ON THE TESSERAL-HARMONICS RESONANCE PROBLEM IN ARTIFICIAL-SATELLITE THEORY

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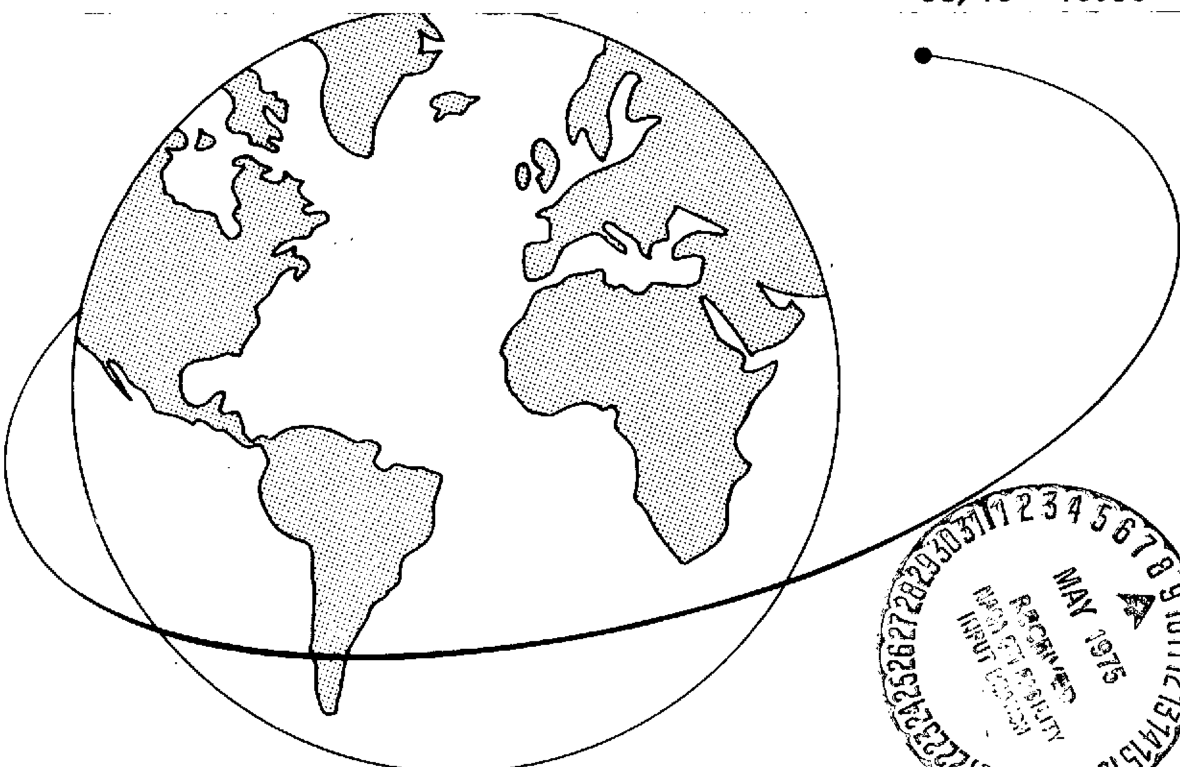

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ABSTRACT

The longitude-dependent part of the geopotential usually gives rise only to short-period effects in the motion of an artificial satellite. However, when the motion of the satellite is commensurable with that of the earth, the path of the satellite repeats itself relative to the earth and perturbations build up at each passage of the satellite in the same spot, so that there can be important long-period effects.

In order to take these effects into account in deriving a theoretical solution to the equations of motion of an artificial satellite, it is necessary to select terms in the longitude-dependent part of the geopotential that will contribute significantly to the perturbations. We have tried to make a selection that is valid in a general case, regardless of the initial eccentricity of the orbit and of the order of the resonance.

The solution to the equations of motion of an artificial satellite, in a geopotential thus determined, is then derived by using Hori's method by Lie series, which, by its properties regarding canonical invariance, has proved advantageous in the classical theory.

RESUME

La partie du géopotentiel, dépendant de la longitude, ne provoque généralement que des effets à courte période dans le mouvement d'un satellite artificiel. Cependant, quand le mouvement du satellite est commensurable avec celui de la terre, le parcours du satellite se répète par rapport à la terre et les perturbations s'ajoutent à chaque passage du satellite au même endroit, et peuvent ainsi créer d'importants effets à longue période.

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Pour tenir compte de ces effets en dérivant une solution théorique des équations du mouvement d'un satellite artificiel, il est nécessaire de choisir ceux des termes de la partie du géopotentiel dépendant de la longitude, qui contribueront d'une façon significative aux perturbations. On a essayé de faire un choix qui soit valable pour le cas général, quelques soient la première excentricité de l'orbite et l'ordre et l'ordre de la résonance.

On a ensuite dérivé la solution des équations du mouvement d'un satellite artificiel dans un géopotentiel ainsi déterminé, en employant la méthode de Hori par la série de Lie, qui a été prouvée avantageuse dans la théorie classique, grâce à ses propriétés concernant l'invariance canonique.

КОНСПЕКТ

Зависящая от долготы составляющая геопотенциала обычно вызывает лишь короткопериодные изменения в движении искусственного спутника земли. Однако, если движение искусственного спутника соизмеримо с движением земли, то путь искусственного спутника относительно земли и пертурбаций, построенных на каждом отрезке пути искусственного спутника в той же точке, повторяется, так что воздействия уже могут быть долгопериодными.

Для учета этих воздействий при получении теоротического решения уравнений движения искусственного спутника необходимо выбрать те члены зависящей от долготы составляющей геопотенциала, которые являются важной частью пертурбаций. Мы пытались осуществить выбор таким образом, чтобы он годился для общего случая, в независимости от начального эксцентриситета и порядка резонанса.

Затем в геопотенциале, определенном таким способом, выводится решение уравнений движения искусственного спутника по методу Хори с помощью серии Ли. Этот метод оказался полезным в классической теории благодаря своей установившейся неизменности.

ON THE TESSERAL-HARMONICS RESONANCE PROBLEM IN ARTIFICIAL-SATELLITE THEORY

Barbara A. Romanowicz

1. INTRODUCTION

If the gravitational potential of the earth is expanded in terms of Legendre polynomials and functions, then in order to obtain a good approximation in the determination of the orbit of an artificial satellite, it is usually sufficient to consider the zonal, longitude-free terms of the expansion. However, the influence of the tesseral terms becomes important in the case when the mean motion of the satellite and the rate of rotation of the earth around its axis are in a simple ratio.

This problem has been examined from different points of view. Morando (1963) considered the particular, important case of a 24-hour satellite; Allan (1967, 1973) worked out perturbations due to resonance in a more general case and to the first order, by using Lagrange's equations. On the other hand, Garfinkel (1974) considered the abstract, mathematical problem of "ideal resonance." We have attempted to derive expressions for the perturbations in the motion of an artificial satellite due to the most general tesseral-harmonics resonance. The calculations are carried out to order 3/2 in the small parameter of the expansion of the perturbing function, and the perturbation method used is that of Hori, which is canonically invariant and avoids mixing of old and new variables, as opposed to the generally used method of Von Zeipel.

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2. CHOICE OF THE DISTURBING FUNCTION AND EQUATIONS OF MOTION

The gravitational potential of the earth at exterior points can be expressed as follows:

$$U(r, \theta, \lambda) = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta) + \sum_{n=2}^{\infty} \sum_{m=1}^{n} J_{n,m} \left(\frac{R}{r}\right)^n P_{n,m}(\cos\theta) \cos m(\lambda - \lambda_{n,m})\right], \quad (1)$$

where (r, θ, λ) are spherical polar coordinates relative to the center of mass of the earth, the axis of rotation being the pole of coordinates; μ is the gravitational constant G times the mass of the earth; R is the mean equatorial radius of the earth; $P_n(z)$ is the n^{th} Legendre polynomial; $P_{n,m}(z)$ are associated Legendre functions:

$$P_{n,m}(z) = (1 - z^2)^{m/2} \frac{d^n P_n(z)}{dz^n} \quad ;$$

and J_n and $J_{n,m}$ are dimensionless coefficients related to the normalized coefficients $\overline{C_{n,m}}$, $\overline{S_{n,m}}$ by:

$$J_{n,m} = \overline{J_{n,m}} \sqrt{2(2n+1)\,(n-m)!/(n+m)!} \quad , \qquad m \neq 0 \quad ,$$

$$-J_n = C_{n,0} = \sqrt{2n+1}\,\overline{C_{n,0}} \quad , \qquad (2)$$

$$\overline{J_{n,m}} \cos m(\lambda - \lambda_{n,m}) = \overline{C_{n,m}} \cos m\lambda + \overline{S_{n,m}} \sin m\lambda \quad .$$

It can be assumed that the largest perturbing forces are due to the leading zonal harmonic, which contains J_2 and corresponds to the oblateness of the earth. The remaining zonal harmonics have much smaller effects and are not considered here. The tesseral harmonics, containing $J_{n,m}$, are the longitude-dependent terms and usually give rise only to short-period effects because they all contain the mean anomaly

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and the sidereal angle in their arguments. In the case of resonance, however, a certain number of these terms can produce effects of large amplitude and very long period. It is therefore necessary to select appropriately those tesseral harmonics to be included in our disturbing function.

The general tesseral harmonic can be developed in terms of the osculating elliptic elements $(a, e, I, \Omega, \omega, M)$, referred to the equator of the earth, in the form (Kaula, 1966):

$$V_{n,m} = \left(\frac{\mu}{R}\right)\left(\frac{R}{a}\right)^{n+1} J_{n,m} \left\{ \sum_{p=0}^{p=n} F_{n,m,p}(I) \sum_q G_{n,p,q}(e) \cos\left[(n-2p)\omega + (n-2p+q)M + m(\Omega - \nu t - \lambda_{n,m})\right] \right\} , \tag{3}$$

where ν is the angular velocity of the rotation of the earth, t is the time, and $F_{n,m,p}(I)$ and $G_{n,p,q}(e)$ are, respectively, the inclination and eccentricity functions as defined by Kaula (1966). The functions $G_{n,p,q}(e)$ are of order $|q|$ in eccentricity.

Resonance occurs when a pair (α, β) of mutually prime integers exists such that the satellite performs β nodal periods while the earth rotates α times relative to the precessing satellite's orbit plane. This can be expressed by

$$\alpha(\dot{\omega} + \dot{M}) = \beta(\nu - \dot{\Omega}) \quad , \tag{4}$$

where $\dot{\omega}$, $\dot{M}$, and $\dot{\Omega}$ are the rates of change with time of ω, M, and Ω, respectively. The corresponding slowly varying arguments are of the form

$$k_1 \Phi_{\alpha,\beta} + \text{const} \quad ,$$

where

$$\Phi_{\alpha,\beta} = \alpha(\omega + M) + \beta(\Omega - \nu t)$$

is called the resonant variable and $k_1 = 1, 2, 3, \ldots$.

Considering that $\dot{\omega}$ is a small quantity and that the general argument in a tesseral harmonic is

$$\phi = (n-2p)\omega + (n-2p+q)M + m(\Omega - \nu t - \lambda_{n,m}) \quad ,$$

we shall select the tesseral harmonics by keeping those containing arguments such that

$$\begin{aligned} & n - 2p + q = k_1 \alpha \quad , \\ & m = k_1 \beta \quad , \end{aligned} \qquad k_1 = 1, 2, 3, \ldots \quad . \tag{5}$$

Since $n \geq m$ and since lower order tesseral harmonics are bound to have larger effects because of the factor $(R/r)^n$, where $R/r < 1$, it can be assumed that it is enough to consider only the cases $k_1 = 1$, 2, and 3.

If $k_1 = 1$, then $m = \beta$, and the tesseral harmonics to be considered are

$$V_{\beta+k_0, \beta} \quad , \qquad k_0 = 0, 1, 2, \ldots \quad .$$

In a general manner, the tesseral harmonics to be kept are

$$V_{k_1\beta + k_0, k_1\beta} \quad ,$$

where $k_1 = 1, 2, 3, \ldots$ and for each k_1, the index k_0 takes values $0, 1, 2, \ldots$.

2.1 Resonant Part of a Tesseral Harmonic

Once we have decided which tesseral harmonics to keep for our disturbing function, we then need to extract for each its most important "resonant" part.

