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A NEW FORMULATION OF THE CONSERVATION EQUATIONS OF FLUID DYNAMICS

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16. Abstract

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A NEW FORMULATION OF THE CONSERVATION EQUATIONS OF FLUID DYNAMICS

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SUMMARY

The computation of time-dependent flows has inspired a new, higher-dimensional formulation of the conservation equations of fluid dynamics in which time is treated as a fourth coordinate. The formulation is derived for a constant-density flow, and then extended to a variable-density flow by introducing a fifth, fictitious coordinate. This new coordinate can also act as a source coordinate, so that external source terms can be included. The analysis is carried out for both incompressible, stratified flow, and compressible equilibrium flow. The results are then extended to non-equilibrium and magnetohydrodynamic flows. Several applications of the new formulation to the computation of time-dependent flows are discussed.

1. INTRODUCTION

An important concept in computational fluid dynamics is the conservation-law form of the flow equations [1]. These are differential forms of the basic integral relations expressing the conservation of mass, momentum, and energy. Since the latter are valid in the presence of discontinuities, appropriate methods of differencing the differential equations will result in the correct jump conditions being satisfied for the computed, smeared-out discontinuities. In Cartesian coordinates, the equations consist only of sums of derivatives - called a strong conservation-law form [2]. One can then difference the equations so as to assure exact numerical conservation of mass, momentum, and energy for the total flow region.

If one employs curvilinear coordinates, in order to satisfy conditions on general boundaries, undifferentiated terms due to the derivatives of the base vectors appear in the scalar decomposition of the momentum equation. This weak conservation-law form [2] prevents the achievement of overall numerical conservation of mass, momentum, and energy, and can thus lead to loss of accuracy in "capturing" sharply curved discontinuities. To avoid this, Anderson, Preiser, and Rubin [3] introduced Cartesian base vectors and achieved a scalar decomposition in the strong conservation-law form. A similar idea was used by MacCormack and Paullay [4], although they employed the integral formulation of the equations. Calculations using this latter formulation have been carried out by Rizzi and Inouye [5] for blunt-body flows, and by Rizzi, Klavins, and MacCormack [6] for supersonic flows. Recently, the author [2] proposed a new, simpler, local scalar decomposition in the strong conservation-law form, using the curvilinear base vectors at the fixed, computational mesh points.

There are two important classes of time-dependent problems in which it is useful to introduce new independent variables which depend on both time and the spatial coordinates. One class involves time-varying boundaries such as flexible walls, detached shock waves, jet boundaries, etc. If the unsteady motion of these boundaries is coupled strongly to the time-dependent phenomena being studied, coordinate transformations making these boundaries coordinate surfaces will greatly simplify the satisfying of boundary conditions. The other class of problems is flow past (or inside) a rigid body undergoing arbitrary translational or rotational motion. It is then useful to employ a non-inertial reference frame fixed with respect to the body. In both cases, the time-dependent coordinate transformations result in undifferentiated "source" terms and a weak conservation law. The situation here is analogous to that discussed in the previous paragraph, where the replacement of Cartesian by curvilinear coordinates led to a weak conservation law. It is therefore natural to seek a strong conservation-law form of the equations resulting from the time-dependent coordinate transformations, analogous to the strong conservation-law form derived by the author for curvilinear coordinates. If one does, one quickly finds that one must introduce a base vector associated with the time coordinate, and that the continuity and energy equations must be coupled with the momentum equation in a single conservation law. The analogy would then also require a fifth, fictitious coordinate (and corresponding base vector). The time-dependent coordinate transformation then becomes a simple coordinate transformation in the higher-dimensional space, and the five equations in strong conservation form are given by the local scalar decomposition of a single vector equation.

The author has developed a higher-dimensional formulation of the conservation equations of fluid dynamics, motivated by the above considerations, and has applied it in Ref. [2] to extend the strong-conservation form of the equations to time-dependent coordinate transformations. The purpose of the present paper is twofold. One is to give a detailed derivation of the formulation (not presented in Ref. [2]), and where possible, to give physical justification for the mathematical assumptions that are made. The other purpose is to generalize the results to a wider class of flow equations, in the hope that the new formulation will find other fields of application.

While the primary application is to generalized coordinates in compressible flow, in order to present the ideas as simply as possible the formulation is first derived in §2 for orthogonal curvilinear coordinates in a constant-density flow. The flow equations are shown to result from a single four-dimensional equation in Euclidean space-time. The components in the spatial direction yield the momentum equation, while the component in the time direction gives the continuity equation. (By contrast, in the space-time of special relativity, time components are related to an energy equation). A physical interpretation of the time component of velocity is proposed. Finally, an examination of steady-state flow shows that the continuity equation can still be considered as the component of a four-dimensional equation in a fourth direction, even though the fourth coordinate is ignorable.

The ideas in the previous sentence are used to consider variable-density flow in §3. An ignorable, fictitious fifth coordinate is introduced, whose directions yield the additional conservation equation which is required. This coordinate can also be employed to include real source terms in the analysis. Both compressible equilibrium flow and incompressible stratified flow are treated. In §4 the analysis is extended to non-equilibrium and magnetohydrodynamic flows. The restriction to orthogonal coordinates is also removed in this section, where results are given in general, curvilinear coordinates. The discussion in §5 centers on the closure problem of completing the set of conservation equations by auxiliary equations. As an illustration of the use of the formulation presented in this paper, this section concludes with a derivation of the scalar decomposition in strong conservation-law form of the ideal gas equations for a general class of time-dependent coordinate transformations. A brief discussion of the conservation laws in a non-inertial reference frame is also presented.

