tx\, tdt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y} p(x) \ , \qquad |x| < a$$

$$v_1(x,0) = \frac{2}{\pi}\Delta_{16}\int_0^\infty E(t)\cos tx\,dt = 0 \ , \qquad a < |x| < h \tag{4.19}$$

Now, in a similar way, we will define the new unknown by

$$\frac{\partial v_1(x,0)}{\partial x} = -\frac{2}{\pi}\Delta_{16}\int_0^\infty E(t)\sin tx\,t\,dt = \phi(x) \tag{4.20}$$

such that $\phi(x) = 0$ for $|x| > a$.

Inverting (4.20) we obtain

$$-\Delta_{16} tE(t) = \int_0^a \phi(x)\sin tx\,dx \tag{4.21}$$

Substitution from (4.21) into the first equation of (4.19) yields

$$\Delta_{27}\int_{-a}^{a}\frac{\phi(t)\,dt}{t-x} + \int_0^\infty \{2A(t)[-\Delta_6\sin(\omega_2 tx)\sinh(\omega_1 tx)$$

$$+ \Delta_5\cos(\omega_2 tx)\cosh(\omega_1 tx)]$$

$$+ 2C(t)[\Delta_5\sin(\omega_2 tx)\sinh(\omega_1 tx) + \Delta_6\cos(\omega_2 tx)\cosh(\omega_1 tx)]\}tdt$$

$$= -\frac{\pi(1-\nu_{xy}\nu_{yx})}{2E_y}\,p(x) \ , \qquad |x| < a \tag{4.22}$$

where the extension of the definition of $\phi(t)$ into the $(-a,0)$ range requires the single-valuedness of the displacements, i.e.,

$$\int_{-a}^{a}\phi(t)\,dt \tag{4.23}$$

$A(t)$ and $C(t)$ are determined from (3.16).

From (3.16), (3.14) and (3.15) after some manipulations and an inversion we have

$$-2A(t)\cos(\omega_2 th)\sinh(\omega_1 th) - 2C(t)\sin(\omega_2 th)\cosh(\omega_1 th)$$

$$+ K(t)\cos(\omega_2^* th)e^{-|\omega_1^*|th} + M(t)\sin(\omega_2^* th)e^{-|\omega_1^*|th}$$

$$= \frac{1}{\pi}\int_0^\infty E(\xi)\sin\xi h\, d\xi \left[\frac{\frac{\Delta_{28}}{\sqrt{\beta_5}}\xi - \frac{\Delta_{12}}{\Delta_{11}}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} + t)^2} + \frac{\frac{\Delta_{28}}{\sqrt{\beta_5}}\xi + \frac{\Delta_{12}}{\Delta_{11}}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} - t)^2}\right] = H_1(t)$$

$$-2A(t)[\beta_7'\cos(\omega_2 th)\cosh(\omega_1 th) - \beta_7''\sin(\omega_2 th)\sinh(\omega_1 th)]$$

$$- 2C(t)[\beta_7''\cos(\omega_2 th)\cosh(\omega_1 th) + \beta_7'\sin(\omega_2 th)\sinh(\omega_1 th)]$$

$$+ K(t)[-\beta_7^{*'}\operatorname{sign}(\omega_1^*)\cos(\omega_2^* th) - \beta_7^{*''}\sin(\omega_2^* th)]e^{-|\omega_1^*|th}$$

$$+ M(t)[\beta_7^{*''}\cos(\omega_2^* th) - \beta_7^{*'}\operatorname{sign}(\omega_1^*)\sin(\omega_2^* th)]e^{-|\omega_1^*|th}$$

$$= \frac{1}{\pi}\int_0^\infty E(\xi)\cos\xi h\, d\xi \left[\frac{\frac{\Delta_{33}}{\sqrt{\beta_5}}\xi + \Delta_{16}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} + t)^2} + \frac{-\frac{\Delta_{33}}{\sqrt{\beta_5}}\xi + \Delta_{16}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} - t)^2}\right] = H_2(t)$$

$$\{-2A(t)[-\Delta_2\sin(\omega_2 th)\sinh(\omega_1 th) + \Delta_1\cos(\omega_2 th)\cosh(\omega_1 th)]$$

$$- 2C(t)[\Delta_1\sin(\omega_2 th)\sinh(\omega_1 th) + \Delta_2\cos(\omega_2 th)\cosh(\omega_1 th)]$$

$$- K(t)[\Delta_{21}\sin(\omega_2^* th) + \Delta_{22}\cos(\omega_2^* th)]e^{-|\omega_1^*|th}$$

$$+ M(t)[\Delta_{21}\cos(\omega_2^* th) - \Delta_{22}\sin(\omega_2^* th)]e^{-|\omega_1^*|th}\}t$$

$$= \frac{1}{\pi}\int_0^\infty E(\xi)\cos\xi h\, \xi d\xi \left[\frac{\Delta_{18}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} + t)^2} - \frac{\Delta_{18}t}{\frac{\omega_1^2\xi^2}{\beta_5} + (\frac{\omega_2\xi}{\sqrt{\beta_5}} - t)^2}\right] = tH_3(t)$$

$$\{-2A(t)[\Delta_9\cos(\omega_2 th)\sinh(\omega_1 th) - \Delta_{10}\sin(\omega_2 th)\cosh(\omega_1 th)]$$

$$-2C(t)[\Delta_9\sin(\omega_2 th)\cosh(\omega_1 th) + \Delta_{10}\cos(\omega_2 th)\sinh(\omega_1 th)]$$

$$+ K(t)[\Delta_{24}\cos(\omega_2^* th) + \Delta_{25}\sin(\omega_2^* th)]e^{-|\omega_1^*|th}$$

$$+ M(t)[\Delta_{24}\sin(\omega_2^* th) - \Delta_{25}\cos(\omega_2^* th)]e^{-|\omega_1^*|th}\}t$$

$$= \frac{1}{\pi}\int_0^\infty E(\xi)\sin\xi h\,\xi d\xi\left[\frac{\Delta_{19}\dfrac{|\omega_1|\xi}{\sqrt{\beta_5}}}{\dfrac{\omega_1^2\xi^2}{\beta_5} + \left(\dfrac{\omega_2\xi}{\sqrt{\beta_5}} - t\right)^2} - \frac{\Delta_{19}\dfrac{|\omega_1|\xi}{\sqrt{\beta_5}}}{\dfrac{\omega_1^2\xi^2}{\beta_5} + \left(\dfrac{\omega_2\xi}{\sqrt{\beta_5}} + t\right)^2}\right] = tH_4(t) \tag{4.24}$$

From (4.21) and (4.24) and the integrals given in Appendix C $H_i(t)$ can be obtained in terms of $\phi(t)$, i.e.,

$$H_1(t) = \frac{\Delta_{31}}{t}\int_{-a}^{a}\{\Delta_{32}\cos[\omega_2(h-\eta)t] + \sin[\omega_2(h-\eta)t]\}e^{-\omega_1(h-\eta)t}\phi(\eta)d\eta$$

$$H_2(t) = \frac{-1}{2t}\int_{-a}^{a}\{\cos[\omega_2(h-\eta)t] + \Delta_{36}\sin[\omega_2(h-\eta)t]\}e^{-\omega_1(h-\eta)t}\phi(\eta)d\eta$$

$$H_3(t) = \frac{\Delta_{37}}{t}\int_{-a}^{a}\sin[\omega_2(h-\eta)t]e^{-\omega_1(h-\eta)t}\phi(\eta)d\eta$$

$$H_4(t) = \frac{\Delta_{38}}{t}\int_{-a}^{a}\{\frac{\omega_2}{\omega_1}\cos[\omega_2(h-\eta)t] - \sin[\omega_2(h-\eta)t]\}e^{-\omega_1(h-\eta)t}\phi(\eta)d\eta \tag{4.25}$$

Solving (4.24) for A(t), C(t) and substituting into (4.22) we obtain the following singular integral equation