If $V_{n,m}$ is a selected tesseral harmonic, with $n = k_1\beta + k_0$ and $m = k_1\beta$, we look for terms containing "resonant" arguments by solving equation (5); that is, since we already know $m = k_1\beta$, we have

$$n - 2p + q = k_1\alpha \quad ,$$

or, in this case,

$$k_1(\beta - \alpha) + k_0 = 2p - q \quad .$$

As the condition $0 \le p \le k_1\beta + k_0$ must be satisfied, the following pairs (p, q) are solutions:

$$\begin{array}{ll} p = 0 \quad , & q = k_1(\alpha - \beta) - k_0 \quad , \\ p = 1 \quad , & q = k_1(\alpha - \beta) - k_0 + 2 \quad , \\ \vdots & \vdots \\ p = x \quad , & q = k_1(\alpha - \beta) - k_0 + 2x \quad , \\ p = k_1\beta + k_0 \quad , & q = k_1(\alpha + \beta) + k_0 \quad , \end{array} \tag{6}$$

and the resonant part of the tesseral harmonic $V_{n,m}$ is

$$R_{n,m} = \mu \frac{R^n}{a^{n+1}} J_{n,m} \sum_{x=0}^{\beta k_1 + k_0} F_{n,m,p_x}(I)\, G_{n,p_x,q_x}(e) \cos\left(k_1\Phi_{\alpha,\beta} - q_x\omega - m\lambda_{n,m}\right) \quad , \tag{7}$$

where

$$q_x = k_1(\alpha - \beta) - k_0 + 2x$$

and

$$p_x = x \quad .$$

Thus,

$$V_{n,m} = R_{n,m} + V'_{n,m} \quad ,$$

where $V'_{n,m}$ is the residual, the effect of which is much smaller than that of $R_{n,m}$.

It will be convenient to express the sum of the resonant parts of all the selected tesseral harmonics in the following manner:

$$\sum R_{n,m} = \sum_{k_1=1}^{3} D(k_1) \cos 2\theta_1(k_1) \quad ,$$

where

$$D(k_1) \cos 2\theta_1(k_1) = \sum_{k_0(k_1)} R_{k_1\beta+k_0,\, k_1\beta} \quad ,$$

the sum over $k_0(k_1)$ meaning that we have taken into account all the values of k_0 when k_1 has a given value. In order to obtain $D(k_1)$ and $\theta_1(k_1)$, we can write

$$S(k_1, k_0) = \frac{\mu R^n J_{n,m}}{a^{n+1}} \quad , \qquad n = k_1\beta + k_0 \quad , \qquad m = k_1\beta \quad ,$$

$$A(k_1, k_0, x) = F_{n,m,x}(I)\, G_{n,x,q_x}(e) \quad , \tag{8}$$

$$D(k_1) \exp 2i\theta_1(k_1) = \sum_{k_0(k_1)} S(k_1, k_0) \sum_{x=0}^{x=n} A(k_1, k_0, x) \exp i(k_1\Phi_{\alpha,\beta} - q\omega - k_1\beta\lambda_{n,m}) \quad ,$$

where

$$q_x = k_1(\alpha - \beta) - k_0 + 2x \quad .$$

If we write

$$\Lambda_{k_1,k_0} = k_1\beta\, \lambda_{k_1\beta+k_0,\, k_1\beta}$$

and

$$2\theta_1(k_1) = k_1\Phi_{\alpha,\beta} - k_1(\alpha-\beta)^{\omega} + \psi(k_1) \quad ,$$

then

$$D(k_1) \exp i\psi(k_1) = \sum_{k_0(k_1)} S(k_1, k_0) \sum_{x=0}^{n} A(k_1, k_0, x) \exp i\left[(k_0 - 2x)\omega - \Lambda_{k_1, k_0}\right] \quad .$$

Thus,

$$B(k_1) = D(k_1) \cos\psi(k_1) = \sum_{k_0} \sum_{x=0}^{x=k_1\beta+k_0} S(k_1, k_0)\, A(k_1, k_0, x) \cos\left[(k_0 - 2x)\omega - \Lambda_{k_1, k_0}\right] \quad , \tag{9a}$$

$$C(k_1) = D(k_1) \sin\psi(k_1) = \sum_{k_0} \sum_{x=0}^{x=k_1\beta+k_0} S(k_1, k_0)\, A(k_1, k_0, x) \sin\left[(k_0 - 2x)\omega - \Lambda_{k_1, k_0}\right] \quad , \tag{9b}$$

$$D(k_1)^2 = B(k_1)^2 + C(k_1)^2 \quad , \tag{9c}$$

in which $D(k_1)$ and $\psi(k_1)$ are functions of a, e, I, and Ω. Appendix A gives expressions for the first and second derivatives of $D(k_1)$ and $\psi(k_1)$ with respect to the canonical set of Delaunay variables.

Finally, the potential in which the satellite moves is taken to be

$$V = \frac{\mu}{r} - J_2 \frac{R^2}{r^3} P_2(\cos\theta) + \sum R_{n,m} + \sum V'_{n,m} \quad , \tag{10}$$

the sums being taken over all the tesseral harmonics selected.

2.2 The Equations of Motion

If we consider the canonical set of Delaunay variables,

$$\begin{aligned} L_D &= \sqrt{\mu a} \ , & G_D &= \sqrt{\mu a(1-e^2)} \ , & H_D &= G_D \cos i \ , \\ \ell_D &= M \ , & g_D &= \omega \ , & h_D &= \Omega \ , \end{aligned} \tag{11}$$

then the equations of motion of the satellite are

$$\frac{dL_D}{dt} = \frac{\partial F_D}{\partial \ell_D} \quad , \qquad \frac{dG_D}{dt} = \frac{\partial F_D}{\partial g_D} \quad , \qquad \frac{dH_D}{dt} = \frac{\partial F_D}{\partial h_D} \quad ,$$
$$\frac{d\ell_D}{dt} = -\frac{\partial F_D}{\partial L_D} \quad , \qquad \frac{dg_D}{dt} = -\frac{\partial F_D}{\partial G_D} \quad , \qquad \frac{dh_D}{dt} = -\frac{\partial F_D}{\partial H_D} \quad , \tag{12}$$

where F_D is the Hamiltonian of the problem:

$$F_D = \frac{\mu^2}{2L_D^2} + F_1 + F_2 \quad , \tag{13a}$$

where

$$F_1 = -J_2 \, \mu \frac{R^2}{r^3} P_2 (\cos \theta) \quad , \tag{13b}$$

which can be expressed in Delaunay variables, and

$$F_2 = \sum V_{n,m} \quad . \tag{13c}$$

Here, the Hamiltonian F_D depends explicitly on time. To avoid this, we can perform a canonical transformation so that the new angular variables will be

$$\ell = \ell_D \quad , \qquad g = g_D \quad , \qquad h = h_D - \nu t \quad .$$

To find the new momenta and the new Hamiltonian, we have to solve

$$L \, d\ell + G \, dg + H \, dh - L_D \, d\ell - G_D \, dg - H_D \, dh - (F_D - F) \, dt = dV \quad ,$$

where dV is the differential of a function. We keep the solution corresponding to $dV \equiv 0$ as follows:

$$L = L_D \quad ,$$
$$G = G_D \quad ,$$
$$H = H_D \quad ,$$
$$F = F_D + \nu H \quad .$$

The equations of motion then become

$$\frac{dL}{dt} = \frac{\partial F}{\partial \ell} \quad , \qquad \frac{dG}{dt} = \frac{\partial F}{\partial g} \quad , \qquad \frac{dH}{dt} = \frac{\partial F}{\partial h} \quad ,$$
$$\frac{d\ell}{dt} = -\frac{\partial F}{\partial L} \quad , \qquad \frac{dg}{dt} = -\frac{\partial F}{\partial G} \quad , \qquad \frac{dh}{dt} = -\frac{\partial F}{\partial H} \quad , \tag{14}$$

with $F = F_0 + F_1 + F_2$:

$$F_0 = \frac{\mu^2}{2L^2} + \nu H \quad ,$$
$$F_1 = -J_2 \mu \frac{R^2}{r^3} \left[\frac{1}{4} \frac{H^2}{G^2} - \frac{3}{4} \left(1 - \frac{H^2}{G^2} \right) \cos 2u \right] , \tag{15}$$
$$F_2 = \sum V_{n,m} \quad ,$$

in the above, $u = g + f$, and f is the true anomaly.

The Hamiltonian is now expanded in terms of powers of the small parameter J_2:

$$F_1 = O(J_2) \quad ,$$
$$F_2 = O(J_2^2) \quad .$$

3. HORI'S PERTURBATION METHOD BY LIE SERIES

In order to remove the variables ℓ and h from the Hamiltonian, we shall apply Hori's method by the use of Lie series (Hori, 1966; Aksnes, 1970).

The greatest advantage of this method is the fact that the Poisson brackets that appear are canonically invariant, which means that each bracket can be calculated in terms of the most convenient set of canonical variables. Also, this method avoids mixing old and new variables.

Let us consider a system of canonical equations

$$\frac{dx_j}{dt} = \frac{\partial F}{\partial y_j} \quad , \qquad \frac{dy_j}{dt} = -\frac{\partial F}{\partial x_j} \quad , \qquad j = 1, \ldots, n \quad , \tag{16}$$

where the Hamiltonian F is developed in powers of a small parameter ϵ,

$$F = \sum_{k=0}^{\infty} F_k \quad , \qquad F_k = O(\epsilon^k) \quad ;$$

and let us also consider a Lie transformation (Hori, 1966)

$$(x_j, y_j) \longrightarrow (x_j', y_j') \quad ,$$

$$x_j = x_j' + \frac{\partial S}{\partial y_j'} + \frac{1}{2}\left\{\frac{\partial S}{\partial y_j'}, S\right\} + \frac{1}{6}\left\{\left\{\frac{\partial S}{\partial y_j'}, S\right\}, S\right\} + \ldots \quad ,$$

$$y = y_j' - \frac{\partial S}{\partial x_j'} - \frac{1}{2}\left\{\frac{\partial S}{\partial x_j'}, S\right\} - \frac{1}{6}\left\{\left\{\frac{\partial S}{\partial x_j'}, S\right\}, S\right\} + \ldots \quad ,$$

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where $\{\ ,\ \}$ are Poisson brackets and S is a function of (x', y') of order 1 in ϵ:

$$S = \sum_{K=1}^{\infty} S_K \quad .$$

Then, if $f(x, y)$ is a function of (x_j, y_j), $j = 1, n$, it can be expressed in terms of the new variables (x_j', y_j') as follows:

$$f(x, y) = f(x', y') + \{f, S\} + \frac{1}{2}\{\{f, S\}, S\} + \frac{1}{6}\{\{\{f, S\}, S\}, S\} + \ldots \quad .$$

The equations of motion then become

$$\frac{dx_j'}{dt} = \frac{\partial F'}{\partial y_j'} \quad , \qquad \frac{dy_j'}{dt} = -\frac{\partial F'}{\partial x_j'} \quad , \tag{17}$$

where F' is the new Hamiltonian:

$$F' = \sum_{k=0}^{\infty} F_k' \quad .$$

If we assume that F does not depend explicitly on time t, we can write the energy integral

$$F(x, y) = F'(x', y') \quad .$$

Expanding in terms of ϵ and collecting terms of the same order of magnitude, we obtain

$$\begin{aligned} &F_0 = F_0' \quad , \\ &F_1 + \{F_0, S_1\} = F_1' \quad , \\ &\{F_0, S_2\} + \frac{1}{2}\{F_1 + F_1', S_1\} + F_2 = F_2' \quad . \end{aligned} \tag{18}$$

In order to average these equations, we introduce a pseudo time t':

$$\frac{dx'_j}{dt'} = \frac{\partial F_0}{\partial y'_j} \quad , \qquad \frac{dy'_j}{dt'} = -\frac{\partial F_0}{\partial x'_j} \quad .$$

If this system has a solution, then

$$\{F_0, S_k\} = -\frac{dS_k}{dt'} \quad , \qquad k \geq 1 \quad ,$$

and if $A(t')$ is a periodic function of t', with period T, then

$$A_s = \frac{1}{T} \int_0^T A(t')\, dt' \tag{19a}$$

is the secular part of A and

$$A_p = A - A_s \tag{19b}$$

is the periodic part of A.

We can remove t' from F' by applying the averaging technique [eqs. (19)] to equations (18) (Hori, 1966; Aksnes, 1970). Then S_k and F'_k are uniquely determined by the following set of equations:

$$\begin{aligned}
F'_0 &= F_0 \quad , \\
F'_1 &= F_{1s} \quad , \\
S_1 &= \int F_{1p}\, dt' \quad , \\
F'_2 &= F_{2s} + \frac{1}{2} \{F_1 + F'_1, S_1\}_s \quad , \\
S_2 &= \int F_{2p} + \frac{1}{2} \{F_1 + F'_1, S_1\}_p \; dt' \quad , \\
&\vdots
\end{aligned} \tag{20}$$

In the present case,

$$F_2 = \sum_{k_1} D(k_1) \cos 2\theta_1(k_1) + \sum V'_{n,m}$$

and the pseudo time t' is defined by

$$\frac{d\ell'}{dt'} = -\frac{\partial F_0}{\partial L'} = \frac{\mu^2}{L'^3} \quad ,$$

$$\frac{dh'}{dt'} = -\frac{\partial F_0}{\partial H'} = -\nu \quad ,$$

so that

$$dt' = \frac{L'^3}{\mu^2} d\ell' - \frac{1}{\nu} dh' \quad . \tag{21}$$

In the process of calculating S_2, we have to evaluate the following integral:

$$\int D(k_1) \cos 2\theta_1(k_1)\, dt' = \frac{D(k_1) \sin 2\theta'(k_1)}{2\delta} \quad ,$$

where

$$\delta = \left[\left(\frac{\mu^2}{L'^3}\right)\alpha - \nu\beta\right] k_1 \quad .$$

In the case of resonance of order (α, β),

$$n = \frac{\mu^2}{L'^3} \simeq \frac{\nu\beta}{\alpha} \quad ,$$

where n is the mean motion of the satellite; thus, δ is a small quantity appearing in the denominator, and the preceding method cannot be used.