2. CONSERVATION EQUATION FOR CONSTANT-DENSITY FLOW

2.1 Three-Dimensional Formulation

Consider the non-relativistic motion of a fluid with constant density ρ. The continuity equation

$$\text{div}\ (\rho\vec{u}) = 0 \tag{2.1}$$

expresses the conservation of mass, where $\vec{u}$ is the velocity vector. The momentum equation relates the time rate of change of momentum per unit volume to the net convection of momentum and the net force acting on the fluid contained in that volume. The surface force can be expressed as the divergence of a stress tensor. If the pressure p characterizes the state of stress existing in a fluid in uniform motion, the stress tensor for a fluid in non-uniform motion can be written as

$$\vec{S} - p\vec{I} ,$$

where the symmetric tensor $\vec{S}$ is called the extra stress tensor or viscous stress tensor, and $\vec{I}$ is the identity tensor. In an inertial reference frame, the external body force in the absence of electromagnetic effects is due to the gravity. (An external force due to a magnetic field is discussed in §4.2.) Introducing the gravitational potential ϕ, the body force per unit mass is

$$\vec{f} = - \text{grad } \phi = - \text{div } (\phi\vec{I}) . \tag{2.2}$$

If one defines a generalized stress tensor

$$\vec{P} \equiv \vec{S} - (p + \rho\phi)\,\vec{I} , \tag{2.3}$$

and the convection tensor

$$\vec{C} \equiv \rho\,\vec{u}\,\vec{u} , \tag{2.4}$$

the vector conservation-law form of the momentum equation can be written as

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \text{div } \vec{T} = 0 , \tag{2.5}$$

where the flow tensor $\vec{T}$ is defined as

$$\vec{T} \equiv \vec{C} - \vec{P} . \tag{2.6}$$

In Cartesian coordinates x_1, x_2, x_3, equations (2.1) and (2.5) are written as

$$\frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3) = 0 , \tag{2.7}$$

$$\frac{\partial}{\partial t}(\rho u_1) + \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 0 , \tag{2.8a}$$

$$\frac{\partial}{\partial t}(\rho u_2) + \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = 0 , \tag{2.8b}$$

$$\frac{\partial}{\partial t}(\rho u_3) + \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0 . \tag{2.8c}$$

As written, the continuity equation and the three components of the momentum equation all consist of sums of derivatives - called a strong conservation-law form [2]. The equivalence in structure of these four equations is only valid in Cartesian coordinates. In curvilinear, orthogonal coordinates, with scale factors h_1, h_2, h_3, and unit base vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, equations (2.1) and (2.5) become

$$\frac{\partial}{\partial x_1}(h_2 h_3 \rho u_1) + \frac{\partial}{\partial x_2}(h_1 h_3 \rho u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 \rho u_3) = 0 , \qquad (2.9)$$

$$\frac{\partial}{\partial t}(h_1 h_2 h_3 \sum_{j=1}^{3} \rho u_j \vec{e}_j) + \frac{\partial}{\partial x_1}(h_2 h_3 \sum_{j=1}^{3} T_{j1}\vec{e}_j) + \frac{\partial}{\partial x_2}(h_1 h_3 \sum_{j=1}^{3} T_{j2}\vec{e}_j) + \frac{\partial}{\partial x_3}(h_1 h_2 \sum_{j=1}^{3} T_{j3}\vec{e}_j) = 0 \qquad (2.10)$$

The decomposition of (2.10) into three scalar equations would introduce undifferentiated terms (weak conservation-law form [2]) due to the terms $\partial\vec{e}_j/\partial x_i$. Actually, a natural scalar decomposition of (2.10) in the strong form is possible [2]. Since this paper is concerned with the formulation of the conservation equations as a single vector equation in a higher-dimensional space, the vector form of the momentum conservation law (2.10) will be retained. The continuity equation (2.9), which is a scalar equation, is thus clearly distinct in character from the momentum equation.

2.2 Four-dimensional formulation

The time derivative in (2.10) has an appearance similar to that of the spatial derivatives. This suggests that the momentum equation can be put completely in divergence form if one considers time as a fourth coordinate in a four-dimensional space. Letting $x_4 \equiv t$, consider the equation

$$\mathrm{div}_4\, \vec{T} = 0 , \qquad (2.11)$$

where $\vec{T}$ is now a four-dimensional flow tensor. In terms of the components of $\vec{T}$, equation (2.11) can be written as

$$\frac{\partial}{\partial x_1}(h_2 h_3 h_4 \sum_{\alpha=1}^{4} T_{\alpha 1}\vec{e}_\alpha) + \frac{\partial}{\partial x_2}(h_1 h_3 h_4 \sum_{\alpha=1}^{4} T_{\alpha 2}\vec{e}_\alpha) + \frac{\partial}{\partial x_3}(h_1 h_2 h_4 \sum_{\alpha=1}^{4} T_{\alpha 3}\vec{e}_\alpha)$$
$$+ \frac{\partial}{\partial x_4}(h_1 h_2 h_3 \sum_{\alpha=1}^{4} T_{\alpha 4}\vec{e}_\alpha) = 0. \qquad (2.12)$$

Greek indexes will indicate quantities defined over the four-dimensional space, while Latin indexes will be reserved for quantities defined over the physical, three-dimensional space. The independence of time and physical space leads to the conditions

$$\vec{e}_4 \cdot \vec{e}_i = \partial \vec{e}_i / \partial x_4 = \partial h_i / \partial x_4 = 0 \qquad (i = 1 \text{ to } 3). \tag{2.13}$$

It follows that $\vec{e}_4$ is constant, so that x_4 is a straight coordinate, and that h_4 is at most a function of x_4. One can thus uncouple the equation for $\vec{e}_4$ in (2.12). The equation for the remaining components is

$$\frac{\partial}{\partial x_1}\left(h_2 h_3 \sum_{j=1}^{3} T_{j1}\vec{e}_j\right) + \frac{\partial}{\partial x_2}\left(h_1 h_3 \sum_{j=1}^{3} T_{j2}\vec{e}_j\right)$$

$$+ \frac{\partial}{\partial x_3}\left(h_1 h_2 \sum_{j=1}^{3} T_{j3}\vec{e}_j\right) + \frac{1}{h_4}\frac{\partial}{\partial x_4}\left(h_1 h_2 h_3 \sum_{j=1}^{3} T_{j4}\vec{e}_j\right) = 0. \tag{2.14}$$

Comparing (2.14) with (2.10), one obtains agreement if

$$T_{j4} = \rho u_j h_4 \qquad (j=1 \text{ to } 3), \tag{2.15}$$

where h_4 is now <u>constant</u>.

