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A FINITE-ELEMENT METHOD FOR LIFTING SURFACES IN STEADY INCOMPRESSIBLE SUBSONIC FLOW

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ABSTRACT

The problem of potential steady subsonic flow for lifting surfaces is considered. This problem requires the solution of an integral equation relating the values of the potential discontinuity on the lifting surface and its wake to the values of the normal derivative of the potential which are known from the boundary conditions. The lifting surface is divided into small (quadrilateral hyperboloidal) surface elements, Σ_i, which are described in terms of the Cartesian components of the four corner points. The values of the potential discontinuity and the normal derivative of the potential are assumed to be constant within each element and equal to their values at the centroids of the elements. This yields a set of linear algebraic equations. Numerical results are in good agreement with existing ones.

FOREWORD

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TABLE OF CONTENTS

LIST OF SYMBOLS

$\bar{a}_i$	base vectors, defined by Eq. 2.2
$\bar{a}^i$, $\bar{a}_i$, $i = 1, 2, 3$	See Eqs. 2.13 and 2.14
A_{hk}	Eqs. 1.10 and 1.11
B_h	Eqs. 1.10 and 1.11
C_{L_α}	Lift coefficient per unit angle of attack
D_k, D_h	Defined by Eq. 1.4
$\bar{I}v(\xi,\eta)$	Defined by Eq. 2.9
$I_D(\xi,\eta)$	See Eqs. 2.9 and 2.10
$\bar{n}$	normal to the surface Σ at P_h
NX,NY	number of wing boxes along x and y directions, respectively
$\bar{P}(x,y,z)$	control point
$\bar{P}{++}$, $\bar{P}{+-}$, $\bar{P}{-+}$, $\bar{P}{--}$	See Eq. 2.7
$\bar{P}_C$, $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$	See Eq. 2.7
$\bar{q}$	See Eq. 2.5
$\bar{Q}_1$, $\bar{Q}_2$, $\bar{Q}_3$, $\bar{Q}_4$	See Eq. 2.40
	See Eq. 2.5
$\bar{v}_h$, $\bar{v}_k$	Velocity at point $\bar{P}_h$ or $\bar{P}_k$
$\bar{v}_{hk}$	See Eq. 2.8
x,y,z	Cartesian coordinates
ξ^1, ξ^2	Defined by Eq. 2.1
ξ, η	Defined by Eq. 2.4

List of Symbols, continued

Σ_B	Surface of the body
Σ_W	Surface of the wake
φ	Perturbation aerodynamic potential
φ_k	Value of φ at $\overline{P}_k$

SPECIAL SYMBOLS

$\overline{\nabla}$	Gradient operator in x, y, z coordinates
T.E.	Trailing edge

SECTION I

FORMULATION OF THE PROBLEM

1.1 Introduction

References 1 and 2 present a general theory for compressible unsteady potential aerodynamic flow around lifting bodies having arbitrary shapes and motions. Reference 3 presents a general numerical formulation for complex configurations in steady subsonic flow. Results are presented in Ref. 4. However, such a formulation is not applicable to zero-thickness configurations (lifting surfaces). The present work introduces a formulation suitable for use with lifting surfaces.

The distribution of the perturbation aerodynamic potential φ, around a body of arbitrary shape is given by the following integral expression

$$4\pi E \varphi = -\oint\!\!\oint_{\Sigma} \left[\frac{\partial \varphi}{\partial n} \frac{1}{r} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] d\Sigma \tag{1.1}$$

where

E = 0 inside the body

E = 1 outside the body

E = 1/2 on the body

(1.2)

Σ is a surface surrounding the body and its wake, and $\vec{n}$ represents the normal to the surface.

If the distance between the upper and lower sides of the surface goes to zero (zero-thickness body), one obtains a lifting surface formulation

$$\varphi = \iint_{\Sigma_B + \Sigma_W} D \frac{\partial}{\partial n_u} \left(\frac{1}{r} \right) d\Sigma \tag{1.3}$$

where

$$D = \frac{\varphi_u - \varphi_\ell}{4\pi} \tag{1.4}$$

The subscript u stands for upper and ℓ stands for lower. Equation (1.3) shows that the potential can be represented in terms of doublets on the body and on the wake. On the wake, the value of D is constant along a streamline and equal to D at the trailing edge.

1.2 Discretization

By dividing the lifting surface into small elements (see also Ref. 3) and applying the mean value theorem for Eq. (1.3), one obtains

$$\varphi = \sum_{k=1}^{N} D_k \iint_{\Sigma_k} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_k \tag{1.5}$$

where D_k are suitable mean values within the element, and the summation is performed over the elements of the lifting surface and of the wake, which is approximated by straight vortex lines starting at the lifting surface trailing edge. The perturbation velocity, $\vec{v} = \vec{\nabla}_0 \varphi$, at the point P_h, is given by

$$\vec{v}_h = \left[\vec{\nabla}_0 \varphi\right]_{P=P_h} = \sum D_k \vec{v}_{hk} \tag{1.6}$$

where

$$\vec{v}_{hk} = \left[\iint_{\Sigma_k} \vec{\nabla}_0 \; \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_k \right]_{P=P_h} \tag{1.7}$$

is the velocity created by the element Σ_k. The normal derivative at the point P_h of the surface is given by

$$\left(\frac{\partial \varphi}{\partial n}\right)_{P=P_h} = \vec{n}_h \cdot \vec{\nabla}_0 \varphi = \sum_{k=1}^{N} D_k \; \vec{n}_h \cdot \vec{v}_{hk} \tag{1.8}$$

The boundary condition to be satisfied at L points (L is the number of lifting surface elements) is

$$\vec{V} \cdot \vec{n} = (\vec{i} + \vec{v}) \cdot \vec{n} = 0 \tag{1.9}$$

which, when combined with Eq. (1.8) becomes:

$$\sum A_{hk} D_k = B_h \tag{1.10}$$

for the L unknown D_h. In Eq. (1.10)

$$A_{hk} = \vec{v}_{hk} \cdot \vec{n}_h$$

$$B_h = \left(\frac{\partial \varphi}{\partial n}\right)_{P=P_h} = -\vec{n}_h \cdot \vec{i} \tag{1.11}$$

where $\vec{i}$ is the unit vector in the direction of the x-axis. The contribution of the wake elements adds only to the row of lifting surface elements in contact with the trailing edge. Once Eq. (1.10) is solved, the velocity $\vec{v}_h$ can be evaluated through Eq. (1.6) using the <u>same</u> coefficients, $\vec{v}_{hk}$.

SECTION II

HYPERBOLOIDAL QUADRILATERAL ELEMENT

2.1 Introduction

Reference 3 introduces a new type of surface element, the hyperboloidal quadrilateral element, a short description of which will be given here. Then, the gradient of Eq. (1.3) (the integral is obtained in analytical form in Ref. 3) will be computed and further, the result will be put in a simple vector form.

2.2 Surface Geometry with Hyperboloidal Quadrilateral Element

Let the geometry of the element Σ_k be described by the vector

$$\vec{P} = \vec{P}(\xi^1, \xi^2) \tag{2.1}$$

where ξ^1 and ξ^2 are the generalized curvilinear coordinates (Fig. 1).

The two base vectors are given by

$$\vec{a}_i = \frac{\partial \vec{P}}{\partial \xi^i} \tag{2.2}$$

and the unit normal to the surface is obtained as

$$\vec{n} = \frac{\vec{a}_1 \times \vec{a}_2}{|\vec{a}_1 \times \vec{a}_2|} \tag{2.3}$$

The surface element $d\Sigma$ is

$$d\Sigma = |\vec{a}_1 d\xi^1 \times \vec{a}_2 d\xi^2| = |\vec{a}_1 \times \vec{a}_2| d\xi^1 d\xi^2 \tag{2.4}$$

The expression for r is

$$\vec{r} = \begin{Bmatrix} x - x_h \\ y - y_h \\ z - z_h \end{Bmatrix} \tag{2.5}$$

Now, consider the equation

$$\vec{P} = \vec{P}_c + \vec{P}_1 \xi + \vec{P}_2 \eta + \vec{P}_3 \xi\eta ; \begin{pmatrix} -1 \le \xi \le 1 \\ -1 \le \eta \le 1 \end{pmatrix} \tag{2.6}$$

The above equation represents a hyperboloid (Fig. 2). $\vec{P}_c$ represents the centroid of the element Σ_k, with $\xi = \eta = 0$.

The corner points of the element Σ_k, are $\vec{P}_{++}$, $\vec{P}_{+-}$, $\vec{P}_{-+}$, $\vec{P}_{--}$, and they are fed in as geometry inputs in the computer program implementing the theoretical formulation. The relationship between the corner points and $\vec{P}_c$, $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$, is

$$\begin{Bmatrix} \vec{P}_c \\ \vec{P}_1 \\ \vec{P}_2 \\ \vec{P}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \vec{P}_{++} \\ \vec{P}_{+-} \\ \vec{P}_{-+} \\ \vec{P}_{--} \end{Bmatrix} \tag{2.7}$$

2.3 The Doublet Integral

Looking again at Eq. (1.13), it can be written in the following form

$$\vec{v}_{hk} = \vec{I}_v(1,1) - \vec{I}_v(1,-1) - \vec{I}_v(-1,1) + \vec{I}_v(1,1) \tag{2.8}$$

where

$$\vec{I}_v(\xi,\eta) = -2\pi \nabla_0 I_D(\xi,\eta) \tag{2.9}$$

where the doublet integral $I_D(\xi,\eta)$ was obtained in analytical form (Ref. 3, Eq. 6.6) as

$$I_D(\xi,\eta) = \tan^{-1}\left[\frac{\vec{q}\times\vec{a}_1\cdot\vec{q}\times\vec{a}_2}{|\vec{q}|(\vec{q}\cdot\vec{a}_1\times\vec{a}_2)}\right] \tag{2.10}$$

