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FINAL TECHNICAL REPORT

RADAR OPTIMIZATION FOR SEA SURFACE AND GEODETIC MEASUREMENTS

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EFFICIENT GEOID AND SEA STATE ESTIMATION BY SATELLITE ALTIMETRY

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ABSTRACT

The efficient estimation of geoid and sea state parameters (waveheight standard deviation and asymmetry and scattering cross-section density) is considered: the optimum processing structures, including maximum likelihood estimators, and their accuracy limits are given for a model accounting for random surface reflectivity, sea height, and additive noise. and allowing for arbitrary radar system parameters, based on the assumption the received signal is a sample function of a normal random process.

The integral equation associated with the "Gaussian signal in Gaussian noise" inference problem was solved previously in "pulse-determined" and "threshold" cases and is here solved approximately in the quasi-stationary case of pulse resolution small relative to the returned signal's extent and exactly in a specific case. It is shown that the optimum processing is generally a mixture of coherent and incoherent integrations which may be viewed as a weighted summation of received power of the match-filtered received data. There is a signal-to-noise ratio dependent best pulse bandwidth; approximate and exact accuracy limits are given. The efficient estimates are generally correlated, the most strong coupling between geoid and asymmetry estimates and between wave height standard deviation and reflectivity estimates. A brief exemplification of system design usage is given.

INTRODUCTION

Radar altimetry has been proposed and is being used to obtain measurements, over the sea from a satellite, of geoid and sea state parameters which are expected to be highly useful in geodesy, oceanography, and meteorology both for scientific uses and applications such as traffic routing. The accuracy required of geoid measurement for "dynamic oceanography" is about 0.1 meter; since tides, currents, and wind-driven waves cause sea surface perturbations much greater than this and since the reflection from the sea surface is a random phenomena, the extraction of geoid information from the "smeared" and random returned pulse requires (at least implicitly) joint estimation of tides, currents, and sea state parameters. It does not appear possible for altimetry itself to separate tidal, current, and geoidal variations: exterior data must be provided for this. But sea state parameters do directly influence the nature of the altimeter's returned signal from the sea and must be estimated, at least implicitly, whether or not desired in themselves, in order to obtain a good geoid estimate.

We consider here the best single pulse joint measurement of the geoid parameter - the delay of the returned pulse from the mean sea level (which here inseparably includes tide and current changes) - and sea state parameters which are here taken as the radar scattering cross-section density, standard deviation of the wave-height distribution (which is assumed zero mean since any nonzero mean cannot be distinguished from the geoid by definition) and an asymmetry parameter accounting for the well-known asymmetry of ocean waves. Such measurements can then be combined with adjunct data - e.g., laser measurements of orbit - to estimate the geoid; there are well-known methods (e.g., least-squares, extended Kalman filtering) to do this and they are being so used. Since it appears that practical altimeter designs are possible that provide accurate measurements even on a single pulse, a linear observation model in a recursive estimation scheme may be attractive; a general statement of a recursive estimation problem appears to be very involved, hence the emphasis on optimizing a single pulse's estimates.

The present discussion is an extension of [1] where estimation of geoid only was considered. The introductory remarks of [1] with respect to theoretical context and desirability of efficient design apply here also and we adopt the model for the returned signal and the notation of [1]. In particular the signal random process covariance function in the symmetric case is

$$R_s(t_1, t_2) = K \int_0^\infty dy\, q(y) \int_{-\infty}^\infty dx\, p_h(x)\, F(t_1 - \tau_o - y - \frac{2x}{c})\, \bar{F}(t_2 - \tau_o - y - \frac{2x}{c}) \qquad (1)$$

where F is the complex modulation of the transmitted pulse, τ_o the delay to geoid, c the velocity of light, p_h the univariate probability density function of wave height which depends on the unknown sea-state parameter σ_h, $q(y) \equiv g_o(\sqrt{cR_o y})\, H(\sqrt{cR_o y})$ where g_o is the reflectivity density and H is the fourth power of the modulus of the far-field antenna pattern, both a function of viewing angle from the vertical, and $K \equiv 2\pi\, (k/4\pi R_o)^4 \;\; (cR_o/2)$ where $k \equiv 2\pi/\lambda$, λ the mean wavelength of the radiated pulse and $R_0 = c\tau_0/2$. It has been assumed that the reflectivity of the sea is a random field with a reflectivity density $g(x, y) = g_1(x, y)/\sec\gamma(x, y)$ where $\gamma(x, y)$ is the angle between the normal to the gross roughness surface of the sea and the altimeter boresight direction and $g_1(x, y)$ is a complex random function of zero mean and spatially incoherent - that is, its covariance function is of the form $g_0(x_1, y_1)$ $\delta(x_1-x_2, y_1-y_2)$. (See Fig. 1 for the geometry of the problem.)

For the narrow antenna patterns of interest here, $g_o(\sqrt{cR_o y}) \approx g_o(0) \equiv g_o$, generally an unknown parameter* that must be estimated along with the unknown geoid delay (τ_o) and sea state parameter σ_h. It is then convenient to define $q_1(y) = H(\sqrt{cR_o y})$ and rewrite Eq. 1 in the form

$$R_s(t_1, t_2) = g_o \int_{-\infty}^{\infty} f(\xi - \tau_o; \sigma_h^2) F(t_1 - \xi) \overline{F}(t_2 - \xi)\, d\xi \qquad \text{(1a)}$$

where the convolution

$$f(\xi - \tau_o; \sigma_h^2) \equiv \frac{Kc}{2} \int_0^{\infty} q_1(\eta)\, p_h\left[\frac{c}{2}(\xi - \tau_o - \eta)\right] d\eta \qquad \text{(2)}$$

Note the unknown parameters (τ_o, σ_h^2, g_o) enter via $g_o f(\xi - \tau_o; \sigma_h^2)$ - which is denoted more briefly $f_*(\xi)$.

As the received, scattered signal is modelled as a sample function of a nonstationary, zero mean, complex Gaussian random process - an assumption with some theoretical and experimental support - whose covariance function depends on geoid and sea state parameters, and is received along with additive, white thermal (Gaussian) noise, the likelihood ratio - necessary for statistical inference - involves the quadratic functional of the received data appropriate to the "Gaussian signal in Gaussian noise problem" [1, 2] having a kernel function determined by an integral equation that cannot, unfortunately, be solved in the generality desired here. Two special cases were solved in [1], the "threshold" case and "pulse determined" case, and those results are here extended to the present joint estimation problem: however, the pulse determined case would a priori seem to be possibly unattractive for sea state estimation.

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*The relation between g_o and σ_o, the radar scattering cross-section density, is $\sigma_o = \pi g_o / \lambda^2$.

We show here that in the threshold case the pulse resolution to best estimate sea state should be somewhat smaller than to best estimate geoid but still of the order of the received (smeared) pulse duration. To adopt this result is generally disquieting because there might well be a best "SNR per sample" (resolution element) analogously to the diversity communication problem, and if so, a SNR-dependent, possibly finer resolution would be dictated.

Therefore we are led to consider an approximation appropriate to the case when the pulse resolution is small relative to the received pulse duration as spread by the antenna pattern and wave height distribution. We are able to approximately solve the required integral equation in this "quasi-stationary" case, observe the structure of the optimum processor, and calculate bounds on estimation accuracy - which are attainable asymptotically by maximum likelihood estimation. As an example application to altimeter system design, there is shown to be a best pulse resolution for estimating geoid and sea state, dependent upon SNR. These results can be used for efficient altimeter system design as exemplified in [1].

Also the integral equation is solved exactly under a specific set of assumptions; the results generally agree with the results of the quasi-stationary approximations when the signal bandwidth (roughly-see below for more precise statement) is relatively large.

Since the received signal is modeled as a sample function of a random process, the better known theory appropriate to a signal known except for a set of parameters (which leads to the matched filter, etc.) is not applicable here. For comparison, the estimate of delay and spread for a non-random signal is discussed in the Appendix.

SUMMARY OF RESULTS

The data-dependent part of the log-likelihood ratio, which describes the operations on the data Z_t, $t \epsilon T$, performed by the receiver/processor is [1, 2] the quadratic functional

$$Q(Z) = \frac{1}{\eta} \int_T dt_1 \overline{Z(t_1)} \int_T dt_2 \, Z(t_2) \, k(t_1, t_2) \tag{3}$$

where the kernel $k(t_1, t_2)$ is the solution to the integral equation

$$\eta_o k(t_1, t_2) + \int_T R_s(t_1, t_3) \, k(t_3, t_2) \, dt_3 = R_s(t_1, t_2), \; t_1, t_2 \; \epsilon \; T. \tag{4}$$

T is the time during which the scattered pulse is received and will be assumed large relative to the support of R_s; η_o is the spectral density of the complex white noise.

Solutions for k. - We have already solved the integral equation (4) in [1] in two special cases. First, in the "threshold case" (see [1] for definition) of, roughly, low signal-to-noise ratio,

$$k(t_1, t_2) \approx \eta_0^{-1} R_s(t_1, t_2) \tag{4a}$$

which gives

$$Q(z) \approx \frac{K}{2\eta_0^2} q(-t) * p_h\left(-\frac{c}{2} t\right) * \left| \overline{F}(-t) * z(t) \right|^2 \Big|_{t=\tau_o} \tag{3a}$$

which can be calculated by the processor shown in Fig. 2. As discussed in [1] this processor performs both coherent and incoherent integration in an intuitively satisfying manner; since the scale of q involves g_o and p_h depends on σ_h (and the asymmetry parameter introduced below) these matched filters will depend on sea state parameter estimates.

Second, in the "pulse determined" case (see [1] for definition) where the radiated pulse has insufficient bandwidth to resolve the wave height and antenna spot smearing,

$$k(t_1, t_2) = k_o\, F(t_1-\tau_o)\, \overline{F}(t_2-\tau_o) \tag{4b}$$

where $k_o \equiv \alpha/(\eta_o + \alpha\mathcal{E}_F)$, $\mathcal{E}_F$ the energy of F and $\alpha \equiv K\int_o^\infty q(y)dy$. Now

$$Q(Z) \approx \frac{k_o}{\eta_o} \,|\, \overline{F}(-t)*Z(t)|^2 \tag{3b}$$

which, aside from scale, agrees with the results of the threshold case with $q(-t) \equiv p_h(-\frac{c}{2}\,t) \equiv \delta(t)$ - reasonably because the transmitted pulse does not "see" the waves and antenna spot in this case. The calculation of 3b is accomplished by the processor of Fig. 2 with the matched filters of impulse responses $q(-t)$ and $p_h(-ct/2)$ omitted. Such a processor is therefore independent of the sea state parameters (aside from g_o) which is an advantage if only geoid estimation is required.

We here solve this integral equation in two more cases. Thus, thirdly, in the "quasi-stationary" case where the extent of F (after matched filtering if dispersed modulation is employed by the transmitter) is small relative to the distance over which $q(t) * p_h(\frac{c}{2}\,t)$ varies significantly - that is, the gross waves and antenna spot are relatively well resolved - we find that $k(t_1, t_2)$ is approximable by a slowly varying impulse response $k_1(t_1, t_1-t_2)$. That is, for each t_1, in a region of the order of the transmitted pulse resolution, k looks like the impulse response of a linear, time-invariant filter. Its "quasi-stationary" transfer function is shown below to be approximately

$$\tilde{k}_1(t_1, \omega) = \frac{f_*(t_1)|\tilde{F}(\omega)|^2}{\eta_o + f_*(t_1)|\tilde{F}(\omega)|^2} \tag{4c}$$

and

$$Q(z) \approx \frac{1}{\eta_o} \int_T |Z(t)|^2 \cdot \frac{f_*(t)}{\eta_o/\beta + f_*(t)} \cdot dt \tag{3c}$$

which is a weighted sum of the received instantaneous power after matched filtering to the transmitted pulse. (Here $\beta \equiv 2\pi \mathcal{E}_F / \Omega_F$ where Ω_F is the bandwidth of F which, for the specific form (3c), was assumed constant over its spectrum.) The calculation of Q(z) is again accomplished by the processor configuration of Fig. 3 except the matched filters q and p_h are replaced by a filter of impulse response $f_*(t)/[\eta_o/\beta + f_*(t)]$. This form of processing has in fact been a starting point for practical designs [5]. A sketch of the weighting function $w(t) \equiv f_*(t) \cdot [\eta_o/\beta + f_*(t)]^{-1}$ for a crudely approximated f_* is shown in Fig. 3 ; exact calculations are easily made for a given f_*.

It should be noted that w(t) is relatively broad in its extent. The weighting function does depend on the sea state parameter estimates.