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\phi(t)dt}{t-x} + \int_{-a}^{a}\theta(x,t)\phi(t)dt = -\frac{(1-\nu_{xy}\nu_{yx})}{2\Delta_{27}E_y}\,p(x)\ , \qquad |x| < a \qquad (4.26)$$

subject to

$$\int_{-a}^{a}\phi(t)dt = 0$$

where

$$\theta(x,t) = \frac{1}{\pi\Delta_{27}}\int_{0}^{\infty}\{\theta_1(x,\eta)\cos[\omega_2(h-t)\eta] + \theta_2(x,\eta)\sin[\omega_2(h-t)\eta]\}e^{-\omega_1(h-t)\eta}d\eta$$

$$\theta_1(x,\eta) = \frac{1}{g_o(\eta)}[g_5(\eta)g_9(x,\eta) + g_7(\eta)g_{10}(x,\eta)]$$

$$\theta_2(x,\eta) = \frac{1}{g_o(\eta)}[g_6(\eta)g_9(x,\eta) + g_8(\eta)g_{10}(x,\eta)] \qquad (4.27)$$

$$g_o(\eta) = g_1(\eta)g_4(\eta) - g_2(\eta)g_3(\eta)$$

$$g_1(\eta) = -\frac{\Delta_{40}}{2}\cos(\omega_2\eta h) + [\frac{\Delta_{41}}{2}\sin(\omega_2\eta h) + \cos(\omega_2\eta h)]\tanh(\omega_1\eta h)$$

$$g_2(\eta) = -\frac{\Delta_{41}}{2}\cos(\omega_2\eta h) + [-\frac{\Delta_{40}}{2}\tanh(\omega_1\eta h) + 1]\sin(\omega_2\eta h)$$

$$g_3(\eta) = -\Delta_{44}\cos(\omega_2\eta h) + [\Delta_{45}\tanh(\omega_1\eta h) - 1]\sin(\omega_2\eta h)$$

$$g_4(\eta) = -\Delta_{45}\cos(\omega_2\eta h) + [-\Delta_{44}\sin(\omega_2\eta h) + \cos(\omega_2\eta h)]\tanh(\omega_1\eta h)$$

(4.28)

$$g_5(\eta) = \Delta_{52}g_1(\eta) + \Delta_{53}g_2(\eta) + \Delta_{55}g_3(\eta) + \Delta_{56}g_4(\eta)$$

$$g_6(\eta) = \Delta_{58}g_1(\eta) + \Delta_{59}g_2(\eta) + \Delta_{61}g_3(\eta) + \Delta_{62}g_4(\eta)$$

$$g_7(\eta) = \Delta_{64}g_1(\eta) + \Delta_{65}g_2(\eta) + \Delta_{67}g_3(\eta) + \Delta_{68}g_4(\eta)$$

$$g_8(\eta) = \Delta_{70}g_1(\eta) + \Delta_{71}g_2(\eta) + \Delta_{73}g_3(\eta) + \Delta_{74}g_4(\eta) \qquad (4.29)$$

$$g_9(x,\eta) = \frac{\sinh(\omega_1\eta x)}{\cosh(\omega_1\eta h)} \sin(\omega_2\eta x)$$

$$g_{10}(x,\eta) = \frac{\cosh(\omega_1\eta x)}{\cosh(\omega_1\eta h)} \cos(\omega_2\eta x) \tag{4.30}$$

Changing the variables

$$t = a\tau \ , \qquad x = a\chi$$

$$\phi(t) = \phi_o(\tau) \ , \qquad p(x) = p_o(\chi)$$

$$\theta_o(\chi,\tau) = a\theta(x,t)$$

$$r(\chi) = -\frac{(1-\nu_{xy}\nu_{yx})}{2\Delta_{27}E_y} p_o(\chi) \tag{4.31}$$

we then have from (4.26)

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi_o(\tau)}{\tau-\chi}\, d\tau + \int_{-1}^{1} \theta_o(\chi,\tau)\phi_o(\tau)d\tau = r(\chi) \ , \qquad |\chi| < 1$$

$$\int_{-1}^{1} \phi_o(\tau)d\tau = 0 \tag{4.32}$$

Expressing the solution as in (4.15) we arrive at

$$\sum_{k=1}^{N} \frac{F(\tau_j)}{N} \left[\frac{1}{\tau_j - \chi_i} + \pi\theta_o(\chi_i,\tau_j)\right] = r(\chi_i) \ , \qquad i=1,\ldots,N-1$$

$$\sum_{k=1}^{N} \frac{\pi}{N} F(\tau_j) = 0 \tag{4.33}$$

where τ_j and χ_i are defined by the equations (4.17).

5. STRESS INTENSITY FACTOR

Due to the symmetry $\tau_{xy1}(x,0) = 0$, hence the stress intensity factor will be defined by

$$K = \lim_{x\to a} \sqrt{2(x-a)}\ \sigma_{y1}(x,0) \tag{5.1}$$

5.1 For the Material of Type I

From the equation (4.9), the dominant part of $\sigma_{y1}(x,0)$ which contributes to the stress singularity at the crack tip can be expressed as

$$\sigma_{y1}(x,0) = \frac{2\gamma_{20}E_y}{\pi(1 - \nu_{xy}\nu_{yx})} \int_{-a}^{a} \frac{\phi(t)dt}{t - x} + \sigma^o_{y1}(x,0) \qquad (5.2)$$

where from (4.15)

$$\phi(t) = \frac{aF(t/a)e^{\pi i/2}}{(t-a)^{1/2}(t+a)^{1/2}} \qquad (5.3)$$

and $\sigma^o_{y1}(x,0)$ is a function which is bounded at the end points.

The behavior of the Cauchy Integral in (5.2) near the end points $x = \pm a$ will be determined by following the method given by Muskhelishvili [8].

Hence, defining a sectionally holomorphic function

$$\Phi(z) = \frac{1}{\pi} \int_{-a}^{a} \frac{\phi(t)dt}{t - z} \qquad (5.4)$$

from (5.3) and (5.4) we can write [8]

$$\Phi(z) = \frac{\sqrt{a/2}\ F(-1)e^{\pi i/2}}{(z+a)^{1/2}} - \frac{\sqrt{a/2}\ F(1)}{(z-a)^{1/2}} + \Phi_o(z) \qquad (5.5)$$

Here, $\Phi_o(z)$ is bounded everywhere except the end points where

$$|\Phi_o(z)| < \frac{C}{|z\pm a|^{\alpha_o}}, \qquad \alpha_o < 1/2$$

C and α_o being real constants.

Hence from (5.1), (5.2), (5.4) and (5.5)

$$K = - \frac{2\gamma_{20}E_y\sqrt{a}}{(1 - \nu_{xy}\nu_{yx})} F(1) \qquad (5.6)$$

which is also the same at the tip $x = -a$. (5.6) can be derived also by using a different approach (see Appendix D).

$F(1)$ can be obtained in terms of $F(\tau_j)$, $j = 1,\ldots,n$ either by an extrapolation or even better by using the formulas given in [11].

5.2 For the Material of Type II

In this case $\sigma_{y1}(x,0)$ can similarly be found by replacing γ_{20} in (5.2) by Δ_{27} [see (4.26)]. Hence,

$$K = -\frac{2\Delta_{27}E_y\sqrt{a}}{(1 - \nu_{xy}\nu_{yx})} F(1) \tag{5.7}$$

6. DISCUSSION

From the point of view of the analysis, the important problem is the one with nonzero crack surface tractions and vanishing loads at infinity. The solution to the actual problem, with the stress-free crack surfaces and the loads applied at infinity, can be obtained by a simple superposition. Hence, in the following, we will assume

$$\sigma_{y1}(x,0) = -p(x) = -p_o = \text{constant} \tag{6.1}$$

and plot $K/p_o\sqrt{a}$ vs. a/h for various material combinations.

Two different materials and their elastic constants will be given below as examples of materials of Type I.

Boron Epoxy

$$E_x = 3.5\times10^6 \text{ psi}, \qquad E_y = 3.24\times10^7 \text{ psi}$$

$$\nu_{yx} = 0.23, \qquad G_{xy} = 1.23\times10^6 \text{ psi}$$

and for the plane strain

$$E_z = 3.5\times10^6 \text{ psi}, \qquad \nu_{zx} = \nu_{zy} = 0.23 \tag{6.2}$$

Glass-Fiber (20% volume fraction)

$$E_x = 6.6\times10^5 \text{ psi}, \qquad E_y = 2.52\times10^6 \text{ psi}$$

$$\nu_{yx} = 0.32, \qquad G_{xy} = 2.9\times10^5 \text{ psi}$$

and for the plane strain

$$E_z = 6.6\times10^5 \text{ psi}, \qquad \nu_{zx} = \nu_{zy} = 0.32 \tag{6.3}$$

In Figure 1, the results in the case of generalized plane stress are plotted for two material combinations consisting of boron-epoxy and glass-fiber. As may be expected, the lower curve illustrates the stiffening effect of a boron-epoxy matrix, resulting in possible crack arrest. Both curves tend to unity as $a/h \to 0$ which corresponds to the solution of an infinite orthotropic plate with a crack. At the other end, as $a/h \to 1$, the effect of the material dissimilarity becomes more apparent. If the crack propagates and reaches the interfaces, then the resulting problem, for which $a = h$, requires a separate treatment.

