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BROADBAND NOISE GENERATED BY TURBULENT INFLOW TO ROTOR OR STATOR BLADES IN AN ANNULAR DUCT

Frank Lane

Prepared by
KLD ASSOCIATES, INCORPORATED
Huntington, N.Y. 11743

for Langley Research Center

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16. Abstract

The Green's function relating the radiated pressure field to the fluctuating forces on rotor or stator blades is developed in the presence of dissipation due to turbulent velocity fluctuations and sound speed fluctuations. The resonances in the output power spectrum which would occur at the cut-off frequencies in the absence of dissipation should be removed and smeared out by the incorporation of dissipation. Wave number dependence is developed for an effective eddy viscosity due to the aforementioned fluctuations in the background medium.

The space-time correlation function $R_{\perp\perp}$ for blade-normal velocity fluctuations on a single or on two different blades is developed in terms of the velocity correlation tensor for the inflow under the assumptions of isotropy and (Taylor) frozen behavior. The correlation function $R_{\perp\perp}$ is then simplified under certain approximations and the behavior of the blade-force correlation function is inferred from that of $R_{\perp\perp}$.

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SYMBOLS

$\delta a/a_o$	fluctuation (relative) sound speed in background medium
a_o	sound speed, ambient
$E(k)$	energy spectrum for velocity fluctuations
f	blade force/unit volume
f_x	axial blade force/unit volume
f_θ	torque blade force/unit volume
f_ν	force distribution, ν th blade
$\langle f_\nu^2 \rangle$	mean square force fluctuation
f, g	scalar correlation functions in R_{ij} (isotropic form)
$H(\ )$	unit step function
$k,\ k_o$	wave number
$\vec{\ell}$	scattering-direction (unit) vector
$M_o = \dfrac{U_o}{a_o}$	free stream Mach Number
N	number of blades
p	pressure
r, θ, x	coordinates (cylindrical)
$r_{in},\ r_{out}$	inside and outside wall radii
$R_{\perp\perp}$	normal velocity correlation
R_{ij}	velocity correlation tensor
$R_m(\ )$	cylinder functions
s	energy flux
t	time

u^i	velocity vector
$u_\perp$	normal velocity
U_o	free stream velocity
x,y,z	coordinates (Cartesian)
Ω	rotor angular velocity
$\varepsilon(r)$	blade local twist angle
ν	blade identification number
ν_T	eddy viscosity
φ_ν	$= 2\pi\nu/N$
Δ	$= \Omega t_d + 2\pi\nu_d/N$
μ_{mq}	eigenvalues of $R_m(\mu r)$ corresponding to vanishing derivative at r_{in} and r_{out}
λ	Fourier transform variable for x
ω	Fourier transform variable for t
ω	generic frequency
$\Phi(k)$	wave number power spectrum for scalars in background medium
$\delta(\ )$	Dirac delta function
δ_{ij}	Kronecker delta
$\chi(\ )$	blade force correlation function
$\Phi_{ij}(k)$	fluctuation velocity power spectral tensor for background medium
ρ	density, also distance in fluid-fixed coordinates
$\delta\rho/\rho_o$	density (relative) fluctuation in background medium
Λ	turbulent velocity integral scale

ξ, η, ζ	coordinates in fluid-fixed system
τ	time in fluid-fixed system
$\rho_d \; \rho_\perp \; \rho_\parallel$	components of distance ρ , defined in eq. (39)
$(\;)_1$	evaluated at $\vec{x}_1, t_1$
$(\;)_2$	evaluated at $\vec{x}_2, t_2$
$(\;)_m$	$1/2\left((\;)_1 + (\;)_2\right)$
$(\;)_d$	$(\;)_1 - (\;)_2$
$\langle\;\rangle$	expectation or ensemble average

I. INTRODUCTION

This report presents the results obtained for the first three tasks under a program whose objective is to analyze aspects of the broadband noise generated by rotor or stator blades subject to inflow turbulence, in an annular duct. The first technical section (Section II) introduces dissipation, in the form of a generic eddy viscosity, into the Green's function relating radiated pressure to blade fluctuation forces. This adds an element of physical realism to the analysis and serves to eliminate the singularities or resonance effects which would otherwise occur in the output power spectrum. These singularities occur at the cutoff frequencies and become dense at higher modes, thereby completely dominating the output power spectral behavior. The introduction of dissipation eliminates these singularities and smears out the neighboring peaks so that the resulting output power spectrum should be well-behaved.

The next section (III) then relates the aforementioned eddy viscosity to actual dissipation losses experienced by waves propagating in a turbulent medium. This is carried out in two ways; first by means of a perturbation technique of J. Keller(1) for waves propagating in a medium with randomly variable index of refraction, and second by means of scattering losses in a single-scattering treatment of waves in a random medium. The second method is carried out by integrating over-all scattering angles the single-scattering cross-section corresponding first to fluctuations in density and sound speed in the background medium and, next, to velocity fluctuations in the background medium. These scattering cross sections are due to the work of Batchelor(2) The results from the two appraoches are shown to agree in the scalar-fluctuating case where the background fluctuations are the same. They give a wave-number dependence to the eddy viscosity, involving the energy spectrum of the background fluctuations at all wave numbers less than twice the incident wave number. The method of incorporating these results into the Green's function for radiated pressure is developed at the end of Section III.

Section IV develops the space-time correlation function for the normal component of the turbulent inflow velocity fluctuations, assuming an isotropic, frozen turbulent inflow velocity correlation tensor. The normal-velocity correlation $R_{\perp\perp}$ is derived for distinct times and distinct points on a single blade or on two different blades. The derivation is performed admitting radially varying blade twist angle.

Then the last two sections V and VI are aimed at developing an insight into the qualitative behavior of the blade force fluctuation correlations in terms of the normal velocity correlation developed in Section IV. Section V derives simplified approximate expressions for R_{11} in terms of three components of the distance function in the case of zero or small radial variation of blade twist angle. Then Section VI utilizes these results to infer the temporal stationarity of the blade-force fluctuations, the homogeneity with respect to blade number, and the inhomogeneity with respect to radius. These conclusions should then serve as the basis for subsequent developments of output power spectra for radiated pressure.

II. INTRODUCTION OF DISSIPATION INTO ANALYSIS OF ACOUSTIC PRESSURE FIELD

We consider the linearized (acoustic) compressible flow equations for an annular domain in the presence of a uniform flow U_o in the axial (x) direction, a force distribution f_i (per unit volume), and a dissipation in the form of a simple eddy viscosity ν_T. Later the eddy viscosity will be considered in detail and made wave-number dependent. We retain the simple density-pressure relation

$$dp = a_o^2 d\rho \tag{1}$$

despite the presence of dissipation.

The linearized equations are then as follows:

$$D\rho + \rho_o \nabla \cdot \vec{u} = 0 \tag{2}$$

$$\rho_o Du^i + \frac{\partial p}{\partial x^i} = \rho_o \nu_T \Delta u^i + f_i \tag{3}$$

where $$D = \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x} \tag{4}$$

Then, eliminating u^r and ρ, we find the following equation for the pressure perturbation p:

$$(1-M_o^2)p_{xx} + p_{yy} + p_{zz} - \frac{p_{tt}}{a_o^2} - 2\frac{M_o}{a_o}p_{xt}$$

$$+ \nu_T \frac{\Delta}{a_o^2} Dp = \nabla \cdot \vec{f} \tag{5}$$

We then introduce the cylinder functions

$$R_{mq}(r) = R_m(\mu_{mq} r) \tag{6}$$

which are linear combinations of $J_m(\mu_{mq}r)$ and $Y_m(\mu_{mq}r)$ such that the normal (radial) derivative vanishes at both inner (r_{in}) and outer (r_{out}) radii of the annulus. Further, we normalize such that

$$\int_{r_{in}}^{r_{out}} r\, R_{mq_1}(r)\, R_{mq_2}(r)\, dr = \delta_{q_1 q_2} \tag{7}$$

The R_{mq} then constitute an orthonormal system with respect to weighting function r, over the domain (r_{in}, r_{out}).