The first case overlaps the second and third cases which are mutually exclusive; taken together they lend great weight to using a processor of the an envelope detector and filtering (incoherent integration), the exact filter shape dependent on signal-to-noise ratio and sea state parameters. Such a processor is eminently practical also. Again if only geoid estimation is desired this filter may be omitted and satisfactory performance is practically achievable [1] without need for sea state parameter estimation.

Fourthly and finally, if it is assumed that the transmitted pulse spectrum is constant over its bandwidth and that $\tilde{q}\,\bar{\varphi}_h$ is the Fourier transform of a rational function, an integral equation equivalent to (4) can be solved exactly. The method of solution is well known and has been reduced to routine calculation which gets sufficiently involved to impede easy understanding. A relatively simple exponential form for $\tilde{q}\,\tilde{\varphi}_h$ was assumed and the calculations performed to find k exactly (Eq. 25 below). It can be expected in practice that the system parameters will be such (see below) that the resulting processor has the realization shown in Fig. 2 where the filters q and p_h are replaced by a filter transfer function

$$\tilde{k}(w) = \tilde{k}_o\, e^{-\hat{\beta}|\omega|}, \quad -\infty < \omega < \infty \quad . \tag{3d}$$

Note that its bandwidth is about $\hat{\beta}^{-1} = (\eta_o/2\alpha\beta\,\phi)^{1/2}$ where ϕ is a measure of the spread or smear of the returned pulse: this can be seen to be in agreement with the result under the quasi-stationary case.

<u>Maximum likelihood estimates.</u> - By definition the maximum likelihood estimates ('MLE's') are those values of the unknown parameters that maximize the likelihood ratio or, equivalently, the natural logarithm of the likelihood ratio which here has the form $[-\Phi + Q(z)]$; Φ is independent of the data and here can be shown [1] to be independent of geoid parameter τ_o but generally will depend on the sea state parameters.* Thus the maximum

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*Finding useful "closed" forms for Φ corresponds in difficulty to find k, the kernel of Q.

likelihood estimate of τ_o is easily seen to be found by choosing the time of maximum output of the low pass filter for any and all of the special cases discussed. Said filter must be adjusted over a trial set of sea state parameters to maximize the output: in principal - but not necessarily practically - this may be done "in parallel" on one pulse return.

Necessary conditions for the MLE's are found, of course, by setting the derivatives of the log-likelihood ratio with respect to each unknown parameter equal to zero. We show below that in the quasi-stationary case and when F has constant spectrum over its bandwidth Ω_F, that these conditions are

$$0 = \int_T dt \, \frac{\partial f_*(t)/\partial \theta_i}{[\eta_o/\beta + f_*(t)]^2} \cdot \left\{ \frac{|Z_t|^2}{\beta B_F} - [\eta_o/\beta + f_*(t)] \right\} . \tag{5}$$

These likelihood equations can have solutions for each stationary point including the true maximum and are therefore meaningful in suggesting processors useful after initial acquisitions to a close approximation of the estimates $\hat{\bar{\theta}}$ to the true values $\bar{\theta}_o \equiv (\theta_1, \theta_2, \theta_3, \theta_4) \equiv (\tau_o, \sigma_h, g_o, \theta_4)$, θ_4 the asymmetry parameter. The functions $v_i(t) \equiv (\partial f_* | \partial \theta_i) [\eta_o/\beta + f_*(t)]^{-2}$ may be regarded as weighting functions and are sketched in Fig. 4 for a crude approximation to a typical f_* when p_h is symmetric.

Such likelihood equations have often been used to suggest feedback structures to find estimates by operating on a sequence of received pulses and driving certain error signals to zero. For example, the form of $v_2(t)$ suggests the use of two time "gates", one with the weighting function equal to $v_2(t)$ for $-\sigma < t < o$ and the other with the weighting function equal to the

the negative of $v_2(t)$ for $o < t < \sigma + \phi$. Each gate output -weighted instantaneous power-is summed and the difference formed as an error signal which readjusts the position of the gating pair. The integral of $v_2(t)$ times $[\eta_o/\beta + f_*(t)]$ is computed separately and applied as a bias to the difference. Similarly for v_1 and v_3.

<u>Performance.</u> - It is well known [4] that the covariance matrix of efficient estimates - those unbiased estimates of minimum possible error variance which are, when they exist, MLE's and whose performance is achieved asymptotically by MLE's - is the inverse of the matrix C of elements $c_{ij} \equiv - E\{\partial^2 \ell n\, \Lambda(z)/\partial\theta_i \theta_j\}$ where $\Lambda(z)$ is the likelihood ratio and $\{\theta_i\}$ the unknown parameters to be estimated. Here the $\{c_{ij}\}$ can be expressed in an integral involving R_s and k (eq. 19 below, e. g.). For example, in the quasi-stationary case we have the approximation.

$$c_{ij} = \int \frac{\partial}{\partial\theta_i} f_*(t) \cdot \frac{\partial}{\partial\theta_j} f_*(t)\, dt$$

$$= \frac{1}{2\pi} \int \frac{\partial}{\partial\theta_i} \tilde{f}_*(\omega) \cdot \frac{\partial}{\partial\theta_j} \tilde{\bar{f}}_*(\omega)\, d\omega. \tag{6}$$

Here $\mathbb{C} \equiv B_F/(\eta_o/\beta + g_o f(o))$ and we have assumed $|\tilde{F}(\omega)|^2 = 2\pi \mathcal{E}_F/\Omega_F \equiv \beta$ for $|\omega| < \Omega_F/2$ and zero otherwise ($B_F \equiv \Omega_F/2\pi$); also we have made a certain approximation with regard to the general shape of f_* - see below. Here the Fourier transform of f_* is $\tilde{f}_*(\omega) = g_o K\, \tilde{q}_1(\omega)\varphi_h[\omega/(c/2)]\cdot \exp(-i\omega\tau_o)$, where φ_h is the univariate characteristic function (the Fourier transform of p_h) of the gross wave height.

An often used [4] approximation to the gross wave distribution is normal; but it is usually still better approximated by incorporating a parameter that accounts for the observed asymmetry. This is conveniently done by using a truncated Gram-Charlier series [3]:

$$\varphi_h(\xi) = e^{-\sigma_h^2\xi^2/2} \cdot (1 - i\theta_{4a}\xi^3)$$

when $\theta_{4a} \equiv E\{h^3\}/3$ is proportional to the (assumed small) third moment of the gross wave height distribution; note $\theta_{4a} \equiv 0$ if p_h is symmetric.

We can now compute the $\{c_{ij}\}$ and invert C to find the desired covariance matrix; its diagonal elements are the error variances of the efficient estimates $(\tau_o^*, \sigma^*, g_o^*, \theta_4^*)(\theta_4 \equiv \theta_{4a}/(c/2)^3,\ \sigma \equiv \sigma_h/(c/2))$:

$$\mathrm{Var}\{\tau_o^*\} = \frac{1}{\mathcal{C}_{g_o}^2} \cdot \frac{1}{1-\rho_{14}^2} \cdot \frac{1}{M(1)}, \tag{7a}$$

$$\mathrm{Var}\{\theta_4^*\} = \frac{1}{\mathcal{C}_{g_o}^2} \cdot \frac{1}{1-\rho_{14}^2} \cdot \frac{1}{M(3)}, \tag{7b}$$

$$\mathrm{Var}\{\sigma^*\} = \frac{1}{\mathcal{C}_{g_o}^2} \cdot \frac{1}{1-\rho_{23}^2} \cdot \frac{1}{\sigma^2 M(2)}, \tag{7c}$$

and

$$\mathrm{Var}\{g_o^*\} = \frac{1}{\mathcal{C}_{g_o}^2} \cdot \frac{1}{1-\rho_{23}^2} \cdot \frac{g_o^2}{M(0)}, \tag{7d}$$

where

$$\rho_{14}^2 \equiv \frac{M(2)^2}{M(1)M(3)}, \tag{7e}$$

$$\rho_{23}^2 \equiv \frac{M(1)^2}{M(0)M(2)}, \tag{7f}$$

and

$$M(k) \equiv \frac{1}{2\pi} \int \omega^{2k} |\tilde{f}(\omega)|^2 \, d\omega, \tag{7g}$$

$\tilde{f}(\omega) = K\tilde{g}_1(\omega)\varphi_h(\omega/(c/2))/_{\theta_{4a} \equiv 0}$. The correlation coefficient of the efficient estimates of geoid delay τ_o and asymmetry parameter θ_4 is ρ_{14}; the correlation coefficient of the efficient estimates of gross wave height standard deviation σ and sea reflectivity density g_o is ρ_{23}; all other correlations are zero to this first order (in θ_{4a}) approximation; also ρ_{14} and ρ_{23} are independent of θ_{4a} to first order.

Consider, for example, the error variance of the efficient estimate τ_o^* of the delay τ_o to the geoid. It is uncorrelated with the efficient estimate g_o^* of reflectivity density though, because additive noise is present, its magnitude depends on the true reflectivity density g_o: this is agreeable. It is also uncorrelated with the efficient estimate σ^* of the standard deviation σ of the gross wave height distribution - a second moment - though its magnitude depends on this measure of (symmetric) spread as is agreeable because greater spreading means greater returned energy which, if processed properly (as it is), should enhance the geoid estimate. Finally it is correlated with the efficient estimate of the asymmetry parameter: errors in estimating the asymmetry of the gross wave height distribution would be expected to lead to geoid location errors. Finally, we may write $M(1) \equiv \mathcal{E}_f^2 \Omega_f^2$ where Ω_f is interpretable as a radius of gyration measure of the bandwidth of f (provided the centroid of $|\tilde{f}|^2$ is zero): this too is agreeable - though not with the superficial thought that the bandwidth of F, the transmitted modulation, determines geoid measurement accuracy.

The correlation between the efficient estimates σ^* and g_o^* is also agreeable: increased total energy is scattered by increasing either σ or g_o, e.g.

More specifically, assume $|\tilde{q}(\omega)|^2 = |\tilde{q}(0)|^2 \exp(-\phi^2\omega^2)$ so that

$|\tilde{f}(\omega)|^2 = \alpha_1{}^2 \exp(-\omega^2/2\nu^2)$ where $\alpha_1 \equiv K|\tilde{q}_1(0)|$ and $\nu^2 \equiv [2(\sigma^2 + \phi^2)]^{-1}$. Then one calculates $\rho_{14}{}^2 = 9/15$ and $\rho_{23}{}^2 = 1/3$, leading to $(1-\rho_{14}{}^2)^{-1} = 15/6$ and $(1-\rho_{23}{}^2)^{-1} = 3/2$, these being the factors by which the various efficient estimates' error variances are increased by their being correlated. Or, to say it another way, the variance of the geoid delay efficient estimate (e.g.) is increased by 250% by the lack of knowledge of asymmetry. Further, after choosing an optimal pulse bandwidth as discussed below,

$$\mathrm{Var}\{\tau_o{}^*\} = \frac{10}{\pi\sqrt{2}} \left(\frac{\eta_o}{g_o \alpha_1 \mathcal{E}_F} \right) \frac{1}{\nu^2} \tag{8a}$$

$$\mathrm{Var}\{\theta_4{}^*\} = \frac{2}{3\pi\sqrt{2}} \left(\frac{\eta_o}{g_o \alpha_1 \mathcal{E}_F} \right) \frac{1}{\nu^{\cancel{2}6}} , \tag{8b}$$

$$\frac{\mathrm{Var}\{\sigma^*\}}{\sigma^2} = \frac{4}{\pi\sqrt{2}} \left(\frac{\eta_o}{g_o \alpha_1 \mathcal{E}_F} \right) \frac{1}{\nu^4 \sigma^4} , \tag{8c}$$

and

$$\frac{\mathrm{Var}\{g_o{}^*\}}{g_o{}^2} = \frac{6}{\pi\sqrt{2}} \left(\frac{\eta_o}{g_o \alpha_1 \mathcal{E}_F} \right) . \tag{8d}$$

Because of the aforementioned approximation to the shape of f_*, it was of interest to compute the $\{c_{ij}\}$ for a specific, reasonable f_* with a symmetric gross wave height distinction - still under the quasi-stationary approximation and the specific, reasonable assumption of the shape of $|\tilde{F}|$. The results are given below (eqs. 22), the result of tedious residue integrations; for comparison the approximate forms (Eq. 6) are also evaluated with results shown in Eq. 23. (Both 22 and 23 result after an optimum choice of signal bandwidth Ω_F - discussed below.) We see that agreement is very close.