To avoid this situation, we shall introduce a procedure that is commonly used in different resonance problems (Garfinkel, 1973; Hori, 1960). The method consists of developing the Hamiltonian and the determining function S in powers of the square root of the small parameter:

$$S = S_{1/2} + S_1 + S_{3/2} + S_2 + \dots \quad .$$

If F' is the new Hamiltonian, then

$$F' = F'_0 + F'_1 + F'_{3/2} + F'_2 + \dots$$

and equations (18) now become

$$\begin{aligned} F' = F &+ \{F_0, S_{1/2}\} + \{F_0, S_1\} + \{F_0, S_{3/2}\} + \{F_1, S_1\} + \{F_1, S_{1/2}\} + \dots \\ &+ \frac{1}{2}\{\{F_0, S_{1/2}\}, S_{1/2}\} + \frac{1}{2}\{\{F_0, S_{1/2}\}, S_1\} + \frac{1}{2}\{\{F_0, S_1\}, S_{1/2}\} \\ &+ \frac{1}{6}\{\{\{F_0, S_{1/2}\}, S_{1/2}\}, S_{1/2}\} + \frac{1}{2}\{\{F_0, S_1\}, S_1\} + \frac{1}{2}\{\{F_1, S_{1/2}\}, S_{1/2}\} + \dots \quad . \end{aligned} \tag{22}$$

Equation (22) immediately yields to order 0:

$$F'_0(L', G', H') = F_0(L', G', H') \quad .$$

We shall group the different terms in equation (22) according to their order, but first we need to develop some of the Poisson brackets in the following manner:

First consider

$$\{F_0, S_{1/2}\} = -\frac{\mu^2}{L'^3}\frac{\partial S_{1/2}}{\partial \ell'} + \nu \frac{\partial S_{1/2}}{\partial h'} \quad .$$

As $S_{1/2}$ arises strictly from the existence of a resonance and the resonant terms are functions of the resonant parameter $\Phi'_{\alpha,\beta} = \alpha(\ell' + g') + \beta h'$, we look for a solution $S_{1/2} = S_{1/2}(L', G', H', g', \Phi'_{\alpha,\beta})$ such that

$$\frac{\partial S_{1/2}}{\partial \ell'} = \frac{\alpha}{\beta} \frac{\partial S_{1/2}}{\partial h'} \quad .$$

Thus,

$$\{F_0, S_{1/2}\} = \left(-\frac{\mu^2}{L'^3} + \frac{\nu\beta}{\alpha}\right) \frac{\partial S_{1/2}}{\partial \ell'} = \gamma \frac{\partial S_{1/2}}{\partial \ell'} \quad ,$$

where $\gamma = (\nu\beta/\alpha) - (\mu^2/L'^3)$ is a small parameter that can be assumed to be of order 1/2, so that $\{F_0, S_{1/2}\}$ is of order 1 in equation (22).

Let us write

$$S_1 = SJ_1 + SR_1 \quad ,$$

where SJ_1 arises only from the effect of J_2, which can be obtained by solving

$$F_1' = \{F_0, SJ_1\} + F_1 \Longrightarrow \begin{cases} F_1' = F_{1s} \\ SJ_1 = \int F_{1p}\, dt' \end{cases} \quad , \tag{23}$$

and SR comes from the existence of the resonant terms that combine their influence with that of J_2. We shall look for a solution of the form

$$SR_1 = SR_1(L', G', H', g', \Phi'_{\alpha,\beta})$$

such that

$$\{F_0, SR_1\} = \gamma \frac{\partial SR_1}{\partial \ell'}$$

is of order 3/2.

We can then write

$$\gamma = \gamma(L') \quad ,$$

$$\frac{1}{2}\{\{F_0, S_{1/2}\}, S_{1/2}\} = \frac{1}{2}\left\{\gamma\ \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}$$

$$= \underbrace{\frac{1}{2}\frac{\partial \gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^2}_{\text{of order 1}} + \underbrace{\frac{1}{2}\gamma\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}}_{\text{of order 3/2}} \quad ; \tag{24}$$

$$\frac{1}{2}\{\{F_0, S_1\}, S_{1/2}\} = \frac{1}{2}\{\{F_0, SJ_1\}, S_{1/2}\} + \frac{1}{2}\left\{\gamma\ \frac{\partial SR_1}{\partial \ell'}, S_{1/2}\right\}$$

$$= \frac{1}{2}\{\{F_0, SJ_1\}, S_{1/2}\} + \frac{1}{2}\frac{\partial \gamma}{\partial L'}\frac{\partial SR_1}{\partial \ell'}\frac{\partial S_{1/2}}{\partial \ell'}$$

$$+ \frac{1}{2}\gamma\left\{\frac{\partial SR_1}{\partial \ell'}, S_{1/2}\right\} \quad , \tag{25}$$

where

$$\frac{1}{2}\{\{F_0, SJ_1\}, S_{1/2}\} \qquad \text{and} \qquad \frac{1}{2}\frac{\partial \gamma}{\partial L'}\frac{\partial SR_1}{\partial \ell'}\frac{\partial S_{1/2}}{\partial \ell'}$$

are of order 3/2 and

$$\frac{1}{2}\gamma\left\{\frac{\partial SR_1}{\partial \ell'}, S_{1/2}\right\}$$

is of order 2;

$$\frac{1}{6}\{\{\{F_0, S_{1/2}\}, S_{1/2}\}, S_{1/2}\} = \frac{1}{6}\frac{\partial^2 \gamma}{\partial L'^2}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^3 + \frac{1}{2}\frac{\partial \gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}$$

$$+ \frac{1}{6}\gamma\left\{\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\} \quad , \tag{26}$$

where

$$\frac{1}{6}\,\frac{\partial^2\gamma}{\partial L'^2}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^3 + \frac{1}{2}\,\frac{\partial\gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}$$

is of order 3/2 and

$$\frac{1}{6}\,\gamma\left\{\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\}$$

is of order 2. In the expressions above,

$$\frac{\partial\gamma}{\partial L'} = \frac{3\mu^2}{L'^4} \quad , \qquad \frac{\partial^2\gamma}{\partial L'^2} = -\frac{12\mu^2}{L'^5} \quad .$$

We write F_2 in the following manner:

$$F_2 = 2\sum_{k_1} D(k_1)\,\sin^2\theta(k_1) - \sum_{k_1} D(k_1) + \sum V'_{n,m} \quad ,$$

where

$$\theta(k_1) = \frac{\pi}{2} - \theta_1(k_1) \quad ,$$

and $D(k_1)$ is a function of (L', G', H', g'); we assume here that g' is reduced to its constant part g'_0. The secular part of g' should be taken into account when the long-period perturbations are removed.

We shall assume for the time being that our Hamiltonian contains terms for only one particular value of k_1, and thus we can drop the k_1 index for convenience.

We now proceed to splitting equation (22) by grouping terms of the same order.

A. We shall allow the first approximation of the determining function, $S_{1/2}$, to contain the resonant part of the Hamiltonian, which means that it will contain secular terms. We then have, to order 1,

$$\{F_0, S_{1/2}\} + \frac{1}{2}\frac{\partial\gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^2 + 2D\sin^2\theta = 0 \quad ,$$

or

$$\gamma\left(\frac{\partial S_{1/2}}{\partial \ell'}\right) + \frac{3}{2}\frac{\mu^2}{L'^4}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^2 + 2D\sin^2\theta = 0 \quad . \tag{27}$$

This is called the "resonance equation."

If

$$A = \frac{2\gamma L'^4}{3\mu^2 a k_1} \quad ,$$

$$k^2 = \frac{12 D \mu^2}{L'^4 \gamma^2} \quad , \tag{28}$$

and we assume that $S_{1/2} = S_{1/2}(\theta)$, then by solving the quadratic equation (27), we get

$$\frac{\partial S_{1/2}}{\partial\theta} = -A \pm A\sqrt{1 - k^2\sin^2\theta} \quad , \tag{29a}$$

or

$$\frac{\partial S_{1/2}}{\partial\theta} = A(\Delta - 1) \tag{29b}$$

if

$$\Delta = \pm\sqrt{1 - k^2\sin^2\theta} \quad . \tag{29c}$$

We discuss this in detail in Section 4.

B. To order 3/2, we have the following:

$$F'_{3/2} = \gamma \frac{\partial SR_1}{\partial \ell'} + \frac{\gamma}{2} \left\{ \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2} \right\} + \frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'} \frac{\partial SR_1}{\partial \ell'}$$

$$+ \frac{1}{2} \{\{F_0, SJ_1\}, S_{1/2}\} + \frac{1}{2} \frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'} \frac{\partial SJ_1}{\partial \ell'} + \{F_1, S_{1/2}\}$$

$$+ \frac{1}{6} \frac{\partial^2 \gamma}{\partial L'^2} \left(\frac{\partial S_{1/2}}{\partial \ell'} \right)^3 + \frac{1}{2} \frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'} \left\{ \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2} \right\} ,$$

or

$$F'_{3/2} = \gamma\Delta \frac{\partial SR_1}{\partial \ell'} + \frac{\gamma\Delta}{2} \left\{ \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2} \right\} + \frac{1}{6} \frac{\partial^2 \gamma}{\partial L'^2} \left(\frac{\partial S_{1/2}}{\partial \ell'} \right)^3$$

$$+ \{F_1, S_{1/2}\} + \frac{1}{2} \{\{F_0, SJ_1\}, S_{1/2}\} + \frac{1}{2} \frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'} \frac{\partial SJ_1}{\partial \ell'} \quad . \tag{30}$$

We notice that in equation (30), some terms contain only $S_{1/2}$ and are therefore due only to the resonant tesseral harmonics, while others express the interaction between the resonant terms and J_2. Let us therefore write

$$SR = Z_1 + Z_2 \quad ,$$

where Z_1 contains only those terms that are functions of $S_{1/2}$, and Z_2 contains all the others. If we write $F'_{3/2} = X + Y$, we then obtain two equations:

$$X = \gamma\Delta \frac{\partial Z_1}{\partial \ell'} + \frac{\gamma\Delta}{2} \left\{ \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2} \right\} + \frac{1}{6} \frac{\partial^2 \gamma}{\partial L'^2} \left(\frac{\partial S_{1/2}}{\partial \ell'} \right)^3 \tag{31}$$

and

$$Y = \{F_1, S_{1/2}\} + \frac{1}{2} \{\{F_0, SJ_1\}, S_{1/2}\} + \frac{1}{2} \frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'} \frac{\partial SJ_1}{\partial \ell'} + \frac{\gamma\Delta \partial Z_2}{\partial \ell'} \quad . \tag{32}$$

C. Finally, to order 2, we have

$$F'_2 = F_{2s} = -D(L', G', H', g') \quad , \tag{33}$$

and

$$
\begin{aligned}
0 = &\sum V'_{n,m} + \{F_0, S_{3/2}\} + \{F_0, S_2\} + \{F_0, S'_2\} \\
&+ \frac{1}{2}\gamma \left\{\frac{\partial SR}{\partial \ell'}, S_{1/2}\right\} + \frac{1}{2}\{\{F_0, SJ_1\}, SJ_1\} \\
&+ \frac{1}{6}\gamma\left\{\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\} + \{F_1, S_1\} + \frac{1}{2}\{\{F_1, S_{1/2}\}, S_{1/2}\} \\
&+ \frac{1}{6}\left\{\left\{\frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_1\right\} + \frac{1}{6}\left\{\left\{\frac{\partial \gamma}{\partial L'} \frac{\partial S_{1/2}}{\partial \ell'}, S_1\right\}, S_{1/2}\right\} \\
&+ \frac{1}{4!}\left\{\left\{\frac{\partial \gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^2, S_{1/2}\right\}, S_{1/2}\right\} + \frac{1}{4!}\left\{\frac{\partial \gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\} \\
&+ \frac{1}{4!}\frac{\partial \gamma}{\partial L'}\left\{\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right) \quad .
\end{aligned} \tag{34}
$$