Next we operate on eq. (5) with the operator,

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\, dx \int_{-\infty}^{\infty} e^{i\omega t}\, dt \int_{0}^{2\pi} e^{-im\theta}\, d\theta \int_{r_{in}}^{r_{out}} r\, R_{mq}(r)\, dr \tag{8}$$

and define $\bar{p}_{mq}(\lambda,\omega)$ as

$$\bar{p}_{mq}(\lambda,\omega) = \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} \int_{\theta=0}^{2\pi} \int_{r=r_{in}}^{r_{out}} r\, R_{mq}(r)$$

$$p(x,r,\theta,t)\, e^{-i\lambda x}\, e^{-im\theta}\, e^{i\omega t}\, dx\, dt\, d\theta\, dr \tag{9}$$

The result of this operation is as follows:

$$-\lambda^2(1-M_0^2)\bar{p}_{mq} - \mu_{mq}^2\bar{p}_{mq} + \frac{\omega^2}{a_0^2}\bar{p}_{mq} - 2\frac{M_0}{a_0}\lambda\omega\,\bar{p}_{mq}$$

$$+\frac{\nu_T}{a_0^2}(-\lambda^2-\mu_{mq}^2)(-i\omega+U_0 i\lambda)\bar{p}_{mq} =$$

$$\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dt_0\int_0^{2\pi}d\theta_0\int_{r_{in}}^{r_{out}}dr_0\; e^{-i\lambda x_0}e^{-im\theta_0}e^{i\omega t_0}$$

$$r_0 R_{mq}(r_0)\,\nabla\cdot\vec{f}(x_0 r_0 \theta_0 t_0) \tag{10}$$

Now we recognize that $\vec{f}$ has only axial and torque components f_x, f_θ, respectively, and we perform the integration on the right hand side of eq. (10) by parts with respect to x_0 and θ_0. The right hand side then becomes

$$\int dx_0\int dt_0\int d\theta_0\int dr_0\; e^{-i\lambda x_0}e^{-im\theta_0}e^{i\omega t_0}R_{mq}(r_0)$$

$$\left\{i\lambda f_x r_0 + im f_\theta\right\}_{(x_0,r_0,\theta_0,t_0)} \tag{11}$$

With the geometry as shown in Fig. 1 and with light loading, the resultant force $\vec{f}$ is closely normal to the blade. Hence

$$f_x \cong f\sin\varepsilon(r)$$

$$f_\theta \cong f\cos\varepsilon(r) \tag{12}$$

where $\varepsilon(r)$ is the local blade twist angle. Then the bracketed term in eq. (11) takes the form

$$\{ \quad \} = f(x_o, r_o, \theta_o, t_o)\left(i\lambda r_o \sin\varepsilon(r_o) + im\cos\varepsilon(r_o)\right) \tag{13}$$

In the absence of the dissipation term, we can write the left-hand side of eq. (10) in the form

$$-(1-M_o^2)(\lambda-\lambda_1)(\lambda-\lambda_2) \tag{14}$$

where λ_1, λ_2 are the roots of

$$(1-M_o^2)\lambda^2 + \mu_{mq}^2 - \frac{\omega^2}{a_o^2} + 2\frac{M_o\omega\lambda}{a_o} = 0$$

or

$$\lambda_{1,2} = \frac{-M_o\frac{\omega}{a_o} \pm \sqrt{M_o^2\frac{\omega^2}{a_o^2} + (1-M_o^2)\left(\frac{\omega^2}{a_o^2} - \mu_{mq}^2\right)}}{(1-M_o^2)}$$

$$= \frac{-M_o\frac{\omega}{a_o} \pm \sqrt{\frac{\omega^2}{a_o^2} - \mu_{mq}^2(1-M_o^2)}}{(1-M_o^2)} \tag{15}$$

Then p would result from the four-fold inversion of expression (11) divided by (14). That is, in the nondissipative case,

$$p(x,r,\theta,t)\Big|_{\nu_T=0} = \frac{-1}{(2\pi)^3(1-M_0^2)} \sum_{m=-\infty}^{\infty} \sum_{q=1}^{\infty} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dt_0$$

$$\int_0^{2\pi} d\theta_0 \int_{-\infty}^{\infty} dx_0 \int_{r_{in}}^{r_{out}} dr_0 \; e^{-i\omega(t-t_0)}$$

$$e^{i\lambda(x-x_0)} e^{im(\theta-\theta_0)} R_{mq}(r_0) R_{mq}(r)$$

$$f(x_0, r_0, \theta_0, t_0) (\lambda-\lambda_1)^{-1} (\lambda-\lambda_2)^{-1}$$

$$\left(i\lambda r_0 \sin\varepsilon(r_0) + im\cos\varepsilon(r_0)\right) \tag{16}$$

Now if the integration with respect to λ is carried out by residues, it is clear that the difference forms

$$(\lambda_1-\lambda_2) = \frac{2\sqrt{\frac{\omega^2}{a_0^2} - \mu_{mq}^2(1-M_0^2)}}{(1-M_0^2)} \tag{17}$$

will occur in the denominator for each ω,m,q.

For each combination m,q the value ω_{mq} of ω which annihilates the radicand in (17) is the well-known cut-off frequency which separates propagation from non-propagation of the m,q mode. These zeros ω_{mq} become extremely dense as m and q get large. Furthermore, when we proceed to form power spectra of pressure p, the cross terms (involving m_1, m_2, q_1, q_2) are integrable with respect to ω, but the self-terms are not integrable. Instead, they lead to infinities or resonances in the output or response power spectrum. These resonances become so dense at large m and q that they completely dominate any output power spectral considerations.

It is for this reason that the dissipation (ν_T) term is introduced. With the dissipation included, we expect that these resonances, which occur at the cut-off frequencies ω_{mq}, will be rounded off and, for large m,q smeared together, leading to a finite, continuous output power spectrum. Moreover, since the origin of the blade fluctuation forces is presumed to arise from the turbulent character of the inflow to the blade row, therefore it is consistent to assume that the dissipation should be of a turbulent character, and ν_T is to be considered in the sense of an eddy viscosity. Subsequently, ν_T will be related directly to the dissipation experienced by acoustic waves in a turbulent medium and will be given wave-number dependence. For the present, however, we treat ν_T as a parameter and return to the full expression (eq. 10) for evaluation of the poles in the λ plane in the presence of nonzero ν_T .

Now the full equation for the roots of λ is as follows:

$$(1-M_0^2)\lambda^2 + \mu_{m\nu}^2 - \frac{\omega^2}{a_0^2} + 2M_0\frac{\omega}{a_0}\lambda$$

$$- \frac{i\,\nu_T}{a_0^2}\left(\omega - U_0\lambda\right)\left(\lambda^2 + \mu_{m\nu}^2\right) = 0 \qquad (18)$$

As it stands this is a cubic in λ . However, we make use of the fact that the dissipation ν_T is small and that the only places where ν_T plays any significant role are at the cut-off points (or singularities in the nondissipative spectrum). Thus we solve eq. (18) for two roots as follows:

We treat the ν_T term as a small correction to the quadratic equation for λ and lump it in with the constant terms, giving λ (where ever it appears in this term) its nondissipative cut-off value, since these are the only points where this term matters. This may be formalized somewhat as follows: Consider the generic equation for λ,

$$a\lambda^2 + b\lambda + c + \nu_T f(\lambda) = 0 \tag{19}$$

where $f(\lambda)$ is any regular nonlinear function of λ . Assume ν_T is small and expand λ in powers of ν_T . (Actually, this should all be done in dimensionless form, but the argument would proceed in the same fashion.)