Also derived below are the error variances of the efficient estimators of geoid delay τ_o and gross wave height spread σ_h^2 in the threshold case. For example, for a normal gross wave height distribution, $|\tilde{F}|$ constant over its spectrum, and $|\tilde{q}(u)|^2 = |\tilde{q}(0)\exp(-\phi_1 u^2/2)|^2$, then setting $\tilde{\nu}^2 \equiv 8\sigma_h^2/c^2 + \phi_1^2$ and $\hat{a} \equiv \nu\Omega_F$, we find

$$\mathrm{Var}\{\tau_o^*\} = \left(\frac{\eta_o\tilde{\nu}}{2g_o\alpha_1\mathcal{E}_F}\right)^2 \cdot \frac{1}{r_{11}(\hat{a})} \tag{9a}$$

and

$$\frac{\mathrm{Var}\{\sigma_h^2\}}{(c/2)^4} = \left(\frac{\eta_o\tilde{\nu}}{2g_o\alpha_1\mathcal{E}_F}\right)^2 \cdot \frac{1}{r_{22}(\hat{a})} \quad . \tag{9b}$$

The forms $r_{11}(\hat{a})$ and $r_{22}(\hat{a})$ are graphed in Fig. 6 as a function of $\hat{a}$. The efficient estimates τ_o^* and σ_h^{2*} are uncorrelated - as is proved true for any symmetric p_h.

Finally we recall from [1] the variance of the efficient geoid delay estimate τ_o^* in the pulse determined case is

$$\mathrm{Var}\{\tau_o^*\} = \frac{\eta_o(\eta_o + 2g_o\alpha_1\mathcal{E}_F)}{(2g_o\alpha_1\mathcal{E}_F)^2} \cdot \frac{1}{\Delta_F^2} \tag{10}$$

where Δ_F is the radius of gyration measure of the bandwidth of F. (When $|\tilde{F}|$ is constant over its support then $\Delta_F = \Omega_F/2\sqrt{3}$.)

III. APPLICATIONS

Modulation bandwidth. - One of the fundamentally interesting questions about system design is the choice of transmitted pulse bandwidth to minimize the estimation error variances.

Taking up first the quasi-stationary case where the bandwidth Ω_F of the pulse F is such that the "smeared" returned pulse f is highly resolved, we note - Eqs. 7 - that Ω_F enters only through $\mathbb{C}$ and each of the error variances are inversely proportional to $\mathbb{C}$: thus Ω_F should be chosen to maximize $\mathbb{C} = B_F/[\eta_o B_F/\mathcal{E}_F + g_o f(0)]$ where $B_F \equiv \Omega_F/2\pi$. It is easily seen that the maximizing B_F is

$$B_{F_1} = \frac{g_o f_o \mathcal{E}_F}{\eta_o} \tag{11a}$$

and

$$\max_{B_F} \mathbb{C} = \frac{\mathcal{E}_F}{4 g_o f_o \eta_o} \quad . \tag{12}$$

To interpret this result note that, from Eq. 1b, the mean power of the received signal at time t_1 (said mean power will slowly vary in this quasi-stationary case) is $\sigma_s^2(t_1) = R_s(t_1, t_1) = f_*(t_1)\,\mathcal{E}_F$, where $\mathcal{E}_F$ is the energy of the complex modulation (which is twice the energy of the real pulse modulation); also the mean power of the complex noise in the bandwidth $B_{F_{opt}}$ is $\sigma_n^2 = \eta_o B_{F_{opt}}$: our result states that B_F should be chosen so that the signal-to-noise ratio $\sigma_s^2(0)/\sigma_n^2$ is unity (recall we set $\tau_o \equiv 0$ for present purpose). Since f(t) generally will decrease as $|t|$ increases from 0 and since we nevertheless can only choose one B_F, we may expect that a more exact calculation would result in a somewhat smaller $B_{F_{opt}}$, corresponding to a ratio $\sigma_s^2(0)/\sigma_n^2$ somewhat greater than unity.

In fact this last is borne out by examining the results of the exact calculation, Eqs. 21e, 21f, 21g, for the symmetric case. Each variance depends on a different factor dependent on $a \equiv (1 + \tilde{a}/\Omega_F)^{1/2}$, only through which Ω_F enters: a can equivalently be chosen, then, to minimize (e.g.) one of these three forms-call them $\mathbb{C}_i(a)$, $i = 1, 2, 3$. It is easily found algebraically that a=2 minimizes both Var(σ^*) and Var(g_o^*) and $a=(3+\sqrt{17})/4 \approx 1.8$ minimizes Var(τ_o^*); further, a choice of a=2 increases Var(τ_o^*) by less than 2%. Thus the choice a=2 is a very good one; then $\tilde{a}/\Omega_F = (2\pi f_*(0)\mathcal{E}_F/\eta_o)/2\pi B_F = \sigma_s^2(0)/\sigma_n^2$ is seen to be (very nearly simultaneously) optimally 3.

This choice of a is allowable, provided, roughly, the signal-to-noise ratio is large enough so that the quasi-stationary approximation remains valid: for this we must have $1/\Omega_F << \theta$. Since $\tilde{\theta}/\Omega = 3$ (for a = 2), $1/\Omega = 3/\tilde{a} = (3/2)(\eta_o/\alpha\mathcal{E}_F)\theta$: thus, more precisely, we must have

$$\frac{\eta_o}{\alpha\mathcal{E}_F} << 1 .$$

As will be seen below, this condition is easily met in practice even for a satellite vehicle.

As will be seen in examples below the optimum choice of a can imply very large pulse modulation bandwidths. It is therefore of interest to see how the $\{\mathbb{C}_i(a)\}$ depend on a for a greater than the optimum a - that is, bandwidths less than the optimum bandwidth: this dependence is sketeched in Fig. 5. To conform to the quasi-stationary approximation only a's below a bound are permissible: that is, for a given $4\pi\alpha\mathcal{E}_F/\eta_0$, since $\rho_F/\phi << 1$, only values of $(4\pi\alpha\mathcal{E}_F/\eta_o)(\rho_F/\phi) << (4\pi\alpha\mathcal{E}_F/\eta_o)$ are allowable - which is to say only $a^2 << (4\pi\alpha\mathcal{E}_F/\eta_o) + 1$ are allowable. For example, in an application discussed below ($4\pi\alpha\mathcal{E}_F/\eta_o$) is greater than 10^6 so a's somewhat less than, say, 300 are allowable.

We see that the error variances increase by an order of magnitude as a increases by an order of magnitude: fortunately, roughly, $a \sim (\Omega_F)^{-1/2}$ so that an order of magnitude change in a corresponds to two orders of magnitude change in Ω_F.

The best choice of bandwidth Ω_F in the threshold case is also examined: Ω_F enters the error variances (Eqs. 9) only through $\gamma_{11}(\hat{a})$ and $\gamma_{22}(\hat{a})$, $\hat{a} \equiv \nu\Omega_F$, which are graphed in Fig. 6.

We note that the best choice of $\Omega_F = \hat{a}/\tilde{\nu}$ (fixed $\tilde{\nu}$) is $\Omega_F \approx 4/\tilde{\nu}$, degrading only a little the accuracy of the geoid estimate and would represent a reasonable compromise.

As $\tilde{\nu}^2 = 8\sigma_h^2/c^2 + \phi_1^2$, and σ_h^2 is unknown and to be estimated, we may have a problem unless $\phi_1^2 >> 8\sigma_h^2/c^2$ - that is, unless the antenna "spot" determines $\tilde{\nu}$. Since, from Fig. 1, too small a bandwidth results in serious degradation, we may wish to choose $\Omega_F \approx 4/\nu_{min}$, where ν_{min} is the minimum possible $\tilde{\nu}$ - very likely determined practically by the antenna spot (scaled into time and smeared by pointing errors) as the sea could be smooth ($\sigma_n^2 = 0$) occassionally. Presumeably generally better performance could be obtained by an adaptive system which utilized an initial estimates of σ_n to more optimally set Ω_F on the subsequent pulse.

System design. - We briefly indicate how the above results may be used in determining reasonable system parameters.

Recall (Eq. 22 of [1])

$$\alpha = \frac{A \sigma_o Q}{(4\pi)^3 R_o^2} \tag{13}$$

where A is the antenna aperture area, the scattering cross-section density $\sigma_o = \pi g_o/\lambda^2$ (a measured and tabulated parameter), and Q depends on aperture illumination function shape: e. g., Q = 1/6 for uniform illumination.

Also (App. III of I)

$$\phi_1 \approx \frac{\lambda^2 R_o}{Ac} \quad . \tag{14}$$

Since ϕ_1 decreases and α increases with increasing physical antenna aperture area A, clearly we want A as large as possible - limited by vehicle size and stability as discussed in [1].

Suppose certain measurement occuracies are required of geoid and sea state estimates. Rescaling to spatial distance instead of time extent by setting $\tau_o = R_o/(c/2)$ and $\sigma = \sigma_h/(c/2)$, we have

$$\mathrm{Var}\,(R_o^*) = (4.51)\frac{\eta_o}{\alpha \mathcal{E}_F}\left[1 + \left(\frac{\phi c/2}{\sigma_h}\right)^2\right]\sigma_h^2$$

and

$$\mathrm{Var}\,(\sigma_h^*) = (3.60)\frac{\eta_o}{\alpha \mathcal{E}_F}\left[1 + \left(\frac{\phi\, c/2}{\sigma_h}\right)^2\right]^2 \sigma_h^2 .$$

Assume parameters $\sigma_o = \pi$, $A = \pi^2 m^2$, $R_o = 10^6$m, and $Q = 1/6$: then we calculate $\alpha = (3.84 \times 10^{14})^{-1}$. If $\lambda = 2$ cm, then $\phi\, c/2 = 25.4$m. If $\eta_o/2 = kT_o = 1.4 \times 10^{-21}$, then $\eta_o/\alpha = 5.38 \times 10^{-7}$. Thus (Eqs. 8)

$$\mathrm{Var}\,(R_o^*) = 1.56 \times 10^{-3}/\mathcal{E}_F$$

and

$$\mathrm{Var}\,(\sigma_h^*) = 8.03 \times 10^{-1}/\mathcal{E}_F \sigma_h^2$$

provided that $\sigma_h^2 << (25.4)^2$, as would be true even in the roughest possible seas. If $\sqrt{\mathrm{Var}(R_o^*)} = 10^{-1}$ m. accuracy is specified then at least $\mathcal{E}_F = 1.56 \times 10^{-1}$ Joule is required, which is rather large; then $\sqrt{\mathrm{Var}(\sigma_h^*)} = 2.27/\sigma_h$ m. which of course will exceed any specified value for sufficiently small σ_h.

If B_F is chosen so that $\sigma_S^2(0)/\sigma_n^2 = 3$, then we find $B_F \approx 1$ KMHz., an impractically large value. But in order even to be in the fine resolution, or quasi-stationary case, the resolution $1/B_F$ must be small relative to $\phi = 0.17$ usec, implying $B_F >> 5.9$ MHz.

Suppose a more practical bandwidth of $B_F = 33$ MHz, which is a thirtieth of the optimum bandwidth: then it is easily calculated that now

$a = 10$. (We already saw that $a_{opt} = 2$ would be increased roughly by $\sqrt{30}$, near optimum a little less.) From Fig. 5 we see that now $\mathcal{C}_1(a) = 135$ and $\mathcal{C}_2(a) = 148$, increases from the minimum values of 18 and 27, resp., corresponding to increases in the error standard deviations by the factors $\sqrt{135}/18 \approx 2.74$ and $\sqrt{148}/27 \approx 2.34$, resp., which would have to be made up by changing other system parameters to meet the same specified accuracy. This calculation points out that the error standard deviations are relatively mildly dependent on pulse modulation bandwidth, within the quasi-stationary approximation.

It is of considerable interest to compare this achievable performance, $\mathrm{Var}(\tau_o^*) = (4.51)(\eta_o/\alpha\mathcal{E}_F)(\sigma^2 + \phi_1^2)$ with that achievable in the "pulse-determined" opposite extreme pulse bandwidth Ω_F where q is not resolved at all, namely ([1], Eq. 24.b) $\mathrm{Var}(\tau_o^*) = 6(\eta_o/\alpha\mathcal{E}_F)\Omega_F^{-2}$ where (see [1], Eqs. 9 and 10) Ω_F^{-2} is larger than $(\sigma^2 + \phi_1^2)$. Thus, set $\Omega_F^{-2} = \delta^2(\sigma^2 + \phi_1^2)$: a general conclusion of [1] was that, in the threshold case, $\delta = 4/3$ was optimum, resulting in a pulse that did not quite resolve the returned signal; adopting this δ, we see that in the pulse determined case the achievable accuracy is at best approximately $(24)\ (\eta_o/\alpha\mathcal{E}_F)(\sigma^2 + \phi_1^2)$.

Thus the best performance, measured by error standard deviation, in the fine resolution extreme is only about $\sqrt{24/4.51} \approx 2$ times better than that obtainable in the pulse determined case. Since the processing required in the latter case is markedly simpler and a priori determined, whenever only geoid estimation is desired the pulse determined case seems markedly preferable: such design examples were considered at length in [1].

