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BOSTON UNIVERSITY

COLLEGE OF ENGINEERING

BOSTON, MASS. 02215

SUBSONIC AND SUPERSONIC INDICIAL AERODYNAMICS AND AERODYNAMIC TRANSFER FUNCTION FOR COMPLEX CONFIGURATIONS

by

Luigi Morino

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E. Carson Yates, Jr., Technical Advisor

ABSTRACT

A general theory for indicial potential compressible aerodynamics around complex configurations is presented. The motion is assumed to consist of constant subsonic or supersonic speed for $t \leq 0$ (steady state) and of small perturbations around the steady state for $t > 0$. Using the finite-element method to discretize the space problem one obtains a set of differential-difference equations in time relating the potential to its normal derivative on the surface of the body. The aerodynamics transfer function is then obtained by using standard method of operational calculus.

FOREWORD

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LIST OF SYMBOLS

a_∞	Speed of sound in undisturbed flow
$\tilde{A}_{JM}$	Subsonic matrix transfer function, Eq. (24)
$\tilde{A}'_{JM}$	Supersonic matrix transfer function, Eq. (36)
B_L	Eq. (18)
B'_L	Eq. (32)
B_{JL}	Eq. (21)
B'_{JL}	Eq. (34)
C_L	Eq. (18)
C'_L	Eq. (32)
C_{JL}	Eq. (21)
C'_{JL}	Eq. (34)
D_L	Eq. (18)
D'_L	Eq. (32)
D_{JL}	Eq. (21)
D'_{JL}	Eq. (34)
E	Eq. (21)
F_N	Eq. (18)
F_{JN}	Eq. (21)
G_N	Eq. (18)
G_{JN}	Eq. (21)
H	Eq. (28)
L_0	Number of nodes on body
L_N	Shape-functions for $\Delta\Phi$, Eq. (14)
M	Mach number, U_∞ / a_∞

M_o	Number of nodes on trailing edge
M_L	Shape-functions for Ψ , Eq. (12)
$\vec{n}$	Normal to σ
$\vec{N}$	Normal to Σ
N_o	Number of nodes on wake
N_L	Shape-functions for Φ , Eq. (12)
P	Point having coordinates (X,Y,Z)
P_*	Control point, (X_*, Y_*, Z_*)
R	Eq. (3)
R'	Eq. (27)
s	Complex frequency (for Laplace transform)
$S(P,T)$	Function for definition of Σ_B
S_{NL}	Eq. (16)
t	Time
T	Nondimensional time, Eq. (2)
U_∞	Velocity of undisturbed flow
x, y, z	Space Coordinates
X, Y, Z	Nondimensional space coordinates, Eq. (2)
β	$\sqrt{1-M^2}$
$\Delta\Phi$	Discontinuity of Φ across the wake
$\Delta\Phi_N$	Nodal values of $\Delta\Phi$
Θ	Eq. (5)
Θ^+	Eq. (30)
Θ^-	Eq. (30)
Θ_L	Θ for $P \equiv P_L$
Θ_L^+	Θ^+ for $P \equiv P_L$

Θ^-_L	Θ^- for $P = P_L$
Θ_{JL}	Eq. (21)
Θ^+_{JL}	Eq. (34)
Θ^-_{JL}	Eq. (34)
Π	Convection time of wake vortices, Eq. (11)
Π_N	Value of Π for $P = P_N$, Eq. (15)
Σ	Surface surrounding body and wake in X, Y, Z space
Σ_B	Surface of body, defined by $S(P, T) = 0$
Σ_W	Surface of wake
φ	Velocity perturbation potential
ϕ	Velocity potential, $U_\infty x + \varphi$
Φ	Nondimensional velocity perturbation potential, $\varphi/U_\infty \ell$
Φ_J	Nodal values of Φ
$\tilde{\Phi}_J$	Laplace's transform of Φ_J
ψ	Normal-wash in x,y,z space,
Ψ	Normal-wash in x,y,z space
Ψ_J	Nodal value of Ψ
$\tilde{\Psi}_J$	Laplace's transform of Ψ_J
∇	Gradient in X,Y,Z space
$\tilde{(\ )}$	Laplace's transform of $(\ )$

Subscripts:

B	Body
J, L	Nodes on the body (range from 1 to L_o)
M	Nodes on trailing edge (ranges from 1 to M_o)
N	Nodes on wake (ranges from 1 to N_o)
W	Wake
TE	Trailing edge

INTRODUCTION

Considered in this note is the problem of unsteady subsonic and supersonic potential aerodynamics for an aircraft having arbitrary shape. The motion of the aircraft is assumed to consist of small perturbations with respect to the constant speed motion. The objective of this paper is to describe the time functional relationship between aerodynamic potential and its normal derivative (normal-wash, $\psi = \partial\varphi/\partial n$) in a form which can be easily used for the flight-dynamics analysis in both time and frequency-domain. The finite element method is used for space-discretization.