$$\lambda = \lambda_0 + \nu_T \lambda_1 + \nu_T^2 \lambda_2 + --- \tag{20}$$

Then the nondissipative roots are

$$\lambda_{(c)_{1,2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{21}$$

And the first order (in ν_T) corrections are given by

$$2a\,\lambda_{(1)}\lambda_{(0)} + b\,\lambda_{(1)} + f(\lambda_{(0)}) = 0$$

or

$$\lambda_{(1)_{\frac{1}{2}}} = \frac{-f(\lambda_{(0)_{\frac{1}{2}}})}{2a\,\lambda_{(0)_{\frac{1}{2}}} + b} = \frac{\mp f(\lambda_{(0)_{\frac{1}{2}}})}{\sqrt{b^2-4ac}} \qquad (22)$$

Then

$$\lambda_{\frac{1}{2}} \cong \frac{-b \pm \sqrt{b^2-4ac} \mp \dfrac{2a\nu_T f(\lambda_{(0)_{\frac{1}{2}}})}{\sqrt{b^2-4ac}}}{2a}$$

$$\cong \frac{-b \pm \sqrt{b^2-4ac-4a\nu_T f(\lambda_{(0)_{\frac{1}{2}}})}}{2a} \qquad (23)$$

We see that the ν_T term enters just as it would if it were lumped with the c term and given the argument $\lambda_{(o)}$. 'Now we simply give $\lambda_{(o)}$ its values at cut-off inside the function $f(\lambda_{(o)})$, since these are the only points where the ν_T term enters significantly.

$$\lambda_{\frac{1}{2}} \cong \frac{-b \pm \sqrt{b^2 - 4ac - 4a\nu_r f\left(\frac{-b}{2a}\right)}}{2a} \tag{24}$$

When we apply this recipe to eq. (18) we find

$$\lambda_{\frac{1}{2}} \cong \frac{-M_0 \frac{\omega}{a_0} \pm \sqrt{\frac{M_0^2\omega^2}{a_0^2} - (1-M_0^2)\left(\mu_{mq}^2 - \frac{\omega^2}{a_0^2} - \nu_T f(\lambda_0)\right)}}{(1-M_0^2)} \tag{25}$$

where

$$f(\lambda_0) = \frac{i}{a_0^2}\left(\omega + \frac{U_0 M \omega}{a_0(1-M_0^2)}\right)\left(\mu_{mq}^2 + \frac{M_0^2\omega^2}{a_0^2(1-M_0^2)^2}\right) \tag{26}$$

or

$$f(\lambda_0) = \frac{i}{a_0^2}\,\omega\left(\frac{1}{1-M_0^2}\right)\left(\mu_{mq}^2 + \frac{M_0^2}{a_0^2(1-M_0^2)^2}\right) \tag{27}$$

and the radicand combines into the form

$$\frac{\omega^2}{a_0^2} - (1-M_0^2)\mu_{mq}^2 + \frac{i\nu_T\omega}{a_0^2}\left(\mu_{mq}^2 + \frac{M_0^2\omega^2}{a_0^2(1-M_0^2)^2}\right) \tag{28}$$

Thus our final expression for the roots λ_1, λ_2 becomes

$$\lambda_{\substack{1\\2}} \cong \frac{\dfrac{-M_o\omega}{a_o} \pm \sqrt{\dfrac{\omega^2}{a_o^2} - (1-M_o^2)\mu_{mq}^2 + \dfrac{i\nu_T\omega}{a_o^2}\left(\mu_{mq}^2 + \dfrac{M_o^2\omega^2}{a_o^2(1-M_o^2)^2}\right)}}{(1-M_o^2)} \tag{29}$$

And, at cut-off $\omega = \pm\omega_{pq}$, the difference $(\lambda_1 - \lambda_2)$ becomes

$$\frac{2}{1-M_o^2}\left\{\frac{i\nu_T}{a_o}\ \mathrm{sgn}(\omega)\mu_{mq}^3\left(\frac{1}{1-M_o^2}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{30}$$

Since $\lambda_1 \lambda_2$ are always complex, one of them will be in the upper half plane and the other will be in the lower half plane. In order to avoid confusion we simply call the upper root λ_{up} and the lower, λ_{lo}.

Note that the form (29) makes it easy to locate λ_1 and λ_2 in the upper or lower half planes, depending upon the value of ω. In fact, this form, in the limit $\nu_T \to 0^+$ provides the best way (from a physical standpoint) to evaluate the pole-locations with respect to the real λ axis in any residue calculations for the nondissipative case.

Then we may write the expression for the radiated pressure field (See eq. 10) in the form

$$p(x,r,\theta,t) = \frac{2\pi}{(2\pi)^3(1-M_o^2)} \sum_{m=-\infty}^{\infty} \sum_{q=1}^{\infty} \int_{-\infty}^{\infty} d\omega \;``\int_{-\infty}^{\infty}" d\lambda \int_{-\infty}^{\infty} dt_o$$

$$\int_0^{2\pi} d\theta_o \int_{-\infty}^{\infty} dx_o \int_{r_{in}}^{r_{out}} dr_o \; f(x_o, r_o, \theta_o, t_o)$$

$$(\lambda_{up} - \lambda_{lo})^{-1} \left(H(x-x_o)\,\delta(\lambda-\lambda_{up}) + H(x_o - x)\,\delta(\lambda - \lambda_{lo}) \right)$$

$$e^{im(\theta-\theta_o)}\, e^{-i\omega(t-t_o)}\, e^{i\lambda(x-x_o)}\; R_{mq}(r)$$

$$R_{mq}(r_o)\left(\lambda r_o \sin\varepsilon(r_o) + m\cos\varepsilon(r_o)\right) \tag{31}$$

Here, the notation

$$``\int_{\lambda=-\infty}^{\infty}" d\lambda \left(H(x-x_o)\,\delta(\lambda-\lambda_{up}) + H(x_o-x)\,\delta(\lambda-\lambda_{lo}) \right)$$

is used to signify that all terms involving λ are evaluated at $\begin{pmatrix} \lambda = \lambda_{up} \\ \lambda = \lambda_{lo} \end{pmatrix}$, when $\begin{pmatrix} x > x_o \\ x < x_o \end{pmatrix}$ respectively.

If we concentrate the blade forces onto rotating lifting lines located in the plane $x_o = 0$, then

$$f(x_o, r_o, \theta_o, t_o) = \sum_{\nu=1}^{N} f_\nu(r_o, t_o)\, \delta(x_o)$$

$$\frac{1}{r_o}\, \delta\left(\theta_o - \frac{2\pi\nu}{N} - \Omega t_o\right)$$

(32)

where $f_\nu\,(r_o, t_o)$ is the (radius-and time-dependent) force per unit span on the ν^{th} blade and N is the number of blades. Then eq. (31) takes the form

$$p(x, r, \theta, t) = \frac{1}{(2\pi)^2 (1 - M_o^2)} \sum_{m=-\infty}^{\infty} \sum_{q=1}^{\infty} \sum_{\nu=1}^{N} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} dt_o$$

$$\int_{r_{in}}^{r_{out}} dr_o \quad r_o^{-1}\, f_\nu(r_o, t_o)\, (\lambda_{up} - \lambda_{lo})^{-1}$$

$$\left(H(x)\, \delta(\lambda - \lambda_{up}) + H(-x)\, \delta(\lambda - \lambda_{lo})\right)\, e^{i\lambda x}$$

$$e^{i\omega(t - t_o)}\, e^{im\left(\theta - \Omega t_o - \frac{2\pi\nu}{N}\right)}$$

$$R_{mq}(r)\, R_{mq}(r_o)\, \left(\lambda r_o \sin \varepsilon(r_o)\right.$$

$$\left. + m \cos \varepsilon(r_o)\right) \qquad (33)$$

Here λ_{up} and λ_{lo} are simply computed from the λ_1, λ_2 of eq. (29), depending upon which lies in the (upper, lower) half plane of complex λ.

