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A MODEL FOR JET-NOISE ANALYSIS USING PRESSURE-GRADIENT CORRELATIONS ON AN IMAGINARY CONE

by Thomas D. Norum

Langley Research Center

Hampton, Va. 23665

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16. Abstract

The technique for determining the near and far acoustic field of a jet through measurements of pressure-gradient correlations on an imaginary conical surface surrounding the jet is discussed. The necessary analytical developments are presented, and their feasibility is checked by using a point source as the sound generator. The distribution of the apparent sources on the cone, equivalent to the point source, is determined in terms of the pressure-gradient correlations.

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A MODEL FOR JET-NOISE ANALYSIS USING PRESSURE-GRADIENT CORRELATIONS ON AN IMAGINARY CONE

By Thomas D. Norum
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SUMMARY

To determine the energy flux and far-field intensity of a jet through measurements of pressure-gradient correlations on a conical surface surrounding the jet, the appropriate Green function is derived. The feasibility of this representation is checked by using a point source as the noise generator. Excellent agreement is obtained between exact results for the intensity normal to the cone and results from the Green function technique, which requires only a small amount of computing time. The representation of the far-field intensity at low frequency in terms of apparent sources on the cone shows the apparent sources to be spread almost symmetrically about the actual location of the point source as projected on the conical surface. The interpretation of this representation at higher frequencies is less clear because of the presence of apparent sinks in the source distribution.

INTRODUCTION

A method for determining the flux of acoustic energy from a jet by measurement of pressure-gradient correlations on an imaginary surface close to the jet is described in references 1 and 2. In this method the velocity field surrounding the jet is measured to establish the region where the velocity fluctuations satisfy the homogeneous wave equation. Once this acoustic region is determined, the imaginary surface over which correlations are to be measured is chosen to lie completely within this region, as close to the jet boundary as possible. At this inner limit of the acoustic field the pressure is entirely acoustical, and the measured pressure-gradient correlations can be related to the near-field and far-field acoustic energy through the wave equation. Hence a good estimate of the regions where noise leaves the jet and the contributions to the far field from these regions can be obtained from acoustic measurements alone.

A plane surface near the jet was used for measurements of a subsonic jet in reference 2. However, since the inner limits of the acoustic field of a circular jet form a conical surface enclosing the jet (fig. 1), the natural choice of a surface to give the most complete description of the radiation is a cone.

The purpose of this report is to develop the mathematical model needed to determine the near and far acoustic fields of a jet from pressure-gradient correlations on a conical surface enclosing the jet. Although the use of this model is aimed at jet-noise experiments, its development is applicable to any source distribution enclosed by an imaginary surface. Hence a general formulation for the acoustical characteristics of the source in terms of the near-field correlation measurements is presented along with the necessary analytical developments for use with a conical surface. The intensity flux through the surface determined from this formulation is compared with the exact results for a known point source. The apparent source distribution on the surface (introduced in ref. 2) due to the known source is also obtained. These results validate the mathematical model and yield information concerning the interpretation of the apparent source distribution.

SYMBOLS

A_ν	defined in equation (54)
b_s	frequency parameter, $k_s r_s$
$B_{\nu m}$	Green function constants (eq. (37))
c	ambient speed of sound
$\vec{e}_i$	unit vector in ith direction
g	time-dependent Green function
G	Green function in frequency domain
$h_\nu^{(2)}$	spherical Bessel function of third kind of order ν
H	defined in equation (47)
I	acoustic intensity
j_ν	spherical Bessel function of first kind of order ν
J_n	nondimensionalized intensity component normal to cone (eq. (43))

k	wave number, ω/c
ℓ,m	integers
$\vec{n}$	normal vector to surface
n	direction normal to surface measured positive into region U
p	pressure
P_{ν}^{m}	associated Legendre function of first kind of degree ν and order m
q_I	apparent source distribution for far-field intensity
Q_{ν}^{m}	associated Legendre function of second kind of degree ν and order m
$\vec{r}$	position vector
r	radial distance from origin
R	distance; also separation function in radial direction
Re	real part of
S	surface
t	time
T	averaging time
U	sourceless region
V	velocity
w	spherical coordinate, $\cos\theta$
$w_0 = \cos\theta_0$	
W	separation function in w-direction

y_ν	spherical Bessel function of second kind of order ν
α,β	degrees of Legendre function; also nondimensional distance
δ	Dirac delta function
ϵ_m	1 if $m = 0$, 2 if $m > 0$
θ	zenith angle in spherical coordinates
θ_0	supplement of cone half-angle, see figures 1, 3, and 4
ν	degree of Legendre function
ρ	density
σ_s	strength of point source
τ	separation time
ϕ	spectral-density function
φ	azimuth angle in spherical coordinates
Φ	separation function in φ-direction
ψ	correlation function
ω	angular frequency
∇	gradient operator
∇^2	Laplacian operator

Subscripts:

n	direction normal to surface
s	pertaining to point source

0, 1, 2 distinguish variables over which integration is performed

Superscripts:

' Green function source coordinate

* complex conjugate

The notation $\langle\ \rangle$ denotes mean value. Arrows over symbols denote vector quantities.

REVIEW OF GENERAL THEORY

Since an expression is desired for the acoustical radiation from a jet in terms of near-field measurements on an imaginary surface near the jet, it is instructive to present the general formulation for the acoustical field in a sourceless region of space. Specific cases of this formulation have been given for the regions outside a cylinder (ref. 1) and opposite a plane surface (ref. 2).

Let U denote this sourceless region which has as its inner boundary the surface S separating it from the region that contains acoustic sources (fig. 2). At any point $\vec{r}$ of U the pressure satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) p\left(\vec{r},t\right) = 0 \tag{1}$$

Upon introducing a Green function in U which satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) g\left(\vec{r},\vec{r}\,',t-t'\right) = -\delta\left(\vec{r}-\vec{r}\,'\right)\ \delta(t-t') \tag{2}$$

and which satisfies an appropriate boundary condition on S, it can be shown by the procedure of reference 3 that

$$p\left(\vec{r},t\right) = \int_{-\infty}^{\infty} dt' \int_S dS_0 \left[p\left(\vec{r}_0,t'\right)\ \frac{\partial g\left(\vec{r},\vec{r}_0,t-t'\right)}{\partial n_0} - g\left(\vec{r},\vec{r}_0,t-t'\right)\ \frac{\partial p\left(\vec{r}_0,t'\right)}{\partial n_0}\right] \tag{3}$$

where n denotes the direction normal to the surface S and is measured positive into the region U.

The boundary condition on g depends on whether measurements of pressure or pressure gradient are to be made. When the latter is chosen, the appropriate condition on g is

$$\frac{\partial g(\vec{r},\vec{r}\,',t-t')}{\partial n} = 0 \tag{4}$$

for all field points $\vec{r}$ located on the surface S.