The analysis presented here is based upon a new integral formulation, derived by the author,[1,2] which includes completely arbitrary motion. However, the numerical implementation (Refs. 3 and 4) is limited to steady and oscillatory flows. On the other hand, in order to perform a linear-system analysis of the aircraft it is convenient to use more general aerodynamic formulations i.e. fully transient response for time-domain analysis, and the aerodynamic transfer function (Laplace's transform of the fully unsteady operator) for frequency-domain analysis (see for instance Ref. 5). Consistently with this type of analysis the unsteady contribution is assumed to start at time t=0, so that for time $t \leq 0$ the flow is in steady state. Furthermore the motion of the aircraft is assumed to consist of small (infinitesimal) perturbation around the steady state motion. Since the initial work by Wagner[6] on unsteady incompressible two-dimensional flow, several problems have been considered. Detailed analysis of the various methods available are given for instance in Refs. 5 and 7. Finite wings can be solved only for particular planform (such as elliptical wings) for subsonic flows, and for general planform in supersonic flows. Strip theory is used for slender wings in subsonic flow.

For subsonic and supersonic flows around arbitrary complex configurations no tool has been available for either time- or frequency-domain analysis. Such a tool is presented in this note, for both time and frequency domain. For the sake of brevity, only the relationship potential-downwash is outlined since the full transfer function (generalized force per unit generalized coordinate) can be easily obtained by including the relation downwash-generalized coordinate (boundary conditions on the body) and the relation between potential and generalized forces.

SUBSONIC INDICIAL AERODYNAMICS

Consider the subsonic case first. Within the small perturbation assumption, the motion of the surface of the aircraft with respect to a frame of reference traveling at uniform subsonic speed with respect to the undisturbed air can be assumed to be negligible. Thus the Green theorem for the equation of the aerodynamic potential is given by[1,2]

$$4\pi\, E(P_*)\, \Phi(P_*, T) = \oiint_\Sigma [\Psi]^\Theta \frac{1}{R}\, d\Sigma + \oiint_\Sigma [\Phi]^\Theta \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma - \oiint_\Sigma \left[\frac{\partial \Phi}{\partial T}\right]^\Theta \frac{1}{R}\frac{\partial R}{\partial N}\, d\Sigma \quad (1)$$

where Σ is a surface surrounding the body and the wake, $\Psi = \partial\Phi/\partial N$ is the normal derivative (normal-wash) of Φ on Σ, $\vec{N}$ is the normal to the surface Σ,

$$\Phi = \varphi/U_\infty \ell,\quad X = x/\beta\ell,\quad Y = y/\ell,\quad Z = z/\ell,\quad T = a_\infty \beta t/\ell \quad (2)$$

and

$$R = \left[(X - X_*)^2 + (Y - Y_*)^2 + (Z - Z_*)^2\right]^{\frac{1}{2}} \quad (3)$$

while

$$[\quad]^\Theta \equiv [\quad]_{T_1 = T - \Theta} \quad (4)$$

where

$$\Theta = M\left(X - X_*\right) + R \tag{5}$$

is the time necessary for a disturbance to propagate from P to P_*. Furthermore

$$\begin{aligned} E &= 1 && \text{outside } \Sigma \\ &= 1/2 && \text{on } \Sigma \\ &= 0 && \text{inside } \Sigma \end{aligned} \tag{6}$$

In Eq. 1 the surface Σ is assumed to be fixed with respect to the frame of reference. However, the effect of the motion of the surface is retained in the boundary condition, which gives the normal-wash approximately as

$$\Psi = \frac{\partial \Phi}{\partial N} = -\left(\frac{1}{M}\frac{\partial S}{\partial T} + \frac{\partial S}{\partial X}\right) \Big/ \left|\nabla S\right| \tag{7}$$

(where $S(P,T) = 0$ is the equation of the surface Σ). This approximation is consistent with the hypothesis of small perturbation which has been invoked also in writing Eq. (1). Also consistent with the hypothesis of small perturbation with respect to the constant-speed motion, is the assumption that the surface of the wake is the one of the steady-state case.

