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APPLICATION OF A FINITE DIFFERENCE TECHNIQUE TO THERMAL WAVE PROPAGATION

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ABSTRACT

A finite difference formulation is presented for thermal wave propagation resulting from periodic heat sources. The numerical technique can handle complex problems that might result from variable thermal diffusivity, such as heat flow in the earth with ice and snow layers. In the numerical analysis, the continuous temperature field is represented by a series of grid points at which the temperature is separated into real and imaginary terms. Next, computer routines previously developed for acoustic wave propagation are utilized in the solution for the temperatures. The calculation procedure is illustrated for the case of thermal wave propagation in a uniform property semi-infinite medium.

INTRODUCTION

Because of increasing concern over environmental heat losses, environmental heat balances are now an integral part of many design procedures. In certain applications, environmental heat transfer calculations can be concerned with the effect of periodic solar temperature variations. An example of such a problem might be the impact of oil pipe lines on the periodic temperature distribution beneath the arctic tundra. In order to analyze problems of this type as well as more general thermal wave propagation problems, a finite difference formulation is developed in this paper.

At present, analytical solutions of thermal wave propagation (refs. 1 and 2) are limited to relatively simple geometries with constant thermal properties or lumped constant temperature regions. The present finite difference technique, however, can handle the heat flow field complications which might arise from (1) variations in thermal properties, (2) complex geometries, or (3) complex boundary conditions. For example, the flow of heat into the earth with covering ice and snow layers could be readily calculated.

The finite difference technique presented herein represents a direct extension of a difference theory developed for noise propagation in ducts (refs. 3 and 4). First, the temperature field is represented by a series of grid points in which the temperature at each grid point is separated into real and imaginary terms. Next, the governing energy equation and the appropriate boundary conditions are present and written in difference form. Finally, computer routines previously developed for acoustic wave propagation are utilized for the solution of the resulting thermal wave difference equations. The calculational procedure is illustrated for the case of thermal wave propagation in a uniform property semi-infinite medium.

SYMBOLS

A	coefficient matrix, eq. (34)
a	thermal diffusivity
C	submatrix, eq. (35)
c	specific heat
F	matrix eq. (34)
i	$\sqrt{-1}$
k	thermal conductivity
m	total number of grid rows
n	total number of grid columns
T	temperature
T_o	amplitude of temperature boundary condition
T_∞	temperature in far field
t	time
X	axial position of exit plane
x	dimensionless axial coordinate, eq. (9)
Δx	axial grid spacing
Y	transverse width of temperature field
y	dimensionless transverse coordinate, eq. (10)
Δy	transverse grid spacing
Z_{exit}	exit impedance, eq. (23)
θ	dimensionless temperature
θ_o	boundary condition
ρ	density

ϕ	spatial temperature field
ϕ_o	boundary condition
ω	circular frequency

Superscripts:

'	dimensional quantity
J	1 or 2
K	2 or 1, $K=J-(-1)^J$
(1)	real part
(2)	imaginary part

Subscripts:

i, j	i, axial index, j, transverse index, see Figure 2
r	reference quantity
1	material #1
2	material #2

GOVERNING EQUATION AND BOUNDARY CONDITIONS

Conduction Equations

The governing differential equation describing the conduction of heat energy in its two dimensional form is given by

$$\frac{\partial}{\partial x'} k\frac{\partial T}{\partial x'} + \frac{\partial}{\partial y'} k\frac{\partial T}{\partial y'} = \rho c\frac{\partial T}{\partial t} \tag{1}$$

where the prime, ', is used to denote a dimensional quantity. Define a dimensionless temperature

$$\theta = \frac{T - T_\infty}{T_o - T_\infty} \tag{2}$$

where T_∞ is the initial temperature of the medium assumed to be uniform and T_o is the amplitude of the temperature variation at the surface or other appropriate convenient temperature. Therefore, the governing equation (1) becomes

$$\frac{\partial}{\partial x'} k\frac{\partial \theta}{\partial x'} + \frac{\partial}{\partial y'} k\frac{\partial \theta}{\partial y'} = \rho c\frac{\partial \theta}{\partial t} \tag{3}$$