Since the three-dimensional flow tensor has symmetric components T_{ij}, it is reasonable to assume that the components of the four-dimensional flow tensor, $T_{\alpha\beta}$, are also symmetric. Therefore let

$$T_{4j} = \rho h_4 u_j \qquad (j = 1 \text{ to } 3). \tag{2.16}$$

The $\underset{\sim}{e}_4$ component of (2.12) thus becomes

$$\frac{\partial}{\partial x_1}(h_2 h_3 \rho u_1) + \frac{\partial}{\partial x_2}(h_1 h_3 \rho u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 \rho u_3) + \frac{h_1 h_2 h_3}{h_4{}^2}\frac{\partial T_{44}}{\partial x_4} = 0. \tag{2.17}$$

One quickly recognizes (2.17) as the continuity equation (2.9), provided that

$$\partial T_{44} / \partial x_4 = 0. \tag{2.18}$$

Conditions (2.15), (2.16) and (2.18) can be shown to follow logically from the assumption that (2.4) defines a <u>four-dimensional</u> convection tensor

$\vec{C}$, with the fourth (time) component of the four-dimensional velocity $\vec{u}$ satisfying

$$u_4 = h_4. \tag{2.19}$$

Equations (2.15), (2.16) and (2.18) can then be replaced by the relation

$$T_{\alpha 4} = T_{4\alpha} = C_{\alpha 4} \qquad (\alpha = 1 \text{ to } 4). \tag{2.20}$$

If (2.6) is assumed to be valid in the four-dimensional space, it follows that

$$P_{\alpha 4} = P_{4\alpha} = 0 \qquad (\alpha = 1 \text{ to } 4). \tag{2.21}$$

Relation (2.19) is made plausible by considering the total derivative as a four-dimensional convective derivative, i.e.,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial x_4} + \sum_{j=1}^{3} \frac{u_j}{h_j} \frac{\partial}{\partial x_j} = \sum_{\alpha=1}^{4} \frac{u_\alpha}{h_\alpha} \frac{\partial}{\partial x_\alpha} \tag{2.22}$$

Alternatively, extending the definition of velocity as the total derivative of the position vector to four dimensions, one obtains

$$u_\alpha = h_\alpha \frac{Dx_\alpha}{Dt} = h_\alpha \left(\frac{\partial x_\alpha}{\partial x_4} + \sum_{j=1}^{3} \frac{u_j}{h_j} \frac{\partial x_\alpha}{\partial x_j}\right) \qquad (\alpha = 1 \text{ to } 4). \tag{2.23}$$

Equation (2.19) follows from (2.23) by setting $\alpha = 4$.

The analysis, so far, does not provide a natural constant velocity to assign to u_4. It can be obtained from a different interpretation of the velocity. In terms of the conservation laws, mass and momentum are considered fundamental. The velocity can then be defined as the specific momentum, or momentum per unit mass. The components of the four-dimensional conservation equation in the three spatial directions give the conservation of momentum, while the component in the fourth (time) direction has been shown to yield the continuity equation expressing the conservation of mass. It is thus reasonable to define the fourth (time) component of velocity as the specific mass, or unity.

In summary, it is natural to introduce time as the fourth, straight coordinate in a four-dimensional Euclidean space, with the scale factor and velocity component in the fourth direction satisfying

$$u_4 = h_4 = 1. \tag{2.24}$$

The conservation of mass and momentum are then embodied in the single four-dimensional vector equation

$$\mathrm{div}_4\, \vec{T} = 0, \tag{2.11}$$

with the four-dimensional flow tensor T given by

$$\vec{T} = \vec{C} - \vec{P}, \tag{2.25}$$

where the four-dimensional convection tensor components are

$$C_{\alpha\beta} = \rho u_\alpha u_\beta \qquad (\alpha, \beta = 1 \text{ to } 4), \tag{2.26}$$

and the four-dimensional generalized stress tensor components are

$$P_{\alpha\beta} = \begin{bmatrix} S_{11}-p-\rho\phi & S_{12} & S_{13} & 0 \\ S_{21} & S_{22}-p-\rho\phi & S_{23} & 0 \\ S_{31} & S_{32} & S_{33}-p-\rho\phi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.27}$$

The S_{ij} in (2.27) are components of the three-dimensional extra stress tensor $\vec{S}$. Thus, for a constant density flow, a single vector equation expresses the conservation of mass and momentum. It is because the time coordinate is always straight, and orthogonal to the spatial coordinates, that the component in the time direction can be uncoupled to produce a scalar equation for the conservation of mass. Since the spatial coordinates can be curved, the momentum equation in conservation-law (or divergence) form must in general remain a vector equation.

2.3 Steady-State Flow

An important special case is that of steady-state flow, in which x_4 is an ignorable coordinate, i.e.,

$$\partial/\partial x_4 = 0. \tag{2.28}$$

There is a fundamental difference between this case and that in which a spatial coordinate is ignorable. The simplest example of the latter is plane flow, in which the velocity component in the ignorable direction is zero. The component of the momentum equation in the ignorable direction then vanishes identically. The number of equations is reduced by one, so that the remaining equations can be embodied in a single three-dimensional (two space and one time) vector equation for a three-dimensional flow tensor.

In steady-state flow, the velocity component u_4 in the ignorable direction does not vanish, but remains equal to one. Therefore, the component of the four-dimensional conservation equation in the ignorable direction does not vanish, but actually remains unchanged. Only in the spatial components are the time derivative terms absent. While the number of independent variables has been reduced by one, the number of equations remains the same. One must still use a four-dimensional formulation to describe the conservation laws. While the <u>coordinate</u> is now a fictitious, ignorable coordinate from the steady-state viewpoint, the <u>direction</u> of the fourth coordinate is very real, since it gives rise to the continuity equation. It is appropriate therefore to refer to the x_4 coordinate as the time coordinate, but to think of the direction of the fourth coordinate as the mass direction, since it is connected with the conservation of mass.