It may be noted that, because of the linearity of the problem, the state-steady contribution can be subtracted from Eq. (1). Therefore in the following it is understood that Φ and Ψ are the unsteady parts of the potential and the downwash, which (in line with the concepts of operational calculus) are assumed to be identically equal to zero for $T \leq 0$:

$$\Phi \equiv 0 \, , \quad \Psi \equiv 0 \qquad (T \leq 0) \tag{8}$$

In order to understand the nature of the aerodynamic operator, Eq. (1), it is convenient to isolate the contribution of the wake. This yields

$$4\pi E(P_*)\,\Phi(P_*,T) = -\oiint_{\Sigma_B} [\Psi]^{\ominus} \frac{1}{R}\, d\Sigma_B + \oiint_{\Sigma_B} \left\{ [\Phi]^{\ominus} \frac{\partial}{\partial N}\left(\frac{1}{R}\right) - \left[\frac{\partial \Phi}{\partial T}\right]^{\ominus} \frac{1}{R}\frac{\partial R}{\partial N} \right\} d\Sigma_B + \iint_{\Sigma_W} \left\{ [\Delta\Phi]^{\ominus} \frac{\partial}{\partial N}\left(\frac{1}{R}\right) - \left[\frac{\partial}{\partial T}\Delta\Phi\right]^{\ominus} \frac{1}{R}\frac{\partial R}{\partial N} \right\} d\Sigma_W \tag{9}$$

where Σ_B is the (closed) surface of the body, while Σ_W is the (open) surface of the wake, and $\Delta\Phi$ is the potential-discontinuity across of the wake, evaluated in the direction of the normal (i.e. $\Delta\Phi \equiv \Phi_u - \Phi_\ell$ if the upper normal is used). It should be noted that the value of $\Delta\Phi$ is not an additional unknown, since it can be easily shown from the Bernoulli theorem that (see Appendix B)

$$\Delta\Phi(P,T) = \Delta\Phi(P_{TE}, T - \Pi) \tag{10}$$

where Π is the nondimensional time necessary for the vortex-point to travel (within the steady flow) from the point, P_{TE} (origin of the vortex-line at the trailing edge), to the point P.

If small perturbation hypothesis can be used for the steady state flow, then Π can be approximated by

$$\Pi \simeq \beta^2 (X - X_{TE})/M \tag{11}$$

Equations (9) and (10) fully describe the problem of linearized unsteady subsonic potential aerodynamics around complex configurations. In order to solve this problem, it is necessary to obtain a numerical approximation for Eq. (9). This can be obtained by using for instance the finite-element method. While other methods can be used, the finite element one appears to be at present time the only method sufficiently general and flexible to be used here. Consider the integrals on Σ_B first. Using a typical finite-element representation, it is possible to write

$$\Psi(P, T-\Theta) = \sum_{L=1}^{L_0} \Psi_L(T-\Theta_L)\, M_L(P)$$
$$\Phi(P, T-\Theta) = \sum_{L=1}^{L_0} \Phi_L(T-\Theta_L)\, N_L(P) \tag{12}$$

where L_0 is the total number of nodes on the body, $\Psi_L(T-\Theta_L)$ and $\Phi_L(T-\Theta_L)$ are time dependent values of Ψ and Φ at the node, L, at the time $T-\Theta_L$ (where Φ_L is the disturbance-propagation time from P_* to P_L); furthermore $M_L(P)$ and $N_L(P)$ are prescribed global shape-functions, obtained by standard assembly of the element shape-function (see for instance Ref. 8). For instance for the hyperboloidal element of Ref. 4 the element shape functions are

$$M_\Delta^E \equiv N_\Delta^E = \frac{1}{4}\left(1+\xi/\xi_\Delta\right)\left(1+\eta/\eta_\Delta\right) \tag{13}$$

where $\xi_\Delta = \pm 1$ and $\eta_\Delta = \pm 1$ define the locations of the corner, Δ , of the element, E.

Next consider the integrals on the wake. In order to facilitate the use of Eq. (10), it is convenient to divide the wake into strips defined by (steady-state) vortex-lines emanating from the nodes on the trailing edge. The strips are then divided into quadrilateral elements. The potential discontinuity can then be expressed as

$$\Delta\Phi(P, T_* - \Theta) = \sum_{N=1}^{N_o} \Delta\Phi_N(T_* - \Theta_N) L_N(P) \tag{14}$$

where N_o is the number of nodes on the wake, $\Delta\Phi_N(T_* - \Theta_N)$ is the value of $\Delta\Phi$ at the N-th node on the wake at time $T - \Theta_N$ (where Θ_N is the propagation time from P to P_*), and $L_N(P)$ is the global shape function relative to the N-th node of the wake. Note that according to Eq. (10)

$$\Delta\Phi_N(T) = \Delta\Phi^{TE}_{M(N)}(T - \Pi_N) \tag{15}$$

where M=M(N) identifies the trailing-edge node which is on the same vortex-line as the node N. Furthermore Π_N is the time necessary for the vortex-point to be convected from the trailing-edge node M to the wake-node N. It may be worth noting that $\Delta\Phi^{TE}_M = \phi_{L_u} - \phi_{L_\ell}$, where L_u and L_ℓ identify the upper and lower trailing-edge nodes on the body corresponding to the M node on the trailing edge. Therefore it is possible to write

$$\Delta \Phi^{TE}_{M(N)} = \sum_{L=1}^{L_1} S_{NL} \Phi_L \tag{16}$$

where $S_{NL} = 1$ ($S_{NL} = -1$), if L identifies the upper (lower) node of the body corresponding to the N-th node of the wake, and $S_{NL} = 0$ otherwise.

