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UNIFIED ANALYSIS OF DUCTED TURBOMACHINERY NOISE

by

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of the

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ABSTRACT AND SUMMARY

Hanson (J. Acoust. Soc. Am., 54, 6, 1973) has modeled turbomachinery noise in a free-field as a sequence of pulsed-dipole acoustic sources with randomly modulated amplitudes and arrival times. This approach produced a far-field noise spectra with a broad-band background in addition to the usual discrete noise component. A systematic method for calculating the internal and external acoustical fields of aircraft turbofan engines has been developed by Zorumski (NASA TR R-419). This work shows that the duct acoustic problem can be reduced to a linear matrix equation $WA = Q$ where A and Q are vectors of modal amplitudes and W is a square transmission/reflection matrix. In this paper the methodologies of Hanson and Zorumski are combined to yield a unified analysis of ducted turbomachinery noise. The essential features of Hanson's analysis are unchanged by the addition of duct acoustic effects. It is shown that the far-field broad-band and discrete noise spectral components can be expressed in terms of modal cross-spectral matrices and directivity vectors which are derivable from the duct analysis.

INTRODUCTION

Aircraft turbomachinery noise has been analyzed by various authors with the aid of either of two possible simplifying assumptions. The noise originates inside the engine at the rotating and fixed blades of the turbomachinery stages, consequently it is possible to study the ducted sound field assuming that the engine duct is infinitely long. This was the main viewpoint used in the classic analysis of "spinning mode" generation by Tyler and Sofin (ref. 1). For these authors, the radiation of the noise was considered as an independent problem. Using an alternate simplifying viewpoint, Lowson (ref. 2) argued that the main features of radiation from engine fans could be found by ignoring the duct effects entirely and considering the noise sources to be in the free-field. Both of these views are valid up to a point and may be used when only qualitative effects are needed. It should be recognized, however, that such drastic simplifications do not make sense in some situations. For example, if one is studying the effect of duct acoustic treatment on fan noise, it would hardly be sensible to neglect the duct; or, if it is desired to predict noise on the ground during an aircraft flyover, it is not sensible to ignore the radiation from the engine inlet. Recently Hanson (ref. 3) has used the no-duct assumption to great advantage to develop his "Unified Analysis of Fan Stator Noise." Hanson assumed that the essential features of his analysis would not be changed by the addition of duct effects and it will be shown in the present paper that his assumption is correct. In order to do this we will employ the results developed by Zorumski (ref. 4) for the radiated field due to ducted noise sources. Thus, the purpose of this paper is to combine the

analyses by Hanson and Zorumski to develop a unified analysis of ducted turbomachinery noise.

SYMBOLS

a	random variable with unit mean
A	mode amplitude
b	random variable with zero mean
B	number of rotor blades
c	ambient speed of sound
$D(\phi), \hat{D}(\phi)$	mode directivity functions
$E\{ \}$	expected value of $\{ \}$
$f(\tau)$	force pulse
$F(\omega)$	Fourier transform of force pulse
G	power spectral density
I	intensity
k	wave number, ω/c
m	circumferential mode number
n	harmonic number
p	acoustic pressure
Q	source mode amplitude

r, θ, z	cylindrical coordinates
r, θ, ϕ	spherical coordinates
R	mode reflection coefficient
s	stator vane number
t	time
T	mode transmission coefficient, also rotor period $2\pi/\Omega$
V	number of stator vanes
W	wave coefficient
x, y, z	rectangular coordinates
Y	broadband noise directivity function
Z	harmonic noise directivity function
$\delta(\omega)$	Dirac delta function
ρ	ambient density
σ	standard deviation
ψ	radial mode function

ω radian frequency

Ω rotor speed

$\begin{bmatrix} \end{bmatrix}$ row matrix

$\begin{Bmatrix} \end{Bmatrix}$ column matrix

$\begin{bmatrix} \\ \end{bmatrix}$ square matrix

REVIEW OF HANSON'S ANALYSIS

Hanson (ref. 3) considered the noise radiated from an aircraft engine fan stage due to the viscous wakes from the upstream rotor. Each stator vane source was modeled as a simple dipole (concentrated force on the fluid) which "pulsed" as each viscous wake from the rotor passed over the vane. The stator noise source model was therefore a circular dipole array, such as in figure 1, in the free-field. A typical pulse from one of these dipoles is $f(t)$ and the transform of this pulse is $F(\omega)$. The pressure spectrum due to a single typical pulse at time $t=0$ on stator vane s is

$$p(r_s; \omega) = - \frac{i k \vec{r}_s \cdot \vec{F}(\omega) e^{i k r_s}}{4 \pi r_s^2} \quad (1)$$

Hanson recognized that the viscous wakes from the rotor blades vary in a random fashion and modeled the actual pulses on the stator vanes as having random amplitudes, with unit mean and standard deviation σ_a . The time of the pulse occurrence was modeled as a Gaussian process with zero mean and standard deviation σ_b . Figure 2 shows a typical sequence of pulses with this random amplitude and arrival time property. With this model, the intensity per unit bandwidth $\frac{dI}{d\omega}$ at the observer position 0 in the far field was found to be

$$\frac{dI}{d\omega} = \frac{B\Omega\omega^2}{8\pi^2} \frac{|\vec{F}(\omega)|^2}{c^3 r^2} \left\{ \left[\sigma_a^2 + 1 - e^{-(\omega T \sigma_b)^2} \right] V Y_V(\phi, \theta) \right.$$

broadband term

$$+ B\Omega \sum_{n=-\infty}^{\infty} e^{-(\omega T \sigma_b)^2} \int (\omega - nB\Omega) Z(\phi, \theta; \omega) \Big\} \quad (2)$$

harmonic terms

In equation (2), B is the number of rotor blades, V is the number of stator vanes, Ω is the rotor angular speed, and $T = 2\pi/\Omega$. The term containing σ_a in equation (2) is the broadband noise due to pulse amplitude modulation (PAM) and the term remaining $1 - e^{-(\omega T \sigma_b)^2}$ is the broadband noise due to pulse position modulation (PPM). Note that PAM generates broadband noise without changing the harmonic noise, but that PPM decreases the harmonic noise by the factor $e^{-(\omega T \sigma_b)^2}$.