With $\tau = t - t'$, the pressure field is then

$$p(\vec{r},t) = -\int_{-\infty}^{\infty} d\tau \int_S dS_0\, g(\vec{r},\vec{r}_0,\tau)\, \frac{\partial p(\vec{r}_0,t-\tau)}{\partial n_0} \tag{5}$$

The notation for correlation and spectral density is now introduced. For covariant stationary processes, the cross correlation of quantity A measured at the point $\vec{r}_a$ with quantity B at point $\vec{r}_b$ is denoted by

$$\psi_{A,B}(\vec{r}_a,\vec{r}_b,\tau) = \left\langle A^*(\vec{r}_a,t)\, B(r_b,t+\tau)\right\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\, A^*(\vec{r}_a,t)\, B(\vec{r}_b,t+\tau) \tag{6}$$

The corresponding cross spectral density is

$$\phi_{A,B}(\vec{r}_a,\vec{r}_b,\omega) = \int_{-\infty}^{\infty} d\tau\, \psi_{A,B}(\vec{r}_a,\vec{r}_b,\tau)\, e^{-i\omega\tau} \tag{7}$$

If $\vec{A}$ is a vector quantity, the notation $\vec{\psi}_{\vec{A},B}(\vec{r}_a,\vec{r}_b,\tau)$ is used, where

$$\vec{\psi}_{\vec{A},B}(\vec{r}_a,\vec{r}_b,\tau) \equiv \sum_{i=x,y,z} \psi_{A_i,B}(\vec{r}_a,\vec{r}_b,\tau)\, \vec{e}_i \tag{8}$$

Subscripts x, y, and z denote directions along the Cartesian axes. The notation $\vec{\phi}_{\vec{A},B}(\vec{r}_a,\vec{r}_b,\omega)$ is expressed as in equation (7) in terms of $\vec{\psi}_{\vec{A},B}(\vec{r}_a,\vec{r}_b,\tau)$.

The correlation of pressure with velocity at the point $\vec{r}$ is

$$\vec{\psi}_{\vec{V},p}(\vec{r},\vec{r},\tau) = -\int_{-\infty}^{\infty} dt \int_S dS_0\, g(\vec{r},\vec{r}_0,t)\, \vec{\psi}_{\vec{V},\frac{\partial p}{\partial n}}(\vec{r},\vec{r}_0,\tau-t) \tag{9}$$

Taking the Fourier transform of this result and manipulating the resulting integrals yield

$$\vec{\phi}_{\vec{V},p}\left(\vec{r},\vec{r},\omega\right) = -\int_S dS_0\ \vec{\phi}_{\vec{V},\frac{\partial p}{\partial n}}\left(\vec{r},\vec{r}_0,\omega\right)\ G\left(\vec{r},\vec{r}_0,\omega\right) \tag{10}$$

where G is the Fourier transform of the Green function g, that is,

$$G\left(\vec{r},\vec{r}_0,\omega\right) = \int_{-\infty}^{\infty} d\tau\ e^{-i\omega\tau}\ g\left(\vec{r},\vec{r}_0,\tau\right) \tag{11}$$

From the equation of motion for the acoustic field,

$$\frac{\partial\vec{V}}{\partial t} = -\frac{1}{\rho}\nabla p \tag{12}$$

the following relation between cross spectral densities is obtained:

$$\vec{\phi}_{\vec{V},\frac{\partial p}{\partial n}}\left(\vec{r},\vec{r}_0,\omega\right) = \frac{1}{i\omega\rho}\ \vec{\phi}_{\nabla p,\frac{\partial p}{\partial n}}\left(\vec{r},\vec{r}_0,\omega\right) \tag{13}$$

Hence

$$\vec{\phi}_{\vec{V},p}\left(\vec{r},\vec{r},\omega\right) = \frac{i}{\omega\rho}\int_S dS_0\ \vec{\phi}_{\nabla p,\frac{\partial p}{\partial n}}\left(\vec{r},\vec{r}_0,\omega\right)\ G\left(\vec{r},\vec{r}_0,\omega\right) \tag{14}$$

This relationship gives the spectral distribution of acoustic intensity at the point $\vec{r}$ in terms of the spectrum of the cross correlation of the pressure gradient at $\vec{r}$ with the pressure-gradient component normal to the surface S. Upon choosing $\vec{r}$ to be a point on S and taking the component of equation (14) normal to S, the spectrum of the normal acoustic intensity at $\vec{r}$ becomes

$$\phi_{V_n,p}\left(\vec{r},\vec{r},\omega\right) = \frac{i}{\omega\rho}\int_S dS_0\ \phi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}\left(\vec{r},\vec{r}_0,\omega\right)\ G\left(\vec{r},\vec{r}_0,\omega\right) \tag{15}$$

The spectrum is thus obtained in terms of the spectral distribution of pressure-gradient correlations along the surface S.

The acoustic intensity normal to S is then obtained as

$$I_n(\vec{r}) = \mathrm{Re}\left\{\psi_{V_n,p}(\vec{r},\vec{r},0)\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \phi_{V_n,p}(\vec{r},\vec{r},\omega)\right\} \tag{16}$$

The distribution of radiated acoustic energy may be determined from the autocorrelation of the far-field pressure. For any far-field point $\vec{r}$ in U, use of equation (5) gives this correlation as

$$\psi_{p,p}(\vec{r},\vec{r},\tau) = \int_S dS_1 \int_S dS_2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_1$$

$$\times\, g^*(\vec{r},\vec{r}_1,t_1)\ g(\vec{r},\vec{r}_2,t_2)\ \psi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}(\vec{r}_1,\vec{r}_2,\tau+t_1-t_2) \tag{17}$$

where $\vec{r}_1$ and $\vec{r}_2$ are points on S. Taking the Fourier transform of this relation and again manipulating the resulting integrals give

$$\phi_{p,p}(\vec{r},\vec{r},\omega) = \int_S dS_1 \int_S dS_2\ G^*(\vec{r},\vec{r}_1,\omega)\ G(\vec{r},\vec{r}_2,\omega)\ \phi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}(\vec{r}_1,\vec{r}_2,\omega) \tag{18}$$

Hence the spectral distribution of the far-field intensity is also expressed in terms of the pressure-gradient correlations on S. The total radiated acoustic intensity at the point $\vec{r}$ is thus

$$I(\vec{r}) = \frac{1}{\rho c}\,\psi_{p,p}(\vec{r},\vec{r},0) = \frac{1}{2\pi\rho c}\int_{-\infty}^{\infty} d\omega\ \phi_{p,p}(\vec{r},\vec{r},\omega) \tag{19}$$

The far-field intensity is now represented in terms of an apparent source distribution on the surface. This distribution determines the contribution to the far field due to apparent sources located on the incremental area dS of the surface and is defined by

$$I(\vec{r}) = \int_S dS_1\ q_I(\vec{r}_1;\vec{r}) \tag{20}$$

Combining equations (18) to (20) gives this distribution as

$$q_I\left(\vec{r}_1;\vec{r}\right) = \frac{1}{2\pi\rho c}\int_{-\infty}^{\infty} d\omega\, G^*\left(\vec{r},\vec{r}_1,\omega\right) \int_S dS_2\, G\left(\vec{r},\vec{r}_2,\omega\right)\, \phi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}\left(\vec{r}_1,\vec{r}_2,\omega\right) \tag{21}$$

GREEN FUNCTION FOR A CONE

To take advantage of the formulation given in the previous section, one needs to find the appropriate Green function. With $t - t' = \tau$ equation (2) can be rewritten as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right) g\left(\vec{r},\vec{r}\,',\tau\right) = -\delta\left(\vec{r}-\vec{r}\,'\right)\,\delta(\tau) \tag{22}$$

Taking the Fourier transform with $k = \omega/c$ yields

$$\left(\nabla^2 + k^2\right) G\left(\vec{r},\vec{r}\,',\omega\right) = -\delta\left(\vec{r},\vec{r}\,'\right) \tag{23}$$

where for all points $\vec{r}$ on S, G must satisfy

$$\frac{\partial G\left(\vec{r},\vec{r}\,',\omega\right)}{\partial n} = 0 \tag{24}$$

and must satisfy the condition for outgoing waves

$$\lim_{r\to\infty} r\left(\frac{\partial G}{\partial r} + ikG\right) = 0 \tag{25}$$