Figure 3, like Figure 2, illustrates the variation of $K/p_o\sqrt{a}$ vs. a/h in the case of plane strain.

Similar curves are given in Figures 4, 5 and 6 for the materials of Type II. For this purpose the following materials are chosen:

For Figures 4 and 5:

(I): $E_x = 2\times10^6$ psi, $E_y = 2.52\times10^7$ psi

$\nu_{yx} = 0.20$, $G_{xy} = 4\times10^6$ psi

and for the plane strain

$E_z = 2\times10^6$ psi, $\nu_{zx} = \nu_{zy} = 0.20$

(II): $E_x = 2.40\times10^6$ psi, $E_y = 8\times10^6$ psi

$\nu_{xy} = 0.25$, $G_{xy} = 3\times10^6$ psi

and for the plane strain

$E_z = 2.40\times10^6$ psi, $\nu_{zx} = \nu_{zy} = 0.25$ (6.4)

For Figure 6:

(III): $E_x = 2.40\times10^6$ psi, $E_y = 8\times10^6$ psi

$\nu_{yx} = 0.25$, $G_{xy} = 3\times10^6$ psi

(IV): $E_x = 1.5\times10^6$ psi, $E_y = 4\times10^7$ psi

$\nu_{yx} = 0.20$, $G_{xy} = 4\times10^6$ psi (6.5)

The discussion of these curves can be made similarly.

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APPENDIX A

It can be shown that s_1 and s_2 as defined by (3.4) are either real or complex conjugates. Hence, two possibilities exist:

1 - Material of type I:

$$s_1 = \omega_1, \quad s_2 = \omega_3 \qquad \omega_2 = \omega_4 = 0 \tag{A-1}$$

from which it follows that

$\beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}$ are real

2 - Material of type II:

$$s_1 = \omega_1 + i\omega_2$$

$$s_2 = \omega_1 - i\omega_2 = \bar{s}_1 , \quad \omega_1 = \omega_3, \quad \omega_2 = -\omega_4 \tag{A-2}$$

from which we obtain

$$\beta_8 = \bar{\beta}_7, \quad \beta_{10} = \bar{\beta}_9, \quad \beta_6 = \text{pure imaginary} \tag{A-3}$$

In the case of generalized plane stress from (2.20) and (2.17) we have

$$\beta_1 = \frac{E_x}{G_{xy}(1 - \nu_{xy}\nu_{yx})} , \quad \beta_2 = \frac{E_y}{G_{xy}(1 - \nu_{xy}\nu_{yx})}$$

$$\beta_3 = 1 + \frac{\nu_{yx}E_x}{G_{xy}(1 - \nu_{xy}\nu_{yx})} \tag{A-4}$$

Hence from (3.4) and (A-4)

$$\beta_4 = 2\nu_{yx} - \frac{E_y}{G_{xy}} , \qquad \beta_5 = \frac{E_y}{E_x} \tag{A-5}$$

Therefore the equation to determine s_1 and s_2 takes the form [see (3.4)]

$$s^4 - \frac{E_y}{E_x}\left(\frac{E_x}{G_{xy}} - 2\nu_{xy}\right)s^2 + \frac{E_y}{E_x} = 0 \quad \text{(A-6)}$$

or by change of variables

$$\frac{1}{s} = i\mu \quad \text{(A-7)}$$

we have

$$\mu^4 + \left(\frac{E_x}{G_{xy}} - 2\nu_{xy}\right)\mu^2 + \frac{E_x}{E_y} = 0 \quad \text{(A-8)}$$

It has been proved by Lekhnitskii [6] and [12] that equation (A-8) has no real roots. The possible roots are

$$\text{a)}\quad \mu_1 = \delta_1 i, \quad \mu_2 = \delta_2 i, \quad \bar{\mu}_1, \quad \bar{\mu}_2 \quad \text{or}$$

$$\text{b)}\quad \mu_1 = \delta_3 + \delta_4 i, \quad \mu_2 = -\delta_3 + \delta_4 i, \quad \bar{\mu}_1, \quad \bar{\mu}_2 \quad \text{(A-9)}$$

Hence from (A-7) and (A-9) we conclude that the roots in terms of s are

$$\text{a)}\quad s_1, \quad -s_1, \quad s_2, \quad -s_2 \quad \text{all being real, or}$$

$$\text{b)}\quad s_1, \quad -s_1, \quad \bar{s}_1, \quad -\bar{s}_1 \quad s_1 = \text{complex} \quad \text{(A-10)}$$

Hence the propositions in (A-1) and (A-2) are proved.

In the case of plane strain the same results are obtained [6], [12].

APPENDIX B

B.1 Material Type I:

Dimensionless bielastic constants γ_i:

$$\gamma_1 = 1 + \frac{\nu_{yx}\omega_1\beta_9}{\sqrt{\beta_5}} , \qquad \gamma_2 = 1 + \frac{\nu_{yx}\omega_3\beta_{10}}{\sqrt{\beta_5}}$$

$$\gamma_3 = \omega_1 + \nu_{yx}\beta_7 , \qquad \gamma_4 = \omega_3 + \nu_{yx}\beta_8$$

$$\gamma_5 = \nu_{xy} + \frac{\omega_1\beta_9}{\sqrt{\beta_5}} , \qquad \gamma_6 = \nu_{xy} + \frac{\omega_3\beta_{10}}{\sqrt{\beta_5}}$$

$$\gamma_7 = \nu_{xy}\omega_1 + \beta_7 , \qquad \gamma_8 = \nu_{xy}\omega_3 + \beta_8$$

$$\gamma_9 = -1 + \omega_1\beta_7 , \qquad \gamma_{10} = -1 + \omega_3\beta_8$$

$$\gamma_9^* = -1 + \omega_1^*\beta_7^* , \qquad \gamma_{10}^* = -1 + \omega_3^*\beta_8^*$$

$$\gamma_{11} = -\frac{|\omega_1|}{\sqrt{\beta_5}} + \beta_9\text{sign}(\omega_1) , \qquad \gamma_{12} = -\frac{|\omega_3|}{\sqrt{\beta_5}} + \beta_{10}\text{sign}(\omega_3)$$

$$\gamma_{13}^* = -|\omega_1^*| - \nu_{yx}^*\beta_7^*\text{sign}(\omega_1^*) , \qquad \gamma_{14}^* = -|\omega_3^*| - \nu_{yx}^*\beta_8^*\text{sign}(\omega_3^*)$$

$$\gamma_{15}^* = -\nu_{xy}^*|\omega_1^*| - \beta_7^*\text{sign}(\omega_1^*) , \qquad \gamma_{16}^* = -\nu_{xy}^*|\omega_3^*| - \beta_8^*\text{sign}(\omega_3^*)$$

$$\gamma_{17} = \frac{E_x^*}{E_x}\,\frac{1 - \nu_{xy}\nu_{yx}}{1 - \nu_{xy}^*\nu_{yx}^*} , \qquad \gamma_{18} = \frac{G_{xy}^*}{G_{xy}}$$

$$\gamma_{19} = \text{sign}(\omega_1)\beta_9 - \frac{\gamma_{11}}{\gamma_{12}}\,\text{sign}(\omega_3)\beta_{10}$$

$$\gamma_{20} = \frac{\gamma_5 - \gamma_6\dfrac{\gamma_{11}}{\gamma_{12}}}{2\gamma_{19}}$$

$$\gamma_{21}^* = \text{sign}(\omega_1^*)\beta_7^* - \text{sign}(\omega_3^*)\beta_8^*$$

$$\gamma_{22}^* = 1 + \text{sign}(\omega_3^*)\ \frac{\beta_8^*}{\gamma_{21}^*}$$

$$\gamma_{23} = -\ \gamma_3 + \frac{\gamma_{17}}{\gamma_{21}^*}\ \beta_7(\gamma_{14}^* - \gamma_{13}^*)$$