In the following sections, we will develop a formula, analogous to equation (1), for the far-field pressure due to a dipole pulse in a finite length duct and then use this result to extend Hanson's analysis to include duct effects. It will be shown that the essential results of the unified analysis are unchanged by the addition of these duct effects.

DIPOLE IN INFINITE DUCT

In order to develop a formula for the far-field pressure due to a dipole pulse, we first consider the sound field due to a dipole in an infinite duct. In this paper, only heuristic arguments will be presented. The analysis of reference 4 is being followed here, so that the reader who is interested in analytical details may refer to that work.

Consider the infinite duct which is formed by an imaginary extension of the finite duct as shown in figure 3. The planes z_1 and z_2 denote ends of a duct section, and a single dipole is located at $\theta=0$, $r=r_s$ in the source plane z_s . In axisymmetric ducts, it is always convenient to develop the pressure spectrum in a Fourier series

$$p(r, \theta, z; \omega) = \frac{\rho c}{k} \sum_{m=-\infty}^{\infty} e^{im\theta} p_m(r, z, \omega) \quad (3)$$

and to treat each term in this series independently. Without bothering with a lengthy derivation, it is assumed that the sound field at z_1 is made up of circular duct modes traveling away from the source to the right. These modes, which are denoted by $\psi_{m\mu}(r)$ are known to be proportioned to Bessel functions of the first kind of order m . The subscript m is the radial mode index. At z_1 then, the m^{th} pressure harmonic in equation (3) is

$$p_m(r, z_s; \omega) = \frac{k^3 |\vec{F}(\omega)|}{\rho_c} \sum_{m=1}^{\infty} Q_{m\mu}^{+1} \psi_{m\mu}(kr) \quad (4)$$

In equation (4) $Q_{m\mu}^{+1}$ is the source mode amplitudes at z_1 and the positive superscript is used to denote the positive direction of travel. In practice the infinite series in equation (4) is always truncated so that it may be replaced by the matrix formula

$$p_m(r, z_1; \omega) = \left[\psi_{m\mu}(kr) \right] \left\{ Q_{m\mu}^{+1} \right\} \frac{k^3 |\vec{F}(\omega)|}{\rho_c} \quad (5)$$

The inner product of the row matrix and the column matrix in equation (5) replaces the summation in equation (4).

Equation (5) gives the pressure at z_1 due to a dipole if the duct is infinite. After considering some elementary transmission and reflection concepts, we will show that this information can be used to predict the pressure in a finite duct.

TRANSMISSION AND REFLECTION

It is known that the propagation of waves in uniform ducts is described by their axial wave number $k_{zm\mu}$ through the function $e^{ik_{zm\mu}z}$; that is, the amplitude of a mode at z_2 is related to the amplitude of that same mode (in a uniform duct) at z_1 by the equation

$$A_{m\mu}^{+2} = e^{ik_{zm\mu}(z_1 - z_2)} A_{m\mu}^{+1} \quad (6)$$

In general $k_{zm\mu}$ is a complex number so that equation (6) describes the phase shift and the attenuation of the wave between planes z_1 and z_2 . The attenuation, an exponential decay of the amplitude, is depicted graphically in figure 4 which shows the relative amplitudes of the waves at planes 1 and 2. All transmission effects in ducts, both uniform and nonuniform, are described by the matrix equation

$$\begin{Bmatrix} A_{m\mu}^{+2} \end{Bmatrix} = \begin{bmatrix} T_{m\mu\nu}^{+2+1} \end{bmatrix} \begin{Bmatrix} A_{m\nu}^{+1} \end{Bmatrix} \quad (7)$$

In the simple case of a uniform duct, the transmission matrix $\begin{bmatrix} T_{m\mu\nu}^{+2+1} \end{bmatrix}$ is a diagonal matrix whose non-zero elements are given by the exponential factors in equation (6). In nonuniform ducts, a more general expression

must be derived for the transmission matrix, however its basic form will be as shown here. Of course, a similar equation is possible for waves traveling from right to left in the duct.

When acoustic modes travel to the end of a finite duct, there is partial radiation of the sound to the far-field and partial reflection of the sound as waves traveling backward from the duct opening. This reflection process is shown graphically in figure 5 where a positive moving wave A_{mv}^{+2} is reflected as a set of negatively moving waves $A_{m\mu}^{-2}$. The analysis of this radiation and reflection process is complex and cannot be treated here. A general solution for radiation from a flanged annular duct is given in reference 5 and the problem of radiation from an unflanged duct is treated in reference 6. Here, we only note that the solution to the radiation problem also yields the reflection equation which must take the form

$$\begin{Bmatrix} A_{m\mu}^{-2} \end{Bmatrix} = \begin{bmatrix} R_{m\mu\nu}^{-2+2} \end{bmatrix} \begin{Bmatrix} A_{mv}^{+2} \end{Bmatrix} \quad (8)$$