The surface S is now chosen to be a cone enclosing the jet, as shown in figure 1. In terms of spherical coordinates (r,w,ϕ), where $w = \cos\theta$ in the usual notation (fig. 3), equation (23) becomes

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial w}\left[\left(1 - w^2\right)\frac{\partial}{\partial w}\right] + \frac{1}{1 - w^2}\frac{\partial^2}{\partial\varphi^2} + k^2r^2\right\} G = -\frac{1}{r^2}\,\delta\left(r-r'\right)\,\delta\left(w-w'\right)\,\delta\left(\varphi-\varphi'\right) \tag{26}$$

The Green function G is constructed by considering the homogeneous form of this differential equation. By the usual separation-of-variable technique, solutions of the form

$$y(r,w,\varphi) = R(r)\,W(w)\,\Phi(\varphi) \tag{27}$$

are assumed, where y is a solution to the homogeneous form of equation (26). With the resulting separation constants set equal to m^2 and $\nu(\nu+1)$, the following set of ordinary differential equations is obtained:

$$\left.\begin{aligned} &\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0 \\ &\frac{d}{dw}\left[\left(1 - w^2\right)\frac{dW}{dw}\right] + \left[\nu(\nu + 1) - \frac{m^2}{1 - w^2}\right]W = 0 \\ &\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[k^2r^2 - \nu(\nu + 1)\right]R = 0 \end{aligned}\right\} \tag{28}$$

The three sets of linearly independent solutions to each of these equations are

$$\left.\begin{aligned} &\Phi(\varphi) : \cos(m\varphi), \sin(m\varphi) \\ &W(w) : P_\nu^m(w), Q_\nu^m(w) \\ &R(r) : j_\nu(kr), y_\nu(kr) \end{aligned}\right\} \tag{29}$$

where P_ν^m and Q_ν^m are the associated Legendre functions of the first and second kinds, respectively, and j_ν and y_ν are the spherical Bessel functions of the first and second kinds, respectively (ref. 4).

The Green function is now constructed from these solutions. Applying the condition of periodicity of φ over 2π shows that m must be an integer. Also, G must be symmetric with respect to φ'; therefore, the φ dependence is

$$\Phi(\varphi) \propto \cos\left[m(\varphi-\varphi')\right] \tag{30}$$

Since G must be finite at $w = 1$ $(\theta = 0)$, Q_ν^m is eliminated from the solution because this function becomes infinite. The normal derivative on the cone is zero only if the degree ν satisfies the condition

$$\left.\frac{dP_\nu^m(w)}{dw}\right|_{w=w_0} = 0 \tag{31}$$

Hence

$$W(w) \propto P_{\nu}^{m}(w) \tag{32}$$

where ν satisfies equation (31) and m is an integer.

The condition of finiteness at $r = 0$ eliminates $y_{\nu}(kr)$ from the solution for $r < r'$. At large values of r the outgoing wave condition requires

$$R(r) \propto h_{\nu}^{(2)}(kr) \equiv j_{\nu}(kr) - iy_{\nu}(kr) \tag{33}$$

The solution that is continuous across the source point is thus

$$R(r) \propto \left\{ \begin{array}{ll} j_{\nu}(kr)\, h_{\nu}^{(2)}(kr') & (r \leqq r') \\ j_{\nu}(kr')\, h_{\nu}^{(2)}(kr) & (r \geqq r') \end{array} \right\} \tag{34}$$

Combining the relations (30), (32), and (34) over all possible indices ν and m gives the solution for the Green function as

$$G = \sum_{m=0}^{\infty} \sum_{\nu} B_{\nu m} \cos\left[m(\varphi-\varphi')\right] P_{\nu}^{m}(w)\, j_{\nu}(kr<)\, h_{\nu}^{(2)}(kr>) \tag{35}$$

where m must be integral, ν satisfies the condition specified in equation (31), and the notation

$$\left. \begin{array}{l} F(x<) = \left\{ \begin{array}{ll} F(x) & (x \leqq x') \\ F(x') & (x \geqq x') \end{array} \right. \\ \\ F(x>) = \left\{ \begin{array}{ll} F(x') & (x \leqq x') \\ F(x) & (x \geqq x') \end{array} \right. \end{array} \right\} \tag{36}$$

is used.

The constants $B_{\nu m}$ are obtained in the usual manner by substituting equation (35) into the differential equation (26) and successively: (a) multiplying by $\left[\cos \ell(\varphi-\varphi')\right] d\varphi$ and integrating over $(0,2\pi)$; (b) multiplying by $P_{\alpha}^{\ell}(w)dw$ and integrating over $\left(w_0,1\right)$, (the orthogonality of the associated Legendre functions being utilized as shown in the appendix); and (c) multiplying by $r^2\,dr$ and integrating over the source point, that is, over $(r'-0,r'+0)$. Performing these calculations results in

$$B_{\nu m} = \frac{k\epsilon_m P_{\nu}^{m}(w')}{2\pi i \int_{w_0}^{1} dw\left[P_{\nu}^{m}(w)\right]^2} \tag{37}$$

Macdonald (ref. 5) shows that the values of ν satisfying equation (31) form a discrete set of real numbers. Hence, upon utilizing the result (eq. (A7)) of the appendix, the Green function becomes

$$G = \frac{k}{2\pi i\left(1 - w_0^2\right)} \sum_{m=0}^{\infty} \epsilon_m \cos\left[m(\varphi-\varphi')\right]$$

$$\times \sum_{\nu} (2\nu + 1) \frac{P_{\nu}^{m}(w')\,P_{\nu}^{m}(w)}{\left[P_{\nu}^{m}(w)\frac{\partial}{\partial\nu}\frac{dP_{\nu}^{m}(w)}{dw}\right]_{w=w_0}} j_{\nu}(kr_<)\,h_{\nu}^{(2)}(kr_>) \tag{38}$$

where the notation of equation (36) is used. Different representations for this function have been obtained by Carslaw (ref. 6) and Felsen (ref. 7) using more involved methods.

CALCULATION OF GREEN FUNCTION

The feasibility of using the conical Green function depends on fast and accurate calculation of Legendre functions of real degree and spherical Bessel functions of real order. The Legendre functions were determined via the Mehler-Dirichlet integral representation as described by Waterman (ref. 8). Results accurate to at least eight significant figures for the Legendre functions and their derivatives were obtained by this method. The degrees ν needed in the Green function were determined so that the calculations in equation (38) are accurate to six significant digits.

The Bessel functions were also calculated from an integral representation, with asymptotic forms used for both large argument and large order (ref. 4). Overlap between these methods of calculation assured an accuracy of six significant figures except near the zeros of the Bessel functions, where accuracy to at least the sixth decimal place was obtained.

APPLICATION TO A POINT SOURCE

To check the validity of the formulation and the practicability of its application with the minimum amount of mathematical detail, the jet was replaced by the simplest deterministic acoustic source, a point monopole of angular frequency ω_s located on the axis of the cone as shown in figure 4. The resulting intensity normal to the cone is computed by equations (15) and (16) and is compared with the result obtained by the direct method, which is described in the following paragraphs.