IV. DERIVATIONS

Quasi-stationary case. - The data-dependent part of the log-likelihood ra which describes the operations on the data Z_t, $t \in T$, performed by the receiver/ processor is [1,2] the quadratic functional

$$Q(z) = \frac{1}{\eta_o} \int_T dt_1 \overline{Z(t_1)} \int_T dt_2\, Z(t_2) k(t_1, t_2) \tag{15}$$

where the kernel $k(t_1, t_2)$ is the solution to the integral equation

$$\eta_o k(t_1, t_2) + \int_T R_s(t_1, t_3) k(t_3, t_2) dt_3 = R_s(t_1, t_2),\ t_1, t_2 \in T. \tag{16}$$

T is the time during which the scattered pulse is received and will be assumed large relative to the support of R_s; η_o is the spectral density of the complex white noise.

Suppose the support of F is small relative to the distance over which f changes significantly: then, from (1a),

$$R_s(t_1,t_2) \approx g_o f(t_1 - \tau_o; \sigma_h^2) \; \mathcal{J}(t_1-t_2) \tag{17}$$
$$= f_*(t_1) \, \mathcal{J}(t_1-t_2)$$

where $\mathcal{J} \equiv F * \bar{F}$, $*$ denoting convolution.

Note that if in fact $f_*(t)$ were independent of t then the signal process would be wide sense stationary and the integral equation (16) would be easily solved by the Fourier transform: $k(t_1, t_2)$ becomes $k(t_1-t_2)$ with Fourier transform

$$\tilde{k}(\omega) = \frac{f_* |\tilde{F}(\omega)|^2}{\eta_o + f_* |\tilde{F}(\omega)|^2} .$$

(The tilda will denote Fourier transform throughout.)

This suggests a "quasi-stationary" solution form under (17): assume $k(t_1, t_2) = k_1(t_1, t_1-t_2)$ where, Fourier transforming on the second argument $\tau \equiv t_1 - t_2$,

$$\tilde{k}_1(t_1, \omega) = \frac{f_*(t_1) \, |\tilde{F}(\omega)|^2}{\eta_o + f_*(t_1) \, |\tilde{F}(\omega)|^2} . \tag{18}$$

It may be verified by substitution into (16), using the assumed slow variation of f_* with respect to F, that (18) is an approximate solution.

It is to be expected, in view of the discussion in [1], that if dispersed transmitted pulses are used, the first step upon reception would be pulse compression and this data is then available for further processing. If the compressed pulse support is small relative to the distance over which g changes, we can then reasonably apply the results of this case.

Realization - The quadratic functional Q(Z) given by Eq.15 can be written in a more special form now using Eq. 18

$$Q(Z) = \frac{1}{2\pi \eta_o} \int_T dt\ \overline{Z(t)} \int d\omega\ e^{-i\omega t} \tilde{Z}_T(\omega) . \frac{f_*(t) |\tilde{F}(\omega)|^2}{\eta_o + f_*(t) |\tilde{F}(\omega)|^2} .$$

($Z_T(t) \equiv Z(t)$, $t \in T$, and zero otherwise.) Suppose $|\tilde{F}(\omega)|^2 = 2\pi \mathcal{E}_F / \Omega_F \equiv \beta$, $|u| < \Omega/2$, and zero otherwise; then

$$Q(Z) = \frac{1}{\eta_o} \int_T dt\ \overline{Z(t)}\ (Z_t * h_t) . \frac{f_*(t)}{\eta_o/\beta + f_*(t)}$$

where $\tilde{h}_\omega \equiv 1$, $|\omega| < \Omega_F/2$, and 0 otherwise. It is likely the originally received data is already approximately so filtered and hence $Z_t \approx Z_t * h_t$: if, e.g., filtering matched to the transmitted pulse had already been done then $S_t = Z_t * h_t$; with this assumption

$$Q(Z) = \frac{1}{\eta_o} \int_T dt\ |Z(t)|^2 . \frac{f_*(t)}{\eta_o/\beta + f_*(t)} \tag{15a}$$

which is a weighted sum of the received instantaneous power after filtering matched to the transmitted pulse.

By definition the maximum likelihood estimator ("MLE") of the unknown parameter $\vec{\theta}$ is that $\tilde{\theta}$ maximizing the log likelihood ratio which here has the form $\{-\underline{\overline{\theta}} + Q(Z)\}$ where [2, p. 177] presently we have

$$\frac{\partial}{\partial\theta_i}\underline{\overline{\theta}} = \frac{1}{\eta_o}\iint_{TT} R_Z(t_1,t_2)\frac{\partial}{\partial\theta_i}k(t_1,t_2)\,dt_1dt_2$$

$$= \frac{1}{2\pi\eta_o}\int_T dt\int dw\,[f_*(t)\tilde{\mathcal{J}}(\omega)+\eta_o]\cdot\frac{\partial}{\partial\theta_i}k(t,\omega).$$

The necessary conditions for the MLE's $\tilde{\theta}_i$ are found by setting the derivative $\partial/\partial\theta_i$ of the log likelihood ratio equal to zero:

$$0 = \int_T dt\,\frac{\partial f_*(t)/\partial\theta_i}{[\eta_o/\beta+f_*(t)]^2}\cdot\left\{\frac{|Z_t|^2}{\beta B_F} - [\eta_o/\beta+f_*(t)]\right\}.$$

It is well known [4] that the correlation matrix of efficient estimates - those unbaised estimates of minimum possible error variance whose performance can be achieved asymptotically by MLE's - is the inverse of the matrix C of elements

$$C_{ij} \equiv -E\left\{\frac{\partial^2}{2\theta_i\partial\theta_j}\ \ell n\,\Lambda(z)\right\}$$

where $\Lambda(z)$ is the likelihood ratio and $\vec{\theta}\equiv(\theta_1,\ldots,\theta_M)$ is the vector of unknown parameters. In the present problem (Gaussian signal in white Gaussian noise) it may be shown [2, p. 179 et. f.] that

$$C_{ij} = \frac{1}{\eta_o}\int_T dt_1\int_T dt_2\,\frac{\partial}{\partial\theta_i}R_s(t_1,t_2)\cdot\frac{\partial}{\partial\theta_j}k(t_1,t_2). \tag{19}$$

$f_*(t_1) \; \mathcal{J}(t_1-t_2)$ and $k(t_1,t_2) = k_1(t_1, t_1-t_2)$: the change of variables ($\tau \equiv t_1-t_2$, $u = t_1$) followed by a Fourier transform on the τ variable gives the representations $\tilde{R}_s(t,\omega) = f_*(t) \; \tilde{\mathcal{J}}(\omega)$ and $\tilde{k}_1(t,\omega)$ as given by Eq. (5), yielding the form

$$C_{ij} = \frac{1}{2\pi\eta_o} \int_T dt \int d\omega \, \frac{\partial}{\partial\theta_i} \left\{ f_*(t) \, \tilde{\mathcal{J}}(\omega) \right\} \frac{\partial}{\partial\theta_j} \left\{ \tilde{k}(t,\omega) \right\}$$

where $\tilde{\mathcal{J}}(\omega) = |\tilde{F}(\omega)|^2$. Recalling only f_* depends on $\vec{\theta}$, carrying out the indicated differentiations gives

$$C_{ij} = \frac{1}{2\pi} \int_T dt \int d\omega \left[\frac{|\tilde{F}(\omega)|^2}{\eta_o + f_*(t)|\tilde{F}(\omega)|^2} \right]^2 \frac{\partial f_{*t}}{\partial\theta_i} \cdot \frac{\partial f_{*t}}{\partial\theta_j} .$$

To proceed assume that, reasonably,

$$|\tilde{F}(\omega)|^2 = \begin{cases} 2\pi \mathcal{E}_F/\Omega_F , & |\omega| < \Omega_F/2 , \\ 0 , & |\omega| > \Omega_F/2 , \end{cases} \tag{20}$$

where $\mathcal{E}_F$ is the energy of the complex modulation (which is twice the energy of the real, bandpass transmitted waveform). Then

$$C_{ij} = \frac{\Omega_F}{2\pi} \int_T dt \; \frac{(\partial f_{*t}/\partial\theta_i)(\partial f_{*t}/\partial\theta_j)}{[\eta_o/\beta + f_*(t)]^2} , \tag{19a}$$

$\beta \equiv 2\pi \mathcal{E}_F | \Omega_F$. We now examine the forms (20): they are somewhat involved though in specific instances numerical integration is straightforward.

As an example of a useful approximation, note f_* results from the convolution of, essentially, the unvariate density function of the gross height variation and the antenna pattern, as scaled into delay. A reasonable approximation in the denominator of (20)is arrived at by noting that, ignoring the sidelobes of the antenna pattern, we can take $f_*(t)$ as "pulse-like", approximately of constant value - say $g_o f(0)$ (Note (20)does not depend on τ_c).

Hence (6) becomes setting $B_F \equiv \Omega_F/2\pi$ and $\mathbb{C} \equiv B_F./(\eta_o/\beta + g_o f_o)^2$,

$$C_{ij} = \mathbb{C}\int \frac{\partial f_{*t}}{\partial \theta_i} \cdot \frac{\partial f_{*t}}{\partial \theta_j}\, dt$$

$$= \frac{\mathbb{C}}{2\pi}\int \frac{\partial \tilde{f}_{*\omega}}{\partial\theta_i} \cdot \frac{\partial \overline{\tilde{f}}_{*\omega}}{\partial\theta_j}\, d\omega$$

where $\tilde{f}_*(\omega) = g_o K \tilde{q}_1(\omega)\varphi_h[\omega/(c/2)]\exp(-i\omega\tau_o)$, and φ_h is the univariate character function corresponding to p_h.

Consider first finding the covariance matrix of the efficient estimates of amplitude, delay, and sea state; assume the univariant distribution of the gross waveheight is normal with zero mean and variance σ_h^2: then

$$\tilde{f}_*(\omega) = g_o K\tilde{q}_1(\omega)\, e^{-\sigma_h^2 \omega^2/2\,(c/2)^2 - i\omega\tau_o}$$

$$= g_o\, \tilde{f}(\omega)\, e^{-i\omega\tau_o}$$

Define $\sigma \equiv \sigma_h/(c/2)$ and the vector of unknown parameters $\vec{\theta} \equiv (\tau_o, \sigma, g_o)$.

Then we calculate:

$$C_{11} = \mathbb{C}\, g_o^2 \cdot M(1)\,,$$

$$C_{12} = C_{13} = 0\,,$$

$$C_{22} = \mathbb{C}\, g_o^2 \sigma^2 \cdot M(2)\,,$$

$$C_{23} = -\mathbb{C}\, g_o \sigma \quad M(1)\,,$$

$$C_{33} = \mathbb{C} M(0)\,.$$

Here we have defined $M(k) \equiv \frac{1}{2\pi}\int \omega^{2k}/\tilde{f}(\omega)/^2\, d\omega$; of course $M(0) \equiv \mathcal{E}_f$ is the energy of the function f and, setting $M(1) \equiv \mathcal{E}_f \Omega_f^2$, we may interpret Ω_f as the radius of gyration measure of the bandwidth of f_t (provided the centroid of $|\tilde{f}|^2$ is zero). Inverting this matrix we find the variance of the efficient estimators:

$$\mathrm{Var}(\tau_o^*) = \frac{1}{\mathcal{Q}\, g_o^2 M(1)} \tag{21a}$$

$$\mathrm{Var}(\sigma^*) = \frac{1}{1-\rho^2} \cdot \frac{1}{\mathcal{Q} g_o^2 \sigma^2 M(2)} \tag{21b}$$

$$\mathrm{Var}(g_o^*) = \frac{1}{1-\rho^2} \cdot \frac{1}{\mathcal{Q} M(0)} \tag{21c}$$

where the correlation coeficient ρ of the efficient estimates of sea state and amplitude is

$$\rho^2 = \frac{C_{23}^2}{C_{22}C_{33}} = \frac{M(1)^2}{M(2)M(0)} \; ; \tag{21d}$$

the efficient estimate τ_o^* of delay is uncorrelated with the other two efficient estimates.

The zero correlation between the delay and sea state efficient estimates holds more generally. Suppose $\varphi_h(u;\sigma_h) = \varphi_h(\sigma_h u)$: then the correlation is zero if and only if the univariate distribution of the gross sea height is even. For then $\partial\varphi_h(u)/\partial\sigma = u\varphi_h'(u)$ and

$$C_{12} = \frac{-i\mathcal{Q}\, g_o^2}{2\pi} \int \omega^2 |K\tilde{q}_1(\omega)|^2 \, \bar{\varphi}_h(\frac{\omega}{c/2})\, \varphi_h'(\frac{\omega}{c/2})\, d\omega \, .$$

First note that, as q_1 is real, $|\tilde{q}_1|^2$ is the sum of the square of an even function and the square of an odd function and therefore even. Second, as p_h is real,

φ_h is real and even if and only if p_h is even; more generally $\mathrm{Re}\{\overline{\varphi}_h \varphi_h^1\}$ is odd - and hence always contributes zero integral - and $d\omega\{\overline{\varphi}_h \varphi_h\}$ is even and hence can make C_{12} non-zero.

