A SIMPLIFIED SIGNAL ANALYSIS TECHNIQUE FOR OBTAINING OPTIMAL ESTIMATES OF SYSTEM DYNAMICS

by

R. F. Whitbeck
Calspan Corporation

Abstract

Given an unknown system with n inputs and m outputs, one possible modeling approach is to compute estimates of transfer functions using the mathematical methods of Wiener-Hopf (i.e., the steady state Kalman estimators). Unfortunately, all the computational methods for implementing the Wiener-Hopf approach require (or assume) an analytical expression for the determinant of the spectral matrix. These assumptions, in turn, pre-ordain the final form of the Wiener estimates and hence the approach tends to lose the "Free-Minimization" methodology which makes it attractive. Clearly, what is needed is an algorithm which operates on the finite length input and output signals in a manner which does not bias the results through the use of possibly unwarranted assumptions. The procedure set forth in this paper furnishes such an algorithm for the special case where n finite-length realizations, for each of the n input signals, are available.

1. Introduction

The Wiener-Hopf method leads to the formalization of a spectral matrix which must be factored into the product of two matrices, one of which is analytic in the right half of the complex frequency plane; the other analytic in the left half plane. Unfortunately, all methods for accomplishing this factorization, at one particular step or another, require (or assume) an analytical expression for the determinant of the spectral matrix. This assumption, in turn, pre-ordains the final form of the Wiener estimates.

For example, if it is assumed that the determinant of the spectral matrix is a rational polynomial in S, then the estimators are forced to be rational polynomials in S also. Clearly what is needed is a computational algorithm which operates on the finite-length input and output signals and furnishes optimal estimates without prejudicing the results through the use of possibly unwarranted assumptions. The procedure set forth in this paper furnishes such an algorithm for the special case where the unknown system has n inputs and n finite length realizations for each of the n input signals are available.

2. Theory

In the interests of brevity, the results will be developed for the case where the unknown system has the two-input and one-output configuration shown in Figure 1. After this, the extension to the case of n inputs and m outputs will be obvious.

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FIGURE 1. TWO INPUT CASE

The Wiener-Hopf minimization procedure leads, for the ideal case of an infinite number of infinite length realizations of the signals, to Equation (1):

$$\begin{bmatrix} \phi_{XX} & \phi_{XY} \\ \phi_{YX} & \phi_{YY} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} - \begin{bmatrix} \phi_{XZ} \\ \phi_{YZ} \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \quad (1)$$

In Equation (1),

$$\phi_{XY} = \lim_{T \to \infty} \frac{1}{2T} E\left\{ X_{(-s)} Y_{(s)} \right\}, \text{ etc.} \quad (2)$$

where, for example,

$$Y_{(s)} = \int_{-T}^{T} y(t) e^{-st} dt \quad (3)$$

The ξ vector is unknown but must have the property of being analytic in the L.H.P. The $E\{\ \}$ denotes the ensemble average of the various auto and cross power spectra.

The solution to this equation, given that $\phi = \theta_p \theta$ in

$$\phi H - \psi = \xi$$

is

$$H = \theta^{-1} \left[\theta_p^{-1} \psi \right]_+ \quad (4)$$

$$H=\frac{\begin{bmatrix}\frac{Y_2}{2\sqrt{T}} & -\frac{Y_1}{2\sqrt{T}}\\ -\frac{X_2}{2\sqrt{T}} & \frac{X_1}{2\sqrt{T}}\end{bmatrix}}{\frac{1}{4T}\left[X_1Y_2-Y_1X_2\right]}\left\{\frac{\begin{bmatrix}\frac{\bar{Y}_2}{2\sqrt{T}} & -\frac{\bar{X}_2}{2\sqrt{T}}\\ -\frac{\bar{Y}_1}{2\sqrt{T}} & \frac{\bar{X}_1}{2\sqrt{T}}\end{bmatrix}}{\frac{1}{4T}\left[\bar{X}_1\bar{Y}_2-\bar{X}_2\bar{Y}_1\right]}\begin{bmatrix}\frac{1}{4T}\left(\bar{X}_1Z_1+\bar{X}_2Z_2\right)\\ \frac{1}{4T}\left(\bar{Y}_1Z_1+\bar{Y}_2Z_2\right)\end{bmatrix}\right\}_+ \tag{8}$$