In order to perform the gradient derivative in Eq. (2.8), it is convenient to consider the directional derivative in the arbitrary direction $\vec{\nu}$. By noting that only $\vec{q} = \vec{P} - \vec{P}_o$ depends upon $\vec{P}_o$, or

$$\frac{\partial\vec{q}}{\partial\nu} = (\vec{\nu}\cdot\vec{\nabla}_o)\vec{q} = -\vec{\nu}$$

$$\frac{\partial\vec{a}_k}{\partial\nu} = 0 \qquad (k = 1, 2) \tag{2.11}$$

one obtains

$$\frac{\partial I_D}{\partial\nu} = \frac{\partial}{\partial\nu}\tan^{-1}\left(\frac{-\vec{q}\times\vec{a}_1\cdot\vec{q}\times\vec{a}_2}{|\vec{q}|\,\vec{q}\cdot\vec{a}_1\times\vec{a}_2}\right)$$

$$= \frac{1}{1+\dfrac{(\vec{q}\times\vec{a}_1\cdot\vec{q}\times\vec{a}_2)^2}{|\vec{q}|^2(\vec{q}\cdot\vec{a}_1\times\vec{a}_2)^2}}\times$$

$$\left\{\frac{\vec{\nu}\times\vec{a}_1\cdot\vec{q}\times\vec{a}_2+\vec{q}\times\vec{a}_1\cdot\vec{\nu}\times\vec{a}_2}{|\vec{q}|\,\vec{q}\cdot\vec{a}_1\times\vec{a}_2} - (\vec{q}\times\vec{a}_1\cdot\vec{q}\times\vec{a}_2)\times\right.$$

$$\left.\left(\frac{\vec{q}\cdot\vec{\nu}}{|\vec{q}|^3}\frac{1}{\vec{q}\cdot\vec{a}_1\times\vec{a}_2}+\frac{1}{|\vec{q}|}\frac{\vec{\nu}\cdot\vec{a}_1\times\vec{a}_2}{(\vec{q}\cdot\vec{a}_1\times\vec{a}_2)^2}\right)\right\} =$$

$$= \frac{|\bar{q}|^2(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2}{\bar{q}\cdot\bar{q}\,(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2+(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)}\;\frac{1}{|\bar{q}|^3(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)}\times$$

$$\Big\{(\bar{\nu}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2+\bar{q}\times\bar{a}_1\cdot\bar{\nu}\times\bar{a}_2)(\bar{q}\cdot\bar{q})(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)-$$

$$(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{\nu}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2)+\bar{\nu}\cdot\bar{a}_1\times\bar{a}_2\;\bar{q}\cdot\bar{q}\Big\}=$$

$$= \frac{1}{|\bar{q}\times\bar{a}_1|^2|\bar{q}\times\bar{a}_2|^2|q|}\Big\{(\bar{\nu}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2+\bar{q}\times\bar{a}_1\cdot\bar{\nu}\times\bar{a}_2)\,\bar{q}\cdot\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2-$$

$$(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{\nu}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2+\bar{\nu}\cdot\bar{a}_1\times\bar{a}_2\;\bar{q}\cdot\bar{q})\Big\} \tag{2.12}$$

Next, it is convenient to introduce some classical concepts of tensor analysis. Consider the relationships between the two sets of conjugate base vectors

$$\bar{a}^1=\frac{\bar{a}_2\times\bar{a}_3}{\bar{a}_1\cdot\bar{a}_2\times\bar{a}_3};\quad \bar{a}^2=\frac{\bar{a}_3\times\bar{a}_1}{\bar{a}_1\cdot\bar{a}_2\times\bar{a}_3};\quad \bar{a}^3=\frac{\bar{a}_1\times\bar{a}_2}{\bar{a}_1\cdot\bar{a}_2\times\bar{a}_3} \tag{2.13}$$

and

$$\bar{a}_1=\frac{\bar{a}^2\times\bar{a}^3}{\bar{a}^1\cdot\bar{a}^2\times\bar{a}^3};\quad \bar{a}_2=\frac{\bar{a}^3\times\bar{a}^1}{\bar{a}^1\cdot\bar{a}^2\times\bar{a}^3};\quad \bar{a}_3=\frac{\bar{a}^1\times\bar{a}^2}{\bar{a}^1\cdot\bar{a}^2\times\bar{a}^3} \tag{2.14}$$

where $\bar{a}_3 = \bar{n}$. Note that

$$\bar{a}^3=\bar{a}_3=\bar{n} \tag{2.15}$$

and

$$\bar{a}^1\cdot\bar{a}^2\times\bar{a}^3=\frac{1}{\bar{a}_1\cdot\bar{a}_2\times\bar{a}_3}=\frac{1}{|\bar{a}_1\times\bar{a}_2|} \tag{2.16}$$

Furthermore, using classical notations, it is possible to write

$$\bar{v} = v^1 \bar{a}_1 + v^2 \bar{a}_2 + v^3 \bar{a}_3 = v^i \bar{a}_i$$

$$= v_1 \bar{a}^1 + v_2 \bar{a}^2 + v_3 \bar{a}^3 = v_i \bar{a}^i \qquad (2.17)$$

with

$$v^i = \bar{v} \cdot \bar{a}^i$$

$$v_i = \bar{v} \cdot \bar{a}_i \qquad (2.18)$$

Moreover, it is convenient to consider the three derivatives $\partial I_D/\partial \bar{a}_1$, $\partial I_D/\partial \bar{a}_2$ and $\partial I_D/\partial \bar{n}$. Since $\bar{v}$ is an arbitrary vector, Eq. (2.12) with $\bar{v} = \bar{a}_1$, yields

$$\frac{\partial I_D}{\partial \bar{a}_1} = \frac{1}{|\bar{q}\times\bar{a}_1|^2 |\bar{q}\times\bar{a}_2|^2} \frac{1}{|\bar{q}|} \times$$

$$\Big\{ (\cancel{\bar{a}_1\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2} + \bar{q}\times\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2)\, \bar{q}\cdot\bar{q}\;\; \bar{q}\cdot\bar{a}_1\times\bar{a}_2 \; -$$

$$(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_1\, \bar{q}\cdot\bar{a}_1\times\bar{a}_2 + \cancel{\bar{a}_1\cdot\bar{a}_1\times\bar{a}_2}\, \bar{q}\cdot\bar{q}) \Big\}$$

$$= \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}\times\bar{a}_1|^2 |\bar{q}\times\bar{a}_2|^2} \frac{1}{|\bar{q}|} \Big\{ (\bar{q}\cdot\bar{a}_1\, \bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_2\, \bar{a}_1\cdot\bar{a}_2)\, \bar{q}\cdot\bar{q} \; -$$

$$(\bar{q}\cdot\bar{q}\, \bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\, \bar{q}\cdot\bar{a}_2)\, \bar{q}\cdot\bar{a}_1$$

$$= \frac{-\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}\times\bar{a}_1|^2 |\bar{q}\times\bar{a}_2|^2 |\bar{q}|}\; \bar{q}\cdot\bar{a}_2 \left[\bar{q}\cdot\bar{q}\;\; \bar{a}_1\cdot\bar{a}_1 - (\bar{q}\cdot\bar{a}_1)^2 \right]$$

$$= -\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|} \; \frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} \qquad (2.19)$$

Similarly, for $\bar{\nu} = \bar{a}_2$

$$\frac{\partial I_D}{\partial \bar{a}_2} = \frac{1}{|\bar{q}\times\bar{a}_1|^2|\bar{q}\times\bar{a}_2|^2|\bar{q}|}\Big\{(\bar{a}_2\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2 + \bar{q}\times\bar{a}_1\cdot\bar{a}_2\times\bar{a}_2)\bar{q}\cdot\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 -$$

$$(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{a}_2\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 + \bar{a}_2\cdot\bar{a}_1\times\bar{a}_2\;\bar{q}\cdot\bar{q})\Big\}$$

$$= \frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}\times\bar{a}_1|^2|\bar{q}\times\bar{a}_2|^2|\bar{q}|}\Big\{(\bar{a}_2\cdot\bar{q}\;\bar{a}_1\cdot\bar{a}_2 - \bar{a}_2\cdot\bar{a}_2\;\bar{q}\cdot\bar{a}_1)\bar{q}\cdot\bar{q} -$$

$$(\bar{q}\cdot\bar{q}\;\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\bar{q}\cdot\bar{a}_2)\bar{q}\cdot\bar{a}_2 =$$

$$\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|}\;\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2} \tag{2.20}$$

Finally, for $\bar{\nu} = \bar{n}$ and using Eqs. (B.1) and (3.45) of Ref. 3, yields,

$$\frac{\partial I_D}{\partial \bar{n}}\,|\bar{q}\times\bar{a}_1|\,|\bar{q}\times\bar{a}_2|^2|\bar{q}| =$$

$$(\bar{n}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2 + \bar{q}\times\bar{a}_1\cdot\bar{n}\times\bar{a}_2)\bar{q}\cdot\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 -$$

$$(\bar{q}\times\bar{a}_1\cdot\bar{q}\times\bar{a}_2)(\bar{q}\cdot\bar{n}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 + \bar{n}\cdot\bar{a}_1\times\bar{a}_2\;\bar{q}\cdot\bar{q}) =$$

$$(\bar{q}\cdot\bar{n}\;\bar{a}_1\cdot\bar{a}_2 - \bar{n}\cdot\bar{a}_2\;\bar{q}\cdot\bar{a}_1 + \bar{q}\cdot\bar{n}\;\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_2\;\bar{a}_1\cdot\bar{n})\times$$

$$\bar{q}\cdot\bar{q}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 - (\bar{q}\cdot\bar{q}\;\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\;\bar{q}\cdot\bar{a}_2)\bar{q}\cdot\bar{n}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 -$$

$$(\bar{q}\cdot\bar{q}\;\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\;\bar{q}\cdot\bar{a}_2)\bar{n}\cdot\bar{a}_1\times\bar{a}_2\;\bar{q}\cdot\bar{q} =$$