Equation (3) can be rewritten as

$$\frac{1}{\rho c a_r} \frac{\partial}{\partial x'} k\frac{\partial \theta}{\partial x'} + \frac{1}{\rho c a_r} \frac{\partial}{\partial y'} k\frac{\partial \theta}{\partial y'} = \frac{1}{a_r} \frac{\partial \theta}{\partial t} \tag{4}$$

where the thermal diffusivity is defined as

$$a = \frac{k}{\rho c} \tag{5}$$

and the reference diffusivity

$$a_r = \frac{k_r}{\rho_r c_r} \tag{6}$$

For a periodic heat source (neglecting the initial transient), the solution for the temperature can be assumed to be of the form

$$\theta(x',y',t) = \phi(x,y)e^{i\omega t} \tag{7}$$

Substituting equations (5) and (7) into equation (4) gives

$$\frac{a}{ka_r}\frac{\partial}{\partial x'}k\frac{\partial\phi}{\partial x'} + \frac{a}{ka_r}\frac{\partial}{\partial y'}k\frac{\partial\phi}{\partial y'} - i\frac{\omega}{a_r}\phi = 0 \tag{8}$$

Finally, the spatial coordinates x' and y' are non-dimensionalized by assuming

$$x' = \frac{2\pi}{\left(\frac{\omega}{2a_r}\right)^{1/2}} x \tag{9}$$

$$y' = \frac{2\pi}{\left(\frac{\omega}{2a_r}\right)^{1/2}} y \tag{10}$$

Thus,

$$\frac{a}{ka_r}\frac{\partial}{\partial x}k\frac{\partial\phi}{\partial x}+\frac{a}{ka_r}\frac{\partial}{\partial y}k\frac{\partial\phi}{\partial y}-i\ 8\pi^2\phi=0 \tag{11}$$

For uniform properties (a equals a_r), equation (11) reduces to

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-i\ 8\pi^2\phi=0 \tag{12}$$

This equation is independent of the frequency parameter ω. The frequency parameter enters the problem by coordinate stretching the dimensionless variables x and y.

Using the exponential notation displayed in equation (7), the dimensionless temperature ϕ has, in general, both real and imaginary parts.

Thus,

$$\phi(x,y)=\phi^{(1)}+i\phi^{(2)} \tag{13}$$

Consequently, equation (12) can be broken into its real and imaginary parts by substituting equation (13) into equation (12),

$$\frac{\partial^2\phi^{(1)}}{\partial x^2}+\frac{\partial^2\phi^{(1)}}{\partial y^2}+8\pi^2\phi^{(2)}=0 \tag{14}$$

$$\frac{\partial^2\phi^{(2)}}{\partial x^2}+\frac{\partial^2\phi^{(2)}}{\partial y^2}-8\pi^2\phi^{(1)}=0 \tag{15}$$

The above two equations can be expressed as a single equation

$$\frac{\partial^2\phi^{(J)}}{\partial x^2} + \frac{\partial^2\phi^{(J)}}{\partial y^2} - (-1)^J 8\pi^2\phi^{(K)} = 0 \tag{16}$$

where J equals 1 or 2 and K equals 2 or 1 respectively.

The solution for temperatures can be found by either solving the complex equation (12) or the set of real equations represented by equation (16). From the standpoint of computer efficiency, the complex equations are the easiest to handle. Equation (16) will only be used to indicate what type of solution schemes are most advantageous. Therefore, any remaining equations or boundary conditions will be left in complex form. Of course, the true physical temperature is represented by only the $\phi^{(1)}$ component.

Boundary Conditions

Surface temperature. - The surface temperature (see Figure 1) treated in this paper assumes

$$\theta(0,y,t) = \theta_o(y) \cos \omega t \tag{17}$$

which in complex form is written as

$$\theta(0,y,t) = \theta_o(y)e^{i\omega t} \tag{17}$$

Therefore, the surface condition on ϕ becomes

$$\phi(0,y) = \phi_o(y) \tag{18}$$

where ϕ_o is a real quantity.