III. COMPUTATION OF WAVE-NUMBER DEPENDENCE OF DISSIPATION OR EDDY VISCOSITY

If we consider plane waves propagating at frequency ω in a quiescent ($U_0 = 0$) free field, in the presence of a simple eddy viscosity ν_T, we find for an "incident" wave $p_0 e^{i(kx-\omega t)}$ (referring to eq. (5) with $U_0=0$, $\vec{f} = \vec{0}$)

$$\left(-k^2 + \frac{\omega^2}{a_0^2} + \frac{\nu_T}{a_0^2} k^2 i\omega\right) p_0 = 0 \tag{34}$$

or

$$k^2 = \frac{\omega^2/a_0^2}{1 - \frac{i\omega\nu_T}{a_0^2}} \cong \frac{\omega^2}{a_0^2}\left(1 + \frac{i\omega\nu_T}{a_0^2}\right)$$

$$\text{for} \quad \frac{\nu_T\omega}{a_0^2} \ll 1 \tag{35}$$

Hence

$$k_{im} = \mathrm{Im}(k) \cong \frac{\omega}{a_0}\frac{\omega}{2a_0^2}\nu_T = \frac{k_0^2\nu_T}{2a_0}$$

where

$$k_0^2 = \frac{\omega^2}{a_0^2} \tag{36}$$

Hence the wave amplitude exhibits an exponential attenuation as it propagates in the direction of positive x. This is of the form,

$$-\frac{k_0^2\nu_T x}{2a_0} \tag{37}$$

Now J. Keller[1] considers a wave propagating in a random medium whose scalar index of refraction is given by the expression

$$(1 + 2\mu + \mu^2) \tag{38}$$

Here μ is a centered random isotropic process with μ considered small relative to unity, and only weakly time-dependent relative to the time dependence (ωt) in the "incident signal." He finds that the medium can be characterized in terms of an effective wave number k which is complex. Moreover, the imaginary part of k is shown to be given by

$$k_{im} = \frac{(2\pi)^2}{2} k_0^2 \langle \mu^2 \rangle \int_0^{2k_0} k' \Phi_\mu (k') dk' \tag{39}$$

where $\langle \mu^2 \rangle$ is the mean square value or intensity of μ , and Φ_μ is its power spectrum, and isotropy is assumed for μ.

$$\Phi_\mu (k) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \langle \mu^2 \rangle^{-1} \langle \mu(\vec{r}) \mu(\vec{r} + \delta\vec{r}) \rangle e^{i\vec{k}\cdot\delta\vec{r}} \, d\delta\vec{r} \tag{40}$$

Translating Keller's k_{im} into an equivalent eddy viscosity, via eq. (36) leads to

$$\nu_T = \frac{2a_o k_{1m}}{k_o^2} = (2\pi)^2 a_o \langle \mu^2 \rangle \int_0^{2k_o} k' \Phi_\mu (k') dk' \tag{41}$$

Thus we find an equivalent eddy viscosity which depends upon the power spectrum of the random index of refraction at all wave numbers less than twice that of the undistorted incident wave.

Now G. K. Batchelor[2] considers single or Born scattering of an incident plane wave in a medium with random fluctuations in density and sound speed. If we temporarily supress the actual relationship between $\frac{\delta \rho}{\rho_o}$ and $\frac{\delta a}{a_o}$ and consider only the random fluctuation $\frac{\delta a}{a_o}$ in sound speed, Batchelor's result can be interpreted as follows: For a stochastic system satisfying the equation,

$$\Delta p - \frac{1}{a_o^2} p_{tt} + \frac{2\delta a_o}{a_o} \frac{p_{tt}}{a_o^2} = 0 , \tag{42}$$

with $\delta a / a_o$ given as a centered, random process, an incident wave $e^{i(\vec{k}_o \cdot \vec{x} - \omega t)}$ gives rise to a Born or single scattering cross section $\sigma(\vec{l})$, in the direction $\vec{l}$, equal to

$$\sigma(\vec{l}) = 2\pi k_o^4 \Phi_{\frac{\delta a}{a_o}} (\vec{k}_o - k_o \vec{l}) \left\langle \left(\frac{\delta a}{a_o}\right)^2 \right\rangle . \tag{43}$$

Here

$\vec{l}$ is a unit vector,

$\Phi_{\delta a / a_o}$ is the power spectrum of the fluctuations $\frac{\delta a}{a_o}$

and $\sigma(\vec{l})$ is the scattered energy flux per unit volume per unit solid angle at direction $\vec{l}$, normalized by the incident energy flux per unit area.

Thus, since only the 2μ term in Keller's index of refraction (38) contributed to k_{im} (39), we should find a closely related expression derivable from $\sigma(\vec{l})$ (43) with Batchelor's Φ_{sa/a_0} . playing the same role as Keller's Φ_{μ} .

The relation is derived as follows: The integral over the unit sphere of $\sigma(\vec{l})$

$$I = \iint \sigma(\vec{l})\, d\vec{l} \tag{44}$$

gives the energy scattered out in all directions, per unit volume, per unit incident energy flux. Thus we have an energy transport equation for the energy flux S .

$$\frac{dS}{dx} = -IS$$

$$S = S_{inc.} e^{-Ix} \tag{45}$$

But

$$S \propto e^{i(kx-\omega t)} e^{-i(kx-\omega t)} = e^{-2k_{im}x} \tag{46}$$

Therefore we should find

$$k_{im} = \frac{1}{2} I = \frac{1}{2} \iint \sigma(\vec{\ell})\, d\vec{\ell}$$

$$= \frac{1}{2} \left\langle \left(\frac{\delta a}{a_0} \right)^2 \right\rangle 2\pi k_0^4 \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \Phi_{\frac{\delta a}{a_0}} \left(2k_0 \sin\frac{\theta}{2} \right) \sin\theta \, d\theta \, d\varphi$$

(47)

where we now assume; as did Keller, that $\Phi_{\delta a/a_0}$ is isotropic and thus depends only on $|\vec{k}_0 - k_0\vec{\ell}\,| = 2k_0\sin\frac{\theta}{2}$. Here θ is the angle between incident and scattering directions, $\vec{k}_0$ and $\vec{\ell}$.

Integrating (47) with respect to φ and letting $2k_0\sin\frac{\theta}{2} = k'$, we find

$$k_{im} = \left\langle \left(\frac{\delta a}{a_0} \right)^2 \right\rangle \frac{(2\pi)^2 k_0^4}{2} \int_{k'=0}^{2k_0} \Phi_{\frac{\delta a}{a_0}}(k')\, dk' \, \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{k_0\cos\frac{\theta}{2}}$$

(48)

or

$$k_{Im} = \left\langle \left(\frac{\delta a}{a_0} \right)^2 \right\rangle \frac{(2\pi)^2}{2} k_0^2 \int_{k'=0}^{2k_0} k' \Phi_{\frac{\delta a}{a_0}}(k')\, dk' \qquad (49)$$

And this agrees exactly with Keller's result (39), with $\frac{\delta a}{a_o}$ playing the role of Keller's μ .