$$\gamma_{24} = \gamma_{22}^*\gamma_{17}\gamma_{14}^* - \text{sign}(\omega_3^*)\beta_8^*\ \frac{\gamma_{17}\gamma_{13}^*}{\gamma_{21}^*}$$

$$\gamma_{25} = -\ \gamma_4 + \frac{\gamma_{17}}{\gamma_{21}^*}\ \beta_8(\gamma_{14}^* - \gamma_{13}^*)$$

$$\gamma_{26} = -\ \gamma_9 - \frac{\gamma_{18}\gamma_9^*}{\gamma_{21}^*}\ \text{sign}(\omega_3^*)\beta_8^* + \gamma_{18}\gamma_{10}^*\gamma_{22}^*$$

$$\gamma_{27} = \gamma_{18}\beta_7\ \left(\frac{\gamma_{10}^* - \gamma_9^*}{\gamma_{21}^*}\right)$$

$$\gamma_{28} = -\ \gamma_{10} - \frac{\gamma_{18}\gamma_9^*}{\gamma_{21}^*}\ \text{sign}(\omega_3^*)\beta_8^* + \gamma_{18}\gamma_{10}^*\gamma_{22}^*$$

$$\gamma_{29} = \gamma_{18}\beta_8\ \left(\frac{\gamma_{10}^* - \gamma_9^*}{\gamma_{21}^*}\right)$$

$$\gamma_{30} = \frac{\gamma_{17}}{\gamma_{21}^*}\ [(\gamma_{13}^* - \gamma_{14}^*)\beta_8^*\text{sign}(\omega_3^*) - \gamma_{14}^*\gamma_{21}^*] = -\gamma_{24}$$

$$\gamma_{31} = \frac{\gamma_{17}}{\gamma_{21}^*}\ (\gamma_{14}^* - \gamma_{13}^*)$$

$$\gamma_{32} = \frac{\gamma_{18}}{\gamma_{21}^*}\ [(\gamma_9^* - \gamma_{10}^*)\beta_8^*\text{sign}(\omega_3^*) - \gamma_{10}^*\gamma_{21}^*]$$

$$\gamma_{33} = \frac{\gamma_{29}}{\beta_8}\ , \qquad \gamma_{34} = \gamma_{30} + \gamma_{31}\beta_9\text{sign}(\omega_1) - \frac{\gamma_1\sqrt{\beta_5}}{|\omega_1|}$$

$$\gamma_{35} = \frac{\gamma_{11}}{\gamma_{12}}\ [\gamma_{30} + \gamma_{31}\beta_{10}\text{sign}(\omega_3) - \frac{\gamma_2\sqrt{\beta_5}}{|\omega_3|}\ ]$$

$$\gamma_{36} = \gamma_{32} + \gamma_{33}\beta_9 \text{sign}(\omega_1) + \frac{\gamma_{11}\sqrt{\beta_5}}{|\omega_1|}$$

$$\gamma_{37} = \frac{\gamma_{11}}{\gamma_{12}} [\gamma_{32} + \gamma_{33}\beta_{10}\text{sign}(\omega_3) + \frac{\gamma_{12}\sqrt{\beta_5}}{|\omega_3|}]$$

$$\gamma_{38} = \gamma_{36}\gamma_{25} - \gamma_{34}\gamma_{29} , \qquad \gamma_{39} = \gamma_{36}\gamma_{24} - \gamma_{34}\gamma_{28}$$

$$\gamma_{40} = \gamma_{34}\gamma_{27} - \gamma_{36}\gamma_{23} , \qquad \gamma_{41} = \gamma_{34}\gamma_{26} - \gamma_{36}\gamma_{24}$$

$$\gamma_{42} = \gamma_{35}\gamma_{29} - \gamma_{37}\gamma_{25} , \qquad \gamma_{43} = \gamma_{35}\gamma_{28} - \gamma_{37}\gamma_{24}$$

$$\gamma_{44} = \gamma_{37}\gamma_{23} - \gamma_{35}\gamma_{27} , \qquad \gamma_{45} = \gamma_{37}\gamma_{25} - \gamma_{35}\gamma_{26}$$

B.2 Material Type II:

Dimensionless bielastic constants Δ_i:

$$\beta_7 = \beta_7' + i\beta_7'' , \qquad \beta_9 = \beta_9' + i\beta_9''$$

$$\beta_7^* = \beta_7^{*'} + i\beta_7^{*''} , \qquad \beta_9^* = \beta_9^{*'} + i\beta_9^{*''}$$

$$\Delta_1 = \omega_1 + \nu_{yx}\beta_7' , \qquad \Delta_2 = \omega_2 + \nu_{yx}\beta_7''$$

$$\Delta_2^* = \omega_2^* + \nu_{yx}^*\beta_7^{*''} , \qquad \Delta_3 = 1 + \frac{\nu_{yx}}{\sqrt{\beta_5}} (-\omega_2\beta_9'' + \omega_1\beta_9')$$

$$\Delta_4 = \frac{\nu_{yx}}{\sqrt{\beta_5}} (\omega_2 \text{sign}(\omega_1)\beta_9' + |\omega_1|\beta_9'')$$

$$\Delta_5 = \omega_1\nu_{xy} + \beta_7' , \qquad \Delta_6 = \omega_2\nu_{xy} + \beta_7''$$

$$\Delta_6^* = \omega_2^*\nu_{xy}^* + \beta_7^{*''} , \qquad \Delta_7 = \nu_{xy} + \frac{1}{\sqrt{\beta_5}} (-\omega_2\beta_9'' + \omega_1\beta_9')$$

$$\Delta_8 = \frac{\Delta_4}{\nu_{yx}} , \qquad \Delta_9 = -1 - \omega_2\beta_7'' + \omega_1\beta_7'$$

$$\Delta_9^* = -1 - \omega_2^*\beta_7^{*''} + \omega_1^*\beta_7^{*'}$$

$$\Delta_{10} = \omega_2\beta_7' + \omega_1\beta_7'' , \qquad \Delta_{11} = \frac{\omega_2}{\sqrt{\beta_5}} - \beta_9''$$

$$\Delta_{12} = - \frac{|\omega_1|}{\sqrt{\beta_5}} + \beta_9' \mathrm{sign}(\omega_1)$$

$$\Delta_{13}^* = |\omega_1^*| + \nu_{yx}^*\beta_7^{*\prime} \mathrm{sign}(\omega_1^*)$$

$$\Delta_{14}^* = \nu_{xy}^*|\omega_1^*| + \beta_7^{*\prime} \mathrm{sign}(\omega_1^*)$$

$$\Delta_{15}^* = \omega_2^*\beta_7^{*\prime} \mathrm{sign}(\omega_1^*) + |\omega_1^*|\beta_7^{*\prime\prime}$$

$$\Delta_{16} = - \beta_9' \mathrm{sign}(\omega_1) - \frac{\Delta_{12}}{\Delta_{11}} \beta_9''$$

$$\Delta_{17} = - \beta_9'' + \frac{\Delta_{12}}{\Delta_{11}} \beta_9' \mathrm{sign}(\omega_1)$$

$$\Delta_{18} = \Delta_4 - \frac{\Delta_{12}}{\Delta_{11}} \Delta_3 , \qquad \Delta_{19} = - \Delta_{11} - \frac{\Delta_{12}^2}{\Delta_{11}}$$

$$\Delta_{20} = \frac{E_x^*}{E_x} \frac{(1 - \nu_{xy}\nu_{yx})}{(1 - \nu_{xy}^*\nu_{yx}^*)} , \qquad \Delta_{21} = \Delta_{20}\Delta_2^*$$

$$\Delta_{22} = \Delta_{20}\Delta_{13}^* , \qquad \Delta_{23} = \frac{G_{xy}^*}{G_{xy}}$$

$$\Delta_{24} = \Delta_{23}\Delta_9^* , \qquad \Delta_{25} = \Delta_{23}\Delta_{15}^*$$

$$\Delta_{26} = \Delta_7 + \Delta_8 \frac{\Delta_{12}}{\Delta_{11}} , \qquad \Delta_{27} = - \frac{1}{2} \frac{\Delta_{26}}{\Delta_{16}}$$

