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SIDESLIP OF WING-BODY COMBINATIONS

By Paul E. Rubbert

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16. Abstract

A small-disturbance theory is developed for predicting the aerodynamics of an airplane in sideslip. Second-order terms involving the interaction between sideslip angle and angle of attack, sideslip angle and wing camber, etc., are retained. It is found that the second-order terms can produce the dominant sideslip effects when the dihedral of the lifting surfaces is small. Numerical implementation of the theory requires a solution procedure capable of producing accurate velocity *gradients* in the first-order solution.

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SIDESLIP OF WING-BODY COMBINATIONS

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1.0 SUMMARY

A theory is developed for predicting the aerodynamic properties of an airplane in sideslip. It is developed around a model problem consisting of a wing-body combination.

The basic approach used is perturbation theory. A solution is assumed in the form of an asymptotic expansion in powers of the small parameters governing angle of attack, angle of sideslip, wing camber, wing thickness, body radius variation, and body camber. Second-order terms governing the interaction between sideslip and the other parameters are retained. It is found that when wing dihedral is large, the dominant sideslip effects occur in the first-order terms. When dihedral is small, however, the dominant sideslip effects occur at second order and depend on products of sideslip and angle of attack, sideslip and wing camber, etc.

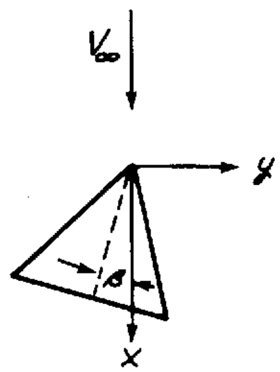
Particular integrals for the inhomogeneous second-order problems are given which automatically satisfy the wake boundary condition. Hence, the remaining homogeneous part can be formulated in terms of distributions of sources and elementary horseshoe vortices on the boundary surfaces in a manner entirely analogous to that commonly done for the first-order problems.

The present formulation is shown to produce results which reduce to known solutions for the infinite yawed wing and for a general planar wing in sideslip.

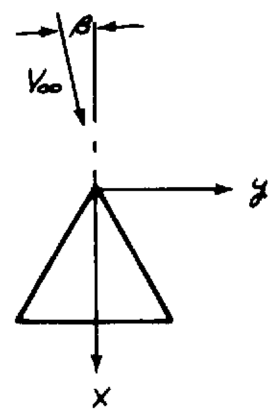
2.0 INTRODUCTION

The development of a small-disturbance theory for the aerodynamics of an airplane in sideslip entails a description of the airplane geometric features and the inclination of the on-coming stream in terms of small parameters. The parametric description must be such that the flow field reduces to a uniform stream in the limit as all parameters vanish.

For planar wings the problem can be formulated using coordinate systems aligned either with the freestream or along the plane of symmetry of the wing as shown in figure 1.



(a) Skewed Planform



(b) Skewed Freestream

FIGURE 1.–ALTERNATE FORMULATONS

In the former case the planform is skewed with respect to the coordinate system by the sideslip angle β, which remains constant in the limit as the angle of attack and wing thickness parameters approach zero. Thus, β does not appear directly in the problem as a small parameter and is not subject to a limit process. However, in the case of the skewed stream formulation, β appears directly as a small parameter for which a limit processs may be defined.

Lowest order solutions from the skewed planform formulation display the dominant effects of sideslip. However, since sideslip does not appear explicitly as a separate small parameter, the rates of change of flow properties with angle of sideslip, and lateral stability derivatives, must be computed by finite difference from solutions obtained at different angles of skew.

The lowest order solutions in the skewed freestream formulation do not produce the dominant sideslip effects. The first order in β solution for a flat wing with no dihedral produces nothing, since the sideslip component of the freestream, which is directed along the y axis, remains undisturbed by the presence of the planar wing. The dominant sideslip effects occur in second-order terms involving products of sideslip and angle of attack, sideslip and thickness, etc. These terms contain the sideslip parameter explicitly, and provide directly the rates of change of flow properties with sideslip and the interaction between sideslip and angle of attack, etc. Thus, the results produced by this formulation appear in a form that is more directly applicable for stability and control work and other aerodynamic applications.

For more complex configurations containing a fuselage, nacelles, dihedral, etc., a formulation using the skewed planform approach becomes very unwieldly, because of the difficulties involved in defining parametrically the geometrical shape such that the flow reduces to a uniform stream in the limit as the small parameters vanish. The skewed stream approach, on the other hand, can use the same parametric geometry definition normally used for symmetric conditions. Hence, for the present work a decision was made to use the latter formulation, even though it entails a higher order development.

3.0 SYMBOLS

A_∞	freestream sonic speed
$aR(x)$	body radius variation from the mean body surface (eq. 1)
B	$\sqrt{\lvert 1 - M^2 \rvert}$
C_p	complete pressure coefficient (eq. 10)
C_{p_a}	pressure coefficient due to body radius variation (eq. 28)
$C_{p_{a\beta}}$	pressure coefficient due to body radius variation and sideslip interaction (eq. 28)
C_{p_c}	pressure coefficient due to body camber (eq. 28)
$C_{p_{c\beta}}$	pressure coefficient due to body camber and sideslip interation (eq. 28)
C_{p_α}	pressure coefficient due to angle of attack (eq. 28)
$C_{p_{\alpha\beta}}$	pressure coefficient due to angle of attack and sideslip interaction (eq. 28)
C_{p_β}	pressure coefficient due to sideslip (eq. 28)
C_{p_θ}	pressure coefficient due to wing camber (eq. 28)
$C_{p_{\theta\beta}}$	pressure coefficient due to wing camber and sideslip interation (eq. 28)
C_{p_τ}	pressure coefficient due to wing thickness (eq. 28)
$C_{p_{\tau\beta}}$	pressure coefficient due to wing thickness and sideslip interaction (eq. 28)
$cG(x)$	body camber distribution (eq. 2)
K^s	potential induced by a point source (eq. 36)
K^v	potential induced by an elementary horseshoe vortex (eq. 34)
ℓ	lateral coordinate along a surface
M	freestream Mach number
m	source density

n	coordinate normal to a surface
$\vec{n}$	unit vector normal to a surface
P	radius of mean body surface (eq. 1)
q	local flow speed
r	body radius
$\vec{r}$	radius vector from a surface point to a field point
S	denotes a surface
s	spanwise coordinate
u, v, w	perturbation velocity components
V_∞	freestream speed
x, y, z	Cartesian coordinates (fig. 2)
α	angle of attack
β	angle of sideslip
γ	vorticity density (also the ratio of specific heats)
Δq_n	discontinuity in the normal velocity component across a surface (fig. 11)
Δq_t	discontinuity in the tangential velocity component in the lateral direction across a surface (fig. 11)
ε	lateral width of a surface strip (fig. 12)
$\theta H(x,s)$	wing camber form (eq. 3)
Λ	sweepback angle
μ	see figure 6
ξ, η, ζ	local coordinates (figs. 12, 13, 14)
ρ	density

$\tau F(x,s)$	wing thickness form (eq. 3)
Φ	complete velocity potential
φ^{P}	denotes a particular solution
φ^{S}	portion of a perturbation potential contributed by sources
φ^{V}	portion of a perturbation potential contributed by vortices
ψ	wing dihedral angle (fig. 4)
$\vec{\nabla}$	gradient operator
[ ]	denotes discontinuity of a function across a surface

Perturbation velocity potentials (eq. 5)

φ_0	angle-of-attack effect.
φ_1	first order sideslip effect
φ_2	axial flow over uncambered body with a thin, uncambered wing.
φ_3	body camber effect.
φ_4	wing camber effect
φ_5	wing thickness effect.
φ_6	angle-of-attack and sideslip interaction.
φ_7	body thickness and sideslip interaction.
φ_8	body camber and sideslip interaction
φ_9	wing camber and sideslip interaction.
φ_{10}	wing thickness and sideslip interaction.

4.0 GEOMETRY DEFINITION

The basic theory is developed around a model problem consisting of a wing with arbitrary dihedral attached to an infinitely long fuselage. The results can be readily extended to include multiple lifting surfaces such as horizontal and vertical tails, as well as nacelles, struts, and similar appendages. The effects of body truncation in the form of a pointed nose and tail can be added in a manner similar to that described by Woodward (ref. 1), wherein the classical slender body solution representing the pointed nose and tail is patched into the results of the present analysis.

Let all length dimensions be scaled by the average wing chord and assigned to be of order unity. The X axis of the coordinate system is aligned lengthwise with the configuration, as shown in figure 2.

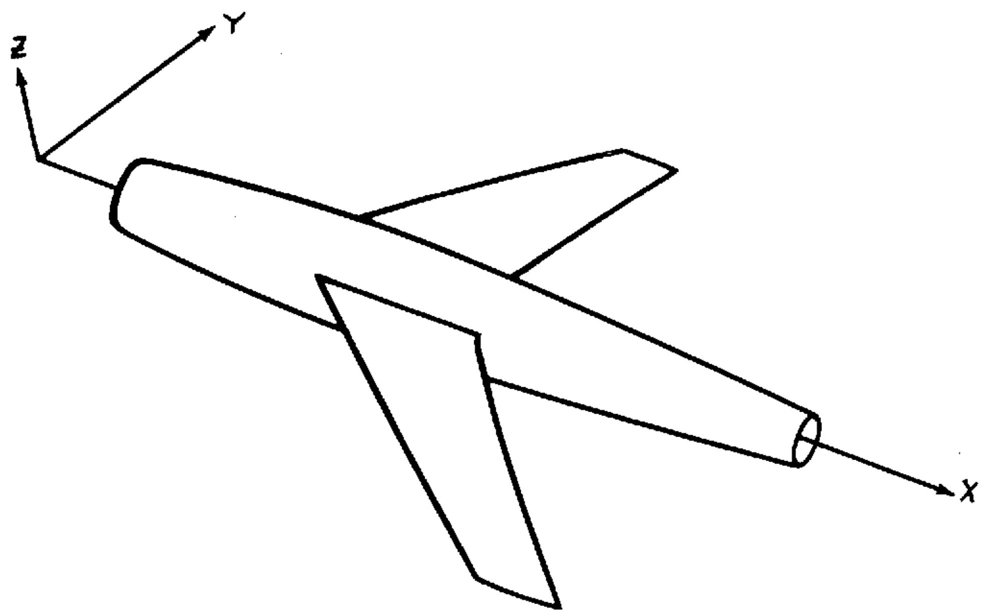


FIGURE 2.—COORDINATE SYSTEM

The y axis points to the right of an observer seated in the body, and the z axis is directed upward.

The freestream velocity $\vec{V}_\infty$ is inclined at an angle α to the xy plane. The component of $\vec{V}_\infty$ in the xy plane is inclined at the sideslip angle β to the x axis. When the component of $\vec{V}_\infty$ along the y axis is directed in the positive y direction, β shall be defined as positive.

The body shall be of circular cross section, and may be cambered. The body radius may vary slightly along its length. The nomenclature chosen for the body is shown in figure 3.

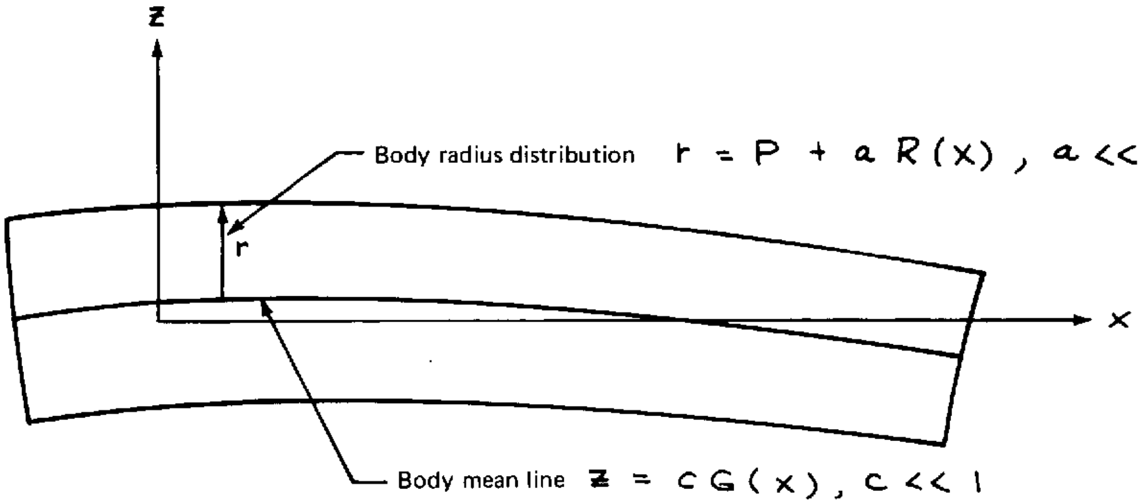


FIGURE 3.—BODY GEOMETRY

The radius distribution of the body is denoted as

$$r = P + a\,R(x) \tag{1}$$

where

P = constant = 0(1)

$R(x) = 0(1)$

$a \ll 1$

The body camber distribution is

$$z = c\,G(x) \tag{2}$$

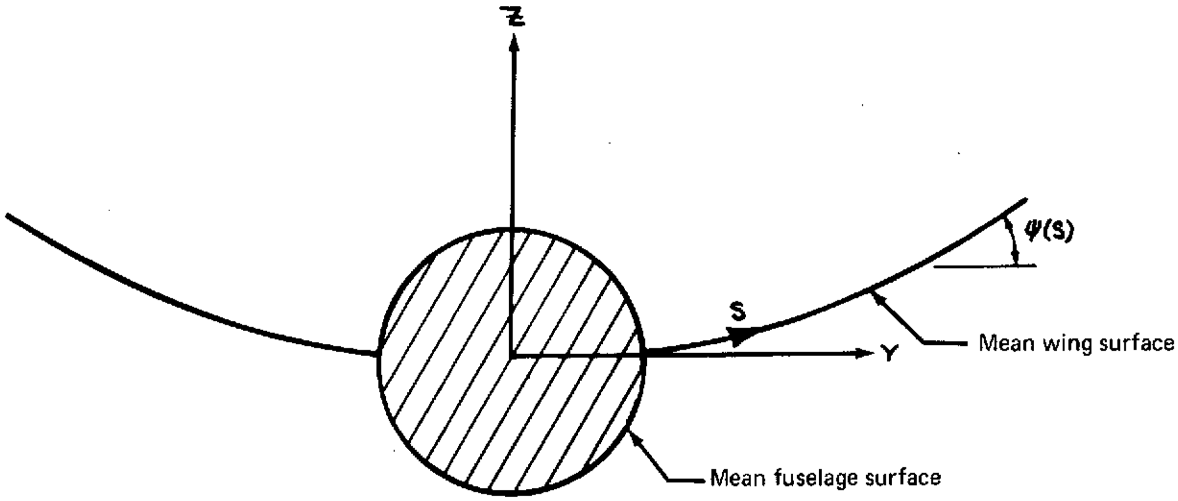
where

$G(x) = 0(1)$

$c \ll 1$

With this notation, the body approaches a cylinder in the limit as $a,\ c \rightarrow 0$.