Adiabatic. - The temperature distribution at y = 0 and Y is chosen such that the heat flux is zero. Thus,

$$\left.\frac{\partial\phi}{\partial y}\right|_{o} = \left.\frac{\partial\phi}{\partial y}\right|_{y} = 0 \tag{19}$$

Exit temperature. - The exit condition chosen sets the mean level about which the temperature oscillation can occur. For example, the exit temperature could be assumed to be of the form

$$T = T_{\infty} \quad \text{at} \quad x = X \tag{20}$$

$$\theta = 0 \quad \text{at} \quad x = X \tag{21}$$

Exit impedance. - Another possible exit condition is very useful in treating a problem in which the medium is assumed to be semi-infinite in the x direction. Consider the one dimensional problem in which the exit plane is at infinity. The analytical solution is derived in reference 1 (pg. 329) and shown to be (for uniform properties)

$$\phi = e^{-2\pi(1+i)x} \tag{22}$$

Borrowing an analogy from acoustics, the ratio between the temperature ϕ and the temperature gradient $\frac{\partial\phi}{\partial x}$ is defined as the impedance Z_{exit};

$$Z_{exit} = \frac{\phi}{\frac{\partial \phi}{\partial x}} \tag{23}$$

For a one-dimensional semi-infinite medium,

$$Z_{exit} = \frac{i - 1}{4\pi} \tag{24}$$

The ratio defined by equation (24) is not a function of x. Thus, the temperature field of a semi-infinite medium can be closely approximated by a finite medium, if equation (24) is used as the exit boundary condition. Of course, for a one-dimensional semi-infinite problem, the application of equation (24) at any position will give an exact agreement (within numerical accuracy) with analysis, as will be shown later in an example.

SOLUTION TECHNIQUE

Finite Difference Equations

Instead of a continuous solution for temperature, the temperature will be determined at isolated grid points by means of the finite difference approximations, as shown in Figure 2. This enables changing the differential equations to a system of algebraic equations for the temperature at each grid point. The governing equations can be approximated in difference form (ref. 5) by using either a Taylor series expansion, a variational, or an integral formulation. In this problem, where a gradient is specified along the exit boundary, the integral method for generating the finite difference approximation is most convenient.

The energy equation in finite difference form is developed by applying the integration method (ref. 5, pg. 168) to the cells in Figure 3. The cells are enclosed by the dashed lines which are spaced midway between the grid lines (not shown). The grid lines would go directly through the grid points. Thus, the integration of the energy equation, equation (12), over the cell becomes

$$\iint_{cell} \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - i\ 8\pi^2 \phi \right] dxdy \tag{25}$$

By applying Green's theorem for the plane region, equation (25) becomes

$$\int \frac{\partial \phi}{\partial N}\, dS - i\ 8\pi^2 \phi_{i,j} \iint_{cell} dxdy = 0 \tag{26}$$

where dS is a length element along the unit cell boundary, and N is the outward normal to the cell boundary.

As usual, the difference formulation approximates the first derivative

$$\left. \frac{\partial \phi}{\partial x} \right|_{i+\frac{1}{2},j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta X} \tag{27}$$

$$\left. \frac{\partial \phi}{\partial y} \right|_{i,j+\frac{1}{2}} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta y} \tag{28}$$

and so forth.

Cell 1 - central. - The integration equation, (26), is now applied to the central region of the temperature field, labeled cell 1 in Figure 3. In evaluating the surface integrals in equation (26), the gradients are assumed to be constant on each side of the cell. Thus, equation (26) becomes

$$\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j-1} + \phi_{i-1,j} - 2\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i\ (2\pi\Delta x)^2\right]\phi_{i,j} + \phi_{i+1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j+1} = 0 \qquad (29)$$

Some intermediate algebraic steps involved in the derivation of equation (29) and the following equations are presented in Appendixes C and D of reference 3, for a difference equation very similar to equation (29).

Cell 2 - adiabatic boundary. - Now consider the difference equation which applies in Cell 2, which is adjacent to the upper boundary in figure 3. For this unit cell, equation (26) can be expressed as

$$\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,m-1} + \frac{1}{2}\phi_{i-1,m} - \left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i(2\pi\Delta x)^2\right]\phi_{i,m} + \frac{1}{2}\phi_{i+1,m} = 0 \qquad (30)$$

where equation (19) was used to evaluate the $\frac{\partial\phi}{\partial N}$ term along the adiabatic boundary. A similar equation applies at the lower boundary.

