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GRAVITY FIELDS OF THE SOLAR SYSTEM

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

FOREWORD

NASA experience has indicated a need for uniform design criteria for space vehicles. Accordingly, criteria have been developed in the following areas of technology:

Environment
Structures
Guidance and control
Chemical propulsion

Individual components are issued as separate monographs as soon as they are completed. A list of monographs published in this series can be found on the last pages.

These monographs are to be regarded as guides to design and not as NASA requirements, except as may be specified in formal project specifications. It is expected, however, that the monographs will be used to develop requirements for specific projects and be cited as the applicable documents in mission studies and in contracts for the design and development of space vehicle systems.

This monograph was prepared under the cognizance of the Goddard Space Flight Center (GSFC) with Scott A. Mills and John J. Sweeney of GSFC as program coordinators. The principal authors were Alan Zendell, Richard D. Brown, and Samir Vincent of Computer Sciences Corporation.

An Advisory Panel, first chaired by J. P. Murphy (then at GSFC and now at NASA Headquarters) and subsequently chaired by T. L. Felsentreger of GSFC, provided guidance to the authors on the content and scope of the monograph and reviewed it for technical validity. Panel members were J. G. Marsh and C. A. Wagner of GSFC. R. K. Squires of GSFC gave technical direction in the early stages of the effort.

Comments concerning the technical content of these monographs will be welcomed by the National Aeronautics and Space Administration, Goddard Space Flight Center, Systems Reliability Directorate, Greenbelt, Maryland 20771.

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GRAVITY FIELDS OF THE SOLAR SYSTEM

1. INTRODUCTION

Precise knowledge of the gravitational fields in the solar system is often an important ingredient in space mission planning and spacecraft design. The gravitational fields affect the trajectories and orbits of space vehicles and design problems involving propulsion and guidance systems, attitude determination and control systems, and scientific experiments. For example, Earth gravity models are used for calibration of launch vehicle guidance systems, design of scientific experiments for geodesy and general relativity, and planning for satellite tracking and telemetry operations.

This monograph briefly discusses the most frequently used formulations of the gravitational field and defines a standard set of models for the gravity fields of the Earth, Moon, Sun, and other massive bodies in the solar system. These models are intended primarily for use by engineers and computer programmers who are not specialists in gravity field-modeling theory. The formulas are presented in standard forms, when possible, with instructions for conversion to other forms in common usage. In section 2, various formulations of the gravitational field are developed. The subsections are arranged in order of the sophistication required in their development. The first model considered is the "point source" or "inverse square" model, which represents the external potential of a spherically symmetrical mass distribution by a mathematical point mass without physical dimensions. The most obvious departure from symmetry in a rotating body is the latitudinal variation commonly referred to in terms of an equatorial bulge or a polar flattening. Accordingly, an oblate spheroid model is presented next. This is accompanied by an introduction to zonal harmonics. The spheroid model is then generalized to a representation of the field resulting from a massive body in terms of a spherical harmonic expansion. The latter formulation is the basis for a number of the spherical harmonic models which have been developed for the Earth and Moon. These models and their application to NASA missions are discussed. In addition, the triaxial ellipsoid model is presented because of its suitability for modeling lunar gravity and use in lunar missions.

Section 3 provides guidance in selection of gravitational models for the Earth and Moon and gives values of the basic parameters used to describe the gravity fields of the other bodies in the solar system.

Appendix A defines the symbols used in this monograph.

2. STATE OF THE ART

The gravitational field in the vicinity of a celestial body may be described in several ways. In general, planets and other massive celestial bodies are not perfectly symmetrical in shape and their mass is not uniformly distributed. Consequently, in each case complete representation of the gravitational field would require an infinite set of orthogonal functions such as spherical harmonics. If the shape and mass distribution are reasonably regular, however, the field may be approximated by much simpler representations. A common simplifying technique is to assign the celestial body a regular geometrical shape that closely approximates the true shape and to assume a uniform density for the material of the body. This approach results in a gravitational field model whose equipotential surfaces are shaped approximately like the body's surface. The usual shapes assumed are a sphere, an ellipsoid of revolution, or a triaxial ellipsoid. A spherical body has spherically-symmetrical equipotential surfaces. The equipotential surfaces external to an ellipsoid of revolution possess axial symmetry but display a latitudinal (zonal) variation in radius similar to that of the body's surface. The equipotential surfaces external to a triaxial ellipsoid vary in radius with longitude and latitude in a manner similar to the meridional and equatorial ellipticities of the body's shape. With the assumption of uniform density, a body of arbitrary shape results in equipotential surfaces which can be represented by an infinite sequence of spherical harmonics.

The gravitational potentials that result from the foregoing models are presented in the following subsections.

2.1 Point-Mass Representation

The simplest way to express the external gravitational field of a celestial body is to treat it as a spherical, uniform body or one whose mass can be considered concentrated at a point. With this assumption, the field is described by a single parameter, the gravitational constant μ, which is the product of the universal gravitational constant G and the mass of the body M. For all the major bodies in the solar system except Earth, Moon, and Mars, the only parameters of the gravitational fields which may be considered well-known are the μ values and the ratios of the mass of the Sun to the masses of the bodies. The mathematical premises for μ and M are given in appendix B.

2.1.1 Application of Point-Mass Model

On the basis of Newton's universal law of gravitation, the point mass representation of a celestial body results in a potential field that is spherically symmetric about the point mass. The location of the point mass is the center of mass of the body, the only point in the body through which the application of any external force will result only in translational motion. The attractive force F that results from such a model is given by the inverse square law. The force on a unit mass located a distance R from the center of mass of the body is given by

$$F = \frac{\mu}{R^2} \tag{1}$$

This force may also be expressed as the magnitude of the gradient of a scalar gravitational potential U by

$$U = \frac{\mu}{R} \tag{2}$$

If the potential of a celestial body is needed at a distance R that is large compared to the body's diameter or if precise knowledge of the effect of small perturbing forces is not required, the potential may be adequately represented with the point-mass model by specifying the values of μ and R.

For a spacecraft traversing cislunar space, the gravitational potential fields of the Earth and Moon may be adequately described with a sum of point-mass models for the Earth and Moon by specifying the value of μ for each body and the distances R_e and R_m from the Earth and Moon, respectively. In interplanetary space, far from any massive body, the point-mass models for the major planets, the asteroids, and the Sun give a sufficient description of the potential. In the case of a planet with natural satellites, a good approximation of its gravitational field at large distances can be obtained by considering the planet and satellites as a single mass point located at the center of their combined mass with a mass equal to their combined mass. In such cases, the consideration of the natural satellites is particularly important in generating accurate planetary ephemerides. When ephemerides are generated over long time periods, even such small corrections are important.

For most of the planets and all the minor celestial bodies, the point-mass model is the only one that can presently be used with confidence. In some cases, notably Pluto, even the ratio of its mass to that of the Sun and the μ value are still quite uncertain. Knowledge of the detailed effects of the gravity fields of Mars and Venus on orbiting vehicles is becoming important, however, with the advent of orbiting and soft-landing missions such as Viking.

2.1.2 Gravitational Constant and Mass

2.1.2.1 Sun

Table 1 gives recent determinations of the gravitational constant of the Sun, μ_s. The values given by references 1 and 2 are extremely close to each other, and both are generally accepted in gravitational work. The Astronomical Ephemeris value is presented for historical perspective. It was obtained without benefit of data from cislunar and interplanetary spacecraft missions. Because the uncertainty in the Jet Propulsion Laboratory (JPL) value (ref. 2) is small, this value is adopted herein.

The value for the Sun's mass, M_s that corresponds to the foregoing adopted value for μ_s is obtained by

$$M_s = \frac{\mu_s}{G}$$

TABLE 1
GRAVITATIONAL CONSTANT FOR THE SUN

Source	μ_s (km^3/s^2)
Astronomical Ephemeris (ref. 3)	1.3246×10^{11}
Anderson et al (ref. 1)	1.327125×10^{11}
JPL Value (ref. 2)	$1.32712499 \times 10^{11} \pm 15 \times 10^3$

where G is the universal gravitational constant, equal to $6.668 \pm .005 \times 10^{-23}$ $km^3/g\ s^2$. The resulting value of 1.9903×10^{33} g is adopted for M_s.

2.1.2.2 Planets

The usual method of presenting the mass of a planet is in terms of the reciprocal mass ratio, $(M'_p)^{-1}$ where M'_p is the observable quantity M_p/M_s (appendix B). Tables 2 through 9 give the best estimates presently available for the reciprocal mass ratios for each of the planets except Earth. The form of the presentation has been adopted from reference 4. The U.S. Naval Observatory (USNO) values are weighted means of all the recent determinations done at the Observatory and elsewhere. The JPL values were adopted in 1968 and taken from reference 2. The Massachusetts Institute of Technology (MIT) values are based on a larger data set than any of the others and include optical and radar observations. Kovalevsky's approach was similar to that of USNO although a smaller set of determinations was used. Kovalevsky's values are presented in terms of a range of equally probable values. Anderson's results are from a recent analysis of Mariner 5 data.

The reciprocal mass ratios for the major planets are given in tables 2 through 9. The uncertainty values therein that were taken from references 2, 4, and 5 are computed from formal statistics. They show the dispersion of data but not necessarily the probable error in derivation of planetary masses.

On the other hand, the uncertainty values accompanying references 6 through 16 in tables 2 through 9 are estimates of the real error in the planetary mass determinations. Accordingly, the real error estimates serve as the basis for the weighted mean values given in the tables for the planetary reciprocal mass ratios.