Combining Eqs. (9, 12, and 14) one obtains

$$2E(P_*)\,\Phi(P_*, T) = \sum_L B_L \Psi_L(T - \Theta_L) + \sum_L C_L \Phi_L(T - \Theta_L) + \sum_L D_L \frac{d}{dT} \Phi(T - \Theta_L) + \sum_N F_N \Delta\Phi_N(T - \Theta_N) + \sum_N G_N \frac{d}{dT} \Delta\Phi_N(T - \Theta_N) \tag{17}$$

where,

$$B_L = -\frac{1}{2\pi} \oint\!\!\oint_{\Sigma_B} M_L(P) \frac{1}{R} \, d\Sigma_B$$

$$C_L = \frac{1}{2\pi} \oint\!\!\oint_{\Sigma_B} N_L(P) \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B$$

$$D_L = -\frac{1}{2\pi} \oint\!\!\oint_{\Sigma_B} N_L(P) \frac{1}{R} \frac{\partial R}{\partial N} \, d\Sigma_B$$

$$F_N = \frac{1}{2\pi} \iint_{\Sigma_W} L_N(P) \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_W \tag{18}$$

$$G_N = -\frac{1}{2\pi} \iint_{\Sigma_W} L_N(P) \frac{1}{R} \frac{\partial R}{\partial N} \, d\Sigma_W$$

and according to Eqs. (15) and (16)

$$\Delta \Phi_N (T-\Theta_N) = \sum_L S_{NL} \Phi_L (T-\Theta_N - \Pi_N) \tag{19}$$

Next consider, in particular, that P_* coincides with the node J of the body. In this case E= 1/2 and, using Eq. (19), Eq. (17) reduces to

$$\Phi_J(T) = \sum_L B_{JL} \Psi_L (T-\Theta_{JL}) + \sum_L C_{JL} \Phi_L (T-\Theta_{JL}) + \sum_L D_{JL} \frac{d}{dT} \Phi(T-\Theta_{JL}) + \sum_N \sum_L F_{JN} S_{NL} \Phi_L (T-\Theta_{JN} - \Pi_N) + \sum_N \sum_L G_{JN} S_{NL} \frac{d}{dT} \Phi_L (T-\Theta_{JN} - \Pi_N) \tag{20}$$

where

$$(B_{JL}, C_{JL}, D_{JL}, F_{JN}, G_{JN}, \Theta_{JL}) \equiv (B_L, C_L, D_L, F_N, G_N, \Theta_L)\Big|_{P_*=P_J} \tag{21}$$

Equation (20) indicates the nature of the aerodynamic operator relating potential and normal-wash as obtained by using finite-element representation to discretize the spacial problem. The operator is a linear differential-difference operator to which the methods of operational calculus can be applied. However before considering the Laplace's transform of Eq. (20) it is convenient to make some remarks about the contribution of the wake. It may be noted that, according to Eq. (8),

Φ is identically equal to zero for $T \leq 0$ therefore according to Eq. (15)

$$\Delta \Phi_N \equiv 0 \qquad (T \leq \Pi_N) \tag{22}$$

Hence if the analysis is limited to $T \leq T_{max}$, the contribution of the elements with $\Pi_N \geq T_{max}$ is identically equal to zero. Therefore the wake can be truncated to eliminate these elements. It may be noted that those elements would contribute to the transfer function and thus to the transform of Φ but not to the final solution in the time domain for $T \leq T_{max}$. The advantage is not only that less computational time is used (since less elements are required) but also that the problem of convergence connected with the infinite wake (factors $e^{-s\Pi_N}$ with Real(s) < 0 and $\Pi_N \rightarrow \infty$) are bypassed.*

Next by taking the Laplace transform of Eq. (20) and solving for $\{\tilde{\Phi}_J\}$ one obtains

$$\{\tilde{\Phi}_J\} = [\tilde{A}_{JL}] \{\tilde{\Psi}_L\} \tag{23}$$

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* A correct analysis implies the evaluation of the limit of the present analysis as the number of elements on the wake goes to infinity.

where

$$\left[\tilde{A}_{JL}\right] = \left[\left[\delta_{JK} - \left(C_{JK} + s \cdot D_{JK}\right) e^{-s\,\Theta_{JK}}\right] - \left[\left(F_{JN} + s\,G_{JN}\right) e^{-s(\Theta_{JN} + \Pi_N)}\right]\left[S_{NK}\right]\right]^{-1} \left[B_{KL}\, e^{-s\,\Theta_{KL}}\right] \qquad (24)$$

Equation (22) indicates that the matrix $[\tilde{A}_{JL}]$ is the desired <u>subsonic matrix transfer function</u> relating the transformed vector of the potential $\{\tilde{\Phi}_J\}$ to the transformed vector of the normal-wash $\{\tilde{\Psi}_M\}$.