$$\Delta_{28} = |\omega_1| - \frac{\Delta_{12}}{\Delta_{11}} \omega_2 , \qquad \Delta_{29} = \frac{1}{\sqrt{\beta_5}} \left(|\omega_1| + \omega_2 \frac{\Delta_{12}}{\Delta_{11}}\right)$$

$$\Delta_{30} = \frac{\Delta_{12}}{\Delta_{11}\sqrt{\beta_5}} , \qquad \Delta_{31} = - \frac{1}{2} \frac{\Delta_{30}\sqrt{\beta_5}}{\Delta_{16}}$$

$$\Delta_{32} = \frac{\Delta_{29}}{\Delta_{30}\omega_1} - \frac{\omega_2}{\omega_1}, \qquad \Delta_{33} = \omega_2\Delta_{16} - |\omega_1|\Delta_{17}$$

$$\Delta_{34} = \frac{1}{\sqrt{\beta_5}}(\Delta_{33} - 2\omega_2\Delta_{16}), \quad \Delta_{35} = \frac{\Delta_{16}}{\sqrt{\beta_5}}$$

$$\Delta_{36} = \frac{\omega_2}{\omega_1} + \frac{\Delta_{34}}{\Delta_{35}\omega_1}, \qquad \Delta_{37} = -\frac{\Delta_{18}\sqrt{\beta_5}}{2\omega_1\Delta_{16}}$$

$$\Delta_{38} = -\frac{1}{2}\frac{\Delta_{19}|\omega_1|}{\Delta_{16}}, \qquad \Delta_{39} = \Delta_{22}\beta_7^{*''} - \Delta_{21}\beta_7^{*'}\text{sign}(\omega_1^*)$$

$$\Delta_{40} = \frac{2}{\Delta_{39}}(\Delta_{21}\beta_7' - \Delta_1\beta_7^{*''}), \quad \Delta_{41} = \frac{2}{\Delta_{39}}(\Delta_{21}\beta_7'' - \Delta_2\beta_7^{*''})$$

$$\Delta_{42} = \frac{2}{\Delta_{39}}[\Delta_{22}\beta_7' - \Delta_1\beta_7^{*'}\text{sign}(\omega_1^*)]$$

$$\Delta_{43} = \frac{2}{\Delta_{39}}[\Delta_{22}\beta_7'' - \Delta_2\beta_7^{*'}\text{sign}(\omega_1^*)]$$

$$\Delta_{44} = \frac{1}{2\Delta_{10}}[\Delta_{40}(\Delta_{24} - \Delta_9) - \Delta_{25}\Delta_{42}]$$

$$\Delta_{45} = \frac{1}{2\Delta_{10}}[\Delta_{41}(\Delta_{24} - \Delta_9) - \Delta_{25}\Delta_{43}]$$

$$\Delta_{46} = \frac{\Delta_9}{2\Delta_{10}}, \qquad \Delta_{47} = \frac{1}{2\Delta_{10}\Delta_{39}}[(\Delta_{24} - \Delta_9)\Delta_{21} - \Delta_{25}\Delta_{22}]$$

$$\Delta_{48} = \frac{1}{2\Delta_{10}\Delta_{39}}[-\beta_7^{*''}(\Delta_{24} - \Delta_9) + \Delta_{25}\beta_7^{*'}\text{sign}(\omega_1^*)]$$

$$\Delta_{49} = -\frac{\beta_7^{*''}}{2\Delta_{39}}, \qquad \Delta_{50} = \frac{\Delta_{21}}{2\Delta_{39}}$$

$$\Delta_{51} = 2\Delta_{31}\Delta_{32}\Delta_{46} - \frac{\Delta_{38}}{\Delta_{10}}\frac{\omega_2}{\omega_1} - \Delta_{47}$$

$$\Delta_{52} = \Delta_{51}\Delta_5, \qquad \Delta_{53} = \Delta_{51}\Delta_6$$

$$\Delta_{54} = \Delta_{31}\Delta_{32} + \Delta_{50}, \qquad \Delta_{55} = \Delta_{54}\Delta_5$$

$$\Delta_{56} = \Delta_{54}\Delta_6$$

$$\Delta_{57} = 2(\Delta_{31}\Delta_{46} + \Delta_{37}\Delta_{48}) + \frac{\Delta_{38}}{\Delta_{10}} - \Delta_{36}\Delta_{47}$$

$$\Delta_{58} = \Delta_{57}\Delta_5 , \qquad \Delta_{59} = \Delta_{57}\Delta_6$$

$$\Delta_{60} = \Delta_{31} + \Delta_{36}\Delta_{50} - 2\Delta_{37}\Delta_{49}$$

$$\Delta_{61} = \Delta_{60}\Delta_5 , \qquad \Delta_{62} = \Delta_{60}\Delta_6$$

$$\Delta_{63} = 2\Delta_{31}\Delta_{32}\Delta_{46} - \Delta_{47} - \frac{\Delta_{38}}{\Delta_{10}} \frac{\omega_2}{\omega_1}$$

$$\Delta_{64} = \Delta_{63}\Delta_6 , \qquad \Delta_{65} = - \Delta_{63}\Delta_5$$

$$\Delta_{66} = - \Delta_{31}\Delta_{32} - \Delta_{50}$$

$$\Delta_{67} = - \Delta_{66}\Delta_6 , \qquad \Delta_{68} = \Delta_{66}\Delta_5$$

$$\Delta_{69} = 2(\Delta_{31}\Delta_{46} + \Delta_{37}\Delta_{48}) + \frac{\Delta_{38}}{\Delta_{10}} - \Delta_{36}\Delta_{47}$$

$$\Delta_{70} = \Delta_{69}\Delta_6 , \qquad \Delta_{71} = - \Delta_{69}\Delta_5$$

$$\Delta_{72} = - \Delta_{31} - \Delta_{36}\Delta_{50} + 2\Delta_{49}\Delta_{37}$$

$$\Delta_{73} = - \Delta_{72}\Delta_6 , \qquad \Delta_{74} = \Delta_{72}\Delta_5$$

APPENDIX C

Calculation of certain integrals.

From [9] we have

$$\int_0^\infty e^{-at} \sin bt\, dt = \frac{b}{a^2 + b^2}, \qquad a > 0$$

$$\int_0^\infty e^{-at} \cos bt\, dt = \frac{a}{a^2 + b^2}, \qquad a > 0 \tag{C-1}$$

$$\int_0^\infty \left[\frac{1}{\beta^2 + (\gamma-x)^2} - \frac{1}{\beta^2 + (\gamma+x)^2}\right] \sin(ax)\,dx = \frac{\pi}{\beta} e^{-a\beta} \sin(a\gamma),$$

$a > 0$, $Re(\beta) > 0$, $(\gamma+i\beta)$ is not real

$$\int_0^\infty \left[\frac{1}{\beta^2 + (\gamma-x)^2} + \frac{1}{\beta^2 + (\gamma+x)^2}\right] \cos(ax)\,dx = \frac{\pi}{\beta} e^{-a\beta} \cos(a\gamma),$$

$a > 0$, $|Im\gamma| < Re(\beta)$

and by properly differentiating with respect to a

$$\int_0^\infty \left[\frac{x}{\beta^2 + (\gamma-x)^2} - \frac{x}{\beta^2 + (\gamma+x)^2}\right] \cos(ax)\,dx$$

$$= \frac{\pi}{\beta} e^{-a\beta}[-\beta\sin(a\gamma) + \gamma\cos(a\gamma)]$$

$$\int_0^\infty \left[\frac{x}{\beta^2 + (\gamma-x)^2} + \frac{x}{\beta^2 + (\gamma+x)^2}\right] \sin(ax)\,dx$$

$$= \frac{\pi}{\beta} e^{-a\beta}[\gamma\sin(a\gamma) + \beta\cos(a\gamma)] \tag{C-2}$$

Special Cases.

$$\int_0^\infty \frac{1}{\beta^2 + x^2} \cos(ax)\,dx = \frac{\pi}{2\beta} e^{-a\beta}$$

$$\int_0^\infty \frac{x}{\beta^2 + x^2} \sin(ax)\,dx = \frac{\pi}{2} e^{-a\beta}$$

$$\int_0^\infty \frac{1}{x(\beta^2 + x^2)} \sin(ax)\,dx = \frac{\pi}{2\beta^2} (1 - e^{-\beta a}) \tag{C-3}$$

$$\frac{a + b/\xi}{c\xi^2 + (d\xi + ft)^2} = \frac{-\frac{b(c + d^2)}{f^2 t^2}\,\xi + a - \frac{2bd}{ft}}{c\xi^2 + (d\xi + ft)^2} + \frac{\frac{b}{f^2 t^2}}{\xi}$$

APPENDIX D

Derivation of the Stress Intensity Factor Expression:

From (5.2) and (5.3) we have

$$\sigma_{y1}(a\chi,0) = \frac{2\gamma_{20}E_y}{\pi(1-\nu_{xy}\nu_{yx})}\int_{-1}^{1}\frac{F(\tau)}{\sqrt{1-\tau^2}}\frac{d\tau}{\tau-\chi} + \sigma_{y1}^{o}(a\chi,0) \qquad \text{(D-1)}$$

where $x = a\chi$, $t = a\tau$.