Equation (34) defines S_2, S'_2, and $S_{3/2}$. The term S_2 arises from the existence of J_2 only:

$$\{F_0, S_2\} + \frac{1}{2}\{\{F_0, SJ_1\}, SJ_1\} + \{F_1, SJ_1\} = 0 \quad ; \tag{35}$$

S'_2, which takes care of the nonresonant tesseral part, is obtained by solving

$$\{F_0, S'_2\} + \sum V'_{n,m} = 0 \tag{36a}$$

or

$$S'_2 = \int \sum V'_{n,m} \, dt' \quad ; \tag{36b}$$

and $S_{3/2}$ is due to the resonant tesseral harmonics, which we assume to be a function of θ, such that

$$\{F_0, S_{3/2}\} = \gamma \frac{\partial S_{3/2}}{\partial \ell'}$$

and

$$0 = \gamma \frac{\partial S_{3/2}}{\partial \ell'} + \frac{1}{2}\gamma \left\{\frac{\partial SJ_1}{\partial \ell'}, S_{1/2}\right\} + \frac{1}{2}\frac{\partial \gamma}{\partial L'}\left(\frac{\partial SJ_1}{\partial \ell'}\right)\frac{\partial S_1}{\partial \ell'}$$
$$+ \frac{1}{6}\gamma\left\{\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_{1/2}\right\} + \frac{1}{2}\{\{F_0, SJ_1\}, SR_1\} + \frac{1}{2}\{\{F_1, S_{1/2}\}, S_{1/2}\}$$
$$+ \frac{1}{6}\left\{\left\{\frac{\partial \gamma}{\partial L'}\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\}, S_1\right\} + \frac{1}{6}\left\{\left\{\frac{\partial \gamma}{\partial L'}\frac{\partial S_{1/2}}{\partial \ell'}, S_1\right\}, S_{1/2}\right\} + \dots \quad . \tag{37}$$

If the Hamiltonian contains more than one value of k_1 (usually $k_1 = 1$ is enough), we can write

$$\begin{aligned} S_{1/2} &= \sum_{k_1} S_{1/2,k_1} \quad , \\ Z_1 &= \sum_{k_1} Z_{1,k_1} \quad , \\ Z_2 &= \sum_{k_1} Z_{2,k_1} \quad , \\ &\vdots \end{aligned} \tag{38}$$

We first obtain $S_{1/2}$ by solving equation (27), replacing $S_{1/2}$ by $S_{1/2,k_1}$ for each value of k_1, and adding the different $S_{1/2,k_1}$ terms. This amounts to neglecting mixed terms of the form

$$\frac{\partial S_{1/2,k_1}}{\partial \ell'} \times \frac{\partial S_{1/2,k_2}}{\partial \ell'}$$

in this theory.

We then solve equations (31) and (32) in the same way, again for each value of k_1, and add the perturbations obtained; the neglected mixed terms are reasonably small.

If (L, G, H, ℓ, g, h) is the set of modified Delaunay variables as defined in Section 2, and $(L', G', H', \ell', g', h')$ are the new variables, then after applying Hori's transformation, the solution of the equations of motion (14) becomes

$$\begin{aligned}
L &= L' + \frac{\partial S}{\partial \ell'} + \frac{1}{2}\left\{\frac{\partial S}{\partial \ell'}, S\right\} + \frac{1}{3!}\left\{\left\{\frac{\partial S}{\partial \ell'}, S\right\}, S\right\} + \dots \quad , \\
G &= G' + \frac{\partial S}{\partial g'} + \frac{1}{2}\left\{\frac{\partial S}{\partial g'}, S\right\} + \dots \quad , \\
H &= H' + \frac{\partial S}{\partial h'} + \frac{1}{2}\left\{\frac{\partial S}{\partial h'}, S\right\} + \dots \quad , \\
\ell &= \ell' - \frac{\partial S}{\partial L'} - \frac{1}{2}\left\{\frac{\partial S}{\partial L'}, S\right\} - \dots \quad , \\
g &= g' - \frac{\partial S}{\partial G'} - \frac{1}{2}\left\{\frac{\partial S}{\partial G'}, S\right\} - \dots \quad , \\
h &= h' - \frac{\partial S}{\partial H'} - \frac{1}{2}\left\{\frac{\partial S}{\partial H'}, S\right\} - \dots \quad ,
\end{aligned} \tag{39}$$

where $S = S(L', G', H', \ell', g', h')$ is the generating function

$$S = S_{1/2} + SJ + SR + \dots \quad .$$

Then, if

$$F' = \frac{\mu^2}{2L'^2} + \nu H' - \sum_{k_1} D(k_1) + F'_1$$

is the new Hamiltonian, we get

$$\frac{dL'}{dt} = \frac{\partial F'}{\partial \ell'} = 0 \quad ,$$

$$\frac{dG'}{dt} = \frac{\partial F'}{\partial g'} = 0 \quad ,$$

$$\frac{dH'}{dt} = \frac{\partial F'}{\partial h'} = 0 \quad ,$$

$$\frac{d\ell'}{dt} = -\frac{\partial F'}{\partial L'} = \frac{\mu^2}{L'^3} - \frac{\partial F_1'}{\partial L'} + \sum_{k_1} \frac{\partial D(k_1)}{\partial L'} = \ell_1' \quad ,$$

$$\frac{dg'}{dt} = -\frac{\partial F'}{\partial G'} = -\frac{\partial F_1'}{\partial G'} + \sum_{k_1} \frac{\partial D(k_1)}{\partial G'} = g_1' \quad ,$$

$$\frac{dh'}{dt} = -\frac{\partial F'}{\partial H'} = -\frac{\partial F_1'}{\partial H'} - \nu + \sum_{k_1} \frac{\partial D(k_1)}{\partial H'} = h_1' \quad ,$$

where ℓ_1', g_1', and h_1' are constants such that

$$\ell' = \ell_1' t + \ell_0' \quad ,$$

$$g' = g_1' t + g_0' \quad ,$$

$$h' = h_1' t + h_0' \quad ,$$

with ℓ_0', g_0', and h_0' also constants.

In Section 4, we will give precise expressions of the perturbations up to order 3/2. The perturbations due to J_2 only are not considered in this paper, but can be found in a classical theory of artificial satellites, for instance Brouwer's. However, the interactions between J_2 and the resonant tesseral harmonics are expressed with the generating function Z_2.

4. DETAILED EXPRESSIONS OF THE PERTURBATIONS

In this section, we drop the primes in the variables $(L', G', H', \ell', g', h')$, since no confusion exists here with the original Delaunay variables, which do not appear.

4.1 Expression of $S_{1/2}$

As we already have seen,

$$\frac{\partial S_{1/2}}{\partial \theta} = A(\Delta - 1) \quad ,$$

where

$$\Delta = \epsilon \sqrt{1 - k^2 \sin^2 \theta} \quad , \qquad \epsilon = \pm 1 \quad ,$$

so that

$$S_{1/2} = -A + \epsilon A \int_0^\theta \sqrt{1 - k^2 \sin^2 x}\; dx \quad . \tag{40}$$

Let us discuss the value of ϵ in equation (40). First, we consider the case when $k < 1$. For any value of θ, $1 - k^2 \sin^2 \theta > 0$, and we are in the circulation case. In order to have continuity with the classical solution, valid away from resonance, it is necessary that $S_{1/2} \to 0$ when $k \to 0$. Since $A \to \infty$ when $k \to 0$, then, necessarily, $\epsilon = +1$. Otherwise, $S_{1/2} \to -\infty$ when $A \to \infty$.

For the case when $k = 1$,

$$\Delta = \epsilon \cos \theta \quad ,$$

$$\int_0^\theta \Delta(x)\, dx = \epsilon \sin \theta \quad .$$

In order to have continuity when $k \to 1$ and $k < 1$, we must have $\epsilon = +1$.

When $k > 1$,

$$1 - k^2 \sin^2 x \geq 0 \Longleftrightarrow - \arcsin \frac{1}{k} \leq x \leq \arcsin \frac{1}{k} \quad ,$$

and we are in the libration case.

We need to know the expressions of the elliptic integrals of the first and second kind – $F(\theta, k)$ and $E(\theta, k)$, respectively – for the cases $k > 1$ and $\pi/2 < \theta$. For these, we have the following formulas (Gradshteyn and Ryzhik, 1966):

$$\begin{aligned} E(m\pi \pm \theta, k) &= 2mE \pm E(\theta, k) \quad , \\ F(m\pi \pm \theta, k) &= 2mk \pm F(\theta, k) \quad , \end{aligned} \tag{41}$$

where E and K are the complete elliptic integrals of

$$E = E\left(\frac{\pi}{2}, k\right) \quad ,$$

$$F = F\left(\frac{\pi}{2}, k\right) \quad .$$

Then, if $k_1 = 1/k$, $k'^2 = 1 - k^2$, and θ_1 is defined by $\sin \theta_1 = k \sin \theta$, we have

$$\begin{aligned} E(\theta, k) &= k_1 \left[k^2 E(\theta', k_1) + k'^2 F(\theta', k_1)\right] \quad , \\ F(\theta, k) &= k_1 F(\theta', k_1) \quad . \end{aligned} \tag{42}$$

Returning to our problem, we see that in order to have continuity when $k \rightarrow 1$ and $k > 1$, we can take $\epsilon = +1$ in this case also, because, from equations (42),

$$\lim_{k \to 1} E(\theta, k) = \lim_{k \to 1} E(\theta', k) = E(\theta, 1) = \sin \theta \quad .$$

In conclusion, whatever the value of k, we can write $S_{1/2}$ in the form

$$S_{1/2} = -A\theta + AE(\theta, k) \quad .$$

To simplify this, we can set

$$I_2 = E(\theta, k) \quad ,$$

$$I_0 = F(\theta, k) \quad .$$

Then the derivatives of $S_{1/2}$ are readily obtained:

$$\frac{\partial S_{1/2}}{\partial \ell} = \alpha c_2 \gamma (\Delta - 1) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial g} = d c_2 \gamma (\Delta - 1) + c_2 \gamma a_4 (I_2 - I_0) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial h} = \beta c_2 \gamma (\Delta - 1) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial L} = \frac{2}{\alpha} (I_0 - \theta) + 8 c_1 \gamma (I_2 - \theta) + c_2 p_1 \gamma (\Delta - 1) + (c_2 \gamma a_1 - 4 c_1 \gamma)(I_2 - I_0) \quad , \tag{43}$$

$$\frac{\partial S_{1/2}}{\partial G} = c_2 \gamma p_2 (\Delta - 1) + c_2 \gamma a_2 (I_2 - I_0) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial H} = \gamma c_2 p_3 (\Delta - 1) + c_2 \gamma a_3 (I_2 - I_0) \quad ,$$

where

$$c_2 = \frac{L^4}{3\mu^2 \alpha} \quad , \qquad c_1 = \frac{c_2}{L} \quad ,$$

and

$$a_1 = \frac{1}{D} \frac{\partial D}{\partial L} \quad , \qquad p_1 = \frac{\partial \psi}{\partial L} \quad ,$$

$$a_2 = \frac{1}{D} \frac{\partial D}{\partial G} \quad , \qquad p_2 = \frac{\partial \psi}{\partial G} \quad , \tag{44}$$

[eq. (44) cont. on next page]

$$a_3 = \frac{1}{D}\frac{\partial D}{\partial H} \quad , \qquad p_3 = \frac{\partial \psi}{\partial H} \quad ,$$

$$a_4 = \frac{1}{D}\frac{\partial D}{\partial g} \quad , \qquad p_4 = \frac{\partial \psi}{\partial g} \quad , \tag{44}$$

$$d = \beta + p_4 \quad .$$

To calculate the elliptic integrals, we use the following expansions (from Gradshteyn and Ryzhik):

$$0 < \theta < \frac{\pi}{2} \quad , \qquad k^2 < 1 \quad ,$$

$$F(\theta, k) = \sum_{m=0}^{\infty} \binom{-1/2}{m} (-k^2)^m \, t_{2m}(\theta) \quad , \tag{45}$$

$$E(\theta, k) = \sum_{m=0}^{\infty} \binom{1/2}{m} (-k^2)^m \, t_{2m}(\theta) \quad ,$$

where

$$t_0(\theta) = \theta \quad ,$$

$$t_2(\theta) = \frac{1}{2}(\theta - \sin\theta\cos\theta) \quad ,$$

$$t_{2m}(\theta) = \frac{2m-1}{2m} t_{2(m-1)}(\theta) - \frac{1}{2m}\cos\theta\sin^{2m-1}\theta \quad , \tag{46}$$

$$\binom{a}{n} = \frac{(-1)^n (a)_n}{n!} \quad ,$$

in which

$$(a)_n = a(a+1)\cdots(a+n-1) \quad , \qquad n = 1, 2, \ldots \quad ,$$

$$(a)_0 = 1 \quad ,$$

and

$$K(k) = \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{(1/2)_m \, (1/2)_m \, k^{2m}}{m! \, m!} \quad ,$$

$$E(k) = \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{1}{(1-2m)} \binom{-1/2}{m}^2 k^{2m} \quad . \tag{47}$$