3. CONSERVATION EQUATIONS FOR VARIABLE-DENSITY FLOW

3.1 <u>General Formulation</u>

The continuity equation for variable-density flow is

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \vec{u}) = 0. \tag{3.1}$$

In the momentum equation, the external body force per unit <u>volume</u>, $\rho\vec{f}$, can no longer be put in divergence form, even if $\vec{f}$ is conservative. It therefore cannot be included as part of a generalized stress tensor, but must appear as an external source term. The momentum equation is thus written as

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \text{div}\,\vec{\vec{T}} - \rho\vec{f} = 0, \tag{3.2}$$

where (2.4) and (2.6) are still valid, but $\vec{\vec{P}}$ is defined as

$$\vec{\vec{P}} \equiv \vec{\vec{S}} - p\vec{\vec{I}}. \tag{3.3}$$

The body force $\vec{f}$ is left unspecified. In addition to a gravitational force, it can also include the fictitious force due to the employment of a non-inertial reference frame. In orthogonal curvilinear coordinates, (3.1) and (3.2) are written as

$$\frac{\partial}{\partial t}(h_1h_2h_3\rho) + \frac{\partial}{\partial x_1}(h_2h_3\rho u_1) + \frac{\partial}{\partial x_2}(h_1h_3\rho u_2) + \frac{\partial}{\partial x_3}(h_1h_2\rho u_3) = 0, \tag{3.4}$$

$$\frac{\partial}{\partial t}(h_1h_2h_3\sum_{j=1}^{3}\rho u_j\vec{e}_j) + \frac{\partial}{\partial x_1}(h_2h_3\sum_{j=1}^{3}T_{j1}\vec{e}_j) + \frac{\partial}{\partial x_2}(h_1h_3\sum_{j=1}^{3}T_{j2}\vec{e}_j)$$

$$+ \frac{\partial}{\partial x_3}(h_1h_2\sum_{j=1}^{3}T_{j3}\vec{e}_j) - h_1h_2h_3\rho\sum_{j=1}^{3}f_j\vec{e}_j = 0, \tag{3.5}$$

The presence of the variable density requires an additional conservation equation to complete the system. The next two sections will treat two different forms of this additional equation. The results obtained in this section will be independent of the specific form of the additional equation. The discussion of steady-state flow in §2.3 showed that a scalar conservation equation (in this case the continuity equation) can be associated with a coordinate direction, even though the coordinate itself is ignorable. This suggests that the additional equation be associated with the direction of a new, ignorable coordinate x_5. It thus must have the property that

$$\partial/\partial x_5 = 0 \tag{3.6}$$

for all physical variables. It is easy to show that $\vec{e}_5$ must be constant, and orthogonal to all the other base vectors. Without any loss of generality, one can let

$$h_5 = 1, \tag{3.7}$$

The expression for u_5 depends on the identity of the additional equation.

In the five-dimensional conservation equation

$$\mathrm{div}_5 \vec{T} = 0, \tag{3.8}$$

the equations for $\vec{e}_4$ and $\vec{e}_5$ can be uncoupled, leaving as the remaining equation

$$\frac{\partial}{\partial x_1}(h_2 h_3 \sum_{j=1}^{3} T_{j1}\vec{e}_j) + \frac{\partial}{\partial x_2}(h_1 h_3 \sum_{j=1}^{3} T_{j2}\vec{e}_j) + \frac{\partial}{\partial x_3}(h_1 h_2 \sum_{j=1}^{3} T_{j3}\vec{e}_j)$$

$$+ \frac{\partial}{\partial x_4}(h_1 h_2 h_3 \sum_{j=1}^{3} T_{j4}\vec{e}_j) + \frac{\partial}{\partial x_5}(h_1 h_2 h_3 \sum_{j=1}^{3} T_{j5}\vec{e}_j) = 0. \tag{3.9}$$

In comparing (3.9) with (3.5), one notes that the source term can be formally generated if one lets $T_{j5} = -\rho f_j x_5$. Thus the x_5 coordinate can be considered a source coordinate. This role for the x_5 coordinate seems appropriate, since the same phenomenon (variable density) requiring an additional conservation equation, and an associated new coordinate direction, also necessitates a body force term to be treated as an external source of momentum. Now T_{j5} is undefined within an arbitrary additive fuction of x_1, x_2, x_3, x_4. Generalizing the concept of the convection tensor to five dimensions, one would expect the term $\rho u_j u_5$. In order to satisfy the additional conservation equation, an additional arbitrary function g_j (x_1, x_2, x_3, x_4) will be necessary. Therefore, let

$$T_{j5} = \rho u_j u_5 - \rho f_j x_5 + g_j \qquad (j = 1 \text{ to } 3), \tag{3.10}$$

where the g_j are as yet unspecified functions of x_1, x_2, x_3, x_4. With $T_{j4} = \rho u_j u_4 = \rho u_j$, as in § 2.2, equation (3.9) is in agreement with the momentum equation (3.5).

If one generalizes (2.20) to five dimensions, i.e., lets

$$T_{4\alpha} = \rho u_4 u_\alpha = \rho u_\alpha \qquad (\alpha = 1 \text{ to } 5), \tag{3.11}$$

then the e_4 component of (3.8) becomes

$$\frac{\partial}{\partial x_1}(h_2 h_3 \rho u_1) + \frac{\partial}{\partial x_2}(h_1 h_3 \rho u_2) + \frac{\partial}{\partial x_3}(h_1 h_2 \rho u_3) + \frac{\partial}{\partial x_4}(h_1 h_2 h_3 \rho) = 0. \tag{3.12}$$

This is seen to be identical to the variable-density continuity equation (3.4).

