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COMPUTER ANALYSIS OF MULTICIRCUIT SHELLS OF REVOLUTION BY THE FIELD METHOD

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16. Abstract

The "field method", presented previously for the solution of even-order linear boundary-value problems defined on one-dimensional open branch domains (i.e., trees), is extended to boundary-value problems defined on one-dimensional domains containing circuits (i.e., multi-circuit graphs). This method converts the boundary-value problem into two successive numerically stable initial-value problems, which may be solved by standard forward integration techniques. In addition, a new method for the treatment of singular boundary conditions (i.e., kinematic constraints) is presented. This method, which amounts to a (partial) interchange of the roles of "force" and "displacement" variables, is problem independent with respect to both accuracy and speed of execution.

The method has been implemented in a computer program to calculate the static response of ring-stiffened orthotropic multicircuit shells of revolution to asymmetric loads. Solutions are presented for sample problems which illustrate the accuracy and efficiency of the method.

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SUMMARY

The "field method", presented previously for the solution of even-order linear boundary-value problems defined on one-dimensional open branch domains (i.e., trees), is extended to boundary-value problems defined on one-dimensional domains containing circuits (i.e., multicircuit graphs). This method converts the boundary-value problem into two successive numerically stable initial-value problems, which may be solved by standard forward integration techniques. In addition, a new method for the treatment of singular boundary conditions (i.e., kinematic constraints) is presented. This method, which amounts to a (partial) interchange of the roles of "force" and "displacement" variables, is problem independent with respect to both accuracy and speed of execution.

The method has been implemented in a computer program to calculate the static response of ring-stiffened orthotropic multicircuit shells of revolution to asymmetric loads. Solutions are presented for sample problems which illustrate the accuracy and efficiency of the method.

INTRODUCTION

This paper is a sequel to an earlier paper (ref. 1) in which it is shown how linear boundary-value problems defined on one-dimensional branched domains without circuits (i.e., trees) may be formulated and solved as two successive initial-value problems. In contrast to the more conventional superposition methods for such problems, the integration of the resulting initial-value problems is numerically stable, thereby eliminating the need for artificial subdivision of the tree into a number of "suitably small" subintervals. This method of solution has been termed the "field method"; however, the author has since become aware of a rapidly expanding body of related literature under the general heading of "invariant imbedding" (ref. 2). There are several variations of the invariant imbedding method for solving boundary-value problems, and the field method (variously known as the method of sweeps, the method of factorization, or the method of generalized Riccati transformations) may be considered to be one of them (ref. 3). Although the alternate forms of invariant imbedding eliminate the second (backward) integration of the field method, they suffer from the need of having to choose a priori the set of response output (or storage) points. In the field method, since the response functions are obtained directly from the backward integration, their storage points may be chosen automatically so as to obtain a uniformly dense description of these functions. Furthermore, in many cases the extra work required in the forward integration of the alternate methods more than compensates for the elimination of the backward integration.

In reference 1 the field method is applied to the solution of the equations of static response of open branch ring-stiffened shells of revolution. Because the field method not only eliminates the "long subinterval" (i.e., numerical instability) problem of superposition methods, but is shown, by example, to execute significantly faster, it appears to be a desirable method for shell problems. The purpose of the present paper is twofold: 1) to generalize the field method to boundary-value problems defined on one-dimensional domains which contain circuits, and 2) to improve the treatment of singular boundary conditions (i.e., kinematic constraints) given in reference 1, in which the numerical solution depends on a scaling matrix, an effective choice of which is problem dependent.

SYMBOLS

a,b,c,d	matrix coefficients of linear differential equations [eqs. (1)]
B,D	matrix coefficients of linear boundary conditions [eq. (2)]
$\tilde{D},\tilde{L}$	effective values of D and L at interior vertices [eqs. (39) and (41)]
$\hat{D},\hat{L}$	effective values of D and L at initial vertex of a continuation arc [eqs. (53) and (59)]
f,g	inhomogeneous vectors of linear differential equations [eqs. (1)]
G.P.	generic point of a closed branch arc
I	identity matrix
I.P.	"initial point" of a closed branch arc (see p. 6)
K	stiffness matrix for part of disconnected graph between the I.P. and G.P.
L	inhomogeneous vector of linear boundary conditions [eq. (2)]
M_1	meridional stress couple

n	circumferential harmonic number
P,Q,S	shell forces per unit of circumferential length in axial, radial, and circumferential directions
p	half-order of boundary-value problem
R_2	circumferential radius of curvature
r	small circle radius
s	arc distance
s,ϕ,z	meridional, circumferential, and normal coordinates of shell reference surface
t	wall thickness
u,w	field function matrices [eqs. (6a) and (7)]
v,δ,γ,λ	additional field function matrices for closed branch arcs [eqs. (6)]
x,y	axial and radial coordinates
y,z	generalized force and displacement response vectors
α,β	diagonal matrices with elements equal to zero or one [eqs. (20) and (21)]
Δ	net change across a vertex [eq. (3)]; also effective value of D at I.P. of arcs in a split circuit [eq. (51d)]
κ	$B^{-1}D$
μ,μ_0,μ_2	inhomogeneous vector and matrix coefficients in equation for z at I.P. of arcs in a split circuit [eqs. (51)]
ξ,η,v	shell displacements in axial, radial, and circumferential directions
υ	effective value of v^- at initial vertex of a continuation arc [eq. (53b)]
χ	meridional rotation

Subscripts:

A,E	value at vertex P on arcs A and E of figure 6(b)
C	value at vertex P on circuits C of figure 6(b)

P	value at vertex P of figure 6(b)
0	value at I.P. of a closed branch arc
1,2	value at vertices P_1 and P_2 of figure 6(a)

Superscripts:

T	matrix transpose
$( )'$	$d( )/ds$
$( )^{+}$	value at vertex on exiting arc
$( )^{-}$	value at vertex on entering arc
$(\bar{\ })$	modified variable for singular arcs

ANALYTICAL FORMULATION

As in reference 1, in order to formulate the field method in a general context, it will be convenient to employ some elementary concepts from the theory of geometric graphs. Figure 1 shows the reference meridian of a hypothethical shell of revolution. The heavy dotted points depict boundaries of the shell, which are defined as one of the following types of points:

(1) branch points
(2) branch extremities
(3) ring or ring load points
(4) shell property or load discontinuity points

The boundaries thus divide the meridian into a number of subintervals. In contrast to other numerical integration methods, no additional (artificial) boundaries are required simply to reduce subinterval length.

A geometric graph (ref. 4) is defined as a set of points, called vertices, and a set of non-self-intersecting curves, called arcs, satisfying the following requirements:

(1) Each closed arc contains precisely one vertex.
(2) Each open arc contains precisely two vertices, viz. its end points.
(3) The arcs have no common points, except for the vertices.

It is clear from figure 1 that if we identify the boundaries as vertices and the subintervals as arcs, the reference meridian of a shell of revolution is nothing more than a geometric graph.

The following graph terminology will be used:

(1) Chain - a continuous sequence of arcs from an initial vertex to a terminal vertex. A non-self-intersecting chain is said to be simple.
(2) Circuit - a simple chain whose initial and terminal vertices coincide.
(3) Connected graph - a graph for which every pair of vertices is joined by at least one chain.
(4) Tree - a connected graph which contains no circuits.

Whereas the analysis of reference 1 is limited to boundary-value problems defined on trees, the present analysis treats a much broader class of connected graphs which may contain multiple circuits. In fact, the sole limitation on the generality of graphs treated is contained in the following statement. If two vertices of the graph are joined by three or more distinct simple chains, then all circuits containing one of these vertices also contain the other. The reason for this limitation, as well as its implications, are discussed later in the paper (see p. 16). For each arc of the graph, the arc distance s will be used as the independent variable. Let us assume that the arcs have been ordered and oriented (with respect to the direction of increasing s), but defer for the time being the manner in which this was done.

Definition of Boundary-Value Problem

A system of ordinary differential equations of order 2p may always be written as a system of 2p first-order equations. If we group one-half of the dependent variables in the $p \times 1$ matrix y and the other half in the $p \times 1$ matrix z, the system of first-order equations may be written, for linear systems, as two matrix differential equations, viz.

$$y' + ay + bz = f \tag{1a}$$

$$z' + cy + dz = g \tag{1b}$$

where prime denotes differentiation with respect to s; a, b, c, and d are $p \times p$ matrix functions of s; and f and g are $p \times 1$ matrix functions of s.

Equations (1) are defined at every interior point of each arc of the graph. They are supplemented by linear boundary conditions, defined at the vertices of the graph, of the form

$$B\Delta y + Dz = L \tag{2}$$

where B and D are $p \times p$ matrices, L is a $p \times 1$ matrix, and

$$\Delta y = \sum y^{+} - \sum y^{-} \tag{3}$$

Here, y^{+} and y^{-} represent the values of y at the vertex on exiting (s increasing away from the vertex) and entering (s increasing towards the vertex) arcs, respectively. As implied by the form of eq. (2), it is assumed that z is continuous at vertices.* A vertex and its boundary condition are said to be singular if the matrix B is singular; otherwise they are called regular.†

The majority of boundary-value problems in mechanics are self-adjoint. This property is synonymous with being derivable from a variational principle. Physically, this corresponds to systems which do not involve energy losses. For the system defined by eqs. (1) and (2), the conditions of self-adjointness may be shown to be

$$b = b^T, \quad c = c^T, \quad d = -a^T \tag{4a}$$

$$\kappa = \kappa^T \tag{4b}$$

where

$$\kappa = B^{-1}D \tag{5}$$

and the superscript T denotes matrix transpose. Although condition (4b) strictly applies only to regular vertices, it may be used also for singular vertices if a singular vertex is viewed as a limiting case of a sequence of regular vertices. Thus, for self-adjoint problems a singular vertex is the limit of a sequence of regular vertices, for each of which eq. (4b) holds true.

Field Relations

An *open branch arc* is characterized by the fact that a cut at any of its points disconnects the graph into two separate parts. Conversely, a (single) cut of a *closed branch arc* (not to be confused with a *closed arc*, defined on p. 4), which, by definition, lies in a circuit, does not disconnect the graph. However, in order to develop *field relations*, it is necessary to visualize disconnecting the graph by a cut at a generic point. For this purpose, the concept of an *initial point*, denoted henceforth by I.P., of a closed branch arc is introduced. For each closed branch arc, the I.P. is chosen so that cuts at the I.P. and a generic point, denoted henceforth by G.P., of the arc disconnect the graph into two separate parts. Note that the I.P. of a given closed branch arc is located in a circuit containing the given arc but not necessarily in that arc itself, and in fact several closed branch arcs lying in the same circuit may have the same I.P. (fig. 2).

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*It is assumed on physical grounds that one-half of the dependent variables are continuous at vertices, and these shall be grouped in the vector z. The vector y may be viewed as a generalized force vector corresponding to the generalized displacement vector z, and $-\Delta y$ represents the net external force entering the vertex.

†A singular boundary condition implies a relationship between the components of the displacement vector z, i.e., kinematic constraint.

Since cuts at the I.P. and G.P. of a closed branch arc disconnect the graph into two separate parts, it is clear that the values $z = z_0$ at the I.P. and $z = z$ at the G.P. can serve as boundary conditions which, along with the differential equations (1) and the boundary conditions (2) over one of the parts, determine y and z over that part independently of the corresponding data over the remaining part.* In particular, the values z_0 and z determine the values of y at the cuts, which for linear problems is expressed by the matrix relations

$$y = uz + vz_0 + w \tag{6a}$$

$$-y_0^+ = \delta z + \gamma z_0 + \lambda \tag{6b}$$

where u, v, δ, and γ are $p \times p$ matrices and w and λ are $p \times 1$ matrices. Equations (6) are the field relations for closed branch arcs, which generalize eq. (6) of reference 1 for open branch arcs, viz.

$$y = uz + w \tag{7}$$

In eq. (6b), $-y_0^+$ represents the force vector at the I.P. on an exiting closed branch arc, which shall be called an *initial closed branch arc.* [All nonsubscripted variables in eqs. (6) are understood to be evaluated at the G.P.] Thus, by definition, the I.P. of an initial closed branch arc coincides with its initial vertex. In order to obtain an initial-value problem for the field functions u, v, w, δ, γ, and λ, closed branch arcs are ordered and oriented so that for every cut pair (at the I.P. and the G.P.) all of one part of the disconnected graph is completely described by s-values smaller than that at the G.P. (fig. 2).†

Several properties of the field functions may be deduced on physical grounds. Because of the generally decaying nature of shell response to a local perturbation, it follows that as the G.P. recedes from the I.P., the coupling between response variables at these two points becomes weak. From eqs. (6), this is expressed by the statement that the matrices v and δ decay in magnitude, or symbolically

$$v \to 0 \tag{8a}$$

$$\delta \to 0 \tag{8b}$$

as the distance between the G.P. and the I.P. becomes large. From eq.

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*It is assumed that giving z_0 and z at the cut pair determines a unique solution, i.e., there exists no nontrivial solution of the homogeneous forms of eqs. (1) and (2) satisfying the cut conditions $z_0 = z = 0$, as is the case for problems derivable from the minimization of a positive-definite functional. Otherwise, a transformation of variables, such as used on singular arcs (see p. 11), may be necessary.

†In so doing, it will be assumed that the points of the fully described part are put into one-to-one correspondence with s-values between the values of s at the I.P. and at the G.P.

(8b) it follows that eq. (6b) cannot be used for a direct calculation of z, once y_0^+ and z_0 (and the field functions) are known. For solution of eq. (6b) for z gives

$$z = -\delta^{-1}(y_0^+ + \gamma z_0 + \lambda) \tag{9}$$

Since z does not grow large with δ^{-1} (and since γ and λ do not decay and y_0^+ and z_0 are constants), it follows that at points remote from the I.P. the calculation of z from eq. (9) would involve the small difference of large numbers.*

For self-adjoint problems, to which our attention is henceforth confined, the matrix K defined by

$$K = \begin{bmatrix} u & v \\ \delta & \gamma \end{bmatrix} \tag{10}$$

represents a stiffness matrix for the part of the disconnected graph which is fully described by s-values smaller than s at the G.P. Since such a stiffness matrix is known to be symmetric, its submatrices satisfy the following reciprocal relations

$$u = u^T, \quad \gamma = \gamma^T, \quad \delta = v^T \tag{11}$$

Thus, for example, for eighth-order boundary-value problems, there are 44 independent (scalar) field functions contained in eqs. (6). However, as shown below, only 30 of them (u, v, and w) need be stored.

Plan of Field Method

Before proceeding to derive the differential equations and initial conditions for the field functions, in order to put the discussion in context, a preview of the field method for a graph with circuits is presented. The calculation procedure consists of the following three basic steps (fig. 3).

I A forward integration is performed over the graph for u and w on

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*Not only do each of the elements of v and δ approach zero but, in general, the matrices themselves become numerically singular. This follows from the fact that the columns of v and the rows of δ are solutions of a linear homogeneous system of differential equations [see eqs. (14c) and (15)]. Each of the columns of v and the rows of δ is therefore a linear combination of a fundamental set of (decaying) solution vectors. As the G.P. recedes from the I.P., the fundamental solution vector which decays the slowest will dominate the other components so that, regardless of their initial values, the columns of v and the rows of δ will lose their numerical independence. Thus, the inversion of δ will, by itself, incur increasing round-off error, and hence *any* scheme of direct calculation of z using δ^{-1} will fail.

open branch arcs and u, w, v, γ, and λ on closed branch arcs. On open branch arcs u and w are stored, and on closed branch arcs u, w, and v are stored.

