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FOR AN ELASTIC WEDGE

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Abstract. In this paper the contact and the crack problems for an elastic wedge of arbitrary angle are considered. The problem is reduced to a singular integral equation which, in the general case, may have a generalized Cauchy kernel. The singularities under the stamp as well as at the wedge apex are studied and the relevant stress intensity factors are defined. The problem is solved for various wedge geometries and loading conditions. The results may be applicable to certain foundation problems and to crack problems in symmetrically loaded wedges in which cracks may initiate from the apex.

1. INTRODUCTION

In conventional contact problems it is generally assumed that the substrate consists of an elastic half space or a layered medium (see [1] for a thorough discussion). With the application in foundation engineering in mind, in these problems the main interest has been mostly in the evaluation of the contact pressure. If the external load is not applied symmetrically, it is also possible to evaluate the "tilt angle" of the rigid stamp simulating the structure [1,2]. In practice one may also encounter a certain group of foundation problems in which because of the nonuniform stiffness of the substrate the structure may again tilt even if the loads are symmetrically distributed. Among these nonsymmetric foundation problems perhaps the simplest one is the frictionless contact problem for a plane elastic wedge-shaped substrate (Figure 1a). Here the main questions of practical interest

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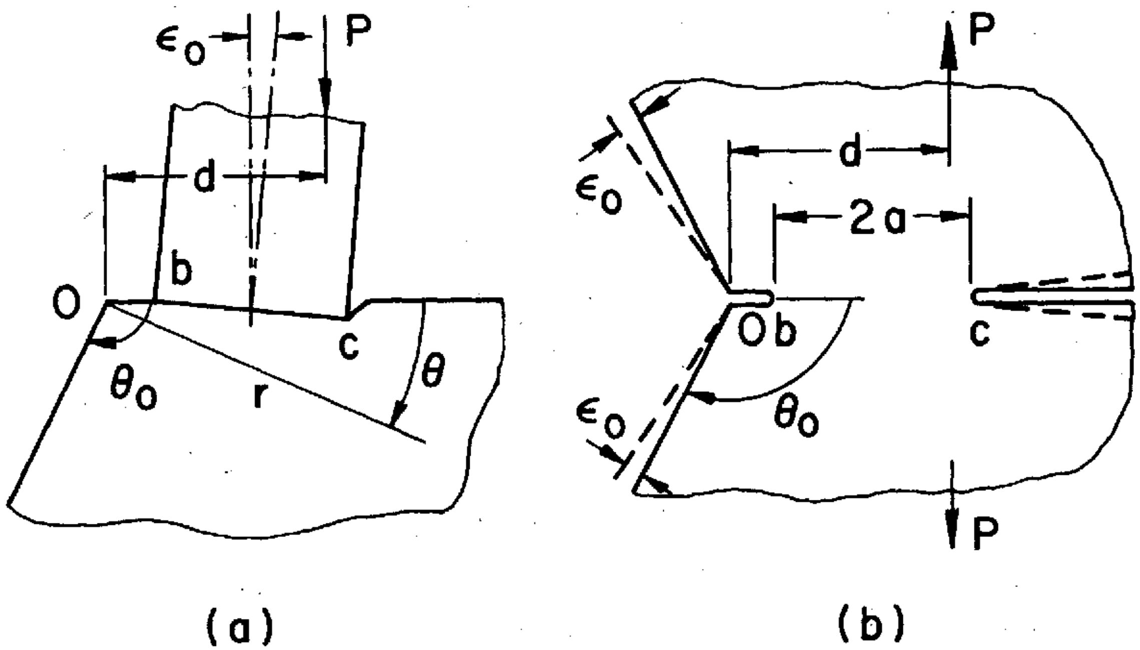

Figure 1 Geometry of stamp and crack problems.

are the contact stress distribution and the value of the possible tilt angle for a given applied load. In particular one may be interested in finding the location of the applied load for which the stamp (or the structure) would remain vertical, or the angle of rotation of the stamp if the load is symmetrically applied.

It may be noted that there is a group of crack problems for plane elastic wedges (Figure 1b) the formulation of which is identical to that of the contact problems (Figure 1a). Here the cracked wedge is loaded perpendicular to and away from the net section $\theta=0$, $b<r<c$ and with the application to fracture problems in mind the main practical interest in the problem is in finding the stress intensity factors and the net section stress. In contact as well as in crack problems the case of $b=0$ (Figure 1) is of special interest. Analytically the problem requires special attention due to the fact that for $b>0$ the formulation leads to a singular integral equation with a simple Cauchy singularity whereas for $b=0$ the kernel is of the generalized Cauchy type. In the crack problem (Figure 1b) if the stress intensity factor at b is negative the crack located on $0<r<b$, $\theta=0$ will close and has to be ignored in the solution. In the rectangular stamp problem shown in Figure 1a if the strength of the stress singularity at b is positive (i.e., if the contact stress becomes "tensile") the problem becomes one of "receding contact" [3] with the contact area being also an unknown (see [4] for a similar phenomenon in the axisymmetric double contact problems).

In this paper the frictionless contact problem for a plane elastic wedge is formulated for an arbitrary stamp profile. As examples the problems for the rectangular (Figure 1a) and the semicircular (insert in Figure 4a) stamps are considered. The solution of the crack problem (Figure 1b) is obtained by simply reinterpreting the results found for the rectangular stamp.