The pressure field due to the point source is given by

$$p(\vec{r},t) = \sigma_s \frac{e^{i\omega_s t} e^{-ik_s R}}{4\pi R} \tag{39}$$

where $R = \left|\vec{r} - \vec{r}_s\right| = \sqrt{r_s^2 + 2wr_s r + r^2}$ and σ_s is the strength of the source. With the velocity field determined through the acoustic equation of motion (eq. (12)), the correlation of pressure and velocity becomes

$$\vec{\psi}_{p,\vec{V}}(\vec{r},\vec{r},\tau) = \frac{\sigma_s^2 \left(k_s R - i\right) \nabla R \, e^{i\omega_s \tau}}{16\pi^2 \rho c k_s R^3} \tag{40}$$

Now choose $\vec{r}$ to be located on the surface of the cone and note that

$$\frac{\partial R}{\partial n} = \frac{r_s}{R}\sqrt{1 - w_0^2} \tag{41}$$

along this surface; then the component of acoustic intensity normal to the cone is

$$\begin{aligned} I_n(\vec{r}) &= \mathrm{Re}\left\{\psi_{p,V_n}(\vec{r},\vec{r},0)\right\} \\ &= \frac{\sigma_s^2 r_s \sqrt{1 - w_0^2}}{16\pi^2 \rho c R^3} \end{aligned} \tag{42}$$

Nondimensionalizing this result as

$$J_n(\vec{r}) = \frac{16\pi^2 \rho c r_s^2}{\sigma_s^2 \sqrt{1 - w_0^2}} I_n(\vec{r}) \tag{43}$$

gives

$$J_n(\vec{r}) = \left[1 + 2w_0\left(\frac{r}{r_s}\right) + \left(\frac{r}{r_s}\right)^2\right]^{-3/2} \tag{44}$$

and hence the nondimensionalized intensity component normal to the cone is a function of only the relative distance along the surface r/r_s, and of course the cone angle.

To compare this result with that from the Green function approach, equations (15) and (16) are combined to give

$$I_n(\vec{r}) = \text{Re}\left\{\int_{-\infty}^{\infty} \frac{d\omega i}{\omega\rho} \int_S dS_0 \, G(\vec{r},\vec{r}_0,\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \, \psi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}(\vec{r},\vec{r}_0,\tau) e^{-i\omega\tau}\right\} \tag{45}$$

where $\vec{r}$ is any point on the cone.

Calculating the pressure-gradient correlation by means of equation (39) yields

$$\psi_{\frac{\partial p}{\partial n},\frac{\partial p}{\partial n}}(\vec{r},\vec{r}_0,\tau) = \frac{\sigma_s^2 \left(1 - w_0^2\right) r_s^2}{16\pi^2} \frac{H(k_s R_0)}{R_0^3} \frac{H^*(k_s R)}{R^3} e^{i\omega_s \tau} \tag{46}$$

where

$$H(x) \equiv (1 + ix)e^{-ix} \tag{47}$$

By using this expression the two infinite integrals are readily calculated; and nondimensionalizing again as was done for equation (43) yields

$$J_n(\vec{r}) = \text{Re}\left\{\frac{i\sqrt{1 - w_0^2}\, r_s^4}{k_s} \frac{H^*(k_s R)}{R^3} \int_S dS_0 \frac{H(k_s R_0)}{R_0^3} G(\vec{r},\vec{r}_0,\omega_s)\right\} \tag{48}$$

If the Green function from equation (38) is introduced, and since an element of surface area of the cone is

$$dS_0 = r_0\sqrt{1 - w_0^{\,2}}\, d\varphi_0\, dr_0 \tag{49}$$

the integration over φ_0 is easily performed to give

$$J_n(\vec{r}) = \mathrm{Re}\left\{\frac{H^*\left[b_s\sqrt{1 + 2w_0\left(\frac{r}{r_s}\right) + \left(\frac{r}{r_s}\right)^2}\right]}{\left[1 + 2w_0\left(\frac{r}{r_s}\right) + \left(\frac{r}{r_s}\right)^2\right]^{3/2}} \sum_{\nu} (2\nu + 1) \frac{P_\nu^0(w_0)}{\left[\frac{\partial}{\partial\nu}\frac{dP_\nu^0(w)}{dw}\right]_{w=w_0}}\right.$$

$$\left.\times \int_0^\infty d\left(\frac{r_0}{r_s}\right)\left(\frac{r_0}{r_s}\right)\frac{H\left[b_s\sqrt{1 + 2w_0\left(\frac{r_0}{r_s}\right) + \left(\frac{r_0}{r_s}\right)^2}\right]}{\left[1 + 2w_0\left(\frac{r_0}{r_s}\right) + \left(\frac{r_0}{r_s}\right)^2\right]^{3/2}} j_\nu\left(b_s\frac{r_<}{r_s}\right) h_\nu^{(2)}\left(b_s\frac{r_>}{r_s}\right)\right\} \tag{50}$$

where $b_s = k_s r_s$. (The notation of eq. (36) is again used.) In contrast to the exact formulation (eq. (44)), J_n here appears to depend also on the frequency through the parameter b_s.

The total radiated acoustic intensity at the far-field point $\vec{r}$ is given by equation (19). For the point source being considered the pressure field from equation (39) yields

$$I_{direct}(\vec{r}) = \frac{\sigma_s^{\,2}}{16\pi^2\rho c r^2} \tag{51}$$

since in the far field $R = r$.

For this simple source it is of interest to determine the relationship expressed in equation (21), that is, the contribution to the far-field intensity from apparent sources located on the cone. Determining the intensity by means of equations (18) and (19) yields

$$I(\vec{r}) = \frac{1}{2\pi\rho c}\int_{-\infty}^{\infty} d\omega \int_S dS_1 \int_S dS_2\, G^*(\vec{r},\vec{r}_1,\omega)G(\vec{r},\vec{r}_2,\omega)\int_{-\infty}^{\infty} d\tau\, \psi_{\frac{\partial p}{\partial n'},\frac{\partial p}{\partial n}}(\vec{r}_1,\vec{r}_2,\tau)e^{-i\omega\tau} \tag{52}$$

Substituting in the same way as was done to obtain the nondimensionalized intensity component normal to the cone (eq. (50)) gives

$$I(\vec{r}) = \left(\frac{r_s}{r}\right)^2 \frac{\sigma_s^2}{16\pi^2\rho c} \sum_\nu A_\nu^*(w) \int_0^\infty dr_1 \frac{r_1 H^*(k_s R_1)}{R_1^3} j_\nu(k_s r_1)$$

$$\times \sum_\nu A_\nu(w) \int_0^\infty dr_2 \frac{r_2 H(k_s R_2)}{R_2^3} j_\nu(k_s r_2) \tag{53}$$

where

$$A_\nu(w) = (2\nu + 1)e^{i\nu\frac{\pi}{2}} \frac{P_\nu^0(w)}{\left[\frac{\partial}{\partial\nu} \frac{dP_\nu^0(w)}{dw}\right]_{w=w_0}} \tag{54}$$

and the asymptotic formula

$$h_\nu^{(2)}(k_s r) \sim \frac{\exp\left[-i\left(k_s r - \frac{1}{2}\pi\nu - \frac{1}{2}\pi\right)\right]}{k_s r} \qquad (\text{as} \quad r \to \infty)$$

has been used. In this expression the intensity appears to depend on the frequency of the source and on the far-field angle $\theta = \cos^{-1}w$, but from the direct expression (eq. (51)), it can be seen that this dependence is only apparent.