Carrying out the inner multiplication gives

$$H=\frac{\begin{bmatrix}\frac{Y_2}{2\sqrt{T}} & -\frac{Y_1}{2\sqrt{T}}\\ -\frac{X_2}{2\sqrt{T}} & \frac{X_1}{2\sqrt{T}}\end{bmatrix}}{\frac{1}{4T}\left[X_1Y_2-Y_1X_2\right]}\left\{\frac{\begin{matrix}\frac{1}{8T^{3/2}}\left[\bar{Y}_2\bar{X}_1Z_1+\cancel{\bar{Y}_2\bar{X}_2}Z_2-\bar{X}_2\bar{Y}_1Z_1-\cancel{\bar{X}_2\bar{Y}_2}Z_2\right]\\ \frac{1}{8T^{3/2}}\left[-\cancel{\bar{Y}_1\bar{X}_1}Z_1-\bar{Y}_1\bar{X}_2Z_2+\cancel{\bar{X}_1\bar{Y}_1}Z_1+\bar{X}_1\bar{Y}_2Z_2\right]\end{matrix}}{\frac{1}{4T}\left(\bar{X}_1\bar{Y}_2-\bar{X}_2\bar{Y}_1\right)}\right\}_+$$

or

$$H=\frac{\begin{bmatrix}\frac{Y_2}{2\sqrt{T}} & -\frac{Y_1}{2\sqrt{T}}\\ -\frac{X_2}{2\sqrt{T}} & \frac{X_1}{2\sqrt{T}}\end{bmatrix}}{\frac{1}{4T}\left[X_1Y_2-Y_1X_2\right]}\left\{\begin{matrix}\frac{Z_1}{2\sqrt{T}}\\ \frac{Z_2}{2\sqrt{T}}\end{matrix}\right\}_+$$

$$\therefore\begin{bmatrix}H_1\\ H_2\end{bmatrix}=\frac{\begin{bmatrix}Y_2 & -Y_1\\ -X_2 & X_1\end{bmatrix}}{X_1Y_2-Y_1X_2}\begin{bmatrix}Z_1\\ Z_2\end{bmatrix}=\frac{\begin{bmatrix}Y_2Z_1-Y_1Z_2\\ X_1Z_2-X_2Z_1\end{bmatrix}}{X_1Y_2-Y_1X_2} \tag{9}$$

The details of solving this equation analytically are discussed, for example, in Reference 1 and 2.

For the case where only a finite number of fixed length records are available, the various power spectra in Equation (1) are replaced by their estimates. For example, the estimate of ϕ_{XY}, call it $\hat{\phi}_{XY}$, is

$$\hat{\phi}_{XY}=\frac{1}{2Tn}\sum_{i=1}^{n}X_{i(-s)}Y_{i(s)} \tag{5}$$

Suppose now that only two realisations of each signal are available. Equation (1) becomes, using $\bar{X}$ to denote $X_{(-s)}$, etc.:

$$\begin{bmatrix}\frac{1}{4T}\left(\bar{X}_1X_1+\bar{X}_2X_2\right) & \frac{1}{4T}\left(\bar{X}_1Y_1+\bar{X}_2Y_2\right)\\ \frac{1}{4T}\left(\bar{Y}_1X_1+\bar{Y}_2X_2\right) & \frac{1}{4T}\left(\bar{Y}_1Y_1+\bar{Y}_2Y_2\right)\end{bmatrix}\begin{bmatrix}H_1\\ H_2\end{bmatrix}-\begin{bmatrix}\frac{1}{4T}\left(\bar{X}_1Z_1+\bar{X}_2Z_2\right)\\ \frac{1}{4T}\left(\bar{Y}_1Z_1+\bar{Y}_2Z_2\right)\end{bmatrix}=\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix} \tag{6}$$

Because we have chosen to work directly in the frequency domain with the transform of the various signals, the factorization of the two input case becomes trivial when only two realizations of the inputs are available. Factor Equation (6) as

$$\hat{\phi}=\begin{bmatrix}\frac{\bar{X}_1}{2\sqrt{T}} & \frac{\bar{X}_2}{2\sqrt{T}}\\ \frac{\bar{Y}_1}{2\sqrt{T}} & \frac{\bar{Y}_2}{2\sqrt{T}}\end{bmatrix}\begin{bmatrix}\frac{X_1}{2\sqrt{T}} & \frac{Y_1}{2\sqrt{T}}\\ \frac{X_2}{2\sqrt{T}} & \frac{Y_2}{2\sqrt{T}}\end{bmatrix}=\theta_{-}\theta \tag{7}$$

Compute the inverses of the θ and θ_{-} given in Equation (7) and substitute into Equation (4)

Thus one need only work with the transforms of the signals when only two realizations are available. Note that no assumptions were necessary - the Wiener-Hopf theory guarantees that the H's computed according to Equation (9) are the best mean square fits possible under the stated conditions.

