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16. Abstract

A function integral equation governing the unsteady motion of a supersonic cascade is derived. Various representations of the Kernal function are derived and discussed.

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ON THE KERNEL FUNCTION FOR THE UNSTEADY SUPERSONIC CASCADE WITH SUBSONIC LEADING EDGE LOCUS

by M. E. Goldstein

INTRODUCTION

In current aircraft engine technology there has been considerable interest in the problem of the unsteady supersonic cascade with subsonic axial velocity. Thus we consider a two-dimensional oscillating cascade with a subsonic leading edge locus in a supersonic flow which is uniform far upstream. We suppose that the blades have small thickness and camber and are undergoing small amplitude harmonic oscillations. Kurosaka[1] has obtained a low frequency analytical solution to this problem and Verdon[2] has obtained a finite difference solution.

In this note we reduce the problem to the solution of a functional integral equation and derive and compare two representations of the kernel function which are useful for computations.

DERIVATION OF INTEGRAL EQUATION

Consider a two-dimensional oscillating cascade with a subsonic leading edge locus in a supersonic flow which is uniform far upstream. The blades which are assumed to have small thickness and camber, are undergoing small amplitude harmonic oscillations. We suppose that all lengths are nondimensionalized by the half-blade chord, $c/2$, the time t is nondimensionalized with respect to $c/2$ divided by the free stream velocity U, the pressure fluctuation p is nondimensionalized by the free-stream density ρ_0 times U^2, the upwash velocity v is nondimensionalized by U and a denotes the free stream speed of sound. Then the pressure fluctuation is governed by the equation (see fig. 1)

$$\frac{\partial^2\Psi}{\partial y^2} - \beta^2 \frac{\partial^2\Psi}{\partial x^2} - \beta^2 k^2 \Psi = 0 \tag{1}$$

where

$$\Psi \equiv pe^{i(\omega t - Mkx)} \tag{2}$$

$$\beta^2 = M^2 - 1$$

$$k = \omega M/\beta^2$$

$M = U/a$ is the free stream Mach number and the upwash velocity is related to the pressure by

$$e^{i\beta^2 kx/M} \frac{\partial}{\partial x}\left(e^{-i\beta^2 kx/M} V\right) = -e^{iMkx} \frac{\partial\Psi}{\partial y} \tag{3}$$

where

$$V = ve^{i\omega t}$$

The upwash velocity on the n^{th} blade is assumed to differ from that of the 0^{th} blade by only a phase factor so that

$$V(x + ns^{\dagger}, ns) = e^{in\sigma}V(x,0) \qquad \text{for } |x| < 1,\ n = 0, \pm 1, \pm 2, \ldots \tag{4}$$

where σ is the interblade phase angle and the upwash velocity on the 0^{th} blade is related to its displacement $W_0 e^{-i\omega t}$ by

$$V(x,0) = \left(-i\omega + \frac{\partial}{\partial x}\right) W_0(x) \qquad \text{for } |x| < 1$$

As is usual we suppose for convenience that the frequency has a small positive imaginary part which we shall set equal to zero at the end of the analysis. Then

$$k = k_r + i\varepsilon \qquad \text{with } 0 < \varepsilon << 1$$

and the outgoing wave boundary condition at infinity is now replaced by a boundedness condition.

Since equation (1) possesses the separation of variables solution

$e^{-i(\alpha x-\beta\gamma y)}$

where

$\gamma \equiv \sqrt{\alpha^2 - k^2}$

the boundary condition (4) suggests that (following Lane and Friedman[3]) we seek a solution in the form of the superposition

$$\Psi = \sum_{n=-\infty}^{\infty} \Psi_n \tag{5}$$

where

$$\Psi_n \equiv \frac{\operatorname{sgn} y_n}{2} \int_{-\infty+i\delta}^{\infty+i\delta} f_n(\alpha) e^{-i(\alpha x_n - \beta\gamma|y_n|)}\, d\alpha \quad , \tag{6}$$

we have put (see fig. 1)

$$\left.\begin{aligned} x_n &= x - ns^{\dagger} \\ y_n &= y - ns \end{aligned}\right\} \qquad n = 0, \pm 1, \pm 2, \; . \; . \; .$$