Introducing for convenience the nondimensional coordinate β measured along the cone axis by

$$\beta = \frac{r_1}{r_s} \cos\left(\pi - \theta_0\right) = -\frac{w_0 r_1}{r_s} \tag{55}$$

and normalizing the intensity by its value found by the direct method yield

$$\frac{I}{I_{direct}} = \int_0^\infty d\beta \, q_I'(\beta; w) \tag{56}$$

where

$$q_I'(\beta;w) = \frac{\beta}{w_0^4} \frac{H^*\left(b_s\sqrt{1 - 2\beta + \frac{\beta^2}{w_0^2}}\right)}{\left(1 - 2\beta + \frac{\beta^2}{w_0^2}\right)^{3/2}} \sum_\nu A_\nu^*(w)\, j_\nu\left(\frac{b_s\beta}{-w_0}\right)$$

$$\times \sum_\nu A_\nu(w) \int_0^\infty d\alpha\, \alpha \frac{H\left(b_s\sqrt{1 - 2\alpha + \frac{\alpha^2}{w_0^2}}\right)}{\left(1 - 2\alpha + \frac{\alpha^2}{w_0^2}\right)^{3/2}} j_\nu\left(\frac{b_s\alpha}{-w_0}\right) \tag{57}$$

The real part of this expression is interpreted as the contribution to the far-field intensity at the angle $\theta = \cos^{-1}w$ from apparent sources located on the conical strip between β and $\beta + d\beta$, and is analogous to equation (21) for the contribution to the far field due to apparent sources located on an incremental area of the surface. Its shape as a function of the two variables β and ω_s gives an indication of how well the source can actually be represented by the apparent sources on the cone, and hence of how well these apparent sources can be retraced to determine an actual source location along the center line of the cone.

Equation (57) also affords an additional check on the formulation, since an integration of the real part over all values of β should equal unity, whereas integration of the imaginary part should yield zero.

The limiting form of this function for low frequency is obtained by setting $b_s = 0$. This yields

$$q_I'(\beta;w)\Big|_{b_s\to 0} = \frac{1 + w_0}{w_0^2} \frac{\beta}{\left(1 - 2\beta + \frac{\beta^2}{w_0^2}\right)^{3/2}} \tag{58}$$

RESULTS

The point-source nondimensionalized intensity component normal to a cone of half-angle 12^o is presented in figure 5. The solid line represents the direct values calculated from equation (44), whereas the plotted points were obtained from the Green function solution (eq. (50)). These points were obtained for the frequency parameter b_s equal to both 1 and 10. (For jet-noise experiments, $b_s = 10$ is estimated to correspond to a source at the nozzle exit of frequency between 400 Hz and 4000 Hz, depending on the location of the cone vertex.) The number of eigenvalues needed to obtain a given degree of accuracy increased with frequency: Accuracy of 0.1 percent was obtained by using about 10 eigenvalues for $b_s = 1$ and 30 eigenvalues for $b_s = 10$. The calculated points averaged about 1 minute of computing time on a CDC 6600 computer.

The contributions to the far-field intensity at an angle of 45^o to the cone axis due to apparent sources on the cone (eq. (57)) are shown in figure 6(a) for small values of the frequency parameter b_s. The result for the case in which the frequency approaches zero ($b_s = 0$, eq. (58)) is given along with results calculated for $b_s = 0.1$ and 1.0. The distribution of apparent sources is spread almost symmetrically about the projected location of the actual source with little variation over this low frequency range.

Figure 6(b) shows this distribution for higher frequencies of the source. The peak of the distribution again occurs at the projected position of the source, whereas the magnitude of the peak and the spread in the "downstream" direction increase with b_s. Increasing frequency also results in negative contributions in the "upstream" direction, which correspond to apparent energy sinks in the interpretation given to equation (57). This result indicates that a high-frequency distributed source would appear to be located farther downstream than its actual location because of cancellation effects. Hence care must be exercised when making this apparent-source interpretation at the higher frequencies.

The integration of the results for the real and imaginary parts of equation (57) does yield unity and zero, respectively, indicating accuracy of these results. The computing time for the far-field intensity was of the same order as for the near-field flux, the time increasing with the frequency of the source.

CONCLUDING REMARKS

The developments necessary for determination of the near and far acoustic fields of a jet in terms of measured pressure gradients on an imaginary conical surface surrounding the jet have been presented. A point monopole source was used in place of the jet to check the mathematical model. Accurate results for the intensity normal to the cone were found from this formulation in a small amount of computer time. At low source frequencies the representation of the far-field intensity in terms of apparent sources on the cone shows the sources to be spread almost symmetrically about the projected location of the actual source. At higher frequencies the apparent sources at the projected position of the actual source contribute an even greater amount to the far-field intensity, although upstream of the source the representation for the apparent sources becomes less clear because of the presence of apparent acoustical sinks. This phenomenon must be recognized when the apparent source distribution calculated from pressure-gradient correlations is interpreted.

The results indicate that the measurement of pressure-gradient correlations on a conical surface surrounding a jet is a practical method for determining the acoustic field of the jet. A more extensive analytic investigation should be made if an apparent-source interpretation of the far-field intensity is to be obtained from the experimental correlations.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., October 22, 1974.

APPENDIX

RELATIONS FOR ASSOCIATED LEGENDRE FUNCTIONS

Let $P_\alpha^m(w)$ and $P_\beta^m(w)$ be the associated Legendre functions of order m and arbitrary degrees α and β. Hence they satisfy

$$\left.\begin{aligned} &\left\{\frac{d}{dw}\left[\left(1 - w^2\right)\frac{d}{dw}\right] + \left[\alpha(\alpha+1) - \frac{m^2}{1 - w^2}\right]\right\}P_\alpha^m(w) = 0 \\ &\left\{\frac{d}{dw}\left[\left(1 - w^2\right)\frac{d}{dw}\right] + \left[\beta(\beta+1) - \frac{m^2}{1 - w^2}\right]\right\}P_\beta^m(w) = 0 \end{aligned}\right\} \tag{A1}$$

Multiplying the first of these equations by $P_\beta^m(w)$ and the second by $P_\alpha^m(w)$, subtracting, and integrating w between w_0 and 1 yield

$$\left[\beta(\beta + 1) - \alpha(\alpha + 1)\right]\int_{w_0}^1 dw\, P_\alpha^m(w)\, P_\beta^m(w)$$

$$= \left\{\left(1 - w^2\right)\left[P_\alpha^m(w)\,\frac{dP_\beta^m(w)}{dw} - P_\beta^m(w)\,\frac{dP_\alpha^m(w)}{dw}\right]\right\}_{w=w_0} \tag{A2}$$

Hence if the following condition is satisfied for $\nu = \alpha$ and β:

$$\left.\frac{dP_\nu^m(w)}{dw}\right|_{w=w_0} = 0 \tag{A3}$$

and if $\alpha \neq \beta$, then

$$\int_{w_0}^1 dw\, P_\alpha^m(w)\, P_\beta^m(w) = 0 \tag{A4}$$

that is, P_α^m and P_β^m are orthogonal on the interval $\left(w_0, 1\right)$.