TABLE 2
RECIPROCAL MASS RATIO OF MERCURY

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	6,021,900	
USNO (ref. 4)	5,987,000	32,000
JPL (ref. 2)	5,983,000	25,000
MIT (ref. 6)	6,020,000	10,000
Kovalevsky (ref. 7)	5,900,000	to 6,100,000
Ash (ref. 8)	6,025,000	15,000
Howard (ref. 9)	6,023,600	600

TABLE 3
RECIPROCAL MASS RATIO OF VENUS

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	408,522.7	
USNO (ref. 4)	408,519	11
JPL (ref. 2)	408,522	3
MIT (ref. 6)	408,522	3
Kovalevsky (ref. 7)	408,512	to 408,532
Mariners 2 and 4 (ref. 10)	408,521.8	1
Anderson (ref. 11)	408,523.5	1
Howard (ref. 12)	408,523.9	1.2

TABLE 4
RECIPROCAL MASS RATIO OF MARS

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	3,098,710	
USNO (ref. 4)	3,098,709	9
JPL (ref. 2)	3,098,700	100
MIT (ref. 6)	3,098,700	30
Kovalevsky (ref. 7)	3,098,650	to 3,098,750
Mariner 4 (ref. 13)	3,098,708	9
Mariner 9 (ref. 14)	3,098,720	70

TABLE 5
RECIPROCAL MASS RATIO OF JUPITER

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	1,047,373.6	
USNO (ref. 4)	1,047.366	0.007
JPL (ref. 2)	1,047.3908	0.0074
MIT (ref. 6)	1,047.4	0.1
Kovalevsky (ref. 7)	1,047.34	to 1,047.39
Jupiter Monograph (ref. 15)	1,047.39	0.04
Pioneer 10 (ref. 16)	1,047.342	0.02

TABLE 6
RECIPROCAL MASS RATIO OF SATURN

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	3,498.5	
USNO (ref. 4)	3,498.1	0.4
JPL (ref. 2)	3,499.2	0.4
MIT (ref. 6)	3,498.5	0.5
Kovalevsky (ref. 7)	3,497	to 3,500

TABLE 7
RECIPROCAL MASS RATIO OF URANUS

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	22,920	
USNO (ref. 4)	22,800	107
JPL (ref. 2)	22,930	6
MIT (ref. 6)	22,900	200
Kovalevsky (ref. 7)	22,600	to 23,000

TABLE 8
RECIPROCAL MASS RATIO OF NEPTUNE

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	19,323	
USNO (ref. 4)	19,325	26
JPL (ref. 2)	19,260	100
MIT (ref. 6)	19,400	100
Kovalevsky (ref. 7)	19,200	to 19,400

TABLE 9
RECIPROCAL MASS RATIO OF PLUTO

Source	$(M'_p)^{-1}$	Uncertainty
Weighted Mean Value	1,934,000	
USNO (ref. 5)	3,000,000	500,000
JPL (ref. 2)	1,812,000	40,000
MIT (ref. 6)	3,500,000	2,000,000
Kovalevsky (ref. 7)	1,500,000	to 2,500,000

The gravitational constant of a planet, μ_p may be computed from

$$\mu_p = GM_s M'_p = \mu_s/(M'_p)^{-1} \tag{3}$$

where $(M'_p)^{-1}$ is found in tables 2 through 9. The advantage of computing μ_p in this way is that it avoids the uncertainty in our knowledge of G.

2.1.2.3 Earth and Moon

The usual means of expressing the mass of the Earth is in terms of the inverse ratio of the combined mass of the Earth and Moon to the mass of the Sun:

$$(M'_{e-m})^{-1} = \left[\frac{M_e + M_m}{M_s}\right]^{-1} \tag{4}$$

Table 10 contains the best available estimates of $(M'_{e-m})^{-1}$. The comment in section 2.1.2.2 concerning the uncertainty ranges of reciprocal mass ratios of planets for different investigators applies also to values from these investigators in table 10.

Table 11 contains values of $\mu_e = GM_e$ that are computed by three different methods. With a spheroid model for the Earth (sec. 2.2), μ_e can be estimated in terms of the Earth's rotation rate and the dynamic flattening. A second method uses radar values for the mean distance of the Moon in a modified form of Kepler's equations. The third determination is from the tracking of lunar probes by JPL. Also included in table 11 are a value recently determined by Esposito and Wong from Mariner 9 data (ref. 17) and the values adopted by JPL in 1968 (ref. 2).

The masses of the Earth and Moon can then be separated using the ratio M_e/M_m, given in table 12.

Prior to the launching of spacecraft into cislunar and interplanetary space, the ratio of the mass of the Moon to that of the Earth was known to three significant figures, $M_e/M_m = 81.3$ (ref. 21). This value may still be used when only three significant figures are required. The Earth-Moon system parameters that can be estimated directly are the gravitational constant $\mu_m = GM_m$ and the mass ratio M_e/M_m. As in the case of the mass of the Sun (sec. 2.1.2.1), the absolute value of M_m is known only as accurately as G, and G is known only to about three significant figures (sec. 3.1).

Recent determinations of μ_m and M_e/M_m are listed in table 12. The values for Rangers 6, 7, 8, and 9 and Mariners 2 and 4 were computed from reference 21; Blackshear's value represents a best fit to his latest gravity model derived from Lunar Orbiter data; and the values derived from Mariners 5, 6, and 7 and Pioneers 8 and 9 were communicated by Null of JPL and represent JPL's adopted values as of 1969.

TABLE 10
RECIPROCAL MASS RATIOS FOR EARTH PLUS MOON

Source	$(M'_{e\text{-}m})^{-1}$	Uncertainty
Weighted Mean Value	328,900.21	
USNO (ref. 4)	328,900.12	0.20
JPL (ref. 2)	328,900.1	0.3
MIT (ref. 6)	328,900	1.0
Kovalevsky (ref. 7)	328,900 to	328,930

TABLE 11
THE EARTH'S GRAVITATIONAL CONSTANT

Source	M_e (km^3/s^2)
Dynamic Flattening (ref. 18)	398,603.2
Mean Distance of the Moon (ref. 19)	398,600.1
Lunar Probes (ref. 20)	398,600.9
Mariner 9 (ref. 17)	398,600.8
JPL Adopted Value (ref. 2)	398,601.2*

*Adopted for this monograph.

TABLE 12
GRAVITATIONAL CONSTANT AND MASS RATIO OF THE MOON

Source	μ_m (km^3/s^2)	M_e/M_m
Weighted Mean Value	4902.78	81.30090
Rangers 6, 7, 8, and 9 (ref. 21)	4902.65 ± 0.16	81.30245 ± .00246
Mariners 2 and 4 (ref. 21)	4902.735 ± 0.21	81.30175 ± .00315
Blackshear[1]	4902.867	81.2994
Mariners 5, 6, and 7 (ref. 22) Pioneers 8 and 9	4902.801 ± .022	81.30071 ± .00036
JPL Adopted Value (ref. 2)	4902.78 + 0.06	81.3010 + .001

[1] Personal Communication, August 28, 1970.

2.2 Spheroid Model

2.2.1 Flattening and Zonal Harmonics, J_2 and J_4

A rotating celestial body may be approximated with good accuracy by an ellipsoid (spheroid) of revolution. The gravitational field of such a body has axial symmetry but varies with latitude because of its ellipsoidal shape. In most formulations, this variation is expressed in terms of the second and fourth even-zonal harmonics of the spherical harmonic expansion having constant coefficients, J_2 and J_4. As shown in appendix C, these coefficients may in turn be expressed in terms of the rotation and the dynamical flattening f which is given by

$$f = \frac{(a_e - b)}{a_e}$$

where a_e = the equatorial radius
b = the polar radius

The departures from the spherically symmetric field that result from the J_2 and J_4 terms may be represented geometrically by relative highs and lows in the equipotential surfaces as compared to a sphere (fig. 1).

The geometrical flattening of the major planets has been estimated from optical observations of their shapes. From these estimates and the expressions relating the rotation, J_2, and J_4 to f, J_2 and J_4 may be estimated. For planets with natural satellites (particularly Mars, Jupiter, and Saturn), optical tracking of the satellites can be used to estimate the magnitude of the secular perturbations from which J_2 and J_4 may be computed.

The gravitational field that results from a spheroidal body is treated in references 23, 24, 25, and 26; references 23 and 27 give a brief mathematical description of the gravitational field of a spheroid body with a discussion of Legendre functions.

2.2.2 Application

The spheroid model is an important tool in formulating the gravitational field of a planet. The parameters that describe it can be derived directly from observations of orbiting satellites which show large secular changes in the right ascension of the ascending node Ω and in the argument of periapsis ω. Because of the large secular motions of close-Earth satellites, the spheroid model is the simplest model that can be recommended for orbit determination; however, more detailed and accurate models are usually used. The ellipsoidal equipotential surface of the spheroid model also serves as the common reference surface for many geodetic applications.

2.2.3 Major Planets

Data from tracking of natural satellites and spacecraft in the vicinity of the nearer planets have yielded estimates of only J_2 and J_4 for some of them. In some cases, the dynamical oblateness can be estimated only from the optically-observed geometrical flattening (sec. 2.2.1). Such estimates, however, require assumptions about the plasticity, viscosity, and modulus of elasticity of the planetary interiors that are often little more than educated guesses.