and in order to insure that the solution remain bounded at infinity we have chosen the branch cut for the square root, γ, and the integration contour, $\alpha_r + i\delta$, in the manner shown in figure 2 (with $\delta > \varepsilon$). This solution possesses the jump discontinuity

$$[\Psi(x)] = [\Psi_n(x)] = \int_{-\infty+i\delta}^{\infty+i\delta} f_n(\alpha) e^{-i\alpha x_n}\, d\alpha \tag{7}$$

across the line $y = ns$ passing through the n^{th} blade since it is only possible to satisfy the requirement that the upwash velocity be continuous by allowing a discontinuity in the pressure. The resulting pressure discontinuity (in front of and behind the blade will be eliminated in the subsequent analysis.

Since the upwash velocity, v, vanishes at infinity, equation (3) can be integrated to obtain

$$V = -e^{i\beta^2 kx/M} \int_{-\infty}^{x} e^{ikx'/M} \frac{\partial \Psi}{\partial y} (x',y)dx'$$

Inserting equations (5) and (6) and carrying out the integration now shows that

$$V = \frac{1}{2i} e^{iMkx} \frac{\partial}{\partial y} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{M}{M\alpha - k} \sum_{n=-\infty}^{\infty} (\mathrm{sgn}\ y_n)\ f_n(\alpha) e^{-i(\alpha x_n - \gamma\beta |y_n|)}\ d\alpha \quad (8)$$

If we put

$$f_n(\alpha) = e^{in\Gamma} f_0(\alpha)$$

where

$$\Gamma = \sigma - Mks^+ \quad (9)$$

It is easy to show from equation (8) that

$$V(x + ns^+,\ y + ns) = e^{in\sigma} V(x,y)$$

Hence the boundary condition (4) is automatically satisfied and

$$V = \frac{1}{i2} \frac{\partial}{\partial y} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{Mf_0(\alpha)}{M\alpha - k} \sum_{n=-\infty}^{\infty} (\mathrm{sgn}\ y_n)\ e^{i[n\sigma - (\alpha - Mk)x_n + \beta\gamma |y_n|]}\ d\alpha \quad (10)$$

On the other hand since $[\Psi(x)] = 0$ for $|x| > 1$ we can invert the Fourier transform in equation (7) (with $n = 0$) to obtain

$$f_0(\alpha) = \frac{1}{2\pi} \int_{-1}^{1} [\Psi] e^{i\alpha x}\ dx$$

Upon inserting this into equation (10) and interchanging the order of integration, we obtain

$$V(x,y) = \int_{-1}^{1} K(x - x',y)[P(x')]dx' \tag{10}$$

where

$$P \equiv pe^{i\omega t} = \psi\, e^{iMkx} \tag{11}$$

and

$$K(x,y) \equiv \frac{M}{i4\pi}\frac{\partial}{\partial y}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{1}{M\alpha - k} \sum_{n=-\infty}^{\infty} (\mathrm{sgn}\ y_n)\, e^{i[n\sigma-(\alpha-Mk)x_n+\beta\gamma|y_n|]}\, d\alpha \tag{12}$$

By letting $y \to 0$ we obtain an integral equation for the pressure ium [P] across the zeroth blade in terms of the known upwash velocity on the blade surface. Namely,

$$V(x,0) = \int_{-1}^{1} K_0(x - x')[P(x')]dx' \tag{13}$$

where

$$K_0(x) \equiv \lim_{y\to 0} K(x,y) \tag{14}$$

EXPRESSIONS FOR THE KERNEL FUNCTION

The form (12) and (14) for the kernel function is not suitable for numerical evaluation because the integral will not converge if we just put $\delta = \varepsilon = 0$ in the integrand. In order to carry out this limit it is convenient to express the kernel in a different form.

