Finite Length Filters With Maximally Confined Spectral Power

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ABSTRACT

The problem of finding a function which, in addition to being zero outside a specified range in x-space, has its spectral power well confined to a certain range in k-space is solved numerically. Properties of the solutions are also discussed.

It is common practice to filter a set of data before performing a spectral analysis. This is done either to remove unwanted noise or in hope of enhancing a weak signal thought to be present in the data. As both the filtering and spectral analysis are almost always performed on a digital computer, the filter "window" is effectively zero outside some range due to the finite precision of computer arithmetic. Therefore we will consider a filtering function $g(x)$ which is defined to be zero except for $x \in [-L,L]$. We have chosen to consider a symmetric range to simplify the analysis; however, we will see that the results are easily generalized. It is well known that if a function is zero outside $[-L,L]$ there is no finite K for which its Fourier transform is zero outside $[-K,K]$. Nonetheless, the aim of the filtering usually is to remove or retain some range of frequencies. Hence we want to consider a filter which is zero outside $[-L,L]$ but also has its Fourier transform concentrated in the range $[-K,K]$. Before going further we must choose some measure of how well a function's Fourier transform is concentrated in $[-K,K]$. The measure we will use is the fraction of the spectral power contained in $[-K,K]$. For a given K and L we may then ask if there is a function which is optimal in the sense of being zero outside $[-L,L]$ and having the largest possible fraction of its spectral power in $[-K,K]$.

We choose the following definition of the Fourier transform:

$$\tilde{g}(k) \equiv (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-ikx} g(x)dx \quad . \tag{1}$$

Then from the Fourier theorem we also know that

$$g(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ikx} \tilde{g}(k)\,dk \quad . \tag{2}$$

Then the quantity we wish to maximize is

$$\frac{\int_{-K}^{K} |\tilde{g}(k)|^2 dk}{\int_{-\infty}^{\infty} |\tilde{g}(k)|^2 dk} = \frac{\int_{-K}^{K} |\tilde{g}(k)|^2 dk}{\int_{-\infty}^{\infty} |g(x)|^2 dx} \quad . \tag{3}$$

We will restrict our attention to real valued filter windows. The problem as presently posed is indeterminate without some normalization constraint, so we make the obvious choice

$$\int_{-\infty}^{\infty} g^2(x)\,dx = 1 \quad . \tag{4}$$

We see now that our choice for a measure of compactness in k-space makes generalizing the range in both k- and x- space very easy. Translating the range in x-space ΔL say to $[-L+\Delta L,\ L+\Delta L]$ will multiply the Fourier transform by $e^{ik\Delta L}$, leaving the spectral power unchanged. Similarly multiplying the filter by $e^{-i\Delta kx}$ will translate the Fourier transform in k-space. Maximizing (3) is equivalent to finding the eigenfunction associated with the largest eigenvalue of the integral equation

$$\alpha\, f(x) = \frac{1}{\pi} \int_{-KL}^{KL} \frac{\sin(x - x')}{x - x'}\, f(x')\,dx' \qquad \text{for} \quad |x| < KL \tag{5}$$

where $f(Kx) = g(x)$ and α is the fraction of the spectral power of g contained in $[-K,K]$. This result is derived in appendix I. In appendix II it is shown that if f satisfies (5) then its Fourier transform satisfies

$$\alpha\, \tilde{f}\big((KL)k\big) = \frac{1}{\pi} \int_{-KL}^{KL} \frac{\sin(k-k')}{k-k'}\, \tilde{f}\big((KL)k'\big)\, dk' \quad . \tag{6}$$

f cannot be its own Fourier transform since $f(x)$ is defined to be zero outside $[-KL, KL]$ while (6) is valid for all k. Nevertheless if the largest eigenvalue of (5) is non-degenerate, f and its Fourier transform are simply related.