The wing shall be positioned along a mean surface described by $z = z_w(s)$, $y = y_w(s)$, whose generators are parallel to the x axis, as shown in figure 4.



Rear View

FIGURE 4.—WING DIHEDRAL SHAPE

The wing dihedral distribution is defined by $\psi(s)$, where s is distance measured along the span. For the initial analysis, $\psi(s)$ is assumed to be 0(1).

The wing surface shape is described parametrically as a function of the coordinates (x, s) . Decomposed into camber and thickness, the wing surface shape becomes

$$
\begin{aligned}
y(x,s) &= y_w(s) \mp \tau F(x,s) \sin\psi - \theta H(x,s) \sin\psi \\
z(x,s) &= z_w(s) \pm \tau F(x,s) \cos\psi + \theta H(x,s) \cos\psi
\end{aligned}
\tag{3}
$$

where

$F(x, s)$ describes the thickness shape

$H(x, s)$ describes the camber shape

τ, θ are small parameters governing the magnitude of the thickness and camber distributions

The upper and lower signs refer to the upper and lower wing surface, respectively. With these definitions, an airfoil section generated by a cutting plane normal to the mean wing surface assumes its usual definition of

$$
z_{\substack{2\text{-}D \\ \text{AIRFOIL}}} = \pm \tau F(x,s) + \theta H(x,s) \tag{4}
$$

5.0 EXPANSION OF THE VELOCITY POTENTIAL

An asymptotic expansion of the velocity potential containing all first-order terms plus the second-order interaction terms involving the sideslip angle β is assumed to be of the form

$$\Phi = V_\infty \left(x + \beta y + \alpha z + \alpha \varphi_0 + \beta \varphi_1 + a \varphi_2 + c \varphi_3 \right.$$

$$+ \theta \varphi_4 + \tau \varphi_5 + \alpha \beta \varphi_6 + a \beta \varphi_7 + c \beta \varphi_8 \tag{5}$$

$$\left. + \theta \beta \varphi_9 + \tau \beta \varphi_{10} \right) + O(\alpha^2, \beta^2, \alpha a, \text{etc.})$$

The various perturbation potentials appearing in this expression can be interpreted in the following manner.

φ_0 angle-of-attack effect.

φ_1 first order sideslip effect

φ_2 axial flow over uncambered body with a thin, uncambered wing.

φ_3 body camber effect.

φ_4 wing camber effect

φ_5 wing thickness effect.

φ_6 angle-of-attack and sideslip interaction.

φ_7 body thickness and sideslip interaction.

φ_8 body camber and sideslip interaction

φ_9 wing camber and sideslip interaction.

φ_{10} wing thickness and sideslip interaction.

The objective in choosing this particular form is to obtain a sequence of simplified problems governing the primary effects of camber, thickness, angle of attack, and sideslip. Second-order terms involving the sideslip angle β are retained in recognition of the fact that the dominant sideslip effect may appear as either a first-order or a second-order effect, depending primarily on the magnitude of the wing dihedral. For the case of a wing with large dihedral, the potential φ_1 will be of order unity (its magnitude is governed primarily by the wing dihedral). In that case, the first-order term $\beta \varphi_1$ gives the dominant sideslip effect and all second-order terms could be neglected. For the case of a wing with no dihedral, however, φ_1

is zero and the dominant sideslip effect is given by the second-order terms. Thus, in the general case it is necessary to include at least some of the second-order terms given in eq. (5) to ensure that the dominant sideslip effect is not overlooked. Arguments will be given later for eliminating some of these terms.

Substitution of eq. (5) into the equations of motion and the boundary conditions will yield, after equating like powers of the small parameters, a series of equations and boundary conditions for the determination of each of the separate perturbation potentials.

6.0 BOUNDARY CONDITIONS

The boundary conditions consist of the requirement that the flow be parallel to all solid surfaces, the Kutta condition for subsonic trailing edges, continuity of pressure across the wake, and the vanishing of disturbances at infinity. The requirement that DB/Dt = 0, where $B(x,y,z,t)$ defines the position of the solid surfaces, leads to the appropriate expressions for the solid surface boundary condition.

6.1 BOUNDARY CONDITIONS ON THE WING

The equation of the wing surface, (3), together with eq. (5) for the velocity potential, are substituted into the expression DB/Dt = 0 to yield the result

$$\begin{aligned}
&[-\tau \tfrac{\partial F}{\partial x} \mp \theta \tfrac{\partial H}{\partial x}][1 + \alpha \varphi_{ox} + \beta \varphi_{1x} + \cdots + \tau\beta \varphi_{1ox}] \\
&+ [\mp \sin\psi - \tau(\tfrac{\partial F}{\partial s}\cos\psi - F \tfrac{\partial \psi}{\partial s}\sin\psi) \mp \theta(\tfrac{\partial H}{\partial s}\cos\psi \\
&- H\tfrac{\partial \psi}{\partial s}\sin\psi)][\beta + \alpha \varphi_{oy} + \beta \varphi_{1y} + a\, \varphi_{2y} + \cdots + \tau\beta \varphi_{1oy}] \qquad (6)\\
&+ [\pm \cos\psi - \tau(\tfrac{\partial F}{\partial s}\sin\psi + F\tfrac{\partial \psi}{\partial s}\cos\psi) \mp \theta(\tfrac{\partial H}{\partial s}\sin\psi \\
&+ H\tfrac{\partial \psi}{\partial s}\cos\psi)][\alpha + \alpha\varphi_{oz} + \beta\varphi_{1z} + a\,\varphi_{2z} + \cdots + \tau\beta\varphi_{1oz}] = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } B = 0
\end{aligned}$$

where higher order terms not considered in the expansion (5) have been deleted.

The various potentials on $B = 0$ must now be expressed in Taylor series about the mean wing surface. Since we are interested in retaining only those second-order terms involving products of β, this operation will introduce extra terms only from $\beta\varphi_{1y}$ and $\beta\varphi_{1z}$ appearing in eq. (6). The proper expansion is

$$\begin{aligned}
\varphi_{1y}(B = \pm 0) = \varphi_{1y}(\eta = \pm 0) + \frac{\partial \varphi_{1y}(\eta = \pm 0)}{\partial \eta}[\pm \tau F(x,s) \qquad (7)\\
+ \theta H(x,s)] + O(\tau^2, \theta^2, \tau\theta)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{1z}(B = \pm 0) = \varphi_{1z}(\eta = \pm 0) + \frac{\partial \varphi_{1z}(\eta = \pm 0)}{\partial \eta}[\pm \tau F(x,s) \\
+ \theta H(x,s)] + O(\tau^2, \theta^2, \tau\theta),
\end{aligned}$$

where η is a normal coordinate measured upward from the mean wing surface. The ± sign denotes the upper or lower surface of the wing, respectively.

Inserting eq. (7) into (6) and equating orders of magnitude finally results in the boundary conditions to be applied on the mean wing surface for the various potentials.

$$O(\alpha): \frac{\partial \varphi_0}{\partial n} = -\cos\psi$$

$$O(\beta): \frac{\partial \varphi_1}{\partial n} = \sin\psi$$

$$O(a): \frac{\partial \varphi_2}{\partial n} = 0$$

$$O(c): \frac{\partial \varphi_3}{\partial n} = 0$$

$$O(\theta): \frac{\partial \varphi_4}{\partial n} = \frac{\partial H}{\partial x}$$

$$O(\tau): \frac{\partial \varphi_5}{\partial n} = \pm\frac{\partial F}{\partial x}$$

$$O(\alpha\beta): \frac{\partial \varphi_6}{\partial n} = 0$$

$$O(a\beta): \frac{\partial \varphi_7}{\partial n} = 0 \tag{8}$$

$$O(c\beta): \frac{\partial \varphi_8}{\partial n} = 0$$

$$O(\theta\beta): \frac{\partial \varphi_9}{\partial n} = (1+\varphi_{1y})\frac{\partial}{\partial s}(H\cos\psi) + \varphi_{1z}\frac{\partial}{\partial s}(H\sin\psi) + \varphi_{1x}\frac{\partial H}{\partial x}$$

$$+ H\sin\psi\frac{\partial \varphi_{1y}}{\partial n} - H\cos\psi\frac{\partial \varphi_{1z}}{\partial n}$$

$$O(\tau\beta): \frac{\partial \varphi_{10}}{\partial n} = \pm\Big\{(1+\varphi_{1y})\frac{\partial}{\partial s}(F\cos\psi) + \varphi_{1z}\frac{\partial}{\partial s}(F\sin\psi)$$

$$+ \varphi_{1x}\frac{\partial F}{\partial x} + F\sin\psi\frac{\partial \varphi_{1y}}{\partial n} - F\cos\psi\frac{\partial \varphi_{1z}}{\partial n}\Big\}$$

where

$$\frac{\partial}{\partial n} = \cos\psi\frac{\partial}{\partial z} - \sin\psi\frac{\partial}{\partial y}$$

Note that the potentials appearing on the right-hand side of the $O(\theta\beta)$ and $O(\tau\beta)$ expressions may be discontinuous across the mean wing surface, indicating that a source-like term may be present in the solution.

6.2 BOUNDARY CONDITIONS ON THE WING WAKE

The condition to be satisfied is that the pressure be continuous across the wake surface. Introducing expansion (5) into the exact expression

$$C_p = \frac{2}{\gamma M^2}\left\{\left[1 - \frac{\gamma - 1}{2A_\infty^2}\left(q^2 - V_\infty^2\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \tag{9}$$

where

M = freestream Mach number

V_∞ = freestream speed

q = local flow speed

A_∞ = sound speed in the undisturbed stream

and expanding in powers of the small parameters yields the following expression for C_p.

$$\begin{aligned} C_p = -2 \{ & \alpha \varphi_{0x} + \beta \varphi_{1x} + a \varphi_{2x} + c \varphi_{3x} + \theta \varphi_{4x} + \tau \varphi_{5x} \\ & + \alpha\beta(\varphi_{6x} + \varphi_{0y} + \varphi_{1z} + \nabla\varphi_0 \cdot \nabla\varphi_1 - M^2 \varphi_{0x}\varphi_{1x}) \\ & + a\beta(\varphi_{7x} + \varphi_{2y} + \nabla\varphi_1 \cdot \nabla\varphi_2 - M^2 \varphi_{1x}\varphi_{2x}) \\ & + c\beta(\varphi_{8x} + \varphi_{3y} + \nabla\varphi_1 \cdot \nabla\varphi_3 - M^2 \varphi_{1x}\varphi_{3x}) \\ & + \theta\beta(\varphi_{9x} + \varphi_{4y} + \nabla\varphi_1 \cdot \nabla\varphi_4 - M^2 \varphi_{1x}\varphi_{4x}) \\ & + \tau\beta(\varphi_{10x} + \varphi_{5y} + \nabla\varphi_1 \cdot \nabla\varphi_5 - M^2 \varphi_{1x}\varphi_{5x})\} \end{aligned} \tag{10}$$

The pressure coefficient on the actual wake surface (whose position is generally unknown) is next expressed in a Taylor-series expansion about the mean wake surface, a cylindrical surface lying directly behind the mean wing surface. Let the deviation of a point on the actual wake from the mean wake surface be denoted as $\Delta y(x)$, $\Delta z(x)$, as shown in figure 5.