II Upon reaching the final vertex, in principle, two cases must be distinguished. If the final arc is an open branch arc or a closed branch arc whose I.P. coincides with its terminal vertex, the field relations and boundary conditions at the final vertex are solved simultaneously to give the value of z, say z_1, at the final vertex. If, on the other hand, the final arc is a closed branch arc whose I.P. is distinct from its terminal vertex, then the field relations and boundary conditions at the final vertex and the I.P. are solved simultaneously to give z_1 and the value of z at the I.P.

III Using z_1 as a final value, and z-continuity at interior vertices, a backward integration of the equation [cf. eqs. (1b) and (7)]

$$z' + (cu + d)z = g - cw \tag{12}$$

is performed over the graph. Before integrating eq. (12) on a closed branch arc, w on that arc is replaced by $w + vz_0$ where z_0 is the value of z at its I.P. [cf. eqs. (6a) and (7)]. As shown later, z_0 for each closed branch arc can be calculated prior to the backward integration on that arc. When a response storage point is reached, y is calculated from the field relation (7).

Differential Equations for Field Functions

The differential equations for u, v, and w are obtained by the same procedure used in reference 1 for u and w on open branch arcs. That is, eq. (6a) is differentiated with respect to s, the differential equations (1) are used to eliminate y' and z', and eq. (6a) is used again to eliminate y. This gives

$$\begin{aligned}(u' - ucu + au - ud + b)z + (v' - ucv + av)z_0 \\ + w' - ucw + aw + ug - f = 0\end{aligned} \tag{13}$$

Since eq. (13) must be satisfied identically with respect to z and z_0 (which depend on data at points of greater s, whereas u, v, and w do not), it follows that

$$u' - ucu + au - ud + b = 0 \tag{14a}$$

$$w' - ucw + aw + ug - f = 0 \tag{14b}$$

$$v' - ucv + av = 0 \tag{14c}$$

Note that eqs. (14a,b) apply to both open and closed branch arcs, and that eq. (14c) is simply the homogeneous form of eq. (14b).*

In a similar manner, to obtain the differential equations for γ and λ, differentiate eq. (6b) with respect to s, and use eq. (1b) to eliminate z' and eq. (6a) to eliminate y. This gives

$$[\delta' - \delta(cu + d)]z + (\gamma' - \delta cv)z_0 + \lambda' + \delta(g - cw) = 0 \qquad (15)$$

Since eq. (15) is also an identity in z and z_0, it follows from eqs. (15) and (11) that [the coefficient of z in eq. (15) being identically zero since by eqs. (4a) and (11) it is the transpose of the left side of eqs. (14c)]

$$\gamma' - v^T cv = 0 \qquad (16a)$$

$$\lambda' + v^T(g - cw) = 0 \qquad (16b)$$

These are remarkably simple differential equations, which may be integrated by quadrature after the integration for v and w is made.

Initial Values of Field Functions

In general, there are three different types of arcs for which initial values of the field functions must be obtained. These shall be called: (1) ordinary arcs, (2) initial closed branch arcs, and (3) continuation arcs.

Ordinary arcs.- An ordinary arc is either: (1) an open branch arc whose initial vertex is incident with no closed branch arcs, or (2) a closed branch arc whose initial vertex is incident with precisely one other closed branch arc which precedes the ordinary arc and enters (i.e., is oriented towards) its initial vertex (fig. 4). For example, in figure 2 arc 3 is an ordinary arc. Note that in the case of an ordinary closed branch arc, its I.P. is also the I.P. for the preceding closed branch arc.

Regular arcs: An ordinary arc is called regular if its initial vertex is regular. The initial values of the field functions u, v, and w on a regular ordinary arc are obtained by substituting the field relation (6a) into the boundary condition (2) for its initial vertex to obtain

$$(\Delta u + B^{-1}D)z + \Delta vz_0 + \Delta w - B^{-1}L = 0 \qquad (17)$$

Since this is an identity in z and z_0, and since there are no v-contributions from open branch arcs, eq. (17) yields the initial values

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*Henceforth, all results pertaining to u and w (or $\bar{u}$ and $\bar{w}$) apply to both open and closed branch arcs. On open branch arcs, the closed branch field functions v, γ, and λ (or $\bar{v}$, $\bar{\gamma}$, and $\bar{\lambda}$) may be taken to be zero and equations for them ignored.

$$u^{+} = -B^{-1}D + \sum u^{-} \tag{18a}$$

$$w^{+} = B^{-1}L + \sum w^{-} \tag{18b}$$

$$v^{+} = v^{-} \tag{18c}$$

Moreover, evaluation of eq. (6b) on either side of the initial vertex, and subtraction of the results gives

$$\gamma^{+} = \gamma^{-} \tag{19a}$$

$$\lambda^{+} = \lambda^{-} \tag{19b}$$

Thus, all of the additional closed branch field functions are continuous at the initial vertex of an ordinary closed branch arc.

Singular arcs: On an ordinary arc which is singular ($|B| = 0$), eqs. (18a,b) break down. This problem is treated in reference 1 by forming on singular arcs a modified z-vector as a linear combination of z and y. In this relation, a scaling matrix, introduced as the coefficient of z, is required for dimensional homogeneity. Since this matrix must be determined empirically, its numerical effectiveness cannot be assured for all problems. In fact, the scaling matrix suggested in reference 1 results in large round-off errors, as well as a slow forward integration, on singular arcs for shells of revolution with very high harmonic loading (viz. circumferential wave numbers $n \simeq 1000$). For this reason, a universal transformation of variables for singular arcs was sought.

In the case of an arc with initial clamping (i.e., the constraint $z = 0$), the problem is resolved simply by an interchange of the roles of y and z. The generalization of this complete interchange of variables for an arbitrary singular arc, which may have at its initial vertex only partial kinematic constraint, is an interchange of some of the corresponding components of y and z. This is expressed by the transformation*

$$\bar{y} = \alpha y + \beta z \tag{20a}$$

$$\bar{z} = -\beta y + \alpha z \tag{20b}$$

where α and β are diagonal $p \times p$ matrices with diagonal elements equal to zero or unity such that

$$\alpha + \beta = I \tag{21}$$

Equations (20) correspond to the interchange of the roles of the i-th components of the y and z vectors if the i-th diagonal element of β is unity. As a rule, the i-th diagonal element of β is chosen to be unity

*The minus sign in eq. (20b) is necessary in general to preserve the self-adjoint form of the differential equations (1) when written in terms of $\bar{y}$ and $\bar{z}$ [cf. eqs. (36) and (37)].

if in order to make B nonsingular, it is necessary to replace the i-th column of B by the i-th column of D (see also p. 15). This is equivalent to interchanging components of y with corresponding components of z in sufficient number so that the coefficient matrix of $\bar{y}^+$ in the boundary equation (2) written in terms of $\bar{y}^+$ and $\bar{z}^+$ is nonsingular.*

For α and β as defined, the following useful relations hold.

$$\alpha\alpha = \alpha \tag{22a}$$

$$\beta\beta = \beta \tag{22b}$$

$$\alpha\beta = \beta\alpha = 0 \tag{22c}$$

Using eqs. (21) and (22), eqs. (20) may be easily inverted to give

$$y = \alpha\bar{y} - \beta\bar{z} \tag{23a}$$

$$z = \beta\bar{y} + \alpha\bar{z} \tag{23b}$$

Thus, the inversion of eqs. (20) is effected simply by changing the sign of β. Substitution of eqs. (23) into the field relations (6) gives the modified field relations

$$\bar{y} = \bar{u}\bar{z} + \bar{v}z_0 + \bar{w} \tag{24a}$$

$$-y_0^+ = \bar{\delta}\bar{z} + \bar{\gamma}z_0 + \bar{\lambda} \tag{24b}$$

where

$$\bar{u} = (\alpha - u\beta)^{-1}(u\alpha + \beta) \tag{25a}$$

$$\bar{w} = (\alpha - u\beta)^{-1}w \tag{25b}$$

$$\bar{v} = (\alpha - u\beta)^{-1}v \tag{25c}$$

and

$$\bar{\delta} = \delta(\beta\bar{u} + \alpha) \tag{26a}$$

$$\bar{\gamma} = \gamma + \delta\beta\bar{v} \tag{26b}$$

$$\bar{\lambda} = \lambda + \delta\beta\bar{w} \tag{26c}$$

Comparing the form of eqs. (20) and (24) with that of eqs (23) and (6), respectively, shows that eqs. (25) and (26) also are inverted simply by changing the sign of β, i.e., barred and unbarred variables in eqs. (25) and (26) may be interchanged if β is replaced by $-\beta$.

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*That such an interchange is always possible is based on two premises: 1) the boundary conditions (2) constitute four linearly independent equations, and 2) the boundary-value problem is self-adjoint. Note that the assumption of linear independence implies that the rank of the $p \times 2p$ matrix $[B \vdots D]$ is p, i.e., it contains a $p \times p$ nonsingular submatrix.

For self-adjoint problems, it is easily shown that the reciprocal relations (11) are satisfied by the modified field functions. To show that $\bar{u}$ is symmetric, start with the identity [see eq. (21)]

$$(\alpha + \beta)u = u(\alpha + \beta) \tag{27}$$

or

$$\alpha u - u\beta = u\alpha - \beta u \tag{28}$$

In view of eqs. (22), eq. (28) can be factored to

$$(\alpha - u\beta)(\alpha u + \beta) = (u\alpha + \beta)(\alpha - \beta u) \tag{29}$$

or

$$(\alpha u + \beta)(\alpha - \beta u)^{-1} = (\alpha - u\beta)^{-1}(u\alpha + \beta) \tag{30}$$

Because of the symmetry of u, α, and β, the left-hand side of eq. (30) is the transpose of its right-hand side, which by eq. (25a) is $\bar{u}$. Hence

$$\bar{u}^T = \bar{u} \tag{31}$$

To show that $\bar{\delta}$ is the transpose of $\bar{v}$, use eqs. (11) and (31) to rewrite eq. (26a) in the form

$$\bar{\delta} = v^T(\beta\bar{u} + \alpha) = [(\alpha + \bar{u}\beta)v]^T = \bar{v}^T \tag{32}$$

where the last equality follows from the inverse of eq. (25c), viz.

$$v = (\alpha + \bar{u}\beta)^{-1}\bar{v} \tag{33}$$

As a consequence of eqs. (11), eqs. (26b,c) may be written as

$$\bar{\gamma} = \gamma + v^T\beta\bar{v} \tag{34a}$$

$$\bar{\lambda} = \lambda + v^T\beta\bar{w} \tag{34b}$$

and, as a consequence of eq. (32), barred and unbarred variables in eqs. (34) may be interchanged if β is replaced by $-\beta$. Thus from eq. (34a) and its inverse, one gets

$$\bar{\gamma} - \gamma = v^T\beta\bar{v} = \bar{v}^T\beta v = (v^T\beta\bar{v})^T \tag{35}$$

which shows that the difference $\bar{\gamma} - \gamma$ is symmetric. Since, by eqs. (11), γ is symmetric, it follows that $\bar{\gamma}$ is also symmetric.

In terms of $\bar{y}$ and $\bar{z}$, the differential equations (1) become

$$\bar{y}' + \bar{a}\bar{y} + \bar{b}\bar{z} = \bar{f} \tag{36a}$$

$$\bar{z}' + \bar{c}\bar{y} + \bar{d}\bar{z} = \bar{g} \tag{36b}$$

where

$$
\begin{aligned}
\bar{a} &= \alpha a\alpha + \alpha b\beta + \beta c\alpha + \beta d\beta \\
\bar{b} &= -\alpha a\beta + \alpha b\alpha - \beta c\beta + \beta d\alpha \\
\bar{c} &= -\beta a\alpha - \beta b\beta + \alpha c\alpha + \alpha d\beta \\
\bar{d} &= \beta a\beta - \beta b\alpha - \alpha c\beta + \alpha d\alpha \\
\bar{f} &= \alpha f + \beta g \\
\bar{g} &= -\beta f + \alpha g
\end{aligned}
\tag{37}
$$

Using eqs. (4a), it is easily shown from eqs. (37) that eqs. (4a) are satisfied also in terms of barred matrices, so that the transformation (20) preserves the self-adjoint property of the differential equations. Equations (24) and (36) can be used to derive differential equations for the modified field functions in exactly the same way as (6) and (1) were used to derive (14) and (16). Since eqs. (24) and (36) are identical in form to (6) and (1), respectively, the differential equations for the modified field functions are in the same form as eqs. (14) and (16) with all variables replaced by corresponding barred variables.

Initial values of the modified field functions on singular ordinary arcs may be obtained from eqs. (18), (19), (25), and (34). As in reference 1, we consider the singular vertex as the limit of a sequence of regular vertices, for each of which [cf. eq. (18a)]

$$
\begin{aligned}
(\alpha - u^{+}\beta)^{-1} &= [\alpha - (-B^{-1}D + \sum u^{-})\beta]^{-1} \\
&= (B\alpha + \tilde{D}\beta)^{-1}B
\end{aligned}
\tag{38}
$$

where
$$\tilde{D} = D - B \sum u^{-} \tag{39}$$

Then, evaluation of eqs. (25) and (34) at the initial vertex of the ordinary arc gives, in view of eqs. (18) and (19), the initial values

$$\bar{u}^{+} = -(B\alpha + \tilde{D}\beta)^{-1}(\tilde{D}\alpha - B\beta) \tag{40a}$$

$$\bar{w}^{+} = (B\alpha + \tilde{D}\beta)^{-1}\tilde{L} \tag{40b}$$

$$\bar{v}^{+} = (B\alpha + \tilde{D}\beta)^{-1}Bv^{-} \tag{40c}$$

$$\bar{\gamma}^{+} = \gamma^{-} + (v^{-})^{T}\beta\bar{v}^{+} \tag{40d}$$

$$\bar{\lambda}^{+} = \lambda^{-} + (v^{-})^{T}\beta\bar{w}^{+} \tag{40e}$$

where
$$\tilde{L} = L + B \sum w^{-} \tag{41}$$

Note that the transformation (20) includes as special cases the identity transformation

$$\bar{y} = y, \quad \bar{z} = z \qquad (\beta = 0) \tag{42}$$

used on regular arcs, and the complete interchange of variables

$$\bar{y} = z, \quad \bar{z} = -y \qquad (\beta = I) \tag{43}$$

used on arcs with initial clamping. Therefore, for the sake of unifying the method, all arcs could be integrated in terms of barred variables using the general equations (36), (37), and (40), depending on an appropriate choice for β. However, because of the extra work involved in setting up the barred matrices of eqs. (37), it is advantageous in programming the method to retain the special transformations (42) and (43) as separate options.* Moreover, the complete interchange (43) is adequate to treat all singular arcs for which the specification of y^+ and, for closed branch arcs, z_0 uniquely determines z^+. For problems derivable from the minimization of a positive-definite functional, this is the case, regardless of boundary conditions, for all closed branch arcs, and also for open branch arcs for which the initial vertex is an interior vertex if rigid body displacements are not admissible. For such problems, in practice, the only time the general (partial interchange) transformation (20) need be used is for singular open branch arcs whose initial vertex is a branch extremity or at whose initial vertex less than complete rigid body constraint is prescribed.†

Initial closed branch arcs.- As discussed previously, an initial closed branch arc is characterized by the fact that its I.P. coincides with its initial vertex. Since the use of a minimum number of I.P.'s leads to the simplest analysis, in general an initial closed branch arc will be any closed branch arc for which no vertex preceding its initial vertex can serve as its I.P. A little reflection will show that a sufficient condition for a closed branch arc to be an initial closed branch arc is that its initial vertex is incident with another closed branch arc which lies in a common circuit and is either 1) an exiting arc or 2) an entering but not preceding arc. For example, in figure 2 arcs 1, 2, 4, and 6 are initial closed branch arcs.