Form 1. - Using the results of appendix A shows that

$$K_0(x) = \frac{\beta}{2}\sum_{n=-\infty}^{\infty} e^{i(n\sigma+Mkx_n)}\,\delta(x_n - \beta s|n|)$$

$$-\frac{\beta k}{2}\sum_{n=-\infty}^{\infty} H(x_n - \beta|ns|)e^{i[n\sigma+(\beta^2k/M)x_n]}\left\{e^{(ik/M)x_n}\left[\frac{x_n J_1(k\sqrt{x_n^2-\beta^2n^2s^2})}{\sqrt{x_n^2-\beta^2n^2s^2}}\right.\right.$$

$$\left.\left.+\frac{i}{M}J_0\left(k\sqrt{x_n^2-\beta^2n^2s^2}\right)\right] - \frac{k}{M^2\beta}\beta^2\int_{\beta|ns|}^{x_n} e^{i(k/M)\xi}\,J_0(k\sqrt{\xi^2-\beta^2n^2s^2})d\xi\right\} \quad (15)$$

Because of the Heaviside function, H, the sum only need to be carried out until the largest n for which $ns^{\dagger} + \beta s|n| < x$. For a subsonic leading edge $s^{\dagger} > s\beta$ hence for $|x| < 1$ no terms with n greater than $1/(s^{\dagger} + \beta s)$ can occur in the sum.

Form 2. - Since $\mathcal{I}m(\alpha - Mk) = 0$ and $\mathcal{I}m\,\gamma > 0$ for $\delta = M\varepsilon$ $(-\infty < \alpha_r < \infty)$ it follows that

$$\left|e^{i[(\alpha-Mk)ns^{\dagger}+\beta\gamma|y_n|]}\right| < 1$$

and we can use the geometric series

$$\sum_{n=0}^{\infty} Z^n = \frac{1}{1-Z}$$

to evaluate the sum in the integrand of equation (12) to obtai

$0 < y < s$

$$\sum_{n=-\infty}^{\infty} e^{i[n\sigma+ns^{\dagger}(\alpha-Mk)+\beta\gamma|y_n|]} = \frac{e^{i\beta\gamma y}}{1-e^{-2i\Delta_-}} + \frac{e^{-i\beta\gamma y+2i\Delta_+}}{1-e^{2i\Delta_+}}$$

$$= \frac{1}{2i}\left[\frac{e^{i(\Delta_-+\beta\gamma y)}}{\sin \Delta_-} - \frac{e^{i(\Delta_+-\beta\gamma y)}}{\sin \Delta_+}\right]$$

where

$$\Delta_{\pm} \cong \frac{1}{2}\,[\sigma - Mks^{\dagger} + \alpha s^{\dagger} \pm \beta\gamma s] \tag{16}$$

Hence

$$K_0(x) = -\frac{1}{8\pi}\lim_{y\to 0}\frac{\partial}{\partial y}\int_{-\infty+i\varepsilon M}^{\infty+i\varepsilon M} \frac{e^{-i(\alpha-Mk)x}}{\left(\alpha-\frac{k}{M}\right)} \times \left[\frac{e^{i(\Delta_-+\beta\gamma y)}}{\sin \Delta_-} + \frac{e^{i(\Delta_+-\beta\gamma y)}}{\sin \Delta_+}\right]d\alpha \tag{17}$$

At first glance it might appear that the integrand in this expression possesses branch points due to the appearance of the radical γ. However, it can easily be verified by replacing γ by $-\gamma$ that this function depends only on γ^2 and the branch points are therefore "canceled," and the integrand possesses only poles.

We can therefore use Jordan's lemma to evaluate the integral in terms of its residues. To this end notice that the poles of the integrand occur at $\alpha = k/M$ and at the points where

$\Delta_{\pm} = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots.$

But it follows from equation (16) that the latter points are given by

$$\alpha_n^{\pm} = \Gamma_n \frac{s^{\dagger}}{d^{2}} \pm \frac{s\beta}{d^{\dagger}}\sqrt{\left(\frac{\Gamma_n}{d^{\dagger}}\right)^2 - k^2} \tag{18}$$

where we have put

$$\Gamma_n \equiv -\sigma + Mks^{\dagger} + 2n\pi \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \tag{19}$$

and

$$d^{\dagger} \equiv \sqrt{s^{\dagger 2} - \beta^2 s^2} \tag{20}$$

Notice that $d^{\dagger}$ is real for the subsonic leading edge. The + sign corresponds to the roots which lie in the upper half plane and the − sign to those in the lower half plane. The locus of roots in the complex α-plane is shown in figure 3.