2. FORMULATION OF THE PROBLEM.

Consider a plane elastic wedge of arbitrary angle θ_0. Let the wedge be subjected to an external load P applied through a frictionless rigid stamp of known profile (Figures 1a). The solution of the problem may be obtained by considering either the related biharmonic equation or the Navier's equations under the following boundary conditions:

$$\sigma_{\theta\theta} = 0, \; \sigma_{r\theta} = 0, \; (\theta = \theta_0, \; 0<r<\infty), \tag{1.a,b}$$

$$\sigma_{r\theta} = 0, \; (\theta=0, \; 0<r<\infty), \tag{2}$$

$$\sigma_{\theta\theta} = 0, \; (\theta=0, \; 0<r<b \; , \; c<r<\infty), \tag{3.a}$$

$$\frac{\partial}{\partial r} u_\theta = g(r), \; (\theta=0, \; b<r<c) \tag{3.b}$$

$$\int_b^c \sigma_{\theta\theta}(r,0)dr = - P, \tag{4}$$

where g(r) is a known function and P is the resultant load per unit thickness. Let σ and v refer to the following complex stress and displacement combinations:

$$\sigma(r,\theta) = \sigma_{r\theta} + i\sigma_{\theta\theta},$$

$$v(r,\theta) = \frac{\partial}{\partial r}(u_r + iu_\theta), \; (0\leq\theta\leq\theta_0, \; 0<r<\infty). \tag{5.a,b}$$

Using the Mellin transform one may easily obtain (see, e.g.[5])

$$M[r^2\sigma] = 2i(s+1)[Ase^{is\theta} + B(s+1)e^{i(s+2)\theta} - \bar{B}e^{-i(s+2)\theta}]$$

$$M[r^2 v] = -\frac{s+1}{\mu}[Ase^{is\theta} + B(s+1)e^{i(s+2)\theta} + \kappa\bar{B}e^{-i(s+2)\theta}], \quad (6.a,b)$$

where A(s) and B(s) are unknown complex functions and the Mellin transform and its inversion are defined by

$$M[F] = \int_0^\infty F(r)r^{s-1}dr,$$

$$\int_0^\infty r^n \frac{d^nF}{dr^n} r^{s-1}dr = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} M[F],$$

$$F(r) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} M[F]r^{-s}ds. \quad (7.a-c)$$

provided the integral in (7a) is convergent and in (7c) the strip of regularity containing the constant γ is selected in such a way that (the physically imposed) regularity conditions at $r=0$ and at $r=\infty$ are satisfied. In (6) μ and κ are the elastic constants of the wedge (μ is the shear modulus, $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν being the Poisson's ratio).

To formulate the problem it is convenient to obtain first various Green's functions by replacing the mixed boundary conditions (3a) and (3b) by the following concentrated force:

$$\sigma_{\theta\theta}(r,0) = f\delta(r-t). \quad (8)$$

Substituting from (1), (2) and (8) into (6) we find

$$sA(s) = \frac{ft^{s+1}}{2(s+1)} - sb_1(s) - i(s+2)b_2(s),$$

$$B(s) = b_1(s) + ib_2(s),$$

$$b_1(s) = \frac{ft^{s+1}}{4(s+1)} \frac{(s+1)(1-\cos 2\theta_o)+1-\cos 2(s+1)\theta_o}{D(s)},$$

$$b_2(s) = \frac{ft^{s+1}}{4(s+1)} \frac{(s+1)\sin 2\theta_o+\sin 2(s+1)\theta_o}{D(s)},$$

$$D(s) = (s+1)^2(1-\cos 2\theta_o) - [1-\cos 2(s+1)\theta_o]. \tag{9.a-e}$$

If the density function $f=f(t)$ is given or determined, substituting from (9) into (6) and inverting, one may obtain the complete solution of the problem. In particular, from (5b), (6b) and (9) it follows that

$$M[r^2 \frac{\partial u_\theta}{\partial r}] = \frac{1+\kappa}{\mu} (s+1)b_2(s). \tag{10}$$

Similarly

$$M[r^2\sigma] = 2i(s+1)[fe^{is\theta} + 2i(b_1+ib_2)(s+1)e^{i(s+1)\theta}\sin\theta + 2i(b_1-ib_2)e^{-i\theta}\sin(s+1)\theta]. \tag{11}$$

Instead of (8) if we now assume that

$$\sigma_{\theta\theta}(r,0) = \begin{cases} f(r), & (b<r<c), \\ 0, & (0<r<b,\ c<r<\infty), \end{cases} \tag{12}$$

from (3b), (10), (9d), (7c) and (12) we obtain the following integral equation to determine the unknown function $f(r)$:

$$\frac{4\mu}{1+\kappa} rg(r) = \int_b^c f(t)dt \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{t}{r}\right)^{s+1} * \frac{(s+1)\sin 2\theta_o+\sin 2(s+1)\theta_o}{D(s)} ds, \quad (b<r<c). \tag{13}$$

In (13) the strip of regularity containing γ is determined from the behavior of stresses or displacement derivatives as $r\to 0$

and $r\to\infty$, namely that they vanish at $r=\infty$ and at most may have an integrable singularity at $r=0$. Let

$$D(s_k) = 0, \quad (k = \mp 1, \mp 2, \ldots),$$

$$Re(s_{-j-1})<Re(s_{-j})<\gamma<Re(s_j)<Re(s_{j+1}), \quad (j=1,2,\ldots). \tag{14}$$

Thus, one way to evaluate the kernel in (13) may be to sum the residues at s_k, $(k=1,2,\ldots)$ for $r>t$ and at s_{-k}, $(k=1,2,..)$ for $r<t$. Noting that (13) gives $\partial u_\theta/\partial r$ along the infinite line $\theta=0$, $0<r<\infty$, and the dominant terms for $\partial u_\theta/\partial r$ are of the order

$$\frac{\partial u_\theta}{\partial r} \sim r^{-(s_{-1}+2)}, (r<t), \quad \frac{\partial u_\theta}{\partial r} \sim r^{-(s_{+1}+2)}, \quad (r>t), \tag{15}$$

the regularity conditions, i.e.,

$$\frac{\partial u_\theta}{\partial r} \sim r^{-\omega}, \quad (\omega<1) \text{for } r\to 0; \quad \frac{\partial u_\theta}{\partial r} \sim r^{-\omega_0}, \quad (\omega_0 \geq 1) \text{for } r\to\infty \tag{16}$$

require that $Re(s_{-1})<-1$ and $Re(s_{+1})\geq -1$. If one also notes that -1 is a root of $D(s)$, it is then clear that the strip of regularity of the inversion integrals for stresses and the displacement derivatives will be

$$Re(s_{-1})<Re(s) = \gamma<-1 = Re(s_{+1}) \tag{17}$$

where s_{-1} is the first root of $D(s)$ to the left of the line $Re(s)=-1$.