Since the field relations (6) break down at the initial vertex of

---

*As an example, for orthotropic shells of revolution without property variation through the wall thickness, the calculation of the derivatives $\bar{u}'$ and $\bar{w}'$ [eqs. (14a,b)] consumes roughly 18 percent more time for the complete interchange (43) than for the identity transformation (42) and 13 percent more time for the general transformation (20) than for the complete interchange, when the complete interchange and identity transformations are programmed as separate options.

†Actually, it suffices to have prescribed complete rigid body constraint at any vertex on the part of the disconnected graph fully described by s-values less than s at a cut on the singular arc.

an initial closed branch arc, such an arc is always treated as a singular arc, i.e., the integration on it is in terms of modified field functions. As for all singular closed branch arcs (see discussion above), the complete interchange of variables, eqs. (43), is used for initial closed branch arcs. The values of the field functions at the I.P. are obtained by inspection of the modified field relations (24) evaluated at the I.P. Substituting eqs. (43) into eqs. (24) and evaluating at the I.P. gives

$$z_0 = -\bar{u}_0 \overset{+}{y}_0 + \bar{v}_0 z_0 + \bar{w}_0 \tag{44a}$$

$$-\overset{+}{y}_0 = -\bar{\delta}_0 \overset{+}{y}_0 + \bar{\gamma}_0 z_0 + \bar{\lambda}_0 \tag{44b}$$

Since eqs. (44) must be satisfied identically with respect to $\overset{+}{y}_0$ and z_0, it follows that

$$\bar{u}_0 = \bar{\gamma}_0 = 0 \tag{45a}$$

$$\bar{v}_0 = I \tag{45b}$$

$$\bar{w}_0 = \bar{\lambda}_0 = 0 \tag{45c}$$

Note that the additional initial value $\bar{\delta}_0 = I$ implied by eq. (44b) is consistent with eq. (32) and is not used since, as noted earlier, for self-adjoint problems there is no need to calculate δ.

Continuation arcs.- A continuation arc is defined simply as any arc not of the preceding two types. It is therefore either: 1) an open branch arc whose initial vertex is incident with a closed branch arc, or 2) a closed branch arc whose initial vertex is incident with more than one other closed branch arc and which is not an initial closed branch arc. For example, in figure 2 arcs 5 and 7 are continuation arcs.

In order to simplify the treatment of continuation arcs, a limitation of graph generality is introduced here. Henceforth our attention is restricted to graphs for which the following statement is true. *If two vertices P_1 and P_2 are joined by three or more distinct simple chains, then all circuits containing P_1 (or P_2) also contain P_2 (or P_1).* Examples of graphs admitted by this limitation, as well as more complex graphs not admitted, are shown in figure 5. (Note that the existence of additional open branches does not affect a graph's admissibility.) The treatment of such graphs reduces to the treatment of the elementary *split circuit* configuration shown in figure 6(a). In this figure, arcs A and E may be open branch, closed branch, or even nonexistent arcs, and chains C_1 and C_2 are representative of an arbitrary number of chains C_i $(i = 1,2,\ldots,m)$ joining P_1 and P_2. According to this limitation, if A and (hence) E are closed branch arcs or C_3 exists, then no circuits, other than those depicted, containing P_1 or P_2 exist. Additional open branch arcs incident with (and oriented towards) P_1 or P_2 may exist and are accounted for in the analysis.

The *loop* configuration shown in figure 6(b) is the limiting case of that in figure 6(a) when one of the chains C_i approaches zero length, so that the vertices P_1 and P_2 collapse into a single vertex P. It is necessary to include this limit configuration as a special case in the analysis since, as will be shown, the calculation of initial conditions for the continuation arc E in figure 6(a) entails ever-increasing round-off errors as P_1 approaches P_2. In figure 6(b) the circuit C is representative of an arbitrary number of such circuits containing the vertex P, and additional open branch arcs may enter P. For graphs consistent with the above-stated limitation, the sufficient condition for an initial closed branch arc given on page 15 is also a necessary condition, i.e., the initial vertex of an initial closed branch arc is incident with another closed branch arc which lies in a common circuit and is either 1) an exiting arc [fig. 6(a)] or 2) an entering but not preceding arc [fig. 6(b)].*

Split circuits: The analysis of the configuration of figure 6(a) has two objectives: 1) to determine the initial values of the field functions on the continuation arc E, and 2) to express z at the I.P. P_1 in terms of z at P_2 in preparation for the backward integration of the chains C_i (cf. step III, p. 9). The field relations available for this purpose are the following [cf. eqs. (6)]:

Arc A evaluated at P_1

$$y_1^- = u_1^- z_1 + v_1^- z_0 + w_1^- \tag{46a}$$

$$-y_0^+ = (v_1^-)^T z_1 + \gamma_1^- z_0 + \lambda_1^- \tag{46b}$$

Chains C_i evaluated at P_2

$$y_2^- = u_2^- z_2 + v_2^- z_1 + w_2^- \tag{47a}$$

$$-y_1^+ = (v_2^-)^T z_2 + \gamma_2^- z_1 + \lambda_2^- \tag{47b}$$

Arc E evaluated at P_2

$$y_2^+ = u_2^+ z_2 + v_2^+ z_0 + w_2^+ \tag{48a}$$

$$-y_0^+ = (v_2^+)^T z_2 + \gamma_2^+ z_0 + \lambda_2^+ \tag{48b}$$

Any open branch arcs incident with P_1 or P_2 are oriented towards these vertices, and their field relations are assumed to be included in eqs. (46a) and (47a), respectively, with $v_1^- = v_2^- = 0$. In addition, the boundary conditions at P_1 and P_2 are [cf. eq. (2)]:

$$B_1(\sum y_1^+ - \sum y_1^-) + D_1 z_1 = L_1 \tag{49a}$$

$$B_2(y_2^+ - \sum y_2^-) + D_2 z_2 = L_2 \tag{49b}$$

---

*Tacitly included in this definition is the special case of a closed arc [e.g., in fig. 6(b) the circuit C if it consists of a single arc], in which case the "other nonpreceding" arc is the initial closed branch arc itself.

The solution of eqs. (46)-(49) may be conveniently divided into four steps. First, eqs. (46a) and (47b) are substituted into eq. (49a) to eliminate all y_1^- and y_1^+. This gives the desired relation

$$z_1 = \mu_2 z_2 + \mu_0 z_0 + \mu \tag{50}$$

where

$$\mu_2 = \Delta^{-1} B_1 \sum (v_2^-)^T \tag{51a}$$

$$\mu_0 = \Delta^{-1} B_1 v_1^- \tag{51b}$$

$$\mu = \Delta^{-1}[L_1 + B_1(\sum w_1^- + \sum \lambda_2^-)] \tag{51c}$$

$$\Delta = D_1 - B_1(\sum u_1^- + \sum \gamma_2^-) \tag{51d}$$

As a check of this result, note that if A is an open branch arc ($v_1^- = 0$) and all C_i are absent ($v_2^- = \lambda_2^- = \gamma_2^- = 0$), eq. (50) reduces to the formula for the terminal value of z for a tree [eq. (12) of ref. 1]. It is noted also that eq. (50) is valid even if P_1 is a singular vertex, since the inverse of B_1 does not appear in eqs. (51). Upon reaching P_2 during the backward integration, both z_0 (the value of z at the I.P. of arc E) and z_2 are known. The value z_1 (at the I.P. of arcs of the chains C_i) is then calculated from eq. (50), whereupon the backward integration of eq. (12) on the chains C_i can be made.

Second, eq. (50) is substituted into eq. (47a) to eliminate z_1, and the result substituted into eq. (49b) to eliminate all y_2^-. The result is

$$B_2 y_2^+ + \hat{D}_2 z_2 - B_2 \upsilon z_0 = \hat{L}_2 \tag{52}$$

where

$$\hat{D}_2 = D_2 - B_2[\sum u_2^- + (\sum v_2^-)\mu_2] \tag{53a}$$

$$\upsilon = (\sum v_2^-)\mu_0 \tag{53b}$$

$$\hat{L}_2 = L_2 + B_2[\sum w_2^- + (\sum v_2^-)\mu] \tag{53c}$$

Note that if A is an open branch arc ($\upsilon = 0$) and E is absent ($y_2^+ = 0$), eq. (52) gives z_2, the terminal value of z, in a form similar to the formula for a tree.

Third, the initial values of the field functions u, v, and w on the continuation arc E are obtained by substituting eq. (48a) into eq. (52) to eliminate y_2^+ and demanding that the resulting equation be an identity in z_0 and z_2. The results are

$$u_2^+ = -B_2^{-1}\hat{D}_2 \tag{54a}$$

$$w_2^+ = B_2^{-1}\hat{L}_2 \tag{54b}$$

$$v_2^+ = \upsilon \tag{54c}$$

Fourth, to obtain the initial values of γ and λ on arc E, eq. (50) is substituted into eq. (46b) to eliminate z_1, and the result is substituted into eq. (48b) to eliminate y_0^+. This result, also being an identity in z_0 and z_2, yields

$$\gamma_2^+ = \gamma_1^- + (v_1^-)^T \mu_0 \tag{55a}$$

$$\lambda_2^+ = \lambda_1^- + (v_1^-)^T \mu \tag{55b}$$

Note that if arc A is an open branch arc, then eqs. (54) and (55) imply that arc E is also an open branch arc, as it should be. For $v_1^- = \gamma_1^- = \lambda_1^- = 0$ imply, through eqs. (51b), (53b), (54c), and (55), that $v_2^+ = \gamma_2^+ = \lambda_2^+ = 0$. Also, it may be easily shown that eqs. (54a) and (55a) maintain the required symmetry of u and γ.

If P_2 is a singular vertex, eqs. (54) and (55) are replaced by formulas for the initial values of modified field functions. These may be obtained directly from eqs. (25), (34), (54), and (55) in a fashion similar to the derivation of eqs. (40). The results are

$$\bar{u}_2^+ = -(B_2\alpha + \hat{D}_2\beta)^{-1}(\hat{D}_2\alpha - B_2\beta) \tag{56a}$$

$$\bar{w}_2^+ = (B_2\alpha + \hat{D}_2\beta)^{-1}\hat{L}_2 \tag{56b}$$

$$\bar{v}_2^+ = (B_2\alpha + \hat{D}_2\beta)^{-1}B_2\upsilon \tag{56c}$$

$$\bar{\gamma}_2^+ = \gamma_2^+ + \upsilon^T\beta\bar{v}_2^+ \tag{56d}$$

$$\bar{\lambda}_2^+ = \lambda_2^+ + \upsilon^T\beta\bar{w}_2^+ \tag{56e}$$

where γ_2^+ and λ_2^+ are given by eqs. (55). As discussed on page 15, if E is a closed branch arc, or if E is an open branch arc with complete rigid body constraint prescribed at P_2, then eqs. (56) can be simplified by choosing $\beta = I$ [and hence by eq. (21) $\alpha = 0$].

Now consider the case where the length h of one of the chains C_i is very small. Assuming this chain to consist of a single arc, say Γ, and P_1 to be a force-free vertex (i.e., $B_1 = I$, $D_1 = L_1 = 0$), we expand the field functions on Γ in Laurent series about P_1. This gives the results

$$\sum u_2^- = -h^{-1}c^{-1} + O(1) \tag{57a}$$

$$\sum v_2^- = h^{-1}c^{-1} + O(1) \tag{57b}$$

$$\mu_2 = I + O(h) \tag{57c}$$

where the matrix c may be evaluated at any point on Γ, and O(1) means terms remaining finite as $h \to 0$. For small h, the term $h^{-1}c^{-1}$ dominates O(1). Since the terms $h^{-1}c^{-1}$ cancel each other in the calculation of $\hat{D}_2$ [cf. eq. (53a)], it is clear that this calculation is subject to ever-increasing round-off errors as h approaches zero. For this reason, it is necessary to treat the loop configuration [fig. 6(b)] as a special case.

Loops: Initial values of the field functions on the continuation arc E of figure 6(b) can be obtained by applying the limiting process described above to eqs. (51)-(56). However, it is more direct to consider the loop configuration as a separate case and follow a procedure similar to that used to derive eqs. (52)-(56). Omitting the details, the results are the following. In place of eqs. (52) and (53)

$$By_E^+ + \hat{D}z_P - Bv_A^- z_0 = \hat{L} \tag{58}$$

where

$$\hat{D} = D - B[\sum u^- + \sum \gamma_C^- + \sum v_C^- + (\sum v_C^-)^T] \tag{59a}$$

$$\hat{L} = L + B(\sum w^- + \sum \lambda_C^-) \tag{59b}$$

In place of eqs. (54) and (55)

$$u_E^+ = -B^{-1}\hat{D} \tag{60a}$$

$$w_E^+ = B^{-1}\hat{L} \tag{60b}$$

$$v_E^+ = v_A^- \tag{60c}$$

$$\gamma_E^+ = \gamma_A^- \tag{60d}$$

$$\lambda_E^+ = \lambda_A^- \tag{60e}$$

If P is a singular vertex, one obtains in place of eqs. (56)

$$\bar{u}_E^+ = -(B\alpha + \hat{D}\beta)^{-1}(\hat{D}\alpha - B\beta) \tag{61a}$$

$$\bar{w}_E^+ = (B\alpha + \hat{D}\beta)^{-1}\hat{L} \tag{61b}$$

$$\bar{v}_E^+ = (B\alpha + \hat{D}\beta)^{-1}Bv_A^- \tag{61c}$$

$$\bar{\gamma}_E^+ = \gamma_A^- + (v_A^-)^T\beta\bar{v}_E^+ \tag{61d}$$

$$\bar{\lambda}_E^+ = \lambda_A^- + (v_A^-)^T\beta\bar{w}_E^+ \tag{61e}$$

CALCULATION PROCEDURE

As outlined previously (fig. 3), the calculation procedure essentially consists of three basic steps: 1) forward integration for the field functions u, w, v, γ, and λ,* 2) calculation of z at the final vertex, and

*Unless otherwise specified, in this section u, w, v, γ, and λ will stand for $\bar{u}$, $\bar{w}$, $\bar{v}$, $\bar{\gamma}$, and $\bar{\lambda}$, respectively, on singular arcs.

3) backward integration to obtain the response functions y and z.

Forward Integration

On regular (ordinary and continuation) arcs, the differential equations (14) and (16) are integrated starting with the initial values given by eqs. (18) and (19) for ordinary arcs, (54) and (55) for continuation arcs exiting from a split circuit, and (60) for continuation arcs exiting from a loop. On singular arcs, the differential equations (14) and (16) written for modified (barred) variables are integrated starting with the initial values given by eqs. (40) for ordinary arcs, (45) for initial closed branch arcs, (56) for continuation arcs exiting from a split circuit, and (61) for continuation arcs exiting from a loop.

As they are calculated, the primary field functions u, w, and v (u and w on open branch arcs), which are used in the subsequent backward integration, are stored. At the terminal vertex of singular arcs, it is convenient to replace the modified functions $\bar{u}$, $\bar{w}$, and $\bar{v}$ by u, w, and v, respectively, using the inverse of eqs. (25). This is done so that the initial values of the field functions for all arcs, as well as the value of z at the final vertex (see below), may be computed the same way, regardless of whether preceding arcs are regular or singular.