We now make use of the fact that the kernel of (5) has a diagonal expansion in spherical Bessel functions (Jackson 1962):

$$\frac{\sin(x-x')}{x-x'} = \sum_{n=0}^{\infty} (2n+1) j_n(x) j_n(x') \quad . \tag{7}$$

Now if we define $\varphi_n(x) = (2n+1)^{\frac{1}{2}} j_n(x)$ and expand the eigenfunctions in these (non-orthogonal) basis functions we can rewrite (5) as

$$\alpha\, f(x) = \alpha \sum_{n=0}^{\infty} a_n \varphi_n(x) = \frac{1}{\pi} \int_{-KL}^{KL} \sum_{n=0}^{\infty} \varphi_n(x)\varphi_n(x') \sum_{m=0}^{\infty} a_m \varphi_m(x')\, dx' \quad . \tag{8}$$

If we define the inner product $(f,g) = \int_{-KL}^{KL} f(x)g(x)\,dx$ we may use the independence of the spherical Bessel functions to write

$$\alpha\, a_n = \frac{1}{\pi} \sum_{m=0}^{\infty} (\varphi_n, \varphi_m) a_m \quad . \tag{9}$$

Since our range of integration is symmetric, the inner products (φ_n, φ_m) are zero unless $m+n$ is even. Thus the problem separates naturally into finding the largest eigenvalue and associated eigenfunction for odd and even functions separately. Therefore the remarks at the end of appendix II apply to all of the eigenfunctions of this kernel. We will leave implicit the fact that there are two separate problems to be solved.

Now we may attempt to approximate the eigenvalues and eigenfunctions by truncating the spherical Bessel function expansion at some point. Thus we need a reasonably efficient method of calculating the spherical Bessel functions and the inner products of (9). The spherical Bessel functions are easily calculated by "backwards recursion" (see, for example, Abramowitz and Stegun 1964a). This method is particularly suited to this case since the value of several spherical Bessel functions will be needed for the same argument. The inner product can be relatively easily calculated from a relation given by Abramowitz and Stegun (1964b) from which one can show

$$\int^{x} j_n(t) j_m(t) dt = \frac{x}{n+m+1} \left\{ j_n(x) j_m(x) + 2 \sum_{k=1}^{\infty} j_{n+k}^{(x)} \, j_{m+k}^{(x)} \right\} . \qquad (10)$$

Therefore we need only evaluate the spherical Bessel functions at the end points of the interval of integration to find the required inner products. For this calculation standard library subroutines were used to evaluate the largest eigenvalue and associated eigenvector a_n of (9) truncated at N (for details of the algorithm used see Wilkinson and Reinsch 1971a,b,c). Then similar results were calculated for (9) truncated at $N+1$. When the difference between succesive approximations to the eigenvalue and the eigenfunction evaluated at zero and the end point of the range was sufficiently small, the approximations were accepted. The fraction of spectral power in [-K,K] is plotted as a function of KL for both the even and odd cases in figure 1. The best eigenfunctions for several values of KL are plotted in figure 2 with the axes scaled so that the functions are non-zero on the same range and have the same value at zero.

The solution of this problem may also be of interest to workers writing computer codes that use a k-space representation truncated to a finite number of harmonics to solve a differential equation in x-space (see, for example, Buneman 1974). In this application a function in x-space is represented by a finite range in k-space (actually by a finite number of harmonics) but it is desired to restrict the function to a suitable range in x-space. Thus we have found a k-space form factor that is in some sense optimal for this problem. It is also possible that the solution to the similar problem in higher dimensions may be of interest. The diagonal kernels for the two and three dimensional cases are derived in appendix III. The expansion of the eigenfunction in (8) also leads to an expansion of the eigenfunction in Legendre polynomials as shown in appendix IV.

ACKNOWLEDGMENTS

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APPENDIX I

We wish to maximize

$$\alpha \equiv \frac{\int_{-K}^{K} |\tilde{g}(k)|^2 dk}{\int_{-\infty}^{\infty} g^2(x)\,dx} = \int_{-K}^{K} |g(k)|^2 dk \qquad (I.1)$$

under the constraint $\int_{-\infty}^{\infty} g^2(x)\ dx = 1$. Now we can rewrite (I.1) using the Fourier theorem and the requirement that g be real and zero outside [-L, L]

$$\alpha = \frac{1}{2\pi} \int_{-K}^{K} \int_{-L}^{L} \int_{-L}^{L} g(x) g(x')\ e^{ikx}\, e^{-ikx'} dx\ dx'\ dk \quad . \qquad (I.2)$$

Interchanging the order of integration and performing the integral over k yields

$$\alpha = \frac{1}{\pi} \int_{-L}^{L} \int_{-L}^{L} \frac{\sin K(x - x')}{x - x'}\, g(x) g(x')\, dx\ dx' \quad . \qquad (I.3)$$