In this problem it is not possible to obtain the roots s_k in the closed form. Hence, the resulting infinite series giving the kernel cannot be properly studied for the nature and separation of possible singular parts. To investigate the singular behavior of the kernel it is more convenient to express the inner integral in

(13) in terms of a real integral by letting $\gamma=-1$ indenting the contour to the left and defining

$$s+1 = iy,\ \log(t/r) = \rho. \tag{18}$$

The integral equation (13) may then be expressed as

$$\frac{4\mu}{1+\kappa} rg(r) = \int_b^c f(t)dt\Big[\frac{2\theta_o+\sin 2\theta_o}{2(2\theta_o^2-1+\cos 2\theta_o)} - \frac{1}{\pi}\int_o^\infty \frac{(y\sin 2\theta_o+\sinh 2\theta_o y)\sin\rho y}{\cosh 2\theta_o y-1-y^2(1-\cos 2\theta_o)}\,dy\Big],$$

$$(b<r<c). \tag{19}$$

As $\rho\to 0$ or $t\to r$ the infinite integral in (19) becomes divergent. Since the integrand is bounded and continuous everywhere in $0\le y<\infty$, the divergence will be due to the behavior of the integrand at infinity and the divergent part of the integral may easily be separated by considering the asymptotic behavior of the integrand as $y\to\infty$. Thus, adding and subtracting the asymptotic part of the integrand to and from the integrand and evaluating the related integral, (19) becomes

$$\frac{4\mu}{1+\kappa} rg(r) = -\frac{1}{\pi}\int_b^c f(t)dt\Big[\frac{1}{\rho} - \frac{\pi(2\theta_o+\sin 2\theta_o)}{2(2\theta_o^2-1+\cos 2\theta_o)} + \int_o^\infty\Big(\frac{y\sin 2\theta_o+\sinh 2\theta_o y}{\cosh 2\theta_o y-1-y^2(1-\cos 2\theta_o)} - 1\Big)\sin\rho y\,dy\Big],$$

$$(b<r<c). \tag{20}$$

It may easily be shown that for $b\to 0$ (20) is an ordinary singular integral equation with a simple Cauchy singularity. This may be seen by observing that

$$\frac{1}{r\log(t/r)} = \frac{1}{r(\frac{t}{r}-1)}[1+\sum_{1}^{\infty}\frac{(-1)^n}{n+1}(\frac{t}{r}-1)^n]^{-1}$$

$$= \frac{1}{t-r}[1+0(\frac{t}{r}-1)]. \qquad (21)$$

In the special case of $\theta_o=\pi$, using the relation

$$\int_o^\infty(\cosh ay-1)\sin\rho y dy = \sum_{n=0}^{\infty} 2e^{-2a(n+1)y}\sin\rho y dy$$

$$= \sum_{m=1}^{\infty}\frac{2\rho}{\rho^2+(2ma)^2} = -\frac{1}{\rho}+\frac{\pi}{2a}\coth\frac{\rho\pi}{2a} \qquad (22)$$

the inner integral in (20) becomes

$$\int_o^\infty(\frac{\sinh 2\pi y}{\cosh 2\pi y-1}-1)\sin\rho y dy = \frac{1}{2}\coth\frac{\rho}{2}-\frac{1}{\rho}. \qquad (23)$$

Noting that $\rho=\log(t/r)$ and substituting from (23), it may easily be seen that (20) reduces to the integral equation for the elastic half plane given by

$$\frac{4\mu}{1+\kappa}g(r) = -\frac{1}{\pi}\int_b^c\frac{f(t)}{t-r}dt, \quad (b<r<c), \qquad (24)$$

which has a closed form solution for any Hölder-continuous g(r) [6].

In another special case where $\theta_o=2\pi$, (i.e., the plane with a semi-infinite crack loaded on one flank) again using (22) it may be shown that (20) reduces to

$$\frac{4\mu}{1+\kappa}\sqrt{r}\,g(r) = -\frac{1}{2\pi}\int_b^c\frac{f(t)dt}{\sqrt{t}-\sqrt{r}}, \quad (b<r<c). \qquad (25)$$

If we let

$$t = p^2,\ r = s^2,\ \sqrt{r}g(r) = G(s),\ \sqrt{t}f(t) = F(p)$$

$$b = d^2,\ c = e^2 \qquad (26)$$

(25) becomes

$$\frac{4\mu}{1+\kappa} G(s) = -\frac{1}{\pi} \int_d^e \frac{F(p)}{p-s} dp \tag{27}$$

which can again be solved in closed form for a given G(s) [6]. For example, if the stamp has a rectangular profile, $g(r)=0=G(s)$, and the solution (27) satisfying

$$\int_b^c f(r)dr = -P \tag{28}$$

becomes

$$f(r) = -\frac{P}{2\pi[r(\sqrt{r}-\sqrt{b})(\sqrt{c}-\sqrt{r})]^{\frac{1}{2}}}, \quad (b<r<c) \tag{29}$$

3. STRESSES AROUND THE APEX OF THE WEDGE.

As indicated before, after obtaining f(t) by solving (20), the stresses and the displacement derivatives in the wedge may be evaluated by means of definite integrals having f(t), $(b<t<c)$ as the density function. From the view point of fracture of the solid of particular interest is the cleavage stress $\sigma_{\theta\theta}(r,\theta)$ which is known to be singular at the wedge apex $r=0$ for wedge angles $\theta_o>\pi$. The inversion of (11) would indicate that for small values of r the dominant term for the stresses is of the order

$$\sigma_{ij} \sim r^{-(s_{-1}+2)}, \quad (i,j=r,\theta),$$

$$D(s_{-1}) = 0, \mathrm{Re}\,(s_{-1})<-1 \tag{30.a,b}$$

where s_{-1} is the first root to the left of $\mathrm{Re}(s)=-1$. By examining (30b) with D(s) as given by (9e) it may easily be shown that s_{-1} is real and