$$(\bar{q}\cdot\bar{q}\;\bar{a}_1\cdot\bar{a}_2 + \bar{q}\cdot\bar{a}_1\;\bar{q}\cdot\bar{a}_2)\bar{q}\cdot\bar{n}\;\bar{q}\cdot\bar{a}_1\times\bar{a}_2 -$$

$$(\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2)\,\bar{q}\cdot\bar{q}\;\bar{n}\cdot\bar{a}_1\times\bar{a}_2 =$$

$$\frac{1}{|\bar{a}_1\times\bar{a}_2|}\Big[(\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2 + \bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2)(\bar{q}\cdot\bar{a}_1\times\bar{a}_2)^2 -$$

$$(\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2)\,|\bar{a}_1\cdot\bar{a}_2|^2\,\bar{q}\cdot\bar{q}\Big] =$$

$$\frac{1}{|\bar{a}_1\times\bar{a}_2|}\Big\{\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2\big[\bar{q}\cdot\bar{q}\,|\bar{a}_1\times\bar{a}_2|^2 - \bar{q}\cdot\bar{a}_2\,\bar{a}_1\times\bar{a}_2\cdot\bar{a}_1\times\bar{q} - \bar{q}\cdot\bar{a}_1\,\bar{a}_2\times\bar{a}_1\cdot\bar{a}_2\times\bar{q}\big]$$

$$+\,\bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2\big[\bar{q}\cdot\bar{q}\,|\bar{a}_1\times\bar{a}_2|^2 - \bar{q}\cdot\bar{a}_2\,\bar{a}_1\times\bar{a}_2\cdot\bar{a}_1\times\bar{q} - \bar{q}\cdot\bar{a}_1\,\bar{a}_2\times\bar{a}_1\cdot\bar{a}_2\times\bar{q}\big] -$$

$$(\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2)\,\bar{q}\cdot\bar{q}\,|\bar{a}_1\cdot\bar{a}_2|^2\Big\} =$$

$$\frac{1}{|\bar{a}_1\times\bar{a}_2|}\Big\{2\,\bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2\,\bar{q}\cdot\bar{q}\,\big[\bar{a}_1\cdot\bar{a}_1 - \bar{a}_2\cdot\bar{a}_2 - (\bar{a}_1\cdot\bar{a}_2)^2\big] -$$

$$\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_2\big[\bar{q}\cdot\bar{a}_2(\bar{a}_1\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\,\bar{a}_1\cdot\bar{a}_2) + \bar{q}\cdot\bar{a}_1(\bar{a}_2\cdot\bar{a}_2\,\bar{q}\cdot\bar{a}_1 - \bar{q}\cdot\bar{a}_2\,\bar{a}_1\cdot\bar{a}_2)\big]$$

$$-\,\bar{q}\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2\big[\bar{q}\cdot\bar{a}_2(\bar{a}_1\cdot\bar{a}_1\,\bar{q}\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_1\,\bar{a}_1\cdot\bar{a}_2) + \bar{q}\cdot\bar{a}_1(\bar{a}_2\cdot\bar{a}_2\,\bar{q}\cdot\bar{a}_1 - \bar{q}\cdot\bar{a}_2\,\bar{a}_1\cdot\bar{a}_2)\big] =$$

$$\frac{1}{|\bar{a}_1\times\bar{a}_2|}\Big\{\bar{q}\cdot\bar{a}_1\big[\bar{q}\cdot\bar{a}_2\,\bar{a}_1\cdot\bar{a}_1 - \bar{q}\cdot\bar{a}_1\,\bar{a}_1\cdot\bar{a}_2\big]\big[\bar{q}\cdot\bar{q}\,\bar{a}_2\cdot\bar{a}_2 - (\bar{q}\cdot\bar{a}_2)^2\big] +$$

$$\bar{q}\cdot\bar{a}_2\big[\bar{q}\cdot\bar{a}_1\,\bar{a}_2\cdot\bar{a}_2 - \bar{q}\cdot\bar{a}_2\,\bar{a}_1\cdot\bar{a}_2\big]\big[\bar{q}\cdot\bar{q}\,\bar{a}_1\cdot\bar{a}_1 - (\bar{q}\cdot\bar{a}_1)^2\big] =$$

$$\frac{1}{|\bar{a}_1\times\bar{a}_2|}\Big\{\bar{q}\cdot\bar{a}_1\,\bar{q}\times\bar{a}_1\cdot\bar{a}_2\times\bar{a}_1\,|\bar{q}\times\bar{a}_2|^2 + \bar{q}\cdot\bar{a}_2\,\bar{q}\times\bar{a}_2\cdot\bar{a}_1\times\bar{a}_2\,|\bar{q}\times\bar{a}_1|^2\Big\} =$$

$$= -\bar{q}\cdot\bar{a}_1\,\bar{q}\times\bar{a}_1\cdot\bar{n}\,|\bar{q}\times\bar{a}_2|^2 + \bar{q}\cdot\bar{a}_2\,\bar{q}\times\bar{a}_2\cdot\bar{n}\,|\bar{q}\times\bar{a}_1|^2 \tag{2.21}$$

or

$$\frac{\partial I_D}{\partial \bar{n}} = - \frac{\bar{q}\times\bar{a}_1\cdot\bar{n}}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2} + \frac{\bar{q}\times\bar{a}_2\cdot\bar{n}}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_2}{|q\times a_2|^2} \tag{2.22}$$

Finally, combining Eqs. (2.13) and (2.18) yields

$$\frac{\partial I_D}{\partial \bar{\nu}} = \bar{\nu}\cdot\bar{\nabla}_0 I_D =$$

$$(\nu^1\bar{a}_1 + \nu^2\bar{a}_2 + \nu^3\bar{a}_3)\cdot\bar{\nabla}_0 I_D =$$

$$\nu^1\frac{\partial I_D}{\partial \bar{a}_1} + \nu^2\frac{\partial I_D}{\partial \bar{a}_2} + \nu^3\frac{\partial I_D}{\partial a_3} =$$

$$\bar{\nu}\cdot\left[\bar{a}_2\times\bar{n}\frac{\partial I_D}{\partial \bar{a}_1} + \bar{n}\times\bar{a}_1\frac{\partial I_D}{\partial a_2} + \bar{a}_1\times\bar{a}_2\frac{\partial I_D}{\partial \bar{n}}\right]\frac{1}{|\bar{a}_1\times\bar{a}_2|}$$

Or (2.23)

$$\frac{\partial I_D}{\partial \bar{\nu}} = \frac{\bar{\nu}}{|\bar{a}_1\times\bar{a}_2|}\cdot\left\{\bar{a}_2\times\bar{n}\left(-\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}\times\bar{a}_2|}\right) + \right.$$

$$\bar{n}\times\bar{a}_1\left(-\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right) +$$

$$\left.(\bar{a}_1\times\bar{a}_2)\left[-\frac{\bar{q}\times\bar{a}_1\cdot\bar{n}}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2} + \frac{\bar{q}\times\bar{a}_2\cdot\bar{n}}{|\bar{q}|}\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2}\right]\right\} \equiv$$

$$\equiv \bar{\nu}\cdot\left\{\bar{q}\cdot\bar{a}_2\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} - \bar{q}\cdot\bar{a}_1\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right\}\frac{1}{|\bar{q}|} \tag{2.24}$$

since, according to Eqs. (2.13) to (2.16), and Eq. (3.44) of Ref. 3,

$$\frac{1}{|\bar{a}_1 \times \bar{a}_2|}\left(-\bar{a}_2 \times \bar{n}\,\bar{q}\cdot\bar{a}_1 \times \bar{a}_2 + \bar{a}_1 \times \bar{a}_2\,\bar{q}\cdot\bar{a}_2 \times \bar{n}\right) =$$

$$|\bar{a}_1 \times \bar{a}_2|\left(-\bar{a}^1\bar{q}\cdot\bar{a}^3 + \bar{a}^3\bar{q}\cdot\bar{a}^1\right) =$$

$$= -\bar{q} \times (a^1 \times a^3)\frac{1}{|\bar{a}^1 \times \bar{a}^2 . \bar{a}^3|} = \bar{q} \times \bar{a}_2 \tag{2.25}$$

and

$$\frac{1}{|\bar{a}_1 \times \bar{a}_2|}\left(-\bar{n} \times \bar{a}_1\,\bar{q}\cdot\bar{a}_1 \times \bar{a}_2 - \bar{a}_1 \times \bar{a}_2\,\bar{q}\cdot\bar{a}_1 \times \bar{n}\right) =$$

$$|\bar{a}_1 \times \bar{a}_2|\left(-\bar{a}^2\bar{q}\cdot\bar{a}^3 + \bar{a}^3\bar{q}\cdot\bar{a}^2\right) =$$

$$= -\bar{q} \times (\bar{a}^2 \times \bar{a}^3)\frac{1}{\bar{a}^1 . \bar{a}^2 \times \bar{a}^3} = -\bar{q} \times \bar{a}_1 \tag{2.26}$$

Equation (2.24) is equivalent to the desired expression for $\bar{\nabla}_0 I_D$:

$$\bar{I}_V = \bar{\nabla}_0 I_D = \left(\bar{q}\cdot\bar{a}_2 \frac{\bar{q} \times \bar{a}_2}{|\bar{q} \times \bar{a}_2|^2} - \bar{q}\cdot\bar{a}_1 \frac{\bar{q} \times \bar{a}_1}{|\bar{q} \times \bar{a}_1|^2}\right)\frac{1}{|q|} \tag{2.27}$$

2.4 Alternative Proof

In order to verify Eq. (2.27), note that according to Eqs. (2.9) and (3.50) of Ref. 3

$$\frac{\partial^2 \bar{I}_V}{\partial\xi\partial\eta} = -2\pi\frac{\partial^2}{\partial\xi\partial\eta}\left[\bar{\nabla}_0 I_D\right] = -2\pi\bar{\nabla}_0\left(\frac{\partial^2 I_D}{\partial\xi\partial\eta}\right) = \bar{\nabla}_0\left(\frac{q\cdot a_1\times a_3}{r^3}\right) =$$

$$= -\bar{\nabla}_0\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\left|\bar{a}_1\times\bar{a}_2\right| \tag{2.28}$$