We want to maximize α under the constraint $\int_{-L}^{L} g^2(x)\,dx = 1$. Therefore we introduce a Lagrange multiplier and take the variation

$$\delta\left(\alpha + \gamma \int_{-L}^{L} g^2(x)\,dx\right)$$

$$= \left(2 \int_{-L}^{L} \delta\ g(x)\,dx \left\{\frac{1}{\pi} \int_{-L}^{L} \frac{\sin K(x - x')}{x - x'}\, g(x')\,dx' - \gamma g(x)\right\}\right) \quad . \qquad (I.4)$$

For an extremal an arbitrary variation must be zero so we may set the expression in curly brackets equal to 0.

$$\frac{1}{\pi}\int_{-L}^{L} \frac{\sin K(x-x')}{x-x'}\, g(x')dx' = \gamma\ g(x) \qquad \text{for} \qquad |x| < L \tag{I.5}$$

multiplying both sides by $g(x)$ and integrating identifies γ as α and setting $f(kx) = g(x)$ results in equation 5.

APPENDIX II

We will demonstrate the validity of equation (6). First we define a function $H(z)$ and write down the integral equation for its Fourier transform

$$\alpha\,\tilde{H}(k) = \frac{1}{\pi}\int_{-KL}^{KL} \frac{\sin(k-k')}{k-k'}\,\tilde{H}(k')\,dk' \quad . \tag{II.1}$$

Now we take the inverse Fourier transform of both sides which gives

$$\alpha\,H(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{ihz}\,\tilde{H}(k)\,dk$$

$$= \frac{1}{\pi}\,(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty}\int_{-KL}^{KL} \frac{\sin(k-k')}{k-k'}\,\tilde{H}(k')\,dk'\;e^{ikz}\,dk \quad . \tag{II.2}$$

Interchanging the orders of integration on the right hand side and performing the integral over k yields

$$\alpha\,H(z) = \frac{1}{\pi}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\Big(\sigma(z+1) - \sigma(z-1)\Big) \int_{-KL}^{KL} \tilde{H}(k')\;e^{ik'z}\,dk' \tag{II.3}$$

where $\sigma(z)$ is the Heaviside step function. Now we replace $\tilde{H}(k')$ by its definition and interchange orders of integration obtaining

$$\alpha\,H(z) = \frac{1}{\pi}\,\frac{1}{2}\Big(\sigma(z+1) - \sigma(z-1)\Big)\int_{-\infty}^{\infty} H(z')\,dz' \int_{-KL}^{KL} e^{-ik'(z-z')}\,dk' \quad . \tag{II.4}$$

Performing the integration over k' and noting that any function satisfying (II.4) will be zero outside $[-1,1]$ we may write

$$\alpha\, H(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sin KL(z-z')}{z-z'}\, H(z')\,dz' \quad . \tag{II.5}$$

Thus letting $z = \frac{x}{KL}$ we see that $H\left(\frac{x}{kL}\right)$ satisfies the same integral equation as $f(x)$ so that $\tilde{f}(k)$ satisfies equation (6). We can say more in the case that the eigenfunction is either even or odd. Since the square integrals of a function and its Fourier transform are equal we can in these cases write

$$\tilde{f}(k) = \left(\frac{KL}{\alpha}\right)^{\frac{1}{2}} e^{i\theta}\, f_{ext}\Big((KL)k\Big) \tag{II.6}$$

where f_{ext} is the analytic continuation of the eigenfunction and $\theta = 0$ or π for even eigenfunctions, while $\theta = \pm\,\pi/2$ for odd eigenfunctions.

APPENDIX III

In this appendix we briefly outline the method of solution for the two- and three-dimensional analogs of the problem treated in the text. The problem to be solved in two (three) dimensions is the following: to select from the set of functions which vanish outside a circle (sphere) of radius L that member which has the largest fraction of its spectral power contained in a circle (sphere) of radius K in k-space. From the symmetry of the problem we assume that the solution is cylindrically (spherically) symmetric, i.e. a function of the radius only. The two cases follow.