The extention to the n input case is now obvious but will require a double subscript notation (refer to Figure 2).

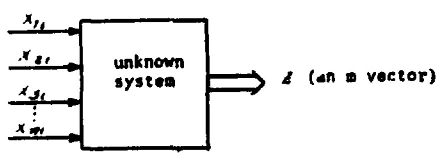

FIGURE 2. DOUBLE SUBSCRIPT. INPUTS

The factorization is now $\theta_* \theta$ where

$$\theta = \begin{bmatrix} \frac{x_{11}}{\sqrt{2\pi T}} & \frac{x_{21}}{\sqrt{2\pi T}} & \cdots & \frac{x_{n1}}{\sqrt{2\pi T}} \\ \frac{x_{12}}{\sqrt{2\pi T}} & \cdots & & \frac{x_{n2}}{\sqrt{2\pi T}} \\ \vdots & & & \\ \frac{x_{1n}}{\sqrt{2\pi T}} & \cdots & & \frac{x_{nn}}{\sqrt{2\pi T}} \end{bmatrix} \qquad (10)$$

3. <u>Experimental Results</u>

To develop some feel for the results given in Equation (9), the simple experiment shown in Figure 3 was set up as a digital simulation.

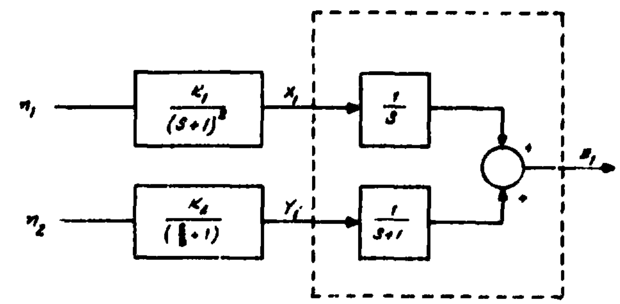

FIGURE 3. BLOCK DIAGRAM OF DIGITAL SIMULATION

In Figure 3, n_1 and n_2 are white noise sources fed through two different filters to produce the input signals X and Y. The estimates of $1/s$ and $1/(s+1)$ were then computed using Equation (9). The results are shown in Figure 4 for three different values of the ratio of K_1/K_2. As can be seen, the estimate of $1/s$ is excellent when the n_1 noise dominates while the estimate of $1/(s+1)$ is excellent when n_2 dominates. Somewhere in between, poorer estimates of both $1/s$ and $1/(s+1)$ are obtained.

The experiment also provided an opportunity to check out the effect of operating on the data with three different time windows. These are denoted as the square window (⊓), the triangular data window (△) and the triangular window convolved with itself (△ * △). The resulting magnitude plots for $1/s$ ($K_1/K_2 = 25$), shown in Figure 5, demonstrates the dramatic effect the time window has on the variability of the estimates.

4. <u>Conclusion</u>

Given an n input system and n experimental records for each input, it has been shown that the optimum Wiener filter is easily computed using only the experimental data. That is, no assumptions concerning the analytical structure of the spectral matrix are necessary.