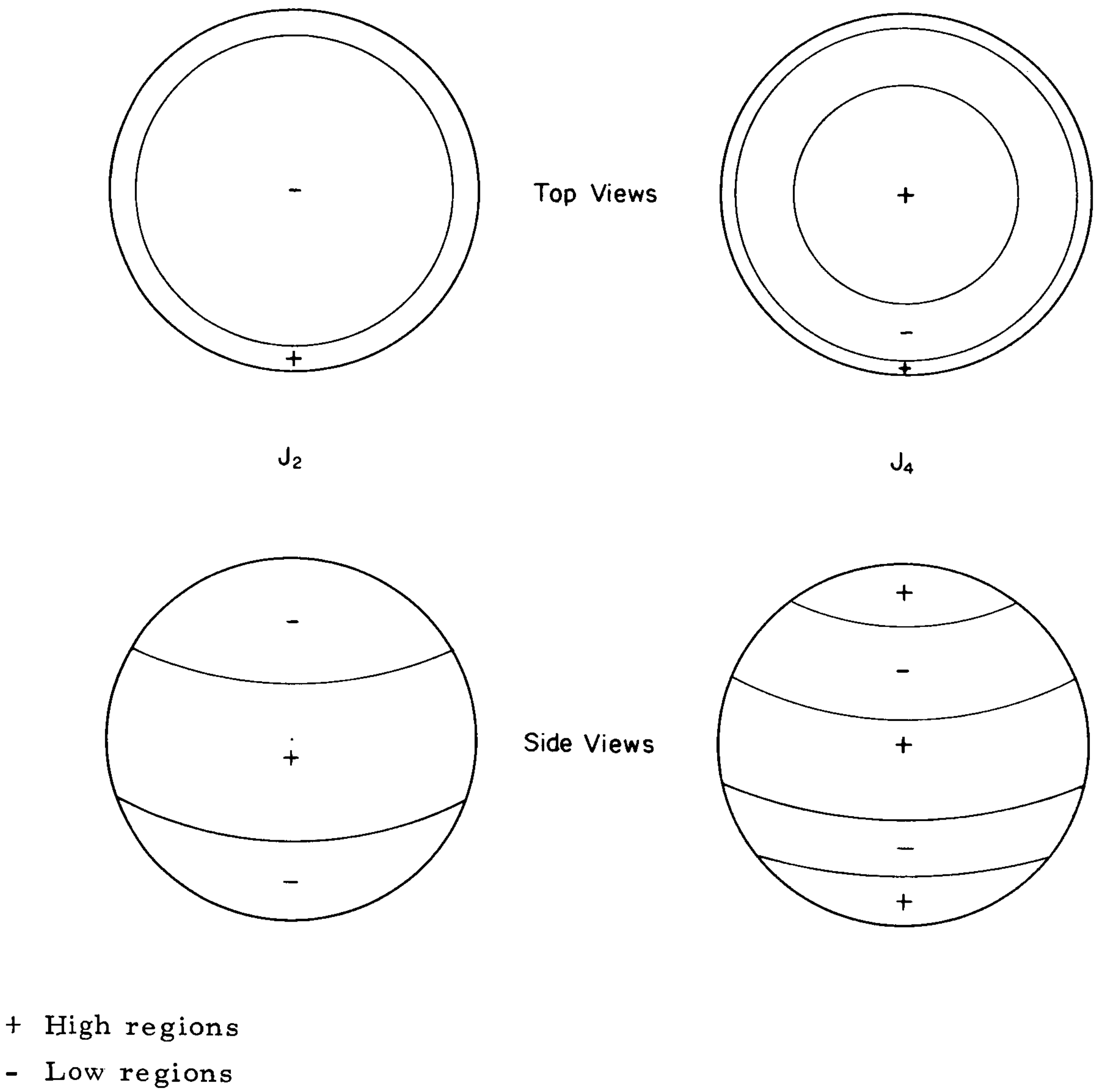

Figure 1.—Highs and Lows of a Spheroidal Equipotential Surface Relative to a Spherical Surface

2.2.3.1 Mercury and Venus

Mercury and Venus have no significant flattening (ref. 3). As Venus has no natural satellites, the only available source of data on Venus' dynamic oblateness is the Mariner 5 flyby from which Anderson (ref. 11) has estimated

$$J_2 \text{ (Venus)} = 2.7 \pm 0.9 \times 10^{-5}$$

No estimates have been made for J_4. No dynamical estimates of J_2 and J_4 have been made for Mercury.

2.2.3.2 Mars

J_2 for Mars has been estimated from the motion of its two natural satellites by Wilkins (ref. 28) and from Doppler tracking of Mariner 4 by Null (ref. 13). Because the actual quantity derived in both cases was not J_2 but the product $J_2 R_m^2$ where R_m is an assumed value for the equatorial radius of Mars, the computed value of J_2 is affected by the determination of R_m. The values for J_2 in table 13 (refs. 13 and 28) assume a value for R_m of 3394 km in contrast to the radius of 3402 ± 8 km for this monograph (sec. 3.2.2). Wilkins' result has been adjusted to reflect a value of 3394 km for R_m instead of his original value of 3409 km.

TABLE 13
J_2 FOR MARS

Source	J_2
Natural Satellites (ref. 28)	.001968 ± .000006
Mariner 4 (ref. 13)	.00187 ± .00007
Lorell & Shapiro (ref. 14)	.00196 ± .00001
Jordan & Lorell (ref. 29)	.001964 ± .000006

It was not feasible to estimate J_2 from the motion of Mariners 6 and 7 because of nongravitational perturbations such as gas venting that acted on the spacecraft. J_4 has not been estimated for Mars. The flattening for Mars has been estimated geometrically to be $(192)^{-1}$ (ref. 3); this compares with Wilkins' estimate of $(190.4 \pm 1.9)^{-1}$ and estimate of $(190.8 \pm .7)^{-1}$ by Lorell and Shapiro for the dynamical flattening (ref. 28).

2.2.3.3 Jupiter

A value for the geometrical and dynamical flattening of Jupiter are computed in reference 13 as $(16.35)^{-1}$. Anderson et al. (ref. 16) calculate dynamical flattening as (15.456 ± 0.24)· from Pioneer 10 results and give best available determinations of J_2 and J_4:

$$J_2 \text{ (Jupiter)} = (147.2 \pm 0.4) \times 10^{-4}$$

$$J_4 \text{ (Jupiter)} = -(6.5 \pm 3.8) \times 10^{-4}$$

2.2.3.4 Saturn

In reference 30, the geometrical and dynamical flattenings of Saturn are computed to be $(9.5 \pm 0.7)^{-1}$ and $(10.3 \pm 0.5)^{-1}$, respectively. The values of J_2 and J_4 computed in reference 26 are

$$J_2 \text{ (Saturn)} = (166.5 \pm 2.0) \times 10^{-4}$$

$$J_4 \text{ (Saturn)} = (9.6 \pm 1.0) \times 10^{-4}$$

Reference 31 notes that these values contain a contribution of undetermined magnitude from Saturn's rings.

2.2.3.5 Uranus, Neptune, and Pluto

Reference 32 contains estimates for the geometrical flattening of Uranus and Neptune of $(33 \pm 33)^{-1}$ and $(50 \pm 50)^{-1}$, respectively. This notation indicates that the correct values may lie anywhere in the range of expected uncertainties with virtually equal probability. Reference 33 gives estimates of the geometrical flattening of $(18)^{-1}$ for Uranus and $(60)^{-1}$ for Neptune, and a value of J_2 for Neptune of approximately 0.005.

No estimates of Pluto's spheroid parameters have been made.

2.2.4 Sun

Determination of the Sun's zonal harmonics presents a different problem from that encountered with the planets. Because of the large mass of the Sun, the effects of general relativity must be accounted for in determining J_2 and J_4 from such observed quantities as the advance of the perihelion of Mercury. The observed rate of advance agrees well with the prediction of general relativity but leaves the contribution from the Sun's oblateness quite uncertain. Anderson (ref. 1) has estimated that J_2 for the Sun has an upper limit of 10^{-5} with an uncertainty of 70 percent. Dicke (ref. 34) has postulated a certain amount of geometrical flattening of the Sun in an attempt to support his own relativistic theory that would require that the Sun's oblateness contribute about 10 percent of the motion of Mercury's perihelion. His value for f was $5.0 \pm 0.7 \times 10^{-5}$. A recent determination by Oesterwinter (ref. 35) did not estimate J_2 precisely for the Sun but gave it an upper limit of 1.4×10^{-5}.

2.3 Spherical Harmonic Model

The representation of a planet's surface topography in spherical coordinates is appropriate because most planets are nearly spherical. The formulation of the planet's gravitational potential in terms of spherical harmonic expansion follows naturally because the gravitational field is strongly related to the planet's shape. In recent years spherical harmonic models have been constructed for the gravity fields of the Earth and Moon; and as data becomes available, attempts will be made to construct models for other planets. Appendix D gives the theoretical formulation of the gravitational potential in terms of spherical harmonics and discusses the physical significance of their mutual orthogonality.

2.3.1 Orthogonal Functions

All irregularities in shape and mass distribution contribute to the total gravitational potential of a body. To model such irregularities precisely, a formulation is required which uses an infinite sequence of orthogonal functions. The sequence of orthogonal spherical harmonics is appropriate for representation of a function over a spherical surface which is analogous to the use of Fourier series for functions in a rectilinear space.

2.3.2 Representation of the Gravitational Potential

2.3.2.1 Zonal, Sectorial, and Tesseral Harmonics

Spherical harmonics may be visualized as small, periodic adjustments to a perfectly spherical shape that result in a surface which oscillates above and below the spherical surface at regular intervals; positive values represent local "highs" relative to the reference sphere and negative values, "lows." The categorization of the harmonics is based on the geometrical pattern of the highs and lows. Zonal harmonics possess longitudinal symmetry and oscillate in sign only with latitude. Sectorial harmonics possess latitudinal symmetry and oscillate in sign only with longitude. Tesseral harmonics comprise all the other harmonics of varying degree and order and change in sign over a latitude/longitude grid of "tesserae." An analytical description of the foregoing concepts is given in appendix E.

2.3.2.2 Resonant Orbits

Certain Earth satellite orbits, described as "resonant," experience large perturbing effects from particular harmonic coefficients (ref. 26). A resonant orbit has a mean motion commensurate with Earth's rotation, i.e., the ratio 24/P is a rational number where P is approximately the satellite period in hours. The most pronounced resonances occur when 24/P is an integer. During each commensurate period, the geographic trace of the satellite repeats, so any anomaly in the geopotential, however small, has an opportunity to build up small perturbations into large displacements as long as the commensurability is maintained. Such orbits have been recognized since 1960 (ref. 36) as the reasonable ones to observe for the determination of certain of the longitude-dependent terms of the geopotential. Appendix E gives a more detailed discussion and a list of harmonic coefficients so determined.

2.3.2.3 Normalization and Conversion

The coefficients C_{nm} and S_{nm} which appear in the expression for the geopotential in spherical harmonics, often are normalized or combined with other coefficients to yield coefficients that have clear physical interpretations. Appendix F gives the generally adopted normalization procedure and relates the various forms of geopotential expression to the form adopted herein (equation E-2).

2.3.3 Current Models

2.3.3.1 Earth

Many spherical harmonic models have been developed for Earth gravity since the first artificial Earth satellite flew in 1959. Each successive satellite launch and each advance in tracking system accuracy have led to more complete and accurate models representing a wider variety of orbits. Notable models that have led to present state of the art models are

- The 4 by 4 and 6 by 6 models of Izsak derived from optical data (refs. 37 and 38)
- The 4 by 4 and 6 by 6 Doppler data models of Guier (ref. 39) and Anderle (ref. 40)
- The 8 by 8 model of Kaula (ref. 41) based on both optical and surface gravity data
- The 8 by 8 Doppler data model of Guier and Newton (ref. 42)
- The SAO-M1 model based on optical and surface gravity data described by Gaposchkin (ref. 43) and Lundquist (ref. 44)
- The SAO-69 model based on optical and surface gravity data (ref. 45)
- The GEM-6 gravity model (ref. 46)

The Goddard Earth Model-6 (GEM-6) (ref. 46) reflects a denser distribution of tracking stations, more different types of tracking data, and a wider range of orbit inclinations than any other such model and includes observations of 26 satellites and considerable amounts of surface gravity data. The GEM-6 model consists of a complete set of tesseral and sectorial harmonics through degree and order 16, zonal harmonics through degree 22, and other selected resonant terms (sec. 2.3.2.2) of orders 9, 11, 12, 13, and 14. In deriving the coefficients of the GEM-6 model, both optical and laser tracking data were used. GEM-6 is recommended for the general applications described in section 3.3.2.1.