Considering the simultaneous effects of sources, transmission, and reflections gives the following equation

$$\begin{bmatrix} I & 0 & -R & 0 \\ -T & I & 0 & 0 \\ 0 & 0 & I-T & \\ 0-R & 0 & I \end{bmatrix} \begin{Bmatrix} A_{mv}^{+1} \\ A_{mv}^{+2} \\ A_{mv}^{-1} \\ A_{mv}^{-2} \end{Bmatrix} = \frac{k^3 |\vec{F}(\omega)|}{pc} \begin{Bmatrix} Q_{mv}^{+1} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (9)$$

Equation (9) will be denoted by

$$\begin{bmatrix} W \end{bmatrix} \begin{Bmatrix} A \end{Bmatrix} = \frac{k^3 |\vec{F}(\omega)|}{\rho_c} \begin{Bmatrix} Q \end{Bmatrix} \quad (10)$$

therefore, the solution for the mode amplitudes is

$$\begin{Bmatrix} A \end{Bmatrix} = \frac{k^3 |\vec{F}(\omega)|}{\rho_c} \begin{bmatrix} W \end{bmatrix}^{-1} \begin{Bmatrix} Q \end{Bmatrix} \quad (11)$$

RADIATION FROM DUCTED DIPOLES

The solution for the sound field inside of the duct has been developed in terms of the mode amplitudes at the duct ends. These mode amplitudes also give the radiated acoustic field as shown in reference 5 and 6. Here, each mode which strikes the duct end is assumed to have a known radiation directivity function $D_{m\mu}(\phi_2)$ which has been determined from the solution to the radiation problem. Thus, the far-field pressure at the observer point 0 due to a single mode at the end of the duct is

$$p_m(r, \phi; \omega) = \frac{\rho c^2}{\omega} \frac{e^{ikr_2}}{kr_2} D_{m\mu}^{+2}(\phi_2) A_{m\mu}^{+2} \quad (12)$$

and the pressure due to all modes at the end of the duct can be found, by superposition, to be

$$p(r_2, \theta, \phi, \omega) = \frac{\rho c^2}{\omega} \frac{e^{ikr_2}}{kr_2} \sum_{m=-\infty}^{\infty} [D_m(\phi_2)] \{A_m\} e^{im\theta} \quad (13)$$

Substituting the solution for the mode amplitudes, equation (11) into equation (13) gives

$$p(r, \theta, \phi, \omega) = kr_2 \frac{|\vec{F}(\omega)| e^{ikr_2}}{r_2^2} \sum_{m=-\infty}^{\infty} [\hat{D}_m(\phi_2)] \{Q_m\} e^{im\theta} \quad (14)$$

where

$$\left[\hat{D}_m(\phi_2) \right] = \left[D_m(\phi_2) \right] \left[W_m \right]^{-1} \quad (15)$$

Equation (14), the far-field pressure spectrum due to a dipole in a duct, replaces equation (1) when there is an array of dipoles in a duct with random amplitude and pulsing times. Although this expression is more complex than the free-field formula, Hanson's analysis can still be carried through with no essential changes.

RESULTS OF UNIFIED ANALYSIS

Essentially, all of Hanson's results are preserved except that the expressions for the directivity effects are more complex. In the expression for the far-field broadband intensity, Hanson's directivity function is

$$Y(\phi) = \sin^2 \beta \cos^2 \phi + (1/2) \cos^2 \beta \sin^2 \phi \quad (16)$$

provided there are more than two stator vanes. The angle β in equation (16) is the mean inflow angle to the stator which determines the orientation of the dipole with respect to the plane of the stator vanes. The analysis with duct effects shows that the broadband directivity for the ducted stator is

$$Y(\Phi; \omega) = (4\pi)^2 \sum_{m=-\infty}^{\infty} [\hat{D}_{mv}(\Phi)] \left[\{Q_{m\mu}\} [Q_{mv}^*] \right] \left\{ \hat{D}_{m\mu}^*(\Phi) \right\} \quad (17)$$

that is, the broadband directivity is axisymmetric but depends on the

directivities of the cylindrical mode orders m and radial mode orders μ or ν as well as the modal source amplitudes $Q_{m\mu}$ which the unit dipole would generate in an infinite duct. The square matrix in equation (17) is the modal cross spectrum for a unit dipole in an infinite duct. Attempts have been made by Bolleter and Crocker (ref. 7) and by Harel and Perulli (ref. 8) to measure this cross spectrum for general sources. These measurement attempts have been partially successful. The present analysis therefore provides a link between the results of nonreflecting duct measurements, which can be made in the laboratory, and the prediction of far-field radiated noise with a finite length duct.

Hanson also found a fairly simple expression for the directivity of the harmonic noise from a free-field dipole array. This was

$$Z(\phi, \theta; \omega_n) = \left\{ \sum_{s=1}^{\nu} \chi_{1s}(\beta, \theta, \phi, \nu) \right\}^2 + \left\{ \sum_{s=1}^{\nu} \chi_{2s}(\beta, \theta, \phi, \nu) \right\}^2 \quad (18)$$

where

$$\begin{Bmatrix} \chi_1 \\ \chi_2 \end{Bmatrix} = \left[\sin \beta \cos \phi - \cos \beta \sin \phi \sin \left(\theta - \frac{2\pi s}{\nu} \right) \right] \begin{Bmatrix} \sin \psi_s \\ \cos \psi_s \end{Bmatrix}, \quad (19)$$

$$\psi_s = nBM \sin \phi \cos \left(\theta - \frac{2\pi s}{V} \right) - 2\pi nB \frac{s}{V}, \quad (20)$$