Since the degrees ν of equation (A3) form a discrete set of real values (ref. 5), then for any root α there exists a value $\beta = \alpha + \epsilon$, with $\epsilon > 0$, such that no roots exist in the interval (α,β). Hence from equation (A2),

$$\int_{w_0}^{1} dw\; P_\alpha^m(w)\; P_{\alpha+\epsilon}^m(w) = \left[\frac{1}{\epsilon}\; \frac{\left(1 - w^2\right) P_\alpha^m(w)}{2\alpha + 1 + \epsilon}\; \frac{dP_{\alpha+\epsilon}^m(w)}{dw}\right]_{w=w_0} \tag{A5}$$

Therefore

$$\int_{w_0}^{1} dw \left[P_\alpha^m(w)\right]^2 = \lim_{\epsilon \to 0} \int_{w_0}^{1} dw\; P_\alpha^m(w)\; P_{\alpha+\epsilon}^m(w)$$

$$= \left[\frac{\left(1 - w^2\right) P_\alpha^m(w)}{2\alpha + 1} \lim_{\epsilon \to 0} \frac{1}{\epsilon}\; \frac{dP_{\alpha+\epsilon}^m(w)}{dw}\right]_{w=w_0} \tag{A6}$$

But the limit in this last expression is simply the rate of change of $dP_\alpha^m(w)/dw$ with respect to the degree α. Hence

$$\int_{w_0}^{1} dw \left[P_\alpha^m(w)\right]^2 = \left\{\frac{\left(1 - w^2\right) P_\alpha^m(w)}{2\alpha + 1}\; \frac{\partial}{\partial\alpha}\left[\frac{dP_\alpha^m(w)}{dw}\right]\right\}_{w=w_0} \tag{A7}$$

for any α satisfying equation (A3).

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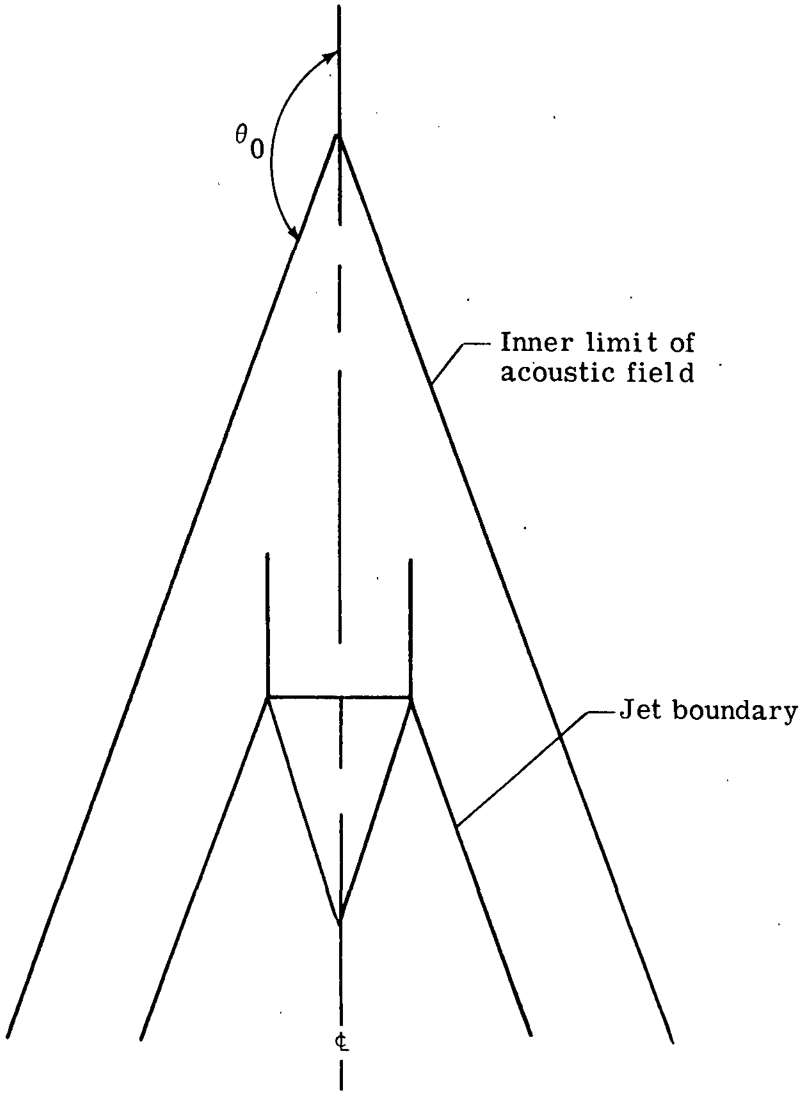


Figure 1.- Schematic of conical surface in acoustic field of jet.

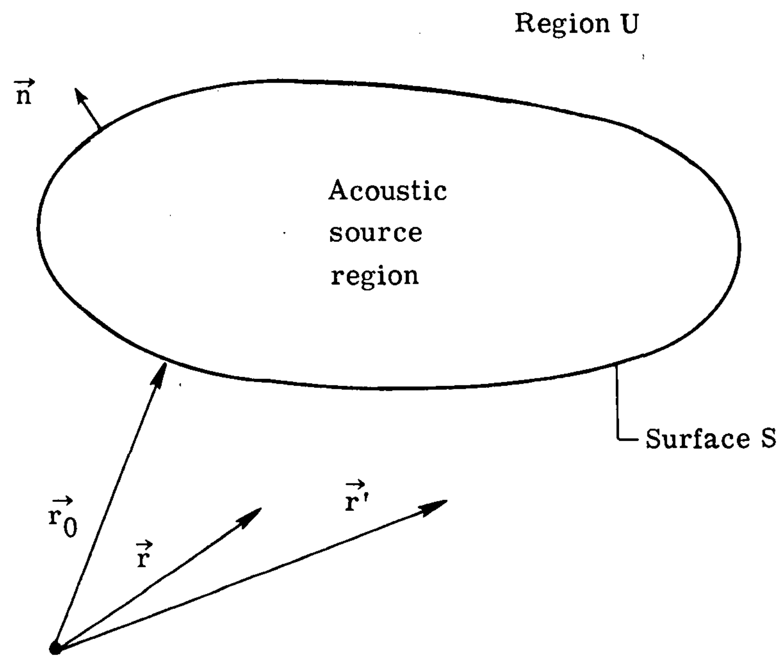


Figure 2.- Geometry for general theory.

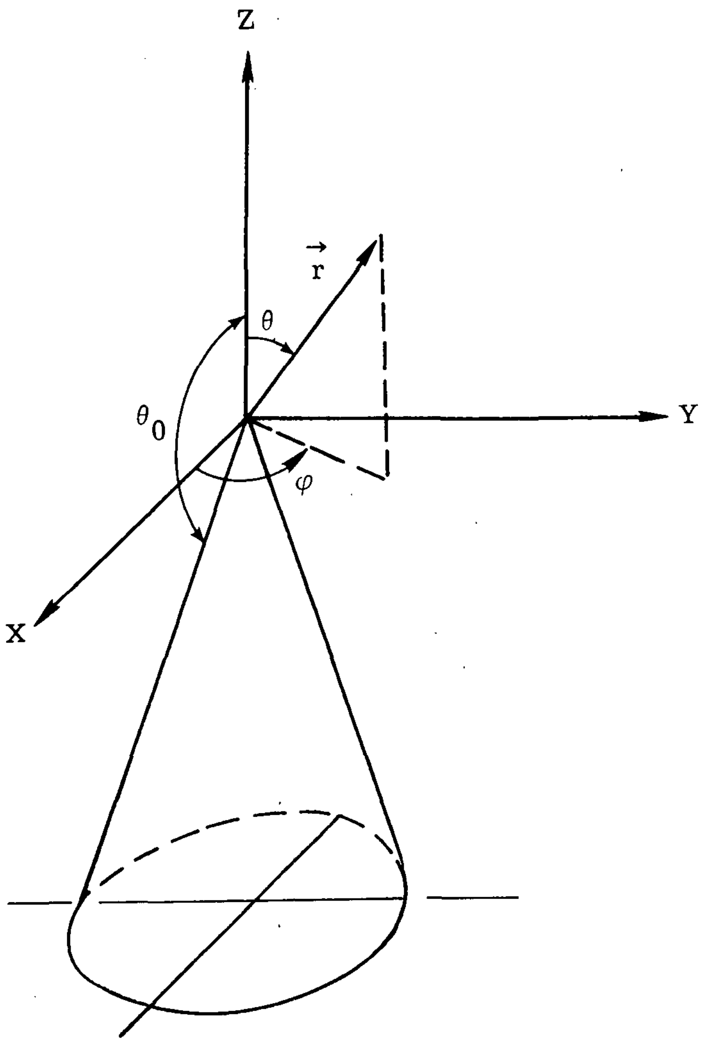


Figure 3.- Coordinates for conical Green function.