Now Batchelor[2] shows the relation between fluctuations in sound speed and those in density,

$$2\frac{\delta a}{a_o} = -\frac{\delta\rho}{\rho_o} \tag{50}$$

and gives a more complete form of the stochastic wave equation (42) which includes both effects simultaneously. The net result, after invoking single scattering, is to modify the scattering cross section $\sigma(\vec{l})$ from the relation given by (43) to the following:

$$\sigma(\vec{l}) = \frac{\pi}{2} k_o^4 \left\langle \left(\frac{\delta\rho}{\rho_o}\right)^2 \right\rangle \cos^2\theta \, \Phi_{\frac{\delta\rho}{\rho_o}}\left(\vec{k}_o - k_o\vec{l}\right) \tag{51}$$

The angle θ is, as before, the angle between $\vec{k}_o$ and $\vec{l}$. The spectrum $\Phi_{\delta\rho/\rho_o}$ is now that of the relative density variation. Repeating the preceding development for this new scattering cross section and again invoking isotropy, we find

$$k_{im} = \frac{\langle (\frac{\delta\rho}{\rho_0})^2 \rangle}{2} \frac{\pi}{2} k_0^4 \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \cos^2\theta \, \Phi_{\frac{\delta\rho}{\rho_0}} (2k_0 \sin\frac{\theta}{2}) \sin\theta \, d\theta \, d\varphi$$

$$= \langle (\frac{\delta\rho}{\rho_0})^2 \rangle \frac{\pi^2}{2} k_0^4 \int_{\theta=0}^{\pi} \Phi_{\frac{\delta\rho}{\rho_0}} (2k_0 \sin\frac{\theta}{2}) \cos^2\theta \, d\theta \sin\theta$$

$$= \langle (\frac{\delta\rho}{\rho_0})^2 \rangle \frac{\pi^2}{2} k_0^4 \int_{k'=0}^{2k_0} \Phi_{\frac{\delta\rho}{\rho_0}} (k') \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{k_0 \cos\frac{\theta}{2}} (1 - 2\sin^2\frac{\theta}{2})^2 dk'$$

$$= \langle (\frac{\delta\rho}{\rho_0})^2 \rangle \frac{\pi^2}{2} k_0^2 \int_{0}^{2k_0} k' \Phi_{\frac{\delta\rho}{\rho_0}} (k') (1 - \frac{k'^2}{2k_0^2})^2 dk'$$

(52)

Hence, finally, the equivalent eddy viscosity for isotropic fluctuations in density, with related fluctuations in sound speed, is given by

$$\nu_T = \frac{2a_0}{k_0^2} k_{im} = \langle (\frac{\delta\rho}{\rho_0})^2 \rangle a_0 \pi^2 \int_0^{2k_0} k' \Phi_{\frac{\delta\rho}{\rho_0}} (k')$$

$$(1 - \frac{k'^2}{2k_0^2})^2 dk'$$

(53)

And again, in this more complete case, the effective eddy viscosity depends on the power spectrum of $\delta\rho/\rho_0$ for all wave numbers less than twice that (k_0) of the incident wave.

The relation (53) for a wave-number dependent equivalent eddy viscosity holds for fluctuations in the related scalars, density and sound speed, in the background turbulent medium. A similar, but more complex treatment can be given for the single scattering approximation to fluctuation-velocity-induced dissipation. Batchelor[2] also treats this case and arrives at a scattering cross section

$\sigma(\vec{\ell})$ given by (in our notation)

$$\sigma(\vec{\ell}) = 2\pi \frac{\langle v'^2 \rangle k_0^2}{a_0^2} \cos^2\theta \, k_0^i k_0^j \, \Phi_{ij}(\vec{k}_0 - k_0\vec{\ell}) \tag{54}$$

where the notation is as before, with the additional quantities

$\langle v'^2 \rangle$ = mean square fluctuation velocity (in one direction)

Φ_{ij} = fluctuation velocity power spectral tensor

We integrate σ over the unit sphere (over $\vec{\ell}$) as previously, to find I for an isotropic velocity fluctuation field in the following sequence of steps:

$$\Phi_{ij}(\vec{k}) = \frac{E(k)}{4\pi k^4}\left(k^2 \delta_{ij} - k^i k^j\right) \tag{55}$$

Here E(k) is the normalized energy spectrum. Then $k_o^i\, k_o^j\, \Phi_{ij}\,(\vec{k_o} - k_o \vec{\ell}\,)$ is given by

$$\frac{E(2k_o \sin\frac{\theta}{2})}{4\pi(2k_o \sin\frac{\theta}{2})^4}\left((2k_o \sin\tfrac{\theta}{2})^2 k_o^2 - (\vec{k_o}\cdot[\vec{k_o} - k_o\vec{\ell}\,])^2\right)$$

$$= \frac{E(2k_o \sin\frac{\theta}{2})}{4\pi(2k_o \sin\frac{\theta}{2})^4}\left((2k_o \sin\tfrac{\theta}{2})^2 k_o^2 - (k_o^2 - k_o^2 \cos\theta)^2\right)$$

$$= \frac{E(2k_o \sin\frac{\theta}{2})}{4\pi(2k_o \sin\frac{\theta}{2})^4}\left((2k_o \sin\tfrac{\theta}{2})^2 k_o^2 - (2k_o^2 \sin^2\tfrac{\theta}{2})^2\right)$$

$$= \frac{k_o^2\, E(2k_o \sin\frac{\theta}{2})}{4\pi(2k_o \sin\frac{\theta}{2})^2}\left(1 - \sin^2\frac{\theta}{2}\right) \tag{56}$$

Thus

$$I = \frac{2\pi \langle v'^2 \rangle k_0^4}{4\pi a_0^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \; \frac{\sin\theta \cos^2\theta \cos^2\frac{\theta}{2}}{\left(2k_0 \sin\frac{\theta}{2}\right)^2} \; E\left(2k_0 \sin\frac{\theta}{2}\right)$$

$$= \frac{(2\pi)^2 \langle v'^2 \rangle k_0^4}{4\pi a_0^2} \int_0^{2k_0} dk' \, \frac{E(k') \, 2\sin\frac{\theta}{2} \cos^3\frac{\theta}{2}}{(k')^2 k_0 \cos\frac{\theta}{2}} \left(1 - 2\sin^2\frac{\theta}{2}\right)^2$$

$$= \frac{\pi \langle v'^2 \rangle k_0^3}{a_0^2} \int_0^{2k_0} dk' \, \frac{E(k') \, \frac{k'}{k_0}}{(k')^2} \left(1 - \frac{k'^2}{4k_0^2}\right)\left(1 - \frac{k'^2}{2k_0^2}\right)^2$$

(57)

Then using

$$k_{im} = \frac{I}{2} \qquad (47)$$

and

$$y_r = \frac{2a_0 k_{im}}{k_0^2} = \frac{a_0 I}{k_0^2}, \qquad (58)$$

we find, for the equivalent eddy viscosity resulting from velocity-induced (single) scattering losses,

$$\nu_T = \frac{\pi \langle v'^2 \rangle}{a_0} \int_0^{2k_0} \frac{E(k')}{k'} \left(1 - \frac{k'^2}{4k_0^2}\right)\left(1 - \frac{k'^2}{2k_0^2}\right)^2 dk'$$

(59)

Thus, once again, we have an equivalent eddy viscosity dependent upon the energy spectrum of the background fluctuations (in velocity, for the present case) for all wave numbers up to twice the incident wave number.

The actual eddy viscosity to be used in our expressions (e.g. eq. 29) for λ should include both contributions (59) and (53). If coupling exists between scalar $\delta\rho/\rho_0$ and vector $\delta\vec{u}$ fluctuations, then there should also be cross spectral terms in ν_T . However under isotropic assumptions if we invoke an additional assumption of nearly solenoidal velocity fluctuations, then velocity fluctuations are uncorrelated with scalar fluctuations

$$\langle \delta\rho \, \delta\vec{u} \rangle = 0 \qquad (60)$$

We invoke this approximation herein and merely add the contributions (53) and(59) to ν_T .

The foregoing has been developed for a quiescent free field. To find the value of k_0 appropriate for our application (eq. 29) we proceed as follows: First in fluid-fixed coordinates

(ξ, η, ζ, τ) the frequency ω_o is related to that ω in our (casing-fixed) (xyzt) coordinate system by the equations,

$$
\begin{aligned}
x &= \xi + U_o\tau \\
y &= \eta \\
z &= \zeta \\
t &= \tau
\end{aligned}
\tag{61}
$$

Therefore we have the equivalence,

$$e^{i(\lambda x - \omega t)} = e^{i(\lambda\xi + \lambda U_o\tau - \omega\tau)} \tag{62}$$

Hence the relation between ω_o and ω is simply,

$$\omega_o = \omega - \lambda U_o \tag{63}$$

and

$$k_o = \frac{\omega_o}{a_o} = \frac{\omega}{a_o} - \lambda M_o \tag{64}$$

At cut-off

$$\lambda = \frac{-M\omega/a_o}{1 - M_o^2} \tag{65}$$

and

$$k_0 = \frac{\omega}{a_0}\left(1 + \frac{M_0^2}{1-M_0^2}\right)$$

$$= \frac{\omega/a_0}{1-M_0^2} \tag{66}$$

This, or rather its absolute value, is the value of k_o to be used in applying eqs. (53) and (59) to the evaluation of λ_1 λ_2 or λ_{up} and λ_{lo} (eq. 29).