and M is the rotational Mach number at the dipole radius. Note that the directivity of the harmonic noise is not axisymmetric. The directivity of the harmonic noise from the ducted dipole array depends on the properties of the Tyler-Sofrin spinning modes. It is shown in the appendix that this expression is

$$Z(\Phi, \theta; \omega) = (4\pi)^2 \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} [\hat{D}_{m\nu}(\Phi)] \left[\{Q_{m\mu}\} \{Q_{m'\nu}^*\} \right] \{\hat{D}_{m'\mu}^*(\Phi)\} e^{i(k'-k)V\theta} \quad (21)$$

$$\text{where } m = nB - kV, \quad (22a)$$

$$\text{and } m' = nB - k'V \quad (22b)$$

The harmonic directivity depends on only the selected mode orders $m=nB-kV$ as predicted by Tyler and Sofrin; however it is not axisymmetric because of the factor $\exp [i(k'-k)V\theta]$ which corresponds to Hanson's result for the free-field array. If the intensity is averaged over the azimuth angle θ ,

the double sum in the directivity expression (21) is reduced to a single sum.

CONCLUDING REMARKS

A unified analysis has been formally carried out for the radiated noise field of a circular array of pulsing dipoles in a finite length duct. The result shows that there is no qualitative difference in the predicted broadband and harmonic spectra between a free-field array and a ducted array; however, the magnitude and directivity of the radiated field is changed by the presence of the duct and a quantitative computational approach has been outlined which will predict the duct effects. The analysis has also shown that measured modal cross spectra from sources in a nonreflecting duct may be used to predict the noise radiated from those same sources in a finite duct. The analysis also shows again that the harmonic noise is made up of the familiar Tyler-Sofrin "spinning modes" of order $m=nB-ky$.

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APPENDIX

Statistical Analysis

In the text a relation is presented (eqs. (14) and (15)) for the far-field pressure spectrum (complex pressure) due to a dipole pulse at position $\theta_s = 0$ in an acoustically lined duct (fig. 6). If the pulse occurs at time $t = 0$, the far-field pressure is

$$p_s(r, \theta, \phi, \omega) = (kr) \frac{|\vec{F}(\omega)| e^{ikr}}{r^2} \sum_{m=-\infty}^{\infty} L D_m(\theta) [W_m]^{-1} \{Q_m\} e^{im(\theta - \theta_s)} \quad (A-1)$$

However, if the pulse occurs at an arbitrary time $t = t_0$ the pressure transform is

$$e^{i\omega t_0} p_s(r, \theta, \phi, \omega)$$

Now consider the complex pressure $p_{sn}(r, \theta, \phi, \omega)$ produced in the far-field due to the n^{th} pulse on the s^{th} vane. The amplitude of this pulse will be written as $a_{sn} |\vec{F}(\omega)|$ where a_{sn} is a random variable with unit mean, that is

$$E\{a_{sn}\} = 1 \quad (A-2)$$

for all values of s and n . Therefore, as discussed by Hanson (ref. 3), $|\vec{F}(\omega)|$ is the mean pulse shape and the random variable a_{sn} accounts for the pulse amplitude modulations. The reader is cautioned that Hanson uses the subscript m in reference 3 as the vane number, whereas, in the present paper s denotes vane number and m denotes the circumferential

wave number in the duct analysis. The arrival time t_{sn} for the n^{th} pulse on the s^{th} vane, as developed by Hanson, is

$$t_{sn} = \left(\frac{s}{V} - \frac{q_s}{B} + \frac{n}{B} + b_{sn} \right) T \quad (\text{A-3})$$

where V is the number of vanes, B is the number of blades, $T = \frac{2\pi}{\Omega}$ is the period of rotation, Ω being the rotor angular velocity. The parameter b_{sn} is a random variable which accounts for the jitter or pulse position modulations. Here it is assumed that

$$E\{b_{sn}\} = 0 \quad (\text{A-4})$$

for all values of s and n .

Hence, the mean arrival time for the n^{th} pulse on the s^{th} vane is

$$E\{t_{sn}\} = \left(\frac{s}{V} - \frac{q_s}{B} + \frac{n}{B} \right) T \quad (\text{A-5})$$

The parameter q_s in equations (A-3) and A-5) is defined by

$$q_s = \left[\frac{sB}{V} \right] \quad (\text{A-6})$$

where the bracket $[\]$ indicates that only the integer part of the quotient is retained. Substituting equation (A-3) into equation (A-1) gives

$$p_{sn}(r, \theta, \phi, \omega) = a_{sn} e^{i\omega T b_{sn}} \left[\frac{(kr) |\vec{F}(\omega)|}{r^2} \right. \\ \left. \cdot e^{i \left\{ kr + \left(\frac{s}{r} - \frac{\theta_s}{B} + \frac{\eta}{B} \right) \omega T \right\}} \right] D(\theta - \theta_s, \phi) \quad (A-7)$$

$$D(\theta, \phi) = \sum_{m=-\infty}^{\infty} [D_m(\theta)] [W_m]^{-1} \{Q_m\} e^{im\theta} \quad (A-8)$$

An observer located at a fixed position in the far field will experience a pressure time-history produced by a sequence of pulses as shown in figure 2. Thus, a particular sequence of realizations for the random variables a_{sn} and b_{sn} determine a single realization of the random far-field pressure. Equation (A-7) gives the Fourier transform of a typical far-field pulse $p_{sn}(r, \theta, \phi, \omega)$ which itself is a random variable. In the following it will be shown that the power spectral density $G(r, \theta, \phi, \omega)$ of the far-field pressure can be determined in terms of statistical averages performed on the random variables $p_{sn}(r, \theta, \phi, \omega)$. An expression is then determined for the acoustical intensity per unit bandwidth.