Noting that

$$\frac{\partial}{\partial\xi}(\bar{q}\times\bar{a}_1) = \frac{\partial}{\partial\xi}(\bar{P}_0+\eta\bar{P}_2)\times(\bar{P}_1+\eta\bar{P}_3) = 0$$

$$\frac{\partial}{\partial\eta}(\bar{q}\times\bar{a}_2) = \frac{\partial}{\partial\eta}(\bar{P}_0+\xi\bar{P}_1)\times(\bar{P}_2+\xi\bar{P}_3) = 0 \tag{2.29}$$

yields

$$\frac{\partial^2}{\partial\xi\partial\eta}\left(\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|}\;\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2}\right) =$$

$$\frac{\partial}{\partial\xi}\left\{\left[\frac{\bar{a}_2\cdot\bar{a}_2}{|\bar{q}|}+\bar{q}\cdot\bar{a}_2\left(-\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|^3}\right)\right]\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2}\right\} =$$

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{|\bar{q}|^3}\left[\bar{q}\cdot\bar{q}\,\bar{a}_2\cdot\bar{a}_2-(\bar{q}\cdot\bar{a}_2)^2\right]\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2}\right\} =$$

$$\frac{\partial}{\partial\xi}\;\frac{\bar{q}\times\bar{a}_2}{|\bar{q}|^3} = \frac{1}{|\bar{q}|^3}(\bar{a}_1\times\bar{a}_2+\bar{q}\times\bar{P}_3)+\bar{q}\times\bar{a}_2\left(-3\,\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|^5}\right) =$$

$$\left[\bar{q}\cdot\bar{q}\,(\bar{a}_1\times\bar{a}_2+\bar{q}\times\bar{P}_3)-3\,\bar{q}\times\bar{a}_2\,\bar{q}\cdot\bar{a}_1\right]\frac{1}{|\bar{q}|^5} \tag{2.30}$$

and, similarly, interchanging indices,

$$\frac{\partial^2}{\partial\xi\partial\eta}\left(\frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|}\;\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right) =$$

$$\left[\bar{q}\cdot\bar{q}\,(\bar{a}_2\times\bar{a}_1+\bar{q}\times\bar{P}_3)-3\,(\bar{q}\times\bar{a}_2\,\bar{q}\cdot\bar{a}_1)\right]\frac{1}{|\bar{q}|^5}$$

(2.31)

and thus

$$\frac{\partial^2\bar{I}_V}{\partial\xi\partial\eta}=\left[2\,\bar{q}\cdot\bar{q}\;\bar{a}_1\times\bar{a}_2+3\,(\bar{q}\times\bar{a}_1\,\bar{q}\cdot\bar{a}_2-\bar{q}\times\bar{a}_2\,\bar{q}\cdot\bar{a}_1)\right]\frac{1}{|\bar{q}|^5} \tag{2.32}$$

On the other hand

$$\bar{\nu}\cdot\bar{\nabla}_0\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)|\bar{a}_1\times\bar{a}_2|=$$

$$=-\bar{\nu}\cdot\bar{\nabla}_0\;\frac{\bar{q}\cdot\bar{a}_1\times\bar{a}_2}{|\bar{q}|^3}=$$

$$\bar{\nu}\cdot\left[\frac{\bar{a}_1\times\bar{a}_2}{|\bar{q}|^3}+\bar{q}\cdot\bar{a}_1\times\bar{a}_2\left(-3\frac{\bar{q}}{|\bar{q}|^5}\right)\right]$$

$$\bar{\nu}\cdot\left[\bar{q}\cdot\bar{q}\;\bar{a}_1\times\bar{a}_2-3\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2\,\bar{q}\right]\frac{1}{|\bar{q}|^5}=$$

$$=-\bar{\nu}\cdot\left[2\,\bar{q}\cdot\bar{q}\,\bar{a}_1\times\bar{a}_2+3\,(\bar{q}\times\bar{a}_1\,\bar{q}\cdot\bar{a}_2-\bar{q}\times\bar{a}_2\,\bar{q}\cdot\bar{a}_1)\right]\frac{1}{|\bar{q}|^5}$$

(2.33)

since

$$\bar{q}\cdot\bar{q}\,\bar{a}_1\times\bar{a}_2\cdot\bar{\nu}-\bar{q}\cdot\bar{\nu}\,\bar{q}\cdot\bar{a}_1\times\bar{a}_2=$$

$$\bar{q}\times(\bar{a}_1\times\bar{a}_2)\cdot\bar{q}\times\bar{\nu}=$$

$$\bar{q}\cdot\bar{a}_2\,\bar{a}_1\cdot\bar{q}\times\bar{\nu}-\bar{q}\cdot\bar{a}_1\bar{a}_2\cdot\bar{q}\times\bar{\nu}=$$

$$=-\bar{q}\cdot\bar{a}_2\bar{\nu}\cdot\bar{q}\times\bar{a}_1+\bar{q}\cdot\bar{a}_1\bar{\nu}\cdot\bar{q}\times\bar{a}_2 \tag{2.34}$$

Since $\overline{\nu}$ is an arbitrary vector, Eq. (2.25) is equivalent to

$$-\overline{V}_0 \frac{\partial}{\partial n}\left(\frac{1}{r}\right) |\overline{a}_1 \times \overline{a}_2| = \left[2\overline{q}\cdot\overline{q}\,\overline{a}_1 \times \overline{a}_2 + 3(\overline{q}\times\overline{a}_1\,\overline{q}\cdot\overline{a}_2 - \overline{q}\times\overline{a}_2\,\overline{q}\cdot\overline{a}_1)\right]\frac{1}{|q|^5} \tag{2.35}$$

Equations (2.32) and (2.35) are the desired proof of the validity of Eq. (2.28).

2.5 Singularity at $\overline{q} = 0$

If the point $\overline{P_0}$ belongs to the surface Σ_k, the integral in Eq. (1.7) is singular.* In the following, the type of singularity is analyzed and it is shown that the principal value of the integral must be used. Consider a small circle of radius ε in the neighborhood of the singularity. Assume that the point $\overline{P_0}$ is at very small distance from the surface Σ_k and consider a small circular element Σ_ε on Σ_k with the center on the normal projection of $\overline{P_0}$ on the surface Σ_k and radius ε.

Assuming the z-axis to be directed along the normal $\overline{n}$, Eq. (1.7) reduces to

$$\overline{v}_h = \overline{V}_0 \iint_{\Sigma_k - \Sigma_\varepsilon} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma + \overline{v}_\varepsilon \tag{2.36}$$

with (for symmetry reasons, the derivatives with respect to x_0 and y_0 are zero)

$$\overline{v}_\varepsilon = \overline{V}_0 \iint_{\Sigma_\varepsilon} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma = \overline{n}\, 2\pi \frac{\partial}{\partial z_0} \int_0^\varepsilon \frac{z_0}{(\rho^2 + z_0^2)^{3/2}}\,\rho\, d\rho\, d\theta =$$

---

* P_0 is <u>not</u> on the boundary of Σ_k.

$$= 2\pi\bar{n}\left(\int_0^{\varepsilon}\frac{\rho}{(\rho^2+z_0^2)^{3/2}}\,d\rho - 3z_0^2\int_0^{\varepsilon}\frac{\rho}{(\rho^2+z_0^2)^{5/2}}\,d\rho =\right.$$

$$-2\pi\bar{n}\left[-\frac{1}{\sqrt{\rho^2+z_0^2}} + z_0^2\frac{1}{(\rho^2+z_0^2)^{3/2}}\right]\Bigg|_0^{\varepsilon} =$$

$$-2\pi\bar{n}\left[-\frac{\rho^2}{(\rho^2+z_0^2)^{3/2}}\right]\Bigg|_0^{\varepsilon} = -2\pi\bar{n}\frac{\varepsilon}{(\varepsilon^2+z_0^2)^{3/2}}$$

(2.37)

As Z_0 approaches zero, one obtains

$$\bar{v}_k = \bar{V}_0\iint_{\Sigma_k}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\Sigma = \oint\!\!\!\oint_{\Sigma_k}\bar{V}_0\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\Sigma \tag{2.38}$$

with

$$\oint\!\!\!\oint_{\Sigma_k}\bar{V}_0\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\Sigma = \iint_{\Sigma_k-\Sigma_\varepsilon}\bar{V}_0\frac{\partial}{\partial n}\left(\frac{1}{r}\right)d\Sigma - 2\pi\frac{1}{\varepsilon^2}\bar{n} \tag{2.39}$$

It may be noted that the first expression in Eq. (2.38) is not singular.* Hence, Eqs. (2.8) and (2.27) (obtained by using the first integral in Eq. (2.38)) are still valid even if the point is on the surface.