We will be able to carry out the integrations required in Eq.(19a) above if we assume an exponential form for $\tilde{q}_1$ and φ_h: let $\tilde{q}_1(\omega) \equiv |\tilde{q}_1(0)| \exp(-\phi_1 |\omega|)$ and $\varphi_h(\omega) = \exp(-\sigma_h|\omega|)$. Then

$$f_*(t) = g_o \cdot \frac{\alpha_1}{\pi \phi} \cdot \frac{1}{\left[1 + \left(\frac{t-\tau_o}{\phi}\right)^2\right]} ,$$

where $\phi \equiv \phi_1 + \sigma$, $\sigma \equiv \sigma_h/(c/2)$ and the parameter $\alpha \equiv g_o \alpha_1 = K|\tilde{q}(0)|$ is given explicity in terms of radar system parameters by Eq. 22 of [1]: $\alpha = A\sigma_o Q/(4\pi)^3 R_o^2$, where $\sigma_o \equiv \pi g_o/\lambda^2$. Also from [1], App. III, $\phi_1 \approx \lambda^2 R_o/$ Ac. The elements $\{c_{ij}\}$ given by (20) may now be straightforwardly, albeit somewhat laboriously, calculated by residue integration:

$$C_{11} = \frac{\Omega_F}{2\pi} \cdot \left(\frac{2 g_o \alpha_1 \beta}{\pi \phi^2 \eta_o}\right)^2 \cdot \phi \cdot \frac{\pi}{2} \cdot \frac{1}{a(a+1)^3} ,$$

$$C_{22} = \frac{\Omega_F}{2\pi} \cdot \left(\frac{g_o \alpha_1 \beta}{\pi \phi^2 \eta_o}\right)^2 \cdot \phi \cdot \frac{\pi}{2} \cdot \frac{a^3 - a^2 + 3a + 1}{a^3 (a+1)^3} ,$$

$$C_{33} = \frac{\Omega_F}{2\pi} \cdot \left(\frac{\alpha_1 \beta}{\pi \phi \eta_o}\right)^2 \cdot \phi \cdot \frac{\pi}{2} \cdot \frac{1}{a^3}$$

$$C_{12} = C_{13} = 0 ,$$

$$C_{23} = \frac{\Omega_F}{2\pi} \cdot \left(\frac{g_o \alpha_1^2 \beta^2}{\pi^2 \phi^3 \eta_o^2}\right) \cdot \phi \cdot \frac{\pi}{2} \cdot \frac{a^2 - 2a - 1}{a^3 (a+1)^2} ,$$

where $a^2 \equiv 1 + g_o \alpha_1 \beta / \pi \phi \eta_o$; recall $\beta = 2\pi \mathcal{E}_F / \Omega_F$.

Defining $\tilde{a} \equiv g_o \alpha_1 \beta \Omega_F / \pi \phi \eta_o = 2 g_o \alpha_1 \mathcal{E}_F / \phi \eta_o$, we may write the $\{c_{ij}\}$ in the form

$$C_{11} = \frac{\tilde{a}}{\phi} \cdot \frac{a^2 - 1}{a(a+1)^3} = \frac{\tilde{a}}{\phi} \cdot \frac{a-1}{a(a+1)^2} ,$$

$$C_{22} = \frac{\tilde{a}}{4\phi} \cdot \frac{(a^2-1)(a^3-a^2+3a+1)}{a^3(a+1)^3} = \frac{\tilde{a}}{4\phi} \cdot \frac{(a-1)(a^3-a^2+3a+1)}{a^3(a+1)^2}$$

$$C_{33} = \frac{\tilde{a}\phi}{4g_o^2} \cdot \frac{a^2-1}{a^3} ,$$

$$C_{12} = C_{13} = 0 ,$$

and

$$C_{23} = \frac{\tilde{a}}{4g_o} \cdot \frac{(a^2-1)(a^2-2a-1)}{a^3(a+1)^2} = \frac{\tilde{a}}{4g_o} \frac{(a-1)(a^2-2a-1)}{a^3(a+1)} .$$

Inverting this matrix to find the covariance matrix of the efficient estimators, the diagonal elements are

$$\mathrm{Var}(\tau_o^*) = \frac{\phi}{\tilde{a}} \cdot \frac{a(a+1)^2}{a-1} , \qquad (21e)$$

$$\mathrm{Var}(\sigma^*) = \frac{\phi}{\tilde{a}} \cdot \frac{(a+1)^3}{a-1} , \qquad (21f)$$

and

$$\mathrm{Var}(g_o^*) = \frac{g_o^2}{\tilde{a}\phi} \cdot \frac{a^3-a^2+3a+1}{a-1} \qquad (21g)$$

As an application of these results, consider the choice of best pulse modulation bandwidth Ω_F, which enters these error variances only via $a^2 = 1 + \tilde{a}/\Omega_F$: a can equivalently be chosen, then, to minimize one of these three forms. It is easily found algebraically that $a=2$ minimizes both $\mathrm{Var}(\sigma^*)$ and $\mathrm{Var}(g_o^*)$ and $a=(3+\sqrt{17})/4\approx 1.8$ minimizes $\mathrm{Var}(\tau_o^*)$; further, a choice of $a=2$ increases $\mathrm{Var}(\tau_o^*)$ by less than 2%. Thus the choice $a=2$ is a very good one; then $\tilde{a}/\Omega_F = (2\pi f_*(0)\mathcal{E}_F/\eta_o)/2\pi B_F = \sigma_s^2(0)/\sigma_n^2$ is seen to be (very nearly simultaneously) optimally 3.

This choice of a is allowable provided, roughly, the signal-to-noise ratio is large enough so that the quasi-stationary approximation remains valid: for this we must have $1/\Omega_F << \theta$. Since $\tilde{\theta}/\Omega = 3$ (for $a=2$), $1/\Omega = 3/\tilde{a} = (3/2)(\eta_o/\alpha\mathcal{E}_F)\theta$: thus, more precisely, we must have

$$\frac{\eta_o}{\alpha\mathcal{E}_F} << 1 .$$

As will be seen below, this condition is easily met in practice even for a satellite vehicle.

With $a = 2$, we find

$$\mathrm{Var}(\tau_o^*) = 9\left(\frac{\eta_o}{\alpha\mathcal{E}_F}\right)(\phi_1+\sigma)^2 , \qquad (22a)$$

$$\mathrm{Var}(\sigma^*) = \frac{27}{2}\left(\frac{\eta_o}{\alpha\mathcal{E}_F}\right)(\phi_1+\sigma)^2 , \qquad (22b)$$

and

$$\frac{\mathrm{Var}(g_o^*)}{g_o^2} = \frac{11}{2}\left(\frac{\eta_o}{\alpha\mathcal{E}_F}\right) ; \qquad (22c)$$

the correlation coefficient of the efficient estimates of sea state σ and amplitude g_o is, in magnitude, $1/\sqrt{33}$, a rather low value in that $\mathrm{Var}(\sigma^*)$ and $\mathrm{Var}(g_o^*)$ are increased by this correlation only by the factor (33/32).

Evaluating the approximate forms given by Eqs. 6 for this particular f_* we find

$$\mathrm{Var}(\tau_o^*) \approx 16 \left(\frac{\eta_o}{\alpha \mathcal{E}_F}\right) (\phi_1 + \sigma)^2 , \tag{23a}$$

$$\mathrm{Var}(\sigma^*) \approx \frac{64}{3} \left(\frac{\eta_o}{\alpha \mathcal{E}_F}\right) (\phi_1 + \sigma)^2 , \tag{23b}$$

$$\frac{\mathrm{Var}(g_o^*)}{g_o^2} \approx \frac{32}{3} \left(\frac{\eta_o}{\alpha \mathcal{E}_F}\right) \tag{23c}$$

and $\rho^2 = 1/4$; here $\mathfrak{C} = \mathfrak{C}_{opt} = \pi \mathcal{E}_F \phi / 4\eta_o \alpha$ corresponding to unity SNR was used. The parameter dependencies enter identically in the exact and approximate forms. Comparing the error standard deviations, the numerical factors, in order, are in the exact calculation (3, 3.7, 2.4) and in the approximate calculation (4, 4.6, 3.3): these are in good agreement.

A Specific Exact Solution. - Suppose that the reception time is large relative to the time over which the returned pulse is non-zero, a good approximation in practice: then in Eq. 4, the integral equation that determines the kernel function $k(t_1, t_2)$ of the optimum processor, we may set $T = (-\infty, \infty)$. Taking the two-dimensional Fourier transform (denoted by a double tilda) of this equation we find

$$\eta_o \,\tilde{\tilde{k}}(u, v) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\tilde{R}}_S(u, -\omega) \tilde{\tilde{k}}(\omega, v) d\omega = \tilde{\tilde{R}}_S(u, v) , -\infty < u, v < \infty . \tag{24}$$

From Eq. 8 of [1] we have (temporarily set $\tau_o \equiv 0$)

$$\tilde{\tilde{R}}_S(u,-v) = \tilde{\rho}(u-v)\tilde{F}(u)\overline{\tilde{F}}(v)$$

where

$$\tilde{\rho}(u-v) \equiv K\,\tilde{q}(u-v)\,\overline{\varphi}_h\left(\frac{u-v}{c/2}\right) .$$

Again assume

$$\tilde{F}(u) = \begin{cases} \dfrac{2\pi\,\mathcal{E}_F}{\Omega_F} & , \ |u| \le \Omega_F/2, \\ 0 & , \ |u| > \Omega_F/2 ; \end{cases} \tag{24a}$$

then Eq. 22 becomes

$$\frac{\eta_o}{\beta}\tilde{\tilde{k}}_1(u,v) + \int_{-\Omega_F/2}^{\Omega_F/2} \tilde{\rho}(u-\omega)\tilde{\tilde{k}}_1(\omega,v)d\omega = \tilde{\rho}(u-v), u, v \in \left(-\frac{\Omega_F}{2}, \frac{\Omega_F}{2}\right) \tag{24a}$$

where $\tilde{\tilde{k}}_1(u,v) \equiv \tilde{\tilde{k}}(u,-v)$.

The integral equation (22a) satisfied by $\tilde{\tilde{k}}_1$ is of a type well known in communication theory (see, e.g. [3]). There is a straightforward method of solution when $\tilde{\rho}$ is the (one-dimensional) Fourier transform of a rational function; thus any reasonably behaved $q * p_h$ can be approximated by a rational function and the corresponding k found - in principle, though the calculations may be tedious.

To apply this method here assume that

$$\tilde{\rho}(u) = \alpha\, e^{-\phi|u|} \quad , \quad -\infty < u < \infty \quad ;$$

then $\rho(t) = (\alpha/\pi\,\phi)[1+(t/\phi)^2]$ which roughly approximates a possible $q * p_h$: it would be desirable perhaps to model the usual asymmetry but this requires more complication. The solution to Eq. 4b is now easily found by adapting the solution for an interval $(0, \Omega_F)$ found in, e.g., [1], p. 388, Eq. 1.46:

$$\tilde{\tilde{k}}_1(u,v) = \begin{cases} K(u)\,K(-v)\,, & -\Omega_F/2 < u < v < \Omega_F/2\,, \\ K(v)\,K(-u)\,, & -\Omega_F/2 < v < u < \Omega_F/2\,, \end{cases} \tag{25}$$

where

$$K(u) \equiv \sqrt{\mathcal{D}}\,[\,(\hat{\beta}+\phi)e^{\hat{\beta}(\Omega_F/2+u)} + (\hat{\beta}-\phi)e^{-\hat{\beta}(\Omega_F/2+u)}\,]\,,$$

$$\mathcal{D} \equiv \phi\alpha\beta^2\{\eta_o\hat{\beta}\,[\,(\hat{\beta}+\phi)^2 e^{\hat{\beta}\Omega_F} - (\hat{\beta}-\phi)^2 e^{-\hat{\beta}\Omega_F}\,]\}^{-1}\,,$$

and $\hat{\beta}^2 \equiv \phi^2 + 2\beta\alpha\phi/\eta_o$. Of course $\tilde{\tilde{k}}$ itself is now readily found and further one can show (by an easy argument using the integral equation) that when $\tau_o \neq 0$,

$$\tilde{\tilde{k}}(u,v) = \begin{cases} e^{-i\tau_o(u+v)}\,K(u)K(v), & -\Omega_F/2 < u < v < \Omega_F/2, \\ e^{-i\tau_o(u+v)}\,K(-v)K(-u), & -\Omega_F/2 < v < u < \Omega_F/2. \end{cases}$$

Suppose $\hat{\beta}\Omega_F >> 1$: then

$$\tilde{\tilde{k}}(u,v) \approx \frac{\phi\alpha\beta^2}{\eta_o\hat{\beta}\,(\phi+\hat{\beta})^2}\cdot e^{-\hat{\beta}|u+v|}\,, -\Omega_F/2 < u,v < \Omega_F/2.$$

Since $\hat{\beta}^2\Omega_F^{\;2} = \phi^2\Omega_F^{\;2} + \dfrac{4\pi\mathcal{E}_F\alpha}{\eta_o}\cdot\phi\Omega_F$, it is sufficient for this case to obtain if either

$$\phi^2\Omega_F^{\;2} \approx 4\pi^2\cdot\frac{\phi^2}{\rho_F^{\;2}} >> 1 -$$

that is, the smeared returned pulse is long relative to the transmitted pulse resolution - or

$$\frac{4\pi\mathcal{E}_F\alpha}{\eta_o}\cdot\phi\Omega_F = 4\pi\cdot\frac{2\pi\mathcal{E}_F\alpha}{\eta_o}\cdot\frac{\phi}{\rho_F} >> 1 \quad -$$

that is, the product of SNR and ϕ/ρ_F is at least the order of unity. (In practical designs contemplated these conditions are met.)