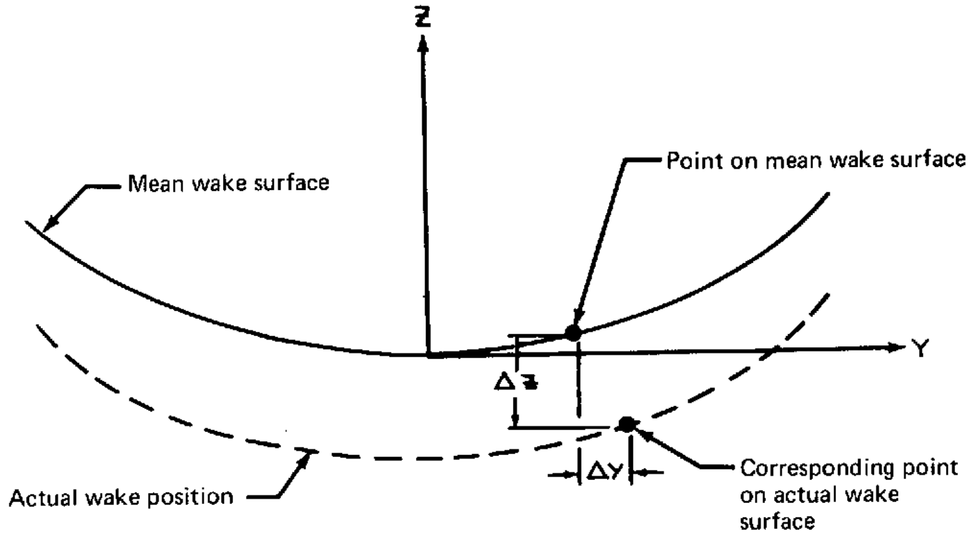


FIGURE 5.–WAKE CROSS SECTION

The distances $\Delta y(x)$, $\Delta z(x)$ are functions of the small parameters and may be expressed as

$$
\begin{aligned}
\Delta y(x) &= \alpha \Delta y_0 + \beta \Delta y_1 + a \Delta y_2 + c \Delta y_3 + \theta \Delta y_4 + \tau \Delta y_5 \\
&\quad + O(\alpha\beta), \text{ etc.} \\
\Delta z(x) &= \alpha \Delta z_0 + \beta \Delta z_1 + a \Delta z_2 + c \Delta z_3 + \theta \Delta z_4 + \tau \Delta z_5 \\
&\quad + O(\alpha\beta), \text{ etc.}
\end{aligned}
\tag{11}
$$

With this notation, the values of the potentials or their derivatives on the actual wake surface, expressed as a series expansion about the mean wake surface, become of the form

$$
\varphi\Big|_{\substack{\text{actual}\\ \text{wake}}} = \varphi\Big|_{\substack{\text{mean}\\ \text{wake}}} + \frac{\partial \varphi}{\partial y}\Big|_{\substack{\text{mean}\\ \text{wake}}} \cdot \Delta y + \frac{\partial \varphi}{\partial z}\Big|_{\substack{\text{mean}\\ \text{wake}}} \cdot \Delta z + O\left\{(\Delta y)^2,\ \Delta y \Delta z,\ (\Delta z)^2\right\} \tag{12}
$$

Equations (11) and (12) substituted into (10) furnish the appropriate expression for the pressure coefficient on the wake. Imposing the requirement that the pressure coefficient must be continuous across the wake to all orders of magnitude then results in the following sequence of boundary conditions to be applied on the mean wake surface. (The notation [ ] is used to denote the discontinuity of the function enclosed in brackets across the surface.)

$$O(\alpha): \quad [\varphi_{0x}] = 0$$

$$O(\beta): \quad [\varphi_{1x}] = 0$$

$$O(a): \quad [\varphi_{2x}] = 0$$

$$O(c): \quad [\varphi_{3x}] = 0$$

$$O(\theta): \quad [\varphi_{4x}] = 0$$

$$O(\tau): \quad [\varphi_{5x}] = 0$$

$$O(\alpha\beta): [\varphi_{6x}] = M^2[\varphi_{0x}\ \varphi_{1x}] - [\varphi_{0y}] - [\varphi_{1z}] - [\nabla\varphi_0 \cdot \nabla\varphi_1]$$
$$- \Delta y_0 [\varphi_{1xy}] - \Delta y_1 [\varphi_{0xy}] - \Delta z_0 [\varphi_{1xz}]$$
$$- \Delta z_1 [\varphi_{0xz}]$$

$$O(a\beta): [\varphi_{7x}] = M^2[\varphi_{1x}\ \varphi_{2x}] - [\varphi_{2y}] - [\nabla\varphi_1 \cdot \nabla\varphi_2] \qquad (13)$$
$$- \Delta y_1 [\varphi_{2xy}] - \Delta z_1 [\varphi_{2xz}]$$

$$O(c\beta): [\varphi_{8x}] = M^2[\varphi_{1x}\ \varphi_{3x}] - [\varphi_{3y}] - [\nabla\varphi_1 \cdot \nabla\varphi_3]$$
$$- \Delta y_1 [\varphi_{3xy}] - \Delta z_1 [\varphi_{3xz}]$$

$$O(\theta\beta): [\varphi_{9x}] = M^2[\varphi_{1x}\ \varphi_{4x}] - [\varphi_{4y}] - [\nabla\varphi_1 \cdot \nabla\varphi_4]$$
$$- \Delta y_1 [\varphi_{4xy}] - \Delta z_1 [\varphi_{4xz}]$$

$$O(\tau\beta): [\varphi_{10x}] = M^2[\varphi_{1x}\ \varphi_{5x}] - [\varphi_{5y}] - [\nabla\varphi_1 \cdot \nabla\varphi_5]$$
$$- \Delta y_1 [\varphi_{5xy}] - \Delta z_1 [\varphi_{5xz}]$$

At first glance it appears that it may be necessary to compute the wake position from the first-order solution in order to evaluate Δy_0, Δy_1, Δz_0, and Δz_1 which appear in the second-order problems. However, it can be shown that the first-order x-velocity components φ_{0x}, φ_{1x}, . . ., φ_{5x} and their derivatives are continuous across the mean wake surface, and hence the terms multiplying Δy_0, Δy_1, Δz_0, and Δz_1 are all zero. The second-order wake boundary conditions are, in fact, independent of the wake deformation.

6.3 BOUNDARY CONDITIONS ON THE BODY

The equation of the body surface is

$$B(x, y, z) = 0 = z - c\,G(x) - \sqrt{\{r(x)\}^2 - y^2} \tag{14}$$

where

$$r(x) = P + a\,R(x)$$

The requirement that DB/Dt = 0, necessary to produce the solid surface boundary condition, leads with the use of eq. (5) to the expression

$$\begin{aligned} &-\left[c\frac{dG}{dx} + \frac{a}{\sin\mu}\frac{dR}{dx}\right]\left[1 + \alpha\varphi_{0x} + \beta\varphi_{1x} + \cdots + \tau\beta\varphi_{10x}\right] \\ &+ \cot\mu\left[\beta + \alpha\varphi_{0y} + \beta\varphi_{1y} + \cdots + \tau\beta\varphi_{10y}\right] \\ &+ \left[\alpha + \alpha\varphi_{0z} + \beta\varphi_{1z} + \cdots + \tau\beta\varphi_{10z}\right] = 0 \\ &\qquad\qquad\qquad\qquad \text{on } B(x, y, z) = 0 \end{aligned} \tag{15}$$

where $\mu = \cot^{-1}\dfrac{y}{\sqrt{r^2 - y^2}}$, as sketched in figure 6.

The next step is to express the velocity components on $B(x, y, z) = 0$ in terms of their values on the mean body surface $r = P$ by means of a Taylor-series expansion about $r = P$. Figure 7 shows the spacial relationship between a point A on the actual fuselage

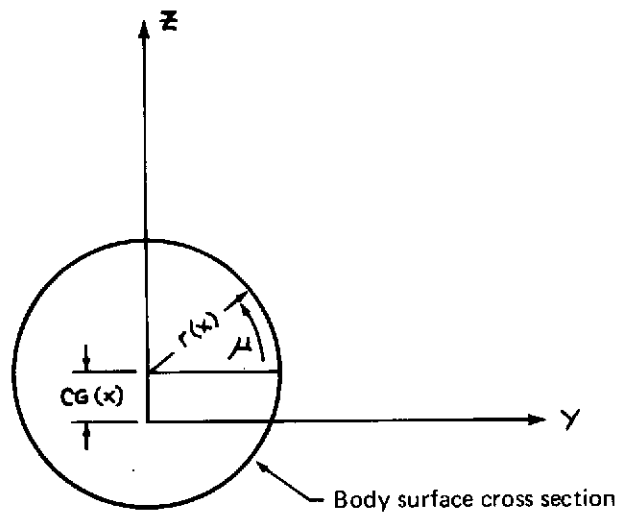


FIGURE 6.—DEFINITION OF μ

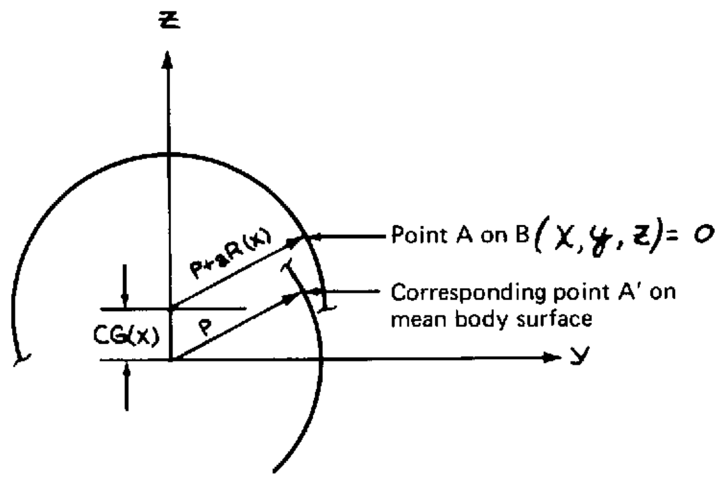


FIGURE 7.—EXPANSION OF BODY BOUNDARY CONDITIONS

surface $B(x, y, z) = 0$ and the corresponding point A' on the mean fuselage surface of radius P about the X axis. The velocity potential at the two points is related by the expression

$$
\begin{aligned}
\varphi(A) &= \varphi(A') + \frac{\partial \varphi(A')}{\partial z} \cdot \left[CG(x) + a R(x) \sin \mu \right] \\
&\quad + \frac{\partial \varphi(A')}{\partial y} \cdot a R(x) \cos \mu + O(c^2, a^2, ac) \qquad (16) \\
&= \varphi(A') + \frac{\partial \varphi(A')}{\partial n} \cdot a R(x) + \frac{\partial \varphi(A')}{\partial z} \cdot CG(x),
\end{aligned}
$$

where n is a coordinate normal to the mean body surface and positive when directed outward.

Each term in eq. (15) is expanded in this manner. Finally, terms of equal order of magnitude are equated to produce the body boundary conditions for the various potentials. For convenience, certain higher order terms can be retained in the first-order boundary conditions to facilitate the numerical application of the Karman-Moore method (ref. 2). In the following equation, which follows from eqs. (15) and (16), these higher order terms, which in principle are not needed but in practice may be convenient, are underlined.

$$
\begin{aligned}
&o(\alpha): && \frac{\partial \varphi_0}{\partial n} = -\sin\mu + \underline{a\frac{dR}{dX}\varphi_{0x}} \\
&o(\beta): && \frac{\partial \varphi_1}{\partial n} = -\cos\mu \\
&o(a): && \frac{\partial \varphi_2}{\partial n} = \frac{dR}{dX} + \underline{a\frac{dR}{dX}\varphi_{2X}} \\
&o(c): && \frac{\partial \varphi_3}{\partial n} = \frac{dG}{dX}\sin\mu + \underline{a\frac{dR}{dX}\varphi_{3X}} \\
&o(\theta): && \frac{\partial \varphi_4}{\partial n} = \underline{a\frac{dR}{dX}\varphi_{4X}} \\
&o(\tau): && \frac{\partial \varphi_5}{\partial n} = \underline{a\frac{dR}{dX}\varphi_{5X}} \qquad (17) \\
&o(\alpha\beta): && \frac{\partial \varphi_6}{\partial n} = 0 \\
&o(a\beta): && \frac{\partial \varphi_7}{\partial n} = \varphi_{1X}\frac{dR}{dX} - R(X)\frac{\partial}{\partial n}\left(\varphi_{1y}\cos\mu + \varphi_{1z}\sin\mu\right) \\
&o(c\beta): && \frac{\partial \varphi_8}{\partial n} = \varphi_{1X}\frac{dG}{dX}\sin\mu - G(X)\frac{\partial}{\partial z}\left(\varphi_{1y}\cos\mu + \varphi_{1z}\sin\mu\right) \\
&o(\theta\beta): && \frac{\partial \varphi_9}{\partial n} = 0 \\
&o(\tau\beta): && \frac{\partial \varphi_{10}}{\partial n} = 0
\end{aligned}
$$

where $\dfrac{\partial \varphi}{\partial n} = \varphi_y\cos\mu + \varphi_z\sin\mu \qquad \text{on } \sqrt{y^2+z^2} - P = 0$

7.0 FLOW EQUATION

The equations of motion and continuity for an irrotational flow can be written as:

$$\vec{\nabla}\left(\frac{1}{2}q^2\right) + \frac{1}{\rho}\vec{\nabla}p = 0$$
$$\vec{\nabla}\cdot\left(\rho\,\vec{\nabla}\varphi\right) = 0 \qquad (18)$$

The assumption of irrotationality is valid to the order of approximation sought. Eliminating the pressure from these equations gives

$$2A^2\,\nabla^2\varphi = \left(\vec{\nabla}q^2\right)\cdot\left(\vec{\nabla}\varphi\right) \qquad (19)$$

where

$$A^2 = A_\infty^2 + \frac{1}{2}(\gamma - 1)(V_\infty^2 - q^2)$$

Substituting the expansion (5) into eq. (19) and equating orders of magnitude produces the governing equations in a straightforward manner. The results are given in the following section, which summarizes the equations and boundary conditions for the various potentials.

8.0 SUMMARY OF THE COMPLETE FIRST- AND SECOND-ORDER PROBLEMS

The following separate problems are obtained by grouping together the results given by eqs. (8), (13), and (17) and the results of the previous section. The boundary conditions listed are to be applied on the mean wing surface, the mean wake surface, and the surface of the mean cylindrical body. The normal coordinate, n, is positive when directed upward from the mean wing surface or outward from the mean body surface.