4.2 Expression of Z_1 and Its Derivatives

We first calculate the Poisson brackets that appear in equation (31). After some algebra,

$$\begin{aligned}\left\{\frac{\partial S_{1/2}}{\partial \ell'}, S_{1/2}\right\} = \alpha c_2 \gamma \Bigg[&(\Delta - 1)\left(\frac{1}{\Delta} - 1\right) + k^2 (I_0 - \theta) \frac{\sin\theta \cos\theta}{\Delta} \\ &+ 4\alpha c_1 \gamma \; (\Delta - 1)^2 + c_2 x \gamma \; (\Delta - 1)\left(\Delta - \frac{1}{\Delta}\right) \\ &+ (I_2 - I_0) \; c_2 \gamma x k^2 \; \frac{\sin\theta\cos\theta}{\Delta} + 4\alpha c_1 \gamma k^2 \; (I_2 - \theta) \frac{\sin\theta\cos\theta}{\Delta} \Bigg] \quad ,\end{aligned} \tag{48}$$

where c_2, γ, k, and Δ are as defined earlier, and

$$c_1 = \frac{L'^3}{3\mu^2 \alpha} \quad ,$$

$$x = \frac{1}{2}\left(d a_2 + \alpha a_1 + \beta a_3 - \frac{4}{L'} \alpha\right) \quad , \tag{49}$$

$$d = \beta + p_4 \quad .$$

The equation that gives X and Z_1 is then

$$X = \gamma\Delta \frac{\partial Z_1}{\partial \ell'} + \frac{\alpha c_2 \gamma^2 \Delta}{2} \left[(\Delta - 1)\left(\frac{1}{\Delta} - 1\right) + k^2 (I_0 - \theta) \frac{\sin\theta\cos\theta}{\Delta} \right.$$

$$+ 4\alpha c_1 \gamma (\Delta - 1)^2 + c_2 x\gamma (\Delta - 1)\left(\Delta - \frac{1}{\Delta}\right) + c_2 x\gamma k^2 (I_2 - I_0) \frac{\sin\theta\cos\theta}{\Delta}$$

$$\left. + \frac{1}{3} \frac{\partial^2 \gamma}{\partial L^2} \alpha^3 c_2^2 \gamma \left(\frac{\Delta - 1}{\Delta}\right)^3 + 4\alpha c_1 \gamma k^2 (I_2 - \theta) \frac{\sin\theta\cos\theta}{\Delta} \right] \quad . \tag{50}$$

Since $(\Delta - 1)$ is periodic in θ, we do not find any constant terms:

$$X = 0 \quad .$$

We assumed earlier that SR_1 was a function of θ, and we will now assume that Z is a function of θ only. Therefore,

$$\frac{\partial Z_1}{\partial \ell'} = \frac{\partial Z_1}{\partial \theta} \frac{\partial \theta}{\partial \ell'} = \frac{\alpha}{2} \frac{\partial Z_1}{\partial \theta} \quad ,$$

and the following equation gives Z_1:

$$\frac{\partial Z_1}{\partial \theta} = -c_2 \gamma \left[(\Delta - 1)\left(\frac{1}{\Delta} - 1\right) + k^2 (I_0 - \theta) \frac{\sin\theta\cos\theta}{\Delta} + 4\alpha c_1 \gamma (\Delta - 1)^2 \right.$$

$$+ c_2 x\gamma (\Delta - 1)\left(\Delta - \frac{1}{\Delta}\right) + c_2 \gamma x k^2 (I_2 - I_0) \frac{\sin\theta\cos\theta}{\Delta}$$

$$\left. + 4\alpha c_1 \gamma k^2 (I_2 - \theta) \frac{\sin\theta\cos\theta}{\Delta} - \frac{4}{3} \alpha c_1 \gamma \left(\Delta^2 - 3\Delta + 3 - \frac{1}{\Delta}\right) \right] \quad . \tag{51}$$

In order to integrate this equation and obtain Z_1, the following expressions are useful:

$$\int \frac{x \sin x \cos x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = - \frac{x \Delta(x)}{k^2} + \frac{I_2(x)}{k^2} \quad ,$$

$$\int \frac{I_2(x) \sin x \cos x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = - \frac{\Delta(x) \, I_2}{k^2} + \frac{1}{k^2} \left(x - \frac{k^2}{2} x + \frac{k^2}{4} \sin 2x \right) \quad , \tag{52}$$

$$\int \frac{I_0(x) \sin x \cos x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = - \frac{I_0 \Delta + x}{k^2} \quad .$$

Thus, by setting

$$q_1 = \alpha c_1 \gamma \qquad \text{and} \qquad q_2 = c_2 \gamma x \quad ,$$

we obtain

$$\begin{aligned} Z_1 = c_2 \gamma \Bigg\{ & I_2 (2 + q_2 + 8 q_1) - \theta \left[3 + \frac{20}{3} q_1 - \frac{k^2}{2} \left(\frac{20}{3} q_1 + 2 q_2 \right) \right] \\ & - \frac{k^2}{4} \sin 2\theta \left(\frac{20}{3} q_1 + 2 q_2 \right) + \Delta I_2 (q_2 + 4 q_1) - \Delta\theta \, (1 + 4 q_1) \\ & + I_0 \left(1 - q_2 - \frac{4}{3} q_1 \right) + \Delta I_0 (1 - q_2) \Bigg\} \quad . \end{aligned} \tag{53}$$

Then the derivatives of Z_1 with respect to the variables L, G, H, ℓ, g, and h are

$$\begin{aligned} \frac{\partial Z_1}{\partial X} = c_2 \gamma \Bigg\{ & \frac{\partial I_2}{\partial X} \left[2 + q_2 + 8 q_1 + \Delta (q_2 + 4 q_1) \right] \\ & - \frac{\partial \theta}{\partial X} \left[3 + \frac{20}{3} q_1 - \frac{k^2}{2} \left(\frac{20}{3} q_1 + 2 q_2 \right) + \Delta (1 + 4 q_1) \right] \\ & + \frac{\partial I_0}{\partial X} \left[1 - q_2 - \frac{4}{3} q_1 + \Delta (1 - q_2) \right] \\ & + \frac{\partial \Delta}{\partial X} \left[I_2 (q_2 + 4 q_1) - \theta (1 + 4 q_1) + I_0 (1 - q_2) \right] \\ & - \left(\frac{20}{3} q_1 + 2 q_2 \right) \left(\frac{k^2}{2} \frac{\partial \theta}{\partial X} \cos 2\theta + \frac{1}{4} \frac{\partial k^2}{\partial X} \sin 2\theta \right) \\ & + \frac{\partial q_2}{\partial X} \left(I_2 + k^2 \theta - \frac{k^2}{2} \sin 2\theta + \Delta I_2 - I_0 - \Delta I_0 \right) \Bigg\} \end{aligned} \tag{54}$$

[eq. (54) cont. on next page]

$$+ \epsilon\left(- c_2 \gamma \frac{\partial q_1}{\partial L'}\left[8 I_2 - \frac{20}{3}\left(1 - \frac{k^2}{2}\right)\theta - \frac{20}{3}\frac{k^2}{4}\sin 2\theta + 4\Delta I_2 - 4\Delta\theta - \frac{4}{3} I_0\right]\right.$$

$$+ \left(4 c_1 \gamma + \frac{1}{\alpha}\right)\left\{I_2(2 + q_2 + 8 q_1) - \theta\left[3 + \frac{20}{3} q_1 - \frac{k^2}{2}\left(\frac{20}{3} q_1 + 2 q_2\right)\right]\right.$$

$$- \frac{k^2}{4}\sin 2\theta\left(\frac{20}{3} q_1 + 2 q_2\right) + \Delta I_2 (q_2 + 4 q_1)$$

$$\left.\left. - \Delta\theta (1 + 4 q_1) + I_0\left(1 - q_2 - \frac{4}{3} q_1\right) + \Delta I_0 (1 - q_2)\right\}\right) \quad (54)$$

for $X = L, G, H, \ell, g,$ and h, where

$$\epsilon = 1 \quad \text{if} \quad X = L' \; ,$$

$$\epsilon = 0 \quad \text{if} \quad X \neq L' \; ;$$

$$\frac{\partial \Delta}{\partial X} = \frac{a_x}{2}\left(\Delta - \frac{1}{\Delta}\right) - \frac{k^2 \sin\theta \cos\theta}{\Delta}\frac{\partial\theta}{\partial X} \quad ; \quad (55a)$$

$$\frac{\partial I_2}{\partial X} = \frac{\partial\theta}{\partial X}\Delta + \frac{a_x}{2}(I_2 - I_0) \quad ; \quad (55b)$$

$$\frac{\partial I_0}{\partial X} = \frac{1}{\Delta}\frac{\partial\theta}{\partial X} + \frac{a_x}{2}(I_{-2} - I_0) \quad ,$$

with

$$I_{-2}(\theta) = \int_0^\theta \frac{dx}{\Delta^3} = \frac{1}{1 - k^2}\left(I_2 - k^2 \frac{\sin\theta\cos\theta}{\Delta}\right) \quad ; \quad (55c)$$

$$\frac{\partial q_2}{\partial X} = c_2 \gamma \frac{\partial x}{\partial X} + \epsilon\left(4 c_1 \gamma + \frac{1}{\alpha}\right) x \quad ; \quad (55d)$$

$$\frac{\partial\theta}{\partial X} = \begin{cases} \dfrac{\partial\theta}{\partial L'} = \dfrac{p_1}{2} \; , & \dfrac{\partial\theta}{\partial \ell'} = \dfrac{\alpha}{2} \; , \\[2ex] \dfrac{\partial\theta}{\partial G'} = \dfrac{p_2}{2} \; , & \dfrac{\partial\theta}{\partial g'} = \dfrac{p_4}{2} \; , \\[2ex] \dfrac{\partial\theta}{\partial H'} = \dfrac{p_3}{2} \; , & \dfrac{\partial\theta}{\partial h'} = \dfrac{\beta}{2} \; ; \end{cases} \quad (55e)$$

$$a_x = \begin{cases} 0 & \text{if} \quad X = \ell' \text{ or } h' \\ a_1 - \frac{4}{L'} - \frac{2}{\alpha} & \text{if} \quad X = L' \\ a_2 & \text{if} \quad X = G' \\ a_3 & \text{if} \quad X = H' \\ a_4 & \text{if} \quad X = g' \quad ; \end{cases} \tag{55f}$$

and

$$\frac{\partial x}{\partial X} = \frac{1}{2}\left(\alpha \frac{\partial a_1}{\partial X} + \beta \frac{\partial a_3}{\partial X} + d \frac{\partial a_2}{\partial X} + \beta \frac{\partial p_4}{\partial X} + 4\epsilon \frac{\alpha}{L^2}\right) \quad . \tag{55g}$$

4.3 Expression of Z_2 and Its Derivatives

From equation (32) and from the fact that $\{F_0, SJ_1\} = -F_{1p}$, we can derive Z_2 as follows:

$$Y = \{F_1, S_{1/2}\} - \frac{1}{2}\{F_{1p}, S_{1/2}\} + \gamma\Delta \frac{\partial Z_2}{\partial \ell} + \frac{1}{2} \frac{\partial \gamma}{\partial L} \frac{\partial S_{1/2}}{\partial \ell} \frac{\partial SJ}{\partial \ell} \quad . \tag{56}$$

Here we can neglect F_{1p}, the periodic part of F_1, as it is of order 1 in eccentricity and therefore very small for the satellites we are dealing with. We then have

$$Y = \left\{F_{1s}, S_{1/2}\right\} + \gamma\Delta \frac{\partial Z_2}{\partial \ell} \quad . \tag{57}$$

Since there are no constant terms,

$$Y = 0 \quad ,$$

and therefore,

$$\gamma\Delta \frac{\partial Z_2}{\partial \ell} = -\left(b_1 \frac{\partial S_{1/2}}{\partial \ell} + b_2 \frac{\partial S_{1/2}}{\partial g} + b_3 \frac{\partial S_{1/2}}{\partial h}\right) \quad , \tag{58}$$

where

$$b_1 = \frac{\partial F_{1s}}{\partial L} = -\frac{3 J_2 \mu^4 R^2}{G^3 L^4}\left(-\frac{1}{4} + \frac{3}{4}\frac{H^2}{G^2}\right) ,$$

$$b_2 = \frac{\partial F_{1s}}{\partial G} = -\frac{3 J_2 \mu^4 R^2}{L^3 G^4}\left(-\frac{1}{4} + \frac{5}{4}\frac{H^2}{G^2}\right) ,$$

$$b_3 = \frac{\partial F_{1s}}{\partial H} = +\frac{3}{2}\frac{J_2 \mu^4 R^2 H}{L^3 G^5} ,$$

and, since Z_2 is assumed to be a function of θ,

$$\frac{\partial Z_2}{\partial \theta} = -\frac{2c_2}{\alpha}\left[(b_1\alpha + b_2\beta + b_3 d)\left(\frac{\Delta - 1}{\Delta}\right) + b_2 a_4\left(\frac{I_2 - I_0}{\Delta}\right)\right] . \tag{59}$$