2.3.3.2 Moon

The spherical harmonic expansions of the gravitational potential of the Moon are incomplete and less reliable than those for Earth because of a lack of tracking data from a broad distribution of orbit inclinations and lack of information from behind the Moon. However, interesting results have come from studies of Apollo flights and Explorer 49 (refs. 47 through 50). One spherical harmonic expansion, the L1 model of NASA Langley Research Center, was used for the Apollo 11 through 17 missions (ref. 51). Although it is difficult to measure quantitatively the accuracy of global models derived from the Lunar

Orbiter data, it may be concluded that they model the data from which they were derived very well. Therefore, the models could be considered in analysis of satelite orbits similar to those flown by the Lunar Orbiters. Data from the Apollo flights have generally not been used for spherical harmonic models because Apollo tracking data are corrupted by effects such as gas venting.

No spherical harmonic expansion models are currently recommended for all Moon missions. The triaxial ellipsoid model (sec. 2.4) is better suited for a wide variety of lunar applications except for low altitude lunar orbiters.

2.3.4 Model Truncation

This section provides guidance for the application of a spherical harmonic model to particular requirements. The spherical harmonic models that have been developed for the Earth's gravitational potential contain a finite number of terms. Such models are usually most effective for orbit prediction when applied to orbits similar to those used in deriving the models.

Because of computational limitations, users of gravity models often require models containing only a small number of terms. Two options may be possible; a model of the desired size may already be available, but more often the user will have to reduce a larger model. In the latter case, the problem of which terms to retain still must be met.

If a user requires a smaller model than the GEM-6 model, the following guidelines may assist in selecting which terms to retain:

- All terms which are at or near resonance for the application orbits

- Only low-degree terms at higher altitudes because of the $1/r^n$ dependence in the potential where n equals the degree of the term

More detailed quantitative rules for determining which terms are to be retained or how much accuracy is lost in truncation are given in appendix G.

2.3.5 Model Accuracy and Gravimetry

Model accuracy in the context of space applications refers to the ability of a model to predict the position of a satellite in a known orbit. A prediction accuracy of ± 30 meters is widely quoted as being generally attainable.

Such accuracy designations, however, are misleading and often lead to misconceptions. The accuracy with which the motion of a satellite can be predicted depends on the length of time over which the prediction is made, parameters of the initial orbit, the method of computation, the way in which the results are expressed, and the manner in which the model is applied. It is, therefore, impossible to assign a single number to represent the accuracy of a model.

For the GEM-6 gravity model, reference 45 provides a satisfactory discussion of model accuracy.

Surface gravimetry has recently been investigated as a possible standard for evaluating the accuracy of satellite models. It has been found that over small areas, gravimetric data can yield far more accurate gravity values than can be deduced from satellite data. Thus a well-measured portion of the geoid can serve as a reference standard for satellite models. The severe limitation on surface gravimetry is that neither the technological nor the economic means have been available to obtain data over much of the ocean surface.

The use of satellite data involves the reverse situation. Although satellite data represent the integrated effects of gravity from every point on the Earth's surface, these measurements lack fine resolution. A model complete to degree 30 has a resolution element of 180/30 or 6 degrees of arc relative to the Earth's center. When applied to surface gravity, such a model can yield average values of gravity over grid elements no smaller than 360 nautical miles (6 degrees of arc) on a side. Where surface gravity data are available, however, it is common to attain accurate average gravity over surface regions extending 1 degree of arc (60 nautical miles) or smaller.

Thus, one of the major problems encountered in using surface gravity is the selection of an appropriate grid size over which comparisons with satellite gravity can be made. A surface gravity grid contains too many details to be approximated by satellite gravity. Moreover, because of the limited coverage of gravimetry data, it cannot be extrapolated realistically over the entire satellite resolution element. Kaula (ref. 52) selected a 300-nautical mile (5 degrees) grid size over which to compare surface measurements with data from some early satellite models.

2.4 The Triaxial Ellipsoid Model (Moon)

2.4.1 Principal Moments of Inertia

The triaxial ellipsoid is a refinement of the spheroid of revolution model (sec. 2.2). The spheroid of revolution yielded a gravitational field which possessed longitudinal symmetry but reflected a latitudinal ellipticity. In physical terms, the principal moments of inertia of the spheroid of revolution, A, B, and C, defined with C measured about the rotation axis, satisfy $A = B \neq C$. The spheroid model potential, as formulated in the general case from MacCullagh's formula (ref. 20), is

$$U = \frac{\mu}{r} + G\frac{A + B + C - 3I}{2r^3} + O\left(\frac{1}{r^4}\right)$$

where r = the distance from the center of mass to an external point P

I = the moment of inertia about the line joining the center of mass and P

If the spheroid potential (sec. 2.2) is compared to this expression for U, J_2 can be written in terms of the principal moments of inertia by

$$J_2 = \frac{C - A}{M\bar{a}_e^2}$$

where $\bar{a}_e$ = the equatorial radius.

For the triaxial ellipsoid where $A \neq B$, the corresponding expression for J_2 is given by

$$J_2 = \frac{C - \frac{1}{2}(A + B)}{M\bar{a}_e^2}$$

Where $\bar{a}_e$ = the mean equatorial radius.

In this model there is an equatorial as well as a polar flattening. As the polar flattening is given principally by the spherical harmonic coefficient C_{20} (equals $-J_2$), the equatorial flattening is represented by the coefficient C_{22}, which is related to the moments of inertia (refs. 26 and 53) by

$$C_{22} = \frac{1}{4}\left(\frac{B - A}{M\bar{a}_e^2}\right)$$

Derivations pertaining to the preceding discussion of the potential from a triaxial ellipsoid may be found in references 24 and 54. The gravity potential of a triaxial ellipsoid may also be expressed in terms of the spherical harmonic expansion by using only three terms, the central force term (μ), C_{20}, and C_{22}.

2.4.2 Application

The triaxial model has been applied to the Earth, but it has not yielded significant improvements over the spheroid model. The main difference between these two models is the C_{22} term. Many investigators, including Wagner (ref. 55), have studied the ellipticity of the Earth's equator and found that C_{22} is nearly three orders of magnitude smaller than C_{20} and is therefore negligible for applications requiring less detail than the spherical harmonic expansion models presented in section 2.3.3.1. For the Earth, this result can be qualitatively expressed in terms of principal moments of inertia by

$$|C - A| \approx |C - B| \gg |A - B|$$

The triaxial ellipsoid is more useful for the Moon. From the moments of inertia ratios, given in reference 56, it can be inferred that $C > B > A$ for the Moon and further that

$$\frac{C - A}{B - A} = 2.79 \pm .04$$

and

$$\frac{C - B}{B - A} = 1.79 \pm .04$$

Both of the foregoing ratios are about 500 for the Earth, i.e., the equatorial flattening is negligible compared to the polar flattening. Because of the more significant equatorial flattening on the Moon, the triaxial ellipsoid is an appropriate model and often has been used to describe the lunar gravitational field. Recent determinations of C_{20} and C_{22} for the Moon are given in table 14.

TABLE 14
TRIAXIAL COEFFICIENTS FOR THE MOON

Source	C_{20}	C_{22}
Melbourne et al. (ref. 2)	-2.0711×10^{-4}	0.20716×10^{-4}
JSC[1]	-2.07108×10^{-4}	0.20716×10^{-4}
Bender et al. (ref. 56)	-2.04×10^{-4}	0.223×10^{-4}

[1] W. Wollenhaupt, JSC, Personal Communications, August 1970.

3. CRITERIA

The descriptive parameters given in this section should be used to establish reference gravity fields for space mission planning and the design of space vehicles, experiments, and instrumentation.

3.1 Sun and Astronomical Unit

The value of the Sun's gravitational constant μ_s recommended for use is

$$\mu_s = 1.327125 \times 10^{11} \text{ km}^3/\text{s}^2$$

The corresponding value of the Sun's mass M_s is

$$M_s = 1.9903 \times 10^{33} \text{ g}$$

which results from

$$M_s = \frac{\mu_s}{G}$$

where G is the universal gravitational constant, taken as $6.668 \pm .005 \times 10^{-23}$ $km^3/g\,s^2$.

The parameter μ_s is the only well-known gravitational parameter for the Sun. Of all the other parameters, only J_2 has been estimated, and then only an upper limit of 10^{-5} is recommended. The same limit may be applied to the geometrical flattening f.

The value recommended for the Astronomical Unit (AU) is 149,597,893 km. The International Astronomical Union (IAU) value of 149,500,000 km is important for interpretation of historical astronomical observations.

3.2 Planets

3.2.1 Masses and Gravitational Constants

The recommended ratios of the mass of the Sun to those of the planets, $(M'_p)^{-1}$, and the gravitational constants of the major planets μ_p are given in table 15.

3.2.2 Planetary Radii and Mean Distances From Sun

Table 16 gives the radii of the major planets and the mean distances from the Sun to each planet.

3.2.3 Harmonic Coefficients

At present, the only harmonic coefficients that have been reliably estimated for the major planets are the spheroidal harmonics J_2 and J_4 which are used in the model presented in section 2.2.2. In some cases, all that is available is an estimate of the observed geometrical flattening. Table 17 gives the recommended values for the parameters that have been estimated.

TABLE 15
RECIPROCAL MASS RATIOS AND GRAVITATIONAL CONSTANTS FOR THE PLANETS

Planet	μ_p (km^3/s^2)	$(M'_p)^{-1}$
Mercury	22,032*	6,021,900
Venus	324,860	408,522.7
Mars	42,828	3,098,707
Jupiter	126,709,801	1,047.3736
Saturn	37,934,115	3,498.5
Uranus	5,790,249	22,920
Neptune	6,868,111	19,323
Pluto	68,621	1,934,000

* Reference 9.

TABLE 16
RADII AND MEAN DISTANCES FROM THE SUN OF THE MAJOR PLANETS

Planet	Equatorial Radius (km)	Mean Distance From Sun[1]	
		(AU)	(km)
Mercury (ref. 9)	2,439 ± 1	.387099	57,909,195
Venus (ref. 57)	6,052 ± 6	.723332	108,208,943
Earth (refs. 2 and 58)	6,378.16 ± .005	1.0	149,597,893
Mars (ref. 57)	3,402 ± 8	1.523691	227,940,963
Jupiter (ref. 27)	71,422 ± 200	5.202803	778,328,366
Saturn (ref. 30)	59,800 ± 350	9.538843	1,426,990,814
Uranus (ref. 32)	27,000 ± 1,000	19.181951	2,869,579,453
Neptune (ref. 32)	25,200 ± 200	30.057779	4,496,580,407
Pluto (ref. 57)	2,250 ± 1,150	39.43871	5,899,947,919

[1] Reference 3.