Expanding $F(\tau)$ into a series of Chebyshev polynomials

$$F(\tau) = \sum_{j=0}^{\infty} C_j T_j(\tau) \qquad \text{(D-2)}$$

and substituting into (D-1), with the use of the formula [13]

$$\int_{-1}^{1}\frac{T_j(\tau)}{\sqrt{1-\tau^2}}\frac{d\tau}{\tau-\chi} = \begin{cases} \pi U_{j-1}(\chi), & |\chi| < 1 \\ \dfrac{\pi[\sqrt{\chi^2-1}-\chi]^j}{(-1)^{j+1}\sqrt{\chi^2-1}}, & |\chi| > 1 \end{cases} \qquad \text{(D-3)}$$

for $|\chi| > 1$, we obtain

$$\sigma_{y1}(a\chi,0) = \frac{2\gamma_{20}E_y}{(1-\nu_{xy}\nu_{yx})\sqrt{\chi^2-1}}\sum_{j=0}^{\infty} C_j \frac{[\sqrt{\chi^2-1}-\chi]^j}{(-1)^{j+1}} + \sigma_{y1}^{o}(a\chi,0), \quad |\chi| > 1 \qquad \text{(D-4)}$$

Hence

$$K = \lim_{x\to a}\sqrt{2(x-a)}\,\sigma_{y1}(x,0) = -\frac{2\gamma_{20}E_y\sqrt{a}}{(1-\nu_{xy}\nu_{yx})}\sum_{j=0}^{\infty} C_j \qquad \text{(D-5)}$$

But from (D-2)

$$F(1) = \sum_{j=0}^{\infty} C_j T_j(1) = \sum_{j=0}^{\infty} C_j \qquad \text{(D-6)}$$

Hence, we finally obtain

$$K = -\frac{2\gamma_{20}E_y\sqrt{a}}{(1-\nu_{xy}\nu_{yx})}F(1) \qquad \text{(D-7)}$$