$O(\alpha)$:

Differential equation: $(1-M^2)\,\varphi_{0xx} + \varphi_{0yy} + \varphi_{0zz} = 0$

$$\text{Boundary conditions}\begin{cases} \text{wing: } \dfrac{\partial \varphi_0}{\partial n} = \varphi_{0z}\cos\psi - \varphi_{0y}\sin\psi = -\cos\psi \\ \text{wake: } \left[\varphi_{0x}\right] = 0 \\ \text{body: } \dfrac{\partial \varphi_0}{\partial n} = \varphi_{0y}\cos\mu + \varphi_{0z}\sin\mu = -\sin\mu + \underline{a\dfrac{dR}{dx}\varphi_{0x}} \end{cases}$$

$O(\beta)$: $(1-M^2)\,\varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} = 0$

wing: $\dfrac{\partial \varphi_1}{\partial n} = \sin\psi$

wake: $\left[\varphi_{1x}\right] = 0$

body: $\dfrac{\partial \varphi_1}{\partial n} = -\cos\mu$

(20)

$O(a)$: $(1-M^2)\,\varphi_{2xx} + \varphi_{2yy} + \varphi_{2zz} = 0$

wing: $\dfrac{\partial \varphi_2}{\partial n} = 0$

wake: $\left[\varphi_{2x}\right] = 0$

body: $\dfrac{\partial \varphi_2}{\partial n} = \dfrac{dR}{dx} + \underline{a\dfrac{dR}{dx}\varphi_{2x}}$

$O(c)$: $(1-M^2)\,\varphi_{3xx} + \varphi_{3yy} + \varphi_{3zz} = 0$

wing: $\dfrac{\partial \varphi_3}{\partial n} = 0$

wake: $\left[\varphi_{3x}\right] = 0$

body $\dfrac{\partial \varphi_3}{\partial n} = \dfrac{dG}{dx}\sin\mu + \underline{a\dfrac{dR}{dx}\varphi_{3x}}$

$O(e):\quad (1-M^2)\,\varphi_{4xx} + \varphi_{4yy} + \varphi_{4zz} = 0$

wing: $\dfrac{\partial \varphi_4}{\partial n} = \dfrac{\partial H}{\partial x}$

wake: $\left[\varphi_{4x}\right] = 0$

body: $\dfrac{\partial \varphi_4}{\partial n} = \underline{a\,\dfrac{dR}{dx}\,\varphi_{4x}}$

$O(\tau):\quad (1-M^2)\,\varphi_{5xx} + \varphi_{5yy} + \varphi_{5zz} = 0$

wing: $\dfrac{\partial \varphi_5}{\partial n} = \pm\dfrac{\partial F}{\partial x}$

wake: $\left[\varphi_{5x}\right] = 0$

body: $\dfrac{\partial \varphi_5}{\partial n} = \underline{a\,\dfrac{dR}{dx}\,\varphi_{5x}}$ (20-cont.)

$$O(\alpha\beta):\quad (1-M^2)\,\varphi_{6xx} + \varphi_{6yy} + \varphi_{6zz} = 2M^2\Big\{\varphi_{oxy} + \varphi_{1xz} + \frac{\partial}{\partial x}\Big(\vec{\nabla}\varphi_o\cdot\vec{\nabla}\varphi_1 + \frac{\gamma-1}{2}M^2\varphi_{ox}\,\varphi_{1x}\Big)\Big\}$$

wing: $\dfrac{\partial \varphi_6}{\partial n} = 0$

wake: $\left[\varphi_{6x}\right] = M^2\left[\varphi_{ox}\ \varphi_{1x}\right] - \left[\varphi_{oy}\right] - \left[\varphi_{1z}\right] - \left[\vec{\nabla}\varphi_o\cdot\vec{\nabla}\varphi_1\right]$

body: $\dfrac{\partial \varphi_6}{\partial n} = 0$

$$O(a\beta):\ (1-M^2)\,\varphi_{7xx} + \varphi_{7yy} + \varphi_{7zz} = 2M^2\Big\{\varphi_{2xy} + \frac{\partial}{\partial x}\Big(\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_2 + \frac{\gamma-1}{2}M^2\,\varphi_{1x}\,\varphi_{2x}\Big)\Big\}$$

wing: $\dfrac{\partial \varphi_7}{\partial n} = 0$

wake: $\left[\varphi_{7x}\right] = M^2\left[\varphi_{1x}\,\varphi_{2x}\right] - \left[\varphi_{2y}\right] - \left[\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_2\right]$

body: $\dfrac{\partial \varphi_7}{\partial n} = \varphi_{1x}\dfrac{dR}{dx} - R(x)\dfrac{\partial}{\partial n}\left(\varphi_{1y}\cos\mu + \varphi_{1z}\sin\mu\right)$

$$O(c\beta): (1-M^2)\,\varphi_{8xx} + \varphi_{8yy} + \varphi_{8zz} = 2M^2\left\{\varphi_{3xy} + \frac{\partial}{\partial x}\left(\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_3 + \frac{\gamma-1}{2}M^2\varphi_{3x}\varphi_{1x}\right)\right\}$$

wing: $\dfrac{\partial\varphi_8}{\partial n} = 0$

wake: $\left[\varphi_{8x}\right] = M^2\left[\varphi_{1x}\ \varphi_{3x}\right] - \left[\varphi_{3y}\right] - \left[\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_3\right]$

body: $\dfrac{\partial\varphi_8}{\partial n} = \varphi_{1x}\dfrac{dG}{dx}\sin\mu - G(x)\dfrac{\partial}{\partial z}\left(\varphi_{1y}\cos\mu + \varphi_{1z}\sin\mu\right)$

$$O(\theta\beta): (1-M^2)\,\varphi_{9xx} + \varphi_{9yy} + \varphi_{9zz} = 2M^2\left\{\varphi_{4xy} + \frac{\partial}{\partial x}\left(\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_4 + \frac{\gamma-1}{2}M^2\varphi_{1x}\varphi_{4x}\right)\right\}$$

wing: $\dfrac{\partial\varphi_9}{\partial n} = (1+\varphi_{1y})\dfrac{\partial}{\partial s}(H\cos\psi) + \varphi_{1z}\dfrac{\partial}{\partial s}(H\sin\psi) + \varphi_{1x}\dfrac{\partial H}{\partial x} + H\dfrac{\partial}{\partial n}\left(\varphi_{1y}\sin\psi - \varphi_{1z}\cos\psi\right)$

wake: $\left[\varphi_{9x}\right] = M^2\left[\varphi_{1x}\ \varphi_{4x}\right] - \left[\varphi_{4y}\right] - \left[\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_4\right]$

body: $\dfrac{\partial\varphi_9}{\partial n} = 0$ (20-cont.)

$$O(\tau\beta): (1-M^2)\,\varphi_{10xx} + \varphi_{10yy} + \varphi_{10zz} = 2M^2\left\{\varphi_{5xy} + \frac{\partial}{\partial x}\left(\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_5 + \frac{\gamma-1}{2}M^2\varphi_{1x}\varphi_{5x}\right)\right\}$$

wing: $\dfrac{\partial\varphi_{10}}{\partial n} = \pm\left\{(1+\varphi_{1y})\dfrac{\partial}{\partial s}(F\cos\psi) + \varphi_{1z}\dfrac{\partial}{\partial s}(F\sin\psi) + \varphi_{1x}\dfrac{\partial F}{\partial x}\right\} + F\dfrac{\partial}{\partial n}\left(\varphi_{1y}\sin\psi - \varphi_{1z}\cos\psi\right)$

wake: $\left[\varphi_{10x}\right] = M^2\left[\varphi_{1x}\ \varphi_{5x}\right] - \left[\varphi_{5y}\right] - \left[\vec{\nabla}\varphi_1\cdot\vec{\nabla}\varphi_5\right]$

body: $\dfrac{\partial\varphi_{10}}{\partial n} = 0$

Note: The underlined terms are of higher order and can be deleted if desired.

9.0 SIMPLIFICATIONS

The complete expressions for the second-order terms summarized in the previous section can be simplified considerably without compromising the basic objective of determining the dominant sideslip term or terms. It was previously pointed out that different terms may dominate, depending on the particular aircraft geometry. Thus, for example, if the wing dihedral is large, the term $\beta\varphi_1$ will give the major sideslip effect, and it is unnecessary to take into consideration the higher order terms. On the other hand, if the aircraft geometry is such that the size of the term $\beta\varphi_1$ is small (for example, $\beta\varphi_1$ is zero for a planar isolated wing), then it is necessary to include the second-order terms. The clue that enables one to effect a simplification is to note that, for the example cited above wherein $\beta\varphi_1$ was zero, all of the terms involving φ_1 vanish in the second-order problems, which simplifies them considerably.

Let us first examine in detail the $o(\beta)$ and $o(\alpha\beta)$ problems, which are reproduced below for convenience.

$$o(\beta):\ (1-M^2)\,\varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} = 0$$

$$\frac{\partial\varphi_1}{\partial n} = \sin\psi \quad \text{on the wing} \tag{21}$$

$$[\varphi_{1x}] = 0 \quad \text{on the wake}$$

$$\frac{\partial\varphi_1}{\partial n} = -\cos\mu \quad \text{on the body}$$

$$o(\alpha\beta):\ (1-M^2)\,\varphi_{6xx} + \varphi_{6yy} + \varphi_{6zz} = 2M^2\Big\{\varphi_{0xy} + \varphi_{1xz} + \frac{\partial}{\partial x}\Big(\vec\nabla\varphi_0\cdot\vec\nabla\varphi_1 + \frac{\gamma-1}{2}M^2\,\varphi_{0x}\,\varphi_{1x}\Big)\Big\}$$

$$\frac{\partial\varphi_6}{\partial n} = 0 \quad \text{on the wing} \tag{22}$$

$$[\varphi_{6x}] = M^2[\varphi_{0x}\,\varphi_{1x}] - [\varphi_{0y}] - [\varphi_{1z}] - [\vec\nabla\varphi_0\cdot\vec\nabla\varphi_1] \quad \text{on the wake}$$

$$\frac{\partial\varphi_6}{\partial n} = 0 \quad \text{on the body}$$

The magnitude of the first-order term φ_1 is governed by the boundary conditions of the $o(\beta)$ problem. The wing contributes an effect proportional to ψ, and the body an effect proportional to ρ^2, the square of the mean body radius. Hence, the magnitude of the $\beta\varphi_1$ term may be denoted as $o(\beta\psi, \beta\rho^2)$. If ψ and ρ^2 are small (of the same order as α, say), then the second-order terms are comparable in magnitude to this one.

Turning now to the $O(\alpha\beta)$ problem, we split the particular solution of the inhomogeneous differential equation into two terms, φ_{P_1} and φ_{P_2}, satisfying the equations

$$(1-M^2)\,\varphi_{P_1xx} + \varphi_{P_1yy} + \varphi_{P_1zz} = 2M^2\left(\varphi_{oxy} + \varphi_{1xz}\right)$$

and (23)

$$(1-M^2)\,\varphi_{P_2xx} + \varphi_{P_2yy} + \varphi_{P_2zz} = 2M^2\frac{\partial}{\partial x}\left(\vec{\nabla}\varphi_o\cdot\vec{\nabla}\varphi_1 + \frac{\gamma-1}{2}M^2\varphi_{ox}\,\varphi_{1x}\right)$$

where

$$\varphi_{6\ \text{particular solution}} = \varphi_{P_1} + \varphi_{P_2} \tag{24}$$

An estimate of the magnitudes of φ_{P_1} and φ_{P_2} is obtained by examining the size of the inhomogeneous terms. The flow due to angle of attack, φ_o, will be 0(1). The term φ_1 is $o(\psi,\ p^2)$. Hence the magnitudes of the first-order term and the second-order particular solutions must be as follows:

$$\begin{aligned}\beta\varphi_1 &= o(\beta\psi,\ \beta p^2)\\ \alpha\beta\,\varphi_{P_1} &= O(\alpha\beta)\\ \alpha\beta\,\varphi_{P_2} &= O(\alpha\beta\psi,\ \alpha\beta p^2)\end{aligned} \tag{25}$$

The simplification that can be achieved is now evident, for note that the last term, $\alpha\beta\varphi_{P_2}$, is always an $O(\alpha)$ smaller than $\beta\varphi_1$, regardless of whether ψ and p^2 are large or small. Furthermore, when ψ and p^2 are small enough to be of the same magnitude as α, , which is the condition for which $\alpha\beta\,\varphi_{P_1}$ must be retained, then the term $\alpha\beta\,\varphi_{P_2}$ may be neglected in comparison with $\alpha\beta\varphi_{P_1}$. Hence $\alpha\beta\,\varphi_{P_2}$ is always an $O(\alpha)$ smaller than the largest term, whether it be $\beta\varphi_1$ or $\alpha\beta\varphi_{P_1}$, and may always be neglected. One could also neglect that portion of φ_{P_1} arising from the term φ_{1xz} on the right-hand side of eq. (23), since it too must always be an $O(\alpha)$ smaller than the largest remaining term. However, it will be retained in the later analysis, since it lends a degree of symmetry to the relationship between α and β.

Turning now to the boundary conditions and applying the same reasoning as above, it becomes apparent that the product terms involving φ_1 or its derivatives that appear in the boundary conditions for φ_6 may also be neglected, for they too will contribute terms which are always an $o(\alpha)$ smaller than the largest remaining term. Hence the simplified problem for the potential governing the interaction between angle of attack and sidelip reduces to

$$O(\alpha\beta):\ (1-M^2)\,\varphi_{6xx} + \varphi_{6yy} + \varphi_{6zz} = 2M^2\left[\varphi_{oxy} + \underline{\varphi_{1xz}}\right].$$

wing: $\dfrac{\partial \varphi_6}{\partial n} = 0$

wake: $\left[\varphi_{6x}\right] = -\left[\varphi_{oy}\right] - \left[\varphi_{1z}\right]$ (26)

body: $\dfrac{\partial \varphi_6}{\partial n} = 0$

where the underlined terms could be neglected if desired.

The remaining second-order problems reduce in a similar manner. For example, consider the $O(\theta\beta)$ problem involving φ_9 , which is reproduced below for convenience.

$$O(\theta\beta):\ (1-M^2)\,\varphi_{9xx} + \varphi_{9yy} + \varphi_{9zz} = 2M^2\left\{\varphi_{4xy} + \frac{\partial}{\partial x}\left(\vec\nabla\varphi_1 \cdot \vec\nabla\varphi_4 + \frac{\gamma-1}{2}M^2\,\varphi_{1x}\,\varphi_{4x}\right)\right\} \quad (27)$$

wing: $\dfrac{\partial \varphi_9}{\partial n} = (1+\varphi_{1y})\dfrac{\partial}{\partial s}(H\cos\psi) + \varphi_{1z}\dfrac{\partial}{\partial s}(H\sin\psi) + \varphi_{1x}\dfrac{\partial H}{\partial x} + H\dfrac{\partial}{\partial n}(\varphi_{1y}\sin\psi - \varphi_{1z}\cos\psi)$

wake: $\left[\varphi_{9x}\right] = M^2\left[\varphi_{1x}\,\varphi_{4x}\right] - \left[\varphi_{4y}\right] - \left[\vec\nabla\varphi_1 \cdot \vec\nabla\varphi_4\right]$

body: $\dfrac{\partial \varphi_9}{\partial n} = 0$

Following the same argument as before, it is apparent that the contribution to the particular solution of $\theta\beta\,\varphi_9$ arising from the product terms involving φ_1 on the right-hand side of the differential equation are always an $o(\theta)$ smaller than the term $\beta\varphi_1$, no matter whether ψ and p^2 are large or small. When ψ and p^2 are small enough to be of the same size as θ, which is the condition for which the second-order term $\theta\beta\varphi_9$ must be retained, then the product terms are an $o(\psi,\ p^2) = o(\theta)$ smaller than the remaining inhomogeneity involving φ_{4xy}. Hence in all cases the product terms are an $o(\theta)$ smaller than the largest remaining term and may be neglected.