2.6 General Element

In this subsection, it is shown how the results obtained thus far can be rewritten in a more expressive fashion. For the sake of simplicity, introduce the following notations (Fig. 3)

---

* P_0 is not on the boundary of Σ_k

$$\bar{q}(1,1) = \bar{P}_{pp} - \bar{P}^{(k)} = \bar{Q}_1$$
$$\bar{q}(-1,1) = \bar{P}_{mp} - \bar{P}^{(k)} = \bar{Q}_2$$
$$\bar{q}(-1,-1) = \bar{P}_{mm} - \bar{P}^{(k)} = \bar{Q}_3$$
$$\bar{q}(1,-1) = \bar{P}_{pm} - \bar{P}^{(k)} = \bar{Q}_4$$

(2.40)

Note that

$$\bar{a}_1(\eta=1) = (\bar{Q}_1 - \bar{Q}_2)/2$$
$$\bar{a}_1(\eta=-1) = (\bar{Q}_4 - \bar{Q}_3)/2$$
$$\bar{a}_2(\xi=1) = (\bar{Q}_1 - \bar{Q}_4)/2$$
$$\bar{a}_2(\xi=-1) = (\bar{Q}_2 - \bar{Q}_3)/2$$

(2.41)

Next, combining Eqs. (2.7), (2.27) (2.40), and (2.41), one obtains

$$\bar{v}_{hk} = \bar{I}_v(1,1) - \bar{I}_v(-1,1) + \bar{I}_v(-1,-1) - \bar{I}_v(1,-1) =$$

$$\left[\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} - \frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right]_{\xi=1,\,\eta=1} -$$

$$\left[\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} - \frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right]_{\xi=-1,\,\eta=1} +$$

$$\left[\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} - \frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right]_{\xi=-1,\,\eta=-1} -$$

$$\left[\frac{\bar{q}\cdot\bar{a}_2}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_2}{|\bar{q}\times\bar{a}_2|^2} - \frac{\bar{q}\cdot\bar{a}_1}{|\bar{q}|}\frac{\bar{q}\times\bar{a}_1}{|\bar{q}\times\bar{a}_1|^2}\right]_{\xi=1,\,\eta=-1}$$

(2.42)

or

$$\bar{v}_{hk} = \frac{\bar{Q}_1\cdot(\bar{Q}_1-\bar{Q}_4)}{|\bar{Q}_1|}\,\frac{\bar{Q}_1\times(\bar{Q}_1-\bar{Q}_4)}{|\bar{Q}_1\times(\bar{Q}_1-\bar{Q}_4)|^2} - \frac{\bar{Q}_1\cdot(\bar{Q}_1-\bar{Q}_2)}{|\bar{Q}_1|}\,\frac{\bar{Q}_1\times(\bar{Q}_1-\bar{Q}_2)}{|\bar{Q}_1\times(\bar{Q}_1-\bar{Q}_2)|^2}$$

$$- \frac{\bar{Q}_2\cdot(\bar{Q}_2-\bar{Q}_3)}{|\bar{Q}_2|}\,\frac{\bar{Q}_2\times(\bar{Q}_2-\bar{Q}_3)}{|\bar{Q}_2\times(\bar{Q}_2-\bar{Q}_3)|^2} + \frac{\bar{Q}_2\cdot(\bar{Q}_1-\bar{Q}_2)}{|\bar{Q}_2|}\,\frac{\bar{Q}_2\times(\bar{Q}_1-\bar{Q}_2)}{|\bar{Q}_2\times(\bar{Q}_1-\bar{Q}_2)|^2}$$

$$+ \frac{\bar{Q}_3\cdot(\bar{Q}_2-\bar{Q}_3)}{|\bar{Q}_3|}\,\frac{\bar{Q}_3\times(\bar{Q}_2-\bar{Q}_3)}{|\bar{Q}_3\times(\bar{Q}_2-\bar{Q}_3)|^2} - \frac{\bar{Q}_3\cdot(\bar{Q}_4-\bar{Q}_3)}{|\bar{Q}_3|}\,\frac{\bar{Q}_3\times(\bar{Q}_4-\bar{Q}_3)}{|\bar{Q}_3\times(\bar{Q}_4-\bar{Q}_3)|^2}$$

$$- \frac{\bar{Q}_4(\bar{Q}_1-\bar{Q}_4)}{|\bar{Q}_4|}\,\frac{\bar{Q}_4\times(\bar{Q}_1-\bar{Q}_4)}{|\bar{Q}_4\times(\bar{Q}_1-\bar{Q}_4)|^2} + \frac{\bar{Q}_4\cdot(\bar{Q}_4-\bar{Q}_3)}{|\bar{Q}_4|}\,\frac{\bar{Q}_4\times(\bar{Q}_4-\bar{Q}_3)}{|\bar{Q}_4\times(\bar{Q}_4-\bar{Q}_3)|^2} \qquad (2.43)$$

or

$$\bar{v}_{hk} = \frac{\bar{Q}_4\times\bar{Q}_1}{|\bar{Q}_4\times\bar{Q}_1|^2}\left[\frac{\bar{Q}_4\cdot\bar{Q}_4-\bar{Q}_1\cdot\bar{Q}_4}{|\bar{Q}_4|}+\frac{\bar{Q}_1\cdot\bar{Q}_1-\bar{Q}_1\cdot\bar{Q}_4}{|\bar{Q}_1|}\right]+$$
$$\frac{\bar{Q}_1\times\bar{Q}_2}{|\bar{Q}_1\times\bar{Q}_2|^2}\left[\frac{\bar{Q}_1\cdot\bar{Q}_1-\bar{Q}_1\cdot\bar{Q}_2}{|\bar{Q}_1|}+\frac{\bar{Q}_2\cdot\bar{Q}_2-\bar{Q}_1\cdot\bar{Q}_2}{|\bar{Q}_2|}\right]+$$
$$\frac{\bar{Q}_2\times\bar{Q}_3}{|\bar{Q}_2\times\bar{Q}_3|^2}\left[\frac{\bar{Q}_2\cdot\bar{Q}_2-\bar{Q}_2\cdot\bar{Q}_3}{|\bar{Q}_2|}+\frac{\bar{Q}_3\cdot\bar{Q}_3-\bar{Q}_2\cdot\bar{Q}_3}{|\bar{Q}_3|}\right]+$$
$$\frac{\bar{Q}_3\times\bar{Q}_4}{|\bar{Q}_3\times\bar{Q}_4|^2}\left[\frac{\bar{Q}_3\cdot\bar{Q}_3-\bar{Q}_3\cdot\bar{Q}_4}{|\bar{Q}_3|}+\frac{\bar{Q}_4\cdot\bar{Q}_4-\bar{Q}_3\cdot\bar{Q}_4}{|\bar{Q}_4|}\right] \qquad (2.44)$$

It may be noted that each of the four terms depends upon two corners of one edge of the element. Hence, Eq. (2.44) is independent of the numbering used (it depends, however, upon the direction of the numbering which is anticlockwise with respect to the normal $\bar{n}$).

Next, consider the limit of Eq. (2.44) when one edge shrinks to zero, that is when the hyperboloidal element reduces to a triangular element. As mentioned, the numbering is inessential. Hence, without loss of generality, it is assumed that $\overline{Q}_4 \to \overline{Q}_3$ (see Fig. 4). By setting

$$\overline{Q}_4 - \overline{Q}_3 = \lambda\, \overline{Q}_{43} \tag{2.45}$$

where $\overline{Q}_{43}$ is a unit vector and λ tends to zero. The last term of Eq. (2.44) yields

$$\begin{aligned}
&\lim_{\lambda \to 0} \frac{\overline{Q}_3 \times \overline{Q}_4}{|\overline{Q}_3 \times \overline{Q}_4|^2} \left(\frac{\overline{Q}_3 \cdot \overline{Q}_3 - \overline{Q}_3 \cdot \overline{Q}_4}{|\overline{Q}_3|} + \frac{\overline{Q}_4 \cdot \overline{Q}_4 - \overline{Q}_3 \cdot \overline{Q}_4}{|\overline{Q}_4|} \right) = \\
&\lim_{\lambda \to 0} \frac{\lambda\, \overline{Q}_3 \times \overline{Q}_{43}}{\lambda^2 |\overline{Q}_3 \times \overline{Q}_{43}|^2} \left(- \frac{\lambda\, \overline{Q}_3 \cdot \overline{Q}_{43}}{|\overline{Q}_3|} + \frac{\lambda\, \overline{Q}_4 \cdot \overline{Q}_{43}}{|\overline{Q}_4|} \right) = \\
&\lim_{Q_4 \to Q_3} \frac{\overline{Q}_3 \times \overline{Q}_{43}}{|\overline{Q}_3 \times \overline{Q}_{43}|^2} \left(\frac{-\overline{Q}_3}{|\overline{Q}_3|} + \frac{\overline{Q}_4}{|\overline{Q}_4|} \right) \cdot \overline{Q}_{43} = 0
\end{aligned} \tag{2.46}$$

Hence, for triangular elements

$$\begin{aligned}
\overline{v}_{hk} = &\frac{\overline{Q}_3 \times \overline{Q}_1}{|\overline{Q}_3 \times \overline{Q}_1|^2} \left[\frac{\overline{Q}_3 \cdot \overline{Q}_3 - \overline{Q}_1 \cdot \overline{Q}_3}{|\overline{Q}_3|} + \frac{\overline{Q}_1 \cdot \overline{Q}_1 - \overline{Q}_1 \cdot \overline{Q}_4}{|\overline{Q}_1|} \right] + \\
&\frac{\overline{Q}_1 \times \overline{Q}_2}{|\overline{Q}_1 \times \overline{Q}_2|^2} \left[\frac{\overline{Q}_1 \cdot \overline{Q}_1 - \overline{Q}_1 \cdot \overline{Q}_2}{|\overline{Q}_1|} + \frac{\overline{Q}_2 \cdot \overline{Q}_2 - \overline{Q}_1 \cdot \overline{Q}_2}{|\overline{Q}_2|} \right] + \\
&\frac{\overline{Q}_2 \times \overline{Q}_3}{|\overline{Q}_2 \times \overline{Q}_3|^2} \left[\frac{\overline{Q}_2 \cdot \overline{Q}_2 - \overline{Q}_2 \cdot \overline{Q}_3}{|\overline{Q}_2|} + \frac{\overline{Q}_3 \cdot \overline{Q}_3 - \overline{Q}_2 \cdot \overline{Q}_3}{|\overline{Q}_3|} \right]
\end{aligned} \tag{2.47}$$

Similarly, for a polygonal element with n corners

$$\bar{\nabla}_{hk} = \bar{T}_{1,2} + \bar{T}_{2,3} + \cdots + \bar{T}_{n,1} \tag{2.48}$$

with

$$\bar{T}_{ij} = \frac{\bar{Q}_i \times \bar{Q}_j}{|\bar{Q}_i \times \bar{Q}_j|^2} \left(\frac{\bar{Q}_i \cdot \bar{Q}_i - \bar{Q}_i \cdot \bar{Q}_j}{|\bar{Q}_i|} + \frac{\bar{Q}_j \cdot \bar{Q}_j - \bar{Q}_i \cdot \bar{Q}_j}{|\bar{Q}_j|} \right) \tag{2.49}$$

Equation (2.48) can be proved as follows. The solid angle is an additive quantity. Hence, $\bar{\nabla}_{hk}$, which is the gradient of the solid angle is an additive quantity. Thus, the general proof is obtained by mathematical induction: assumed to be true for $n = n_0$, it is shown to be true for $n = n_0 + 1$. Thus (see Fig. 5 for the case $n_0 = 4$), noting that $\bar{T}_{ij} = -\bar{T}_{ji}$

$$\begin{aligned}
(\bar{\nabla}_{hk})_{A+B} &= (\bar{\nabla}_{hk})_A + (\bar{\nabla}_{hk})_B = \\
&(\bar{T}_{1,3} + \bar{T}_{3,4} + \bar{T}_{4,5} + \cdots + \bar{T}_{n,1}) + (\bar{T}_{1,2} + \bar{T}_{2,3} - \bar{T}_{1,3}) = \\
&\bar{T}_{1,2} + \bar{T}_{2,3} + \bar{T}_{3,4} + \bar{T}_{4,5} + \cdots + \bar{T}_{n,1}
\end{aligned} \tag{2.50}$$

in agreement with Eq. (2.48).