Cell 3 - exit plane. - In a similar manner, the difference equation which applies to cell 3 is found to be

$$\frac{1}{2}\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{n,j-1} + \phi_{n-1,j} - \left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i\ (2\pi\Delta x)^2 + (1+i)2\pi\Delta x\right]\phi_{n,j}$$

$$+ \frac{1}{2}\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{n,j+1} = 0 \qquad (31)$$

where equation (24) written in the form

$$\left.\frac{\partial\phi}{\partial x}\right|_{n,j} = \frac{4\pi}{i-1}\phi_{n,j} \qquad (32)$$

was used to evaluate $\frac{\partial\phi}{\partial N}$ at the exit.

Cell 4 - corner. - Finally, the difference equation for cell 4 is

$$\frac{1}{2}\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{n,m-1} + \frac{1}{2}\phi_{n-1,m} - \frac{1}{2}\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i\ (2\pi\Delta x)^2\right.$$

$$\left. + (1+i)\ 2\pi\Delta x\right]\phi_{n,m} = 0 \qquad (33)$$

MATRIX SOLUTION

The collection of the various difference equations at each grid point forms a set of simultaneous equations which can be expressed in matrix notation as

$$\{A\} \cdot [\bar{\phi}] = [\bar{F}] \qquad (34)$$

where A is the known coefficient matrix, $\bar{\phi}$ is the unknown column vector containing the unknown complex temperatures, and $[\bar{F}]$ is the known column vector containing the various initial conditions.

Equation (34) can also be expressed in terms of all real quantities. In order to accomplish this, the column vector ϕ is expressed in terms of $\phi^{(1)}$ and $\phi^{(2)}$ and the A matrix is subdivided as follows

$$\left\{\begin{array}{c|c} A_1 & -C \\ \hline +C & A_1 \end{array}\right\} \left[\begin{array}{c} \bar{\phi}^{(1)} \\ \hline \bar{\phi}^{(2)} \end{array}\right] = \left[\begin{array}{c} \bar{F}^{(1)} \\ \hline \bar{F}^{(2)} \end{array}\right] \tag{35}$$

The A_1 has a form typical of those matrices found in two-dimensional steady state heat-conduction problems, while the C matrix represents the coupling that occurs between the $\phi^{(1)}$ and $\phi^{(2)}$ temperatures in the $(-1)^J$ term of equation (16).

Standard computer texts such as reference 5 dealing with the steady state heat conduction problem point out that the A_1 matrix is positive definite in at least one row; consequently, iteration schemes such as Gauss-Seidel can be used to obtain a solution. However, in the matrix equation (35), the C matrix adds to the off-diagonal elements and as a result the matrix will no longer be positive definite. As a result, conventional iteration techniques cannot be used. However, matrices of the form of equation (34) or (35) can be solved by elimination techniques. In particular, a solution of the block tridiagonal form of the complex matrix equation (34) appears to be the most efficient from both operational time and computer storage consideration. Dr. D. W. Quinn of Wright-Patterson Air Force Base, Dayton, Ohio has developed a computer package which will efficiently handle equation (35). This code had to be modified slightly to give results in terms of temperature rather than acoustic pressure for which it was developed.

APPLICATIONS

Semi-Infinite Body

As an example of the use of the numerical technique, the case of one-dimensional heat wave propagation into a semi-infinite medium will now be considered. This case allows a comparison of the numerical and analytical temperature distributions in the medium. The analytical solution for this problem is given by equation (22) in this report.

Because there is no variation of temperature in the y direction, the two-dimensional grid lattice shown in Figures 2 and 3 can be reduced to a one-dimensional lattice as shown in the upper sketch of Figure 4. The calculation was made for a depth of 1 with the exit impedance given by equation (24) to simulate a semi-infinite medium.

As seen in Figure 4, the agreement between the numerical and analytical results is good. By a series of numerical calculations, the number of grid points necessary to get accurate temperature variations for this example was found to be

$$n \approx 12X \qquad (36)$$

where X represents the position of the exit plane as shown in Figure 1.

Composite Structure

Finally, in order to solve problems with varying composition, the material can be assumed to be broken up into many bands each with a uniform composition for which equation (29) applies. However, to complete the problem, it is necessary to develop the difference equation at the junction between two materials as shown in Figure 5.