A. Two Dimensions

For a function $f(r)$ vanishing for $r > L$, the two dimensional Fourier transform is a function of the magnitude of $\vec{k}$ only and is defined as

$$\tilde{f}(k) = \frac{1}{2\pi}\int_0^L r dr \int_0^{2\pi} d\phi \; e^{-ikr \cos\phi} f(r) = \int_0^L r dr \; J_0(kr) f(r) \,, \qquad \text{(III.1)}$$

where $J_0(x)$ is the ordinary Bessel function of the first kind. The inverse transform is given by

$$f(r) = \int_0^\infty k dk \; J_0(kr)\tilde{f}(k) \quad . \qquad \text{(III.2)}$$

From (III.1) and (III.2) we see that

$$\int_0^L \Gamma^2(r) dr = \int_0^\infty \tilde{\Gamma}^2(k) dk \equiv N$$

where $\Gamma(r) = \sqrt{r}\, f(r)$ and $\tilde{\Gamma}(k) = \sqrt{k}\, \tilde{f}(k)$. We wish to find f (or Γ) such that the quantity

$$\alpha = \frac{1}{N}\int_0^K dk\, \tilde{\Gamma}^2(k) \tag{III.3}$$

is maximized. As in the main part of the text, we set $N = 1$ as our normalization constraint. From (III.1), (III.3) then becomes

$$\alpha = \int_0^L dr \int_0^L dr_1\, \Gamma(r)H(r,r')\Gamma(r') \tag{III.4}$$

where

$$H(r,r') = \sqrt{rr'}\int_0^K kdk\, J_0(kr)J_0(kr') \tag{III.5}$$

$$= K\,\frac{\sqrt{rr'}}{r^2 - r'^2}\left[rJ_1(Kr)J_0(Kr') - r'J_1(Kr')J_0(Kr)\right] \quad . \tag{III.6}$$

To maximize α subject to the normalization constraint, we proceed as in appendix I to find that $\Gamma(r)$ must satisfy the homogeneous integral equation

$$\alpha\Gamma(r) = \int_0^L H(r,r')\Gamma(r')dr' \quad . \tag{III.7}$$

The maximum value of α is then the largest eigenvalue of (III.7) and the desired function $\Gamma(r)$ is the corresponding eigenfunction. To calculate α and find the eigenfunction Γ as we did in the text requires that we find an expansion of the kernel $H(r,r')$ analogous to (7). For these purposes the expression of $H(r,r')$ via (III.6) is not particularly useful so we return to (III.5) and note that

$$J_0(kr)J_0(kr') = \frac{1}{2\pi}\int_0^{2\pi} J_0(kR)d\psi \quad , \tag{III.8}$$

where $R^2 = r^2 + r'^2 - 2rr'\cos\psi$. But

$$\int_0^K kdk\ J_0(kR) = \frac{K}{R}\ J_1(KR) \tag{III.9}$$

$$= 2\sum_{n=0}^{\infty} (n+1)\ \frac{J_{n+1}(Kr)}{r}\ \frac{J_{n+1}(Kr')}{r'}\ \frac{\sin(n+1)\psi}{\sin\psi} \tag{III.10}$$

(cf. Gradshteya and Ryzkik 1965a,b). From (III.8)-(III.10), (III.5) becomes

$$H(r,r') = 2\sum_{n=0}^{\infty}(2n+1)\ \frac{J_{2n+1}(Kr)}{\sqrt{r}}\ \frac{J_{2n+1}(Kr')}{\sqrt{r'}} \quad . \tag{III.11}$$

Proceeding in a manner similar to that presented in the text, we then find from (III.7) and (III.11) that $\Gamma(r)$ can be written as

$$\Gamma(r) = \frac{1}{\sqrt{r}}\sum_{n=0}^{\infty} a_n\phi_n(Kr) \tag{III.12}$$

where

$$\phi_n(r) = \sqrt{2(2n+1)}\ J_{2n+1}(Kr)$$

and the column matrix a_n satisfies the eigenvalue equation

$$\alpha a_n = \sum_{n=0}^{\infty} A_{nm}(KL) a_m \tag{III.13}$$

with

$$A_{nm}(KL) = \int_0^{KL} \frac{dx}{x} \phi_n(x)\phi_m(x) = 2\sqrt{(2n+1)(2m+1)} \int_0^{KL} \frac{dx}{x} J_{2n+1}(x) J_{2m+1}(x) \ .$$