Either a Runge-Kutta or a predictor-corrector integration scheme may be used to integrate eqs. (14). Typically, the field functions have a narrow zone of rapid variation (i.e., a boundary layer) immediately following the initial vertex of each arc, but are otherwise slowly varying. In order to take advantage of this behavior, it is desirable that the integration scheme use a variable step size which is automatically controlled to meet a prescribed truncation error tolerance. Then the most efficient storage locations for u, w, and v (to obtain their best description for the least amount of storage) are the end-points of each integration step. In this way, the spacing of the storage points will vary with the rate of variation of the stored functions, the more rapid their variation, the closer together their storage points. In order to obtain the greatest possible accuracy, it is desirable to store, in addition to the primary field functions, their s-derivatives, which are avialable at no extra effort. Then a more accurate "double point" interpolation polynomial can be used to calculate intermediate values of these functions. The author has had good success with a fourth-order Runge-Kutta routine which proceeds in double steps, each of which is composed of two equally spaced basic Runge-Kutta steps.* In order to maintain fourth-order accuracy throughout, fourth-order interpolation of the primary field functions is made during the backward integration. For this purpose, these functions are stored at the end of each basic Runge-Kutta step, whereas their derivatives are stored only at the end of each

*Quadrature of eqs. (16) for γ and λ is made over each double step by Simpson's rule.

double step. Then interpolation at points lying in the range of a double step ($s_1 < s < s_3$) is made according to the formula

$$f(s) = \theta^2(\theta - 2)^2 f_2 - (\theta - 1)(\theta - 2)^2[(1 + 2\theta)f_1 + \theta h f_1']/4$$
$$+ (\theta - 1)\theta^2[(5 - 2\theta)f_3 + (\theta - 2)hf_3']/4 + O(h^5) \qquad (62)$$

where $$\theta = (s - s_1)/h \qquad (63a)$$

$$h = s_2 - s_1 = s_3 - s_2 \qquad (63b)$$

Here, f_1, f_2, and f_3 are function values at the step points s_1, s_2, and s_3, respectively, and f_1' and f_3' are s-derivatives of f at s_1 and s_3.

With such a storage method, for any given problem the number of storage locations for the primary field functions is not known a priori. In order to avoid overflow of any fixed size storage area in core, and also to reduce the core storage requirement, a method of "dynamic" storage is used. In this method, only a relatively small buffer area for field function storage is set aside in core. When, during forward integration, this buffer fills up, it contents are automatically transferred to an external mass storage file. Because in a sequence of problems for which only load changes occur, u and v need not be recalculated, it is convenient to actually use two external files, one for w and the other for u and v. Hence at the end of forward integration, in general, two external files consisting of a sequence of logical records, one for each buffer load, will have been created. During backward integration, these logical records are returned to the core buffer area (in reverse order) when the integration progresses out of the region of applicability of the current buffer contents.

Calculation of z at Final Vertex

There are three types of final vertices at which the calculation of z differs. The final vertex (i.e., the terminal vertex of the final arc) may be: 1) the closure point of a split circuit, such as point P_2 in figure 6(a) with arc E absent, 2) the closure point of a loop, such as point P in figure 6(b) with arc E absent, or 3) incident with no closed branch arcs.

<u>Closure point of a split circuit</u>.- In this case, arc A of figure 6(a) is necessarily an open branch arc, and z is given by eq. (52) with terms in y_2^+ and υ omitted, viz.

$$z_2 = \hat{D}_2^{-1}\hat{L}_2 \qquad (64)$$

where $\hat{D}_2$ and $\hat{L}_2$ are given by eqs. (53a,c).

<u>Closure point of a loop</u>.- In this case, arc A of figure 6(b) is also an open branch arc, and z is given by eq. (58) with terms in y_E^+ and v_A^-

omitted, viz.

$$z_P = \hat{D}^{-1}\hat{L} \tag{64}$$

where $\hat{D}$ and $\hat{L}$ are given by eqs. (59).

No incident closed branch arc.- In this case, the formula for z may be obtained from eq. (50) considering all chains C_i absent, or just as easily from eq. (58) considering all circuits C and arc E absent. The result is

$$z = \tilde{D}^{-1}\tilde{L} = (D - B \sum u)^{-1}(L + B \sum w) \tag{65}$$

Note that this formula is more general than the corresponding formula, $z = (D - Bu)^{-1}(L + Bw)$, given in reference 1 for a tree. This is the case since the ordering rules for open branch arcs given in reference 1 have been relaxed to the extent that the final vertex no longer need be a branch extremity.

Backward Integration

With the value of z at the final vertex as an initial condition, a backward integration of eq. (12) is performed over the graph, using z-continuity at interior vertices. Before integrating each closed branch arc, the stored values of w on that arc are replaced by $w + vz_0$, where z_0 is the value of z at its I.P. Except for this replacement (and, of course, a similar one for w') there is no difference between the backward integration on open and closed branch arcs. Upon reaching point P_2 of a split circuit [fig. 6(a)], the value of z_0 for arcs of the chains C_i is calculated from eq. (50). [In this case z_0 is z_1 and should not be confused with the z_0 in eq. (50), which is the known value of z at the I.P. of arc E.] Upon reaching point P of a loop [fig. 6(b)], the value of z_0 for arcs of the circuits C is known since in this case, by z-continuity, $z_0 = z_P$. From the values of z obtained by integration on regular arcs, y is calculated using eq. (7).

On singular arcs, the backward integration is performed in terms of modified variables, i.e., eq. (12) written for barred variables is integrated. For this purpose, the value of z at the terminal vertex of a singular arc is replaced by $\bar{z}$ by first computing y there from eq. (7) (recall that u and w have been stored there) and then $\bar{z}$ from eq. (20b). After so doing, u and w at the terminal vertex are changed back to $\bar{u}$ and $\bar{w}$ through use of eqs. (25a,b), as required for the integration of the modified form of eq. (12).* From the values of $\bar{z}$ obtained by integration, $\bar{y}$ is computed from eq. (7) written for barred variables, and y and z are

*Note that, because of the similarity of eqs. (25b) and (25c), the transformation from $w + vz_0$ to $\bar{w} + \bar{v}z_0$ on singular closed branch arcs is the same as that from w to $\bar{w}$ on singular open branch arcs.

then given by eqs. (23).

SHELLS OF REVOLUTION

A computer program employing the field method to obtain the linear elastic response of multicircuit ring-stiffened shells of revolution subject to general harmonic mechanical and thermal loads has been written. For this class of problems, the graph over which the boundary-value problem is defined represents the reference meridian of the shell. The differential equations (1) are eighth-order so that the response matrices y and z are 4-element column vectors. The equations are ordered so that y and z are the force and displacement vectors (fig. 7)

$$y^T = r(P,\ Q,\ S,\ M_1) \tag{66a}$$

$$z^T = (\xi,\ \eta,\ v,\ \chi) \tag{66b}$$

where P, Q, and S are forces per unit of circumferential length referred to fixed axial, radial, and circumferential coordinate directions x, y, and ϕ; ξ, η, and v are the corresponding displacement components; M_1 is the meridional bending moment per unit of circumferential length; and χ is the corresponding rotation.

The matrices a, b, c, f, and g [eqs. (1)], as well as $\kappa = B^{-1}D$ and $B^{-1}L$ [eq. (2)] for ring boundaries, are given in the appendix. Since this problem is self-adjoint, b, c, and κ are symmetric and $d = -a^T$ [cf. eqs. (4)]. Corresponding shell and ring equations have been given previously (although not in precisely the same form) in reference 5.

Example Solutions

Solutions of the shell equations by the field method for three example multicircuit meridians are presented in this section. In each case, in order to have a quantitative check on the accuracy of the numerical solution, the structure was devised to have a plane of symmetry normal to the axis of revolution. (In practice, only one-half of such symmetrical structures would be modeled.) Thus, the accuracy of the solution can be checked simply by comparing the calculated response at symmetry points.

Diagrams of the meridians of the three sample shell configurations, showing entry (output station) numbers,* branch point numbers (encircled), and harmonic pressure load amplitudes (underlined), are given in figure 8. In this figure, the arrow on each arc indicates the arc orientation, i.e., the direction in which the forward integration proceeds. Arc ordering is

*Equally spaced output stations corresponding to intervening integral entry numbers are not shown.

indicated by the entry numbers in that during the forward integration the entry points are encountered in the order of increasing entry number. Note that the first and second examples are the same problem set up with different arc ordering and orientation. For this problem the circumferential harmonic number of the pressure loading and response was n = 1000, whereas in the third example n = 10.* For both configurations the following homogeneous, isotropic wall properties were assumed: $E = 10^7$, $\nu = 0.25$, $t = 0.1$. These properties and the dimensions and pressure amplitudes shown in figure 8 may be taken to be in any consistent set of units.

Each of the problems has been run in single precision on the UNIVAC 1108 computer (a 36-bit word machine) with no degradation of accuracy relative to the CDC 6600 computer (a 60-bit word machine). Therefore, it is concluded that the small numerical discrepancies which occur at symmetry points (see table I, which is reproduced from computer print-outs with an added column labeled "SYM. POINT" giving the entry number of the symmetry point for each output station) are due essentially to truncation error, which is controlled by an input truncation error tolerance, and that round-off errors are negligible.

For each calculation the truncation error tolerance was set to 10^{-3}, which means that at the end of each Runge-Kutta double step (see p. 21), in both forward and backward integrations, the relative difference between the values of each (scalar) dependent variable as calculated by the Runge-Kutta and Simpson rules (over the double step) was less than 10^{-3}. Roughly speaking, this means that for each step these two calculations agreed to three significant digits. Since, as noted, round-off errors were negligible, one expects this degree of accuracy in the response functions (except in the immediate vicinity of their zeroes). Inspection of the calculated values of the components of the force and displacement vectors for each of the sample problems (table I) shows that this is indeed the case.

Execution times (central processor times in seconds) and the number of double steps made during the forward and backward integrations for each problem are shown in the following table.

Example	Execution time	Forward steps	Backward steps
a	15.2 (U-1108)	124	110
b	14.8 (U-1108)	208	104
c	12.6 (CDC-6600)	300	218

*In concocting such hypothetical problems, it should be kept in mind that for a given shell the value of n cannot, within the scope of shell theory, be chosen arbitrarily large since nt/r must remain small compared to unity.

Comparison with Other Methods

Conventional methods of numerical analysis of the response of shells of revolution suffer from several practical difficulties. In finite-difference methods one must choose in advance the finite-difference mesh points. The difficulty here is that too few mesh points may result in excessive truncation error, whereas too many mesh points may result in excessive round-off error. This difficulty is illustrated in figure 9, which has been taken from reference 6, giving the results of a bifurcation buckling analysis of an axially loaded cylindrical shell as a function of the number of finite-difference intervals on the shell half-length. Numerical convergence appears to be obtained with 27 intervals since the critical loads calculated with 27 and 28 intervals agree to four significant digits. However, when the number of intervals is increased further, the growth of round-off errors causes divergence.

In the superposition methods, which employ forward integration techniques, the truncation error is controlled automatically by the use of self-checking variable step-size integration routines. However, since the auxiliary solutions to be superimposed exhibit exponential growth characteristics, their superposition involves large round-off errors at points remote from the initial point of integration. To circumvent this problem, the meridian of the shell is artificially subdivided into a number of "suitably small" segments. One difficulty with this approach is that the subdivision requirements are known only very roughly, and depend on factors in addition to the properties of the structure itself, e.g., the circumferential wave number and, for eigenvalue problems, the eigenvalue shift. Also, some problems may not be treatable simply because more segments are required than allowed by computer storage limitations.

The field method not only retains the advantage of automatic control of truncation error, but also produces stable initial value problems whose solutions are not sensitive to small numerical error introduced at each integration step. Consequently, no artificial subdivision of the shell meridian is required, leaving the user free from concerns of numerical convergence (truncation error) or ill-conditioning (round-off error). This is illustrated in figure 10, showing computation times for the calculation of the linear elastic response of a pressurized clamped cylinder by the field method and also by an efficient superposition method, known as the Zarghamee-Robinson method (ref. 7).* The bar graph [fig. 10(a)] compares the computational efficiencies of the two methods in terms of CDC 6600 CPU seconds for the axisymmetric response for several r/t ratios. In each case the field method is significantly more efficient. More important is the fact that, even with 33 segments, the superposition method failed to give meaningful results at $r/t = 10^4$, whereas the field method experienced no difficulty without any subdivision. Similar results are obtained when the circumferential wave number is varied, as shown in figure 10(b).

---

*The author is indebted to Dr. Wendell B. Stephens of the NASA Langley Research Center for providing the results shown in figure 10.

To provide additional verification of the accuracy of the field method for closed branch shells, a comparison of the field and Zarghamee solutions for the shell depicted in figure 11 is presented.* (For comparisons of field and Zarghamee solutions for open branch shells, see reference 1.) The meridian of this shell consists of a single circuit with attached open branches. Only entry numbers at physical boundaries are shown in the figure. The edges at entries 15 and 53 are clamped, whereas the remaining four edges are force-free. The interior boundaries at entries 30, 68, and 76 have attached rings, whereas the remaining three interior boundaries are force-free. Each of the open branch segments (between boundaries) has a unit slant length, whereas each of the closed branch segments has a slant length of 2 units. The segments 25-29 and 48-52 are cylindrical, whereas all others are conical at 45° from the axis. The following homogeneous, isotropic wall properties were assumed: $E = 10^7$, $\nu = 0.25$, $t = 0.1$. The three rings are identical, with the following section properties: $EA = 2. \times 10^6$, $EI_x = 2.1667 \times 10^4$, $EI_y = 1.6667 \times 10^3$, $EI_{xy} = 0$, $GJ = 5.6240 \times 10^2$. For each ring, it is assumed that the section centroid lies on the shell reference (i.e., middle) surface. The only load consists of a uniform second harmonic ($n = 2$) unit pressure loading as shown in the figure. In table II force and displacement components calculated by each of the methods are compared at the boundary points of one-half of the symmetrical structure. For the results of the Zarghamee-Robinson method only those digits which differ from corresponding digits of the field method results are shown.† As the table shows, the agreement is very good, in spite of the use of a truncation error tolerance of only 10^{-2} in each calculation.

CONCLUDING REMARKS

The field method of solution of general even-order linear boundary-value problems defined on one-dimensional domains containing circuits has been formulated. The method has been implemented in a computer program for the static elastic response of ring-stiffened orthotropic shells of revolution with multicircuit meridians. Example solutions have been presented which illustrate the accuracy and efficiency of the method.

The field method has several advantages over more conventional numerical methods such as other numerical integration (superposition) and finite-difference methods. These are:

(1) The field method is essentially free of numerical ill-conditioning problems. All calculations are made in single precision,

*Since the Zarghamee-Robinson method has been extended to closed branch shells with only a single circuit (ref. 5), it could not be used to provide solution comparisons for the shells of figure 8.

†In the table, an exact zero is indicated by 0., whereas 0.000 indicates a number which is zero to three decimal places.

even on 36-bit word computers such as the UNIVAC 1108, and there is no need for artificial subdivision of the domain of integration.

(2) Solution of large systems of algebraic equations is avoided. The method is developed in terms of matrices whose order equals the half-order of the boundary-value problem.

(3) A priori choice of output or storage points for response variables is unnecessary since a uniformly dense set of storage points can be chosen automatically during execution. This is a particularly important feature when combined with iterative methods for nonlinear or eigenvalue problems, for which the response of each iteration is the input to the succeeding iteration. Consequently, there are no convergence problems related to an inadequate choice of response (or finite-difference) points.

Because of these features, the field method is potentially a truly automatic method in that one can expect, with confidence, from computer programs based on it meaningful answers for every meaningful problem for which the program is designed.