Integration of equation (11) over the cell shown in Figure 5 gives

$$\int_{-}^{+}\int_{-}^{+}\left[\frac{a}{ka_r}\frac{\partial}{\partial x}k\frac{\partial\phi}{\partial x}+\frac{a}{ka_r}\frac{\partial}{\partial y}k\frac{\partial\phi}{\partial y}-i\ 8\pi^2\phi\right]dxdy=0 \tag{37}$$

The integration limits - and + stands for the left and right sides (also bottom and top) of the cell respectively. Integration of equation (37) gives

$$\int_{-}^{+}\left[\int_{-}^{0^-}\frac{a_1}{a_r}\frac{\partial^2\phi}{\partial x^2}dx+\int_{0^+}^{+}\frac{a_2}{a_r}\frac{\partial^2\phi}{\partial x^2}dx\right]dy$$

$$+\int_{-}^{0^-}\frac{a_1}{a_r}\int_{-}^{+}\frac{\partial^2\phi}{\partial y^2}dy\ dx+\int_{0^+}^{+}\frac{a_2}{a_r}\int_{-}^{+}\frac{\partial^2\phi}{\partial y^2}dy\ dx$$

$$-i\ 8\pi^2\ \phi_{i,j}\int_{-}^{+}\int_{-}^{+}dxdy=0 \tag{38}$$

It is necessary, as shown above, to break the x integration into two domains (- to 0^- and 0^+ to +), since the integrand is discontinuous at the center of the cell, labeled 0 in Figure 5. Now, equation (38) can be directly integrated to give

$$\int_{-}^{+}\left\{\frac{a_1}{a_r}\left[\frac{\partial\phi}{\partial x}\Big|^{0^-}-\frac{\partial\phi}{\partial x}\Big|_{-}\right]+\frac{a_2}{a_r}\left[\frac{\partial\phi}{\partial x}\Big|^{+}-\frac{\partial\phi}{\partial x}\Big|_{0^+}\right]\right\}dy$$

$$\frac{\partial\phi}{\partial y}\Big|_{-}^{+}\left[\frac{a_1}{a_r}\int_{-}^{0^-}dx+\frac{a_2}{a_r}\int_{0^+}^{+}dx\right]$$

$$-i\ 8\pi^2\phi_{ij}\ \Delta x\Delta y=0 \tag{39}$$

which yields

$$\frac{1}{2}\left(\frac{a_1+a_2}{a_r}\right)\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j-1} + \frac{a_1}{a_r}\phi_{i-1,j} - \left[\left(\frac{a_1+a_2}{a_r}\right) + \left(\frac{a_1+a_2}{a_r}\right)\left(\frac{\Delta x}{\Delta y}\right)^2 + i\ 2(2\pi\Delta x)^2\right]\phi_{i,j} + \left(\frac{a_2}{a_r}\right)\phi_{i+1,j} + \frac{1}{2}\left(\frac{a_1+a_2}{a_r}\right)\left(\frac{\Delta x}{\Delta y}\right)^2\phi_{i,j+1} = 0 \qquad (40)$$

where it has been assumed that

$$a_1 \frac{\partial\phi}{\partial x}\bigg|^{0^-} - a_2 \frac{\partial\phi}{\partial x}\bigg|_{0^+} \approx 0 \qquad (41)$$

The approximation sign is used because the true interface boundary condition is that the heat flux are equal at the interface. Equation (39) can now be used as the interface between two different materials. For the special case where a_1, a_2 and a_r are equal, equation (39) reduces to equation (29).

Other approximations could be used to develop the difference equations at the interface. For example,

$$a_1 \frac{\partial\phi}{\partial x}\bigg|^{0^-} - a_2 \frac{\partial\phi}{\partial x}\bigg|_{0^+} \approx (a_1-a_2)\left.\frac{\partial\phi}{\partial x}\right|_0 = \frac{(a_1-a_2)}{2}\ \frac{(\phi_{i+1,j}-\phi_{i-1,j})}{\Delta x} \qquad (42)$$