From here on a method analogous to that employed in the one-dimensional case can be used to evaluate the entries in the matrix A_{nm} and to solve the truncated form of the matrix equation (III.13) for the largest eigenvalue and corresponding eigenvector a_n. From (III.12), then, our desired function $f(r)$ is given by

$$f(r) = \frac{1}{\sqrt{r}} \Gamma(r) = \frac{1}{r} \sum_{n=0}^{\infty} a_n \phi_n(Kr) = \sum_{n=0}^{\infty} 2\sqrt{2n+1}\, a_n \frac{J_{2n+1}(Kr)}{r} \ .$$

B. Three Dimensions

Again in this case, the function $f(r)$, vanishing for $r > L$, has a three-dimensional transform, which is a function of $|\vec{k}|$ ($\equiv k$) only, given by

$$k\tilde{f}(k) = \frac{1}{(2\pi)^{3/2}} \int_0^L r^2 dr\ f(r) \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu\ e^{-ikr\mu} = \sqrt{\frac{2}{\pi}} \int_0^L r dr \sin kr\, f(r) \ . \tag{III.14}$$

The inverse transform is

$$rf(r) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} k dk \sin kr\, \tilde{f}(k) \ . \tag{III.15}$$

From (III.14) and (III.15) we see that

$$\int_0^L \Gamma^2(r)\,dr = \int_0^\infty \tilde{\Gamma}^2(k)\,dk \equiv N$$

where $\Gamma(r) = rf(r)$ and $\tilde{\Gamma}(k) = k\tilde{f}(k)$.

The quantity to be maximized is

$$\alpha = \frac{1}{N}\int_0^K \tilde{\Gamma}^2(k)\,dk \quad . \qquad \text{(III.16)}$$

Again setting $N = 1$ as the normalization constraint and using (III.14) in (III.16) we find

$$\alpha = \int_0^L dr \int_0^L dr' \;\Gamma(r)H(r,r')\Gamma(r') \quad , \qquad \text{(III.17)}$$

where

$$H(r,r') = \frac{2}{\pi}\int_0^K dk \sin kr \sin kr' = \frac{1}{\pi}\,\frac{\sin K(r-r')}{r-r'} - \frac{1}{\pi}\,\frac{\sin K(r+r')}{r+r'} \;. \qquad \text{(III.18)}$$

The extremal condition for α with the normalization constraint is then the homogeneous integral equation

$$\alpha\Gamma(r) = \int_0^L H(r,r')\Gamma(r')\,dr' \quad ,$$

or, from (III.18),

$$\alpha\Gamma(r) = \frac{1}{\pi}\int_0^L dr' \;\frac{\sin K(r-r')}{r-r'}\;\Gamma(r') - \frac{1}{\pi}\int_{-L}^0 dr' \;\frac{\sin K(r-r')}{r-r'}\;\Gamma(-r') . \qquad \text{(III.19)}$$

Thus if we define $\Gamma(r)$ for $r < 0$ according to

$$\Gamma(r) = -\Gamma(-r) \quad , \tag{III.20}$$

we see that $\Gamma(r)$ must satisfy

$$\alpha\Gamma(r) = \frac{1}{\pi} \int_{-L}^{L} dr' \, \frac{\sin K(r-r')}{r-r'} \, \Gamma(r') \quad . \tag{III.21}$$

We note that (III.21) is exactly the same equation as that for the one-dimensional case (I.5). However we have the added restriction, (III.20), that $\Gamma(r)$ be odd. Thus the solution in this case is the same for the one-dimensional case, except that we must find the largest eigenvalue corresponding to an odd eigenfunction $\Gamma(r)$. The solution for $f(r)$ which has the largest fraction of its power spectrum confined to a sphere of radius K in k-space is then given by

$$f(r) = \frac{1}{r} \Gamma(r) \quad .$$

The largest value of α as a function of KL for three dimensions is plotted in figure 1 as the curve corresponding to the best odd function.

APPENDIX IV

We have from equation (6) the Fourier transform of an eigenfunction

$$\tilde{f}(k) = \left(\frac{KL}{\alpha}\right)^{\frac{1}{2}} \begin{Bmatrix} \pm 1 \\ \pm i \end{Bmatrix} \sum_{n=0}^{\infty} a_n \begin{Bmatrix} (4n+1)^{\frac{1}{2}} j_{2n}(KLk) \\ (4n+3)^{\frac{1}{2}} j_{2n+1}(KLk) \end{Bmatrix} \qquad \text{(IV.1)}$$

where the upper quantities in curly brackets refer to even eigenfunctions, the lower quantities to odd eigenfunctions. We have the general form for the Fourier transform of a spherical Bessel function from Abramowitz and Stegun (1964c)