APPENDIX

MATRICES FOR AXISYMMETRIC SHELLS AND RINGS

Symbols

A	ring section area
E	ring elastic modulus
e_x, e_y	ring centroidal eccentricities
F_x, F_y, F_ϕ	harmonic amplitudes of ring force loads per unit of circumferential length
GJ	ring torsional stiffness
I_x, I_y, I_{xy}	ring section moments of inertia
k	ring stiffness matrix
L_1, L_2	harmonic amplitudes of shell moment loads per unit of surface area in meridional and circumferential directions
ℓ_e, ℓ_f, ℓ_t	ring load vectors
N_x, N_y, N_ϕ	harmonic amplitudes of ring moment loads per unit of circumferential length
X_1, X_2, X_3	harmonic amplitudes of shell force loads per unit of surface area in meridional, circumferential, and normal directions
ε	ring eccentricity matrix
$\Theta_1^{(0)}, \Theta_2^{(0)}$	harmonic amplitudes of meridional and circumferential thermal force loads
$\Theta_1^{(1)}, \Theta_2^{(1)}$	harmonic amplitudes of meridional and circumferential thermal moment loads
θ	harmonic amplitude of ring free thermal strain
λ_{ij}	orthotropic shell wall normal stiffness coefficients
μ_{ij}	orthotropic shell wall shear stiffness coefficients
ρ	ring centroidal radius

Shell

For elastic orthotropic shells of revolution subjected to symmetric n-th harmonic loads, the coefficient and load matrices of eqs. (1) are given below. For the definitions of the stiffness (λ_{ij}, μ_{ij}) and load [X_i, L_i, $\Theta_i^{(m)}$] variables in these matrices, see reference 5.

$$
a = \frac{1}{r}\begin{bmatrix}
n^2\lambda_{23}r'/R_2 & n^2\lambda_{23}r'^2/r & n(r/R_2-\mu_{12}/r) & n^2\lambda_{24}r'/r \\
-(\lambda_{13}+n^2\lambda_{23}/R_2)(r/R_2) & -(\lambda_{13}+n^2\lambda_{23}/R_2)r' & nr' & -(\lambda_{14}+n^2\lambda_{24}/R_2) \\
-n(\lambda_{13}+\lambda_{23}/R_2)(r/R_2) & -n(\lambda_{13}+\lambda_{23}/R_2)r' & r' & -n(\lambda_{14}+\lambda_{24}/R_2) \\
(1-\lambda_{23}/R_2)rr' & -r^2/R_2-\lambda_{23}r'^2 & n\mu_{12} & -\lambda_{24}r'
\end{bmatrix}
$$

$$
b = -\frac{1}{r}\begin{bmatrix}
n^2(n^2\lambda_{22}r'^2+\mu_{11})/r^2 & -n^2(\lambda_{12}+n^2\lambda_{22}/R_2)r'/r & -n^3(\lambda_{12}+\lambda_{22}/R_2)r'/r & -n^2(\lambda_{22}r'^2+\mu_{11})/r \\
 & \lambda_{11}+2n^2\lambda_{12}/R_2+n^4\lambda_{22}/R_2^2 & n[\lambda_{11}+(n^2+1)\lambda_{12}/R_2+n^2\lambda_{22}/R_2^2] & (\lambda_{12}+n^2\lambda_{22}/R_2)r' \\
 & & n^2(\lambda_{11}+2\lambda_{12}/R_2+\lambda_{22}/R_2^2) & n(\lambda_{12}+\lambda_{22}/R_2)r' \\
\text{Symmetric} & & & \lambda_{22}r'^2+n^2\mu_{11}
\end{bmatrix}
$$

$$
c = -\frac{1}{r}\begin{bmatrix}
\lambda_{33}(r/R_2)^2 & \lambda_{33}r'(r/R_2) & 0 & \lambda_{34}(r/R_2) \\
 & \lambda_{33}r'^2 & 0 & \lambda_{34}r' \\
 & & \mu_{22} & 0 \\
\text{Symmetric} & & & \lambda_{44}
\end{bmatrix}
$$

$$
f = \begin{Bmatrix}
r[-(r/R_2)X_1+r'(X_3-nL_1/r)]+n^2(r'/r)[\Theta_2{}^{(1)}-\lambda_{23}\Theta_1{}^{(0)}-\lambda_{24}\Theta_1{}^{(1)}] \\
-r[r'X_1+(r/R_2)(X_3-nL_1/r)]+[\lambda_{13}+n^2\lambda_{23}/R_2]\Theta_1{}^{(0)}+[\lambda_{14}+n^2\lambda_{24}/R_2]\Theta_1{}^{(1)}-\Theta_2{}^{(0)}-n^2\Theta_2{}^{(1)}/R_2 \\
-rX_2+(r/R_2)L_1+n[(\lambda_{13}+\lambda_{23}/R_2)\Theta_1{}^{(0)}+(\lambda_{14}+\lambda_{24}/R_2)\Theta_1{}^{(1)}-\Theta_2{}^{(0)}-\Theta_2{}^{(1)}/R_2] \\
-rL_2+r'[\lambda_{23}\Theta_1{}^{(0)}+\lambda_{24}\Theta_1{}^{(1)}-\Theta_2{}^{(1)}]
\end{Bmatrix}
$$

$$
g = \begin{Bmatrix} (r/R_2)[\lambda_{33}\Theta_1{}^{(0)}+\lambda_{34}\Theta_1{}^{(1)}] \\ r'[\lambda_{33}\Theta_1{}^{(0)}+\lambda_{34}\Theta_1{}^{(1)}] \\ 0 \\ \lambda_{34}\Theta_1{}^{(0)}+\lambda_{44}\Theta_1{}^{(1)} \end{Bmatrix}
$$

Rings

For elastic isotropic ring boundaries, the coefficient and load matrices $\kappa = B^{-1}D$ and $B^{-1}L$ of eq. (2) are (ref. 5)

$$
\kappa = -\varepsilon^T k \varepsilon
$$

$$
B^{-1}L = -\varepsilon^T(\ell_f + \ell_t - k\ell_e)
$$

where

$$
k = \frac{1}{\rho}\begin{bmatrix} n^2(n^2EI_y+GJ)/\rho^2 & n^4EI_{xy}/\rho^2 & n^3EI_{xy}/\rho^2 & -n^2(EI_y+GJ)/\rho \\ & EA+n^4EI_x/\rho^2 & n(EA+n^2EI_x/\rho^2) & -n^2EI_{xy}/\rho \\ & & n^2(EA+EI_x/\rho^2) & -nEI_{xy}/\rho \\ \text{Symmetric} & & & EI_y+n^2GJ \end{bmatrix}
$$

$$
\varepsilon = \begin{bmatrix} 1 & 0 & 0 & e_y \\ 0 & 1 & 0 & -e_x \\ ne_x/r & ne_y/r & \rho/r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\ell_f{}^T = (\rho F_x + nN_y,\ \rho F_y - nN_x,\ \rho F_\phi - N_x,\ \rho N_\phi)
$$

$$
\ell_t{}^T = EA\theta(0,\ 1,\ n,\ 0)
$$

$$
\ell_e{}^T = \theta(e_x,\ e_y,\ 0,\ 0)
$$

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TABLE I.- CALCULATED FORCE AND DISPLACEMENT AMPLITUDES

(a) Split Circuit in Loop

SYM. POINT	ENTRY	S	P	Q	CAP S	MI	XI	ETA	V	CHI
31	1	0.0000	3.1078+02	-1.3701-03	2.2391-02	-1.5140+00	9.8569-09	-1.0752-03	-3.5032-06	-1.0213-07
30	2	1.1107+00	2.3659+02	9.2529+01	-1.0557+02	2.1948+00	4.5550-04	-1.7270-03	-4.3788-05	-5.2489-04
29	3	2.2214+00	3.5997+01	8.5735+01	-1.8035+02	-6.9778+00	1.2168-04	-1.0245-03	4.0172-05	3.8299-04
28	4	2.2214+00	9.7335+01	2.8108+01	-4.8407+01	6.6247+00	1.2168-04	-1.0245-03	4.0172-05	3.8299-04
27	5	3.0068+00	4.6749+01	4.7739+01	-5.0880+01	-1.7142+00	-1.6899-04	-1.7255-04	4.3005-05	8.3368-04
26	6	3.7922+00	-4.8165-03	4.2711+01	-1.4928-01	-4.0156-01	-3.4259-04	-6.8510-08	9.8198-05	4.8871-07
25	7	4.5776+00	-4.6707+01	4.7724+01	5.0527+01	-1.7117+00	-1.6916-04	1.7260-04	4.2157-05	-8.3360-04
24	8	5.3630+00	-9.7105+01	2.8028+01	4.8042+01	6.6113+00	1.2189-04	1.0242-03	3.8445-05	-3.8341-04
23	9	5.3630+00	-6.1335+01	5.7634+01	-1.3196+02	-1.3600+01	1.2168-04	-1.0245-03	4.0172-05	3.8299-04
22	10	6.1484+00	1.2580+01	-1.3814+01	-1.0727+02	2.8219+00	5.1937-04	-7.3955-05	8.2880-05	-8.5801-04
21	11	6.9338+00	8.4554-03	-6.7498+01	-1.8100-01	-2.3092-01	6.2279-04	-1.8187-07	1.4441-04	-1.2323-06
20	12	7.7192+00	-1.2643+01	-1.3855+01	1.0697+02	2.8245+00	5.2031-04	7.3278-05	8.1973-05	8.5764-04
19	13	8.5046+00	6.1270+01	5.7642+01	1.3183+02	-1.3607+01	1.2189-04	1.0242-03	3.8475-05	-3.8341-04
18	14	8.5046+00	-3.5834+01	8.5670+01	1.7988+02	-6.9954+00	1.2189-04	1.0242-03	3.8445-05	-3.8341-04
17	15	9.6153+00	-2.3583+02	9.2610+01	1.0577+02	2.2071+00	4.5718-04	1.7289-03	-4.8603-05	5.2704-04
16	16	1.0726+01	-3.1151+02	3.5245-04	6.3405-03	-1.5224+00	-1.8873-08	1.0729-03	-5.5312-06	9.4098-08
15	17	1.1837+01	-2.3583+02	-9.2806+01	-1.0576+02	2.2071+00	-4.5709-04	1.7287-03	-4.8602-05	-5.2772-04
14	18	1.2948+01	-3.5815+01	-8.5666+01	-1.7987+02	-6.9971+00	-1.2194-04	1.0242-03	3.8435-05	3.8331-04
13	19	1.2948+01	6.1280+01	-5.7645+01	-1.3185+02	-1.3606+01	-1.2194-04	1.0242-03	3.8435-05	3.8331-04
12	20	1.3733+01	-1.2640+01	1.3853+01	-1.0700+02	2.8247+00	-5.1987-04	7.3359-05	8.2025-05	-8.5839-04
11	21	1.4518+01	7.7581-03	6.7499+01	1.6648-01	-2.3096-01	-6.2281-04	-3.2116-07	1.4442-04	3.4890-07
10	22	1.5304+01	1.2580+01	1.3814+01	1.0725+02	2.8220+00	-5.1981-04	-7.3910-05	8.2877-05	8.5895-04
9	23	1.6089+01	-6.1325+01	-5.7631+01	1.3194+02	-1.3601+01	-1.2163-04	-1.0244-03	4.0189-05	-3.8314-04
8	24	1.6089+01	-9.7099+01	-2.8026+01	-4.8028+01	6.6106+00	-1.2194-04	1.0242-03	3.8435-05	3.8331-04
7	25	1.6875+01	-4.6710+01	-4.7727+01	-5.0532+01	-1.7117+00	1.6868-04	1.7287-04	4.2131-05	8.3406-04
6	26	1.7660+01	-2.6580-03	-4.2711+01	9.8543-02	-4.0153-01	3.4255-04	2.7275-07	9.8193-05	4.8755-07
5	27	1.8445+01	4.6743+01	-4.7734+01	5.0870+01	-1.7141+00	1.6939-04	-1.7231-04	4.3036-05	-8.3316-04
4	28	1.9231+01	9.7338+01	-2.8107+01	4.8421+01	6.6248+00	-1.2163-04	-1.0244-03	4.0189-05	-3.8314-04
3	29	1.9231+01	3.6013+01	-8.5737+01	1.8036+02	-6.9766+00	-1.2163-04	-1.0244-03	4.0189-05	-3.8314-04
2	30	2.0341+01	2.3659+02	-9.2530+01	1.0561+02	2.1951+00	-4.5563-04	-1.7273-03	-4.3829-05	5.2474-04
1	31	2.1452+01	3.1080+02	-1.0581-03	-4.8722-03	-1.5135+00	9.8569-09	-1.0752-03	-3.5032-06	-1.0213-07

*FIN

TABLE I.- CALCULATED FORCE AND DISPLACEMENT AMPLITUDES - Continued

(b) Split Circuit in Split Circuit

SYM. POINT	ENTRY	S	P	Q	CAP S	MI	XI	ETA	V	CHI
21	1	0.0000	9.7331+01	2.8107+01	-4.8403+01	6.6246+00	1.2168-04	-1.0245-03	4.0175-05	3.8308-04
22	2	7.8540-01	4.6748+01	4.7740+01	-5.0872+01	-1.7144+00	-1.6900-04	-1.7249-04	4.3008-05	8.3387-04
23	3	1.5708+00	-2.6878-03	4.2715+01	-1.5187-01	-4.0169-01	-3.4247-04	-3.2087-08	9.6169-05	1.5466-07
24	4	2.3562+00	-4.6705+01	4.7723+01	5.0520+01	-1.7115+00	-1.6911-04	1.7262-04	4.2146-05	-8.3361-04
25	5	3.1416+00	-9.7095+01	2.8024+01	4.8029+01	6.6106+00	1.2193-04	1.0242-03	3.8441-05	-3.8344-04
16	6	3.1416+00	-6.1340+01	5.7636+01	-1.3196+02	-1.3600+01	1.2168-04	-1.0245-03	4.0175-05	3.8308-04
17	7	3.9270+00	1.2578+01	-1.3813+01	-1.0728+02	2.8221+00	5.1931-04	-7.4052-05	8.2680-05	-8.5771-04
18	8	4.7124+00	8.9391-03	-6.7505+01	-1.9449-01	-2.3137-01	6.2274-04	-2.3260-07	1.4440-04	-1.3442-06
19	9	5.4978+00	-1.2639+01	-1.3853+01	1.0699+02	2.8249+00	5.2038-04	7.3204-05	8.2018-05	8.5755-04
20	10	6.2832+00	6.1275+01	5.7645+01	1.3184+02	-1.3607+01	1.2193-04	1.0242-03	3.8441-05	-3.8344-04
15	11	6.2832+00	-3.5979+01	-8.5719+01	1.8037+02	6.9733+00	1.2168-04	-1.0245-03	4.0175-05	3.8308-04
14	12	7.3939+00	-2.3657+02	-9.2519+01	1.0563+02	-2.1964+00	4.5534-04	-1.7267-03	-4.3808-05	-5.2608-04
13	13	8.5046+00	-3.1080+02	3.2243-03	4.9967-03	1.5130+00	1.5605-07	-1.0748-03	-3.4381-06	-1.2270-06
12	14	9.6153+00	-2.3661+02	9.2539+01	-1.0549+02	-2.1949+00	-4.5527-04	-1.7267-03	-4.3587-05	5.2457-04
11	15	1.0726+01	-3.5998+01	8.5739+01	-1.8034+02	6.9781+00	-1.2167-04	-1.0245-03	4.0154-05	-3.8304-04
6	16	1.0726+01	6.1337+01	5.7637+01	-1.3195+02	1.3601+01	-1.2167-04	-1.0245-03	4.0154-05	-3.8304-04
7	17	1.1511+01	-1.2578+01	-1.3812+01	-1.0728+02	-2.8221+00	-5.1938-04	-7.3972-05	8.2671-05	8.5803-04
8	18	1.2297+01	-8.4911-03	-6.7503+01	-1.9489-01	2.3111-01	-6.2279-04	-2.0201-07	1.4440-04	1.2482-06
9	19	1.3082+01	1.2644+01	-1.3855+01	1.0697+02	-2.8244+00	-5.2029-04	7.3288-05	8.1972-05	-8.5764-04
10	20	1.3868+01	-6.1267+01	5.7640+01	1.3183+02	1.3606+01	-1.2188-04	1.0242-03	3.8447-05	3.8336-04
1	21	1.3868+01	-9.7339+01	2.8112+01	-4.8402+01	-6.6254+00	-1.2167-04	-1.0245-03	4.0154-05	-3.8304-04
2	22	1.4653+01	-4.6752+01	4.7742+01	-5.0886+01	1.7145+00	1.6904-04	-1.7250-04	4.2998-05	-8.3377-04
3	23	1.5438+01	3.6420-03	4.2719+01	-1.3743-01	4.0210-01	3.4271-04	3.4683-08	9.6193-05	-3.4837-07
4	24	1.6224+01	4.6705+01	4.7722+01	5.0823+01	1.7116+00	1.6909-04	1.7262-04	4.2161-05	8.3362-04
5	25	1.7009+01	9.7107+01	2.8028+01	4.8045+01	-6.6115+00	-1.2188-04	1.0242-03	3.8447-05	3.8336-04
30	26	1.7009+01	3.5839+01	8.5669+01	1.7988+02	6.9947+00	-1.2188-04	1.0242-03	3.8447-05	3.8336-04
29	27	1.8120+01	2.3684+02	9.2612+01	1.0577+02	-2.2070+00	-4.5714-04	1.7289-03	-4.6599-05	-5.2703-04
28	28	1.9231+01	3.1152+02	2.4131-03	1.5466-02	1.5221+00	4.6501-08	1.0730-03	-5.5293-06	6.8881-08
27	29	2.0341+01	2.3683+02	-9.2606+01	-1.0576+02	-2.2070+00	4.5712-04	1.7288-03	-4.6602-05	5.2763-04
26	30	2.1452+01	3.5820+01	-8.5668+01	-1.7987+02	6.9966+00	1.2193-04	1.0242-03	3.8441-05	-3.8344-04