When $x < 0$ we must close the contour in the upper half plane and when $x > 0$ in the lower half plane. Hence

$$K_0(x) = \begin{cases} K^+(x) & x < 0 \\ K^-(x) & x > 0 \end{cases} \tag{21}$$

where

$$K^{\pm}(x) \equiv \pm \sum \text{Res} \quad \text{in} \begin{pmatrix} \text{upper} \\ \text{lower} \end{pmatrix} \text{half plane}$$

Then upon using the results of appendix B to evaluate the residues we find that

$$K^+(x) = \frac{1}{2i} \lim_{y \to 0} \frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty} \frac{(\Gamma_n - \alpha_n^+ s^{\dagger})\, e^{-i[(\alpha_n^+ - Mk)x + (\Gamma_n - \alpha_n^+ s^{\dagger})\frac{y}{s}]}}{(\alpha_n^+ - \frac{k}{M})\ (s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n^+)} \tag{22}$$

and

$$K^-(x) = \frac{\omega}{2} \frac{\sinh(\omega s)\, e^{i\omega x}}{\cosh(\omega s) - \cos(\sigma - s^{\dagger}\omega)}$$

$$- \frac{1}{2i} \lim_{y \to 0} \frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty} \frac{(\Gamma_n - \alpha_n^- s^{\dagger})\, e^{-i[(\alpha_n^- - Mk)x + (\Gamma_n - \alpha_n^- s^{\dagger})\frac{y}{s}]}}{(\alpha_n^- - \frac{k}{M})\ (s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n^-)} \tag{23}$$

These series are only conditionally convergent and will not converge at all if we take the derivative. In order to obtain convergent series notice that

$$\alpha_n^{\pm} \sim \frac{\Gamma_n}{s^{\dagger} \mp \beta s} \mp \frac{k^2 s\beta}{2\Gamma_n} + O(n^{-2}) \qquad \text{as} \quad n \to \infty$$

Hence the n^{th} term of these sums behaves like

$$\frac{e^{-i\Gamma_n\left(\frac{x\mp\beta y}{s^{\dagger}\mp\beta s}\right)}}{\Gamma_n} \qquad \text{as} \quad n \to \infty$$

The series composed of these terms will converge to a row of step functions. Hence its derivatives will converge to a row of delta function. We can evaluate the latter series by using the theory of distributions to show that (Lighthill[4])

$$\lim_{y\to 0} \frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma_n} e^{-i\frac{\Gamma_n}{s^{\dagger}\mp\beta s}(x\mp\beta y)}$$

$$= \pm\frac{i\beta}{s^{\dagger} \mp \beta s} e^{i\left[\frac{\sigma-Mks^{\dagger}}{s^{\dagger}\mp\beta s}\right]x} \sum_{n=-\infty}^{\infty} e^{-\frac{2in\pi x}{s^{\dagger}\mp\beta s}}$$

$$= \pm\, i\beta e^{i\left[\frac{\sigma-Mks^{\dagger}}{s^{\dagger}\mp\beta s}\right]x} \sum_{n=-\infty}^{\infty} \delta[x - n(s^{\dagger} \mp \beta s)]$$

$$= \pm i\beta \sum_{n=-\infty}^{\infty} e^{in(\sigma-Mks^{\dagger})}\, \delta(x_n \pm \beta sn)$$

Hence,

$$K^{\pm}(x) = \tilde{K}^{\pm} + \frac{\beta}{2} \sum_{n=-\infty}^{\infty} e^{i(n\sigma + Mkx_n)} \delta(x_n \pm \beta sn) \tag{24}$$

where

$$\tilde{K}^{+} \equiv -\frac{e^{iMkx}}{2s} \sum_{n=-\infty}^{\infty} \left[\frac{(\Gamma_n - \alpha_n^{+} s^{\dagger})^2 e^{-i\alpha_n^{+} x}}{(\alpha_n^{+} - \frac{k}{M})(s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n)} + \frac{s\beta e^{-i\Gamma_n x/(s^{\dagger} - \beta s)}}{s^{\dagger} - \beta s} \right] \tag{25}$$