In this case, the coefficients in front of the $\phi_{i-1,j}$ and $\phi_{i+1,j}$ terms in equation (40) would be identically equal to $\frac{1}{2}\left(\frac{a_1+a_2}{a_r}\right)$. This approximation would average the temperature gradient at the interface. It would, in effect, give the same results had the interface between materials 1 and 2 been positioned midway between the grid points. With the interface positioned midway between the grid points, the difference equations to the left of the interface would be written as

$$\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j-1} + \phi_{i-1,j} - 2\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i\,\frac{a_r}{a_1}\,(2\pi\Delta x)^2\right]\phi_{i,j}$$

$$+ \phi_{i+1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j+1} = 0 \tag{43}$$

while to the right of the interface

$$\left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j-1} + \phi_{i-1,j} - 2\left[1 + \left(\frac{\Delta x}{\Delta y}\right)^2 + i\,\frac{a_r}{a_2}\,(2\pi\Delta x)^2\right]\phi_{i,j}$$

$$+ \phi_{i+1,j} + \left(\frac{\Delta x}{\Delta y}\right)^2 \phi_{i,j+1} = 0 \tag{44}$$

If equation (43) were multiplied by $\frac{a_1}{a_r}$, equation (44) multiplied by $\frac{a_2}{a_r}$ and the resulting equations averaged, then $\frac{1}{2}\left(\frac{a_1+a_2}{a_r}\right)$ would be the coefficient on the ϕ terms, which would be consistant with equation (42).

With sufficient number of grid points, either approximation should yield approximately the same temperature distribution except very near to the interface.

CONCLUDING REMARKS

A finite difference solution for heat wave propagation in a two-dimensional medium has been presented. A special exit impedance is derived which allows a finite heat field to approximate a semi-infinite medium. The numerical theory is shown to be in good agreement with the corresponding exact analytical results for a semi-infinite one-dimensional medium.

The finite difference formulation is flexible and should be a useful tool in the solution of complex heat conduction problems.

REFERENCES

1. Arpaci, V. S., Conduction Heat Transfer, Addison-Wesley Publ. Co., Mass. (1966).
2. Carslaw, H. S. and Jaeger, J. C., Conduction of Heat in Solids, Oxford Press, London (1959).
3. Baumeister, K. J. and Bittner, E. C., "Numerical Simulation of Noise Propagation in Jet Engine Ducts," TN D-7339, National Aeronautics and Space Administration (Oct. 1973).
4. Baumeister, K. J., "Analysis of Sound Propagation in Ducts Using the Wave Envelope Concept," TN D-7719, National Aeronautics and Space Administration (July 1974).
5. Vargo, R. S., Matrix Iterative Analyses, Prenctice-Hall, Cleveland (1962).

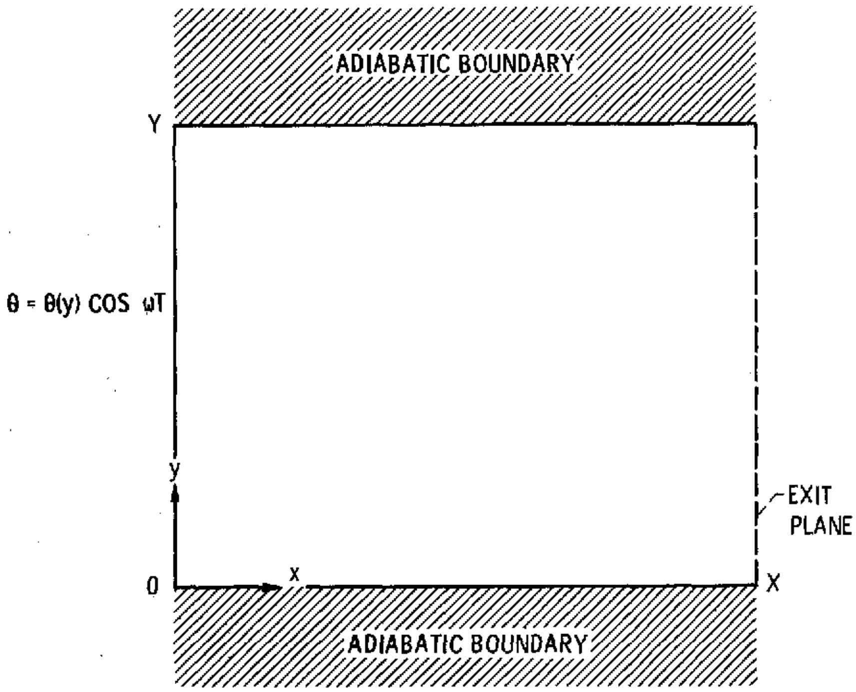

Figure 1. - Schematic of temperature field.