SUPERSONIC INDICIAL AERODYNAMICS

In this section the formulation for the supersonic case is briefly outlined. For simplicity only supersonic trailing edges are consideres so that the contribution of the wake can be ignored.* Under small perturbation assumption the Green theorem for potential supersonic flow is given by

$$4\pi E(P_*)\, \Phi(P_*, T) = -\oint\!\!\oint_{\Sigma_B} \left([\Psi']^{\Theta^+} + [\Psi']^{\Theta^-} \right) \frac{H}{R}\, d\Sigma \; +$$
$$\oint\!\!\oint_{\Sigma_B} \left([\Phi]^{\Theta^+} + [\Phi]^{\Theta^-} \right) \frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right) d\Sigma \; -$$
$$\oint\!\!\oint_{\Sigma_B} \left(\left[\frac{\partial \Phi}{\partial T}\right]^{\Theta^+} + \left[\frac{\partial \Phi}{\partial T}\right]^{\Theta^-} \right) \frac{H}{R'}\, \frac{\partial R'}{\partial N^c}\, d\Sigma \qquad (25)$$

where $\Psi' = \partial\Phi/\partial N^c$ ($\frac{\partial}{\partial N^c}$ is the conormal derivative[4]) is the conormal-wash,

$$X = x/\beta \ell \qquad Y = y/\ell \qquad Z = z/\ell \qquad T = a_\infty \beta t/\ell \qquad (26)$$

(with $B = \sqrt{M^2-1}$) and

$$R' = \left[(X - X_*)^2 - (Y - Y_*)^2 - (Z - Z_*)^2 \right]^{1/2} \qquad (27)$$

while

$$\begin{aligned} H &= 1 \qquad \text{for} \quad X_* - X > \left[(Y - Y_*)^2 + (Z - Z_*)^2 \right]^{1/2} \\ &= 0 \qquad \text{for} \quad X_* - X \le \left[(Y - Y_*)^2 + (Z - Z_*)^2 \right]^{1/2} \end{aligned} \qquad (28)$$

and

$$[\quad]^{\Theta^\pm} = [\quad]_{T_1 = T - \Theta^\pm} \qquad (29)$$

with

$$\Theta^\pm = M(X_* - X) \pm R' \qquad (30)$$

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* If the trailing edge is not fully supersonic then the contribution of the wake can be treated similarly to the subsonic case, with the only difference that the device of truncating the wake at finite distance is not necessary in the supersonic case, since only a finite portion of the wake can have an effect on the aircraft.

Using Eq. (12) and following the same procedure used for the subsonic case one obtains the supersonic indicial aerodynamic operator

$$2\,E(P_*)\,\Phi(P_*,T) = \sum_L B'_L\left[\Psi'_L\left(T-\Theta^+_L\right)+\Psi'\left(T-\Theta^-_L\right)\right]+$$
$$\sum_L C'_L\left[\Phi_L\left(T-\Theta^+_L\right)+\Phi_L\left(T-\Theta^-_L\right)\right]+ \tag{31}$$
$$\sum_L D'_L\left[\frac{d}{dT}\Phi_L\left(T-\Theta^+_L\right)+\frac{d}{dT}\Phi_L\left(T-\Theta^-_L\right)\right]$$

where

$$B'_L = -\frac{1}{2\pi}\oint\!\!\oint_{\Sigma_B} M_L(P)\,\frac{H}{R'}\,d\Sigma_B$$
$$C'_L = \frac{1}{2\pi}\oint\!\!\oint_{\Sigma_B} N_L(P)\,\frac{\partial}{\partial N^c}\left(\frac{H}{R'}\right)d\Sigma_B \tag{32}$$
$$D'_L = -\frac{1}{2\pi}\oint\!\!\oint_{\Sigma_B} N_L(P)\,\frac{H}{R'}\,\frac{\partial R'}{\partial N}\,d\Sigma_B$$