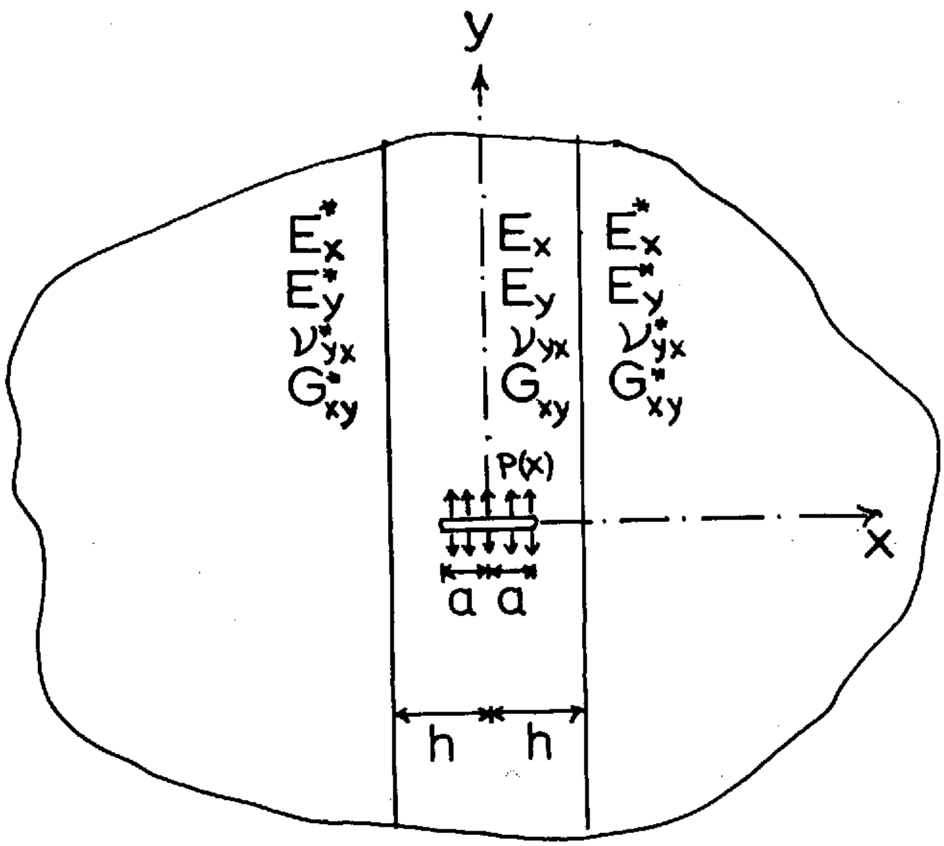

Figure 1. Geometry of the Problem

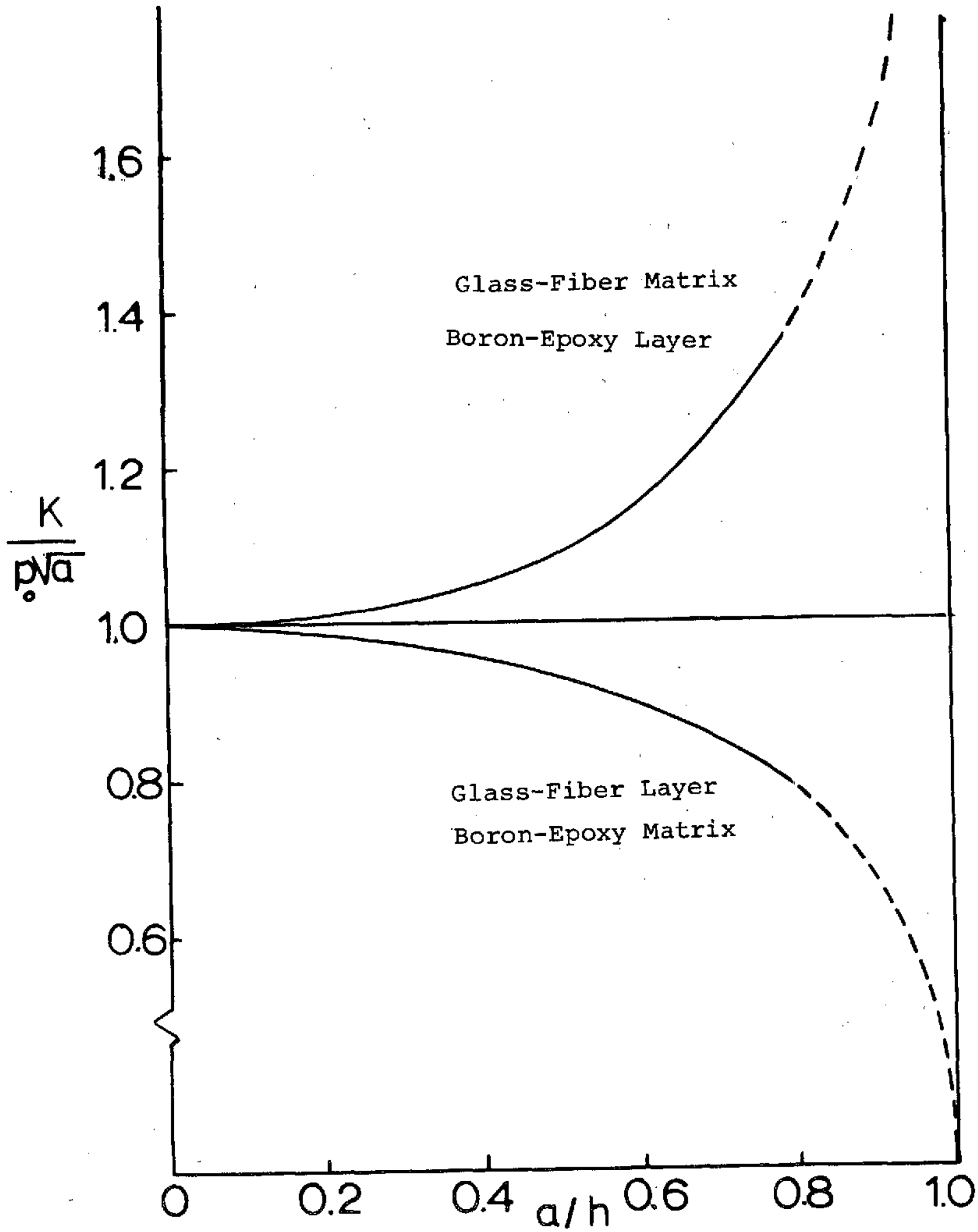

Figure 2. $K/p\sqrt{a}$ vs. a/h
Material of Type I, Generalized Plane Stress

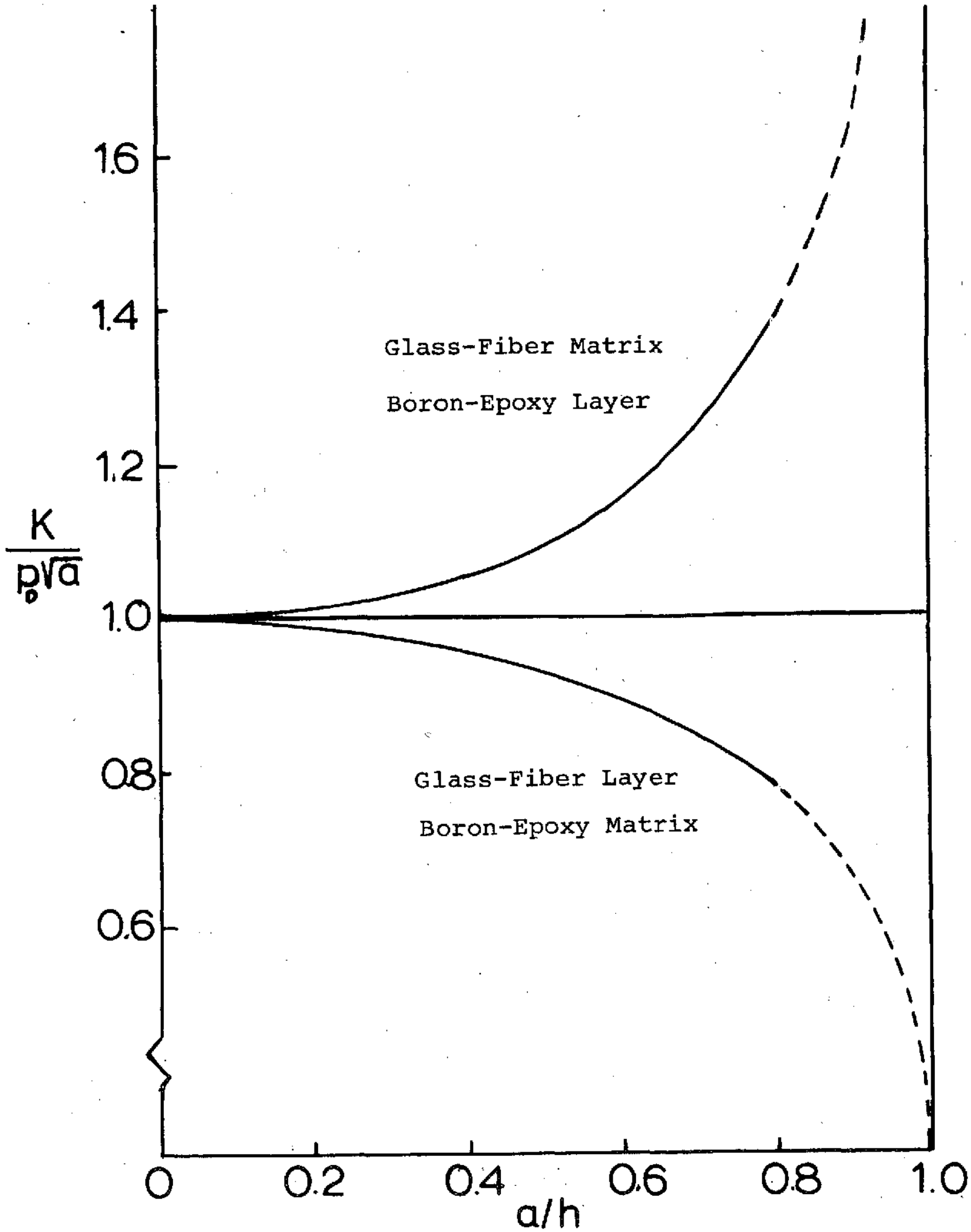

Figure 3. $K/p\sqrt{a}$ vs. a/h
Material of Type I , Plane Strain

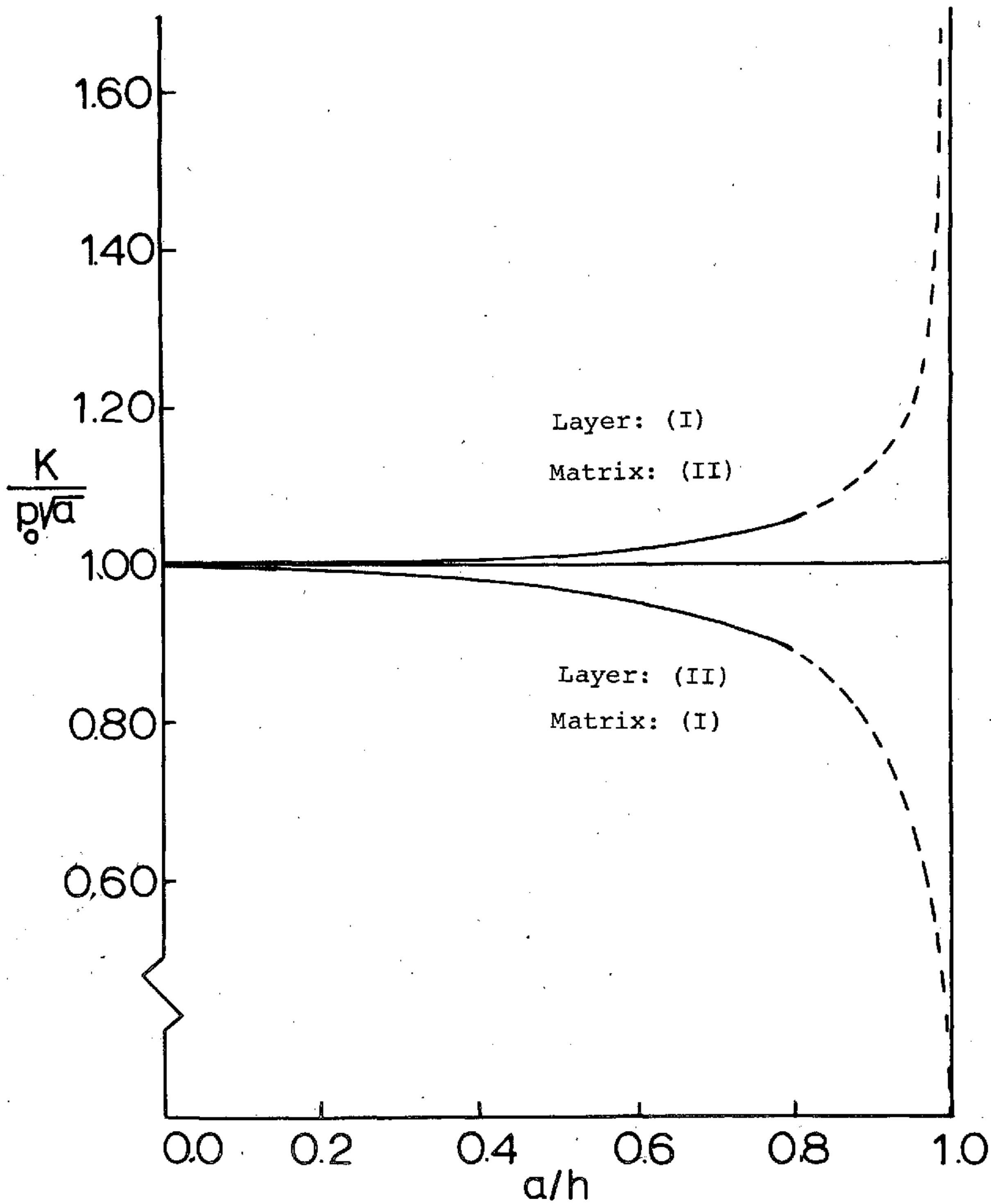