Assuming that the five-dimensional flow tensor is symmetric, and letting

$$\vec{g} = \sum_{j=1}^{3} g_j \vec{e}_j, \tag{3.13}$$

the $\vec{e}_5$ component of (3.8) takes the form

$$\mathrm{div}\ (\rho u_5 \vec{u}) - x_5\ \mathrm{div}\ (\rho \vec{f}) + \mathrm{div}\ \vec{g} + \frac{\partial}{\partial x_4} (\rho u_5) + \frac{\partial T_{55}}{\partial x_5} = 0. \tag{3.14}$$

Equation (3.14) must be identified with the additional conservation equation. The next two sections will consider two such equations of technical interest.

3.2 Incompressible Stratified Flow

One case of a variable-density flow important in oceanography is that of an incompressible, but stratified fluid. The incompressibility condition is stated as

$$\mathrm{div}\ \vec{u} = 0, \tag{3.15}$$

which is equivalent to the conservation of volume. Following the interpretation of a velocity component as the specific value of the conserved quantity in the corresponding component of the conservation equation, it follows that u_5 should be defined as the specific volume, i.e.,

$$u_5 = 1/\rho, \tag{3.16}$$

Equation (3.14) can then be reduced to (3.15) if $\vec{g}$ is set equal to zero, and if one defines T_{55} as

$$T_{55} = \rho u_5^2 + \mathrm{div}\ (\rho \vec{f})\ x_5^2/2\ . \tag{3.17}$$

The five-dimensional flow tensor $\vec{T}$ appearing in (3.8) is thus given by

$$\vec{T} = \vec{C} - \vec{P}, \tag{3.18}$$

where the convection and generalized stress tensors are defined by

$$C_{\alpha\beta} = \rho\ u_\alpha u_\beta \qquad (\alpha, \beta = 1 \text{ to } 5), \tag{3.19}$$

and

$$P_{\alpha\beta} = \begin{bmatrix} S_{11} - p & S_{12} & S_{13} & 0 & \rho f_1 x_5 \\ S_{21} & S_{22} - p & S_{23} & 0 & \rho f_2 x_5 \\ S_{31} & S_{32} & S_{33} - p & 0 & \rho f_3 x_5 \\ 0 & 0 & 0 & 0 & 0 \\ \rho f_1 x_5 & \rho f_2 x_5 & \rho f_3 x_5 & 0 & -\operatorname{div}(\rho\vec{f})\, x_5^2/2 \end{bmatrix}, \quad (3.20)$$

together with equations (2.24), (3.7) and (3.16).

3.3 Compressible Equilibrium Flow

The more common case of a variable-density flow is that of a compressible gas. If the flow is in equilibrium, the energy equation is the only additional conservation equation. Let ε be the specific internal energy, so that the total energy per unit volume is

$$e = \rho\, [\varepsilon + \frac{1}{2}\,(u_1^2 + u_2^2 + u_3^2)]\,. \quad (3.21)$$

Define a heat-flux vector $\vec{q}$ to include both conduction and thermal radiation. For completeness, let $\dot{e}$ be the net rate of addition of energy per unit volume from a non-thermal source, such as a nuclear reaction. The conservation of energy can then be expressed as

$$\frac{\partial e}{\partial t} + \operatorname{div}\,(e\vec{u} + p\vec{u} - \vec{u}\cdot\vec{S} + \vec{q}) - \rho\vec{u}\cdot\vec{f} - \dot{e} = 0\,. \quad (3.22)$$

The velocity component u_5 defined by this conservation law is

$$u_5 = e/\rho = \varepsilon + \frac{1}{2}\,(u_1^2 + u_2^2 + u_3^2)\,. \quad (3.23)$$

Equation (3.14) reduces to (3.22) if one defines

$$\vec{g} = p\vec{u} - \vec{u}\cdot\vec{S} + \vec{q}\,, \quad (3.24)$$

and

$$T_{55} = \rho u_5^2 + \operatorname{div}\,(\rho\vec{f})\, x_5^2/2 - (\rho\vec{u}\cdot\vec{f} + \dot{e})\, x_5\,. \quad (3.25)$$

The five-dimensional formulation of compressible, equilibrium flow is thus defined by (3.8), (3.18), (3.19), (2.24), (3.7), and (3.23), with $P_{\alpha\beta}$ given by

$$
P_{\alpha\beta} = \begin{bmatrix}
S_{11} - p & S_{12} & S_{13} & 0 & \sum_{k=1}^{3} u_k S_{k1} - pu_1 - q_1 + \rho f_1 x_5 \\
S_{21} & S_{22} - p & S_{23} & 0 & \sum_{k=1}^{3} u_k S_{k2} - pu_2 - q_2 + \rho f_2 x_5 \\
S_{31} & S_{32} & S_{33} - p & 0 & \sum_{k=1}^{3} u_k S_{k3} - pu_3 - q_3 + \rho f_3 x_5 \\
0 & 0 & 0 & 0 & 0 \\
P_{15} & P_{25} & P_{35} & 0 & (\rho \sum_{k=1}^{3} u_k f_k + \dot{e})\, x_5 - \mathrm{div}(\rho \vec{f})\, x_5^2/2
\end{bmatrix} \quad (3.26)
$$

This completes the formulation of the conservation equations for variable-density flow. The results will be generalized in § 4.

4. GENERALIZATIONS

4.1 Non-Equilibrium Flow

If the flow of a compressible gas is out of equilibrium, in general one must introduce n independent non-equilibrium variables to specify the state of the gas. Let such a variable be designated by α_ν, where α_ν represents the mass fraction of species ν, or the specific internal energy (per mass of mixture) of the νth internal mode of some species. Equations (3.1), (3.2), and (3.22) are still valid, provided that $\vec{u}$ is the global velocity of the mixture, and the definitions of ε and $\vec{q}$ are appropriately modified to take into account the internal molecular structures and interdiffusion of the various species.