SECTION III

SUMMARY AND RESULTS

Consider Eq. (1.3); dropping the subscript 0 yields

$$\varphi = \iint_{\Sigma_B} D \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma + \iint_{\Sigma_W} D_{T.E.} \frac{\partial}{\partial n_u}\left(\frac{1}{r}\right) d\Sigma \tag{3.1}$$

Assume that the geometry of the wake is prescribed as straight vortex lines or from the preceeding iterations (See Fig. 5; for a description of the iteration procedure see Ref. 5). Divide the wake into L strips, Σ_ℓ , each bounded by two streamlines. Divide the surface of the body into small polygonal elements (hyperboloidal quadrilateral, or triangular, for instance).

Then, Eq. (3.1) can be approximated by

$$\varphi = \sum_{h=1}^{N} D_h \iint_{\Sigma_h} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_h + \sum_{\ell=1}^{L} \Delta D_{TE,\ell} \iint_{\Sigma_\ell} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_\ell \tag{3.2}$$

Next, assume that, in virtue of the Kutta condition it is possible to replace $\Delta D_{TE,\ell}$ with the values of D at the centroid Σ_h of the element having an edge in common with the strip Σ_ℓ. Then, Eq. (3.2) can be rewritten as

$$\varphi = \sum_{h=1}^{N} D_h \iint_{\Sigma_h^*} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_h \tag{3.3}$$

where

$$\Sigma_h^* = \Sigma_h \tag{3.4}$$

if Σ_h has no edge in contact in contact with the wake, while

$$\Sigma_h^* = \Sigma_h + \Sigma_\ell \tag{3.5}$$

if Σ_h has an edge in contact with the strip Σ_ℓ.

The perturbation velocity, $\bar{v}_h$, at the centroid, $\bar{P}_h$ of the element Σ_h is given by

$$\bar{v}_h = \left[\bar{\nabla}_0 \varphi\right]_{\bar{P}=\bar{P}_h} = \sum_1^N D_k \bar{v}_{hk} \tag{3.6}$$

where

$$\bar{v}_{hk} = \left(\bar{\nabla}_0 \iint_{\Sigma_k} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\Sigma_k\right)_{\bar{P}=\bar{P}_k} \tag{3.7}$$

Finally, imposing the boundary condition at the centroid of the elements, Σ_h, yields the system

$$[A_{hk}]\{D_k\} = \{B_h\} \tag{3.8}$$

where

$$A_{hk} = \bar{v}_{hk} \cdot \bar{n}_h \tag{3.9}$$

while

$$B_h = \left(\frac{\partial \varphi}{\partial n}\right)_{\bar{P}=\bar{P}_h} \tag{3.10}$$

is prescribed from the boundary conditions. Solving Eq. (3.8) yields the coefficient D_k: then, it is possible to evaluate $\bar{v}_h$ through Eq. (3.6).

The integral in Eq. (3.4) can be evaluated by using Eq. (2.48) for a general polygonal element, or Eq.(2.47) for triangular elements, or Eq. (2.44) for hyperboloidal quadrilateral elements . Note that, if the element Σ_h^* includes a strip $\sum_\ell$, it will be convenient to approximate it with a series of quadrilateral subelements. Then $\sum_h^*$ can be treated as a single polygonal element: in this way the contribution of the edges (which would eventually eliminate each other) need not be evaluated.

This formulation has been implemented into a computer program, ILSA , (acronym for Incompressible Lifting Surface Aerodynamics). See also Ref. 5. Figure 7 shows the lift coefficient distribution per unit angle of attack for a rectangular wing of AR = 8, at Mach Number M = 0. A convergence study for various numbers of wing elements is also shown and compared to the result obtained by Yates (Ref. 6). The results obtained with ILSA indicate good agreement with existing ones and a fast rate of convergence. As mentioned before, a better wake geometry can be obtained by an iteration process. This process is shown in detail in Reference 5.

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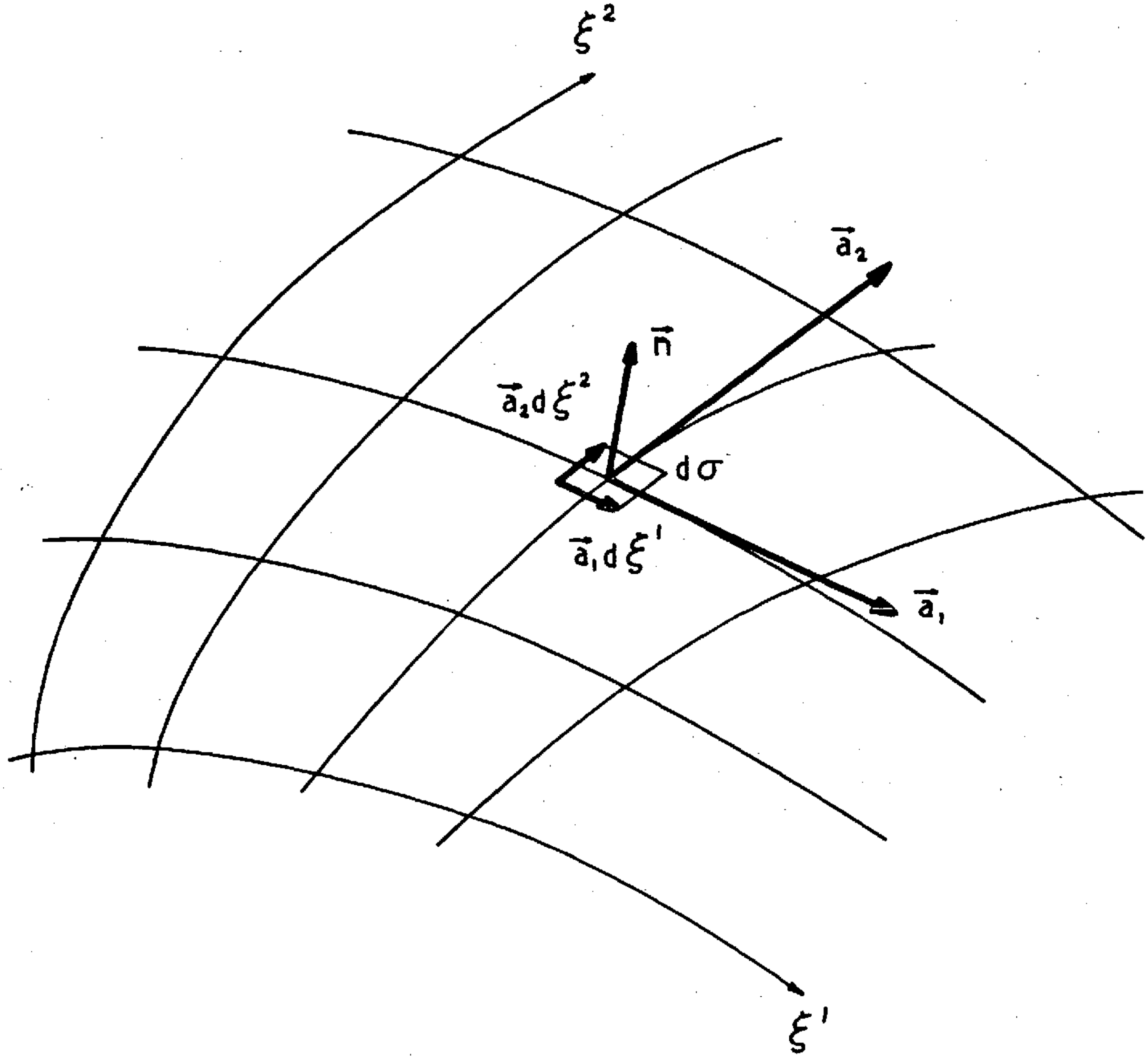