First rewrite equation (A-7) as

$$\begin{aligned} p_{sn}(r, \theta, \phi, \omega) &= X_{sn} + i Y_{sn} \\ &= a_{sn} |D_s| e^{i\omega(C_{sn} + b_{sn}T)} \end{aligned} \quad (A-9)$$

in which

$$\left. \begin{aligned} D_s &= \frac{(r r_s) |\vec{F}(\omega)|}{r^2} D(\theta - \theta_s, \phi) \\ A_n &= \frac{nT}{E} \\ B_s &= \frac{Kr}{\omega} + \left(\frac{s}{V} - \frac{\theta_s}{B} \right) T + \frac{1}{\omega} \text{ARG}(D_s) \\ C_{sn} &= A_n + B_s \end{aligned} \right\} \quad (A-10)$$

The power spectral density for the far-field pressure $p(r, \theta, \phi, t)$, consisting of an infinite train of pulses, is given by (ref. 3)

$$G(r, \theta, \phi, \omega) = \lim_{N \rightarrow \infty} \frac{B}{(2N+1)T} \bar{E}_N(r, \theta, \phi, \omega) \quad (A-11)$$

where:

$$\bar{E}_N(r, \theta, \phi, \omega) = E \left\{ 2\pi \left| \sum_{n=-N}^N \sum_{s=1}^V X_{sn} + i Y_{sn} \right|^2 \right\}$$

which can immediately be rewritten as follows:

$$\begin{aligned} \bar{E}_N(r, \theta, \phi, \omega) = & 2\pi E \left\{ \left(\sum_{n=-N}^N \sum_{s=1}^V X_{sn} \right)^2 \right\} + \\ & + 2\pi E \left\{ \left(\sum_{n=-N}^N \sum_{s=1}^V Y_{sn} \right)^2 \right\} \end{aligned} \quad (A-12)$$

Herein it will be assumed that the $2V(2N + 1)$ random variables a_{sn} and b_{sn} are independent. It then follows that the $2V(2N + 1)$ random variables X_{sn} and Y_{sn} are uncorrelated. An additional consequence of the assumption of independence is that

$$f_{sn}(a_{sn}, b_{sn}) = f_{sn}(a_{sn}) g_{sn}(b_{sn}) \quad (A-13)$$

where $f_{sn}(a_{sn}, b_{sn})$ is the set of $2V(2N + 1)$ joint probability density functions for the a_{sn} 's and b_{sn} 's and $f_{sn}(a_{sn})$ and $g_{sn}(b_{sn})$ are the respective individual densities. It is further assumed that these densities are independent of s and n (vane number and pulse number). Therefore, the a_{sn} 's and b_{sn} 's are associated with two classes of random variables a and b , respectively. The required probability density functions will be assumed as

$$\left. \begin{aligned} f_{sn}(a_{sn}) &= f(a) \\ g_{sn}(b_{sn}) &= \frac{1}{\sigma_b \sqrt{2\pi}} e^{-b^2/2\sigma_b^2} \end{aligned} \right\} \quad (A-14)$$

for all n and s . It is not necessary to specify a density for the random variable a . However, from equation (A-2), $E\{a\} = 1$ and its standard deviation will be noted at ∇_a . A Gaussian density is assumed for the random variable b . As indicated by equations (A-4) and (A-14), $E\{b\} = 0$ and its standard deviation is ∇_b .

Continuing the development from equation (A-12) the following well-known result will be useful:

$$E\left\{\left(\sum_{n=-N}^N \sum_{s=1}^V X_{sn}\right)^2\right\} = \text{VAR}\left\{\sum_{n=-N}^N \sum_{s=1}^V X_{sn}\right\} + \left(E\left\{\sum_{n=-N}^N \sum_{s=1}^V X_{sn}\right\}\right)^2 \quad (\text{A-15})$$

where $\text{VAR}\{ \}$ denotes the variance of the expression within the brackets. Since the X_{sn} 's are uncorrelated

$$\text{VAR}\left\{\sum_{n=-N}^N \sum_{s=1}^V X_{sn}\right\} = \sum_{n=-N}^N \sum_{s=1}^V \text{VAR}\{X_{sn}\} \quad (\text{A-16})$$

where:

$$\text{VAR}\{X_{sn}\} = E\{X_{sn}^2\} - \left(E\{X_{sn}\}\right)^2 \quad (\text{A-17})$$

Relations similar to equations (A-15) - (A-17) can also be written for the Y_{sn} variables.

Substituting these results (eqs. (A-15) to (A-17)) into equation (A-12) gives:

$$\begin{aligned} \bar{E}_N(r, \theta, \phi, \omega) = & 2\pi \left[\left(E\left\{ \sum_{n=-N}^N \sum_{s=1}^V X_{sn} \right\} \right)^2 + \right. \\ & + \left(E\left\{ \sum_{n=-N}^N \sum_{s=1}^V Y_{sn} \right\} \right)^2 + \sum_{n=-N}^N \sum_{s=1}^V E\{X_{sn}^2\} + \\ & + \sum_{n=-N}^N \sum_{s=1}^V E\{Y_{sn}^2\} - \sum_{n=-N}^N \sum_{s=1}^V \left(E\{X_{sn}\} \right)^2 + \\ & \left. - \sum_{n=-N}^N \sum_{s=1}^V \left(E\{Y_{sn}\} \right)^2 \right] \quad (\text{A-18}) \end{aligned}$$