In particular if P* coincides with the node J, Eq. (31) reduces to

$$\Phi_J(T) = \sum_L B'_{JL}\left[\Psi'_L\left(T-\Theta^+_{JL}\right)+\Psi'_L\left(T-\Theta^-_{JL}\right)\right]$$
$$+\sum_L C'_{JL}\left[\Phi_L\left(T-\Theta^+_{JL}\right)+\Phi_L\left(T-\Theta^-_{JL}\right)\right] \tag{33}$$
$$+\sum_L D'_{JL}\left[\frac{d}{dt}\Phi_L\left(T-\Theta^+_{JL}\right)+\frac{d}{dT}\Phi_L\left(T-\Theta^-_{JL}\right)\right]$$

where

$$\left(B'_{JL},\,C'_{JL},\,D'_{JL},\,\Theta^+_{JL},\,\Theta^-_{JL}\right) = \left(B'_L,\,C'_L,\,D'_L,\,\Theta^+_{JL},\,\Theta^-_{JL}\right)\Big|_{P_*=P_J} \tag{34}$$

Finally, taking the Laplace transform of Eq. (33) one obtains

$$\left\{\tilde{\Phi}_J\right\} = \left[\tilde{A}'_{JM}\right]\left\{\Psi'_M\right\} \tag{35}$$

where

$$\left[\tilde{A}'_{JL}\right] = \left[\delta_{JK} - \left(C'_{JK} + s D'_{JK}\right)\left(e^{-\ \Theta^{+}_{JK}} + e^{-s\Theta^{-}_{JK}}\right)\right]^{-1}$$

$$\left[B'_{KL}\left(e^{-s\Theta^{+}_{KL}} + e^{-s\Theta^{-}_{KL}}\right)\right] \tag{36}$$

Equation (35) indicates that the matrix $\left[\tilde{A}'_{JL}\right]$ is the desired <u>supersonic matrix transfer function</u> relating the transformed vector of the potential $\left\{\tilde{\Phi}_J\right\}$ to the transformed vector of the conormal-wash $\left\{\tilde{\Psi}'_M\right\}$, for supersonic trailing edge configurations. If the trailing edge is not fully supersonic, the formulation may be modified following the same ideas used for the subsonic case. It may be worth noting that according to Eq. (29)

$$e^{-s\Theta^{+}_{JK}} + e^{-s\Theta^{-}_{JK}} = \tag{37}$$

$$= 2\, e^{-sM(X_J - X_K)} \cosh\left\{ s\left[(X_J - X_K)^2 - (Y_J - Y_K)^2 - (Z_J - Z_K)^2\right]^{\frac{1}{2}}\right\}$$

CONCLUDING REMARKS

A general theory for unsteady compressible potential aerodynamics has been presented. The motion is assumed to consist of small perturbation starting at time t=0) around a steady-state constant-velocity motion. In this case the relationship between the velocity potential and the normal-wash is given by an integral operator in space and a differential-difference operator in time. Using the finite element method to solve for the spacial problem one is left with a differential-difference equation in time. This can be solved numerically for time-domain or by using the Laplace transform and thus obtaining the matrix transfer function for frequency-domain analysis.

The results presented here represent a considerable improvement with respect to the formulation available thus far since complex configurations could be analyzed only for steady and oscillatory flows (see, for instance, Refs. 3 and 4) while unsteady flows could be analyzed only for simple configurations such as zero-thickness wing with special planforms. With the method presented here unsteady flows around complex configurations can be analyzed for most cases of practical interest. For, the linear equations of flight dynamics implies small perturbations around a steady state motion, usually constant speed horizontal flight. The unsteady aerodynamic analysis presented here does not require any additional limitation and therefore is the most general formulation within the above framework.

In addition it should be noted that the increase in generality of the formulation is obtained at no additional increase in computational complexity. For, if simply oscillatory problems are considered the only advantage is to replace s with $i\Omega$ (Ref. 3 and 4), with no particular computational saving.

Another advantage of the present approach is that the dependence of the matrix A_{JM} upon the complex frequency, s, is given in a very simple explicity analytic form. This is a considerable computational advantage since once the coefficients B_{LM}, C_{JL}, D_{JL}, F_{JN}, G_{JN}, S_{NL}, and Θ_{JN} have been evaluated (and these are necessary even for the evaluation of the potential at one single reduced frequency), it is a trivial matter (essentially the inversion of one matrix) to obtain the results at different values of s. Also advantage can be taken of the analytic dependence upon s to obtain approximate expressions for the matrix $[\tilde{A}_{JM}]$. For instance, at low frequencies a Taylor series expansion for Eq. (24) can be taken to yield

$$\left[\tilde{A}_{JL}\right] = \Bigg[\Big(\left[\delta_{JK}\right]-\left[C_{JK}\right]-\left[F_{JN}\right]\left[S_{NK}\right]\Big) + s\Big(\left[C_{JK}\,\Theta_{JK}\right]-\left[D_{JK}\right]+$$