$$\tilde{j}_n(k) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (-i)^n \; P_n(k) \qquad \text{(IV.2)}$$

where $P_n(k)$ is the Legendre polynomial of order n. When we take the inverse Fourier transform of both sides of (IV.1) we find

$$f(x) = \pm\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n a_n \begin{Bmatrix} (4n+1)^{\frac{1}{2}} \; P_n\left(\frac{x}{KL}\right) \\ (4n+3)^{\frac{1}{2}} \; P_n\left(\frac{x}{KL}\right) \end{Bmatrix} . \qquad \text{(IV.3)}$$

We see that the value of the first neglected term gives an upper bound on the error in truncating (IV.3).

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FIGURE CAPTIONS

Figure 1. Fraction of spectral power confined (α) and $\log_{10}(1-\alpha)$ as a function of KL. Curve 1 is the fraction of spectral power contained in $[-K,K]$ for the best even eigenfunction on $[-L,L]$ as a function of KL. Curve 2 is the same quantity for the best odd eigenfunction. Curve 3 is the logarithm of $(1-\alpha)$, the fraction of the spectral power outside $[-K,K]$, for the best even eigenfunction on $[-L,L]$ as a function of KL. Curve 4 is the same quantity for the best odd eigenfunction.

Figure 2. Eigenfunctions associated with the largest eigenvalue for various values of KL. The eigenfunctions associated with the largest eigenvalue plotted for KL varying from $\pi/2$ (uppermost curve) to 4π (lowest curve) in increments of $\pi/2$. As the functions are in these cases symmetric, they are plotted only for $x > 0$. The ordinate is arbitrary with all functions normalized to have the same value at $x = 0$.