$$s_{-1} = -2 \text{ for } 0<\theta_o\leq\pi,$$

$$-2\leq s_{-1}\leq -3/2 \text{ for } \pi\leq\theta_o\leq 2\pi. \qquad (31)$$

Thus, for $\theta_o\leq\pi$ the stresses at r=0 are bounded and are of no particular interest. For $\theta_o>\pi$ they are singular at r=0, (30) gives the power of singularity, and the residue of the related inversion integral at s_{-1} gives the asymptotic values. After routine manipulations from (11), (12), and (9) we find

$$\lim_{r\to 0} r^{2+s_{-1}}(\sigma_{r\theta}+i\sigma_{\theta\theta}) = -\frac{F(s_{-1})}{D'(s_{-1})}\int_b^c t^p f(t)dt, \qquad (32)$$

$$p = s_{-1}+1,$$

$$D'(s_{-1}) = 2[p(1-\cos 2\theta_o) - \theta_o \sin 2p\theta_o],$$

$$F(s_{-1}) = pe^{-i\theta p}(c_1+ic_2)\sin\theta + (c_1-ic_2)e^{i\theta}\sin p\theta,$$

$$c_1 = p(1-\cos 2\theta_o) + (1-\cos 2p\theta_o),$$

$$c_2 = p\sin 2\theta_o + \sin 2p\theta_o. \qquad (33)$$

If we define

$$2 + s_{-1} = \omega,\; F(s_{-1}) = F_1(\theta) + iF_2(\theta), \qquad (34)$$

the "stress intensity factor" for the cleavage stress at r=0 may be expressed as

$$k(\theta) = \lim_{r\to 0} r^{\omega}\sigma_{\theta\theta}(r,\theta) = -\frac{F_2(\theta)}{D'(s_{-1})}\int_b^c t^{\omega-1}f(t)dt \qquad (35)$$

Thus, for a given wedge angle θ_o the integral in (35) will be the measure of the intensity of stresses at the apex of the wedge.

4. THE CASE OF b=0.

In the problem of flat-based stamp shown in Figure 1a and in the crack problem shown in Figure 1b, the limiting case of $b=0$ is of special interest. In this case, as the crack problem would indicate, the singularity at the end point $r=b=0$ is that of a symmetrically loaded wedge of angle $2\theta_o$. In fact, it may also be shown that in the integral equation (20) part of the kernel corresponding to the inner integral is not bounded for all values of r and t in the closed interval [0,c], and becomes unbounded when r and t approach the end point $r=0$ simultaneously. This means that for $b=0$ the kernel of the singular integral equation (20) is of generalized Cauchy type and the related fundamental function is of the following form:

$$w(t) = t^{\beta}(c-t)^{\alpha}, \quad (-1<\mathrm{Re}(\alpha,\beta)\leq 0). \tag{36}$$

The characteristic equations giving the powers α and β are found to be

$$\cot\pi\alpha = 0,$$

$$(1+\beta)\sin 2\theta_o + \sin 2(1+\beta)\theta_o = 0. \tag{37.a,b}$$

From (37) it is seen that α and β are real, $\alpha=-0.5$, $\beta=0$ for $0<\theta_o\leq\pi/2$, and $-1<\beta<0$ for $2\pi>\theta_o>\pi/2$. For example, $\beta=-0.5$ for $\theta_o=\pi$, $\beta=-2/3$ for $\theta_o=3\pi/2$, and $\beta\to-1$ for $\theta_o\to 2\pi$. In the crack problem (i.e., $\theta_o=2\pi$) $\beta=-1$ and it appears that the stresses have a nonintegrable singularity. Since the load is applied at the crack tip (i.e., $b=0$) this result is expected.

5. SOLUTION AND NUMERICAL RESULTS.

The integral equation (20) subject to the condition (4) can easily be solved numerically once the stamp profile $g(r)$ is specified (for the numerical technique see, for example, [7,8]). Most of the results given in this section refer to flat stamp and crack problems shown in Figures 1a and 1b, which are considered to be of greater practical interest. However, some results on the half-cylinder stamp problems shown in the insert of Figures 4a will also be given.

In the crack and flat stamp problems, the problem may be posed in one of two ways. In the first one may specify the external load P and the distance of its line of application d. In this case, generally there would be a small rotation ε_0 of the stamp (or the half wedge away from the net section bc). This quantity may be computed by using the following moment equilibrium condition:

$$\int_b^c f(r)r\,dr = \pm Pd \qquad (38)$$

where P is the magnitude of the applied load and noting that $f(r)=\sigma_{\theta\theta}(r,0)$, + sign refers to the crack and - refers to the stamp problem.

In the second group of problems one may specify the load P and the angle of rotation ε_0. In this case d is unknown and is again determined from (38). The practical problem here of course is the determination of d for a given load and no rotation (i.e., ε_0=0). In both of these problems the input function in (20) is (see (3b))

$$g(r) = \frac{\partial}{\partial r} u_\theta(r,0) = \varepsilon_o, \quad (b<r<c). \tag{39}$$

In the flat stamp and crack problems for b>0 the fundamental function of the integral equation and the solution may be expressed as

$$w(t) = [(c-t)(t-b)]^{-\frac{1}{2}}, \quad f(t) = F(t)w(t), \quad (b<t<c) \tag{40.a,b}$$

where F(t) is bounded in $b \le t \le c$. In this case, in addition to the contact pressure -f(r) in stamp and the net section stress f(r) in crack problems, the quantities of physical interest are d or ε_o, the stress intensity factors defined by

$$k(b) = \lim_{r \to b} \sqrt{2(r-b)}\, f(r), \quad k(c) = \lim_{r \to c} \sqrt{2(c-r)}\, f(r), \tag{41}$$

and the power ω and the strength $k(\theta)$ of the stress singularity at the wedge apex for $\theta_o > \pi$ in stamp problems.