and

$$\tilde{K}^{-} \equiv \frac{\omega \sinh(\omega s)\, e^{i\omega x}}{2[\cosh(\omega s) - \cos(\sigma - s^{\dagger}\omega)]}$$

$$+ \frac{e^{iMkx}}{2s} \sum_{n=-\infty}^{\infty} \left[\frac{(\Gamma_n - \alpha_n^{-} s^{\dagger})^2 e^{-i\alpha_n^{-} x}}{(\alpha_n^{-} - \frac{k}{M})(s^{\dagger}\Gamma_n - d^{\dagger 2}\bar{\alpha}_n)} - \frac{s\beta e^{-i\Gamma_n x/(s^{\dagger} + \beta s)}}{s^{\dagger} + \beta s} \right] \tag{26}$$

are now convergent series.

COMPARISON OF KERNEL FUNCTION REPRESENTATIONS

When the kernel function given by equations (21), (25), and (26) or that given by equation (15) is substituted into the integral equation (13) we obtain a functional integral equation (and not an ordinary integral equation) due to the introduction of terms of the form

$[P(x_n + |n|s\beta)]$

caused by the integration of the delta functions. The second form of the kernel function has two advantages over the first. Namely its series converge like e^{inx}/n rather than like $e^{inx}/\sqrt{n}$ and it is much simpler and faster to evaluate the terms of its series than evaluating Bessel functions.

The series in (25) and (26) are only conditionally convergent. However, it is shown in appendix C that the same device which was used to make the original converge can also be used to replace the series (25) and (26) by absolutely convergent series. Although the results, which are given by equations (C1) and (C2) are now more complicated, they are definitely more suitable for numerical computation. They show that the removal of the slowly convergent part of the series results in a row of step functions which represent the discontinuities of $\tilde{K}^{\pm}$. The series which appear in (C1) and (C2) represent continuous function.

APPENDIX A

It follows from reference 5 that

$$\frac{i}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{e^{i(a\gamma-b\alpha)}}{\gamma}\,d\alpha = J_0(k\sqrt{b^2 - a^2})H(b \quad - a \ )$$

where a and b are real numbers, $\gamma = \sqrt{\alpha^2 - k^2}$ and

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Hence

$$-\frac{i}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \gamma e^{i(a\gamma-b\alpha)}\,d\alpha = \left[\frac{\partial^2}{\partial a^2} J_0(k\sqrt{b^2 - a^2}\right]H(b \quad - a \ ) - \frac{d\delta(b \quad - a \ )}{da_r} - \frac{k^2 a}{2}\,\delta(b \quad - a \ )$$

$$-\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{e^{i[a\gamma+(c-\alpha)b]}}{c - \alpha}\,d\alpha = H(b \quad - a \ )\int_a^b \left[e^{icb}\,\frac{\partial^2}{\partial a^2} J_0(k\sqrt{b^2 - a^2})\right]db + e^{icb}\delta(b \quad - a \ ) - \left(\frac{k^2 a}{2} + ic\right)e^{ica}H(b \quad - a \ )$$

where $c = c_r + i(\epsilon/M)$ (see fig. 2). But using the identity

$$\frac{\partial^2}{\partial a^2} J_0\left(k\sqrt{b^2 - a^2}\right) = \frac{\partial^2}{\partial b^2} J_0\left(k\sqrt{b^2 - a^2}\right) + k^2 J_0(k\sqrt{b^2 - a^2})$$

shows that

$$
\begin{aligned}
-\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{\gamma e^{i[a\gamma+(c-\alpha)b]}}{c-\alpha}\,d\alpha = H(b \quad - a \;)\Bigg[-e^{icb}\Bigg\{ & \frac{kbJ_1\left(k\sqrt{b^2-a^2}\right)}{\sqrt{b^2-a^2}} \\
& + icJ_0\left(k\sqrt{b^2-a^2}\right)\Bigg\} \\
& + \left(k^2-c^2\right)\int_a^b e^{icb}J_0\left(k\sqrt{b^2-a^2}\right)db\Bigg] \\
& + e^{icb}\delta(b \quad - a\;)
\end{aligned}
$$