We know from [1] that the calculation of Q(z) is realized by the configuration shown in Figure 1 - because $\tilde{\tilde{k}}$ has the some functional form as $\tilde{\tilde{R}}_S$ which is $\tilde{\tilde{k}}$ (aside from the multiplier η_o^{-1}) in the threshold case which was discussed there. The final linear, time-invariant filter has a transfer function

$$\tilde{k}(\omega) \approx \frac{\phi\alpha\beta^2}{\eta_o\hat{\beta}(\hat{\beta}+\phi)^2} \cdot e^{-\hat{\beta}|\omega|}, \quad -\infty < \omega < \infty ;$$

in particular its bandwidth is about

$$\hat{\beta}^{-1} \approx \left(\frac{\eta_o}{2\alpha\phi\beta}\right)^{1/2} .$$

The result agrees with the same results obtained under the quasi-stationary approximation.

It would be of interest to use this precise result for calculation of performance limits and hence the best bandwidth setting for any SNR.

Nonsymmetric Wave Height Distribution. - Although it is often reasonable to assume a normal univariate density function for the random gross wave height, it is known even in simple cases that the density function is only approximately symmetric. It is clear that a nonzero mean can be completely ambigous with the geoid parameter: see Eq. (1)! Such a nonzero mean may be associated with a large area current. We therefore consider a zero mean, nonsymmetric distribution that is a small perturbation from normal: the Gram-Charlier series [3] when truncated gives a convenient representation: thus assume

$$\varphi_{ha}(t) = e^{-\sigma_h^2 t^2/2}\,(1 - i\theta_{4a} t^3)$$

where $\theta_{4a} \equiv E\{h^3\}/3!$ will generally also be an unknown parameter. We therefore consider the efficient joint estimation of $\vec{\theta} \equiv (\tau_o, \sigma, g_o, \theta_4)$, $\theta_4 \equiv \theta_{4a}/(c/2)^3$. It will be convenient to write $\varphi_{ha}(t) = \varphi_h(t)\,(1 - i\theta_4 t^3)$ where $\varphi_h(t)$ as before is the normal characteristic function $(\exp(-\sigma_h^2 t^2/2)$ and keep f and f_* unchanged.

The matrix of c_{ij}'s is straightforwardly calculated to be the symmetric matrix

$$C = ¢\, g_o^2 \begin{pmatrix} M(1)+\theta_4^2 M(4) & & & \\ 0 & \sigma^2[M(2)+\theta_4^2 M(5)] & & \cdots \\ 0 & -\frac{\sigma}{g_o}[M(1)+\theta_4^2 M(4)] & \frac{1}{g_o^2}[1+\theta_4^2 M(3)] & \\ M(2) & -\theta_4 \sigma M(4) & \frac{\theta_4}{g_o} M(3) & M(3) \end{pmatrix}$$

Note that if the true (unknown) value of θ_4 is zero then C becomes

$$C_o \equiv ¢\, g_o^2 \begin{pmatrix} M(1) & & & \\ 0 & \sigma^2 M(2) & \cdots & \\ 0 & -\frac{\sigma}{g_o} M(1) & 1/g_o^2 & \\ M(2) & 0 & 0 & M(3) \end{pmatrix}$$

Then the correlation matrix of the efficient estimates is

$$C_o^{-1} = \frac{1}{\alpha g_o^2}\begin{pmatrix} \frac{1}{1-\rho_{14}^2}\cdot\frac{1}{M(1)} & & & \\ 0 & \frac{1}{1-\rho_{23}^2}\cdot\frac{1}{M(2)\sigma^2} & & \\ 0 & \frac{-g_o\,\rho_{23}^2/M(1)\sigma}{1-\rho_{23}^2} & \frac{1}{1-\rho_{23}^2}\cdot g_o^2 & \\ \frac{\rho_{14}^2/M(2)}{1-\rho_{14}^2} & 0 & 0 & \frac{1}{1-\rho_{14}^2}\cdot\frac{1}{M(3)} \end{pmatrix}$$

where

$$\rho_{14}^2 \equiv \frac{M(2)^2}{M(1)M(3)}$$

and

$$\rho_{23}^2 \equiv \frac{M(1)^2}{M(0)M(2)} \quad .$$

The variances of the efficient estimates are, of course, the diagonal elements.

In the more interesting case when the true value of θ_4 is not zero we may use the assumption that θ_4 is small by ignoring the θ_4^2 terms to ease the labor of calculating the inverse C^{-1} which may then be approximated as $C_o^{-1} + \theta_4 C_o^{-1} C_1 C_o^{-1}$ where

$$C_1 \equiv \alpha g_o^2 \begin{pmatrix} 0 & & & \\ 0 & 0 & & \cdots \\ 0 & 0 & 0 & \\ 0 & -\sigma M(4) & M(3)/g_o & 0 \end{pmatrix}$$

It is found that $C_0^{-1} C_1 C_0^{-1}$ has zero diagonal elements and hence the variances of the efficient estimates are unchanged; that (1, 4) and (2, 3) elements are zero and hence the correlation of the efficient estimates τ_o^* and θ_4^* are unchanged as with the efficient estimates σ^* and g_o^*; and that the (1, 2), (1, 3) and (2, 4) elements are nonzero so that there is at least a weak correlation between all efficient estimates.

Summarizing,

$$\mathrm{Var}(\tau_o^*) = \frac{1}{\mathbb{C} g_o^2} \cdot \frac{1}{1-\rho_{14}^2} \cdot \frac{1}{M(1)} ,$$

$$\mathrm{Var}(\theta_4^*) = \frac{1}{\mathbb{C} g_o^2} \cdot \frac{1}{1-\rho_{14}^2} \cdot \frac{1}{M(3)} ,$$

$$\mathrm{Var}(\sigma^*) = \frac{1}{\mathbb{C} g_o^2} \cdot \frac{1}{1-\rho_{23}^2} \cdot \frac{1}{M(2)\,\sigma^2} ,$$

and

$$\mathrm{Var}(g_o^*) = \frac{1}{\mathbb{C} g_o^2} \cdot \frac{1}{1-\rho_{23}^2} \cdot \frac{g_o^2}{M(0)} .$$

<u>Example</u> - Suppose that $|\tilde{q}(\omega)|^2 = |\tilde{q}(0)| \,.\, \exp(-\phi^2\omega^2)$ so that

$$|\tilde{f}(\omega)|^2 = \alpha_1^2\, e^{-\omega^2/2\nu^2}$$

where $\alpha_1 = K|\tilde{q}_1(0)|$ and $\nu^2 \equiv [2(\sigma^2 + \phi^2)]^{-1}$. We then have $M(0) = \alpha_1^2 \sqrt{2\pi}\,\nu$, $M(1) = \alpha_1^2\sqrt{2\pi}\,\nu \cdot \nu^2$, $M(2) = 3\,\alpha_1^2\sqrt{2\pi}\,\nu \cdot \nu^4$, and $M(3) = 15\,\alpha_1^2\sqrt{2\pi}\,\nu \cdot \nu^6$. The correlation coefficients are then $\rho_{14}^2 = 9/15$ and $\rho_{23}^2 = 1/3$ and $(1-\rho_{14}^2)^{-1} = 15/6$ and $(1-\rho_{23}^2) = 3/2$: these are the factors by which the various efficient estimates variances are increased by their correlation.

If the signal pulse bandwidth is chosen optimally we have seen $\mathcal{A} = \mathcal{A}_{opt} = (\mathcal{E}_F / 4 g_o f_o \eta_o)$: but here

$$f(0) = \frac{1}{2\pi}\int \tilde{f}(\omega) dw = \frac{1}{\sqrt{\pi}} \alpha_1 \nu$$

so $(\mathcal{A} g_o^2)^{-1} = 4\eta_o \alpha_1 \nu / \sqrt{\pi}\, \mathcal{E}_F g_o$. Then

$$\mathrm{Var}(\tau_o^*) = \frac{4\eta_o \alpha_1 \nu}{\sqrt{\pi}\,\mathcal{E}_F g_o} \cdot \frac{15}{6} \cdot \frac{1}{\sqrt{2\pi}\alpha_1^2 \nu^3} = \frac{20}{2\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha \nu^2} ,$$

$$\mathrm{Var}(\theta_4^*) = \frac{4\eta_o \alpha_1 \nu}{\sqrt{\pi}\,\mathcal{E}_F g_o} \cdot \frac{15}{6} \cdot \frac{1}{15\alpha_1^2 \sqrt{2\pi}\, \nu^7} = \frac{4}{6\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha \nu^6} ,$$

$$\mathrm{Var}(\sigma^*) = \frac{4\eta_o \alpha_1 \nu}{\sqrt{\pi}\,\mathcal{E}_F g_o} \cdot \frac{3}{2} \cdot \frac{1}{\sigma^2} \cdot \frac{1}{\sqrt{2\pi} \cdot 3\alpha_1^2 \nu^5} = \frac{4}{\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha \nu^4 \sigma^2} ,$$

and

$$\mathrm{Var}(g_o^*) = \frac{4\eta_o \alpha_1 \nu}{\sqrt{\pi}\,\mathcal{E}_F g_o} \cdot \frac{3}{2} \cdot \frac{g_o^2}{\sqrt{2\pi}\, \alpha_1^2 \nu} = \frac{12}{2\pi\sqrt{2}} \cdot \frac{\eta_o g_o^2}{\mathcal{E}_F \alpha} .$$

These results can also be written in the form (in part using the definition of ν)

$$\mathrm{Var}(\tau_o^*) = \frac{20}{2\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha} \cdot [2(\sigma^2 + \phi_1^2)] ,$$

$$\mathrm{Var}(\theta_4^*) = \frac{4}{6\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha} \cdot [2(\sigma^2 + \phi_1^2)]^3 ,$$

$$\frac{\mathrm{Var}(\sigma^*)}{\sigma^2} = \frac{4}{\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha} \cdot [2(1 + \phi_1^2/\sigma^2)]^2 ,$$

and

$$\frac{\mathrm{Var}(g_o^*)}{g_o^2} = \frac{12}{2\pi\sqrt{2}} \cdot \frac{\eta_o}{\mathcal{E}_F \alpha} .$$

Threshold case. - In the "threshold case" (in which the estimates can have arbitrarily good accuracy) we can readily find the structure of the best estimators and calculate the (Cramer-Rao) lower bound on the estimator's accuracy. From [1] the best processor calculates the form

$$Q(z) = \frac{K}{2\eta_0^2}\, q(-t) * p_h(-\frac{c}{2} t) * \left|\bar{F}(-t) * Z(t)\right|^2 \Big|_{t=\tau_o} \quad ;$$

the realization of this calculation by a "mixed integrator" processor is discussed in [1]. We find, by direct calculation, assuming geoid and sea state unknown,

$$C_{11} = \frac{1}{(2\pi\eta_0)^2} \int du \int dv\, |\tilde{R}_S(u, v)|^2 (u+v)^2 , \tag{26a}$$

$$C_{12} = \frac{-4i}{(2\pi\eta_0} \int du \int dv\, |\tilde{R}_S(u, v)|^2 (u+v)^3 , \tag{26b}$$

and

$$C_{22} = \frac{4}{(2\pi\eta_0 c)^2} \int du \int dv\, |\tilde{R}_S(u, v)|^2 \cdot \frac{4}{c^2} (u+v)^4 . \tag{26c}$$

Here we also assumed that the univariate sea height distribution is normal [4], of zero mean and variance σ_h^2; $\tilde{\tilde{R}}_S$ is the two-dimensional Fourier transform of the signal covariance function R_S. Thus [1]

$$\tilde{\tilde{R}}_S(u, v) = K\, \tilde{F}(u)\, \bar{\tilde{F}}(-v)\, \tilde{q}(u+v)\, \bar{\varphi}_n\!\left(\frac{u+v}{c/2}\right) e^{-i(u+v)\tau_o} ,$$

where, then $\tilde{F}$ is the signal pulse spectrum (Fourier transform) and φ_n is the univariate characteristic function of the random sea height.