$$\left[F_{JN}\left(\Theta_{JN}+\Pi_{N}\right)\right]\left[S_{NL}\right]-\left[G_{JN}\right]\left[S_{NK}\right]\Big)+\ldots\Bigg]^{-1}\Big[\left[B_{KL}\right]-s\left[B_{KL}\,\Theta_{KL}\right]+\ldots\Big]=$$

$$\left[\Gamma_{JK}\right]\left[B_{KL}\right] - s\Bigg[\left[\Gamma_{JK}\right]\left[B_{KL}\,\Theta_{KL}\right] +$$

$$\left[\Gamma_{JK}\right]\Big(\left[C_{KH}\,\Theta_{KH}\right]-\left[D_{KH}\right]+\left[F_{KN}\left(\Theta_{JN}+\Pi_{N}\right)+G_{KN}\right]\left[S_{NH}\right]\Big)\left[\Gamma_{HP}\right]\left[B_{PL}\right]\Bigg] \tag{38}$$

where

$$\left[\Gamma_{JK}\right] = \Big[\left[\delta_{JK}\right]-\left[C_{JK}\right]-\left[F_{JN}\right]\left[S_{NL}\right]\Big]^{-1} \tag{39}$$

It is essential to note the difference between the method presented here and the classical approach for unsteady (oscillatory) aerodynamics. In this case the solution is assumed to be oscillatory (i.e. of the type $e^{i\Omega T}$) and then the problem is solved in space.[4] Here the problem is solved first in space and then in time. This inversion of the time and space solutions might appear to be irrelevant, but is not. For, in the classical formulation the convergence of the space solution is analyzed on the time-transformed unknown which is highly oscillating in space. On the other hand, here the finite-element method is applied to the untransformed equation, where the unknown is smooth and therefore fewer elements are required for convergence. The transform applies to the discrete system and therefore high frequency components do not involve a change in the number of elements. This question is analyzed more in detail in Appendix A, where the Laplace transform is used first and then the transformed equation is solved by finite elements. This process yields Eq. (A.3) which should be compared to Eq. (20). Note that the integrals in Eq. (A.4) are not frequency independent and therefore a large number of elements is required for convergence. On the other hand the integrals in Eq. (18) are frequency independent and the convergence must be evaluated on the time-domain solution.

Another advantage of the present approach is the already mentioned possibility of truncating the wake at finite distance if the analysis is limited to $T \leq T_{max}$. (since $\Delta\Phi_N \equiv 0$ for $T < T_{max} < \Pi_N$). This eliminates the problem of convergence as the length of the wake goes to infinity.

Other questions which have not been discussed here are the Kutta condition and the role of the diaphragms in supersonic flow. These points are analyzed in Refs. 3 and 4 for a zeroth-order finite-element solution (i.e. potential, Φ , and normal-wash, Ψ , constant within each element). Further investigations for higher-order solutions are now under way.

In conclusion a new approach for small-perturbation unsteady aerodynamics has been presented. Taking full advantage of the finite element method and the operational calculus the problem is simplified considerably and the relationship potential-normalwash is reduced to a system of algebraic equation, with explicit dependence of the coefficients upon the complex frequency, s .

APPENDIX A

In this Appendix it is shown how interchanging the order of the time and space solution, a different type of equation is obtained. Taking the Laplace transform of Eq. (9) one obtains

$$4\pi E(P_*)\tilde{\Phi}(P_*) = -\oint\!\!\oint_{\Sigma_B} \tilde{\Psi}\,\frac{e^{-s\Theta}}{R}\,d\Sigma_B +$$

$$\oint\!\!\oint_{\Sigma_B} \tilde{\phi}\,e^{-s\Theta}\,\frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B -$$

$$\oint\!\!\oint_{\Sigma_B} \tilde{\phi}\,s\,e^{-s\Theta}\,R\,\frac{\partial R}{\partial N}\,d\Sigma_B +$$

$$\iint_{\Sigma_W} \Delta\tilde{\phi}\,e^{-s\Theta}\,\frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_W \qquad \text{A.1}$$

$$\iint_{\Sigma_W} \Delta\tilde{\phi}\,s e^{-s\Theta}\,R\,\frac{\partial R}{\partial N}\,d\Sigma_W$$

Next using the finite-element method, i.e. setting

$$\tilde{\Psi}(P) = \sum_L \tilde{\Psi}_L\,M_L(P)$$
$$\tilde{\Phi}(P) = \sum_L \tilde{\Phi}_L\,N_L(P)$$
$$\Delta\tilde{\phi}(P) = \sum_N \Delta\tilde{\Phi}_N\,L_N(P) \qquad \text{A.2}$$

yields, at $P = P_L$,

$$\{\tilde{\Phi}_J\} = [\tilde{B}_{JL}]\{\Psi_L\} + [\tilde{C}_{JL}]\{\tilde{\Phi}_L\} + s[D_{JL}]\{\tilde{\Phi}_L\}$$
$$+ [\tilde{F}_{JN}]\{\Delta\tilde{\Phi}_N\} + s[\tilde{G}_{JN}]\{\Delta\tilde{\Phi}_N\}$$