Fig. 1 Surface geometry

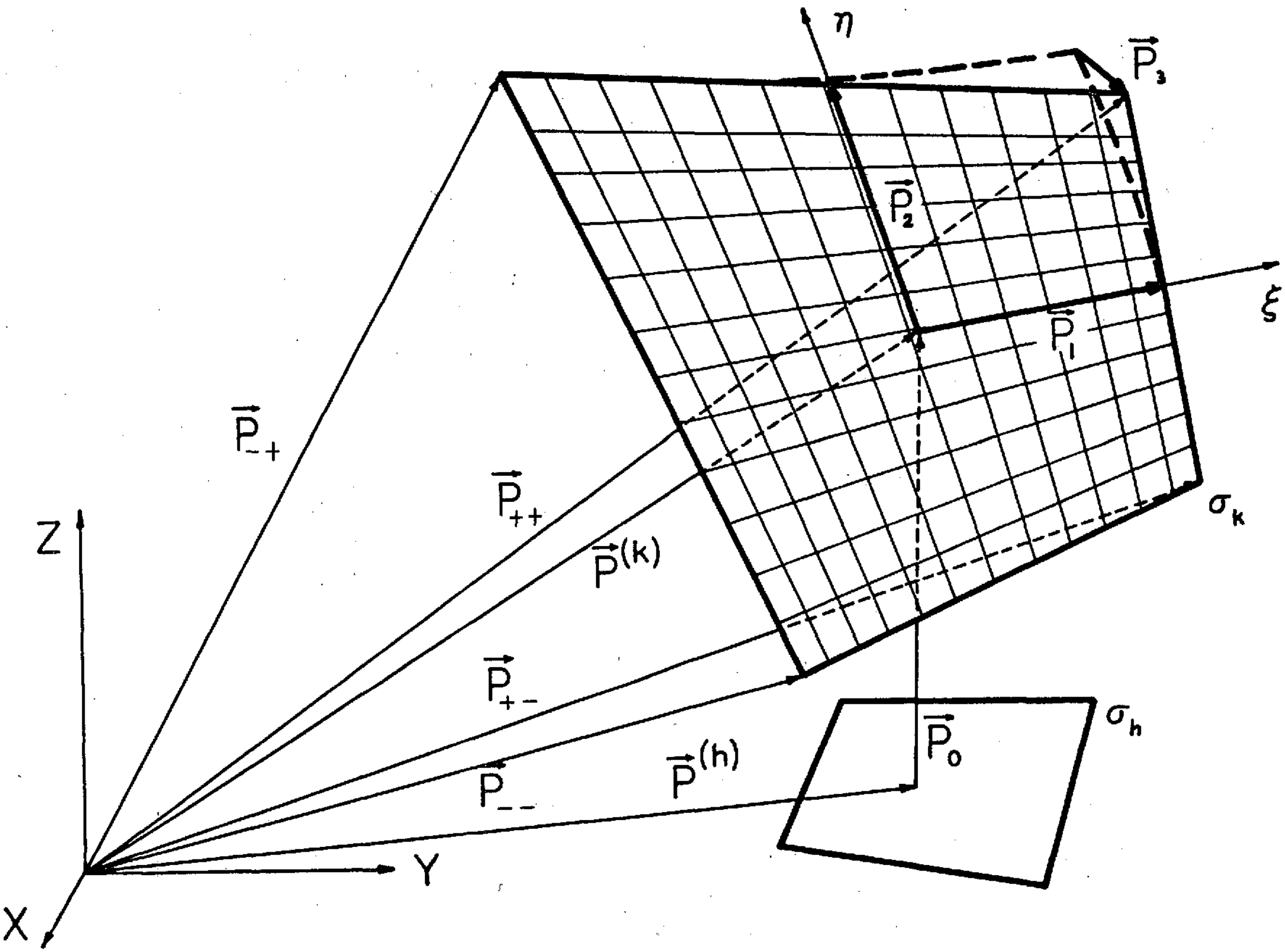

Fig. 2 Geometry of hyperboloidal element

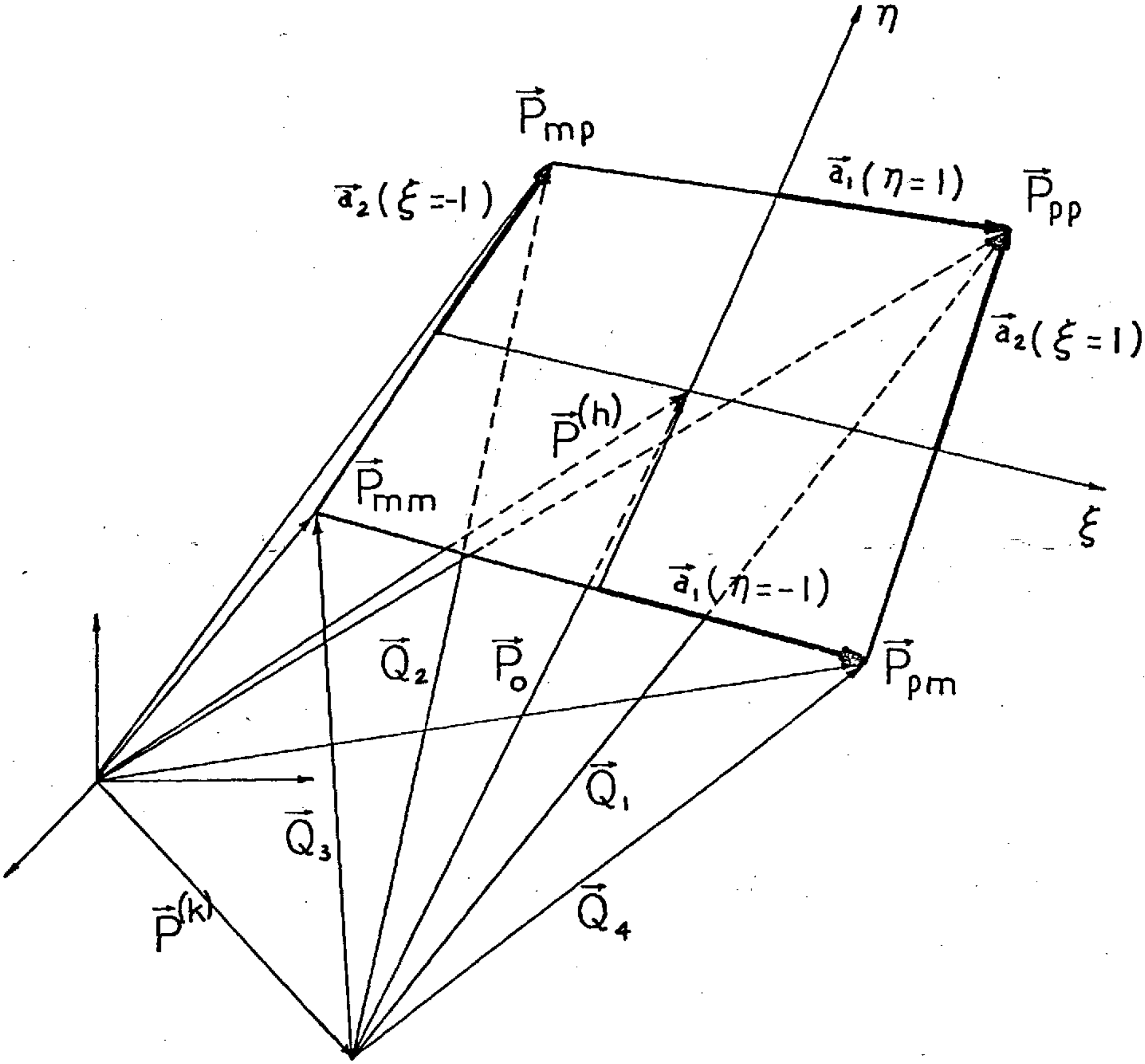

Fig. 3. Hyperboloidal-Element Geometry with Definition of the Vectors $\vec{Q}_h$