TABLE 17
SPHEROID MODEL COEFFICIENTS FOR THE PLANETS

Planet	Geometrical Flattening, f	J_2	J_4
Mercury	0.0		
Venus	0.0	$\sim 8.4 \times 10^{-6}$	
Mars	$(192)^{-1}$	1.9×10^{-3}	
Jupiter	$(15.456)^{-1}$	147.2×10^{-4}	-6.5×10^{-4}
Saturn	$(9.5)^{-1}$	0.017	9.6×10^{-4}
Uranus	$(25)^{-1}$		
Neptune	(50)	~ 0.005	

3.3 EARTH

3.3.1 Mass and Gravitational Constant

In planning for deep space trajectories (far from the Earth and Moon), the Earth and Moon should be treated as a single body whose mass $M_{e\text{-}m}$ is given by

$$M_{e\text{-}m} = \frac{M_s}{328{,}900}$$

where M_s is the mass of the Sun. The gravitational constant of the Earth-Moon $\mu_{e\text{-}m}$ is 403,504 km^3/s^2.

In or near cislunar space the Earth and Moon should be treated as separate bodies. The mass and gravitational constant of the Earth are given by

$$M_e = \frac{M_s}{332{,}945.4}$$

$$\mu_e = 398{,}601.2 \text{ km}^3/\text{s}^2$$

3.3.2 Spherical Harmonic Models

The gravitational field of a rotating celestial body is generally that of a sphere modified by effects of rotation and irregularities in mass density and topography. Such a field can be described effectively by a spherical harmonic expansion model.

3.3.2.1 General Application

The GEM-6 model of the Earth's gravitational potential should be used when there is a requirement involving many different satellite orbits or an undefined or unspecified satellite orbit.

Table 18 gives the values of the coefficients of the GEM-6 model. Because of its size, the GEM-6 model may be unwieldy or inefficient for certain applications. In such cases, its size may be reduced in accordance with the guidelines presented in section 2.3.4.

3.3.2.2 Orbits at Resonant Altitudes

When a satellite is to be flown at or near one of the resonant altitudes (sec. E.4, appendix E), the mission planner should select the associated resonant coefficients from an appropriate model and use them in the GEM-6 model. For example, if the period of an orbit indicates resonance of order m and there is an accurate independent determination of these

TABLE 18
NORMALIZED COEFFICIENTS FOR THE GEM-6 MODEL (x 10^6)

ZONALS

INDEX N	M	VALUE	INDEX N	M	VALUE	INDEX N	M	VALUE	INDEX N	M	VALUE	INDEX N	M	VALUE
2	0	−484.1661	3	0	0.9607	4	0	0.5362	5	0	0.0661	6	0	−0.1451
7	0	0.0961	8	0	0.0426	9	0	0.0264	10	0	0.0606	11	0	−0.0528
12	0	0.0306	13	0	0.0470	14	0	−0.0206	15	0	−0.0045	16	0	−0.0077
17	0	0.0192	18	0	0.0091	19	0	0.0044	20	0	0.0143	21	0	−0.0098
22	0	−0.0138												

SECTORIALS AND TESSERALS

INDEX N	M	VALUE $\bar{C}$	$\bar{S}$	INDEX N	M	VALUE $\bar{C}$	$\bar{S}$	INDEX N	M	VALUE $\bar{C}$	$\bar{S}$
2	1	−0.0009	−0.0032	2	2	2.4251	−1.3883	3	1	2.0021	0.2482
3	2	0.9332	−0.6311	3	3	0.6969	1.4260	4	1	−0.5403	−0.4648
4	2	0.3461	0.6695	4	3	0.9655	−0.2073	4	4	−0.1636	0.3051
5	1	−0.0684	−0.0842	5	2	0.6651	−0.3112	5	3	−0.4656	−0.1947
5	4	−0.2485	0.0360	5	5	0.1845	−0.7119	6	1	−0.0734	−0.0150
6	2	0.0643	−0.3740	6	3	0.0115	0.0098	6	4	−0.0867	−0.4655
6	5	−0.2747	−0.5464	6	6	0.0173	−0.2627	7	1	0.2501	0.1385
7	2	0.3463	0.0876	7	3	0.1988	−0.1844	7	4	−0.2807	−0.1408
7	5	0.0265	0.0228	7	6	−0.3074	0.1213	7	7	0.0624	0.0048
8	1	0.0102	0.0579	8	2	0.0610	0.0860	8	3	−0.0378	−0.0667
8	4	−0.2311	0.0284	8	5	−0.0570	0.0622	8	6	−0.0947	0.2528
8	7	0.0658	0.0819	8	8	−0.0832	0.0701	9	1	0.1426	0.0137
9	2	0.0552	−0.0216	9	3	−0.1299	−0.0727	9	4	−0.0125	−0.0147
9	5	−0.0036	−0.0686	9	6	0.0163	0.1267	9	7	−0.0566	−0.0037
9	8	0.2519	−0.0101	9	9	−0.0275	0.0873	10	1	0.0927	−0.1343
10	2	−0.0419	−0.0703	10	3	−0.0383	−0.0998	10	4	−0.0699	−0.1247
10	5	−0.0525	−0.0377	10	6	−0.0550	−0.1342	10	7	0.0174	−0.0237
10	8	0.0473	−0.1213	10	9	0.0960	−0.0749	10	10	0.1365	−0.0391
11	1	−0.0087	0.0431	11	2	−0.0123	−0.1176	11	3	−0.0334	−0.0841
11	4	−0.0263	−0.1033	11	5	0.0842	0.0406	11	6	−0.0421	−0.0353
11	7	0.0041	−0.1123	11	8	−0.0204	0.0714	11	9	−0.0349	0.0393
11	10	−0.0464	−0.0333	11	11	0.0696	−0.0356	12	1	−0.0717	−0.0477
12	2	−0.0530	0.0621	12	3	0.0094	0.0371	12	4	−0.0488	−0.0158
12	5	0.0624	0.0230	12	6	0.0531	0.0280	12	7	−0.0261	0.0127
12	8	−0.0187	−0.0031	12	9	−0.0002	0.0251	12	10	0.0231	−0.0012
12	11	0.0066	0.0359	12	12	−0.0123	−0.0103	13	1	−0.0157	−0.0216
13	2	−0.0454	−0.0867	13	3	−0.0460	0.0454	12	4	0.0298	−0.0670
13	5	0.0586	0.0469	13	6	−0.0848	0.0577	13	7	−0.0414	−0.0482
13	8	−0.0055	−0.0347	13	9	0.0271	0.0588	13	10	−0.0240	−0.0044
13	11	−0.0576	−0.0830	13	12	−0.0261	0.0991	13	13	−0.0543	0.0722
14	1	−0.0038	0.0480	14	2	−0.0150	0.0429	14	3	0.0653	0.0032
14	4	0.0019	0.0010	14	5	−0.0144	−0.0216	14	6	0.0166	−0.0442
14	7	0.0420	0.0030	14	8	−0.0007	−0.0414	14	9	0.0140	0.0552
14	10	−0.0380	−0.0797	14	11	0.0614	−0.0313	14	12	0.0047	−0.0413
14	13	0.0211	0.0281	14	14	−0.0448	−0.0016	15	1	0.0333	−0.0224
15	2	0.0370	−0.0641	15	3	−0.0457	0.0279	15	4	−0.0070	0.0163
15	5	0.0136	0.0358	15	6	−0.0130	−0.1076	15	7	0.0751	0.0668
15	8	−0.0261	−0.0242	15	9	0.0116	0.0385	15	10	0.0351	−0.0480
15	11	−0.0090	−0.0106	15	12	−0.0338	0.0145	15	13	−0.0191	−0.0000
15	14	0.0036	−0.0189	15	15	−0.0444	0.0356	16	1	0.0321	−0.0091
16	2	−0.0200	0.0639	16	3	−0.0083	−0.0205	16	4	0.0252	0.0306
16	5	0.0120	0.0173	16	6	0.0321	0.0136	16	7	−0.0003	−0.0286
16	8	−0.0456	−0.0046	16	9	−0.0652	−0.0676	16	10	−0.0118	0.0386
16	11	0.0189	−0.0078	16	12	0.0197	−0.0195	16	13	0.0034	−0.0139
16	14	−0.0172	−0.0430	16	15	−0.0475	−0.0378	16	16	−0.0376	−0.0119
17	12	0.0137	−0.0012	17	13	0.0145	0.0204	17	14	−0.0111	−0.0013
18	12	−0.0636	−0.0269	18	13	−0.0137	−0.0580	18	14	−0.0127	−0.0006
19	12	−0.0309	−0.0300	19	13	−0.0205	−0.0291	19	14	−0.0006	0.0011
20	12	−0.0056	−0.0154	20	13	0.0114	−0.0282	20	14	0.0055	−0.0098
21	12	−0.0324	−0.0175	21	13	−0.0241	0.0108	21	14	0.0094	0.0078
22	12	−0.0435	−0.0065	22	13	−0.0324	−0.0151	22	14	−0.0077	−0.0007

resonant coefficients, the GEM-6 model can be used after replacement of the model coefficients by the independent coefficients.

3.3.2.3 12 to 24 Hour Orbital Periods; Low-Order Resonance

When orbital periods are in the 12- to 24-hour range, the GEM-6 model can be reduced considerably in size to accommodate the orbit. The exact size of the model to be used depends on the orbital eccentricity and accuracy requirements. For nearly circular synchronous (24 hour) orbits, coefficients through (3,3) are usually acceptable.

Table 19 gives coefficients which may be used to replace corresponding terms in the GEM-6 model when low-order resonance effects are important. The satellite orbits given in table 20 were used in computing the values in table 19.

3.3.2.4 Polar Orbit Model

For polar orbits (inclination near 90 degrees), the GEM-6 model can be used.

3.3.2.5 Small Models

A requirement for a small model with general applicability can be met by truncating the GEM-6 model.