We note immediately that since the left side of (25b) is real, the right side must be zero: that is, the efficient estimates of geoid and sea state are uncorrelated! Note also that only the modulus $|\tilde{F}|$ of the signal pulse spectrum enters, then, into the accuracies.

If we assume the signal pulse spectrum modulus form given by (20) and $|\tilde{q}(u)|^2 = |\tilde{q}(0)\exp(-\phi_1^2 u^2/2)|^2$, then we can calculate

$$C_{11} = \left(\frac{2\alpha\mathcal{E}_F}{\eta_0\tilde{\nu}}\right)^2 \cdot \gamma_{11}(\hat{a})$$

and

$$C_{22} = \left(\frac{2\alpha\mathcal{E}_F}{\eta_0\tilde{\nu}^2}\right)^2 \cdot \left(\frac{1}{c/2}\right)^4 \cdot \gamma_{22}(\hat{a}),$$

where $\hat{a} \equiv \tilde{\nu}\Omega_F$, $\tilde{\nu}^2 \equiv 8\sigma_n^2/c^2 + \phi_1^2$, $\alpha \equiv K|\tilde{q}(0)|$

$$\gamma_{11}(\hat{a}) \equiv \frac{2}{\tilde{a}^2}\left[\hat{a}\Psi(\hat{a}) - 2(1-e^{-\hat{a}^2/2})\right], \tag{27a}$$

$$\gamma_{22}(\hat{a}) \equiv \frac{1}{\tilde{a}^2}\left\{3\hat{a}\Psi(\hat{a}) - 8\left[1-(1+\hat{a}^2/8)e^{-\hat{a}^2/2}\right]\right\}, \tag{27b}$$

and, finally,

$$\Psi(\hat{a}) \equiv \int_0^{\hat{a}} e^{-x^2/2}\,dx .$$

Note that since the matrix of elements c_{ij} is diagonal we have, as aforementioned, the efficient estimates τ^* and $(\sigma_n^2)^*$ uncorrelated and their error variances are just the reciprocals of (26a) and (26c). These forms can now be used to make inferences about good system design just as in [1].

For example, note that the pulse modulation bandwidth Ω_F enters forms (26a) and (26c) only as $\hat{a} = \tilde{\nu}\Omega_F$ in the arguments of γ_{11} and γ_{22}, resp.

Uncorrelated estimates. - We have, in the course of the above calculations, showed that the efficient estimators of geoid and sea state are uncorrelated when the univariate probability density of sea heigh is Gaussian. We now show that it is the property of evenness of the density function that is sufficient for this property of the efficient estimates. (We are still assuming the threshold case.)

Recalculating (3) using (5) for an arbitrary characteristic function $\varphi_n(u) \equiv \hat{\varphi}_n(\sigma_n u)$, the required derivative $\partial\varphi_n(u)/\partial\sigma_n = u\,\hat{\varphi}'(\sigma_n u) = u\,\varphi_n'(u)$. Then we may write

$$C_{12} = i\left(\frac{K}{2\pi\eta_0}\right)^2 \int_{-\infty}^{\infty} du\, \tilde{G}(u) \int_{-\infty}^{\infty} du\, \tilde{G}(v)\tilde{H}(u-v)$$

where $\tilde{G}(\omega) = |\tilde{F}(\omega)|^2$ and $\tilde{H}(u) \equiv |\tilde{q}(u)|^2 \varphi_h\left(\frac{u}{c/2}\right) \varphi_u'\left(\frac{u}{c/2}\right) u^2$. Set I equal to i times the double integral: then by Parseval's theorem we may write

$$I = i \int_{-\infty}^{\infty} |G(t)|^2\, H(t)dt.$$

In what follows denote the real part of a complex-valued function F by F_R and the imaginary part by F_I.

We first note that since, in view of its definition, $\tilde{G}$ is real, G_R is even and G_I is odd and therefore $|G_t|^2 = G_{R_t}^{\ 2} + G_{I_t}^{\ 2}$ is even. Similarly of real implies $|\tilde{q}|^2$ even.

We also note that the realness of p_n implies φ_{n_R} is even and φ_{n_I} is odd ($\varphi_{n_I} \equiv 0$ if and only if p_n is even); hence φ'_{n_R} is odd and φ'_{n_I} is even and so $\{\varphi_n\varphi_n'\}_R$ is odd and $\{\varphi_n\varphi_n'\}_I$ is even ($\{\varphi_n\varphi_n'\}_I \equiv 0$ if and only if p_n is even). Thus $\tilde{H}_R$ is odd and $\tilde{H}_I$ is even, implying H is purely imaginary - thus checking that I is in fact real - and it will be odd if and only if p_n is even.

Thus finally $I = -\int |G(t)|^2 H_I(t)dt$ will be zero if H_I is odd which is true if and only if p_n is even.

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APPENDIX

It is of interest to review the situation when the scattering mechanism is not random so that a signal known except for delay and spread is received in white, thermal noise. The well-known theory [3] for this model gives, e.g.,

$$C_{12} = \frac{-1}{2\pi\eta_0} \mathrm{Re} \int_{-\infty}^{\infty} \bar{\tilde{S}}(\omega) \frac{\partial^2}{\partial\tau_o \partial\sigma_h} \tilde{S}(\omega) d\omega \tag{A.1}$$

where

$$S(t) = \int F(t-\tau_o-\eta)\, p_h\left(\frac{c}{2}\eta\right) dy \quad , \tag{A.2}$$

so that

$$\tilde{S}(\omega) = \tilde{F}(\omega)\varphi_h\left(\frac{\omega}{c/2}\right) e^{-i\omega\tau_o} . \tag{A.2a}$$

Again, as above, assume $\varphi_h(\omega) = \hat{\varphi}_h(\sigma_h\omega)$: Then $\partial\varphi_h(\omega)/\partial\sigma_h = \omega\hat{\varphi}_h'(\sigma_h\omega) = \omega\varphi_h'(\omega)$,

$$\frac{\partial^2}{\partial\tau_o\partial\sigma_h}\tilde{S}(\omega) = -\frac{i\omega^2}{c.2} \cdot \tilde{F}(\omega)\, \varphi_h'\left(\frac{\omega}{c/2}\right) e^{-i\omega\tau_o}$$

and

$$C_{12} = \frac{1}{2\pi\eta_0} \mathrm{Re}\, i\int_{-\infty}^{\infty} |\tilde{F}(\omega)|^2\, \bar{\varphi}_h\left(\frac{\omega}{c/2}\right)\varphi_h'\left(\frac{\omega}{c/2}\right)\frac{\omega^2}{c/2}\, d\omega . \tag{A.1a}$$

It is immediately clear that if φ_h is real - which is true if and only if p_n is even - then $C_{12} = 0$: that is, the efficient estimates of delay and spread are uncorrelated. We have already observed that $\mathrm{Re}\{i\,\bar{\varphi}_h\bar{\varphi}_h'\} = (\bar{\varphi}_h\varphi_h')_I$ is even and zero if and only if p_n is even.

We can also calculate readily

$$C_{11} = -E\left\{\frac{\partial^2}{\partial\tau_o^2}\ell n\Lambda\right\} = \frac{1}{2\pi\eta_0}\int \omega^2 |\tilde{F}(\omega)\ \varphi_h(\omega)|^2\, d\omega \tag{A.3}$$

and

$$C_{22} = -E\{\frac{\partial^2}{\partial(\sigma_h^2)^2} \ell n\Lambda\} = \frac{1}{2\pi\eta_0} \cdot \frac{1}{4}\int \omega^4 |\tilde{F}(\omega)\varphi_h(\omega)|^2 d\omega \qquad (A.4)$$

where we have assumed $\varphi_h(\omega) = \exp(-\sigma_h^2\omega^2/2)$, a normal characteristic function.

If we assume, for ease of calculation,

$$|\tilde{F}(\omega)| = \sqrt{\mathcal{E}_F}\; e^{-\omega^2/2\Omega_F^2}, \qquad (A.3a)$$

then

$$C_{11} = \frac{\mathcal{E}_F}{\sqrt{2\pi}\,\eta_0\sigma_h^3} \cdot \frac{1}{[2(1+1/\Omega_F^2\sigma_h^2)]^{3/2}}$$

and

$$C_{22} = \frac{\mathcal{E}_F}{\sqrt{2\pi}\,\eta_0\sigma_h^5} \cdot \frac{3}{4}\frac{1}{[2(1+1/\Omega_F^2\sigma_h^2)]^{5/2}} \qquad (A.4a)$$

The essential dependence of these forms on Ω_F is shown in Fig. A.1

It is clear that Ω_F a small integer of multiple of $1/\sigma_h$ essentially achieves the maximum value in C_{11} and C_{22} insofar as they depend on Ω_F. In this the results agree with the results for geoid and sea state estimation in the threshold case.

LIST OF CAPTIONS

Figure 1. - Geometry of radar altimeter. The coordinates are (x, y, z), where the x and z directions are the vehicle velocity vector v and the local vertical directions respectively. The antenna aperture is A and the nominal antenna pattern B has nominal beamwidth β ; F is an instantaneous radiated pulse position. The geoid G and sea surface $\mathcal{S}$ are separated by h(x, y); the geoid G has least range $z = R_o$ at $(x = 0, y = 0)$.

Figure 2. - Realization of optimum processor when preprocessing SNR is small. A is a bandpass filter matched to the complex modulation F, B is an envelope-squared detector, and C and D are lowpass filters matched to q and p_h, respectively.

Figure 3. - Weighting function $w(t) \equiv f_*(t)[\eta_0/\beta + f_*(t)]^{-1}$ for specific f_* as a function of $\chi \equiv f_o(0)/(\eta_0/\beta)$.

Figure 4. - The weighting functions $\{v_i(t)\}$ appearing in likelihood equations.

Figure 5. - Dependence of $\mathcal{J}_i(a)$ on a, where $a^2 \equiv 1 + \tilde{a}/\Omega_F$; (see Eqs. 21e, 21f, 21g).

Figure 6. - Dependence of $\gamma_{11}(\hat{a})$ and $\gamma_{22}(\hat{a})$ an $\hat{a} \equiv \tilde{\nu}\Omega_F$.

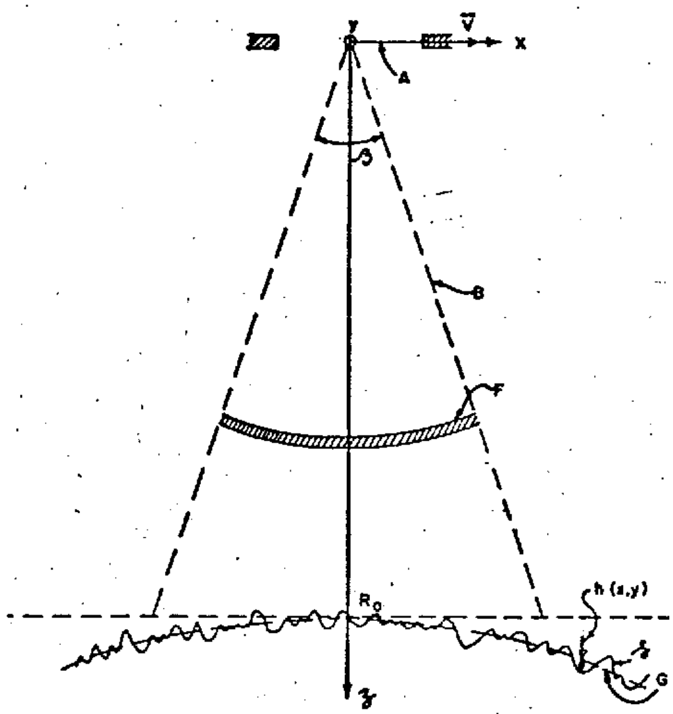

Fig.1. Geometry of radar altimeter. The coordinates are (x,y,z), where the x and z directions are the vehicle velocity vector v and the local vertical directions respectively. The antenna aperture is A and the nominal antenna pattern B has nominal beamwidth β; F is an instantaneous radiated pulse position. The geoid G and sea surface $\mathcal{S}$ are separated by $h(x,y)$; the geoid G has least range $z = R_0$ at $(x = 0, y = 0)$.