This result can be shown in a different manner. Our wave functions are of the form

$$\psi = e^{im\theta} R_{mq}(r) \tag{67}$$

in transverse (y,z) or (r,θ) planes. These satisfy an equation of the form

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} + \mu_{mq}^2\psi = 0 \tag{68}$$

or

$$\Delta_{yz}\psi + \mu_{mq}^2\psi = 0 \tag{69}$$

Thus our three dimensional modes or wave forms

$$\phi = \psi e^{i\lambda x} \tag{70}$$

satisfy the equation

$$\Delta_{xyz}\phi = -\mu_{mq}^2\phi - \lambda^2\phi \tag{71}$$

Hence each of the modes, from which our pressure expression is synthesized, has a wave number k_o satisfying

$$k_o^2 = \lambda^2 + \mu_{mq}^2 \tag{72}$$

But, from the first of eqs. (15) we have, (in the nondissipative case)

$$\begin{aligned} \mu_{mq}^2 + \lambda^2 &= M_o^2\lambda^2 + \frac{\omega^2}{a_o^2} - 2\frac{M_o\lambda\omega}{a_o} \\ &= \left(\frac{\omega}{a_o} - M_o\lambda\right)^2 \end{aligned} \tag{73}$$

This agrees with our expression (64) for the value k_o to be used in evaluating γ_T .

IV. CORRELATION $R_{\perp\perp}$ FOR BLADE-NORMAL VELOCITY FLUCTUATIONS

In forming output power spectra or related quantities from the expressions developed in sections II and III, it will be necessary to have available the space-time correlation of the forces

$$f_{\nu_1}(r_1,t_1)\ ,\quad f_{\nu_2}(r_2,t_2)$$

on blades ν_1, ν_2 at radii r_1, r_2 and at time instants t_1, t_2, respectively. In order to perform an actual computation of such correlations it would be necessary to invert the integral equation for three-dimensional, nonsteady compressible cascade aerodynamics under sinusoidal gust entry conditions. Since this is beyond the present state of the art, we proceed as follows: First, in the present section, we compute the space-time correlation $R_{\perp\perp}$ for blade-normal velocity components $u_\perp$ on one blade or on two distinct blades. Then, since the blade force fluctuations are induced by the inflow normal velocity fluctuations, we use the results for $R_{\perp\perp}$, or simplified approximations thereto, to infer certain facts about the expected general behavior of the required blade force correlations.

We refer to the blade-row geometry as illustrated in Fig. 1, and develop the correlation $R_{\perp\perp}$ in terms of the velocity correlation tensor R_{ij} of the inflow. Moreover, we assume that the inflow correlation tensor is isotropic in fluid-fixed coordinates and frozen in casing-fixed and blade-fixed coordinates. This means that, in fluid-fixed coordinates (ξ,η,ζ,τ), R_{ij} is given by (3)

$$R_{ij} = \left(f(\rho)-g(\rho)\right)\frac{\rho_i\rho_j}{\rho^2} + g(\rho)\,\delta_{ij}$$

with $$\rho = \sqrt{(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2}$$

and $$\rho^i = \vec{\rho} = (\xi-\xi_0,\ \eta-\eta_0,\ \zeta-\zeta_0)$$

(74)

R_{ij} is considered normalized, so that f and g are dimensionless, with unit values at $\rho = 0$.

$$f(0) = g(0) = 1 \tag{75}$$

Now

$$\begin{aligned} x &= \xi + U_0 \tau \\ y &= \eta \\ z &= \zeta \\ t &= \tau \end{aligned} \tag{76}$$

and thus for the ν th blade lifting line located in the plane x = 0,

$$\begin{aligned} \xi &= -U_0 t \\ \eta &= y = r\cos(\Omega t + 2\pi \nu/N) \\ \zeta &= z = r\sin(\Omega t + 2\pi \nu/N) \\ \tau &= t \end{aligned} \tag{77}$$

Therefore, for two blades ν_1 ν_2 and two different times t_1, t_2 and radii r_1, r_2, we find the following:

$$\vec{\rho} = \begin{Bmatrix} -U_0 t_d \\ r_1 c_1 - r_2 c_2 \\ r_1 s_1 - r_2 s_2 \end{Bmatrix} \tag{78}$$

where

$$(\ )_d = (\ )_1 - (\ )_2$$

$$\begin{pmatrix} C_1 \\ S_1 \end{pmatrix} = \begin{pmatrix} \cos \\ \sin \end{pmatrix} (\Omega t_1 + 2\pi \nu_1 / N)$$

$$\begin{pmatrix} C_2 \\ S_2 \end{pmatrix} = \begin{pmatrix} \cos \\ \sin \end{pmatrix} (\Omega t_2 + 2\pi \nu_2 / N) \tag{79}$$

Also

$$\rho = |\vec{\rho}| = U_c^2 t_d^2 + r_1^2 + r_2^2 - 2 r_1 r_2 \cos\Delta \tag{80}$$

where

$$\Delta = \Omega t_d + \frac{2\pi \nu_d}{N} \tag{81}$$

From Fig. 1 the normal fluctuation velocity component $u_\perp$ can be expressed as follows:

$$\begin{aligned} u_\perp &= u_x \sin\varepsilon + u_z \cos\varepsilon \cos(\Omega t + 2\pi\nu/N) \\ &\qquad - u_y \cos\varepsilon \sin(\Omega t + 2\pi\nu/N) \\ &= u_1 \sin\varepsilon + \cos\varepsilon \left[u_3 \cos(\Omega t + 2\pi\nu/N) \right. \\ &\qquad \left. - u_2 \sin(\Omega t + 2\pi\nu/N) \right] \end{aligned} \tag{82}$$

where

$\varepsilon = \varepsilon(v) =$ blade local twist angle as defined by reference to Fig. 1.

Since we are going to form a second order correlation and since (u_x, u_y, u_z) have zero mean values, therefore, it is unnecessary to correct for mean velocities Ωt and U_o.

Now we form $R_{\perp\perp}$ from eq. (82) evaluated at (r_1, t_1) on blade ν_1 and at (r_2, t_2) on blade ν_2.

$$\langle v'^2 \rangle R_{\perp\perp} = \langle v'^2 \rangle \Big\{ R_{11} \sin\varepsilon_1 \sin\varepsilon_2 + \cos\varepsilon_1 \cos\varepsilon_2 \big(R_{33} C_1 C_2 + R_{22} S_1 S_2 - R_{32} C_1 S_2 - R_{23} C_2 S_1 \big)$$

$$+ \sin\varepsilon_1 \cos\varepsilon_2 \big(R_{13} C_2 - R_{12} S_2 \big)$$

$$+ \sin\varepsilon_2 \cos\varepsilon_1 \big(R_{31} C_1 - R_{21} S_1 \big) \Big\} \tag{83}$$

Here the R_{ij} are given by (74) with $\vec{\rho}$ and ρ as given by (78) and (80), respectively. $\langle v'^2 \rangle$ is the (unidirectional) mean square velocity fluctuation.