APPENDIX B

Put

$$I^{\pm}(\alpha) \equiv \frac{e^{-i(\alpha-MK)(x)}}{\left(\alpha - \frac{k}{M}\right)} \frac{e^{i(\Delta_{\pm} \mp \beta\gamma y)}}{\sin \Delta_{\pm}} \tag{B1}$$

Then since

$$\frac{\partial \sin \Delta_{\pm}}{\partial \alpha} = \frac{1}{2}\left(s^{\dagger} \pm \frac{s\beta}{\gamma}\alpha\right)\cos \Delta_{\pm} = \frac{1}{2\gamma}(\gamma s^{\dagger} \pm s\beta\alpha)\cos \Delta_{\pm}$$

and since equation (16) shows that

$$\pm(\Gamma_n - \alpha_n s^{\dagger}) = \beta s\gamma(\alpha_n) \equiv \beta s\gamma_n$$

if α_n is a root of $\Delta_{\pm} = n\pi$, it follows that whenever α is in the neighborhood of a root α_n of $\Delta_{\pm} = n\pi$

$$\sin \Delta_{\pm} \sim \frac{(-1)^n}{2\gamma_n}(\gamma_n s^{\dagger} \pm s\beta\alpha_n)(\alpha - \alpha_n) = \frac{(-1)^n(\alpha - \alpha_n)}{2}\left[\frac{s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n}{\Gamma_n - \alpha_n s^{\dagger}}\right]$$

Hence

$$\lim_{\alpha \to \alpha_n} (\alpha - \alpha_n) I^{\pm}(\alpha) = \frac{2(\Gamma_n - \alpha_n s^{\dagger})e^{-i[(\alpha_n - Mk)x + (\Gamma_n - \alpha_n s^{\dagger})y/s]}}{\left(\alpha_n - \frac{k}{M}\right)\left(s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n\right)}$$

Since

$$\lim_{y \to 0} \frac{\partial}{\partial y}[I^{+}(\alpha) + I^{-}(\alpha)] = \frac{i\beta\gamma}{\left(\alpha - \frac{k}{M}\right)}\left[\frac{\sin(\Delta_{+} - \Delta_{-})}{\sin \Delta_{+} \sin \Delta_{-}}\right]e^{-i(\alpha - Mk)x}$$

$$= \frac{i\beta\gamma}{\alpha - \frac{k}{M}} \frac{2 \sin \beta\gamma s \; e^{-i(\alpha - Mk)x}}{\cos \beta\gamma s - \cos(\sigma - Mks^{\dagger} + \alpha s^{\dagger})}$$

it follows that the residue of the integrand at α_0 is

$$\frac{i\beta^2k^2}{M}\,\frac{\sinh(\beta^2ks/M)e^{i\beta^2kx/M}}{\cosh(\beta^2ks/M)-\cos(\sigma-\beta^2s^{\dagger}k/M)} = 2i\omega\,\frac{\sinh(\omega s)\,e^{i\omega x}}{\cosh(\omega s)-\cos(\sigma-s^{\dagger}\omega)}$$

APPENDIX C

Upon retaining terms of order n^{-1} we find that the n^{th} term of the sums in equations (22) and (23) now behaves like

$$\frac{1}{\Gamma_n}\left\{1+\frac{k(1-\delta_{n,0})}{2n\pi M}\left[s^{\dagger}\mp\beta s\left(1-\frac{iMkx}{2}\right)\right]\right\}\exp\left[\frac{-i\Gamma_n}{s^{\dagger}\mp\beta s}(x\mp\beta y)\right]\qquad \text{as} \quad n\to\infty$$