The same reasoning applies to all terms involving φ_1 which appear in the boundary conditions. When ψ and p^2 are large, the entire second-order term is negligible, and when ψ and p^2 are the same size as θ , then the terms in the boundary conditions involving φ_1 and its derivatives are an $o(\theta)$ smaller than the remaining terms. It follows that all terms in the boundary conditions for φ_9 which involve φ_1 or its derivatives can always be neglected.

10.0 SUMMARY OF THE REDUCED FIRST- AND SECOND-ORDER PROBLEMS

In summary, the problems for the various potentials reduce to those listed below. The boundary conditions listed are to be applied on the mean wing surface, the mean wake surface, and the mean cylindrical body surface of radius P . The normal coordinate, n , is positive when directed upward from the mean wing surface or outward from the mean body surface. The angle μ is as shown in figure 8. The reduced differential equations and boundary conditions arise from application of the reductions derived in section 9.0 to the set of eq. (20). The respective expressions for the pressure coefficients come from eq. (10).

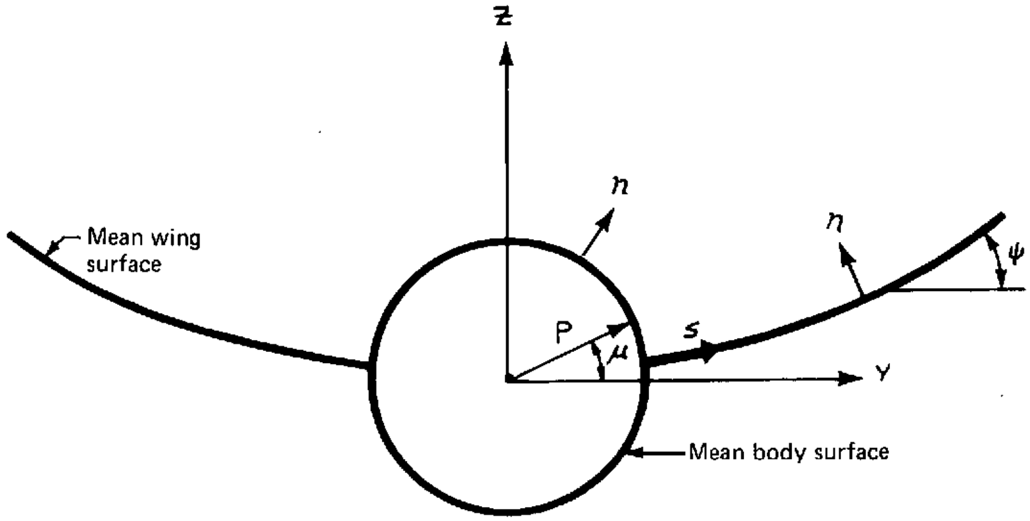


FIGURE 8.–CROSS SECTION COORDINATES

Summary of Reduced Problems

$O(\alpha)$: $(1-M^2)\,\varphi_{oxx} + \varphi_{oyy} + \varphi_{ozz} = 0$

wing: $\dfrac{\partial \varphi_o}{\partial n} = -\cos\psi$

wake: $[\varphi_{ox}] = 0$ (28)

body: $\dfrac{\partial \varphi_o}{\partial n} = -\sin\mu + \underline{a\dfrac{dR}{dx}\varphi_{ox}}$

$C_{p_\alpha} = -2\alpha\,\varphi_{ox}$

$O(\beta)$: $(1-M^2)\,\varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} = 0$

wing: $\dfrac{\partial \varphi_1}{\partial n} = \sin\psi$

wake: $[\varphi_{1x}] = 0$

body: $\dfrac{\partial \varphi_1}{\partial n} = -\cos\mu$

$C_{p_\beta} = -2\beta\,\varphi_{1x}$

$$O(a):\quad (1-M^2)\,\varphi_{2xx} + \varphi_{2yy} + \varphi_{2zz} = 0$$

wing: $\dfrac{\partial \varphi_2}{\partial n} = 0$

wake: $[\varphi_{2x}] = 0$

body: $\dfrac{\partial \varphi_2}{\partial n} = \dfrac{dR}{dx} + \underline{a\,\dfrac{dR}{dx}\,\varphi_{2x}}$

$$C_{p_a} = -2a\,\varphi_{2x}$$

$$O(c):\quad (1-M^2)\,\varphi_{3xx} + \varphi_{3yy} + \varphi_{3zz} = 0$$

wing: $\dfrac{\partial \varphi_3}{\partial n} = 0$

wake: $[\varphi_{3x}] = 0$ (28-cont.)

body: $\dfrac{\partial \varphi_3}{\partial n} = \dfrac{dG}{dx}\sin\mu + \underline{a\,\dfrac{dR}{dx}\,\varphi_{3x}}$

$$C_{p_c} = -2c\,\varphi_{3x}$$

$$O(\theta):\quad (1-M^2)\,\varphi_{4xx} + \varphi_{4yy} + \varphi_{4zz} = 0$$

wing: $\dfrac{\partial \varphi_4}{\partial n} = \dfrac{\partial H}{\partial x}$

wake: $[\varphi_{4x}] = 0$

body: $\dfrac{\partial \varphi_4}{\partial n} = \underline{a\,\dfrac{dR}{dx}\,\varphi_{4x}}$

$$C_{p_\theta} = -2\theta\,\varphi_{4x}$$

$$O(\tau):\quad (1-M^2)\,\varphi_{5xx} + \varphi_{5yy} + \varphi_{5zz} = 0$$

wing: $\dfrac{\partial \varphi_5}{\partial n} = \pm\dfrac{\partial F}{\partial x}$

wake: $[\varphi_{5x}] = 0$

body: $\dfrac{\partial \varphi_5}{\partial x} = \underline{a\,\dfrac{dR}{dx}\,\varphi_{5x}}$

$$C_{p_\tau} = -2\tau\,\varphi_{5x}$$

$$O(\alpha\beta): \quad (1-M^2)\varphi_{6xx} + \varphi_{6yy} + \varphi_{6zz} = 2M^2[\varphi_{0xy} + \underline{\varphi_{1xz}}]$$

wing: $\dfrac{\partial \varphi_6}{\partial n} = 0$

wake: $[\varphi_{6x}] = -[\varphi_{0y}] - \underline{[\varphi_{1z}]}$

body: $\dfrac{\partial \varphi_6}{\partial n} = 0$

$$C_{p_{\alpha\beta}} = -2\alpha\beta(\varphi_{6x} + \varphi_{0y} + \underline{\varphi_{1z}})$$

$$O(a\beta): \quad (1-M^2)\varphi_{7xx} + \varphi_{7yy} + \varphi_{7zz} = 2M^2\varphi_{2xy}$$

wing: $\dfrac{\partial \varphi_7}{\partial n} = 0$

wake: $[\varphi_{7x}] = -[\varphi_{2y}]$

body: $\dfrac{\partial \varphi_7}{\partial n} = \underline{\varphi_{1x}\dfrac{dR}{dx}}$ (28 cont.)

$$C_{p_{a\beta}} = -2a\beta(\varphi_{7x} + \varphi_{2y})$$

$$O(c\beta): \quad (1-M^2)\varphi_{8xx} + \varphi_{8yy} + \varphi_{8zz} = 2M^2\varphi_{3xy}$$

wing: $\dfrac{\partial \varphi_8}{\partial n} = 0$

wake: $[\varphi_{8x}] = -[\varphi_{3y}]$

body: $\dfrac{\partial \varphi_8}{\partial n} = \underline{\varphi_{1x}\dfrac{dG}{dx}\sin\mu}$

$$C_{p_{c\beta}} = -2c\beta(\varphi_{8x} + \varphi_{3y})$$

$$O(\theta\beta): \quad (1-M^2)\varphi_{9xx} + \varphi_{9yy} + \varphi_{9zz} = 2M^2\varphi_{4xy}$$

wing: $\dfrac{\partial \varphi_9}{\partial n} = \dfrac{\partial}{\partial s}(H\cos\varphi)$

wake: $[\varphi_{9x}] = -[\varphi_{4y}]$

body: $\dfrac{\partial \varphi_9}{\partial n} = 0$

$$C_{p_{\theta\beta}} = -2\theta\beta(\varphi_{9x} + \varphi_{4y})$$

$$o(\tau\beta): \quad (1-M^2)\,\varphi_{10xx} + \varphi_{10yy} + \varphi_{10zz} = 2M^2 \varphi_{5xy}$$

wing: $\dfrac{\partial \varphi_{10}}{\partial n} = \dfrac{\partial}{\partial s}(F \cos\psi)$

wake: $[\varphi_{10x}] = -[\varphi_{5y}]$ (28 cont)

body: $\dfrac{\partial \varphi_{10}}{\partial n} = 0$

$$C_{P_{\tau\beta}} = -2\tau\beta(\varphi_{10x} + \varphi_{5y})$$

The solution for the flow about a wing-body configuration in sideslip is given by the sum of the solutions yielding the individual perturbation potentials (c.f. eq. (5)).

It is interesting to attempt a physical interpretation of the second-order problems. Consider first the inhomogeneous terms appearing in the differential equations. They apparently serve to rotate the compressibility axis (the x-axis) through the angle β to correct for the fact that in the first-order formulation the x-axis was not aligned with the freestream. This is apparent from the particular solution which for the $o(\alpha\beta)$ problem may be written down by inspection as

$$\varphi_{6\ \text{particular}} = y\,\varphi_{ox} - x\,\varphi_{oy} \tag{29}$$

with similar solutions applicable for the other problems. The terms in eq. (29) are just the first-order terms in the Taylor-series expansion of φ_o required to construct the analytic continuation of the potential φ_o at any given point to a different point obtained by a rotation through an angle β.

The wake boundary conditions of the second-order problems require the addition of a spanwise component of wake vorticity equal in strength to β multiplied by the axial vorticity component generated by the first-order solutions. The result is to incline the vorticity vector in the wake to an angle β with the x-axis, which is physically correct.

The boundary conditions on the wing for the $o(\theta\beta)$ and $o(\tau\beta)$ problems account for the effect of spanwise camber and thickness variations encountered by the sideslip component of the freestream. For many configurations the spanwise camber and thickness slopes are small enough to be neglected.

11.0 PARTICULAR SOLUTIONS OF THE REDUCED SECOND-ORDER PROBLEMS

The distinguishing feature of the second-order problems that renders them more difficult than the first-order problems is the presence of inhomogeneous terms in the differential equation. The first step in solving these problems is to establish the particular solution of the inhomogeneous equation.

Since the inhomogeneous equations for the perturbation potentials φ_6 —— φ_{10} are all of the same basic form, one can search for a generalized form of the particular integral to be applied to all five cases. Let us describe the general problem in the following manner:

$$(1-M^2)\,\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 2M^2\,\varphi^{(0)}_{xy} \tag{30}$$

where

φ = any of the second-order perturbation potentials

$\varphi^{(0)}$ = the corresponding potential appearing in the inhomogeneity

All of the inhomogeneous equations appearing in (28) are of this general form. The general wake boundary condition is written in the form

$$[\varphi_x] = -[\varphi^{(0)}_y] \quad \text{on the mean wake surface} \tag{31}$$

One form of a particular solution of eq. (30) can be found by inspection to be

$$\varphi^P = y\,\varphi^{(0)}_x - x\,\varphi^{(0)}_y \tag{32}$$

However, this is not the most convenient form, for it introduces additional singularities into the surface integrals appearing in $\varphi^{(0)}$ (which will be expressed in terms of source and vortex distributions on the boundary surfaces). A more suitable form for the particular solution is available, and will be described in section 11.2.

11.1 DISCUSSION OF HOMOGENEOUS SOLUTIONS

As an aid in interpreting the behavior of the particular solution to be described in section 11.2, it is convenient first to review the character and behavior of the homogeneous solutions of eq. (30), which are identical in form to the solutions of the first-order problems and whose behavior is well known. Homogeneous solutions will be expressed in the form of source and vortex singularity distributions on the mean boundary surfaces. Let us introduce a two-dimensional coordinate system x_1, ℓ on a mean boundary surface, where x_1 is the axial coordinate and ℓ the lateral coordinate, as shown in figure 9.

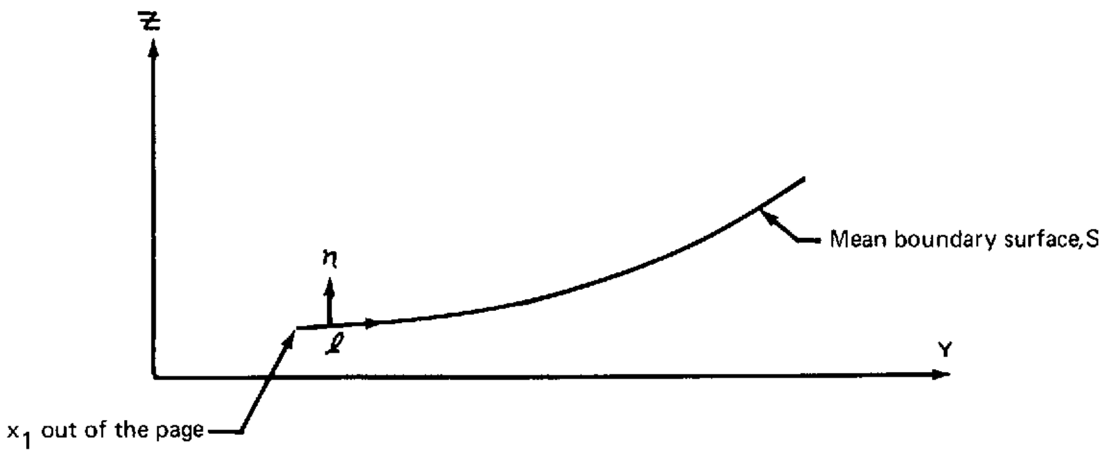


FIGURE 9.—SURFACE COORDINATES

A third coordinate, n, is defined normal to the surface, having positive orientation when directed along $(\vec{x}_1 \times \vec{\ell})$. With this notation, the homogeneous solutions appear in the form of surface integrals, as given below.