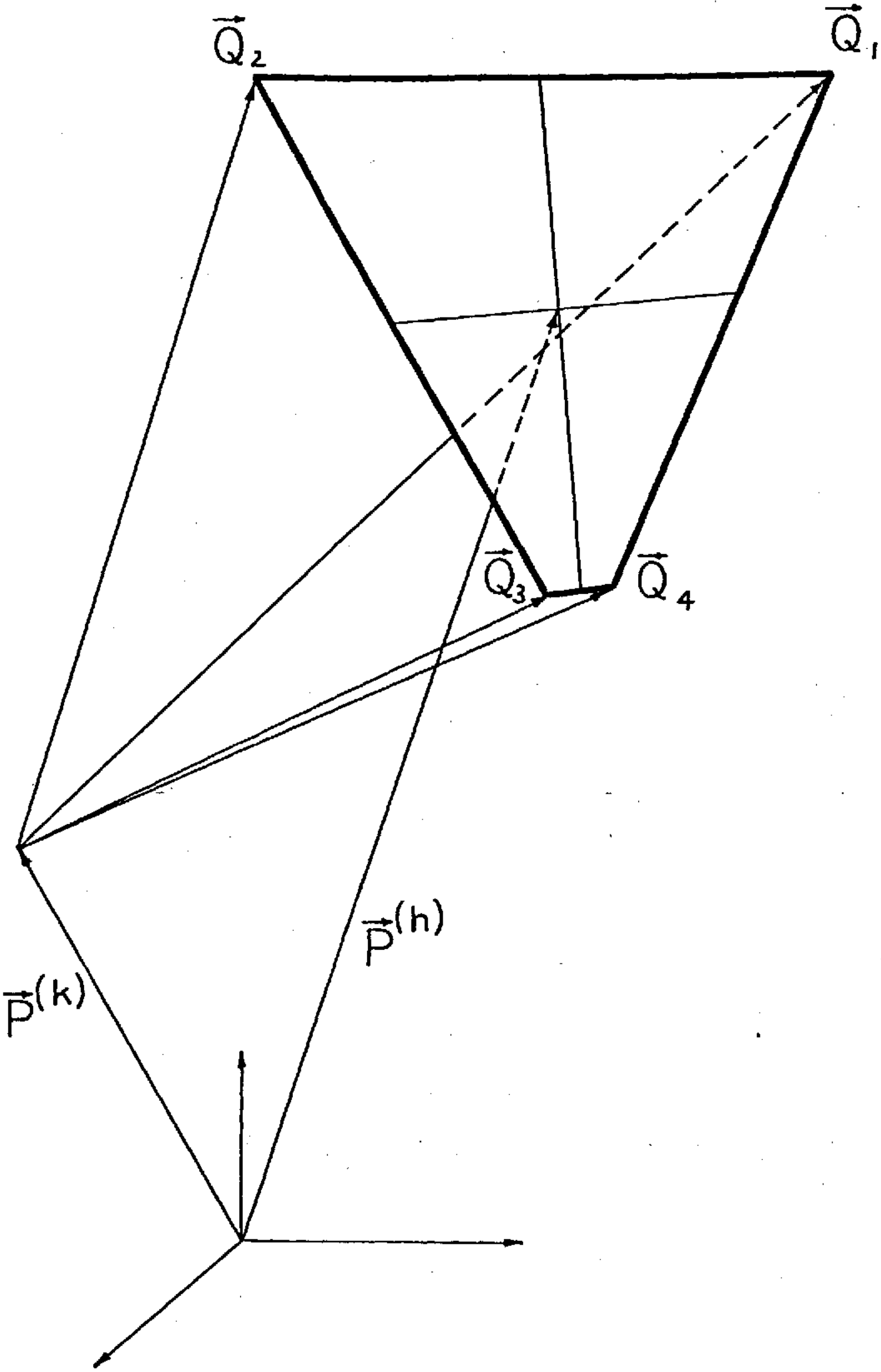

Fig. 4. Triangular element as the limit of a hyperboloidal quadrilateral element

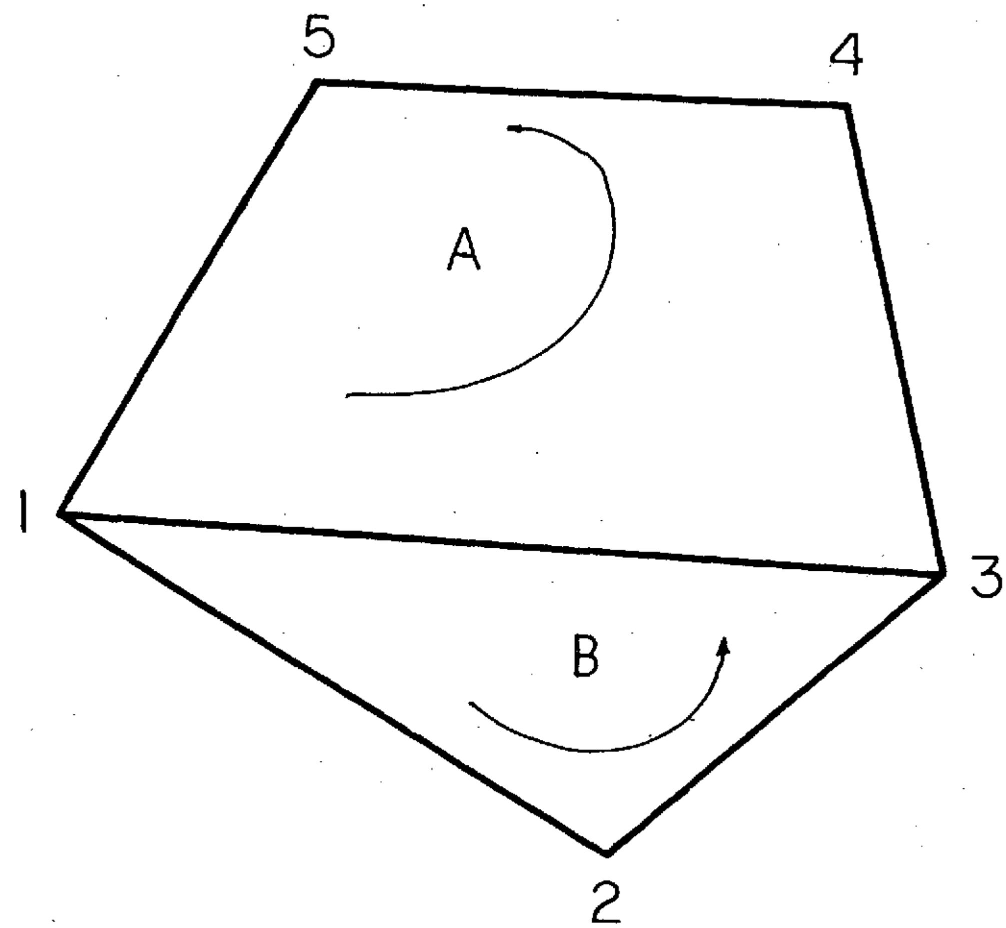

Fig. 5. Pentagonal element

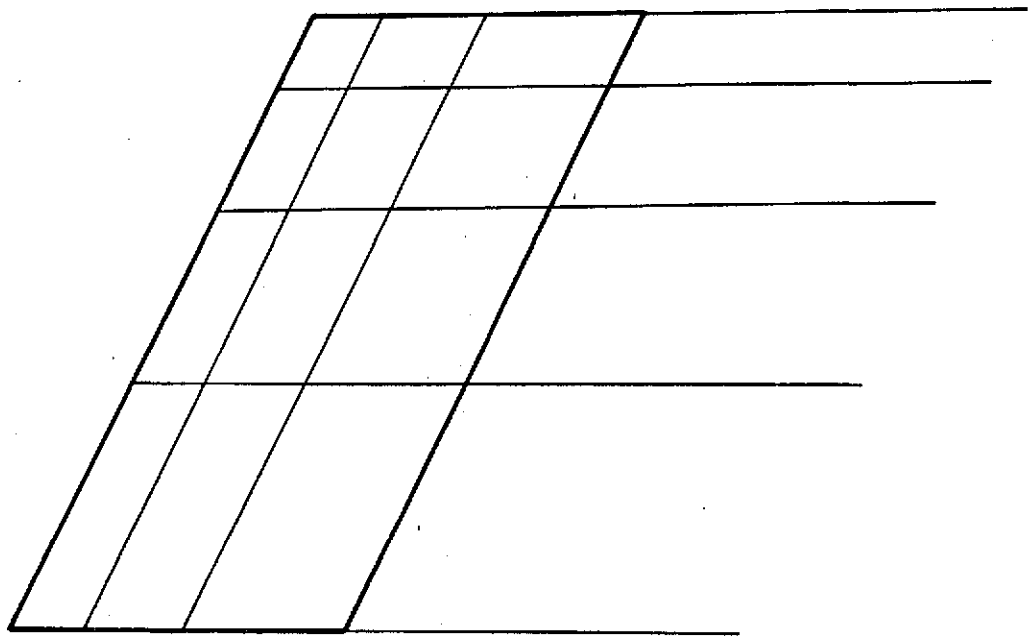

Fig. 6. Lifting surface and wake geometry

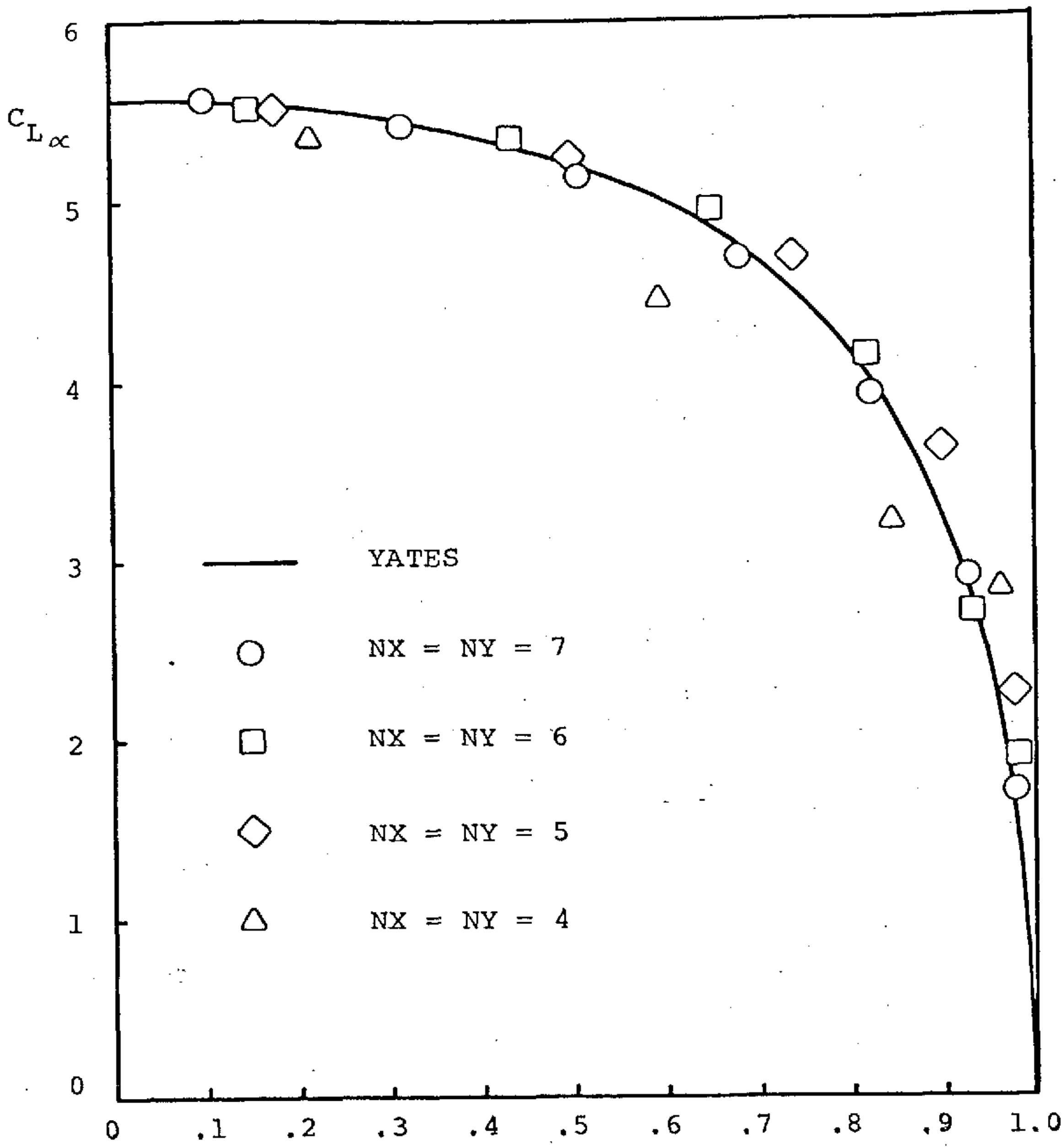

Fig. 7. Convergence problem: section lift coefficient distribution per unit angle of attack for a rectangular wing of AR = 8, M = 0 and comparison with the result obtained by Yates.