*FIN

TABLE I.- CALCULATED FORCE AND DISPLACEMENT AMPLITUDES - Concluded

(c) Loop in Split Circuit

SYM. POINT	ENTRY	S	P	Q	CAP S	MI	XI	ETA	V	CHI
37	1	0.E+00	4.6966E+00	9.2346E+01	-1.6702E+01	1.1177E+00	-7.7628E-07	2.1958E-05	-2.6925E-05	-6.0202E-06
36	2	7.8540E-01	-6.8064E+01	6.8063E+01	-1.2628E+01	-3.2055E-01	6.5584E-05	1.2814E-04	-2.6667E-05	7.5061E-05
35	3	1.5708E+00	-9.9447E+01	-3.5084E-01	-1.3609E+00	-3.5185E-03	5.8907E-07	1.4348E-04	-2.5374E-05	-5.0422E-05
34	4	2.3562E+00	-7.0566E+01	-7.0326E+01	6.8517E+00	1.1503E-02	-6.3112E-05	9.6399E-05	-2.8753E-05	4.9203E-07
33,38	5	3.1416E+00	6.3655E-02	-9.9387E+01	1.0066E+01	-5.7961E-02	-9.1514E-05	2.7534E-05	-2.9588E-05	-1.3636E-05
39	6	3.9270E+00	7.0783E+01	-7.0620E+01	6.7149E+00	7.2900E-02	-5.2810E-05	-4.0263E-05	-2.6468E-05	5.4130E-06
40	7	4.7124E+00	9.9894E+01	-5.7729E-01	1.4161E+00	-4.3847E-02	8.4095E-06	-1.0578E-04	-1.5836E-05	7.5130E-05
41	8	5.4978E+00	6.8004E+01	6.8353E+01	-1.4485E+00	-3.4530E-01	7.4569E-05	-9.0601E-05	-1.6604E-05	-9.5960E-05
42	9	6.2832E+00	-4.2482E+00	9.3188E+01	-4.1802E+00	1.0463E+00	-7.7628E-07	2.1958E-05	-2.6925E-05	-6.0202E-06
43	10	6.2832E+00	-8.9423E+00	8.3822E-01	1.2518E+01	-7.0639E-02	-7.7628E-07	2.1958E-05	-2.6925E-05	-6.0202E-06
44	11	6.7832E+00	-1.2263E+01	-2.6804E-01	1.3231E+00	5.4458E-02	-3.5092E-06	2.3404E-05	-1.9590E-05	1.1563E-05
45	12	7.2832E+00	-1.0207E+01	-1.4214E+00	-1.0168E+01	-3.4806E-01	-6.4917E-06	2.2499E-05	-2.7311E-05	-4.5062E-05
27	13	7.2832E+00	-6.2303E+00	9.1580E+01	-1.3722E+01	-1.3790E+00	-6.4917E-06	2.2499E-05	-2.7311E-05	-4.5062E-05
26	14	8.0686E+00	6.5782E+01	6.5437E+01	-9.7267E+00	6.5112E-01	-1.0130E-04	1.3089E-04	-2.8201E-05	-5.3976E-06
25	15	8.8540E+00	9.7632E+01	-4.1454E+00	-7.6982E+00	-1.0968E+00	-5.0471E-09	6.8142E-05	-4.2072E-05	9.5662E-08
24	16	8.8540E+00	9.2919E+01	-4.1425E+00	-7.6931E+00	1.1414E+00	-5.0471E-09	6.8142E-05	-4.2072E-05	9.5662E-08
23	17	9.6394E+00	6.8579E+01	6.7959E+01	-1.4381E+01	-3.3161E-01	1.2359E-04	-2.3527E-05	-3.1728E-05	1.3862E-04
22	18	1.0425E+01	-5.5256E-01	1.0015E+02	-1.1150E+01	-1.0458E-01	1.5626E-04	3.1952E-05	-2.2923E-05	-7.5297E-05
21	19	1.1210E+01	-7.0741E+01	7.0724E+01	-4.4766E+00	6.4669E-02	7.6222E-05	8.2322E-05	-2.9616E-05	-4.3529E-05
20	20	1.1996E+01	-9.9182E+01	-4.3515E-04	1.7115E-03	1.0815E-02	1.6584E-07	9.9575E-05	-3.3616E-05	-9.6602E-07
19	21	1.2781E+01	-7.0740E+01	-7.0725E+01	4.4706E+00	6.4498E-02	-7.5464E-05	8.2011E-05	-2.9659E-05	4.2036E-05
18	22	1.3566E+01	-5.5293E-01	-1.0016E+02	1.1140E+01	-1.0440E-01	-1.5548E-04	3.2239E-05	-2.2931E-05	7.5328E-05
17	23	1.4352E+01	6.8574E+01	-6.7957E+01	1.4382E+01	-3.3235E-01	-1.2337E-04	-2.3252E-05	-3.1759E-05	-1.3837E-04
16	24	1.5137E+01	9.2912E+01	4.1461E+00	7.6961E+00	1.1415E+00	-5.0471E-09	6.8142E-05	-4.2072E-05	9.5662E-08
15	25	1.5137E+01	9.7625E+01	4.1471E+00	7.6814E+00	-1.0975E+00	-5.0471E-09	6.8142E-05	-4.2072E-05	9.5662E-08
14	26	1.5923E+01	6.5784E+01	-6.5438E+01	9.7150E+00	6.5092E-01	1.0130E-04	1.3087E-04	-2.8186E-05	5.2016E-06
13	27	1.6708E+01	-6.2264E+00	-9.1586E+01	1.3716E+01	-1.3777E+00	6.5204E-06	2.2501E-05	-2.7310E-05	4.5249E-05
32	28	1.6708E+01	-3.9863E+00	-9.2994E+01	3.5377E+00	1.0303E+00	-6.4917E-06	2.2499E-05	-2.7311E-05	-4.5062E-05
31	29	1.7493E+01	6.8662E+01	-6.8926E+01	1.3898E-01	-2.1776E-01	-5.5693E-05	-8.2653E-05	-1.7425E-05	9.7927E-05
30	30	1.8279E+01	1.0085E+02	6.5333E-04	4.9411E-03	-7.8322E-02	-9.3645E-08	-1.3482E-04	-7.6005E-06	-2.9921E-07
29	31	1.9064E+01	6.8663E+01	6.8927E+01	-1.5803E-01	-2.1764E-01	5.5372E-05	-8.2366E-05	-1.7460E-05	-9.8219E-05
28	32	1.9850E+01	-3.9806E+00	9.3008E+01	-3.5539E+00	1.0290E+00	6.5204E-06	2.2501E-05	-2.7310E-05	4.5249E-05
5	33	1.9850E+01	6.1557E-02	9.9381E+01	-1.0088E+01	-5.5735E-02	9.2364E-05	2.7625E-05	-2.9606E-05	1.1819E-05
4	34	2.0635E+01	-7.0568E+01	7.0329E+01	-6.8465E+00	1.3015E-02	6.3435E-05	9.6605E-05	-2.8743E-05	-1.5461E-06
3	35	2.1420E+01	-9.9445E+01	3.5084E-01	1.3594E+00	-3.4288E-03	-5.5277E-07	1.4342E-04	-2.5400E-05	4.9795E-05
2	36	2.2206E+01	-6.8067E+01	-6.8067E+01	1.2612E+01	-3.2040E-01	-6.5358E-05	1.2791E-04	-2.6673E-05	-7.5418E-05
1	37	2.2991E+01	4.6923E+00	-9.2356E+01	1.6693E+01	1.1164E+00	8.0537E-07	2.1979E-05	-2.6903E-05	5.9236E-06
5	38	2.2991E+01	-5.3782E-02	-9.9373E+01	1.0072E+01	5.5466E-02	9.2364E-05	2.7625E-05	-2.9606E-05	1.1819E-05
6	39	2.3777E+01	-7.0777E+01	-7.0616E+01	6.6937E+00	-7.2456E-02	5.3155E-05	-4.0729E-05	-2.6422E-05	-5.9961E-06
7	40	2.4562E+01	-9.9894E+01	-5.7605E-01	1.4174E+00	4.3948E-02	-8.3064E-06	-1.0598E-04	-1.5807E-05	-7.4889E-05
8	41	2.5347E+01	-6.8004E+01	6.8354E+01	-1.4532E+00	3.4513E-01	-7.4400E-05	-9.0440E-05	-1.6625E-05	9.6247E-05
9	42	2.6133E+01	4.2462E+00	9.3193E+01	-4.1786E+00	-1.0460E+00	8.0537E-07	2.1979E-05	-2.6903E-05	5.9236E-06
10	43	2.6133E+01	8.9386E+00	8.3679E-01	1.2515E+01	7.0482E-02	8.0537E-07	2.1979E-05	-2.6903E-05	5.9236E-06
11	44	2.6633E+01	1.2260E+01	-2.6873E-01	1.3275E+00	-5.4190E-02	3.5384E-06	2.3380E-05	-1.9581E-05	-1.1613E-05
12	45	2.7133E+01	1.0207E+01	-1.4220E+00	-1.0162E+01	3.4866E-01	6.5204E-06	2.2501E-05	-2.7310E-05	4.5249E-05

TABLE II.- COMPARISON OF CALCULATED FORCE AND DISPLACEMENT AMPLITUDES

Entry[a]	Method[b]	$P\times10$	$Q\times10$	$S\times10$	$M_1\times10^2$	$\xi\times10^6$	$\eta\times10^5$	$v\times10^6$	$\chi\times10^5$
1	field	-2.738	1.381	2.751	4.803	0.009	1.922	-2.730	0.001
	Z-R	7	79	4	6	0			0
9	field	-4.687	3.628	5.006	-7.219	-9.001	1.164	-5.893	-2.888
	Z-R			12	8	9	5	5	7
10	field	0.	0.	0.	0.	101.84	9.889	-5.586	16.342
	Z-R					6	91	8	4
14	field	-8.208	-2.712	7.996	-31.827	1.596	0.023	-3.560	6.092
	Z-R		1	7	5			1	3
15	field	13.893	9.796	-12.263	-10.318	0.000	0.000	0.000	0.000
	Z-R		5	8	23	0.	0.	0.	0.
19	field	12.800	7.209	-16.619	23.377	1.596	0.023	-3.560	6.092
	Z-R	799		25	1			1	3
20	field	4.592	4.498	-8.622	-8.450	1.596	0.023	-3.560	6.092
	Z-R	1	7	8	4			1	3
24	field	3.411	3.401	-9.955	-7.943	-9.001	1.164	-5.893	-2.888
	Z-R	09	399	61	2	9	5	5	7
25	field	0.	0.	0.	0.	-9.932	8.847	-9.503	-8.929
	Z-R					9		7	
29	field	1.345	-5.913	8.183	20.731	-9.001	1.164	-5.893	-2.888
	Z-R		4	2	2	9	5	5	7
30	field	0.072	1.082	3.158	5.427	-9.001	1.164	-5.893	-2.888
	Z-R	0	1	9	32	9	5	5	7
38	field	0.888	0.000	-0.003	1.701	0.000	0.239	-4.861	0.001
	Z-R	7		0	2			2	0

[a]Entry numbers refer to output stations shown in figure 11.

[b]For the Zarghamee-Robinson method, only digits which differ from corresponding digits of the field method are shown.