Thus

$$R_{11} = (f(\rho) - g(\rho)) \frac{u_0^2 t_d^2}{\rho^2} + g(\rho)$$

$$R_{22} = (f(\rho) - g(\rho)) \frac{(r_1 c_1 - r_2 c_2)^2}{\rho^2} + g(\rho)$$

$$R_{33} = (f(\rho) - g(\rho)) \frac{(r_1 s_1 - r_2 s_2)^2}{\rho^2} + g(\rho)$$

$$R_{12} = R_{21} = (f(\rho) - g(\rho)) \frac{(-u_0 t_d)(r_1 c_1 - r_2 c_2)}{\rho^2}$$

$$R_{13} = R_{31} = (f(\rho) - g(\rho)) \frac{(-u_0 t_d)(r_1 s_1 - r_2 s_2)}{\rho^2}$$

$$R_{23} = R_{32} = (f(\rho) - g(\rho)) \frac{(r_1 c_1 - r_2 c_2)(r_1 s_1 - r_2 s_2)}{\rho^2} \tag{84}$$

Inserting the relations (84) into (83) we find

$$\begin{aligned} R_{\perp\perp} = {} & g(\rho)\{\sin\varepsilon_1 \sin\varepsilon_2 + \cos\Delta \cos\varepsilon_1 \cos\varepsilon_2\} \\ & + \frac{(f(\rho) - g(\rho))}{\rho^2}\Big\{\sin\varepsilon_1 \sin\varepsilon_2 u_0^2 t_d^2 + r_1 r_2 \cos\varepsilon_1 \cos\varepsilon_2 \sin^2\Delta \\ & - r_2 u_0 t_d \sin\varepsilon_2 \cos\varepsilon_1 \sin\Delta \\ & - r_1 u_0 t_d \cos\varepsilon_2 \sin\varepsilon_1 \sin\Delta\Big\} \end{aligned} \tag{85}$$

where ρ and Δ are given in eqs. (80) and (81). For zero or small twist variation

$$\varepsilon(r) = \varepsilon_1 = \varepsilon_2 = \varepsilon \tag{86}$$

and

$$R_{\perp\perp} = g\left(\sin^2\varepsilon + \cos\Delta\cos^2\varepsilon\right)$$
$$+ \frac{(f-g)}{\rho^2}\left(\sin^2\varepsilon\, U_0^2 t_d^2 + \cos^2\varepsilon\, v_1 v_2 \sin^2\Delta\right.$$
$$\left. - \sin\varepsilon\cos\varepsilon\sin\Delta\, U_0 t_d (v_1 + v_2)\right) \tag{87}$$

V. SIMPLIFIED OR APPROXIMATE FORMS FOR $R_{\perp\perp}$

Utilizing the notion

$$
\begin{aligned}
(\ )_d &= (\ )_1 - (\ )_2 \\
(\ )_m &= \tfrac{1}{2}(\ )_1 + \tfrac{1}{2}(\ )_2 \\
(\ )_1 &= (\ )_m + \tfrac{1}{2}(\ )_d \\
(\ )_2 &= (\ )_m - \tfrac{1}{2}(\ )_d
\end{aligned}
\tag{88}
$$

we may rewrite eq. (80) for ρ^2 in the form

$$
\begin{aligned}
\rho^2 &= U_0^2 t_d^2 + r_d^2 + 2 r_1 r_2 (1-\cos\Delta) \\
&= U_0^2 t_d^2 + r_d^2 + 4\left(r_m^2 - \frac{r_d^2}{4}\right)\sin^2\frac{\Delta}{2}
\end{aligned}
\tag{89}
$$

If we assume that $f(\rho)$ $g(\rho)$ decay sufficiently fast with ρ that we may neglect terms in ρ^2 of higher than second order in difference quantities $(\ )d$, then we may approximate ρ^2 by the expression,

$$
\begin{aligned}
\rho^2 &\cong U_0^2 t_d^2 + r_d^2 + 4 r_m^2 \sin^2\frac{\Delta}{2} \\
&\cong U_0^2 t_d^2 + r_d^2 + r_m^2 \Delta^2
\end{aligned}
\tag{90}
$$

lso the constant- ε form (87) for $R_{\perp\perp}$ can be approximated by

$$R_{\perp\perp} \cong g(\rho) + \frac{f(\rho)-g(\rho)}{\rho^2}\left(\sin^2\varepsilon\, U_o^2 t_d^2 + \cos^2\varepsilon\, r_m^2 \sin^2\Delta - 2\sin\varepsilon\cos\varepsilon\, U_o t_d r_m\right)$$

$$\cong g(\rho) + \frac{f(\rho)-g(\rho)}{\rho^2}\left(-U_o t_d \sin\varepsilon + r_m \cos\varepsilon \sin\Delta\right)^2$$

$$\cong g(\rho) + \frac{f(\rho)-g(\rho)}{\rho^2}\left(-U_o t_d \sin\varepsilon + \cos\varepsilon\, r_m \Delta\right)^2$$

(91)

Now we can decompose $\vec{\rho}$ into 3 components, $\rho_\perp$, $\rho_\parallel$, and ρ_d as follows:

$$\rho_d^2 = (r_1 - r_2)^2 = r_d^2 \quad \text{(radial)}$$

$$\left.\begin{aligned} \rho_\perp^2 &= \left(-U_o t_d \sin\varepsilon + 2 r_m \sin\tfrac{\Delta}{2}\cos\varepsilon\right)^2 \\ &\cong \left(-U_o t_d \sin\varepsilon + r_m \Delta \cos\varepsilon\right)^2 \end{aligned}\right\} \quad \text{(blade-normal)}$$

(92)

$$\left.\begin{aligned} \rho_\parallel &= \left(-U_o t_d \cos\varepsilon - 2 r_m \sin\tfrac{\Delta}{2}\sin\varepsilon\right)^2 \\ &\cong \left(-U_o t_d \cos\varepsilon - r_m \Delta \sin\varepsilon\right)^2 \end{aligned}\right\} \quad \text{(chordwise)}$$

Then ρ^2, as given by approximation (92) is equal to

$$\rho^2 = \rho_d^2 + \rho_\perp^2 + \rho_\parallel^2 \tag{93}$$

and

$$R_{\perp\perp} \cong g(\rho) + \left(f(\rho) - g(\rho)\right) \frac{\rho_\perp^2}{\rho^2}$$

$$\cong g(\rho) \frac{\left(\rho_d^2 + \rho_\parallel^2\right)}{\rho^2} + f(\rho) \frac{\rho_\perp^2}{\rho^2} \tag{94}$$

For solenoidal velocity fluctuations,(3)

$$g(\rho) = \frac{\rho f'(\rho)}{2} + f(\rho) \tag{95}$$

Hence, in this case

$$R_{\perp\perp} \cong f(\rho) + \frac{\rho}{2} f'(\rho) \left(1 - \frac{\rho_\perp^2}{\rho^2}\right)$$

$$= f(\rho) + \frac{f'(\rho)}{2\rho} \left(\rho_\parallel^2 + \rho_d^2\right) \tag{96}$$

For example, if $f(\rho)$ is approximately expressible by

$$f(\rho) = e^{-\rho/\Lambda} \tag{97}$$

and if the condition (95) is met,

Then

$$R_{LL} \cong e^{-\rho/\Lambda}\left(1 - \frac{\rho_{\parallel}^{2} + \rho_{d}^{2}}{2\rho\Lambda}\right) \tag{98}$$

where Λ is the integral scale for f. The correlation $e^{-\rho/\Lambda}$ leads to a k^{-2} type power law in the power spectrum E(k) and hence is not a bad approximation in the universal equilibrium range where $k^{-5/3}$ is the correct dependence. The simple exponential fails at $\rho = 0$ where it has a discontinuous slope.

VI. INFERRED BEHAVIOR OF BLADE FORCE CORRELATIONS

In accordance with the plan outlined at the start of Section IV, we use the simplified forms of $R_{\perp\perp}$ developed in Section V to infer something about the general behavior of the blade-force correlation in the absence of a full inversion of the integral equation for three-dimensional sinusoidal gust entry for a cascade in subsonic compressible flow.