Recalling the definition of stress intensity factor at r=0 for $\theta_o > \pi$, (see (32-35)), we define a normalized stress intensity factor k_o by

$$\begin{aligned} k(\theta) &= \lim_{r \to 0} r^{\omega}(\sigma_{r\theta} + i\sigma_{\theta\theta}) \\ &= -\frac{F_1(\theta)+iF_2(\theta)}{D'(s_{-1})} \int_b^c t^{\omega-1} f(t)\,dt \\ &= -\frac{F_1(\theta)+iF_2(\theta)}{\pi D'(s_{-1})}\, P\left(\frac{c-b}{2}\right)^{\omega-1} k_o. \end{aligned} \tag{42}$$

In the case of b=0 the fundamental function of the integral equation (20) is given by (36) where α and β are determined from (37). In this problem too the solution may be expressed by (40b) with F(t) being again a bounded function in $0\leq r<c$. In this case the stress intensity factor at r=0 is defined by

$$\bar{k}(b) = k(0) = \lim_{r\to 0} \sqrt{2}r^{-\beta}f(r). \quad (43)$$

Some calculated results for $b\geq 0$ and $\varepsilon_o=0$ for geometries shown in Figures 1a and 1b are given in Table 1. In this table the stress intensity factors k(c) and k(b) defined by (41) and (43) are positive in the crack problem and are negative in the stamp problem. The normalizing factor for k(c) and k(b) are defined by

$$k_n = \frac{P}{\pi\sqrt{a}}\ , \text{ for } b>0,$$

$$k_n = \frac{P}{\pi a^{1+\beta}}\ , \text{ for } b=0,\ a=(c-b)/2. \quad (44.a,b)$$

The table also shows the powers of singularity ω and β (see (43), (40b) and (36)). The values of ω refer only to b>0 case where $\omega=0$ for $0<\theta_o\leq\pi$, $\omega=0.5$ for $\theta_o=2\pi$, and $0\leq\omega\leq 0.5$ for $\pi\leq\theta_o\leq 2\pi$. The values of β refer only to b=0 case. Note that as required by the physics of the problem $\beta(\theta_o)=-\omega(2\theta_o)$. Thus, the stress intensity measure k_o is nonzero only for $\omega>0$ (i.e., for $\theta_o>\pi, b>0$), and k(b) is nonzero only for $\beta<0$(i.e., for $\theta_o>\pi/2$, b=0).

For $\varepsilon_o=0$ the values given for the distance d indicates that, as expected, $d\to(c+b)/2$ as $(c+b)\to\infty$ for a constant $a=(c-b)/2$, $d>(c+b)/2$ for $\theta_o<\pi$, $d=(c+b)/2$ for $\theta_o=\pi$ and $d<(c+b)/2$ for $\theta_o>\pi$, where d is the distance at which the resultant load P should be applied for zero rotation of the structure (Figure 1a) or the half-wedge (Figure 1b). Another physically expected important phenomenon may be observed from the results given for $\theta_o=60^o$. It is

Table 1. The results for flat stamp and crack problems, for $b \geq 0$, $\varepsilon_o = 0$.

θ_o, ω, β	b/a	c/a	d/a	$k(c)/k_n$	$k(b)/k_n$	$-k_o$
	10	12	11.277	1.5486	0.4414	
$\theta_o=60°$	8	10	9.335	1.6632	0.3225	
$\omega=0$	6	8	7.424	1.8356	0.1428	
	5	7	6.486	1.9579	0.0151	
	4	6	5.569	2.1183	-.1518	
	4	6	5.117	1.2265	0.7583	
$\theta_o=90°$	2	4	3.192	1.3634	0.5958	
$\omega=0$	1	3	2.279	1.5151	0.4003	
	0.5	2.5	1.858	1.6433	0.2238	
$\beta=0$	0	2	1.476	1.8436	$\to 0$	
	2	4	3.037	1.0684	0.9206	
$\theta_o=120°$	1	3	2.056	1.1004	0.8731	
$\omega=0$	0.5	2.5	1.577	1.1319	0.8154	
	0.2	2.2	1.301	1.1636	0.7350	
$\beta=-0.38427$	0	2	1.136	1.2005	0.8993	
$\theta_o=150°$	0.5	2.5	1.507	1.0121	0.9817	
$\omega=0$	0.2	2.2	1.210	1.0153	0.9720	
$\beta=-0.48778$	0	2	1.014	1.0198	0.9958	
$\theta_o=180°$	0	2	1.0	1.0	1.0	
$\omega=0, \beta=-0.5$						
$\theta_o=210°$	0.5	2.5	1.496	0.9930	1.0118	2.821
$\omega=0.28572$	0.2	2.2	1.194	0.9911	1.0190	4.774
$\beta=-0.61749$	0	2	0.988	0.9852	0.4892	
$\theta_o=240°$	1	3	1.982	0.9696	1.0446	2.215
$\omega=0.38427$	0.5	2.5	1.475	0.9595	1.0705	2.862
	0.2	2.2	1.165	0.9483	1.1166	3.835
$\beta=-0.66061$	0	2	0.939	0.9255	0.4940	
	2	4	2.974	0.9540	1.0592	1.779
$\theta_o=270°$	1	3	1.960	0.9323	1.1007	2.315
$\omega=0.45552$	0.5	2.5	1.443	0.9099	1.1607	2.917
	0.2	2.2	1.120	0.8842	1.2716	3.815
$\beta=-2/3$	0	2	0.860	0/8313	0.7361	

seen that at a value $b=Ka$, $4<K<5$ the sign of the stress intensity factor $k(b)$ changes (meaning that in the stamp problem the wedge "bends" and the contact stress around $r=b$ becomes "tension" and in the crack problem it becomes compression). The technique described in this paper is also applicable for this case. The solution of the crack problem is quite straightforward and may be obtained by using the formulation given in this paper provided one lets $b=0$ (i.e., the crack $0<r<b$ is now closed) and $\beta=0$ at $r=0$, with the support of the integral equation (20) being $(0,c)$. The stamp problem on the other hand, is somewhat different and has to be treated as a "receding contact problem" with the contact area $c>r>b_o$ as an additional unknown determined from the condition that the contact stress at $r=b_o$, $(b_o>b)$ vanishes (see the last example given in this paper).

Some sample results for the contact stress (or the net section stress) $f(r)$ are given in Figures 2 and 3. In both cases the angle of rotation ε_o is zero and the corresponding distance d of the line of load application may be found in Table 1. Figure 2 shows the contact stress for $\theta_o=90^\circ$, $b\geq 0$, and Figure 3 show the same for $b=0$ and for various values of the wedge angle θ_o.