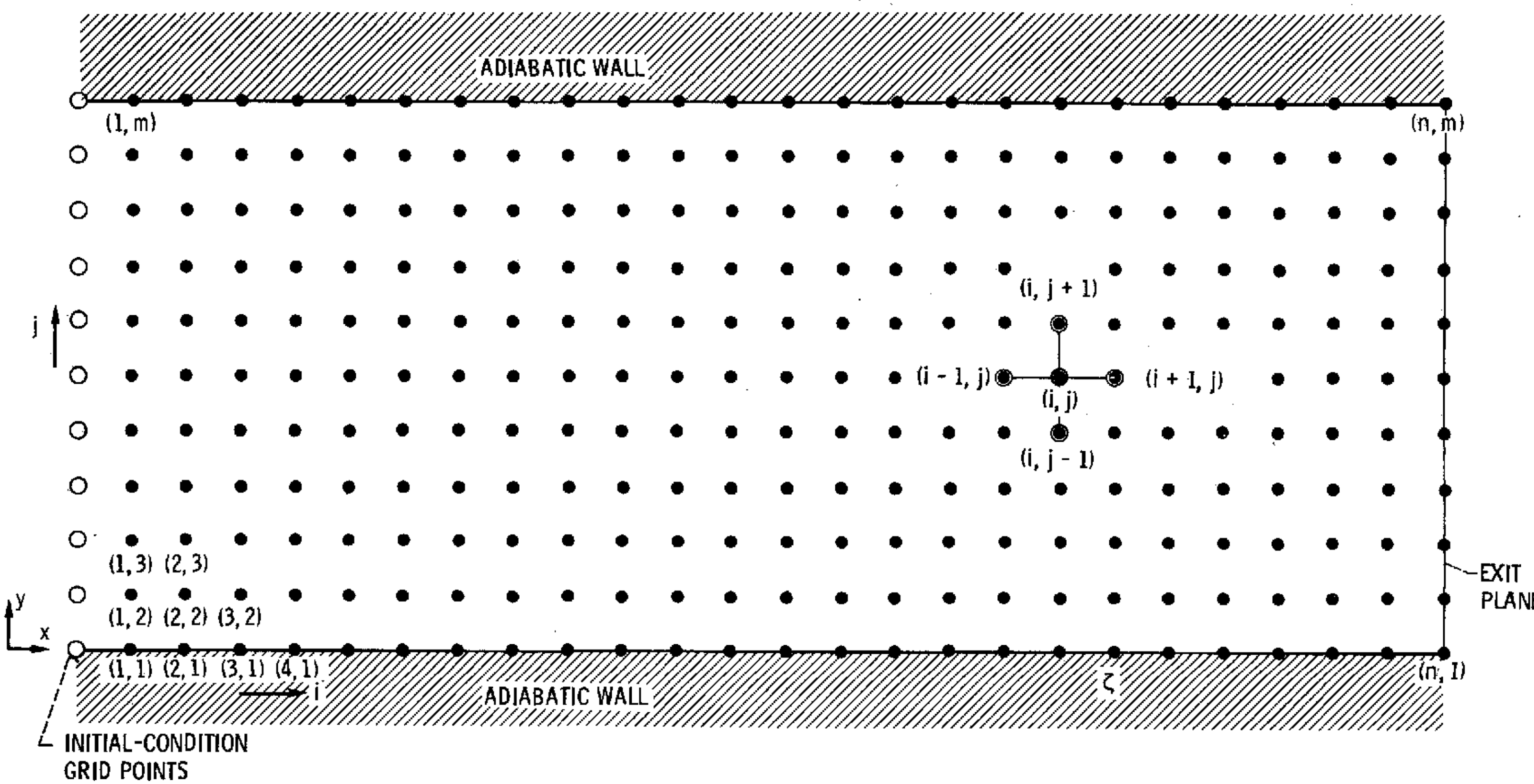

Figure 2. - Grid-point representation of two-dimensional heat field.