Vortex Distribution

$$\varphi^v(x, y, z) = \iint_S \gamma(x_1, \ell)\, K^v(x, y, z, x_1, \ell)\, ds \tag{33}$$

where

x, y, z = coordinates of a field point

$\varphi^v(x,y,z)$ = portion of the perturbation potential contributed by vortices

$\gamma(x_1, \ell)$ = vorticity density, on the surface, of strength established by the boundary conditions

$ds = dx_1\, d\ell$

The function $K^v(x, y, z, x_1, \ell)$ denotes the potential of an elementary horseshoe vortex located at (x_1, ℓ). It is defined as:

$$M < 1\ ;\quad K^v = \frac{1}{4\pi}\left[\frac{\vec{n}\cdot\vec{r}}{\left[(y-y_1(s))^2 + (z-z_1(s))^2\right]}\right]\cdot \left[1 + \frac{x - x_1(s)}{\sqrt{(x-x_1(s))^2 + B^2\left\{(y-y_1(s))^2 + (z-z_1(s))^2\right\}}}\right] \tag{34}$$

$$M > 1 \; ; \; K^V = \frac{1}{2\pi}\left[\frac{(\vec{n}\cdot\vec{r})(x - x_1(s))}{[(y-y_1(s))^2 + (z-z_1(s))^2]}\right] \cdot \tag{34 cont.}$$

$$\left[\frac{H\left((x-x_1(s)) - B\sqrt{(y-y_1(s))^2 + (z-z_1(s))^2}\right)}{\sqrt{(x-x_1(s))^2 - B^2\left\{(y-y_1(s))^2 + (z-z_1(s))^2\right\}}}\right]$$

where

$[x_1(s), y_1(s), z_1(s)]$ = a point on the surface S

$\vec{r}$ = radius vector from $(x_1(s), y_1(s), z_1(s))$ to a field point (x, y, z)

$B = \sqrt{|1 - M^2|}$

$$H(\omega) = \begin{Bmatrix} 1 & \text{for} & \omega > 0 \\ 0 & \text{for} & \omega < 0 \end{Bmatrix}$$

Source Distribution

$$\varphi^s(x, y, z) = \iint_S m(x_1, \ell)\, K^s(x, y, z, x_1, \ell)\, ds \tag{35}$$

where

$\varphi^s(x, y, z)$ = portion of the perturbation potential contributed by sources

$m(x_1, \ell)$ = source density on the surface S.

The function $K^s(x, y, z, x_1, \ell)$ denotes the potential of an elementary source located at (x_1, ℓ). It is defined as

$$M < 1: \quad K^s = -\frac{1}{4\pi}\,\frac{1}{\sqrt{(x-x_1(s))^2 + B^2\left\{(y-y_1(s))^2 + (z-z_1(s))^2\right\}}}$$

$$M > 1: \quad K^s = -\frac{1}{2\pi}\,\frac{H\left((x-x_1(s)) - B\sqrt{(y-y_1(s))^2 + (z-z_1(s))^2}\right)}{\sqrt{(x-x_1(s))^2 - B^2\left\{(y-y_1(s))^2 + (z-z_1(s))^2\right\}}} \tag{36}$$

The entire homogeneous solution is given by the sum of these surface integrals as

$$\varphi^{(o)}(x, y, z) = \varphi^{V}(x, y, z) + \varphi^{S}(x, y, z) \tag{37}$$

Let us now review the behavior of these surface integrals in the limit as the point approaches the surface. In particular, it is of interest to examine the resulting velocity discontinuities which appear across the surface. For this purpose we decompose the ℓ integration into several parts, separating out for special study a segment of length ε around the point on the surface approached by (x, y, z) as shown in figure 10. It can be shown that all discontinuities across the surface arise from the singularities on the small segment ε and that the effect of the remaining part of the surface may be disregarded. Also, the effect of surface curvature over the small segment ε can be disregarded, thereby reducing the problem to that of an integral over a planar strip of width ε inclined at an angle ψ to the y -axis.

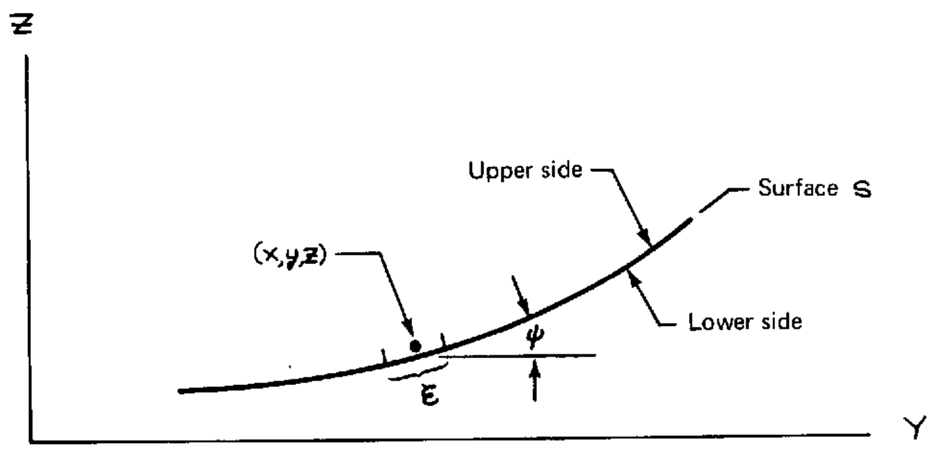


FIGURE 10.–REGION OF INTEGRATION

After carrying out the integration over the segment with the aid of a Taylor-series expansion of $m(x_1, \ell)$ and $\gamma(x_1, \ell)$ about a point on the surface and letting the field point (x, y, z) approach that point on the surface, one finds the following results for the discontinuities across the surface, which are well known.

$$\begin{aligned}
\Delta \varphi(x, y, z) &= \varphi(x, y, z)^{\text{UPPERSIDE}} - \varphi(x, y, z)^{\text{LOWERSIDE}} = \int_{-\infty}^{x} \gamma(x_1, \ell)\, dx_1 \\
\Delta u(x, y, z) &= \gamma(x, y, z) \\
\Delta q_t(x, y, z) &= \int_{-\infty}^{(x,y,z)} \frac{\partial \gamma}{\partial \ell}\, dx_1 \\
\Delta q_n(x, y, z) &= m(x, y, z)
\end{aligned} \tag{38}$$

where

Δq_t = discontinuity in the tangential velocity component in the lateral direction

and

Δq_n = discontinuity in the normal velocity component

as shown in figure 11.

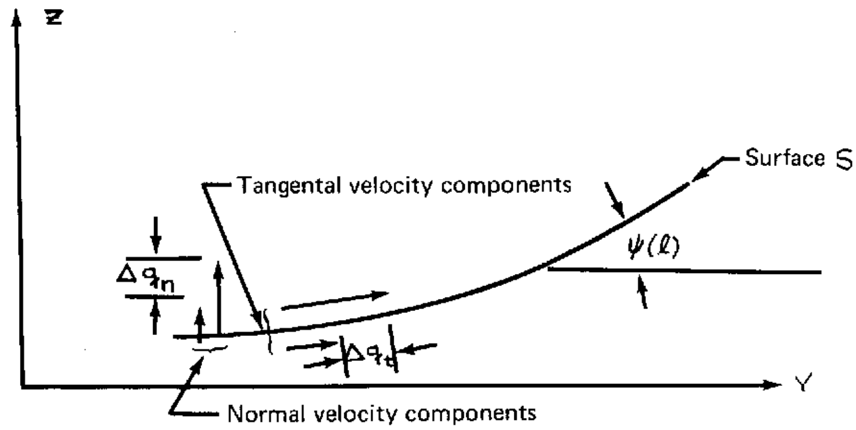


FIGURE 11.–DISCONTINUITIES IN VELOCITY COMPONENTS

The lateral discontinuities may be decomposed into components in the y and z directions as

$$\begin{aligned}
\Delta v(x,y,z) &= \cos\psi \int_{-\infty}^{(x,y,z)} \frac{\partial\gamma}{\partial\ell}\,dx_1 - m(x,y,z)\sin\psi \\
\Delta w(x,y,z) &= \sin\psi \int_{-\infty}^{(x,y,z)} \frac{\partial\gamma}{\partial\ell}\,dx_1 + m(x,y,z)\cos\psi
\end{aligned} \tag{39}$$

It should be noted that in the wake behind a lifting surface only the lateral discontinuities arising from the vortex distribution are nonzero.

11.2 PREFERRED FORM OF THE PARTICULAR SOLUTION

With these results in mind, we now proceed to write a preferred form of particular solution and examine its behavior. The particular solution selected is

$$\varphi^{P}(x,y,z) = \iint_S (y-y_1(s))\,\gamma(x_1,\ell)\frac{\partial}{\partial x}\left(K^V(x,y,z,x_1,\ell)\right)ds$$

$$-\iint_S (x-x_1(s))\,\gamma(x_1,\ell)\frac{\partial}{\partial y}\left(K^V(x,y,z,x_1,\ell)\right)ds$$

$$+\iint_S (y-y_1(s))\,m(x_1,\ell)\frac{\partial}{\partial x}\left(K^S(x,y,z,x_1,\ell)\right)ds \qquad (40)$$

$$-\iint_S (x-x_1(s))\,m(x_1,\ell)\frac{\partial}{\partial y}\left(K^S(x,y,z,x_1,\ell)\right)ds$$

Let us first verify that this is a particular solution of eq. (30). To do this we decompose eq. (40) into homogeneous and particular parts. The homogeneous part is

$$-\iint_S y_1\gamma\frac{\partial K^V}{\partial x}ds + \iint_S x_1\gamma\frac{\partial K^V}{\partial y}ds - \iint_S y_1 m\frac{\partial K^S}{\partial x}ds + \iint_S x_1 m\frac{\partial K^S}{\partial y}ds$$

$$= \frac{\partial}{\partial x}\left[-\iint_S y_1\gamma K^V ds - \iint_S y_1 m K^S ds\right] + \frac{\partial}{\partial y}\left[\iint_S x_1\gamma K^V ds + \iint_S x_1 m K^S ds\right] \qquad (41)$$

The bracketed terms are of the same form as the homogeneous solutions (33) and (35), with the vortex and source densities interpreted as $(y_1\gamma)$, $(y_1 m)$ or $(x_1\gamma)$, $(x_1 m)$. They are thus homogeneous solutions. The derivatives of these terms appearing in eq. (41) are also homogeneous solutions, for it is easily seen by differentiation of eq. (30) that derivatives of homogeneous solutions also satisfy the homogeneous equation. Thus the terms in eq. (41) are indeed homogeneous solutions.

The remaining part of eq. (40) can be written as

$$\iint_S y\gamma\frac{\partial K^V}{\partial x}ds - \iint_S x\gamma\frac{\partial K^V}{\partial y}ds + \iint_S y m\frac{\partial K^S}{\partial x}ds - \iint_S x m\frac{\partial K^S}{\partial y}ds$$

$$= y\frac{\partial}{\partial x}\left[\iint_S \gamma K^V ds + \iint_S m K^S ds\right] - x\frac{\partial}{\partial y}\left[\iint_S \gamma K^V ds + \iint_S m K^S ds\right]$$

$$= y\frac{\partial}{\partial x}\left(\varphi^V + \varphi^S\right) - x\frac{\partial}{\partial y}\left(\varphi^V + \varphi^S\right) \qquad (42)$$

$$= y\,\varphi_x^{(0)} - x\,\varphi_y^{(0)}$$

The latter steps follow from eqs. (33), (35), and (37). It is readily verified by direct substitution into eq. (30) that eq. (42) constitutes a particular solution. It thus follows that eq. (40) is also a particular solution.

Let us now restrict our view to the small strip of width ε , since it contributes all of the discontinuous behavior. Define a local coordinate system aligned with the strip, as shown in figure 12.

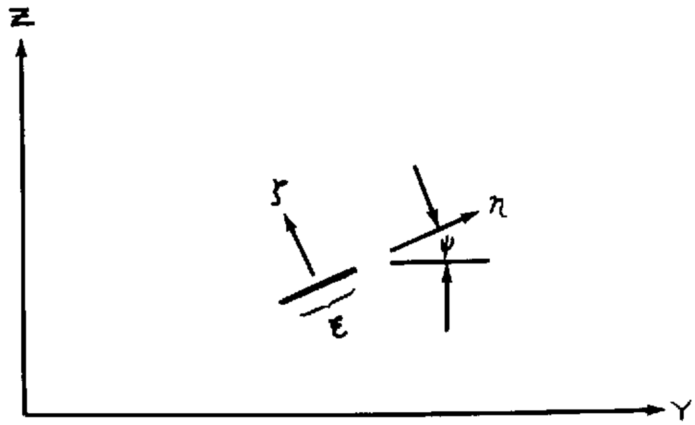


FIGURE 12.—LOCAL COORDINATE SYSTEM

In terms of these coordinates, eq. (40) becomes

$$
\begin{aligned}
\varphi^P = {} & \cos\psi \iint_\varepsilon (\eta-\eta_1)\,\gamma \frac{\partial K^V}{\partial x}\, dx_1 d\eta_1 - \sin\psi \iint_\varepsilon (\zeta-\zeta_1)\,\gamma \frac{\partial K^V}{\partial x}\, dx_1 d\eta_1 \\
& - \cos\psi \iint_\varepsilon (x-x_1)\,\gamma \frac{\partial K^V}{\partial \eta}\, dx_1 d\eta_1 + \sin\psi \iint_\varepsilon (x-x_1)\,\gamma \frac{\partial K^V}{\partial \zeta}\, dx_1 d\eta_1 \\
& + \cos\psi \iint_\varepsilon (\eta-\eta_1)\, m \frac{\partial K^S}{\partial x}\, dx_1 d\eta_1 - \sin\psi \iint_\varepsilon (\zeta-\zeta_1)\, m \frac{\partial K^S}{\partial x}\, dx_1 d\eta_1 \\
& - \cos\psi \iint_\varepsilon (x-x_1)\, m \frac{\partial K^S}{\partial \eta}\, dx_1 d\eta_1 + \sin\psi \iint_\varepsilon (x-x_1)\, m \frac{\partial K^S}{\partial \zeta}\, dx_1 d\eta_1
\end{aligned}
\qquad (43)
$$

where it is understood that φ^P now consists only of the contribution from the strip of width ε .