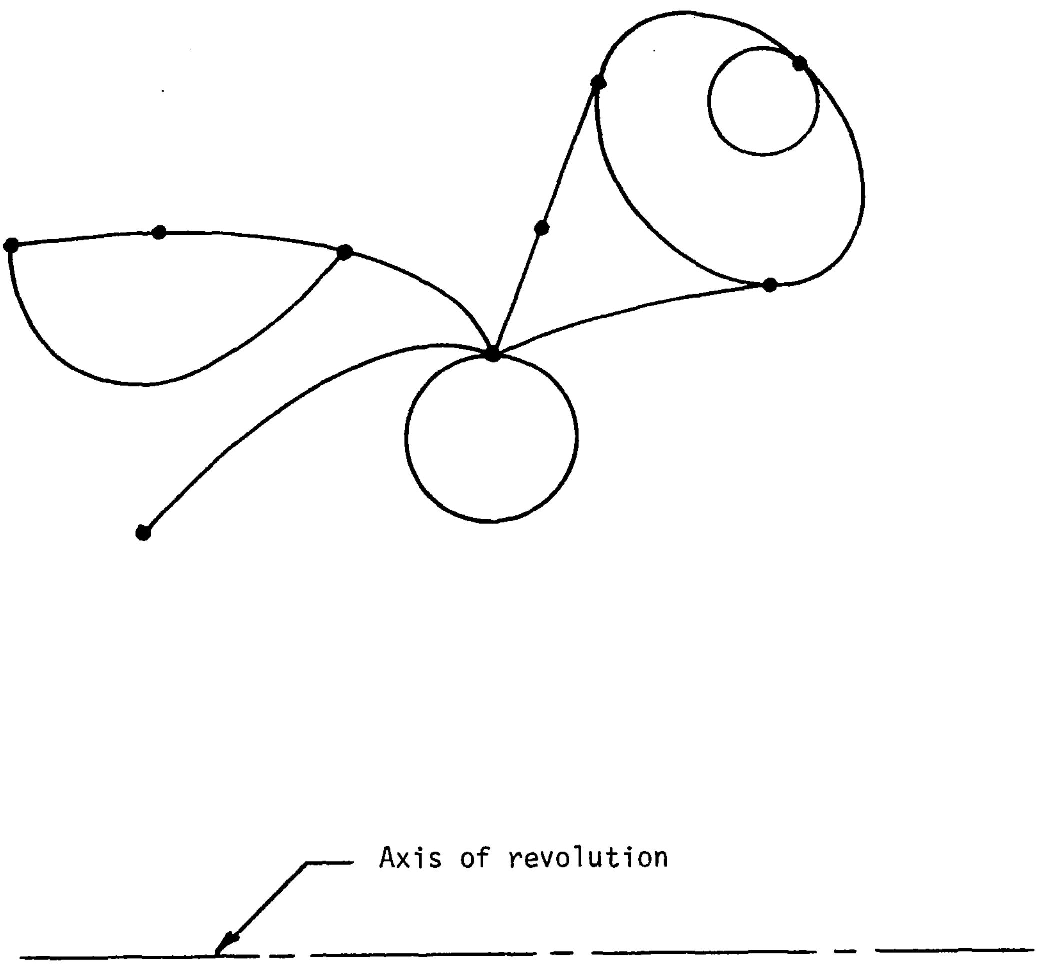
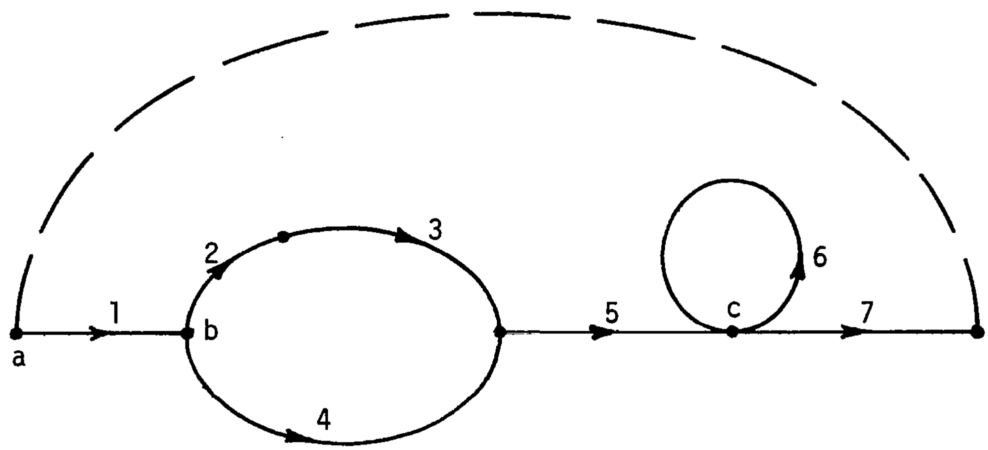


Figure 1.- Reference meridian of a hypothetical shell of revolution



Note: Point a may be the I.P. for arcs 1, 5, and 7.
Point b (arc 2) is the I.P. for arcs 2 and 3.
Point b (arc 4) is the I.P. for arc 4.
Point c is the I.P. for arc 6.

Figure 2.- Illustration of initial points and arc ordering and orientation

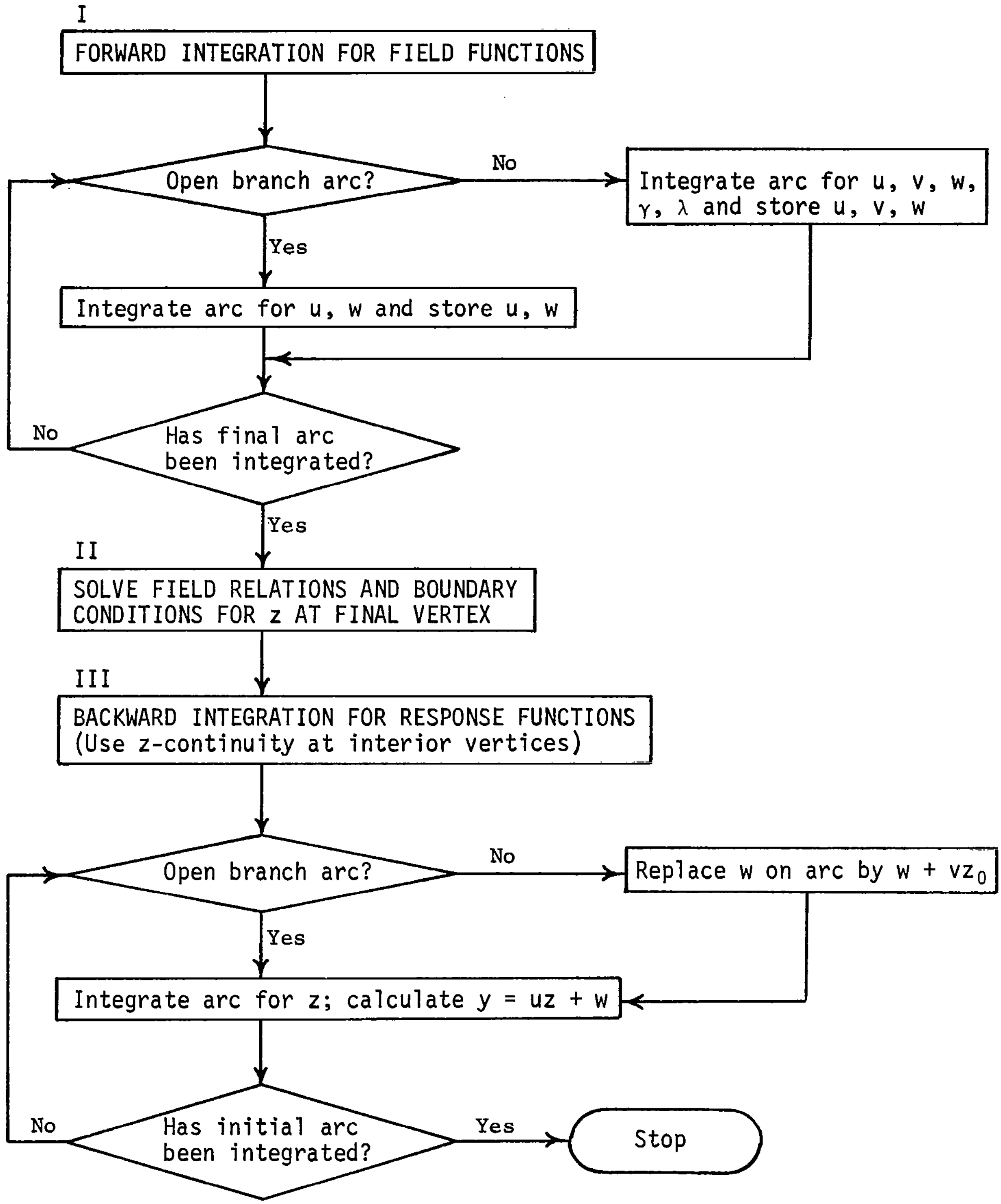


Figure 3.- Flow chart of field method

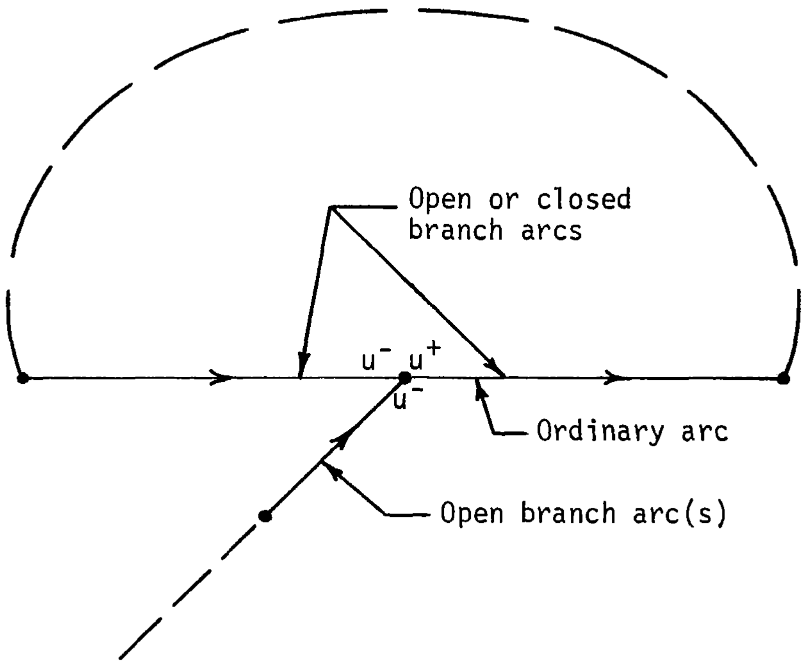


Figure 4.- Illustration of an ordinary arc

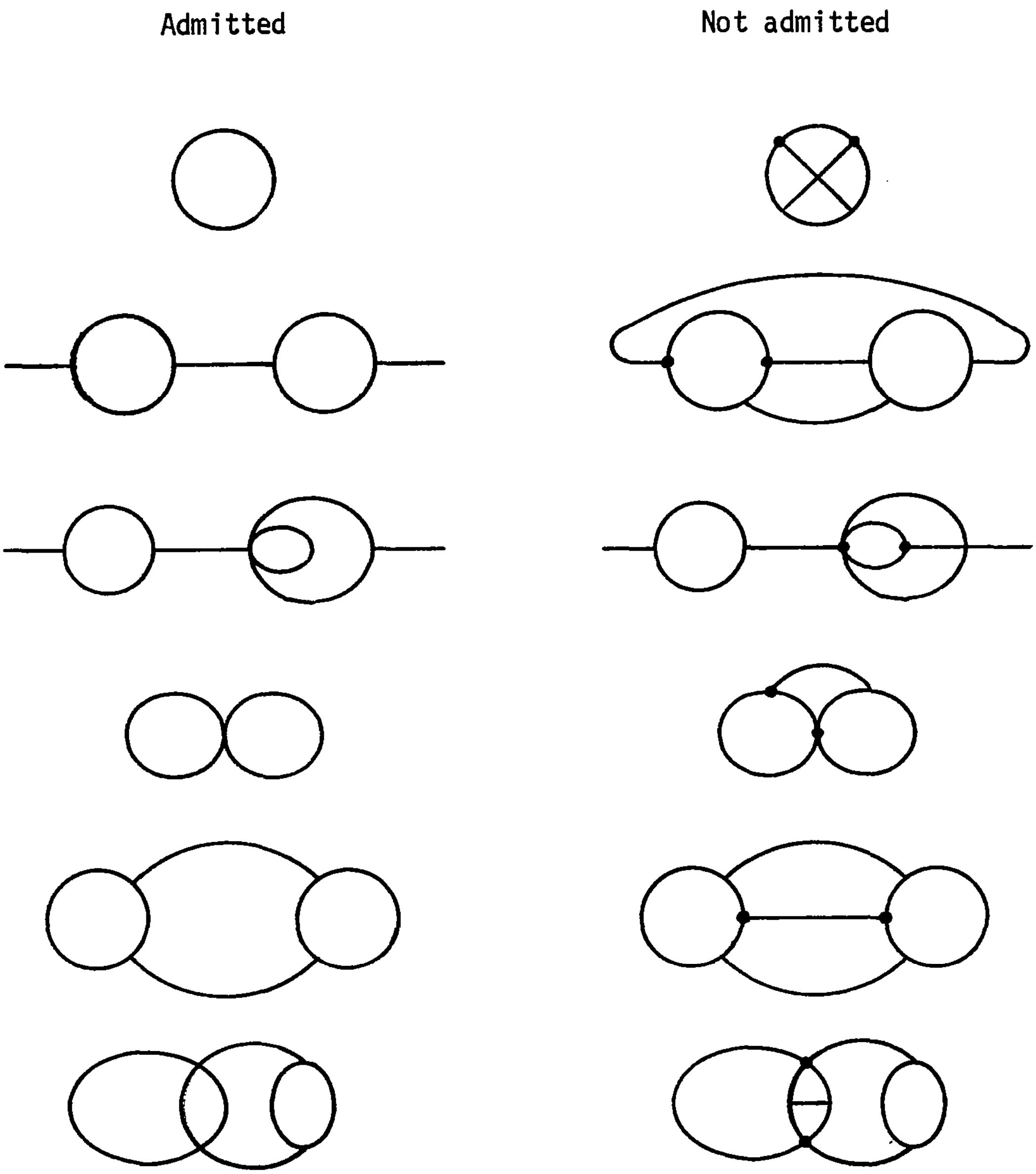
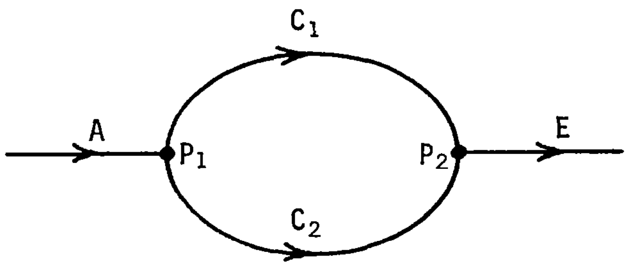
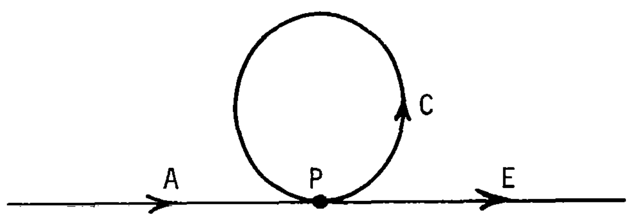


Figure 5.- Examples of graphs admitted by limitation of graph generality. The heavy data depict vertices joined by three distinct chains; one vertex lies in a circuit not containing the other.