Figure 4. $K/p_o\sqrt{a}$ vs. a/h
Material of Type II, Generalized Plane Stress

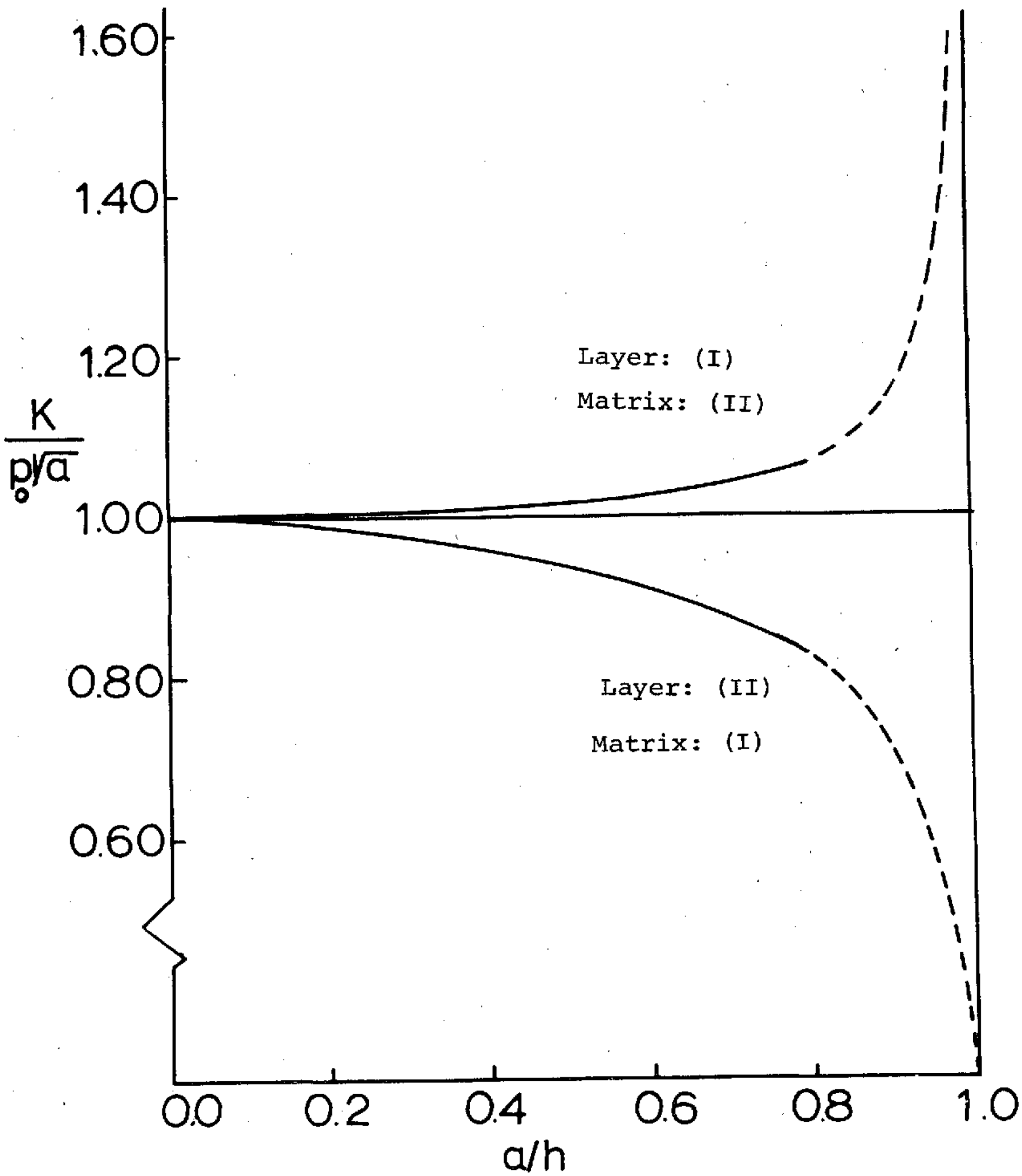

Figure 5. $K/p_o\sqrt{a}$ vs. a/h
Material of Type II, Plane Strain

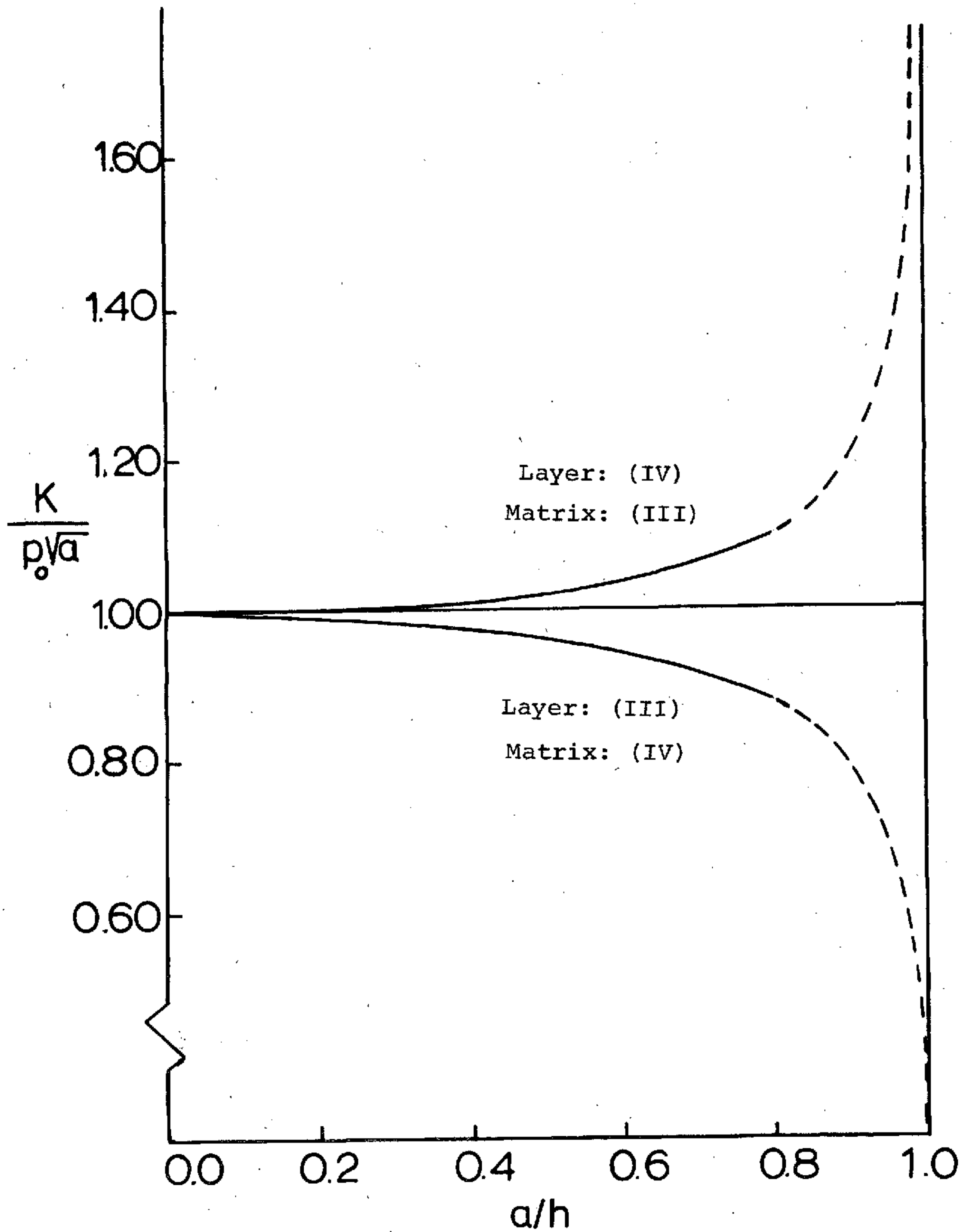

Figure 6. $K/p_o\sqrt{a}$ vs. a/h
Material of Type II, Generalized Plane Stress