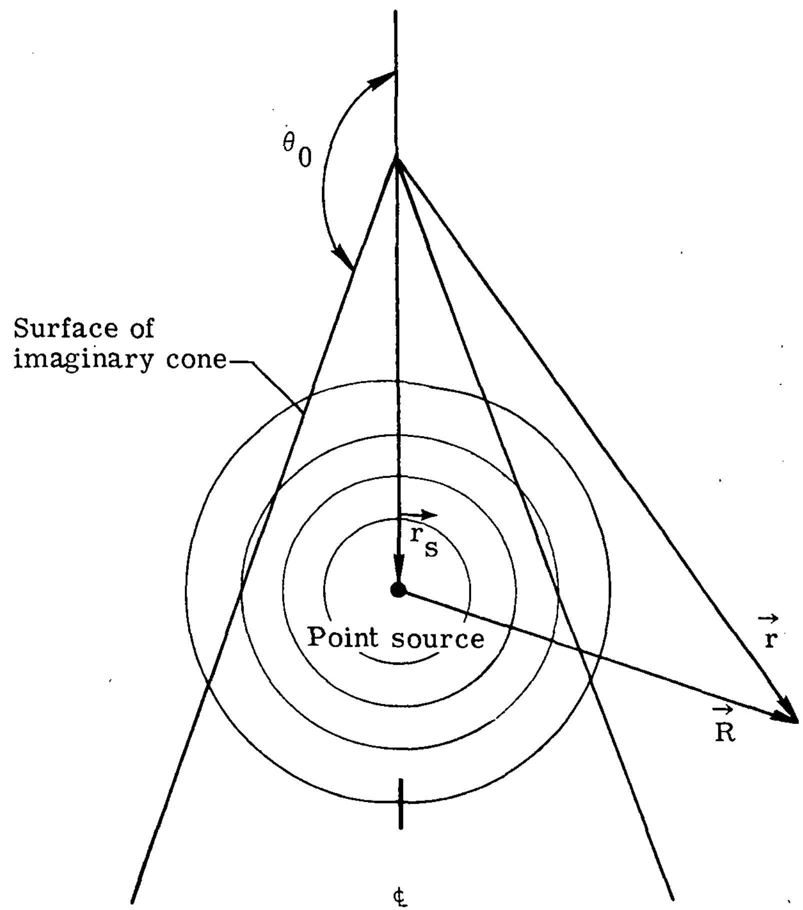


Figure 4.- Point-source location.