If we write

$$B_1 = -\frac{2c_2}{\alpha}(b_1\alpha + b_2 d + b_3\beta) ,$$
$$B_2 = -\frac{2c_2}{\alpha} b_2 a_4 , \tag{60}$$

we can obtain Z_2 from equation (59):

$$Z_2 = B_1(\theta - I_0) + B_2 \int_0^\theta \frac{I_2(x) - I_0(x)}{\Delta(x)}\, dx . \tag{61}$$

Now we can write the integrals (Gradshteyn and Ryzhik, 1966)

$$\int_0^\theta \frac{I_0(x)}{\Delta(x)}\, dx = \frac{I_0^2}{2} ,$$
$$\int_0^\theta \frac{I_2(x)}{\Delta(x)}\, dx = \frac{I_0^2 E}{2K} + \log\frac{\Theta(I_0)}{\Theta(0)} , \tag{62}$$

where E and K are complete elliptic integrals of the first and second kind, respectively, and Θ is the following theta function:

$$\Theta(u) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cos(2mv) \quad ,$$

in which

$$q = \exp\left(\frac{-\pi k'}{k}\right) \quad , \qquad k' = \sqrt{1-k^2} \quad , \tag{63a}$$

and

$$v = \frac{\pi u}{2K} \quad . \tag{63b}$$

The derivatives of Z_2, then, are

$$\begin{aligned} \frac{\partial Z_2}{\partial X} &= B_1 \left(\frac{\partial \theta}{\partial X} - \frac{\partial I_0}{\partial X}\right) + \frac{\partial B_1}{\partial X} (\theta - I_0) \\ &+ \frac{\partial B_2}{\partial X} \int_0^{\theta} \frac{(I_2 - I_0)(x)}{\Delta(x)} dx + B_2 \frac{\partial \theta}{\partial X} \left(\frac{I_2 - I_0}{\Delta}\right) + B_2 \mathscr{A}(X) \quad , \end{aligned} \tag{64}$$

where $X = L', G', H', \ell', g', h'$ and, if $k'^2 = 1 - k^2$,

$$\begin{aligned} \mathscr{A}(X) &= \frac{a_X}{2k'^2} \left[\frac{I_0^2 k'^2}{2} + \frac{I_2^2}{2} - I_0 I_2 - (I_2 - I_0) \frac{k^2 \sin\theta \cos\theta}{\Delta} + \frac{k^2 \sin^2\theta}{2}\right] \quad ; \\ \frac{\partial B_1}{\partial X} &= -\frac{2c_2}{\alpha} \left(\frac{\partial b_1}{\partial X} \alpha + \frac{\partial b_2}{\partial X} d + \frac{\partial p_4}{\partial X} b_2 + \frac{\partial b_3}{\partial X} \beta\right) - \frac{\epsilon}{L'} B_1 \quad ; \\ \frac{\partial B_2}{\partial X} &= -\frac{2c_2}{\alpha} \left(\frac{\partial b_2}{\partial X} a_4 + b_2 \frac{\partial a_4}{\partial X}\right) - \frac{\epsilon}{L'} B_2 \quad , \end{aligned} \tag{65a}$$

where

$$\epsilon = 0 \quad \text{if} \quad X \neq L' \quad ,$$

$$\epsilon = 1 \quad \text{if} \quad X = L' \quad ;$$

$$\frac{\partial p_4}{\partial X} = 0 \quad \text{if} \quad X = \ell' \text{ or } h' \quad ;$$

$$\frac{\partial p_4}{\partial L'}, \frac{\partial p_4}{\partial G'}, \frac{\partial p_4}{\partial H'}, \text{ and } \frac{\partial p_4}{\partial g'} \quad \text{are given in Appendix A} \quad ;$$

$$\frac{\partial b_1}{\partial X} = \frac{\partial b_2}{\partial X} = \frac{\partial b_3}{\partial X} = 0 \quad \text{if} \quad X = \ell' \text{ or } h' \text{ or } g' \quad ;$$

$$\frac{\partial b_1}{\partial L'} = -\frac{4b_1}{L'} \quad ;$$

$$\frac{\partial b_2}{\partial L'} = -\frac{3}{L'} b_2 \quad ;$$

$$\frac{\partial b_3}{\partial L'} = -\frac{3}{L'} b_3 \quad ;$$

$$\frac{\partial b_1}{\partial G'} = -\frac{3b_1}{G'} + \frac{9}{2} \frac{J_2 \mu^4 R^2 H'^2}{L'^4 G'^6} \quad ; \tag{65b}$$

$$\frac{\partial b_2}{\partial G'} = -\frac{4b_1}{G'} + \frac{15}{2} \frac{J_2 \mu^4 R^2 H'^2}{L'^3 G'^7} \quad ;$$

$$\frac{\partial b_3}{\partial G'} = -\frac{5b_3}{G'} \quad ;$$

$$\frac{\partial b_1}{\partial H'} = -\frac{9}{2} \frac{J_2 \mu^4 R^2 H'}{L'^4 G'^5} \quad ;$$

$$\frac{\partial b_2}{\partial H'} = -\frac{15}{2} \frac{J_2 \mu^4 R^2 H'}{L'^3 G'^6} \quad ;$$

$$\frac{\partial b_3}{\partial H'} = \frac{3}{2} \frac{J_2 \mu^4 R^2}{L'^3 G'^5} \quad .$$

We also need the second derivatives of Z_2 that appear in expressions such as $\{\partial Z_2/\partial X, S_{1/2}\}$ for $X = L', G', H', \ell', g', h'$, as they are used in the final expression of the perturbations. Their equations follow:

$$\frac{\partial^2 Z_2}{\partial X\,\partial Y} = \frac{\partial B_1}{\partial Y}\left(\frac{\partial S}{\partial X} - \frac{\partial I_0}{\partial X}\right) + \frac{\partial^2 B_1}{\partial X\,\partial Y}(\theta - I_0) + B_1\left(\frac{\partial^2 \theta}{\partial X\,\partial Y} - \frac{\partial^2 I_0}{\partial X\,\partial Y}\right)$$
$$+ \frac{\partial^2 B_2}{\partial X\,\partial Y}\int_0^\theta \frac{I_2 - I_0}{\Delta}\,dx + \frac{\partial B_2}{\partial X}\mathcal{A}(Y) + \frac{\partial B_2}{\partial Y}\frac{\partial \theta}{\partial X}\left(\frac{I_2 - I_0}{\Delta}\right)$$
$$+ B_2\frac{\partial^2 \theta}{\partial X\,\partial Y}\left(\frac{I_2 - I_0}{\Delta}\right) + B_2\frac{\partial \theta}{\partial X}\left[\frac{(\partial I_2/\partial Y) - (\partial I_0/\partial Y)}{\Delta} - \frac{1}{\Delta^2}\frac{\partial \Delta}{\partial Y}(I_2 - I_0)\right]$$
$$+ \frac{\partial B_2}{\partial Y}\mathcal{A}(X) + B_2\int_0^\theta \frac{\partial^2}{\partial Y\,\partial X}\left(\frac{I_2 - I_0}{\Delta}\right)dx \quad , \tag{66}$$

where

$$\frac{\partial^2 \theta}{\partial X\,\partial Y} = \frac{1}{2}\frac{\partial^2 \psi}{\partial X\,\partial Y} \qquad \text{if} \quad X \text{ and } Y \text{ are } L', G', H', \text{ or } g',$$

[see eqs. (A-9) in Appendix A]

$$\frac{\partial^2 \theta}{\partial X\,\partial Y} = 0 \qquad \text{if} \quad (X \text{ or } Y) \text{ is } (\ell' \text{ or } h') \quad ;$$

$$\frac{\partial^2 B_1}{\partial X\,\partial Y} = -\frac{2c_2}{\alpha}\left(\frac{\partial^2 b_1}{\partial X\,\partial Y}\alpha + \frac{\partial^2 b_2}{\partial X\,\partial Y}d + \frac{\partial b_2}{\partial X}\frac{\partial p_4}{\partial Y} + \frac{\partial p_4}{\partial X}\frac{\partial b_2}{\partial Y}\right.$$
$$\left. + \frac{\partial^2 p_4}{\partial X\,\partial Y}b_2 + \frac{\partial^2 b_3}{\partial X\,\partial Y}\beta\right) - \frac{\epsilon}{L'}\frac{\partial B_1}{\partial Y}$$
$$+ \eta\left[-\frac{4c_1}{\alpha}\left(\frac{\partial b_1}{\partial X}\alpha + \frac{\partial b_2}{\partial X}d + \frac{\partial p_4}{\partial X}b_2 + \frac{\partial b_3}{\partial X}\beta\right) + \frac{\epsilon}{L'^2}B_1\right] \quad ,$$

in which

$$\begin{array}{llll} \epsilon = 0 & \text{if} & X \neq L' & , \\ \epsilon = 1 & \text{if} & X = L' & , \\ \eta = 0 & \text{if} & Y \neq L' & , \\ \eta = 1 & \text{if} & Y = L' & ; \end{array}$$

$$\frac{\partial^2 b_1}{\partial X\, \partial Y} = \frac{\partial^2 b_2}{\partial X\, \partial Y} = \frac{\partial^2 b_3}{\partial X\, \partial Y} = 0 \qquad \text{if} \qquad (X \text{ or } Y) = (\ell' \text{ or } g' \text{ or } h') \quad ;$$

$$\frac{\partial^2 b_1}{\partial L'^2} = \frac{20}{L'^2}\, b_1 \quad ;$$

$$\frac{\partial^2 b_2}{\partial L'^2} = \frac{12}{L'^2}\, b_2 \quad ;$$

$$\frac{\partial^2 b_3}{\partial L'^2} = \frac{12}{L'^2}\, b_3 \quad ;$$

$$\frac{\partial^2 b_1}{\partial L'\, \partial G'} = \frac{\partial^2 b_1}{\partial G'\, \partial L'} = -\frac{4}{L'}\, \frac{\partial b_1}{\partial G'} \quad ;$$

$$\frac{\partial^2 b_1}{\partial L'\, \partial H'} = \frac{\partial^2 b_1}{\partial H'\, \partial L'} = -\frac{4}{L'}\, \frac{\partial b_1}{\partial H'} \quad ;$$

$$\frac{\partial^2 b_1}{\partial G'^2} = \frac{3}{G'^2}\, b_1 - \frac{3}{G'}\, \frac{\partial b_1}{\partial G'} - \frac{27\, J_2\, \mu^4 R^2 H'^2}{L'^4\, G'^7} \quad ; \tag{67}$$

$$\frac{\partial^2 b_1}{\partial G'\, \partial H'} = \frac{\partial^2 b_1}{\partial H'\, \partial G'} = \frac{45}{2}\, \frac{J_2\, \mu^4 R^2 H'}{L'^4\, G'^5} \quad ;$$

$$\frac{\partial^2 b_1}{\partial H'^2} = -\frac{9}{2}\, \frac{J_2\, \mu^4 R^2}{L'^4\, G'^5} \quad ;$$

$$\frac{\partial^2 b_2}{\partial L'\, \partial G'} = \frac{\partial^2 b_2}{\partial G'\, \partial L'} = -\frac{3}{L'}\, \frac{\partial b_2}{\partial G'} \quad ;$$

$$\frac{\partial^2 b_2}{\partial L'\, \partial H'} = \frac{\partial^2 b_2}{\partial H'\, \partial L'} = -\frac{3}{L'}\, \frac{\partial b_2}{\partial H'} \quad ;$$

$$\frac{\partial^2 b_2}{\partial G'^2} = -\frac{105}{2}\, \frac{J_2\, \mu^4 R^2 H'^2}{L'^3\, G'^8} - \frac{4}{G'}\, \frac{\partial b_2}{\partial G'} + \frac{4}{G'^2}\, b_2 \quad ;$$