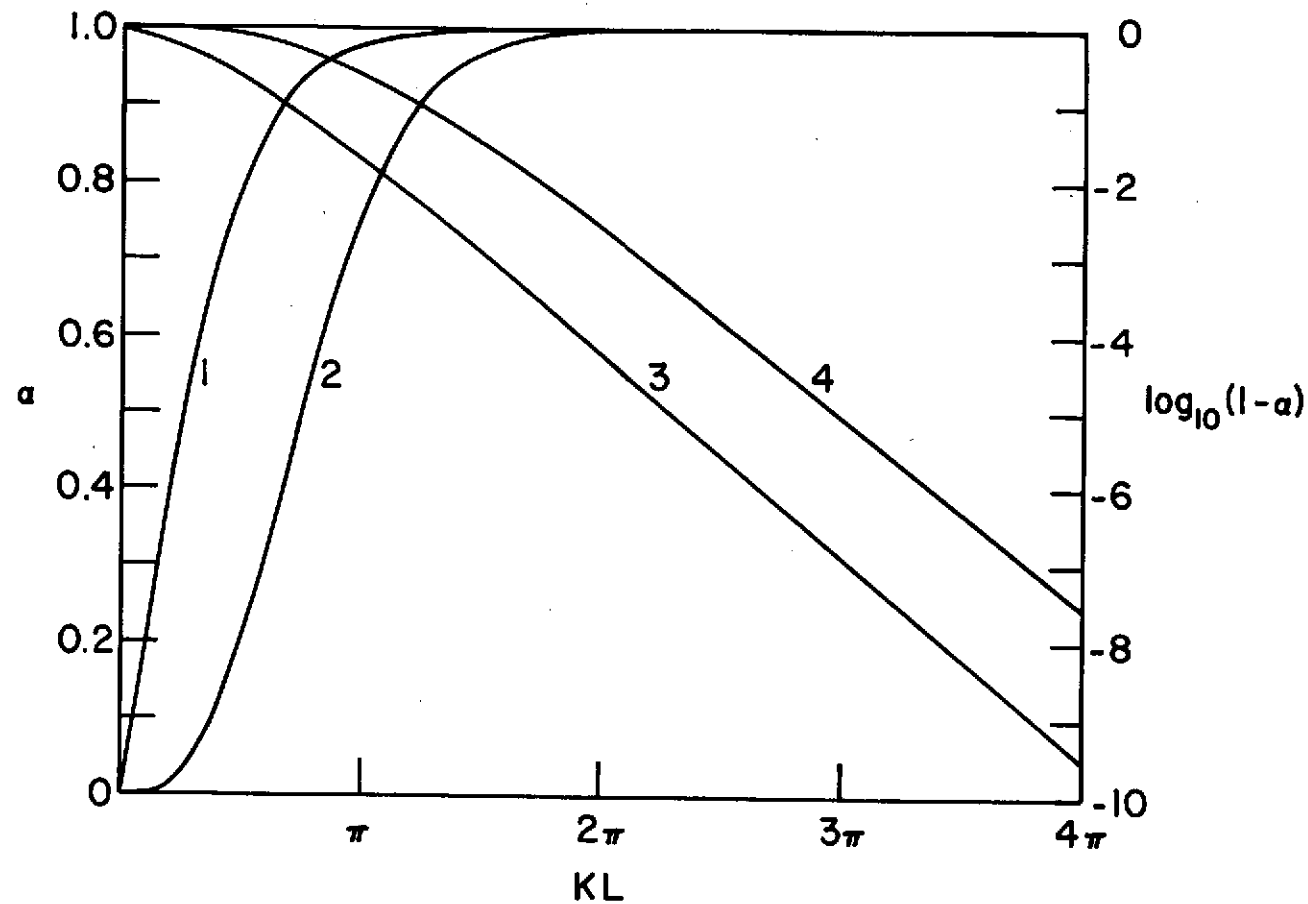

FIGURE 1

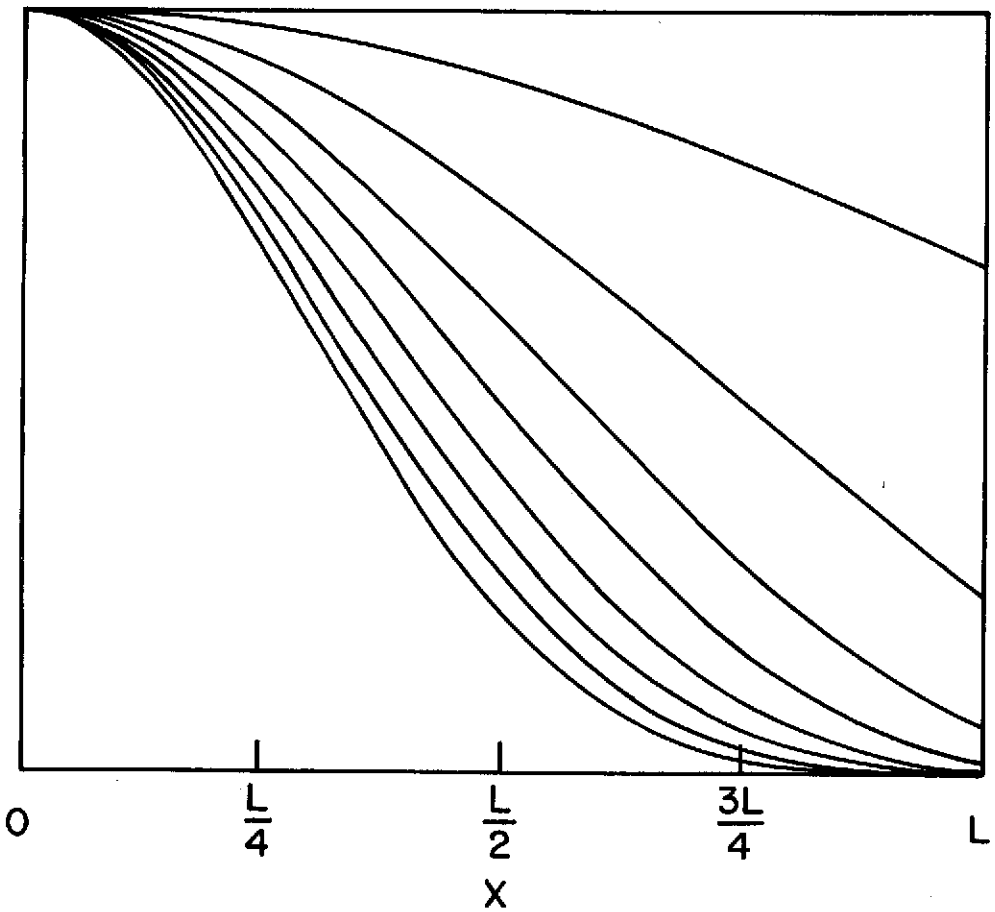

FIGURE 2