With the use of equations (A-13) and (A-14) the various expected values appearing in equations (A-18) can be easily calculated (ref. 3) and are

$$\left. \begin{aligned}
 E\{X_{sn}\} &= |D_s| \cos(\omega C_{sn}) e^{-\frac{1}{2} \omega^2 T^2 \sigma_b^2} \\
 E\{Y_{sn}\} &= |D_s| \sin(\omega C_{sn}) e^{-\frac{1}{2} \omega^2 T^2 \sigma_b^2} \\
 E\{X_{sn}^2 + Y_{sn}^2\} &= |D_s|^2 E\{a^2\} \\
 (E\{X_{sn}\})^2 + (E\{Y_{sn}\})^2 &= |D_s|^2 e^{-\omega^2 T^2 \sigma_b^2}
 \end{aligned} \right\} \quad (A-19)$$

Substituting equations (A-19) into equation (A-18) gives:

$$\begin{aligned}
 \bar{E}_N(r, \theta, \phi, \omega) &= 2\pi \left[(\sigma_a^2 + 1) - e^{-\omega^2 T^2 \sigma_b^2} \right] \sum_{n=-N}^N \sum_{s=1}^V |D_s|^2 + \\
 &+ 2\pi e^{-\omega^2 T^2 \sigma_b^2} \left[\left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \cos(\omega C_{sn}) \right\}^2 + \right. \\
 &\left. + \left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \sin(\omega C_{sn}) \right\}^2 \right] \quad (A-20)
 \end{aligned}$$

In equation (A-20) use has been made of the relation (since the random variable a has unit mean)

$$\sigma_a^2 = E\{a^2\} - 1 \quad (\text{A-21})$$

The power spectral density $G(r, \theta, \phi, \omega)$ (eq. (A-11)) will be determined by separately evaluating the two terms in equation (A-20). Thus, let

$$G(r, \theta, \phi, \omega) = G_1(r, \theta, \phi, \omega) + G_2(r, \theta, \phi, \omega) \quad (\text{A-22})$$

in which:

$$\begin{aligned} G_1(r, \theta, \phi, \omega) = & \lim_{N \rightarrow \infty} \frac{B}{(2N+1)T} \left\{ 2\pi \left[\left(\frac{c}{T_a} + 1 \right) - e^{-\frac{\omega^2}{T_a} T^2 \sigma_b^2} \right] \cdot \right. \\ & \left. \cdot \sum_{n=-N}^N \sum_{s=1}^V |D_s|^2 \right\} \end{aligned} \quad (\text{A-23})$$

$$G_2(r, \theta, \phi, \omega) = \lim_{N \rightarrow \infty} \frac{B}{(2N+1)T} \left\{ 2\pi e^{-\omega^2 T^2 \sigma_b^2} \cdot \right. \\ \left. \cdot \left[\left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \cos(\omega C_{sn}) \right\}^2 + \left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \sin(\omega C_{sn}) \right\}^2 \right] \right\} \quad (A-24)$$

It will be shown that $G_1(r, \theta, \phi, \omega)$ is a broadband component whereas $G_2(r, \theta, \phi, \omega)$ is a harmonic (discrete) component.

The double sum in equation (A-23) can be simplified to

$$\sum_{n=-N}^N \sum_{s=1}^V |D_s|^2 = V(2N+1) \frac{(KT)^2}{r^4} \frac{F(\omega)^2}{\sum_{m=-\infty}^{\infty} L D_m(\theta)} \cdot \\ \cdot [Q_{mm}] \{ \hat{D}_m^*(\theta) \} \quad (A-25)$$

where the asterick * denotes complex conjugate and

$$\left. \begin{aligned} L \hat{D}_m(\theta) &= L D_m(\theta) [W_m]^{-1} \\ [Q_{mm}] &= [Q_m] [Q_m^*] \end{aligned} \right\} \quad (A-26)$$

and the following relation has been applied:

$$\sum_{s=1}^V e^{i(m-m')(\theta-\theta_s)} = \begin{cases} V & m=m' \\ 0 & m \neq m' \end{cases} \quad (A-27)$$

(for $V \neq 2$)
Now by substituting equation (A-25) into equation (A-23) and carrying out the limit as $N \rightarrow \infty$ gives

$$G_1(r, \theta, \phi, \omega) = \left(\frac{2\pi EV}{T} \right) \left[\frac{(Kr)^2 |\vec{F}(\omega)|^2}{r^4} \right] \cdot \left[(\sigma_a^2 + 1) - e^{-\omega^2 T^2 \sigma_b^2} \right] \sum_{m=-\infty}^{\infty} [L_m(\phi)] [Q_{mm}] \{ \hat{D}_m^*(\phi) \} \quad (A-28)$$

which is the broadband component of the far-field spectral density.

In order to determine the discrete component $G_2(r, \theta, \phi, \omega)$ note that (ref. 3)

$$\begin{aligned} & \left[\left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \cos(\omega C_{sn}) \right\}^2 + \left\{ \sum_{n=-N}^N \sum_{s=1}^V |D_s| \sin(\omega C_{sn}) \right\}^2 \right] = \\ & = \left[\left\{ \sum_{n=-N}^N \sin \frac{n\omega T}{B} \right\}^2 + \left\{ \sum_{n=-N}^N \cos \frac{n\omega T}{B} \right\}^2 \right] \cdot \\ & \cdot \frac{(Kr)^2 |\vec{F}(\omega)|^2}{r^4} \left| \sum_{s=1}^V \sum_{m=-\infty}^{\infty} [L_m(\phi)] \{Q_m\} e^{im(\theta-\theta_s)} e^{i\omega T(\frac{s}{V} - \frac{\theta_s}{B})} \right|^2 \quad (A-29) \end{aligned}$$