The results for a symmetrically loaded stamp (i.e., $d=(c+b)/2$) in the more interesting case of $b=0$ are shown in Table 2. In this problem one of the primary unknowns is the rotation ε_o which is given in the third column. Here the normalizing angle ε_n is defined by

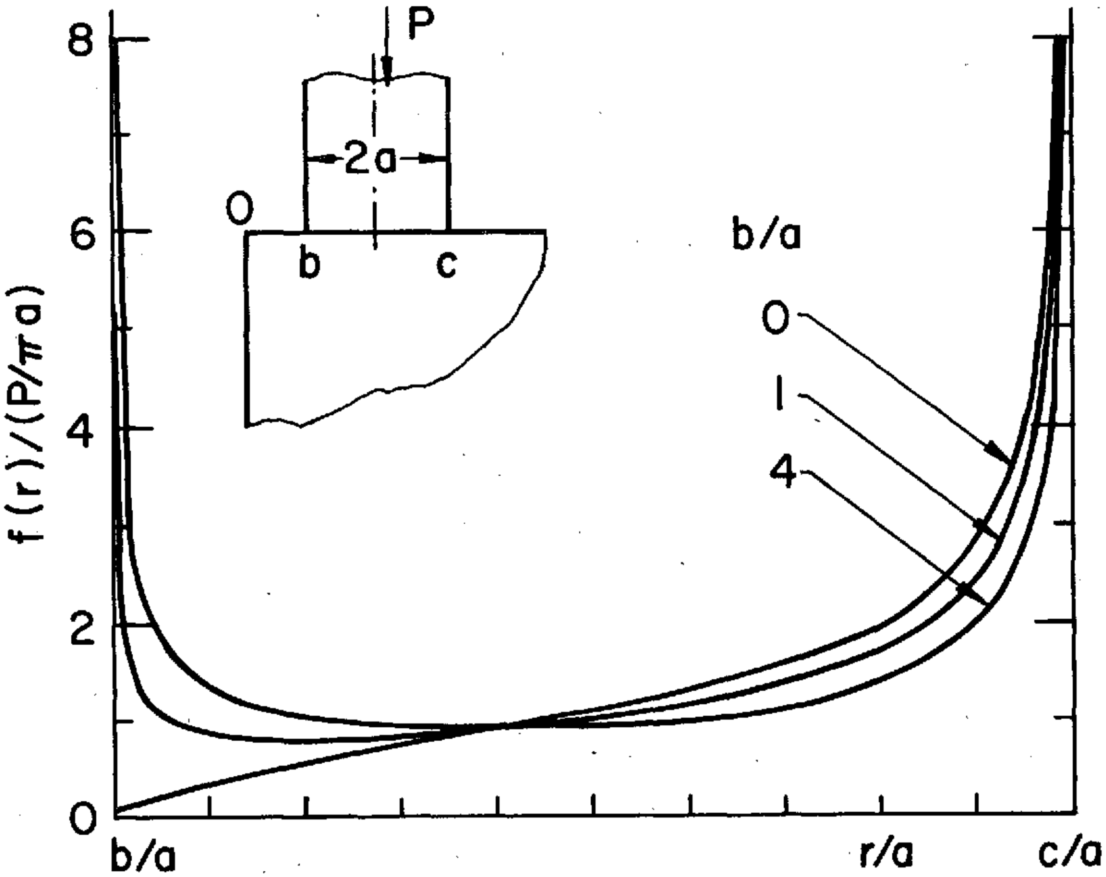

Figure 2 Contact (or net section) stress for a 90-degree wedge (b≥0).