The rate equation for non-equilibrium variable α_ν can be written in conservation form as

$$\frac{\partial(\rho\alpha_\nu)}{\partial t} + \mathrm{div}\ [\rho\alpha_\nu(\vec{u} + \vec{u}_\nu^d)] - \rho\, \dot{\alpha}_\nu = 0 \qquad (\nu = 6 \text{ to } n + 5). \qquad (4.1)$$

Here $\vec{u}_\nu^d$ is the diffusion velocity for a variable α_ν which pertains to a species in a multi-component mixture, and $\dot{\alpha}_\nu$ is the net rate of production of α_ν due to internal processes. One can then introduce a fictitious, ignorable coordinate x_ν for each non-equilibrium variable α_ν, and again define

$$h_\nu = 1 \qquad (\nu = 6 \text{ to } n + 5). \qquad (4.2)$$

The velocity component u_ν defined by (4.1) is

$$u_\nu = \alpha_\nu \qquad (\nu = 6 \text{ to } n + 5). \tag{4.3}$$

The $\vec{e}_\nu$ component of the (n + 5)-dimensional conservation law

$$\mathrm{div}_{n+5}\, \vec{T} = 0 \tag{4.4}$$

is

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 T_{\nu 1}) + \frac{\partial}{\partial x_2}(h_1 h_3 T_{\nu 2}) + \frac{\partial}{\partial x_3}(h_1 h_2 T_{\nu 3})\right] + \frac{\partial T_{\nu 4}}{\partial x_4} + \frac{\partial T_{\nu 5}}{\partial x_5} + \sum_{\beta=6}^{n+5} \frac{\partial T_{\nu\beta}}{\partial x_\beta} = 0 \qquad (\nu = 6 \text{ to } n + 5), \tag{4.5}$$

In orthogonal coordinates, equation (4.1) is written as

$$\frac{\partial(\rho\alpha_\nu)}{\partial t} + \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial x_1}[h_2 h_3 \rho\alpha_\nu (u_1 + u^d_{\nu 1})] + \frac{\partial}{\partial x_2}[h_1 h_3 \rho\alpha_\nu (u_2 + u^d_{\nu 2})] + \frac{\partial}{\partial x_3}[h_1 h_2 \rho\alpha_\nu (u_3 + u^d_{\nu 3})]\right\} - \rho\dot{\alpha}_\nu = 0 \qquad (\nu = 6 \text{ to } n + 5). \tag{4.6}$$

If one again defines

$$T_{\nu\beta} = \rho\, u_\nu u_\beta - P_{\nu\beta} \qquad (\nu = 6 \text{ to } n + 5,\ \beta = 1 \text{ to } n + 5), \tag{4.7}$$

then the following relations follow from a comparison of (4.5) and (4.6):

$$\left.\begin{array}{ll} P_{\nu j} = P_{j\nu} = -\rho\, \alpha_\nu\, u^d_{\nu j} & (j = 1 \text{ to } 3) \\ P_{\nu 5} = P_{5\nu} = \rho\, \dot{\alpha}_\nu x_5 & \\ P_{\nu\beta} = P_{\beta\nu} = 0 & (\beta \neq 1,2,3,5) \end{array}\right\} \quad (\nu = 6 \text{ to } n + 5). \tag{4.8}$$

4.2 Magnetohydrodynamic Flow

If the fluid is electrically conducting, it is subject to external forces and energy transfer due to an electromagnetic field. For non-relativistic fluid motion, displacement currents and charge accumulation can be neglected, and Maxwell's equations are

$$\text{div}\,\vec{b} = 0\,, \tag{4.9}$$

$$\frac{\partial \vec{b}}{\partial t} + \text{curl}\,\vec{e} = 0, \tag{4.10}$$

$$\text{curl}\,\vec{b} - \mu\vec{j} = 0, \tag{4.11}$$

where $\vec{b}$, $\vec{e}$, $\vec{j}$ are the magnetic induction, electric field, and current density, respectively, and μ is the constant magnetic permeability of a vacuum.

The external force per unit volume, $\vec{j} \times \vec{b}$, can be rewritten, with the aid of (4.9) and (4.11), as

$$\vec{j} \times \vec{b} = \frac{1}{\mu}\,\text{curl}\,\vec{b} \times \vec{b} = \frac{1}{\mu}\,\text{div}\,\left(\vec{b}\vec{b} - \frac{|\vec{b}|^2}{2}\,\vec{I}\right). \tag{4.12}$$

The energy transferred to the fluid per unit volume from the electromagnetic field, $\vec{j}\cdot\vec{e}$, becomes

$$\vec{j}\cdot\vec{e} = \frac{1}{\mu}\,\text{curl}\,\vec{b}\cdot\vec{e} = -\frac{1}{\mu}\,\text{div}\,(\vec{e} \times \vec{b}) - \frac{1}{2\mu}\frac{\partial}{\partial t}\,(|\vec{b}|^2), \tag{4.13}$$

The effect of the electromagnetic field is to add to the five-dimensional generalized stress tensor $P_{\alpha\beta}$ the electromagnetic generalized stress tensor $P^m_{\alpha\beta}$ defined by

$$\mu P^m_{\alpha\beta} = \begin{bmatrix} b_1^2 - \frac{1}{2}\sum_{k=1}^{3} b_k^2 & b_1 b_2 & b_1 b_3 & 0 & b_2 e_3 - b_3 e_2 \\ b_2 b_1 & b_2^2 - \frac{1}{2}\sum_{k=1}^{3} b_k^2 & b_2 b_3 & 0 & b_3 e_1 - b_1 e_3 \\ b_3 b_1 & b_3 b_2 & b_3^2 - \frac{1}{2}\sum_{k=1}^{3} b_k^2 & 0 & b_1 e_2 - b_2 e_1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\sum_{k=1}^{3} b_k^2 \\ b_2 e_3 - b_3 e_2 & b_3 e_1 - b_1 e_3 & b_1 e_2 - b_2 e_1 & -\frac{1}{2}\sum_{k=1}^{3} b_k^2 & 0 \end{bmatrix} \tag{4.14}$$

Note that the ten, independent non-zero components of $P^m_{\alpha\beta}$ can be identified with the components of the four-dimensional, symmetric Maxwell stress-energy tensor defined in the relativistic formulation of electrodynamics.