TABLE 19
LOW-ORDER RESONANT COEFFICIENTS ($\times 10^6$)

n	m	$\bar{C}_{nm}$	$\bar{S}_{nm}$
2	2	2.432	−1.407
3	1	1.878	0.247
3	3	0.703	1.470
4	2	0.335	0.671
4	4	−0.137	0.374

[1] References 45 and 37.

TABLE 20
24 HOUR SATELLITE OBSERVATIONS USED IN SOLUTION
OF RESONANT COEFFICIENTS IN TABLE 19

Longitude Span, $\Delta\lambda$		Longitude Span, $\Delta\lambda$	
(Degrees East of Greenwich)	Inclination, i (degrees)	(Degrees East of Greenwich)	Inclination, i (degrees)
302-305	33.0	210-211	1.2
296-301	32.8	288-312	0.6
196-243	32.5	287-288	0.3
72-189	32;3	313-315	0.3
65-68	31.8	349-352	1.0
65-86	29-32.0	346-349	1.0
178-180	0.0	179-194	1.0
174-181	0.0	324-332	0.5
165-172	0.5	321-331	0.8
160-161	1.3	253-256	2.6
146-160	2-3.0		

3.4 Moon

3.4.1 Mass and Gravitational Constant

The mass of the Moon is given accurately only in terms of the mass of the Earth:

$$M_m = \frac{M_e}{81.302}$$

When treating the Moon as a point mass, the recommended value of gravitational constant of the Moon should be assigned the value:

$$\mu_m = 4902.78 \text{ km}^3/\text{s}^2$$

In the more detailed lunar gravity models, however, the value of μ_m is assigned in each case as part of the solution.

3.4.2 Triaxial Model

To define the values of the harmonic coefficients of a triaxial Moon, C_{20} and C_{22} are taken as

$$C_{20} = -2.07108 \times 10^{-4}$$

$$C_{22} = 0.20716 \times 10^{-4}$$

3.4.3 Applications

The triaxial ellipsoid model is recommended for general application except for low altitude lunar satellites. For low altitude orbiters, the considerations of section 2.3.3.2 are pertinent.

3.5 Asteroids and Natural Satellites

3.5.1 Asteroids

Although there is little quantitative information about the physical properties of the asteroids, orbits and ephemerides for about 1800 of them are known in some detail. Ephemerides through the year 2000 have been computed by Duncombe (ref. 59) for the four most widely-observed asteroids, Ceres, Pallas, Juno, and Vesta. Reference 3 lists several references that can be consulted for detailed ephemerides of the asteroids.

The masses of the asteroids can be estimated reliably only by observing the perturbed motions of other bodies in their vicinity. The mass of Ceres, the largest asteroid, has recently been determined from the motion of Pallas to be $6.7 \pm 0.4 \times 10^{-10}$ of the solar mass by Schubart (ref. 60). The mass of Vesta was found by Hertz (ref. 61) to be $1.20 \pm .08 \times 10^{-10}$ of the solar mass from the motion of the asteroid Arete.

Diameters of the four most widely observed asteroids have recently been published by Dollfus (ref. 57) and Gehrels (ref. 62). These values are derived from astronomical observations made by Barnard in 1902. They are presented in table 21.

TABLE 21
DIAMETERS OF PROMINENT ASTEROIDS

Asteroid	Diameter (km)
Ceres	769 ± 20
Pallas	490 ± 25
Juno	196 ± 25
Vesta	400 ± 35

3.5.2 Natural Satellites

The masses and diameters of the natural satellites are presented in table 22. They are based on the estimates of Koslavskaya (ref. 63), Dollfus (ref. 57), De Sitter (ref. 64), and Jeffreys (refs. 65 and 66). The masses are presented in units of the mass of the planet about which each satellite revolves to avoid introducing the uncertainties of the planetary masses. In some cases, a mean value is given with a standard error. In other cases, a range of values is given that should be interpreted as the interval over which any value is equally probable. When no range is given, the mass or diameter value is so uncertain that it was not possible to estimate a range of uniform probability.

TABLE 22
MASSES AND DIAMETERS OF THE NATURAL SATELLITES

Planet	Satellite	Mass (Fraction of Planet)	Diameter (km)
Earth	Moon	0.0123	3476 ± 2
Mars	Phobos (I)	2.7×10^{-8}	22.46 ± 2
	Deimos (II)	4.9×10^{-9}	13.4 ± 2.7
Jupiter	Io (I)	$(4.696 \pm .06) \times 10^{-5}$	3500 ± 150
	Europa (II)	$(2.565 \pm .06) \times 10^{-5}$	3110 ± 150
	Gamymede (III)	$(7.845 \pm .08) \times 10^{-5}$	5550 ± 130
	Callisto (IV)	$(5.603 \pm .17) \times 10^{-5}$	500 ± 150
	V	2.2×10^{-9}	88 to 160
	VI	8.5×10^{-10}	64 to 184
	VII	3.5×10^{-11}	24 to 64
	VIII	7×10^{-13}	6.4 to 18
	IX	1.5×10^{-12}	7 to 20
	X	1.0×10^{-12}	7 to 20
	XI	2.0×10^{-12}	24
	XII	7.0×10^{-13}	6 to 18
Saturn	Mimas (I)	$(6.64 \pm 0.10) \times 10^{-8}$	450 ± 665
	Enceladus (II)	$(1.4 \pm 0.47) \times 10^{-7}$	550 ± 300
	Tethis (III)	$(1.118 \pm 0.015) \times 10^{-6}$	1200 ± 200
	Dione (IV)	$(1.90 \pm 0.07) \times 10^{-6}$	820 ± 400
	Rhea (V)	$(3.8 \pm 3.8) \times 10^{-6}$	1300 ± 300
	Titan (VI)	$(2.425 \pm 0.020) \times 10^{-4}$	4850 ± 300
	Hyperion (VII)	8×10^{-8}	980 to 500
	Iapetus (VIII)	$(3.20 \pm 0.74) \times 10^{-6}$	1150 ± 100
	Phoebe (IX)	5×10^{-8}	190 to 540
Uranus	Ariel (I)	28×10^{-6}	760 to 2170
	Umbriel (II)	8×10^{-6}	500 to 1410
	Titania (III)	49×10^{-6}	910 to 2600
	Oberon (IV)	38×10^{-6}	830 to 2380
	Miranda (V)	1.5×10^{-6}	280 to 820
Neptune	Triton (I)	$(1.34 \pm 0.23) \times 10^{-3}$	3770 ± 1500
	Nereid (II)	3.33×10^{-7}	280 to 800

REFERENCES

1. Anderson, J. D.; et al.: Determination of the Mass of Venus and Other Astronomical Constants From the Radio Tracking of Mariner V. Presented at the 125th Meeting of the American Astronomical Society, (Philadelphia, Pennsylvania), December 4, 1967.

2. Melbourne, W. G.; et al.: Constants and Related Information for Astrodynamic Calculations. National Aeronautics and Space Administration. TR-32-1306, July 15, 1968.

3. U.S. Naval Observatory: Explanatory Supplement to the Astronomical Ephemeris. 1961.

4. Klepczynski, W. J.; et al.: The Masses of the Principal Planets. Presented at the International Astronomical Union Colloquium No. 9, (Heidelberg, Germany), August 12-15, 1970.

5. Seidelmann, P. K.; et al.: Determination of the Mass of Pluto. Astronomical Journal, Vol. 76, 1971, pp. 488-492.

6. Ash, M. E.; Shapiro, I. I.; and Smith, W. R.: Systems of Planetary Masses. Science, Vol. 174(4009), November 1971, pp. 551-556.

7. Kovalevsky, J.: Determination des Masses des Planetes. Presented at the International Astronomical Union Colloquium No. 9, (Heidelberg, Germany), August 12-15, 1970.

8. Ash, M. E.; Shapiro, I. I.; and Smith, W. B.: The System of Planetary Masses. Science, Vol. 174(4009), November 1971, pp. 551-556.

9. Howard, H. T.; et al.: Mercury: Results on Mass, Radius, Ionosphere, and Atmosphere from Mariner 10 Dual Frequency Radio Signals. Science, Vol. 185, July 12, 1974, pp. 179-180.

10. Anderson, J. D.; and Efron, L.: The Mass and Dynamical Oblateness of Venus (Abstract). Bulletin of the American Astronomical Society, Vol. 1, No. 3, 1969.

11. Anderson, J. D.: Geodetic and Dynamical Properties of Planets. EOS Transactions of American Geophysical Union, Vol. 55, No. 5, May 1974.

12. Howard, H. T.; Tyler, G. L.; et. al.: Venus: Gravity Field, Atmosphere and Ionosphere as Measured by the Mariner 10 Dual Frequency Radio System. Science, Vol. 183, March 29, 1974, pp. 1297-1301.

13. Null, G. W.: A Solution for the Mass and Dynamical Oblateness of Mars Using Mariner-IV Doppler Data (Abstract). Bulletin of the American Astronomical Society, Vol. 1, No. 3, 1969.

14. Lorell, J and Shapiro, I. I.: Mariner 9 Celestial Mechanics Experiment, A Status Report. Journal of Geophysical Research, Vol. 78, No. 20, July 10, 1973.

15. National Aeronautics and Space Administration: Space Vehicle Design Criteria Monograph, SP 8069. The Planet Jupiter (1970), December 1971.

16. Anderson, J. D.; Null, G. W.; and Wong, S. K.: Gravity Results From Pioneer 10 Doppler Data. Journal of Geophysical Research, Vol. 79, No. 25, Sept. 1, 1974.

17. Esposito, P. B. and Wong, S. K.: Geocentric Gravitational Constant Determined From Mariner 9 Radio Tracking Data. Presented at the International Symposium on Earth Gravity Models and Related Problems, (St. Louis, Missouri), August 16-18, 1972.

18. Clark, V. C.: Constants and Related Data Used in Trajectory Calculations at the Jet Propulsion Laboratory. Jet Propulsion Laboratory, Technical Report 32-273, May 1962.

19. Delano, E.: The Lunar Equation From Observations of Eros, 1930-1931. Astronomical Journal, Vol. 55, No. 5, August 1950.

20. Sjogren, W. L.; Trask, D. W.; Vegas, C. J.; and Wollenhaupt, W. R.: Physical Constants as Determined From Radio Tracking of the Ranger Lunar Probes. Space Flight Mechanics Specialist Symposium Proceedings, American Astronomical Society, 1967. Also JPL Technical Report 32-1306.