E-8291

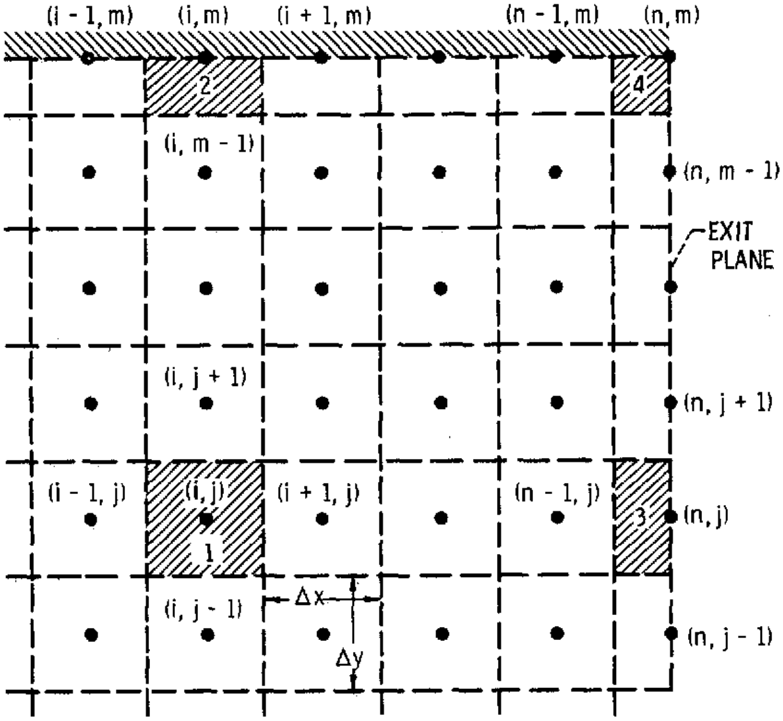

Figure 3. - Integration cells for establishing difference equations.

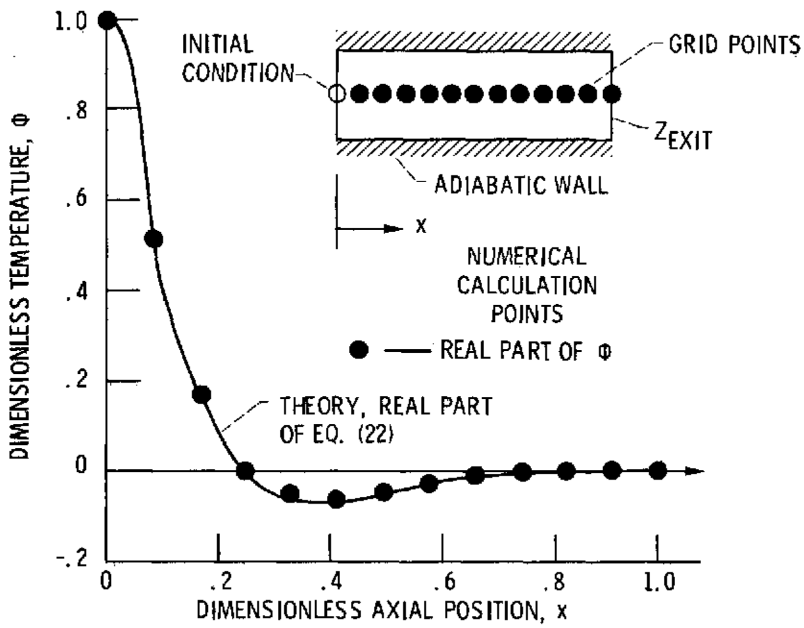

Figure 4. - Analytical and numerical temperature variations for one-dimensional heat wave propagation into a semi-infinite medium.

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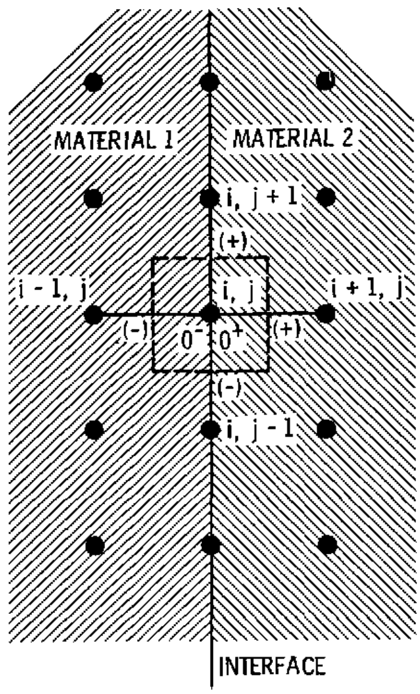

Figure 5. - Grid-point representation of interface between two materials.