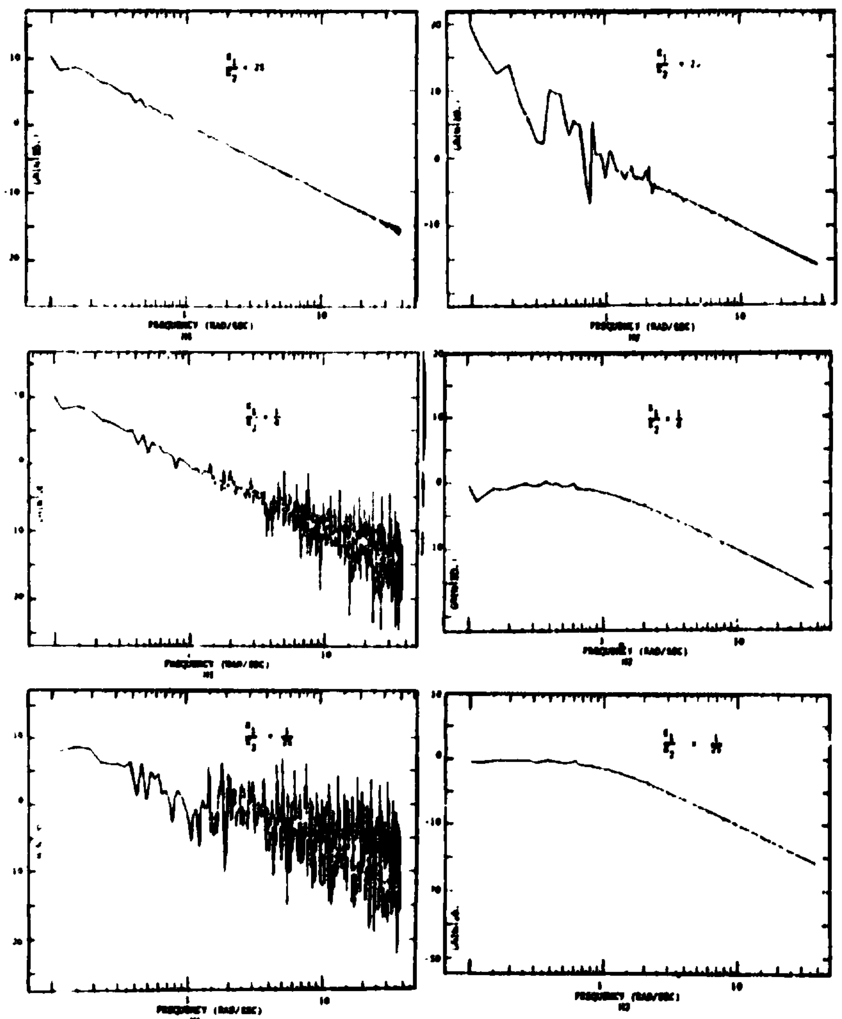

FIGURE 4. ESTIMATION OF H_1, H_2 WITH K_1/K_2 AS A PARAMETER

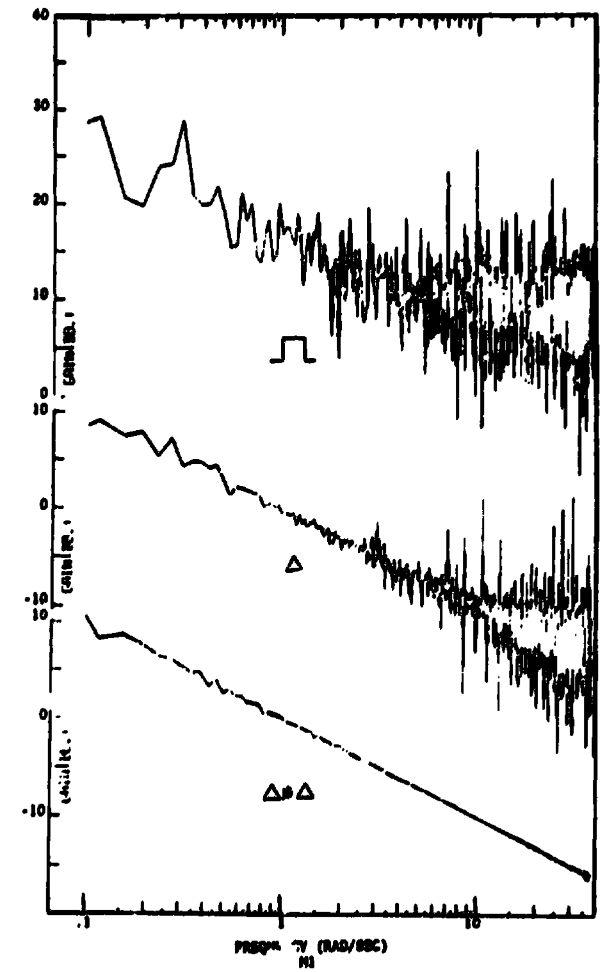

FIGURE 5. EFFECT OF TIME WINDOW ON $1/s$ ESTIMATES

References

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2. Whitbeck, R. F., Knight, J. R., A Study of Pilot Modeling in Multi-Controller Tasks, Cornell Aeronautical Laboratory, Report TH-3037-J-1.