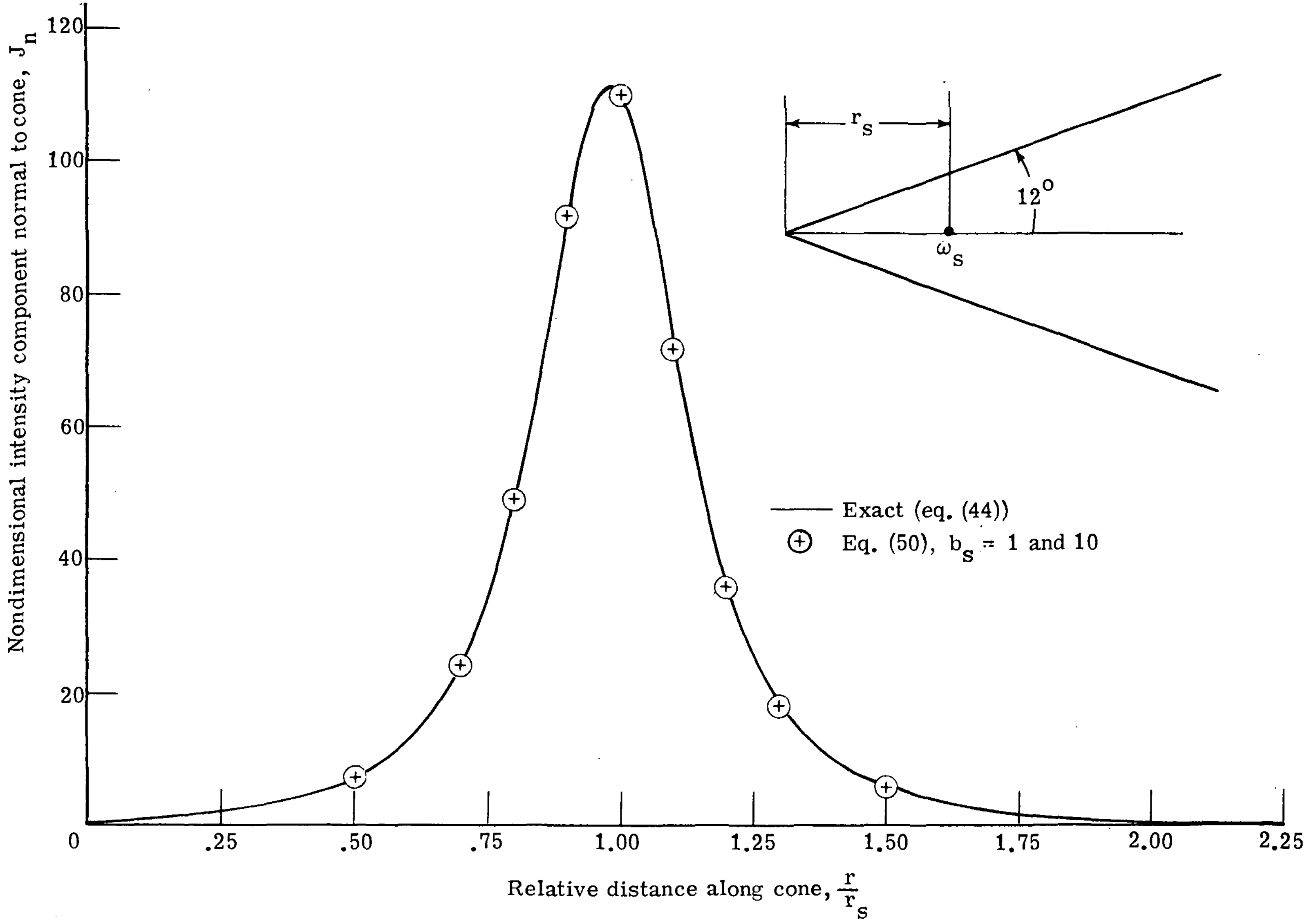
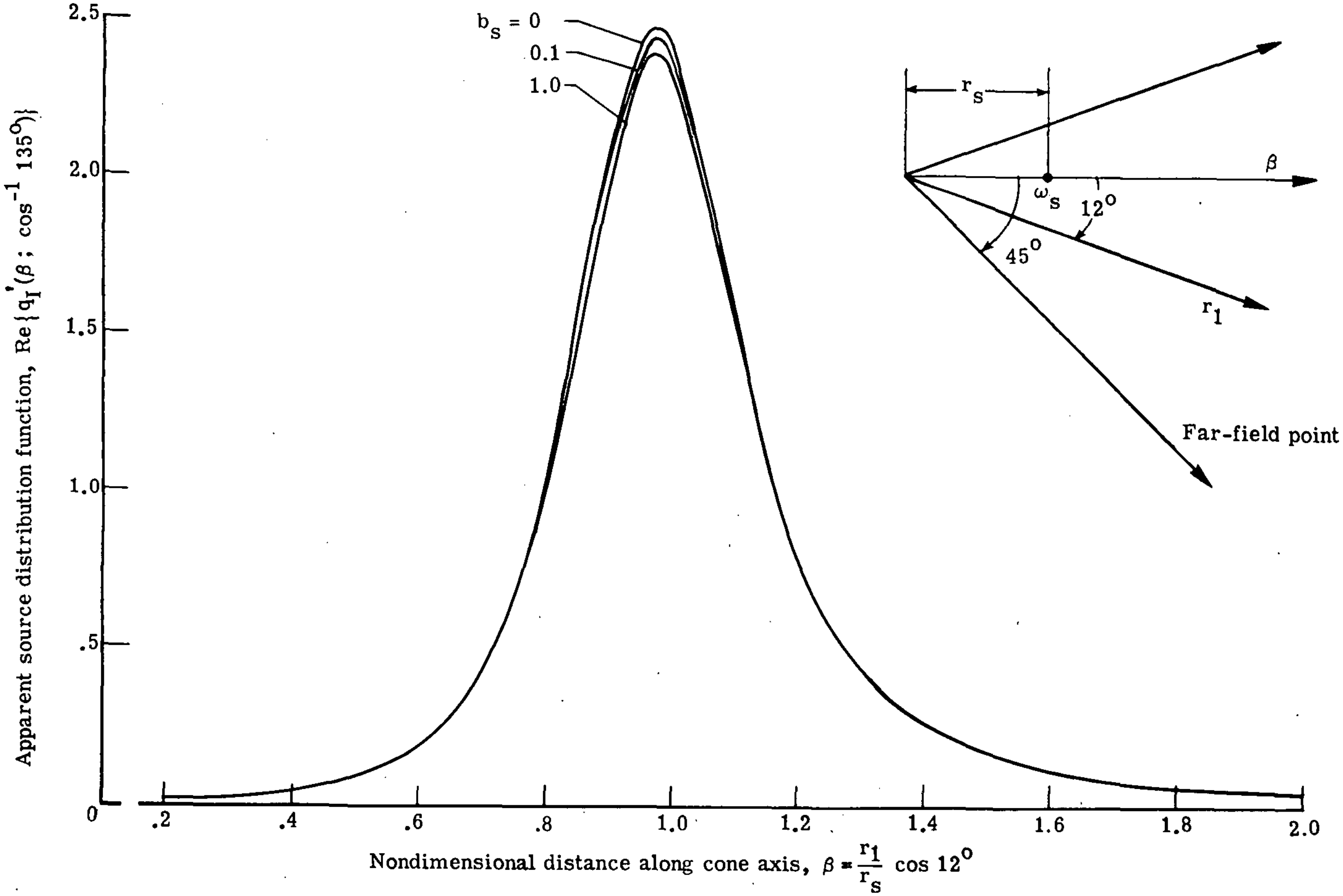
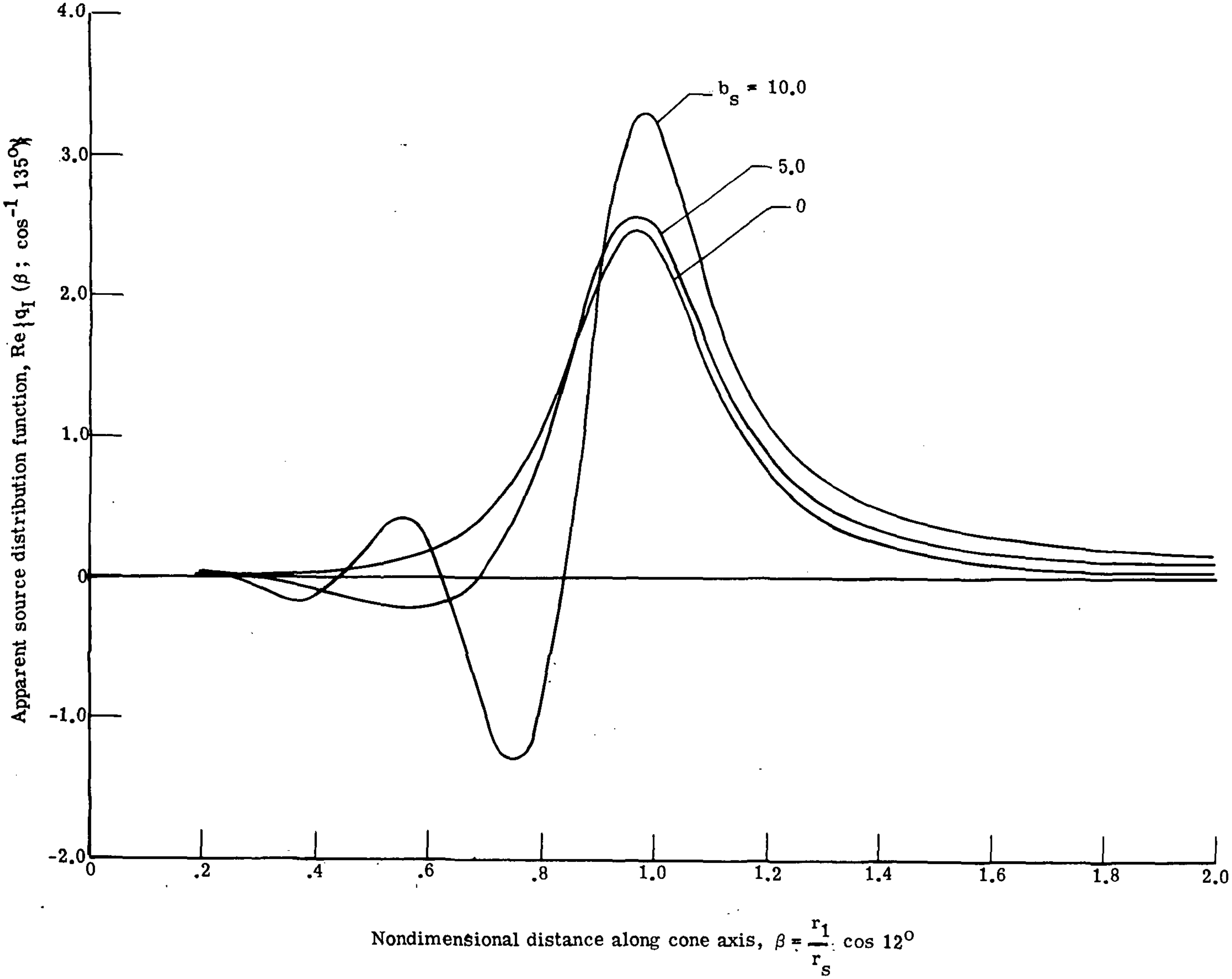


Figure 5.- Intensity component normal to cone (from point source).



(a) $b_s = 0$, 0.1, and 1.0.

Figure 6.- Contribution to far-field intensity from apparent sources on the cone (from point source).



(b) b_s = 10.0, 5.0, and 0.

Figure 6.- Concluded.

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