4.3 General Coordinates

For many important applications, one must use coordinate systems more general than orthogonal coordinates. Let the physical space be described by general, curvilinear coordinates x^1, x^2, x^3. This defines a system of covariant base vectors $\vec{g}_i = \partial\vec{r}/\partial x^i$, where $\vec{r}$ is the position vector. The covariant components of the metric tensor are then given by $g_{ij} = \vec{g}_i \cdot \vec{g}_j$. The formulation of the conservation equations in general coordinates will be illustrated by considering compressible equilibrium flow. One therefore introduces a straight, fourth coordinate $x^4 = t$, and a fictitious, ignorable, straight, source coordinate x^5. The corresponding constant covariant base vectors, which are of unit length, are $\vec{g}_4$ and $\vec{g}_5$. The covariant components of the five-dimensional metric tensor are thus given by

$$g_{\alpha\beta} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{4.15}$$

Let the determinant of $g_{\alpha\beta}$ be designated by $g = |g_{\alpha\beta}| = |g_{ij}|$.

The five-dimensional conservation law

$$\mathrm{div}_5\ T = 0 \tag{3.8}$$

can be written in component form as

$$\frac{\partial}{\partial x^\beta} (\sqrt{g}\ T^{\alpha\beta}\ \vec{g}_\alpha) = 0, \tag{4.16}$$

where summation over repeated indexes is assumed, Greek indexes being summed from one to five, while Latin indexes are summed from one to three. The contravariant components of the flow tensor, $T^{\alpha\beta}$, can be expressed as

$$T^{\alpha\beta} = C^{\alpha\beta} - P^{\alpha\beta} \qquad (\alpha,\beta = 1 \text{ to } 5), \tag{4.17}$$

where the convection tensor is defined by

$$C^{\alpha\beta} \equiv \rho u^{\alpha} u^{\beta} \qquad (\alpha,\beta = 1 \text{ to } 5), \tag{4.18}$$

with

$$u^4 = 1, \tag{4.19}$$

and

$$u^5 = e/\rho = \varepsilon + \frac{1}{2} g_{ij}\, u^i u^j. \tag{4.20}$$

Here u^i ($i = 1$ to 3) are the contravariant components of the real, physical velocity. The contravariant components of the five-dimensional generalized stress tensor take the form

$$P^{\alpha\beta} = \begin{bmatrix} S^{11} - pg^{11} & S^{12} - pg^{12} & S^{13} - pg^{13} & 0 & g_{ij}u^j S^{i1} - pu^1 - q^1 + \rho f^1 x^5 \\ S^{21} - pg^{21} & S^{22} - pg^{22} & S^{23} - pg^{23} & 0 & g_{ij}u^j S^{i2} - pu^2 - q^2 + \rho f^2 x^5 \\ S^{31} - pg^{31} & S^{32} - pg^{32} & S^{33} - pg^{33} & 0 & g_{ij}u^j S^{i3} - pu^3 - q^3 + \rho f^3 x^5 \\ 0 & 0 & 0 & 0 & 0 \\ P^{15} & P^{25} & P^{35} & 0 & (\rho g_{ij} u^i f^j + \dot{e}) x^5 - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\rho f^i) \frac{(x^5)^2}{2} \end{bmatrix}, \tag{4.21}$$

where g^{ij}, S^{ij}, q^i, f^i are the contravariant components of the metric tensor, the extra stress tensor, the heat-flux vector, and the external body force per unit mass, respectively.

The extension to general coordinates of other forms of the conservation equation is carried out in an analogous manner.

5. DISCUSSION

The multi-dimensional conservation equations (2.11), (3.8), or (4.4) do not constitute a complete set of equations. These equations involve two classes of dependent variables. One class consists of the primary variables. For a compressible gas, these are u_i, ρ, and ε. For non-equilibrium flow they also include α_ν. For an incompressible fluid, ε must be replaced by ρ. The other class consists of the secondary variables. The latter include S_{ij}, q_i, f_i (or ϕ), $\dot{e}$, and p (for a compressible gas). For non-equilibrium flow, $u^d_{\nu i}$ and $\dot{\alpha}_\nu$ must also be introduced, while for magnetohydrodynamic flow the field component e_i and b_i appear as secondary variables. All of these secondary variables must be related to the primary variables in order to obtain a complete system of equations.

The relations between the two classes of variables may involve the introduction of a third class of intermediate variables. Expressions for the heat-flux vector components q_i normally involve the temperature. The radiant heat-flux is most generally calculated in terms of the specific intensity. The electromagnetic field components are calculated from Maxwell's equations, which involve the current density vector components j_i. In addition, particular forms of constitutive equations for the extra stress tensor, heat-flux vector, diffusion velocity, and current density require the introduction of various transport coefficients such as viscosity coefficients, diffusion coefficients, thermal conductivities, electrical conductivities, absorption coefficients, etc.

The particular forms that constitutive equations take, as well as expressions for equations of state, transport coefficients, and the internal production terms $\dot{\alpha}_\nu$,are all based on assumed models for a given fluid. These models lie outside the scope of the basic formulation of the conservation equations for the fluid motion. One can, however, categorize these models as to their effect on the completeness of the set of conservation equations. For a simple fluid model the constitutive equations give the secondary variables as explicit functions of the primary and intermediate variables, and their derivatives. The multi-dimensional conservation equations then constitute a complete set of differential equations (supplemented by algebraic relations for the intermediate variables) in the absence of electromagnetic fields, and in the presence of thermal radiation treated within the Rosseland diffusion or the emission-dominated approximation. (In the latter case, the divergence of the radiant heat-flux can be treated as an external energy source term, $\dot{e}$.)

For a complex fluid model, the expressions for the secondary variables could involve integrals of the primary variables, or require the introduction of new variables which are given by auxiliary equations. Thermal radiation generally requires the solution of the radiative transfer equation. It is coupled to the flow equations via the absorption, emission, and scattering coefficients. The presence of electromagnetic fields requires the solution of Maxwell's equations, which are coupled to the flow equations via a generalized Ohm's law. In all of these cases, the conservation equations form only part of the complete system of equations describing the physical situation.