[eq. (67) cont. on next page]

$$\frac{\partial^2 b_2}{\partial G' \, \partial H'} = \frac{\partial^2 b_2}{\partial H' \, \partial G'} = - \frac{4}{G'} \, \frac{\partial b_2}{\partial H'} + \frac{15 \, J_2 \, \mu^4 R^2 H'}{L'^3 \, G'^7} \quad ;$$

$$\frac{\partial^2 b_2}{\partial H'^2} = - \frac{15}{2} \, \frac{J_2 \, \mu^4 R^2}{L'^3 \, G'^5} \quad ;$$

$$\frac{\partial^2 b_3}{\partial L' \, \partial G'} = \frac{\partial^2 b_3}{\partial G' \, \partial L'} = - \frac{3}{L'} \, \frac{\partial b_3}{\partial G'} \quad ;$$

$$\frac{\partial^2 b_3}{\partial L' \, \partial H'} = \frac{\partial^2 b_3}{\partial H' \, \partial L'} = - \frac{3}{L'} \, \frac{\partial b_3}{\partial H'} \quad ;$$

$$\frac{\partial^2 b_3}{\partial G'^2} = \frac{30}{G'^2} \, b_3 \quad ;$$

$$\frac{\partial^2 b_3}{\partial G' \, \partial H'} = \frac{\partial^2 b_3}{\partial H' \, \partial G'} = - \frac{5}{G'} \, \frac{\partial b_3}{\partial H'} \quad ; \tag{67}$$

$$\frac{\partial^2 b_3}{\partial H'^2} = 0 \quad ;$$

$$\begin{aligned} \frac{\partial^2 I_0}{\partial X \, \partial Y} = {} & \frac{\partial^2 \psi}{\partial X \, \partial Y} \left(\frac{1}{2\Delta} \right) - \frac{1}{2\Delta^2} \, \frac{\partial \Delta}{\partial Y} \, \frac{\partial \psi}{\partial X} + \frac{1}{2} \, \frac{\partial a_x}{\partial Y} \, (I_{-2} - I_0) \\ & + \frac{a_x}{2} \left[- \frac{\partial I_0}{\partial Y} + \frac{1}{k'^2} \left(\frac{\partial I_2}{\partial Y} - \frac{\partial k^2}{\partial Y} \, \frac{\sin\theta \cos\theta}{\Delta} + \frac{k^2 \sin\theta \cos\theta}{\Delta^2} \, \frac{\partial \Delta}{\partial Y} \right. \right. \\ & \left. \left. - k^2 \, \frac{\cos 2\theta}{4\Delta} \, \frac{\partial \psi}{\partial Y} \right) + \frac{1}{k'^4} \, \frac{\partial k^2}{\partial Y} \left(I_2 - \frac{k^2 \sin\theta \cos\theta}{\Delta} \right) \right] \quad ; \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 B_2}{\partial X \, \partial Y} = {} & - \frac{2 c_2}{\alpha} \left(\frac{\partial^2 b_2}{\partial X \, \partial Y} \, a_4 + \frac{\partial b_2}{\partial X} \, \frac{\partial a_4}{\partial Y} + \frac{\partial b_2}{\partial Y} \, \frac{\partial a_4}{\partial X} \right) - \frac{\epsilon}{L'} \, \frac{\partial B_2}{\partial Y} \\ & + \eta \left[- \frac{8 c_1}{\alpha} \left(\frac{\partial b_2}{\partial X} \, a_4 + b_2 \, \frac{\partial a_4}{\partial X} \right) + \frac{\epsilon}{L'^2} \, B_2 \right] \quad . \end{aligned}$$

Note: In the expression $\partial^2 Z_2/\partial X\,\partial Y$, we have neglected the term

$$B_2 \int_0^{\theta} \frac{\partial^2}{\partial Y\,\partial X}\left(\frac{I_2 - I_0}{\Delta}\right) dx \quad ,$$

which is a very small quantity.

We also give the second derivatives of $S_{1/2}$, so that we can calculate Poisson brackets:

$$\left\{\frac{\partial S_{1/2}}{\partial X}, S_{1/2}\right\} \quad , \qquad \left\{\frac{\partial S_{1/2}}{\partial X}, Z_1\right\} \quad , \qquad \left\{\frac{\partial S_{1/2}}{\partial X}, Z_2\right\} \quad , \quad \ldots \quad .$$

If the first derivatives of $S_{1/2}$ are in compact form,

$$\frac{\partial S_{1/2}}{\partial X} = 2\,c_2\,\gamma\left(\frac{\partial I_2}{\partial X} - \frac{\partial \theta}{\partial X}\right) + \epsilon(I_2 - \theta)\left(8\,c_1\,\gamma + \frac{2}{\alpha}\right) \quad ,$$

where

$$\epsilon = 0 \;\text{ if } X \neq L' \qquad \text{and} \qquad \epsilon = 1 \;\text{ if } X = L' \quad ,$$

then

$$\frac{\partial^2 S_{1/2}}{\partial X\,\partial Y} = 2\,c_2\,\gamma\left(\frac{\partial^2 I_2}{\partial X\,\partial Y} - \frac{\partial^2 \theta}{\partial X\,\partial Y}\right) + \epsilon\left(\frac{\partial I_2}{\partial Y} - \frac{\partial \theta}{\partial Y}\right)\left(8\,c_1\,\gamma + \frac{2}{\alpha}\right)$$
$$+ \eta\left[\left(8\,c_1\,\gamma + \frac{2}{\alpha}\right)\left(\frac{\partial I_2}{\partial X} - \frac{\partial \theta}{\partial X}\right) + 8\epsilon\,(I_2 - \theta)\left(\frac{L'^2}{\mu_\alpha^2}\,\gamma + \frac{1}{\alpha L'}\right)\right] \quad ,$$

where

$$\eta = 0 \;\text{ if } Y \neq L' \qquad \text{and} \qquad \eta = 1 \;\text{ if } Y = L' \quad ,$$

and

$$\frac{\partial^2 I_2}{\partial X\,\partial Y} = \frac{1}{2}\,\frac{\partial a_x}{\partial Y}\,(I_2 - I_0) + \frac{\Delta}{2}\,\frac{\partial p_x}{\partial Y} + \frac{p_x}{2}\,\frac{\partial \Delta}{\partial Y} + \frac{a_x}{2}\left(\frac{\partial I_2}{\partial Y} - \frac{\partial I_0}{\partial Y}\right) \quad .$$

5. REMOVAL OF LONG-PERIOD TERMS

If we do not assume that g' is constant, as we did in Section 4, then, after eliminating ℓ' and h', we are left with the following system of equations:

$$
\begin{aligned}
\frac{dL'}{dt} &= \frac{\partial F^*}{\partial \ell'} \quad , & \frac{d\ell'}{dt} &= -\frac{\partial F^*}{\partial L'} \quad , \\
\frac{dG'}{dt} &= \frac{\partial F^*}{\partial g'} \quad , & \frac{dg'}{dt} &= -\frac{\partial F^*}{\partial G'} \quad , \\
\frac{dH'}{dt} &= \frac{\partial F^*}{\partial h'} \quad , & \frac{dh'}{dt} &= -\frac{\partial F^*}{\partial H'} \quad ,
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
F^* &= F_0^* + F_1^* + F_2^* \quad , \\
F_0^* &= \frac{\mu^2}{2L'^2} + \nu H' \quad , \\
F_1^* &= F_1^*(L', G', H') \quad , \\
F_2^* &= -\sum_{k_1} D(k_1) = F_2^*(L', G', H', g') \quad .
\end{aligned}
\tag{69}
$$

We now proceed to remove g' from the Hamiltonian, by performing a new canonical transformation using Hori's method.

We introduce a determining function S' and a new Hamiltonian F^{**} such that, if L'', G'', H'', ℓ'', g'', and h'' are the new variables,

$$
\begin{aligned}
F_0^{**} &= F_0^*(L'', H'') \quad , \\
F_1^{**} &= F_1^*(L'', G'', H'') \quad ,
\end{aligned}
$$

and

$$\{F_0^*, S'\} + F_2^* = F_2^{**} \quad .$$

Introducing a pseudotime t^{**}, defined by

$$\frac{dg''}{dt^{**}} = -\frac{\partial F_1^*}{\partial G''}(L'', G'', H'')$$

or

$$dt^{**} = \rho \, dg'' \qquad \text{with} \quad \rho = \left(-\frac{\partial F_1^*}{\partial G''}\right)^{-1} \quad ,$$

we can solve for F_2^{**} and S':

$$F_2^{**} = F_{2s}^* \quad ,$$

$$\{F_0^*, S'\} = -F_{2p}^* \Longrightarrow S' = \int F_{2p}^* \, dt^{**} \quad .$$

Let us find the secular and periodic parts of F_2^{**}. By using expressions for only one fixed value of k_1, the complete expressions of F_2^{**} and S' are then

$$F_2^{**} = \sum_{k_1} F_2^{**}(k_1) \quad , \qquad S' = \sum_{k_1} S'(k_1) \quad .$$

We can assume that k_1 is fixed and can drop the index k_1 in the following derivation. We then have

$$F_2^* = -D \quad ,$$

where

$$D^2 = \left(\sum_i A_i \cos \phi_i\right)^2 + \left(\sum_i A_i \sin \phi_i\right)^2 \quad ,$$

in which

$$A_i = S(k_1, k_0)\, A(k_1, k_0, x) \quad ,$$

$$\phi_i = (k_0 - 2x)\, g - \Lambda_{k_1, k_0} \quad ,$$

and the summations are taken over values of x and k_0 [see eqs. (8) and (9)]. Hence,

$$D^2 = \sum_i A_i^2 + 2 \sum_{i<j} A_i A_j - 4 \sum_{i<j} A_i A_j \sin^2\left(\frac{\phi_i - \phi_j}{2}\right)$$

$$= \left(\sum A_i\right)^2 \left\{1 - \frac{4 \sum A_i A_j \sin^2 [(\phi_i - \phi_j)/2]}{\left(\sum A_i\right)^2}\right\} \quad .$$

To the first order,

$$D = \left(\sum_i A_i\right) \left\{1 - \frac{2 \sum A_i A_j \sin^2 [(\phi_i - \phi_j)/2]}{\left(\sum A_i\right)^2}\right\} \quad ,$$

$$D = \sum_i A_i - \frac{\sum A_i A_j}{\sum_i A_i} + \frac{\sum A_i A_j \cos (\phi_i - \phi_j)}{\sum_i A_i} \quad .$$

Therefore,

$$F_{2s} = \frac{\sum A_i A_j}{\sum A_i} - \sum A_i \quad , \tag{70a}$$

$$F_{2p} = - \frac{\sum A_i A_j \cos (\phi_i - \phi_j)}{\sum A_i} \quad , \tag{70b}$$

and

$$S_2 = -\rho \int \frac{\sum A_i A_j \cos(\phi_i - \phi_j)}{\sum A_i} dg'' = -\frac{\rho}{\sum A_i} \sum_{i<j} \left[\frac{A_i A_j \sin(\phi_i - \phi_j)}{(k_0 - 2x)_i - (k_0 - 2x)_j} \right] . \tag{70c}$$

If we set $(k_0 - 2x)_i - (k_0 - 2x)_j = \alpha_{ij}$, then

$$\frac{\partial S_2}{\partial \ell''} = \frac{\partial S_2}{\partial h''} = 0 \quad ;$$

$$\frac{\partial S_2}{\partial g''} = -\frac{\rho}{\sum_i A_i} \sum_{i<j} A_i A_j \cos(\phi_i - \phi_j) \quad ;$$

and, if $X = L''$, G'', or H'',

$$\frac{\partial S_2}{\partial X} = -\frac{\rho}{\sum A_i} \left[\sum_{i<j} \frac{(A_i A_{j,x} + A'_{i,x} A_j) \sin(\phi_i - \phi_j)}{\alpha_{ij}} \right] + \left[\frac{1}{\sum A_i} \frac{\partial \rho}{\partial X} - \frac{\rho}{\left(\sum A_i\right)^2} \sum A_{i,x} \right] \left[\sum \frac{A_i A_j \sin(\phi_i - \phi_j)}{\alpha_{ij}} \right] , \tag{71}$$

where

$$A_{i,x} = \frac{\partial A_i}{\partial X} = \frac{\partial S(k_1, k_0)}{\partial X} A(k_1, k_0, x) + S(k_1, k_0) \frac{\partial A}{\partial X} (k_1, k_0, x)$$

[see Appendix A, eqs. (A-2)] ;

$$\frac{\partial F_2}{\partial X} = \frac{\sum (A_i A_{j,x} + A_{i,x} A_j)}{\sum A_i} - \sum A_{i,x} - \frac{1}{\left(\sum_i A_i\right)^2} \left(\sum A_i A_j\right) \left(\sum A_{i,x}\right) .$$