Hence we can improve the convergence of the sums in equations (25) and (26) by adding and subtracting

$$-\frac{\beta}{2(s^{\dagger}\mp\beta s)}\,e^{i\left[\frac{\sigma\mp Mk\beta s}{s^{\dagger}\mp\beta s}\right]x}\,\frac{k}{2M\pi}\left[s^{\dagger}\mp\beta s\left(1-\frac{iMk}{2}x\right)\right]\sum_{n=-\infty}^{\infty}\frac{(1-\delta_{n,0})}{n}\,e^{-\frac{2in\pi x}{s^{\dagger}\mp\beta s}}$$

$$=\frac{i\beta k}{4M}\left[\frac{s^{\dagger}\mp\beta s\left(1-\frac{iMkx}{2}\right)}{s^{\dagger}\mp\beta s}\right]e^{i\left[\frac{\sigma\mp\beta sMk}{s^{\dagger}\mp\beta s}\right]x}$$

$$\times\left\{1-\frac{2x}{s^{\dagger}\mp\beta s}+2\sum_{n=1}^{\infty}[H(x_n\pm\beta sn)-H(\mp\beta sn-x_n)]\right\}$$

to obtain

$$\tilde{K}^{+} = \frac{-e^{iMkx}}{2s} \sum_{n=-\infty}^{\infty} \left\{ \frac{(\Gamma_n - \alpha_n^{+} s^{\dagger})^2 e^{-i\alpha_n^{+} x}}{\left(\alpha_n^{+} - \frac{k}{M}\right)\left(s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n^{+}\right)} + \frac{s\beta e^{-\frac{i\Gamma_n x}{s^{\dagger}-\beta s}}}{s^{\dagger} - \beta s} \times \left[1 + \frac{k(1-\delta_{n,0})}{2n\pi M}\left(s^{\dagger} - \beta s\left(1 - \frac{iMkx}{2}\right)\right)\right] \right\}$$

$$- \frac{i\beta k}{4M}\left[\frac{s^{\dagger} - \beta s\left(1 - \frac{iMkx}{2}\right)}{s^{\dagger} - \beta s}\right] e^{i\left[\frac{\sigma - \beta s M k}{s^{\dagger} - \beta s}\right]x}$$

$$\times \left\{1 - \frac{2x}{s^{\dagger}-\beta s} + 2\sum_{n=1}^{\infty}\left[H(x_n + \beta s n) - H(-\beta s n - x_n)\right]\right\} \qquad \text{(C1)}$$

$$\tilde{K}^{-} = \frac{\omega \sinh(\omega s)\, e^{i\omega x}}{2[\cosh(\omega s) - \cos(\sigma - s^{\dagger}\omega)]} + \frac{e^{iMkx}}{2s}\sum_{n=-\infty}^{\infty}\left\{ \frac{(\Gamma_n - \alpha_n^{-} s^{\dagger})^2 e^{-i\alpha_n^{-} x}}{\left(\alpha_n^{-} - \frac{k}{M}\right)\left(s^{\dagger}\Gamma_n - d^{\dagger 2}\alpha_n^{-}\right)} \right.$$

$$\left. - \frac{s\beta e^{-\frac{i\Gamma_n x}{s^{+}+\beta s}}}{s^{+} + \beta s}\left[1 + \frac{k(1-\delta_{n,0})}{2n\pi M}\left(s^{\dagger} + \beta s\left(1 - \frac{iMkx}{2}\right)\right)\right]\right\}$$

$$- \frac{i\beta k}{4M}\left[\frac{s^{\dagger} + \beta s\left(1 - \frac{iMkx}{2}\right)}{s^{\dagger} + \beta s}\right] e^{i\left[\frac{\sigma+\beta s M k}{s^{\dagger}+\beta s}\right]x}$$

$$\times \left\{1 - \frac{2x}{s^{\dagger}+\beta s} + 2\sum_{n=1}^{\infty}\left[H(x_n - \beta s n) - H(\beta s n - x_n)\right]\right\} \qquad \text{(C2)}$$

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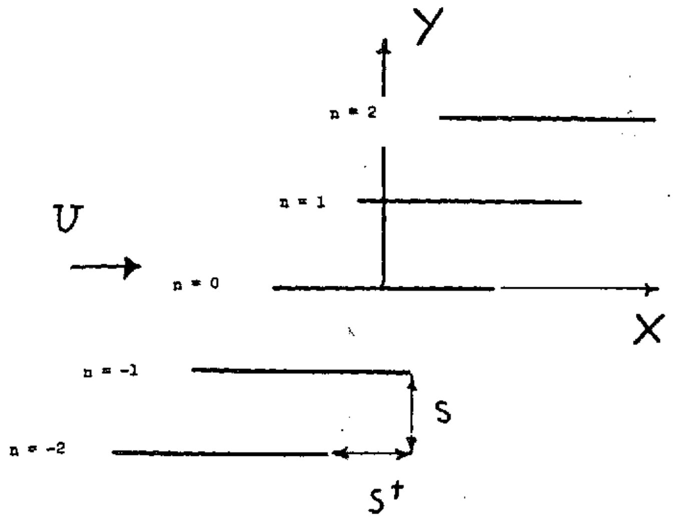