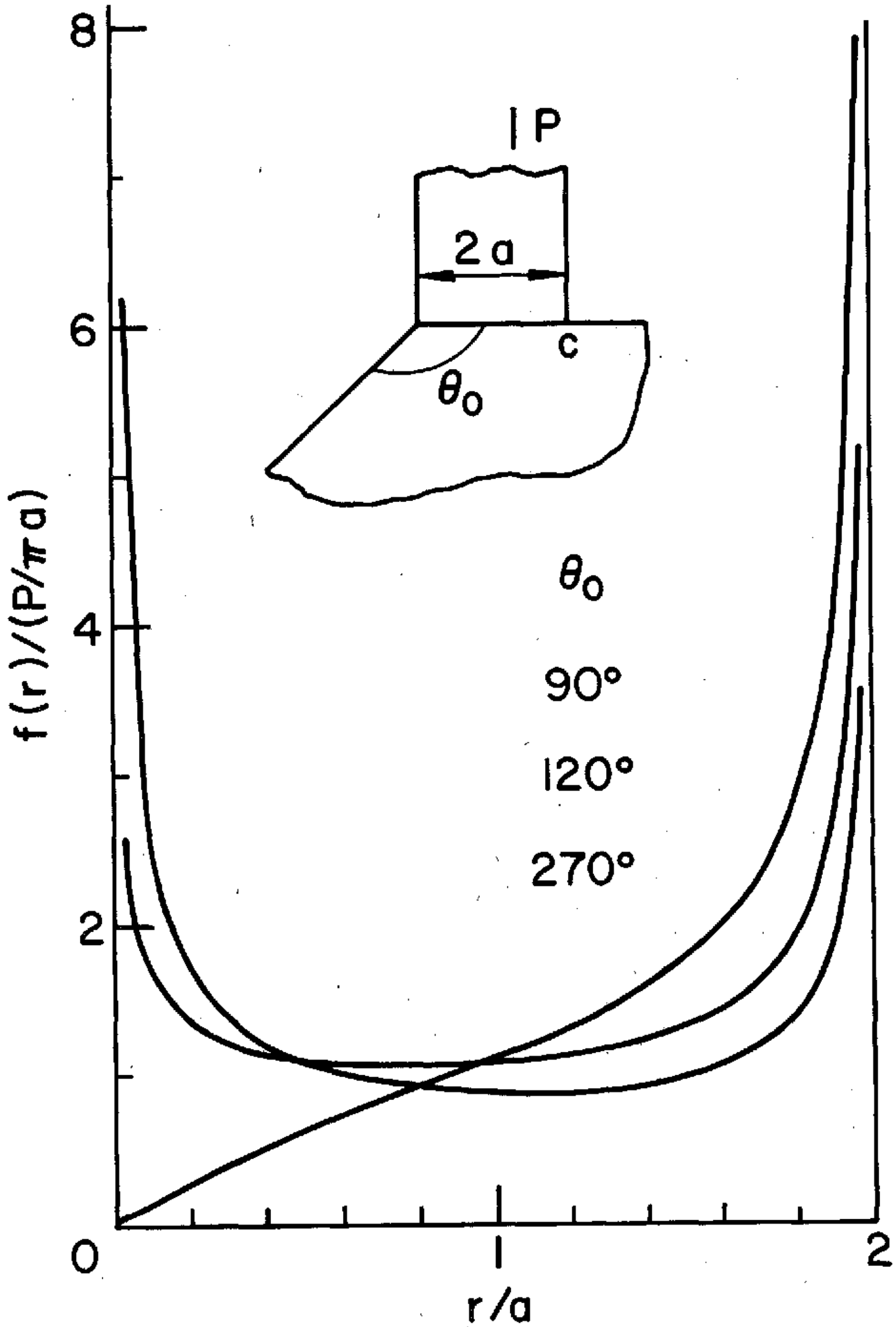

Figure 3 Contact (or net section) stress for an elastic wedge (b=0).

Table 2. The results for the symmetrically loaded stamp or crack problem, b=0, d=c/2=a.

θ_o	$-\beta$	$\varepsilon_o/\varepsilon_n$	$\frac{k(c)}{kn}$	$(a)^{0.5+\beta}\frac{k(0)}{kn}$
90^o	0	-0.4727	0.6486	
120^o	0.38427	-0.0967	0.9132	1.284
240^o	0.66061	0.0373	1.0445	0.449
270^o	0.66667	0.0829	1.1010	0.613

$$\varepsilon_n = \frac{(1+\kappa)P}{4\mu a} . \tag{45}$$

The normalization factor k_n for the stress intensity factors k(0) and k(c) shown in the table is given by (44a) i.e., $k_n=P/\pi\sqrt{a}$. The results given in the table corresponds to the stamp problem where P is the magnitude of the applied (compressive) load, and $\varepsilon_o>0$ means that the stamp rotates in the positive θ direction shown in Figure 1a. Only the results given for $\theta_o<\pi$ are valid for the crack problems. It should be noted that here even though one is dealing with the contact problem for a homogeneous medium, for b=0 and $\theta_o>\pi$ one obtains a stress singularity with a power greater than 0.5 which is the strongest attainable power in notch and crack problems.

The results for a semi-circular rigid stamp are given in Tables 3 and Figure 4. The normalizing stress intensity factor k_n which appears in the table is defined by

$$k_n = \sqrt{2(c-b)}\ 4\mu/(1+\kappa). \tag{46}$$

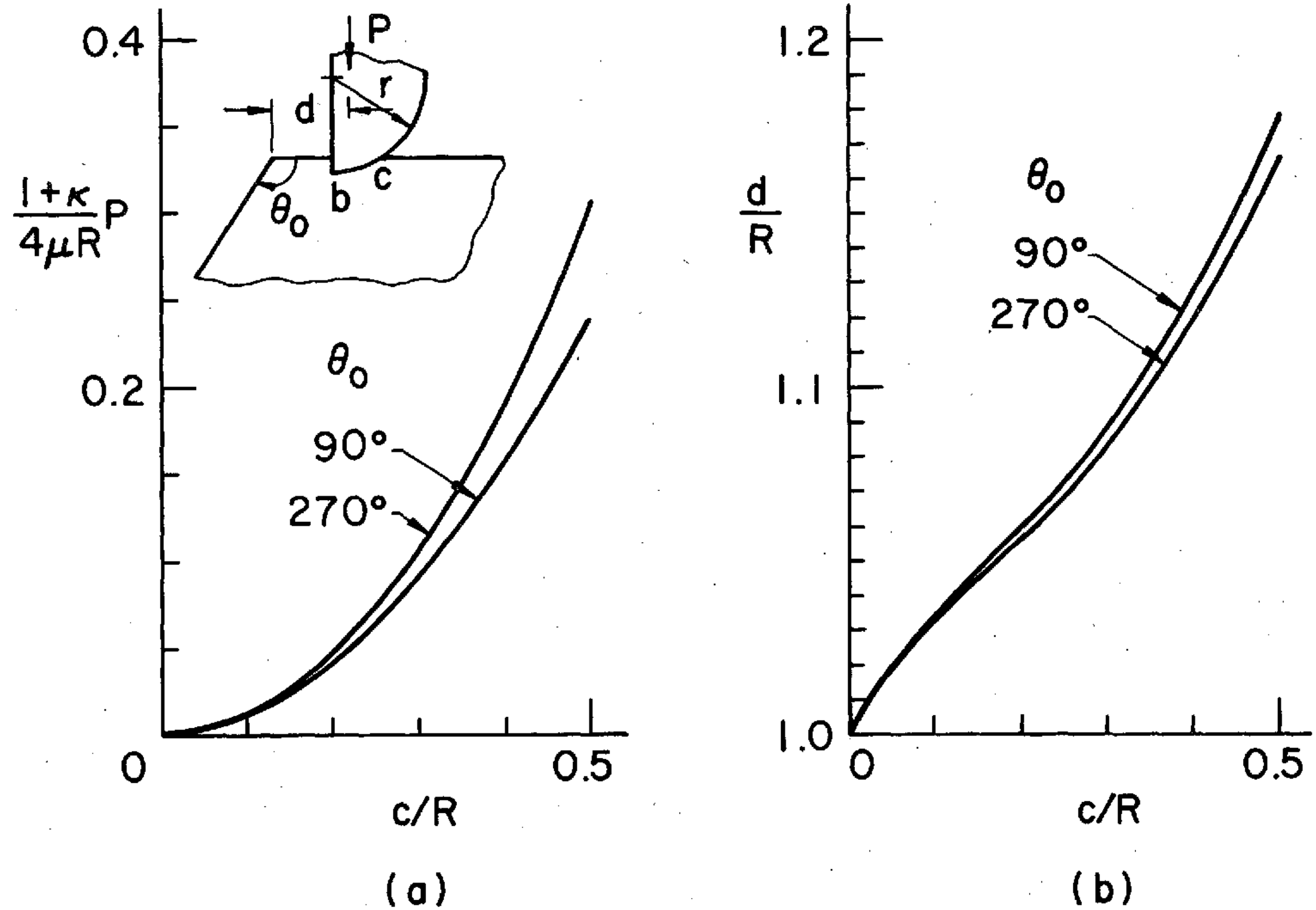

Figure 4 The variation of the resultant load P and its distance d with the contact area for a semi-circular stamp.