The new Hamiltonian becomes

$$F^{**} = F_0^{**} + F_1^{**} + F_2^{**} \quad ,$$

where

$$\begin{aligned}
F_0^{**} &= \frac{\mu^2}{2L''^2} + \nu H'' \quad , \\
F_1^{**} &= F_{1s}\,(L'', G'', H'') \quad , \\
F_2^{**} &= F_{2s}^{*}\,(L'', G'', H'') \quad ;
\end{aligned} \tag{72}$$

and the passage from the old variables $(L', G', H', \ell', g', h')$ to the new ones is given by

$$\begin{aligned}
L' &= L'' + \frac{\partial S'}{\partial \ell''} + \frac{1}{2}\left\{\frac{\partial S'}{\partial \ell''}, S'\right\} + \ldots \quad , \\
G' &= G'' + \frac{\partial S'}{\partial g''} + \frac{1}{2}\left\{\frac{\partial S'}{\partial g''}, S'\right\} + \ldots \quad , \\
H' &= H'' + \frac{\partial S'}{\partial h''} + \frac{1}{2}\left\{\frac{\partial S'}{\partial h''}, S'\right\} + \ldots \quad , \\
\ell' &= \ell'' - \frac{\partial S'}{\partial L''} - \frac{1}{2}\left\{\frac{\partial S'}{\partial L''}, S'\right\} - \ldots \quad , \\
g' &= g'' - \frac{\partial S'}{\partial G''} - \frac{1}{2}\left\{\frac{\partial S'}{\partial G''}, S'\right\} - \ldots \quad , \\
h' &= h'' - \frac{\partial S'}{\partial H''} - \frac{1}{2}\left\{\frac{\partial S'}{\partial H''}, S'\right\} - \ldots \quad .
\end{aligned} \tag{73}$$

The new system of equations is given by

$$\begin{aligned}
\frac{dL''}{dt} &= \frac{\partial F^{**}}{\partial \ell''} \quad , & \frac{d\ell''}{dt} &= -\frac{\partial F^{**}}{\partial L''} \quad , \\
\frac{dG''}{dt} &= \frac{\partial F^{**}}{\partial g''} \quad , & \frac{dg''}{dt} &= -\frac{\partial F^{**}}{\partial G''} \quad , \\
\frac{dH''}{dt} &= \frac{\partial F^{**}}{\partial h''} \quad , & \frac{dh''}{dt} &= -\frac{\partial F^{**}}{\partial H''} \quad ,
\end{aligned} \tag{74}$$

and, since $F^{**} = F^{**}(L'', G'', H'')$, the solution is

$$
\begin{aligned}
L'' &= L''_0 = \text{constant} \quad , \\
G'' &= G''_0 = \text{constant} \quad , \\
H'' &= H''_0 = \text{constant} \quad , \\
\ell'' &= \ell''_0 + \ell''_1 t \quad , \\
g'' &= g''_0 + g''_1 t \quad , \\
h'' &= h''_0 + h''_1 t \quad ,
\end{aligned}
\tag{75}
$$

where ℓ''_0, g''_0, h''_0, ℓ''_1, g''_1, h''_1 are constants and

$$
\begin{aligned}
\ell''_1 &= -\frac{\partial F^{**}}{\partial L''} = \frac{\mu^2}{L''^3} - \frac{\partial F_{1s}}{\partial L''} - \frac{\partial F^*_{2s}}{\partial L''} \quad , \\
g''_1 &= -\frac{\partial F^{**}}{\partial G''} = -\frac{\partial F_{1s}}{\partial G''} - \frac{\partial F^*_{2s}}{\partial G''} \quad , \\
h''_1 &= -\frac{\partial F^{**}}{\partial H''} = -\nu - \frac{\partial F_{1s}}{\partial H''} - \frac{\partial F^*_{2s}}{\partial H''} \quad .
\end{aligned}
\tag{76}
$$

Returning to the initial set of modified Delaunay variables (L, G, H, ℓ, g, h), we now have

$$
\begin{aligned}
L &= L' + \frac{\partial S}{\partial \ell'}(L', G', H', \ell', g', h') + \frac{1}{2}\left\{\frac{\partial S}{\partial \ell'}, S\right\} + \ldots \quad , \\
G &= G' + \frac{\partial S}{\partial g'}(L', G', H', \ell', g', h') + \frac{1}{2}\left\{\frac{\partial S}{\partial g'}, S\right\} + \ldots \quad , \\
H &= H' + \frac{\partial S}{\partial h'}(L', G', H', \ell', g', h') + \frac{1}{2}\left\{\frac{\partial S}{\partial h'}, S\right\} + \ldots \quad , \\
\ell &= \ell' - \frac{\partial S}{\partial L'}(L', G', H', \ell', g', h') - \frac{1}{2}\left\{\frac{\partial S}{\partial L'}, S\right\} - \ldots \quad , \\
g &= g' - \frac{\partial S}{\partial G'}(L', G', H', \ell', g', h') - \frac{1}{2}\left\{\frac{\partial S}{\partial G'}, S\right\} - \ldots \quad , \\
h &= h' - \frac{\partial S}{\partial H'}(L', G', H', \ell', g', h') - \frac{1}{2}\left\{\frac{\partial S}{\partial H'}, S\right\} - \ldots \quad ,
\end{aligned}
\tag{77}
$$

where $S = S_{1/2} + SJ + Z_1 + Z_2$, which, with the help of equations (28), gives the complete solution.

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APPENDIX A

DERIVATIVES OF $D(k_1)$ AND $\psi(k_1)$

A.1 First Derivatives

We recall the following equations from Section 3:

$$D^2(k_1) = B^2(k_1) + C^2(k_1) \quad ,$$

$$D(k_1) \cos \psi(k_1) = B(k_1) \quad , \tag{A-1}$$

$$D(k_1) \sin \psi(k_1) = C(k_1) \quad .$$

Then, if X is any of the Delaunay variables L, G, H, or g,

$$D \frac{\partial D}{\partial X} = B \frac{\partial B}{\partial X} + C \frac{\partial C}{\partial X} \quad .$$

If $X \neq g$,

$$\frac{\partial B}{\partial X} = \sum_{k_0} \sum_{x=0}^{k_1\beta+k_0} \frac{\partial}{\partial X} \left[S(k_1, k_0)\, A(k_1, k_0, x)\right] \cos \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \quad ,$$

$$\frac{\partial C}{\partial X} = \sum_{k_0} \sum_{x=0}^{k_1\beta+k_0} \frac{\partial}{\partial X} \left[S(k_1, k_0)\, A(k_1, k_0, x)\right] \sin \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \quad ,$$

$$\frac{\partial A(k_1, k_0, x)}{\partial X} = \frac{\partial F_{n,m,p}(I)}{\partial I} \frac{\partial I}{\partial X} G(e) + F_{n,m,p}(I) \frac{\partial G_{n,p,q}(e)}{\partial e} \frac{\partial e}{\partial X} \quad , \tag{A-2}$$

$$\frac{\partial S(k_1 k_0)}{\partial X} = \begin{cases} 0 & \text{if} \quad X \neq L \quad , \\ \dfrac{-2\,(\beta k_1 + k_0 + 1)}{L}\, S(k_1, k_0) & \text{if} \quad X = L \quad . \end{cases}$$

If $X = g$,

$$\frac{\partial B}{\partial g} = -\sum_{k_0} \sum_{x=0}^{\beta k_1 + k_0} S(k_1, k_0)\, A(k_1, k_0, x)\, (k_0 - 2x) \sin \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right]$$

$$\frac{\partial C}{\partial g} = \sum_{k_0} \sum_{x=0}^{\beta k_1 + k_0} S(k_1, k_0)\, A(k_1, k_0, x)\, (k_0 - 2x) \cos \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \quad . \tag{A-3}$$

For the derivatives of $\psi(k_1)$, we have

$$C(k_1) = D(k_1) \sin \psi(k_1) \quad ,$$

so that

$$\frac{\partial C(k_1)}{\partial X} = D(k_1) \frac{\partial \psi(k_1)}{\partial X} \cos \psi(k_1) + \frac{\partial D(k_1)}{\partial X} \sin \psi(k_1) \tag{A-4a}$$

and

$$\frac{\partial \psi(k_1)}{\partial X} = \frac{1}{B(k_1)} \left[\frac{\partial C(k_1)}{\partial X} - \frac{1}{D(k_1)} \frac{\partial D(k_1)}{\partial X} C(k_1)\right] \quad . \tag{A-4b}$$

As we did in Section 2, we can now write

$$a_1 = \frac{1}{D_1(k_1)} \frac{\partial D}{\partial L} (k_1) \quad , \qquad a_2 = \frac{1}{D(k_1)} \frac{\partial D}{\partial G} (k_1) \quad ,$$

$$a_3 = \frac{1}{D(k_1)} \frac{\partial D}{\partial H} (k_1) \quad , \qquad a_4 = \frac{1}{D(k_1)} \frac{\partial D}{\partial g} (k_1) \quad ,$$

$$p_1 = \frac{\partial \psi}{\partial L} (k_1) \quad , \qquad p_3 = \frac{\partial \psi}{\partial H} (k_1) \quad ,$$

$$p_2 = \frac{\partial \psi}{\partial G} (k_1) \quad , \qquad p_4 = \frac{\partial \psi}{\partial g} (k_1) \quad .$$

A.2 Second Derivatives

Let us drop the index k_1 for simplicity:

$$\frac{1}{D}\frac{\partial D}{\partial X} = \frac{1}{D^2}\left(B\frac{\partial B}{\partial X} + C\frac{\partial C}{\partial X}\right)$$

$$\frac{\partial}{\partial Y}\left(\frac{1}{D}\frac{\partial D}{\partial X}\right) = -\frac{2}{D^2}\left(\frac{1}{D}\frac{\partial D}{\partial Y}\right)\left(B\frac{\partial B}{\partial X} + C\frac{\partial C}{\partial X}\right)$$

$$+\frac{1}{D^2}\left(\frac{\partial B}{\partial Y}\frac{\partial B}{\partial X} + \frac{\partial C}{\partial Y}\frac{\partial C}{\partial X} + C\frac{\partial^2 C}{\partial X\,\partial Y} + B\frac{\partial^2 B}{\partial X\,\partial Y}\right) \quad . \qquad \text{(A-5)}$$

If $Y \neq g$ and $X \neq g$, then

$$\frac{\partial^2 B}{\partial X\,\partial Y} = \sum_{k_0}\sum_{x}\frac{\partial^2}{\partial X\,\partial Y}(SA)\cos\left[(k_0 - 2x)g - \Lambda_{k_1,k_0}\right] \qquad \text{(A-6a)}$$

and

$$\frac{\partial^2 C}{\partial X\,\partial Y} = \sum_{k_0}\sum_{x}\frac{\partial^2}{\partial X\,\partial Y}(SA)\sin\left[(k_0 - 2x)g - \Lambda_{k_1,k_0}\right] \quad . \qquad \text{(A-6b)}$$

If $X \neq g$ and $Y = g$, we have

$$\frac{\partial^2 B}{\partial X\,\partial g} = -\sum_{k_0}\sum_{x}\frac{\partial SA}{\partial X}(k_0 - 2x)\sin\left[(k_0 - 2x)g - \Lambda_{n,m}\right] \qquad \text{(A-7a)}$$

and

$$\frac{\partial^2 C}{\partial X\,\partial g} = \sum_{k_0}\sum_{x}\frac{\partial SA}{\partial X}(k_0 - 2x)\cos\left[(k_0 - 2x)g - \Lambda_{n,m}\right] \quad . \qquad \text{(A-7b)}$$

If X = g and Y = g, then

$$\frac{\partial^2 B}{\partial g^2} = -\sum_{k_0}\sum_{x} SA\,(k_0 - 2x)^2 \cos\left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \tag{A-8a}$$

and

$$\frac{\partial^2 C}{\partial g^2} = -\sum_{k_0}\sum_{x} SA\,(k_0 - 2x)^2 \sin\left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \;. \tag{A-8b}$$

We find the following second derivative of $\psi(k_1)$ from equations (A-4):

$$\frac{\partial B}{\partial Y}\frac{\partial \psi}{\partial X} + B\frac{\partial^2 \psi}{\partial X\,\partial Y} + \frac{\partial C}{\partial Y}\frac{1}{D}\frac{\partial D}{\partial X} + C\frac{\partial}{\partial Y}\left(\frac{1}{D}\frac{\partial D}{\partial X}\right) = \frac{\partial^2 C}{\partial X\,\partial Y} \;, \tag{A-9a}$$

so that

$$\frac{\partial^2 \psi}{\partial X\,\partial Y} = \frac{1}{B}\left[\frac{\partial^2 C}{\partial X\,\partial Y} - \frac{\partial B}{\partial Y}\frac{\partial \psi}{\partial X} - \frac{1}{D}\frac{\partial D}{\partial X}\frac{\partial C}{\partial Y} - C\frac{\partial}{\partial Y}\left(\frac{1}{D}\frac{\partial D}{\partial X}\right)\right] \;. \tag{A-9b}$$

BIOGRAPHICAL NOTE

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