It is now a simple matter to deduce the limiting behavior of these terms from a knowledge of the basic properties of the integrals which led to the results given by eqs. (38) and (39). For example, let us consider the behavior of the first term arising from the discontinuity in φ^P_x . It is

$$
\left[\varphi^P_x\right] = \cos\psi \left[\frac{\partial}{\partial x} \iint_\varepsilon (\eta-\eta_1)\,\gamma \frac{\partial K^V}{\partial x}\, dx_1 d\eta_1\right] + \text{other terms} \qquad (44)
$$

From eqs. (33) and (38) it follows that

$$\frac{\partial}{\partial x}\left[\iint_{\varepsilon} \gamma \frac{\partial K^{v}}{\partial x} dx_1 d\eta_1\right] = \frac{\partial^2}{\partial x^2}\left[\iint_{\varepsilon} \gamma \, K^{v} dx_1 d\eta_1\right] = \frac{\partial \gamma(x,\eta)}{\partial x} \tag{45}$$

This differs from the term appearing in eq. (44) by the factor $(\eta - \eta_1)$ in the numerator, which approaches zero as $\varepsilon \to 0$. Hence the term in eq. (44) is at most $o(\varepsilon)$ compared to $\frac{\partial \gamma}{\partial x}(x,\eta)$, and must vanish as $\varepsilon \to 0$. Repeating this type of reasoning for all the terms leads to the following conclusions for the velocity discontinuities arising from the particular solution:

<u>Discontinuity in φ_x^p</u>

Upon differentiating eq. (43) with respect to x and examining the behavior of the various terms in the manner described above, one finds only one term that contributes to the discontinuity in φ_x^p . It is

$$\begin{aligned}\left[\varphi_x^p\right] &= -\cos\varphi\left[\frac{\partial}{\partial x}\iint_{\varepsilon}(x-x_1)\,\gamma\,\frac{\partial K^{v}}{\partial \eta}dx_1 d\eta_1\right] \\ &= -\cos\varphi\left[\frac{\partial}{\partial x}\int_{-\infty}^{x}(x-x_1)\frac{\partial \gamma}{\partial \eta}dx_1\right] \\ &= -\cos\varphi\left[\int_{-\infty}^{x}\frac{\partial \gamma}{\partial \eta}dx_1\right]\end{aligned} \tag{46}$$

It follows from eq. (39) and the comment immediately thereafter that the velocity discontinuity across the wake behind the surface is equal to

$$\left[\varphi_x^p\right]_{WAKE} = -\cos\varphi\left[\int_{-\infty}^{x}\frac{\partial \gamma}{\partial \eta}dx_1\right] = -\left[\varphi_y^{(o)}\right] \tag{47}$$

Thus, the particular solution satisfies the wake boundary condition, eq. (31.)

<u>Discontinuity in φ_η^p</u>

The terms from eq. (43) contributing to the discontinuity in φ_η^p are:

$$\left[\varphi_{\eta}^{P}\right] = \cos\psi\left[\frac{\partial}{\partial\eta}\iint_{E}(\eta-\eta_1)\,\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1\,d\eta_1\right]$$

$$-\cos\psi\left[\frac{\partial}{\partial\eta}\iint_{E}(x-x_1)\,\gamma\,\frac{\partial K^{V}}{\partial\eta}\,dx_1\,d\eta\right]$$

$$= \cos\psi\left\{\left[\iint_{E}\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1 d\eta_1\right]+\left[\iint_{E}(\eta-\eta_1)\,\gamma\,\frac{\partial^2 K^{V}}{\partial x\,\partial\eta}\,dx_1 d\eta_1\right]\right.$$

$$\left.-\left[\frac{\partial}{\partial\eta}\iint_{E}(x-x_1)\,\gamma\,\frac{\partial K^{V}}{\partial\eta}\,dx_1\,d\eta_1\right]\right\}$$

$$= \cos\psi\left\{\left[\iint_{E}\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1 d\eta_1\right]+\left[\eta\frac{\partial}{\partial\eta}\iint_{E}\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1\,d\eta_1\right]\right. \tag{48}$$

$$-\left[\frac{\partial}{\partial\eta}\iint_{E}\eta_1\,\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1 d\eta_1\right]-\left[x\frac{\partial}{\partial\eta}\iint_{E}\gamma\,\frac{\partial K^{V}}{\partial\eta}\,dx_1 d\eta_1\right]$$

$$\left.+\left[\frac{\partial}{\partial\eta}\iint_{E}x_1\,\gamma\,\frac{\partial K^{V}}{\partial\eta}\,dx_1\,d\eta_1\right]\right\}$$

$$= \cos\psi\left\{\gamma(x,\eta)+\eta\frac{\partial\gamma(x,\eta)}{\partial\eta}-\frac{\partial}{\partial\eta}(\eta\,\gamma(x,\eta))\right.$$

$$\left.-x\frac{\partial}{\partial\eta}\int_{-\infty}^{x}\frac{\partial\gamma}{\partial\eta}\,dx_1+\frac{\partial}{\partial\eta}\int_{-\infty}^{x}x_1\frac{\partial\gamma}{\partial\eta}\,dx_1\right\}$$

$$-\cos\psi\int_{-\infty}^{x}(x-x_1)\,\frac{\partial^2\gamma}{\partial\eta^2}\,dx_1$$

Discontinuity in φ_{ζ}^{P} –Normal Velocity Component

The terms from eq. (43) contributing to a discontinuity in φ_{ζ}^{P} are:

$$\left[\varphi_{\zeta}^{P}\right] = -\sin\psi\left[\frac{\partial}{\partial\zeta}\iint_{E}(\zeta-\zeta_1)\,\gamma\,\frac{\partial K^{V}}{\partial x}\,dx_1 d\eta_1\right] \tag{49}$$

$$- \sin \psi \left[\iint_E \gamma \frac{\partial K^v}{\partial x} dx_1 d\eta_1 \right] - \sin \psi \frac{\partial}{\partial x} \left[\iint_E (\zeta - \zeta_1) \gamma \frac{\partial K^v}{\partial \zeta} dx_1 d\eta_1 \right] \tag{49 cont.}$$

$$= - \gamma (x, \eta) \sin \psi$$

Thus we find a discontinuity in the normal velocity component that is proportional to the vorticity distribution of the homogeneous solution.

12.0 COMPARISONS WITH KNOWN SOLUTIONS

To gain confidence in the present theory it is desirable to compare results with some known solutions. This can easily be done for the limited class of problems involving planar wings in sideslip, since the present theory can then be compared with a first-order expansion in β of the first-order solution in α obtained with a skewed planform. Two different problems will be considered.

12.1 INFINITE YAWED WING IN INCOMPRESSIBLE FLOW

Consider an infinite yawed planar wing lying in the x-y plane, with coordinate systems and freestream direction as shown in figure 13. The freestream is inclined at the angles

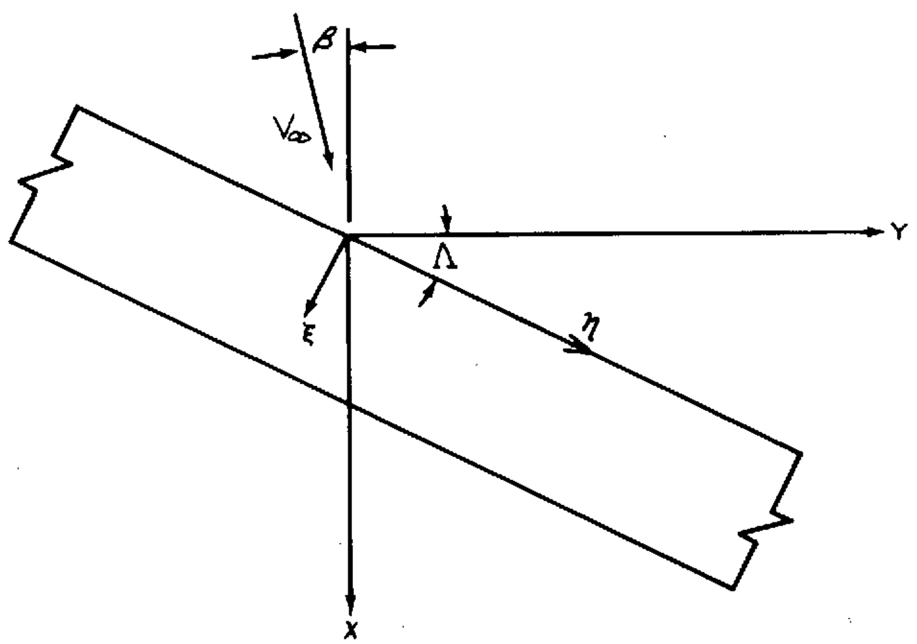


FIGURE 13.—INFINITE YAWED WING ORIENTATION

α and β to the coordinate planes such that the freestream velocity components in the coordinate directions are

$$
\begin{aligned}
V_x &= V_\infty \cos\alpha \cos\beta \\
V_y &= V_\infty \cos\alpha \sin\beta \\
V_z &= V_\infty \sin\alpha
\end{aligned}
\tag{50}
$$

The pressure distribution on this wing can be ascertained by means of simple sweep theory, considering the sweep angle to be $(\Lambda + \beta)$. This result will then be compared to the present theory, which considers a wing of sweep angle Λ subjected to a sideslip velocity βV_∞.

To obtain the simple sweep result, we first establish the freestream velocity components in the ξ, η, and ζ directions. Retaining only first-order terms in α and β, a simplification which is consistent with the present theory, these turn out to be

$$\begin{aligned} v_\xi &= V_\infty \cos\Lambda\,[1 - \beta \tan\Lambda] \\ v_\eta &= V_\infty \cos\Lambda\,[\tan\Lambda + \beta] \\ v_\zeta &= \alpha V_\infty \end{aligned} \tag{51}$$

The pressure distribution is determined by the angle of attack of the normal flow, which is

$$\alpha_{2\text{-}D} = \frac{v_\zeta}{v_\xi} = \frac{\alpha}{\cos\Lambda\,[1 - \beta\tan\Lambda]} = \frac{\alpha}{\cos\Lambda}\,[1 + \beta\tan\Lambda] \tag{52}$$

The latter equality is valid to first order in β.

Now the pressure discontinuity across a two-dimensional flat plate of chord c is known to be (ref. 3)

$$\Delta\left(\frac{p - p_\infty}{\frac{1}{2}\rho_\infty v_\xi^2}\right) = 4\alpha_{2\text{-}D}\sqrt{\frac{c-\xi}{\xi}} \tag{53}$$

where v_ξ and α_{2D} are the normal freestream component and angle of attack. Substituting for v_ξ and α_{2D} results in

$$\frac{p - p_\infty}{\frac{1}{2}\rho_\infty \cos^2\Lambda\,[1 - \beta\tan\Lambda]^2\, V_\infty^2} = \frac{4\alpha}{\cos\Lambda}\,[1 + \beta\tan\Lambda]\sqrt{\frac{c-\xi}{\xi}} \tag{54}$$

which reduces to

$$\Delta C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 4\alpha\cos\Lambda\sqrt{\frac{c-\xi}{\xi}}\,[1 - \beta\tan\Lambda] \tag{55}$$

within first-order accuracy. This is the pressure distribution given by simple sweep theory, valid to first order in β.

Now let us work out a comparable result by means of the present theory. The various problems, from eq. (28), are

$$O(\alpha): \quad \varphi_{o\,xx} + \varphi_{o\,yy} + \varphi_{o\,zz} = 0$$

$$\text{wing: } \frac{\partial \varphi_o}{\partial n} = -1 \qquad (56)$$

$$\text{wake: } \left[\varphi_{ox}\right] = 0$$

$$C_{P_\alpha} = -2\alpha\,\varphi_{ox}$$

$$O(\beta): \quad \varphi_{1\,xx} + \varphi_{1\,yy} + \varphi_{1\,zz} = 0$$

$$\text{wing: } \frac{\partial \varphi_1}{\partial n} = 0 \qquad (57)$$

$$\text{wake: } \left[\varphi_{1x}\right] = 0$$

$$C_{P_\beta} = -2\beta\,\varphi_{1x}$$

$$O(\alpha\beta): \quad \varphi_{6xx} + \varphi_{6yy} + \varphi_{6zz} = 0$$

$$\text{wing: } \frac{\partial \varphi_6}{\partial n} = 0 \qquad (58)$$

$$\text{wake: } \left[\varphi_{6x}\right] = -\left[\varphi_{oy}\right]$$

$$C_{P_{\alpha\beta}} = -2\alpha\beta\left(\varphi_{6x} + \varphi_{oy}\right)$$

The total velocity potential for this simple case consists of

$$\Phi = V_\infty\left(x + \beta y + \alpha z + \alpha\varphi_o + \beta\varphi_1 + \alpha\beta\varphi_6\right) \qquad (59)$$

The solution to the $O(\alpha)$ problem is easily derived from simple sweep theory by setting β equal to zero in eq. (55), giving

$$\Delta C_{P_\alpha} = 4\alpha\cos\Lambda\sqrt{\frac{c-\xi}{\xi}} \qquad (60)$$

The $O(\beta)$ potential, φ_1, is zero, since all of the boundary conditions are zero.

The $o(\alpha\beta)$ potential, φ_{b}, is also zero, since the equation is homogeneous when $M = 0$, and all the boundary conditions are zero. Note that, for this particular example, $[\varphi_{oy}] = 0$ in the wake. Hence the only contribution from the $o(\alpha\beta)$ problem comes from the expression for $C_{P_{\alpha\beta}}$, which reduces to

$$C_{P_{\alpha\beta}} = -2\ \alpha\beta\ \varphi_{oy} \tag{61}$$

Now, from simple sweep theory,

$$\varphi_{oy} = -\varphi_{ox} \tan\Lambda \tag{62}$$

φ_{ox} is related to C_{P_α} by the expression (eq. 56))

$$C_{P_\alpha} = -2\ \alpha\ \varphi_{ox} \tag{63}$$

Eliminating φ_{ox} and φ_{oy} between eqs. (62), (62), and (63) finally gives

$$C_{P_{\alpha\beta}} = 2\alpha\beta\ \varphi_{ox} \tan\Lambda = -\beta \tan\Lambda\ C_{P_\alpha} \tag{64}$$

Combining eq. (60) and (64) to obtain the total pressure coefficient, one finds that

$$\begin{aligned} \Delta C_P = \Delta C_{P_\alpha} + \Delta C_{P_{\alpha\beta}} &= \Delta C_{P_\alpha}\left[1 - \beta \tan\Lambda\right] \\ &= 4\alpha \cos\Lambda \sqrt{\frac{c-\xi}{\xi}}\left[1 - \beta\tan\Lambda\right] \end{aligned} \tag{65}$$

Thus, one obtains a term of $o(\alpha)$ and another of $o(\alpha\beta)$. The entire expression is equal to the simple sweep theory result of eq. (55), which demonstrates that the present theory gives the correct solution for this case.