Fig. 1 - Dimensionless Cascade Configuration.

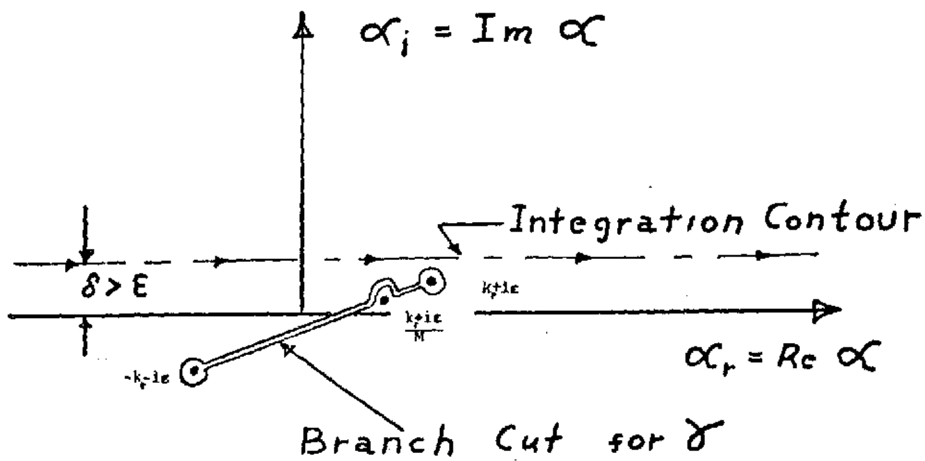

Fig. 2 - Integration Coutour and Branch Cut in Complex α- Plane.

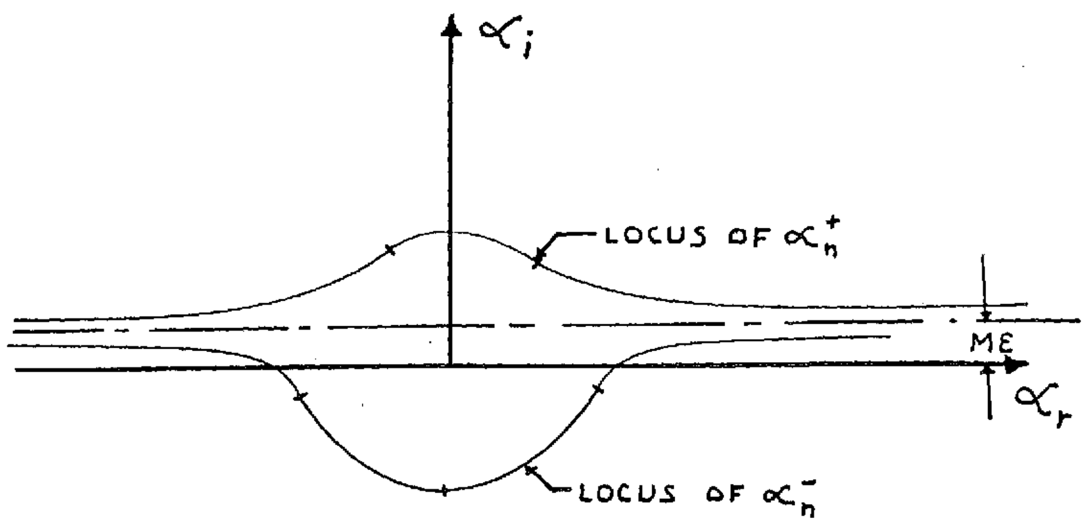

Fig. 3 - Approximate Locus of Roots in Complex α-Plane.