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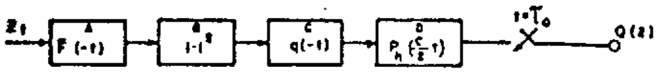


Fig. 2. Realization of optimum processor when preprocessing SNR is small. *A* is a bandpass filter matched to the complex modulation *F*, *B* is an envelope-squared detector, and *C* and *D* are low-pass filters matched to *q* and p_h, respectively.

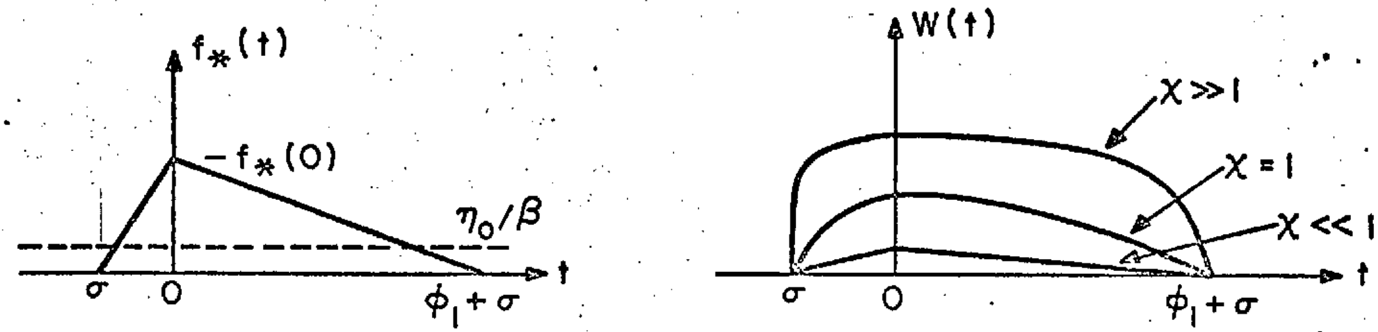
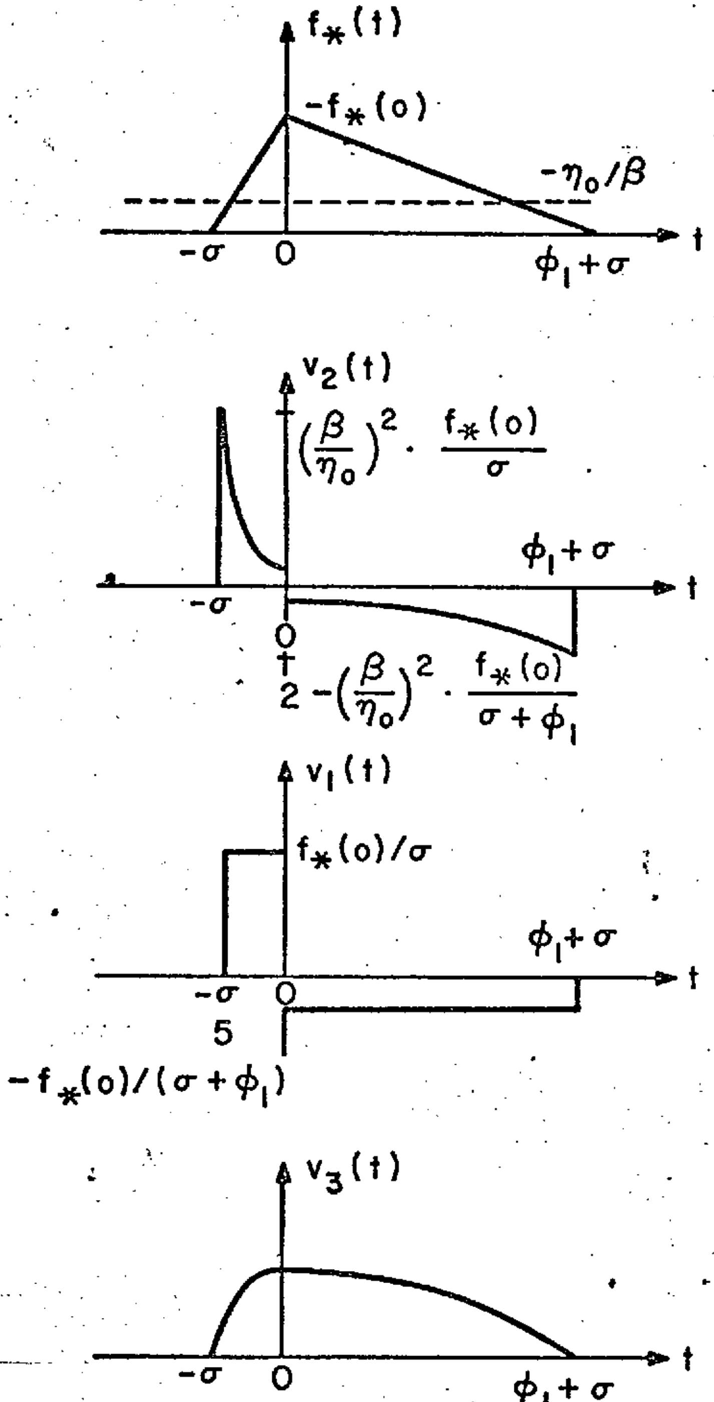


Fig. 3



$f_*(t)$
$-f_*(0)$
$-\eta_0/\beta$
$-\sigma$
0
$\phi_1+\sigma$
t
$v_2(t)$
$\left(\frac{\beta}{\eta_0}\right)^2 \cdot \frac{f_*(0)}{\sigma}$
$\phi_1+\sigma$
$-\sigma$
0
t
$2 -\left(\frac{\beta}{\eta_0}\right)^2 \cdot \frac{f_*(0)}{\sigma+\phi_1}$
$v_1(t)$
$f_*(0)/\sigma$
$\phi_1+\sigma$
$-\sigma$
0
t
5
$-f_*(0)/(\sigma+\phi_1)$
$v_3(t)$
$-\sigma$
0
$\phi_1+\sigma$
t

Fig. 4

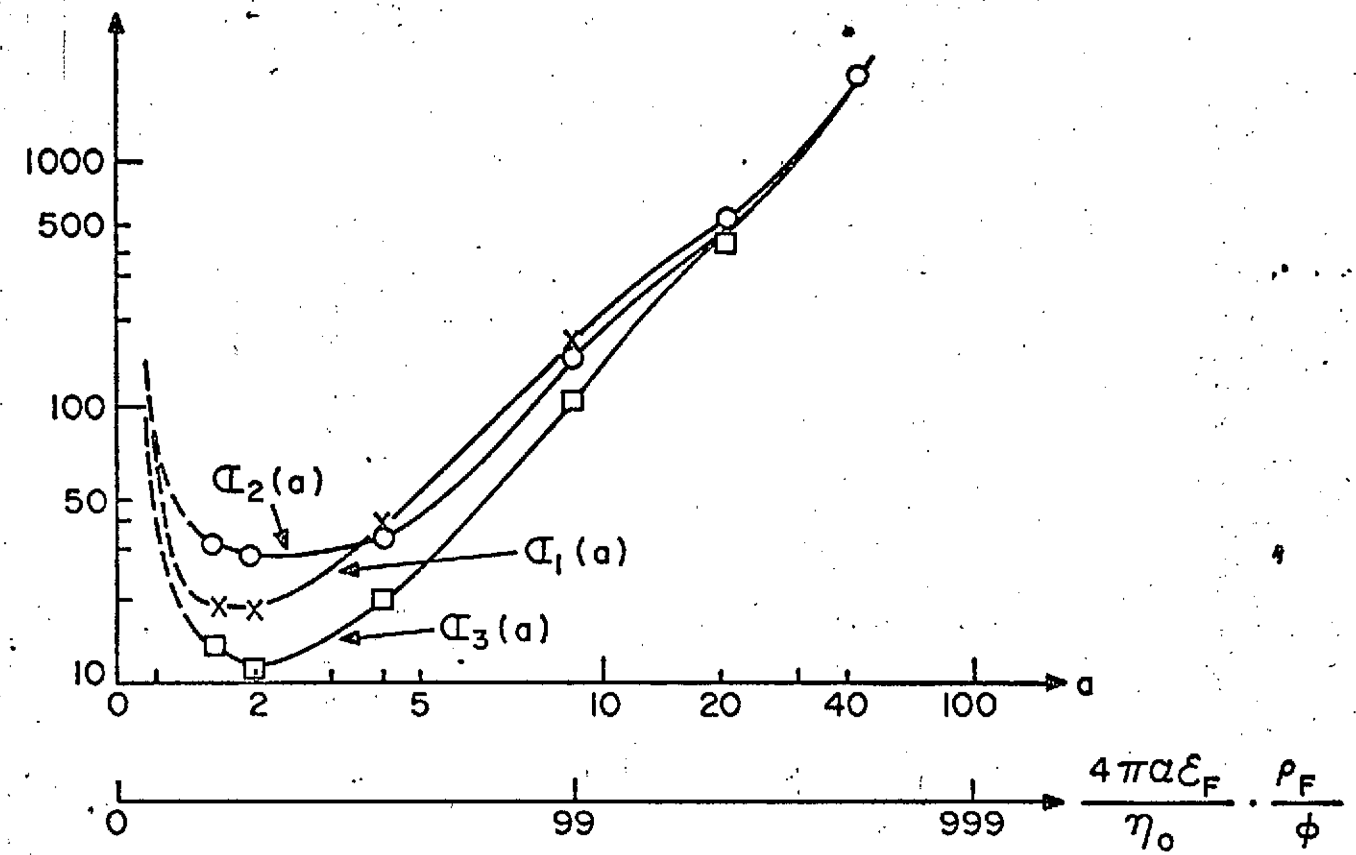

1000
500
100
50
10
$\mathcal{C}_2(a)$
$\mathcal{C}_1(a)$
$\mathcal{C}_3(a)$
0
2
5
10
20
40
100
a
0
99
999
$\frac{4\pi\alpha\mathcal{E}_F}{\eta_0}\cdot\frac{\rho_F}{\phi}$

Fig. 5

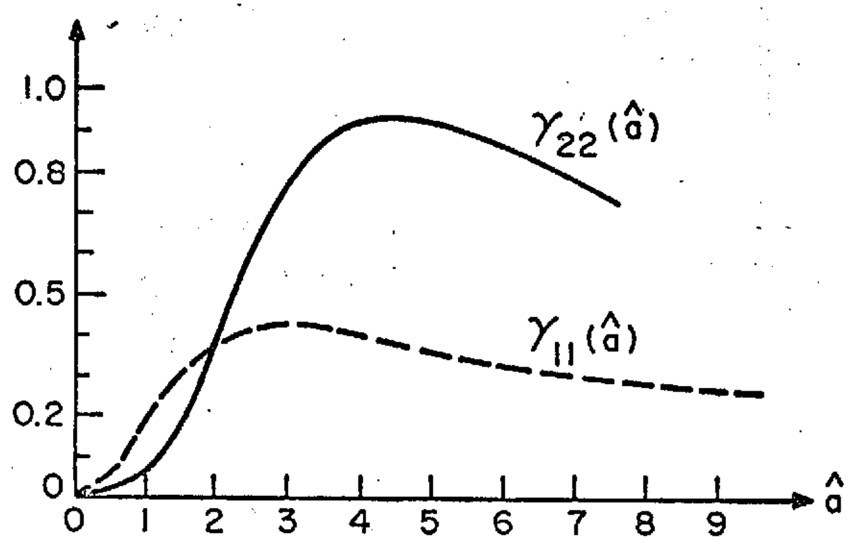


Fig. 6

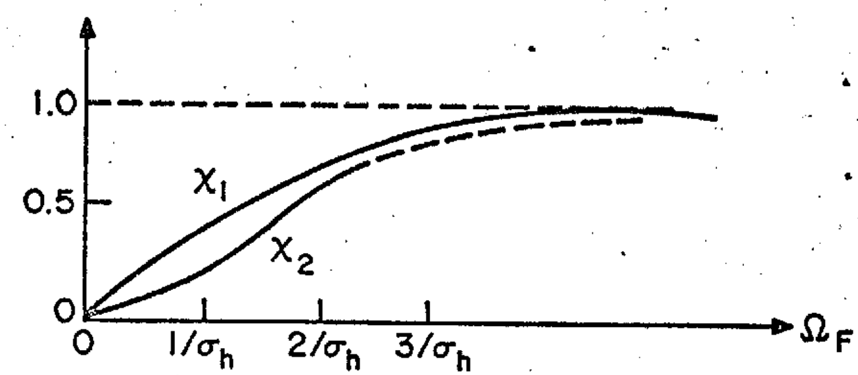


Fig. A.1