We define the normalized correlation Ψ as follows

$$\langle f_{\nu_1}(r_1,t_1)\, f_{\nu_2}(r_2,t_2)\rangle = \langle f^2(r_1)\rangle^{1/2}\langle f^2(r_2)\rangle^{1/2} \Psi(r_1, r_2, \nu_1, \nu_2, t_1, t_2) \tag{99}$$

The forms given in eq. (94), together with the definitions (92) of $\rho_{\parallel}$ $\rho_{\perp}$ and ρ_d , are the most suggestive for purposes of inferring the general behavior of Ψ. First Ψ should obviously depend on r_1 and r_2 through both r_m and r_d, since $\rho_{\perp}$ and $\rho_{\parallel}$ exhibit this behavior. Second, Ψ should depend on t_1 and t_2 only through t_d, since t_m enters nowhere in ρ or $R_{\perp\perp}$ and since the boundary value problem for the nonsteady blade pressure distribution is invariant under translation in time. Thus Ψ is stationary (independent of t_m). Finally, Ψ should depend on blade numbers ν_1 and ν_2 only through the difference ν_d , since only ν_d (and not ν_m) enters into $R_{\perp\perp}$ (through Δ) and since the boundary value problem for pressure is also invariant under translation in θ . Hence we expect

$$\Psi = \Psi(r_m, r_d, \Delta, (-U_0 t_d)) \tag{100}$$

and Δ , in turn, depends on ν_d and t_d . Thus Ψ should be stationary in time and homogeneous in blade number, but not in radius.

In order to invoke the best available estimate of a wave-number dependent transfer function between upwash and blade lift per unit span, we revert temporarily to a two-dimensional cascade geometry and utilize the single-wing transfer function developed by Filotas[4] for airfoil response in a gusty atmosphere, together with the inter-blade cascade effects developed by Lane Friedman[5] for subsonic compressible oscillatory aerodynamics. The Filotas analysis treats three dimensional sinusoidal gust entry in terms of the effective inflow wave number amplitude and direction. The resulting transfer function relates lift per unit span to upwash as follows:

$$\frac{\text{Lift}}{\text{unit span}} = 2\pi \rho_0 b_0 U_0 w(0,0,0,t)\, T$$

Here $w(0,0,0,t)$ is the inflow upwash at midchord at time t. Then T is given as

$$T = \frac{e^{-ik_0 b_0\left(\sin\beta - \frac{\pi\beta\left(1+\frac{1}{2}\cos\beta\right)}{1+2\pi k_0 b_0\left(1+\frac{1}{2}\cos\beta\right)}\right)}}{\left(1+\pi k_0 b_0\left(1+\sin^2\beta+\pi k_0 b_0 \cos\beta\right)\right)^{1/2}} \qquad (101)$$

Here k_o is the magnitude of the blade-plane component of inflow turbulent (Fourier component) wave-number vector $\vec{k}_o$ and β is the angle between $\vec{k}_o$ and the blade mid-chord line. If we express the turbulent inflow velocity $u^i(\vec{x})$ in the usual Fourier-Stieltjes form

$$u^i(\vec{x}) = \iiint e^{i\vec{k}\cdot\vec{x}}\, dZ^i(\vec{k})$$

(102)

then for each Fourier component, $dZ^i(\vec{k})$, $\vec{k}_o$ is given by

$$\vec{k}_o = \vec{k} - \vec{N}(\vec{N}\cdot\vec{k})$$

(103)

where $\vec{N}$ is the unit normal to the blade plane. Then the angle β is the angle between $\vec{k}_o$ and the blade mid-chord line.

The interblade dependence of the transfer function may then be estimated by use of the oscillatory aerodynamic influence coefficients developed by Lane and Friedman[5] using an interblade phase lag angle σ which is expressible as

$$\sigma = \vec{k}\cdot\vec{\ell}$$

(104)

Here $\vec{\ell}$ is the vector from the mid-chord point in the n^{th} blade to the mid-chord point on the $(n+1)^{th}$ blade, in a plane normal to the blade mid-chord lines.

In this way the best available estimate of wave-number dependence and blade-to-blade influence can be incorporated into the blade force correlations (or the power spectrum thereof) for use in subsequent radiated-field power spectral estimates.

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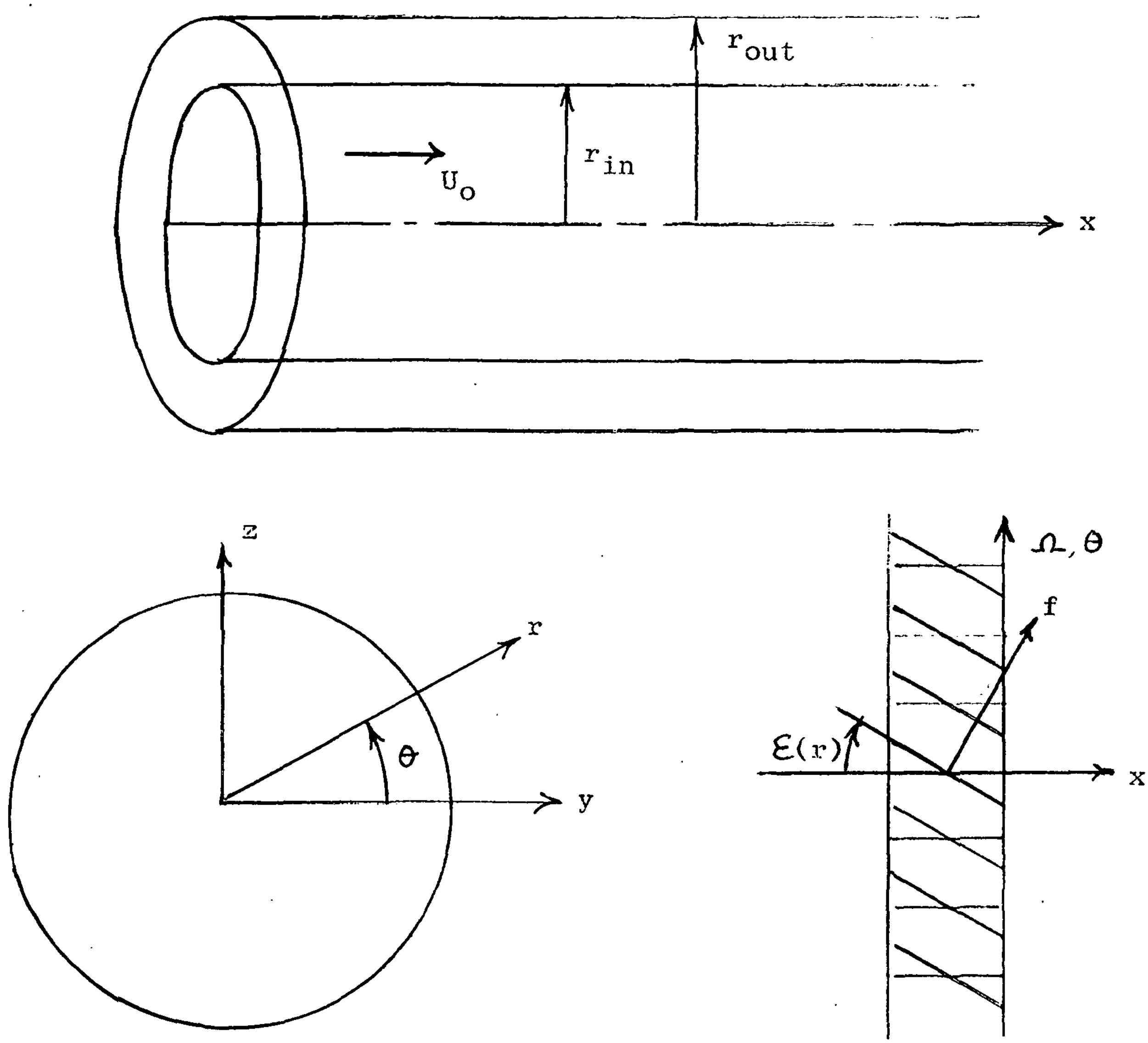

Figure 1

Geometry

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