The constant k_o which is the measure of the strength of stress singularity at the wedge apex for $\theta_o > \pi$ is again defined by (42). In the example it is assumed that b=R=constant and the unknowns c and d are determined from the force and moment equilibrium conditions (4) and (38). For the special case of $\theta_o = \pi$ the solution is given by

Table 3. The results for the semicircular rigid stamp, b=R.

θ_o	b/R	c/R	d/R	$\frac{1+\kappa}{4\mu}\frac{P}{R}$	$k(b)/k_n$	k_o
$\theta_o = 90°$	1	1.1	1.0338	0.01115	0.04596	
	1	1.25	1.0862	0.06505	0.1027	
	1	1.50	1.1777	0.2379	0.1752	
$\theta_o = 270$	1	1.1	1.0333	0.01187	0.05055	0.01166
	1	1.25	1.0830	0.07485	0.1282	0.07179
	1	1.5	1.1652	0.3033	0.2620	0.2806
$\theta_o = 180$	1	1.1	1.0333	0.01178	0.05	

$$p(r) = \frac{2\mu}{1+\kappa}\,\frac{c-b}{R}\left(1+\frac{r-b}{c-b}\right)\left(\frac{c-r}{r-b}\right)^{\frac{1}{2}} \tag{47}$$

The resultant load P and its distance from the apex d are shown in Figures 4a and 4b, respectively. The results do not appear to be significantly different from those obtained for the half plane. Hence no extensive numerical work was carried out regarding this example.

The receding contact problem for small wedge angles is considered as a last example. The results for a 60-degree wedge acted upon by a flat-ended rigid stamp are given in Table 4. The problem has some anomalous features. In this case the index of the singular integral equation is zero and the solution is of the following form

$$f(r) = F(r)[(r-b)/(c-r)]^{\frac{1}{2}}, \quad (b<r<c) \tag{48}$$

where F(r) is again bounded in b<r<c. In the numerical example, in order to avoid the iteration to determine b the problem is solved for a constant stamp rotation, i.e.,

Table 4. The results for a flat stamp on a 60-degree wedge, $f(r) = F(r)[(r-b)/(c-r)]^{\frac{1}{2}}$.

b/c	a/c	d/c	$10^3\varepsilon_o/\varepsilon_n$	$\frac{k(c)}{P/\sqrt{c}}$	P/P_n
0.7	0.15000	0.92401	20.43	1.619	48.956
0.7025	0.14875	0.92466	15.45	1.627	64.708
0.7050	0.14750	0.92530	10.41	1.634	96.088
0.7075	0.14625	0.92595	5.283	1.641	189.27
0.7100	0.14500	0.92660	0.08104	1.649	12339.8
0.7104	0.14480	0.92670	$\to 0$	1.650	$\to\infty$

$g(r)=\varepsilon_o$=constant, and for each given b the load P is determined. The results are shown in Table 4. The normalization constants which appear in the table are defined by

$$\varepsilon_n = \frac{1+\kappa}{\mu c} P, \quad P_n = \frac{\mu c}{1+\kappa} \varepsilon_o. \tag{49}$$

A close examination of the results given in the table would indicate that from the unloaded state P=0, ε_o=0, b/c=0 (c=constant) as the load P is increased for a constant small rotation ε_o b/c increases. It appears that there is a limit b/c$\cong$0.7104 beyond which the contact area cannot be reduced any further. At this value of b,$\varepsilon_o \to 0$ for any fixed load P. From (20) and (28) with $g(r)=\varepsilon_o=0$

it is seen that the unknown function is f(r)/P and $b \cong 0.7104c$ is really independent of the magnitude of the applied load P. This conclusion is typical of the general receding contact problems. From other trial solutions it was also found that it is not possible to obtain any solution in the form of (48) (i.e., with a sharp stamp corner at c and smooth contact at b) for $\varepsilon_0 < 0$. This, of course, is the result one would expect physically.

REFERENCES

1. L.A. Galin, "Contact Problems in the Theory of Elasticity", North Carolina State College, Raleigh, N.C. (1961).

2. L.M. Keer, "Nonsymmetric punch and crack problems", Quart. of Appl. Math., Vol. 23, p. 97, (1965).

3. J. Dundurs and M. Stippes, "Role of elastic constants in certain contact problems", J. Appl. Mech., Vol. 37, Trans. ASME, p. 965, (1970).

4. M.B. Civelek and F. Erdogan, "The axisymmetric double contact problem for a frictionless elastic layer", Int. J. Solids, Structures, Vol. 10, P. 639, (1974).

5. V.L. Hein and F. Erdogan, "Stress singularities in a two-material wedge", Int. J. Fracture Mechanics, Vol. 7, p. 317, (1971).

6. N.I. Muskhelishvili, "Singular Integral Equations" P. Noordhoff, Groningen, Holland, (1953).

7. F. Erdogan, G.D. Gupta, and T.S. Cook, "Numerical solution of singular integral equations" in Methods of Analysis and Solutions of Crack Problems, G.C. Sih, ed. Noordhoff International Publishing, Leyden, Netherlands, (1973).

8. F. Erdogan and G.D. Gupta, "On the numerical solution of singular integral equations", Quart. Appl. Math., Vol. 30, p. 525, (1972).