21. Kopal, Z.: An Introduction to the Study of the Moon. D. Reidell Publishing Co., Dordrecht, Holland, 1966.

22. Null, G. W.: Recent Determinations of the Earth-Moon Mass Ratio Using Radio Tracking Data From Mariner and Pioneer Spacecraft. Bulletin American Astronomical Society, Vol. 2, No. 251, 1970.

23. Caputo, M.: The Gravity Field of the Earth. Academic Press, New York, 1967.

24. Jeffreys, H.: The Earth. Fourth edition, Cambridge University Press, New York, 1959.

25. Heiskanen, W. A.; and Vening-Meinesz, F. A.: The Earth and its Gravity Field. McGraw-Hill Book Company, New York, 1958.

26. Kaula, W.: Theory of Satellite Geodesy. Blaisdell Publishing Company, Waltham, Massachusetts, 1966.

27. Mueller, I.: Introduction to Satellite Geodesy. Frederick Ungar Publishing Company, New York, 1964.

28. Wilkins, G. A.: The Determination of the Mass and Oblateness of Mars From the Orbits of Its Satellites. Her Majesty's Nautical Almanac Office, Technical Note No. 10, London, June 1966.

29. Jordan, J. F.; and Lorell, J.: Mariner 9, An Instrument of Dynamical Science: Paper presented at AAS/AIAA Astrodynamics Conference, American Astronautical Society and American Inst. of Aeronautics and Astronautics, Vail, Colo. July 16-18, 1973.

30. National Aeronautics and Space Administration. Space Vehicle Design Criteria Monograph, SP 8091, The Planet Saturn 1970.

31. Brouwer, D. and Clemence, G. M.: Orbits and Masses of Planets and Satellites. Planets and Satellites, Solar System Series, Vol. 3, eds., Kuiper, G. P.; and Middlehurst, B. M.: University of Chicago Press, Chicago, 1961, Chapter 3.

32. National Aeronautics and Space Administration. Space Vehicle Design Criteria Monograph, SP 8103, The Planets Uranus, Neptune, and Pluto, 1971, November 1972.

33. Gill, J.; and Gault, B.: A New Determination of the Orbit of Triton, Pole of Neptune's Equator, and Mass of Neptune. Astronomical Journal, Vol. 73, 1968.

34. Dicke, R. H.; and Goldenberg, H. M.: Solar Oblateness and General Relativity. Physical Review Letters, Vol. 18, 1967, pp. 313-316.

35. Oesterwinter, C.; and Cohen, C. J.: New Orbit Elements for Moon and Planets. Celestial Mechanics, May 1972.

36. Cook, A. H.: Resonant Orbits of Artificial Satellites and Longitude Terms in the Earth's External Gravitational Potential. The Geophysical Journal, Vol. 4, 1961, p. 53.

37. Izsak, I. G.: Tesseral Harmonics in the Geopotential. Nature, Vol. 199, No. 4889, July 1963.

38. Izsak, I. G.: Tesseral Harmonics of the Geopotential and Corrections to Station Coordinates. Journal of Geophysical Research, Vol. 69, No. 12, June 1964.

39. Guier, W. H.: Determination of the Non-Zonal Harmonics of the Geopotential From Satellite Doppler Data. Nature, Vol. 200, No. 4902, October 1963.

40. Anderle, R. J.: Use of Doppler Observations on Satellites in Geodesy. Presented at the 1965 IEEE, International Space Electronics Symposium, (Miami Beach, Florida), November 1965.

41. Kaula, W. M.: A Review of Geodetic Parameters. Presented at International Astronomical Union Symposium No. 21, (Paris), May 1963.

42. Guier, W. H.; and Newton, R. R.: The Earth's Gravity Field Deduced From the Doppler Tracking of Five Satellites. Journal of Geophysical Research, Vol. 70, No. 18, 1965.

43. Gaposchkin, G. M.: A Dynamical Solution for the Tesseral Harmonics of the Geopotential. . . Transactions of American Geophysical Union, Vol. 47, 1966; and Space Research, Vol. VII, 1967, pp. 683-697.

44. Lundquist, C. A.; and Veis, G.: Geodetic Parameters for a 1966 Smithsonian Institution Standard Earth. Smithsonian Astrophysical Observatory, SAO Special Report No. 200, 1967.

45. Gaposchkin, E. M.; and Lambeck, K.: 1969 Smithsonian Standard Earth (II). Smithsonian Institution, Smithsonian Astrophysical Observatory Special Report No. 315; *and* Earth's Gravity Field to Sixteenth Degree and Station Coordinates from Satellite and Terrestrial Data. Journal of Geophysical Research, Vol. 76, 1971, pp. 4855-83.

46. Henrikson, S. W.: The National Geodetic Satellite Program, U.S. Government Printing Office, Ch. 5, Sec. 6, to be published September 1974.

47. Muller, P. M.; and Sjogren, W. L.: Mascons: Lunar Mass Concentrations. Science, Vol. 161, No. 3842, Aug. 16, 1968, p. 680.

48. Michael, W. H., Jr.; and Blackshear, W. T.: Recent Results on the Mass Gravitational Field and Moments of Inertia of the Moon. The Moon, Vol. 3, D. Reidel Publishing Co., (Dordrecht, Holland), 1972, pp. 388-402.

49. Sjogren, W. L.: Lunar Gravity Estimate; Independent Confirmation. Journal of Geophysical Research, Vol. 76, No. 23, Oct. 10, 1971, pp. 7021-7026.

50. Bryant, W. C., Jr.; and Williamson, R. G.: Lunar Gravity Analysis Results from Explorer 49. AIAA Paper No. 74-810, Given at AIAA Mechanics and Control of Flight Conference (Anaheim, Calif.), Aug. 5-9, 1974.

51. Compton, H. R. and Tolson, R. H.: Study of a Simple Lunar Gravitational Model for Application to Apollo Orbit Determination and Prediction. NASA Langley Research Center letter to NASA JSC, Attn. James C. McPherson, June 11, 1969.

52. Kaula, W. M.: Tests and Combination of Satellite Determinations of the Gravity Field with Gravimetry. Journal of Geophysical Research, Vol. 71, 1966, pp. 5303-13.

53. Lorell, J.: Lunar Orbiter Gravity Analysis. The Moon, Reidell Publishing Co., Dordrecht, Holland, Vol. 1, 1970, pp. 190-231.

54. DeSitter, W.: On the Most Probable Values of Some Astronomical Constants. . . Bulletin of the Astronomical Institutes of the Netherlands, Vol. 4, 1927, pp. 57-63.

55. Wagner, C. A.: Longitude Variations of the Earth's Gravity Field as Sensed by the Drift of Three Synchronous Satellites. Journal of Geophysical Research, Vol. 71, No. 6, March 1966.

56. Bender, P. L.; et al.: The Lunar Laser Ranging Experiment. Science,Vol. 182, Oct. 19, 1973, pp. 229-238.

57. Dollfus, A.: Diameters des Planets et Satellites. Surfaces and Interiors of Planets and Satellites, Academic Press, New York, 1970.

58. American Ephemeris and Nautical Almanac. U.S. Naval Observatory and Her Majesty's Nautical Almanac Office, 1970.

59. Duncombe, R. L.: Heliocentric Coordinates of Ceres, Pallas, Juno, and Vesta, 1928-2000. Astronomical Papers of the American Ephemeris and Nautical Almanac, U.S. Government Printing Office, Vol. 20, Part 2, 1969.

60. Schubart, J.: The Mass of Ceres. Smithsonian Institution, International Astronomical Union, Central Bureau for Astronomical Telegrams, Circular No. 2268, July 1970.

61. Hertz, H. G.: The Mass of Vesta. Science, Vol. 160, 1968, p. 299.

62. Gehrels, T.: Photometry of Asteroids. Surfaces and Interiors of Planets and Satellites, ed., A. Dollfus, Academic Press, New York, 1970.

63. Koslavskaya, S. V.: Masses and Radii of Planets and Satellites. Bulletin of the Institute of Theoretical Astronomy, Academy of Science, USSR, Leningrad, Vol. 9, 1963, p. 330.

64. DeSitter, W.: Jupiter's Galilean Satellites. Royal Astronomical Society Monthly Notice, Vol. 91, 1931.

65. Jeffreys, J.: On the Masses of Saturn's Satellites. Royal Astronomical Society Monthly Notice, Vol. 113, 1953.

66. Jeffreys, H.: Second Order Terms in the Figure of Saturn. Royal Astronomical Society Monthly Notice, Vol. 114, 1954.

67. Danby, J. M. A.: Fundamentals of Celestial Mechanics. MacMillan Company, New York, 1962.

68. Brouwer, D.; and Clemence, G. M.: Methods of Celestial Mechanics. Academic Press, New York, 1961.

69. Heiskanen, W. A.; and Moritz, H.: Physical Geodesy. W. H. Freeman and Co., San Francisco, 1967.

70. Kozai, Y.: The Gravitational Field of the Earth Derived From Motion of Three Satellites. Astronomical Journal, Vol. 66, No. 1, February 1961.

71. King-Hele, D. G.: Evaluation of the Second, Fourth and Sixth Harmonics in the Earth's Gravitational Potential. Nature, Vol. 187, No. 4736, August 1960.

72. Henriksen, S. W.: The Hydrostatic Flattening of the Earth. Annals of the IGY, Pergamon Press, Elmsford, New York, Vol. XII, Part I, 1960.

73. O'Keefe, J. A.; Eckels, A.; and Squires, R. K.: The Gravitational Field of the Earth. Astronomical Journal, Vol. 64, 1959, pp. 245-272.

74. Wagner, C. A.: Combined Solution for Low Degree Longitude Harmonics of Gravity from 12 and 24 Hour Satellites. Journal of Geophysical Research, Vol. 73, 1968, pp. 7651-7660.

75. Yionoulis, S. M.: Improved Coefficients of the 13th Order Harmonics of the Geopotential Derived from Satellite Doppler Data at Three Different Orbital Inclinations. The Johns Hopkins University Applied Physics Laboratory, Technical Memo TG-1003, May 1968.

76. Douglas, B. D.; and Marsh, J. G.: GEOS-II and 13th Order Terms of the Geopotential. Goddard Space Flight Center, X-552-69-291, July 1969.

77. King-Hele, D. G.; Walker, D. M. C.; and Gooding, R. H.: Evaluation of Harmonics in the Geopotential of Order 15 and Odd Degree. Royal Aircraft Establishment Technical Report No. 74026, Farnborough, Hants, England, 1974.