12.2 PLANAR WING IN SIDESLIP–SUBSONIC COMPRESSIBLE FLOW

For this case we shall write the first order in α solution obtained with a skewed planform and then expand this solution in powers of the skew angle β. This will then be compared with the results of the present theory.

Consider the wing of planform S shown in figure 14, which lies in the x-y, ξ-η plane. The freestream is directed according to eq. (50). The problem will first be formulated in the ξ, η, ζ coordinate system, in which the freestream has no sideslip component.

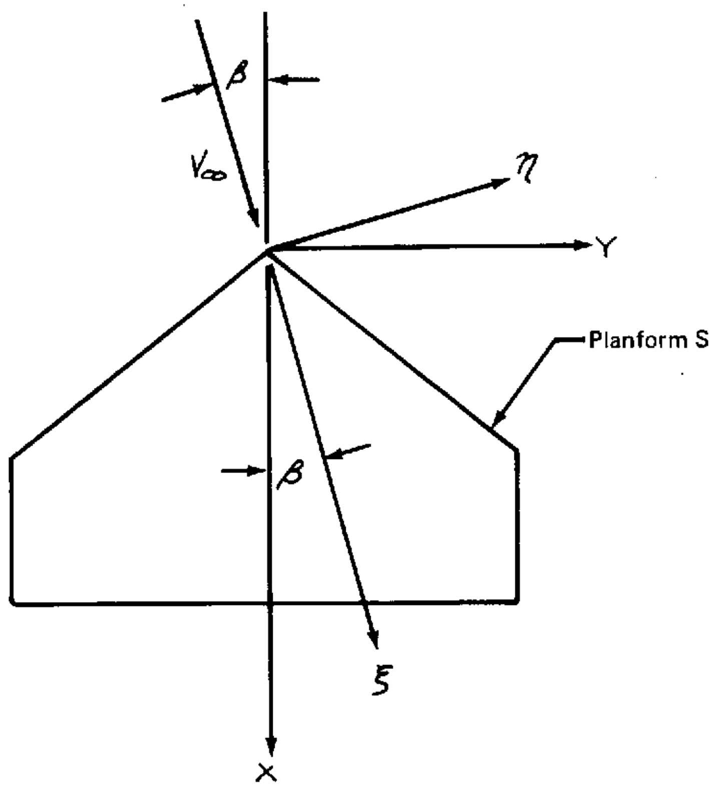


FIGURE 14.–SKEWED WING PLANFORM

This problem is

$$O(\alpha):\quad (1-M^2)\,\varphi^{\alpha}_{\xi\xi} + \varphi^{\alpha}_{\eta\eta} + \varphi^{\alpha}_{\zeta\zeta} = 0$$

$$\text{wing: } \frac{\partial \varphi^{\alpha}}{\partial \zeta} = -1 \qquad (66)$$

$$\text{wake: } \left[\varphi^{\alpha}_{\xi}\right] = 0$$

where the total potential is

$$\Phi = V_\infty\left[\xi + \alpha\zeta + \alpha\varphi^{\alpha}\right] \qquad (67)$$

The solution of this boundary value problem can be expressed in terms of an integral equation expressing the influence of a distribution of elementary horseshoe vortices. This expression is

$$\alpha\varphi^{\alpha}(\xi,\eta,\zeta) = \alpha\iint_S \gamma(\xi_1,\eta_1)\,K^{v}(\xi,\eta,\zeta,\xi_1,\eta_1)\,d\xi_1\,d\eta_1 \qquad (68)$$

where $\gamma(\xi_1, \eta_1)$ is the strength of the bound vorticity and K^V is the potential of the elementary horseshoe, given by eq. (34) as

$$K^V = \frac{1}{4\pi} \frac{\zeta}{(\eta-\eta_1)^2 + \zeta^2} \left\{ 1 + \frac{\xi - \xi_1}{\sqrt{(\xi-\xi_1)^2 + (1-M^2)((\eta-\eta_1)^2 + \zeta^2)}} \right\} \tag{69}$$

We now expand this solution in powers of β by introducing

$$\begin{aligned} \xi &= x \cos\beta + y \sin\beta \approx x + \beta y \\ \eta &= -x \sin\beta + y \cos\beta \approx -\beta x + y \\ \zeta &= z \end{aligned} \tag{70}$$

γ is expanded as

$$\gamma(\xi_1, \eta_1) = \gamma_1(x_1(\xi_1, \eta_1), y_1(\xi_1, \eta_1)) + \beta \gamma_2 (x_1(\xi_1, \eta_1), y_1(\xi_1, \eta_1)) \tag{71}$$

Two terms result from the expansion of K^V. One is the expression (69) with ξ and η replaced by x and y, which will be denoted as K_1^V.

$$K_1^V = \frac{1}{4\pi} \frac{z}{(y-y_1)^2 + z^2} \left\{ 1 + \frac{x - x_1}{\sqrt{(x-x_1)^2 + (1-M^2)((y-y_1)^2 + z^2)}} \right\} \tag{72}$$

The other term, of $O(\beta)$, which will be denoted as βK_2^V, can be written as

$$\beta K_2^V = \beta \left[(y - y_1) \frac{\partial K_1}{\partial x} - (x - x_1) \frac{\partial K_1}{\partial y} \right] \tag{73}$$

Note that this is identical to the form of the particular solution given by eq. (40).

With these expansions, the solution (68) takes the form

$$\begin{aligned} \alpha \varphi^\alpha(x, y, z) &= \alpha \iint_S \gamma_1(x_1, y_1) K_1^V(x, y, z, x_1, y_1) \, dx_1 \, dy_1 \\ &+ \alpha\beta \iint_S \gamma_2(x_1, y_1) K_1^V(x, y, z, x_1, y_1) \, dx_1 \, dy_1 \\ &+ \alpha\beta \iint_S \gamma_1(x_1, y_1) K_2^V(x, y, z, x_1, y_1) \, dx_1 \, dy_1 \end{aligned} \tag{74}$$

The boundary conditions become

$$\text{wing:} \quad \frac{\partial \varphi^{\alpha}}{\partial z} = -1$$

$$\text{wake:} \quad \left[\varphi^{\alpha}_{x}\right] + \beta\left[\varphi^{\alpha}_{y}\right] = 0 \tag{75}$$

Let us now turn to solving the problem by the method presented in this report. The $O(\alpha)$ problem is

$$(1-M^2)\,\varphi_{o\,xx} + \varphi_{o\,yy} + \varphi_{o\,zz} = 0$$

$$\text{wing:} \quad \frac{\partial \varphi_o}{\partial n} = -1 \tag{76}$$

$$\text{wake:} \quad \left[\varphi_{o\,x}\right] = 0$$

The $O(\beta)$ solution is identically zero, because of the zero boundary conditions. The $O(\alpha\beta)$ problem, however, is

$$(1-M^2)\,\varphi_{6\,xx} + \varphi_{6\,yy} + \varphi_{6\,zz} = 2M^2\varphi_{o\,xy}$$

$$\text{wing:} \quad \frac{\partial \varphi_6}{\partial n} = 0 \tag{77}$$

$$\text{wake:} \quad \left[\varphi_{6\,x}\right] = -\left[\varphi_{o\,y}\right]$$

The total velocity potential is

$$\Phi = V_\infty\left[x + \beta y + \alpha z + \alpha\varphi_o + \alpha\beta\varphi_6\right] \tag{78}$$

The solutions to these problems are

$$O(\alpha): \quad \varphi_o = \iint_S \gamma_3(x_1, y_1)\, K_1^V(x, y, z, x_1, y_1)\, dx_1\, dy_1 \tag{79}$$

$$O(\alpha\beta): \varphi_6 = \iint_S \gamma_4(x_1, y_1)\, K_1^V(x, y, z, x_1, y_1)\, dx_1\, dy_1 \tag{80}$$

$$+ \iint_S \gamma_3(x_1, y_1)\, K_2^V(x, y, z, x_1, y_1)\, dx_1\, dy_1$$

where γ_3 and γ_4 are established by the boundary conditions. The latter term is composed of a particular solution involving K_2^V (see eq. (73)), and a homogeneous solution containing K_1^V.

Having obtained a solution by both methods, it remains to compare them. First note that the solution (74) is composed of an $O(\alpha)$ term and two $O(\alpha\beta)$ terms, which would be identical to the solution by the present method, the sum of eqs (79) and (80), provided that it can be shown that

$$\begin{aligned} \gamma_1 &= \gamma_3 \\ \gamma_2 &= \gamma_4 \end{aligned} \tag{81}$$

This can be done as follows. Let us write the term $\alpha\varphi^\alpha$ as

$$\alpha\varphi^\alpha = \alpha\varphi_o^\alpha + \alpha\beta\varphi_6^\alpha \tag{82}$$

where $\alpha\varphi_o^\alpha$ is equal to the first right-hand term of eq. (74) and $\alpha\beta\varphi_6^\alpha$ is equal to the remaining terms. If we can show that $\gamma_1 = \gamma_3$ and $\gamma_2 = \gamma_4$, then φ_o^α and φ_6^α will be equal to their counterparts φ_o and φ_6 obtained by the present method. Substituting the expression (82) into the boundary conditions (75) gives

$$\begin{aligned} &\text{wing: } \frac{\partial\varphi_o^\alpha}{\partial z} + \beta\frac{\partial\varphi_6^\alpha}{\partial z} = -1 \\ &\text{wake: } \left[\varphi_{o\,x}^\alpha\right] + \beta\left[\varphi_{6\,x}^\alpha\right] + \beta\left[\varphi_{o\,y}^\alpha\right] + O(\beta^2) = 0 \end{aligned} \tag{83}$$

Equating orders of β gives

$$O(\alpha):\ \begin{aligned} &\text{wing: } \frac{\partial\varphi_o^\alpha}{\partial z} = -1 \\ &\text{wake: } \left[\varphi_{o\,x}^\alpha\right] = 0 \end{aligned} \tag{84}$$

$$O(\alpha\beta):\ \begin{aligned} &\text{wing: } \frac{\partial\varphi_6^\alpha}{\partial z} = 0 \\ &\text{wake: } \left[\varphi_{6\,x}^\alpha\right] = -\left[\varphi_{o\,y}^\alpha\right] \end{aligned} \tag{85}$$

In comparing these with the boundary conditions given by eqs. (76) and (77), it is obvious that the boundary conditions determining γ_1 and γ_2 are identical to those determining γ_3 and γ_4, and hence that $\gamma_1 = \gamma_3$ and $\gamma_2 = \gamma_4$. Thus the equivalence of the solutions by the two methods has been proved.

13.0 CONCLUDING REMARKS

A theory has been developed for predicting the aerodynamic properties of an airplane in sideslip. The airplane geometry was defined in terms of small parameters governing the angle of attack, angle of sideslip, wing camber, wing thickness, body radius variation, and body camber. These small parameters formed the basis for a perturbation expansion approach wherein the velocity potential and boundary conditions were expanded as asymptotic series in powers of the small parameters. All first-order terms and those second-order terms involving the interaction between sideslip and the other small parameters were retained.

The development produced a set of first-order homogeneous boundary value problems giving the influence of each small parameter, and a set of second-order inhomogeneous problems giving the interactions between sideslip and angle of attack, sideslip and camber, etc. It was found that certain of the inhomogeneous terms appearing in the second-order problems were always of higher order than those producing the dominant effects and hence could be deleted. This led to a simplification of the second-order problems such that generalized particular solutions could be obtained. A particular integral was found which has the property of automatically satisfying the second-order boundary conditions on all trailing vortex sheets. With this, the remaining second-order homogeneous part of the solution can be constructed by the same methods used to solve the first-order problems. The theory was checked by comparing with known solutions for an infinite yawed wing and for a skewed flat wing.

A major conclusion is that the dominant effects influencing the aerodynamics of sideslip can appear either as first-order or as second-order terms, depending primarily on the magnitude of the dihedral of the lifting surfaces. For configurations having large amounts of dihedral, the first-order terms dominate. Sideslip aerodynamics will then depend primarily on the angle of sideslip alone, and will not be strongly influenced by interactions between sideslip and angle of attack or between sideslip and the other parameters. For such configurations, the aerodynamic description given by conventional first-order theory is sufficient.

However, if the dihedral is small, then first-order theory fails to give the dominant effects. For such configurations, the second-order terms involving the interaction between sideslip and angle of attack, etc., are major contributors to sideslip aerodynamics, and it is essential that they be taken into account.

Another outcome is that highly accurate numerical solution procedures are necessary in any calculation of second-order sideslip aerodynamics. This requirement arises from the singular nature of the kernel of the particular integrals for the second-order problems. The kernel is more singular than its homogeneous counterpart, and integration by parts can be used to show that the numerical value of the particular integral depends strongly on the local gradients of the coefficients involving the first-order source and vortex strengths. This behavior can also be seen in the alternate form for the particular integral,

$$\varphi^P = Y\,\varphi_x^{(o)} - x\,\varphi_Y^{(o)}$$

from whence it is obvious that an accurate knowledge of the velocity derivatives of the first-order solution is required for the evaluation of aerodynamic forces and pressures contributed by the second-order particular solution. It is apparent, then, that the present theory must be accompanied by a numerical solution procedure that meets requirements more stringent than those ordinarily needed in a first-order analysis. For an ordinary first-order analysis the aerodynamicist is usually satisfied with a numerical solution procedure that provides accurate forces and pressures (velocities). In contrast, the present second-order theory must be accompanied by a numerical method capable of producing accurate velocity *gradients* in the first-order solution.

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