As an example of the application of the multi-dimensional formulation, consider the compressible equilibrium flow equations written in general coordinates (§4.3). The most general coordinate transformation involving the three physical coordinates and time is

$$x^{\alpha'} = x^{\alpha'}(x^1,x^2,x^3,x^4) \qquad x^{5'} = x^5 \qquad (\alpha' = 1 \text{ to } 4), \tag{5.1}$$

where $x^{4'}$ is the new time like coordinate. In order to achieve a simple form of scalar decomposition, base vectors and tensor components will be defined with respect to the unprimed coordinate system. The transformed Eq. (4.16) thus becomes

$$\frac{\partial}{\partial x^{\alpha'}} \left(\sqrt{g'} \; \frac{\partial x^{\alpha'}}{\partial x^{\nu}} \; T^{\beta\nu} \, \vec{g}_\beta\right) = 0. \tag{5.2}$$

Since $\vec{g}_4$ and $\vec{g}_5$ are constant, one obtains immediately the transformed continuity and energy equations

$$\frac{\partial}{\partial x^{\alpha'}} (\sqrt{g'} \frac{\partial x^{\alpha'}}{\partial x^{\nu}} T^{\beta\nu}) = 0 \qquad (\beta = 4,5) \quad . \tag{5.3}$$

The remaining base vectors $\vec{g}_i$ can be functions of $x^{1'}$, $x^{2'}$, $x^{3'}$ and $x^{4'}$. In any finite-difference numerical procedure, all unknown variables are defined at discrete points fixed in $x^{1'}$, $x^{2'}$,$x^{3'}$,$x^{4'}$ space. Let the subscript P' denote a quantity evaluated at one of these points. A subscript α' appearing to the <u>left</u> of a quantity will denote variation with $x^{\alpha'}$ only, the other coordinates being held fixed at their values at point P'. Introduce unit base vectors by

$$\vec{g}_{(i)} \equiv \vec{g}_i/\sqrt{g_{ii}} \qquad \text{(unsummed)} . \tag{5.4}$$

In any finite-difference algorithm applied to (5.2) at point P', the unit base vector $g_{(j)}$ at a point other than P', appearing in the $x^{\alpha'}$ derivative term, can be expressed as

$$_{\alpha'}\vec{g}_{(j)} = {}_{\alpha'}\gamma^{(k)}_{(j)} \, \vec{g}_{(k)P'} . \tag{5.5}$$

The expansion coefficients $_{\alpha'}\gamma^{(k)}_{(j)}$ are known functions of $x^{\alpha'}$, determined by the original coordinate system and the transformation (5.1). The scalar decomposition of the transformed momentum equation in strong conservation form becomes

$$\frac{\partial}{\partial x^{\alpha'}} (\sqrt{g'} \frac{\partial x^{\alpha'}}{\partial x^{\beta}} \, {}_{\alpha'}\gamma^{(k)}_{(j)} \sqrt{g_{jj}} \, T^{j\beta}) = 0 \qquad (k = 1 \text{ to } 3) . \tag{5.6}$$

If one specialized (5.3) and (5.6) to an ideal gas ($\vec{S} = \vec{q} = \vec{f} = \dot{e} = 0$), and introduces physical contravariant velocity components by

$$u^{(i)} \equiv \sqrt{g_{ii}} \, u^i \qquad \text{(unsummed)}, \tag{5.7}$$

one obtains the following set of strong conservation equations:

$$\frac{\partial}{\partial x^{4'}} [\sqrt{g'} \, \rho(\frac{\partial x^{4'}}{\partial t} + \frac{\partial x^{4'}}{\partial x^j} \frac{u^{(i)}}{\sqrt{g_{jj}}})] + \frac{\partial}{\partial x^{i'}} [\sqrt{g'} \, \rho(\frac{\partial x^{i'}}{\partial t} + \frac{\partial x^{i'}}{\partial x^j} \frac{u^{(j)}}{\sqrt{g_{jj}}})] = 0, \tag{5.8}$$

$$\frac{\partial}{\partial x^{4'}} \{\sqrt{g'}\ {}_{4'}\gamma^{(k)}_{(j)} [\frac{\partial x^{4'}}{\partial t} \rho u^{(j)} + \frac{\partial x^{4'}}{\partial x^m} (\frac{\rho u^{(j)} u^{(m)}}{\sqrt{g_{mm}}} + p\sqrt{g_{jj}}\ g^{jm})]\} \tag{5.9}$$

$$+ \frac{\partial}{\partial x^{i'}} \{\sqrt{g'}\ {}_{i'}\gamma^{(k)}_{(j)} [\frac{\partial x^{i'}}{\partial t} \rho u^{(j)} + \frac{\partial x^{i'}}{\partial x^m} (\frac{\rho u^{(j)} u^{(m)}}{\sqrt{g_{mm}}} + p\sqrt{g_{jj}}\ g^{jm})]\} = 0$$

$$(k = 1 \text{ to } 3),$$

$$\frac{\partial}{\partial x^{4'}} \{\sqrt{g'}\ [\frac{\partial x^{4'}}{\partial t} e + \frac{\partial x^{4'}}{\partial x^j} \frac{(e+p)u^{(j)}}{\sqrt{g_{jj}}}]\}$$
$$+ \frac{\partial}{\partial x^{i'}} \{\sqrt{g'}\ [\frac{\partial x^{i'}}{\partial t} e + \frac{\partial x^{i'}}{\partial x^j} \frac{(e+p)u^{(j)}}{\sqrt{g_{jj}}}]\} = 0. \tag{5.10}$$

Further details and specializations of these results may be found in Ref. [2].

Another application of the higher-dimensional formulation is the calculation of fluid motion in a non-inertial reference frame fixed with respect to a moving body. In the conventional approach, external body force terms appear in the momentum equation, and an external work term is introduced in the energy equation. One can show that the change in reference frame can be treated as a coordinate transformation in a higher-dimensional space, with the external source terms arising from the curvature of the new time coordinate. Using a method analogous to the one just outlined, one can then obtain a scalar decomposition of the equations in strong conservation-law form. The details will appear in a future publication.

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