(a) Split circuit



(b) Loop

Figure 6.- Prototype closed branch elements

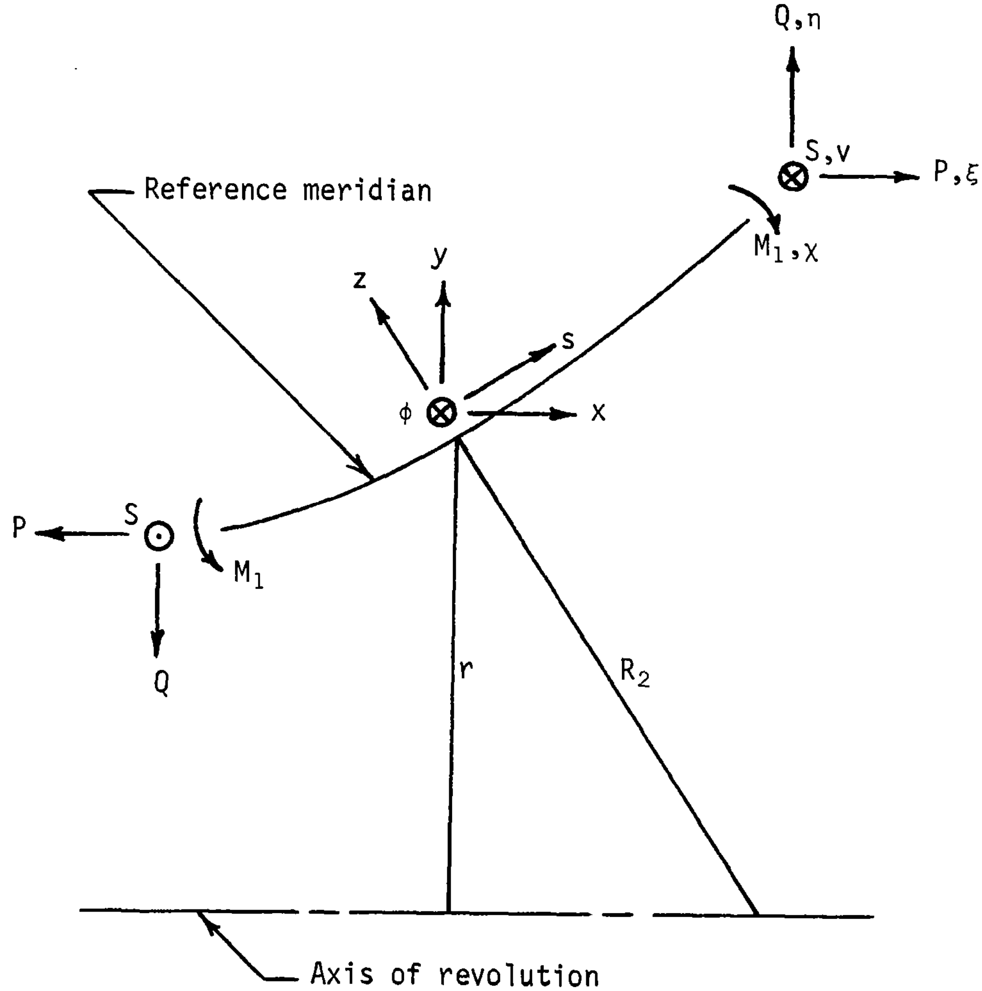
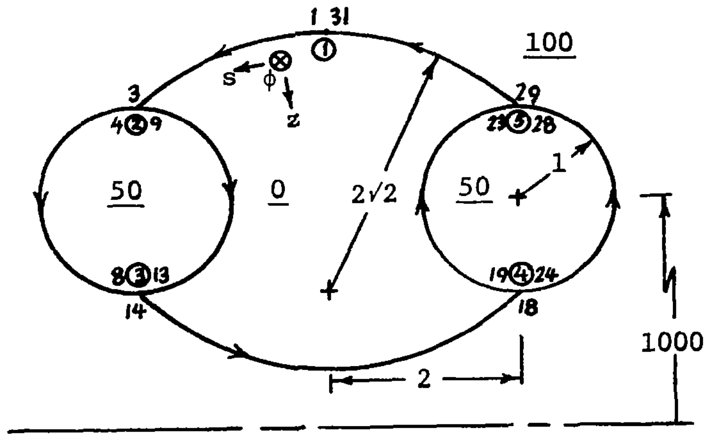
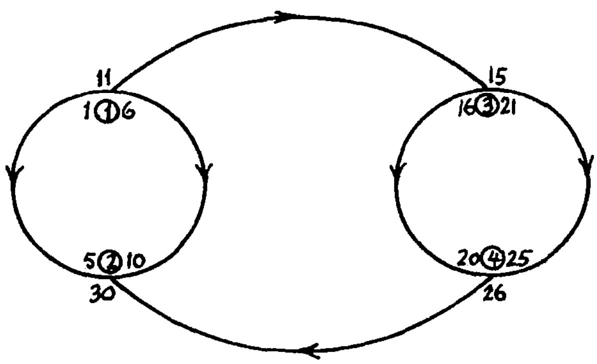


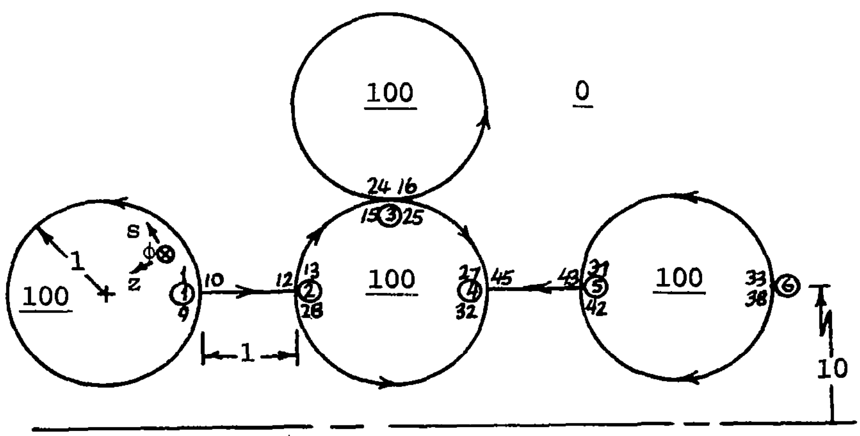
Figure 7.- Shell coordinates and response variables



(a) Split circuit in loop



(b) Split circuit in split circuit



(c) Loop in split circuit

Figure 8.- Example multicircuit meridians

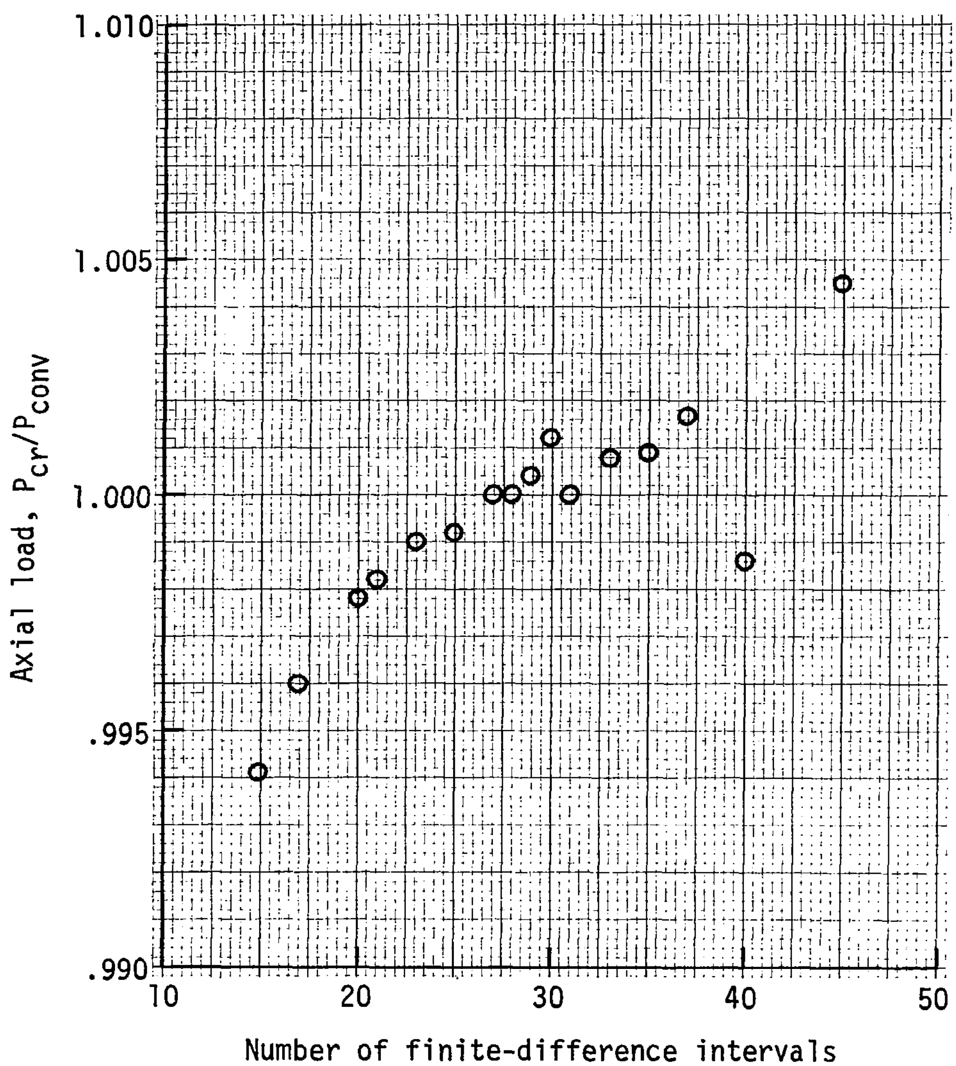
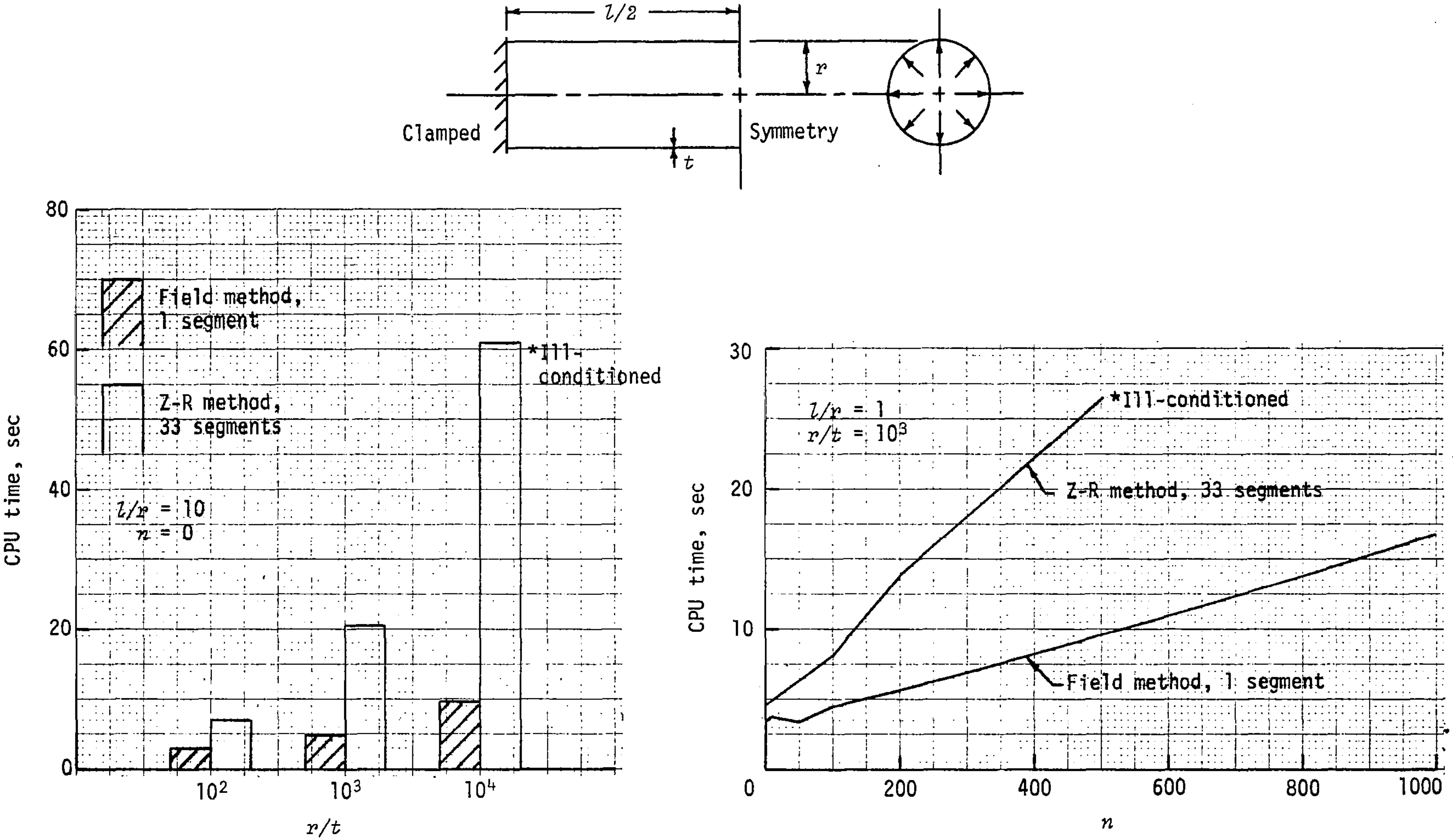


Figure 9.- Buckling of axially loaded cylinder by the finite-difference method
[from ref. 6]



(a) Effect of radius-to-thickness ratio

(b) Effect of circumferential wave number

Figure 10.- Computation times of field and Zarghamee-Robinson methods for pressurized cylindrical shells

CR-3535

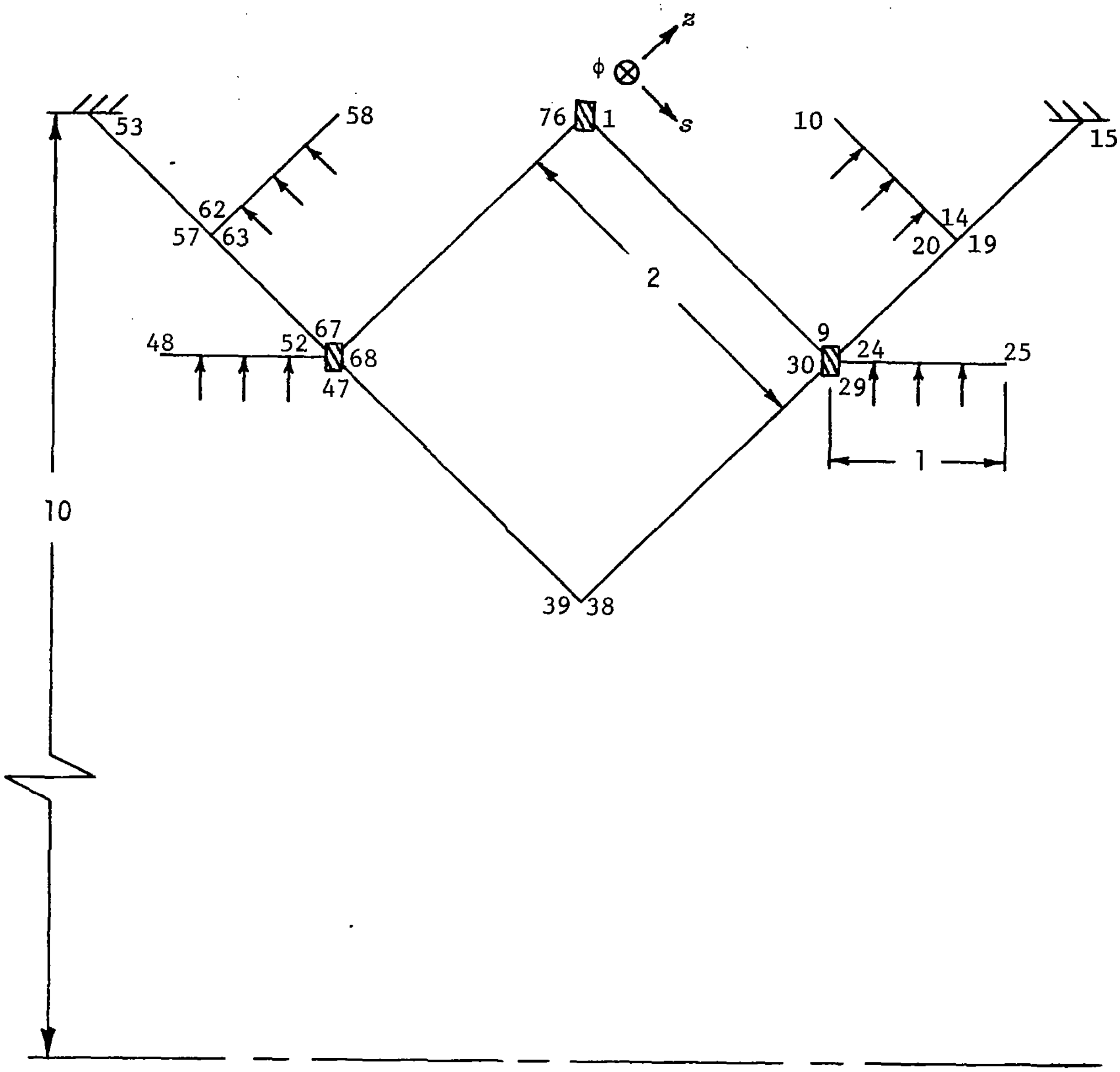


Figure 11.- Closed branch shell used for comparison of field and Zarghamee-Robinson solutions

NASA-Langley, 1975 CR-2535