Now by using the results that (ref. 3)

$$\left. \begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left\{ \sum_{n=-N}^N \cos\left(\frac{n\omega T}{\omega}\right) \right\}^2 &= \frac{2\pi B}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - nB\Omega) \\ \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left\{ \sum_{n=-N}^N \sin\left(\frac{n\omega T}{\omega}\right) \right\}^2 &= 0 \end{aligned} \right\} \quad (A-30)$$

$$\theta_s = \frac{2\pi(s-1)}{V} \quad (A-31)$$

$$T = \frac{2\pi}{\Omega} \quad (A-32)$$

$$\sum_{s=1}^V e^{i \frac{2\pi s}{V} (nB-m)} = \begin{cases} V & \text{FOR } nB-m = KV \\ 0 & \text{FOR } nB-m \neq KV \end{cases} \quad (A-33)$$

$$\omega_n = nB\Omega \quad (A-34)$$

the following result is obtained:

$$G_2(r, \theta, \phi, \omega) = \left(\frac{2\pi E V}{T} \right)^2 e^{-\omega^2 T^2 V_b^2} \frac{(kr)^2 |\vec{F}(\omega)|^2}{r^4} \cdot \quad (A-35)$$

$$\cdot \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} \delta(\omega - \omega_n) [\hat{D}_m(\phi)] [\hat{Q}_{mm'}] \{ \hat{D}_{m'}^*(\phi) \} e^{i(k-k')V\theta}$$

in which

$$m = nB - kV \quad (A-36)$$

$$m' = nB - k'V$$

and

$$[Q_{mm'}] = \left[\{Q_m\} L Q_{m'}^* \right] \quad (A-37)$$

It should be observed that $G_2(r, \theta, \phi, \omega)$ is a discrete spectrum

The radiated intensity per unit bandwidth

$$\frac{dI}{d\omega} = \frac{2}{\rho c} G(r, \theta, \phi, \omega) \quad (A-38)$$

can now be expressed as

$$\begin{aligned} \frac{dI}{d\omega} = \frac{B\Omega\omega^2 |\vec{F}(\omega)|^2}{8\pi^2 \rho c^3 r^2} & \left\{ [(\nabla_a^2 + 1) - e^{-\omega^2 \tau^2 \nabla_b^2}] V Y(\theta, \phi, \omega) + \right. \\ & \left. + B\Omega \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_n) e^{-\omega^2 \tau^2 \nabla_b^2} Z_n(\theta, \phi, \omega) \right\} \end{aligned} \quad (A-39)$$

where

$$Y(\theta, \phi, \omega) = (4\pi)^2 \sum_{m=-\infty}^{\infty} L_{\hat{D}_{mm}(\phi)} [Q_{mm}] \{ \hat{D}_m^*(\phi) \} \quad (A-40)$$

$$\left. \begin{aligned} Z_n(\theta, \phi, \omega) &= (4\pi V)^2 \sum_{k=-\infty}^{\infty} \sum_{k'=-\infty}^{\infty} L_{\hat{D}_m(\phi)} [Q_{mm'}] \{ \hat{D}_{m'}^*(\phi) \} e^{i(k-k')\theta V} \\ m &= nB - kV \\ m' &= nB - k'V \end{aligned} \right\} (A-41)$$

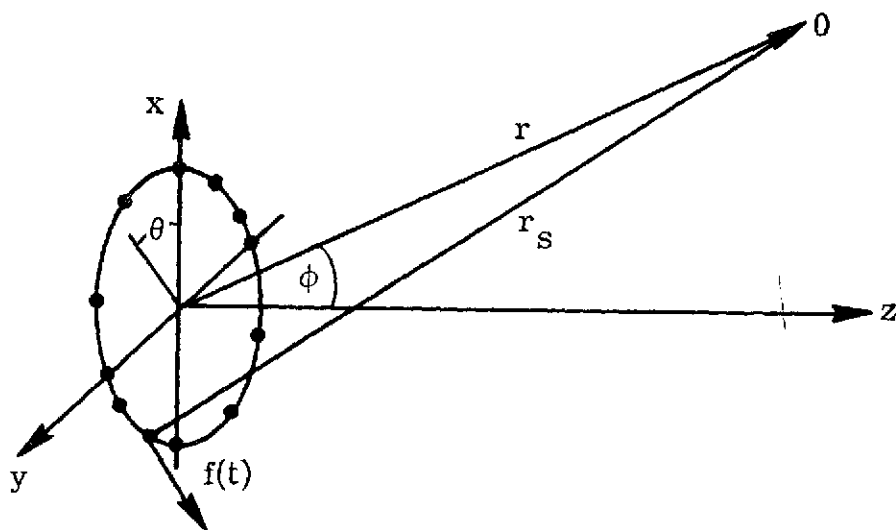


Figure 1.- Circular dipole array.

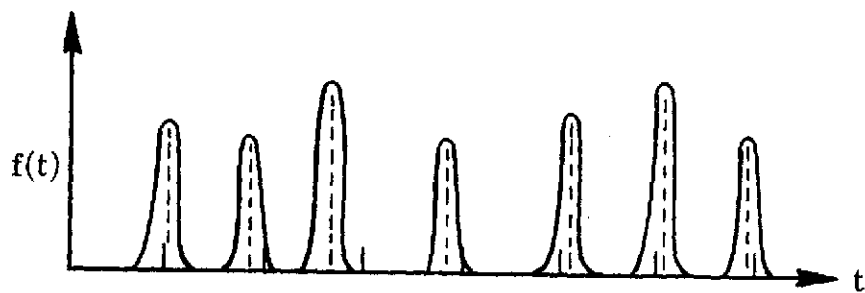


Figure 2.- Dipole pulse amplitude and position modulation.