A.3

where

$$\tilde{B}_{JL} = \left[-\frac{1}{2\pi}\oiint_{\Sigma_B} M_L(P)\, e^{-s\Theta}\, \frac{1}{R}\, d\Sigma_B\right]_{P_* = P_J}$$

$$C_{JL} = \left[\frac{1}{2\pi}\oiint_{\Sigma_B} N_L(P)\, e^{-s\Theta}\, \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_B\right]_{P_* = P_J}$$

$$D_{JL} = \left[-\frac{1}{2\pi}\oiint_{\Sigma_B} N_L(P)\, e^{-s\Theta}\, \frac{1}{R}\, \frac{\partial R}{\partial N}\, d\Sigma_B\right]_{P_* = P_J}$$

$$F_{JN} = \left[\frac{1}{2\pi}\iint_{\Sigma_W} L_N(P)\, e^{-s\Theta}\, \frac{\partial}{\partial N}\left(\frac{1}{R}\right) d\Sigma_W\right]_{P_* = P_J}$$

$$G_{JN} = \left[\frac{1}{2\pi}\iint_{\Sigma_W} L_N(P)\, e^{-s\Theta}\, \frac{1}{R}\, \frac{\partial R}{\partial N}\, d\Sigma_W\right]_{P_* = P_J} \qquad \text{A.4}$$

Finally, according to Eqs. (15) and (16) one obtains

$$\{\Delta\tilde{\Phi}_N\} = \{\Delta\tilde{\Phi}_M^{TE}\, e^{-s\Pi_N}\} = [S_{NL}\, e^{-s\Pi_N}]\{\tilde{\Phi}_L\} \qquad \text{A.5}$$

Combining Eqs. (A.3) and (A.5) and solving for $\{\tilde{\Phi}_J\}$ yields a new expression for the matrix transfer function

$$[\tilde{A}_{JM}] = \Big[[\delta_{JL} - (\tilde{C}_{JL} + s\tilde{D}_{JL})] -$$
$$[\tilde{F}_{JN} + s\tilde{G}_{JN}][S_{NL}\, e^{-s\Pi_N}]\Big]^{-1}[B_{LM}\, e^{-s\Theta_{LM}}]$$

A.6

Comparison with Eq. (24) indicates clearly the advantage of using the space-discretization before the time-transformation.

APPENDIX B

In this Appendix Eq. (10) is derived. Consider the Bernoulli theorem for potential flow

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \int \frac{dp}{\rho} = \text{const} \tag{B.1}$$

Since no pressure difference can exist across the wake, then

$$\left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)_u - \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)_\ell = 0 \tag{B.2}$$

or

$$\frac{\partial}{\partial t} (\phi_u - \phi_\ell) + \frac{1}{2} \left[(\nabla \phi_u \cdot \nabla \phi_u) - (\nabla \phi_\ell \cdot \nabla \phi_\ell) \right] = 0 \tag{B.3}$$

This can be rewritten as

$$\frac{\partial}{\partial t} (\phi_u - \phi_\ell) + \frac{1}{2} \nabla (\phi_u + \phi_\ell) \cdot \nabla (\phi_u - \phi_\ell) = 0 \tag{B.4}$$

or

$$\frac{D_a}{Dt} (\phi_u - \phi_\ell) = 0 \tag{B.5}$$

where

$$\frac{D_a}{Dt} \equiv \frac{\partial}{\partial t} + \vec{Q}_a \cdot \nabla \tag{B.6}$$

is the total time derivative obtained by following a particle having the average velocity.

$$\vec{Q}_a = \frac{1}{2} (\nabla \phi_u + \nabla \phi_\ell) \tag{B.7}$$

Equation (B.5) implies that the wake vortices are simply convected, that is that

$$\Delta\phi(P, t) = \Delta\phi(P_{TE}, t - \pi) \qquad \text{(B.8)}$$

where π is the time necessary for a point of the wake to move from a point P_{TE} of the trailing edge to the point P. If the unsteady flow is infinitesimal π can be evaluated from the steady state solution. If small perturbation apply to steady state flow as well, π is given by the distance $x - x_{TE}$ divided by the velocity, U_∞

$$\pi = (x - x_{TE}) / U_\infty \qquad \text{(B.9)}$$

Using nondimensional variables (Eq. 2), Eqs. (B.8) and (B.9) are replaced by Eqs. (10) and (11) respectively.

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