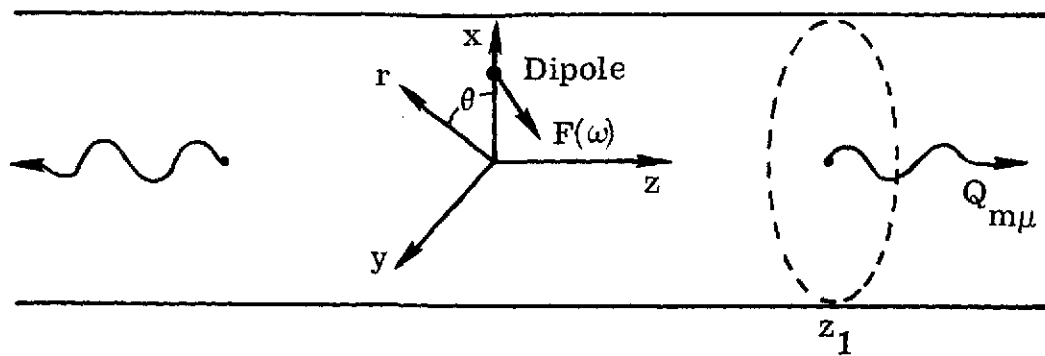


Figure 3.- Dipole in infinite duct.

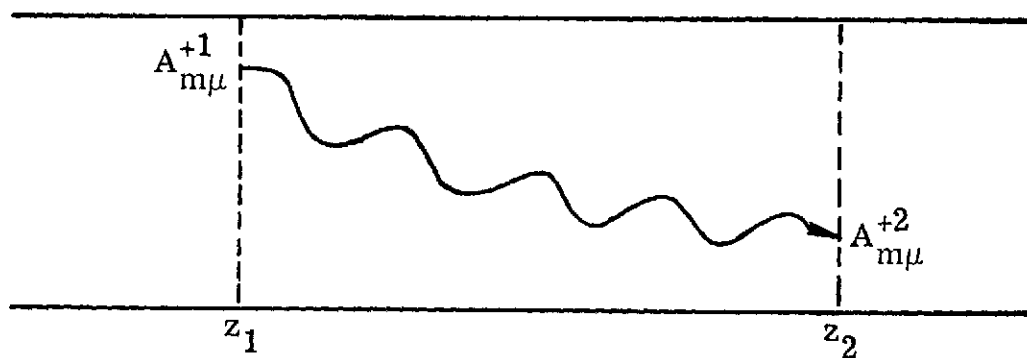


Figure 4.- Forward transmission of mode.

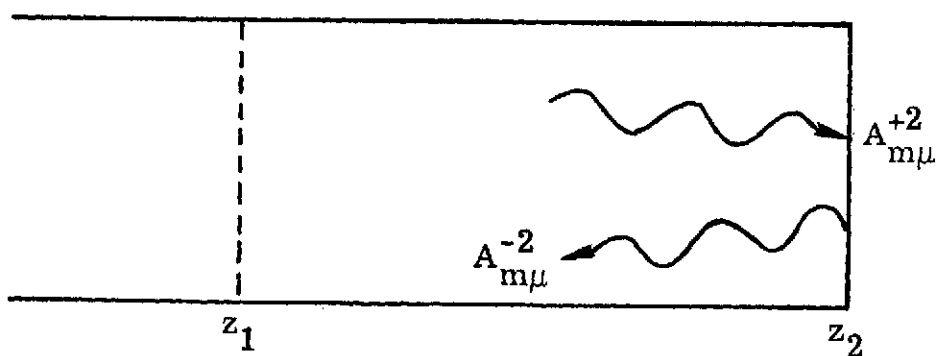


Figure 5.- Reflection from duct end.

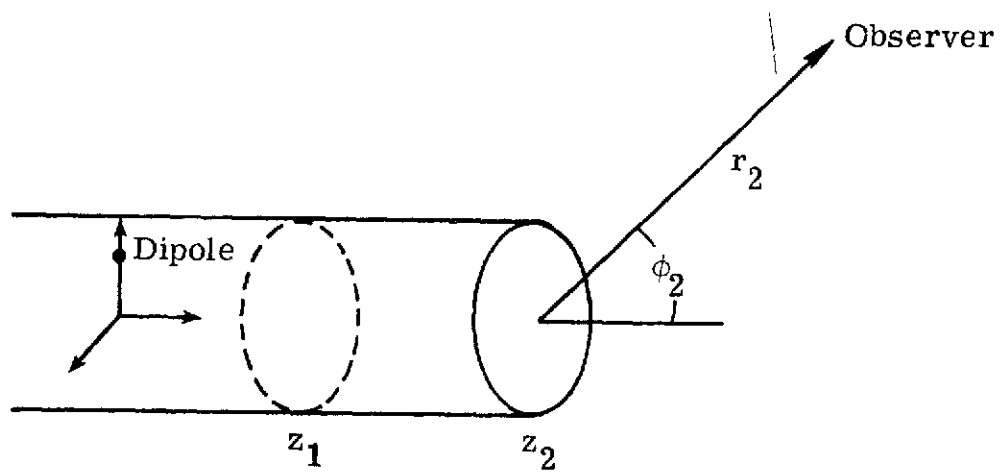


Figure 6.- Radiation from dipole in duct.