78. Hagihara, Y.: Recommendations on Notation of the Earth Potential. Astronomical Journal, Vol. 67, 1962, p. 108.

79. NASA Marshall Space Flight Center – Orbital Flight Handbook. Martin Company, Space Systems Division, 1963.

BIBLIOGRAPHY

Anderle, R. J.: Use of Doppler Observations on Satellites in Geodesy. Presented at the IEEE International Space Electronics Symposium, (Miami Beach, Florida), 1965.

Brouwer, D.; and Clemence, G. M.: Orbits and Masses of Planets and Satellites. Planets and Satellites, Solar System Series, Vol. 3, eds., Kuiper, G. P.; and Middlehurst, B. M.: University of Chicago Press, Chicago, Vol. 3, 1961.

Cayley, A.: Tables of the Development of Functions in the Theory of Elliptic Motion. Memoirs of the Royal Astronomical Society, Vol. 29, 1861, pp. 191-306.

Darwin, G.: The Theory of the Figure of the Earth. . . Royal Astronomical Society Monthly Notices, Vol. 60, 1899.

DeSitter, W.: On the Flattening and Constitution of the Earth. Bulletin of the Astronomical Institution of the Netherlands, Vol. 2, 1924.

Deutsch, R.: Orbital Dynamics of Space Vehicles. Prentice-Hall, New York, 1963.

Dicke, R. H.: The Solar Oblateness and the Gravitational Quadruple Moment. Astrophysical Journal, Vol. 159, 1970, pp. 1-25.

Gaposchkin, E. M.: Future Uses of Laser Tracking. Presented at GEOS 2 Review Meeting, Goddard Space Flight Center, 1970.

Gehrels, T.: Photographic Magnitudes of the Numbered Minor Planets. Transactions of the International Astronomical Union, Vol. 10, 1960.

Gething, P. J.: Dr. Greenwich's Observations of the Horizontal and Vertical Diameters of the Sun. Monthly Notice Royal Astronomical Society, Vol. 115, 1955, p. 558.

Goudas, C. L.: Moments of Inertia and Gravity Field of the Moon. Boeing Scientific Research Laboratories Mathematical Note No. 362, 1964.

Guier, W. H.: Determination of the Non-Zonal Harmonics of the Geopotential From Satellite Doppler Data. Nature, Vol. 200, No. 4902, October 1963.

Ephemerides of the Minor Planets. Institute of Theoretical Astronomy at Leningrad, published annually in Russian and English.

Jeffreys, H.: The Figures of the Earth and the Moon. Royal Astronomical Society Monthly Notices, Geophysics Supplement 5, 1948.

King–Hele, D. G.; Cook, G. E.; and Scott, D. W.: Evaluation of Odd Zonal Harmonics in the Geopotential of Degree Less than 33, from the Analysis of 22 Satellite Orbits. Planetary and Space Science, Vol. 17, 1969, pp. 629-664.

Kozai, Y.: On the Astronomical Constants of the Saturnian Satellites System. Annals of Tokyo, Astronomical Observatory, 2nd Series, Vol. 5, No. 2, 1957-1958.

Lundquist, C. A.; and Veis, G.: The Zonal Harmonic Coefficients. Geodetic Parameters for a 1966 SAO Standard Earth, SAO Special Report No. 200, 1966.

Marsden, B. G.: The Motions of the Galilean Satellites of Jupiter. Yale University doctoral dissertation, University Microfilms Inc., New York, 1966.

Minor Planet Circulars. Minor Planet Center at Cincinnati, Ohio. Published at frequent intervals whenever new information is available.

Murphy, J. P.; and Marsh, J. G.: Derivation and Tests of the Goddard Combined Geopotential Field. GSFC 1.70-C, Goddard Space Flight Center Document X-552-70-104, January 1970.

Newcomb, S.: Astronomical Papers of the American Ephemeris and Nautical Almanac, Vols. 6 and 7, 1898.

Oesterwinter, C.; and Cohen, C. J.: New Orbit Element for Moon and Planets. Celestial Mechanics, May 1972.

O'Keefe, J. A.: Determination of the Earth's Gravitational Field. Space Research, Vol. 1, 1960.

O'Keefe, J. A.; Eckels, A.; and Squires, R. K.: The Gravitational Field of the Earth. Astronomical Journal, Vol. 64, 1959, pp. 245-272.

O'Keefe, J.A.; Eckels, A.; and Squires, R.K.: Vanguard Measurements Give Pear-Shaped Component of Earth's Figure. Science, Vol. 129, No. 3348, February 1959.

Rapp, R. H.: The Equatorial Radius of the Earth and The Zero-Order Undulation of the Geoid. Journal Geophysical Research, Vol. 72, 1967.

Tisserand, A.: Traite de Mecanique Celeste. Gauther-Villars et Fils, 4 Vols., 1889 and 1960.

Wagner, C. A.; Douglas, B. C.; and Fisher, E. R.: Resonant Satellites Geodesy by High Speed Analysis of Mean Kepler Elements. Goddard Space Flight Center Report No. X-552-69-235, 1969.

APPENDIX A

DEFINITION OF SYMBOLS*

a	Semimajor axis
$\bar{a}_e$	Mean equatorial radius of a massive body
a_e	Equatorial radius of a massive body
A	Smallest of a body's principal moments of inertia
b	Polar radius of an ellipsoid of revolution (2.2)
B	Intermediate of a body's principal moments of inertia
C	Largest of a body's principal moments of inertia
C_{nm}	Dimensionless spherical harmonic coefficient
e	Eccentricity
f	Flattening of an ellipsoid of revolution
F	Gravitational force
g	Gravitational acceleration
g_e	Gravitational acceleration at the equator of a massive body
G	Universal gravitational constant (2.1)
i	Orbit inclination relative to equator of attracting body
I	Moment of inertia relative to any specified axis
J_n	Zonal harmonic of degree n (2.3)
K	Factor in computation of K_{nm} (=1 for m=0, =2 for m≠0) (appendix F)
K_{nm}	Normalization factor (appendix F)
m	Index connoting the order of a harmonic
n	Index connoting the degree of a harmonic
n	A unit vector normal to some surface (2.5)
P	Some arbitrary point in space
P_{nm}	Legendre polynomial of degree n and order m
r	Radial distance from center of mass in spherical coordinates
r_o	Equatorial radius of an equipotential surface near an ellipsoidal body (2.2)
S_{nm}	Dimensionless spherical harmonic coefficient

*Numbers in parens give section where symbol used.

t	Time
U	Gravitational potential due to zonal terms
$V(r, \theta, \lambda)$	Spherical harmonic potential function (Appendix D)
$Y_n(\theta, \lambda)$	A surface harmonic of degree n (Appendix D)

<u>Subscripts</u>

e	Earth
e	Equator
m	Moon
m	Order (when preceded by n)
n	Degree
o	Measured at some epoch or reference
P	Planet
s	Sun

<u>Greek Letters</u>

a_n	Coefficients of degree n used in describing a spheroidal surface (appendix C)
θ	Colatitude ($=\pi/2 - \phi$)
λ	Longitude
μ	Gravitational constant (= GM)
π	3.14159265
$\rho(r)$	Radius dependent part of the spherical harmonic potential (appendix D)
ϕ	Latitude
$\Phi^c{}_{nm}, \Phi^s{}_{nm}$	Components of the spherical harmonic potential (Appendix D)
ω	Rotation rate of a massive body

APPENDIX B
ASTRODYNAMIC CONSTANTS AND UNITS

The basis of the definitions of the standard astrodynamic constants is Kepler's laws of planetary motion (ref. 67). Analytically stated, Kepler's Third Law (refs. 67 and 68) says that for any planet, the square of its period P is proportional to the cube of its mean distance from the Sun, or symbolically,

$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM_s\left(1 + \frac{M_p}{M_s}\right)} = \frac{4\pi^2}{k^2 M_s\left(1 + \frac{M_p}{M_s}\right)} \tag{B-1}$$

where a is the semimajor axis of the planet's orbit about the Sun

M_s is the Sun's mass

M_p is the planet's mass, including planet, atmosphere, and satellites

$k = \sqrt{G}$ is the Gaussian gravitational constant

Historically, equation (B-1) is the basis for the definitions of presently adopted astrodynamic units and constants; it contains units of time, mass, and length, and a "universal constant" k whose value can be theoretically derived from perfect determinations of the observable quantities. Therefore, a convenient set of units was adopted:

- The unit of mass was taken to be the Sun's mass, i.e., M_s was set equal to unity
- The unit of time was taken to be a mean solar day, i.e., P was set equal to 365.2563835*
- The unit of length was taken to be the mean distance between the Earth and Sun and was called the astronomical unit (AU)

With these units, a value of k could be inferred from the ratio M_e/M_s where M_e is the mass of the Earth-Moon system. Reference 67 notes that at the time k was computed, M_e/M_s was thought to be $(354{,}710)^{-1}$, and k was computed to be 0.01720209895. If the units of measurement were to be held fixed, then any refinement in M_e/M_s would necessitate changing k as well. For practical astronomy, however, a consistent value of k is virtually a necessity so that k is fixed at the foregoing value by convention. As a result of this convention, the unit of length AU equals the mean radius of the Earth's orbit only if the value of M_e/M_s that was used in deriving k were correct. On the basis of current values of M_e/M_s, the mean radius of the Earth's orbit a_e is 1.00000003 AU (ref. 3).

*P actually also includes the time correction 1.1×10^{-7} T where T is measured in centuries since 1900.

APPENDIX C
GRAVITATIONAL FIELD OF A SPHEROIDAL BODY

C.1 Mathematical Description

The derivations, formulas, and notation used here are essentially those of Kaula (ref. 26). The expressions for the potential and the radius of an equipotential surface are given in terms of the Legendre polynomials P_{20} and P_{40} (sec. C.2) by

$$P_{20} = \frac{3}{2}\sin^2\phi - \frac{1}{2}$$

$$P_{40} = \frac{35}{8}\sin^4\phi - \frac{15}{4}\sin^2\phi + \frac{3}{8}$$

where ϕ is the planetocentric latitude. The potential at a point located on the body's surface at latitude ϕ and distance r from the planet's center is

$$U = \frac{\mu}{r}\left[1 - J_2\left(\frac{a_e}{r}\right)^2 P_{20} - J_4\left(\frac{a_e}{r}\right)^4 P_{40} + \ldots\right] + \frac{1}{2}\omega^2 r^2 \cos^2\phi \quad \text{(C-1)}$$

where a_e is the equatorial radius of the planet

ω is its rotation rate (for an orbiting spacecraft, the rotational term is not present in the potential)

The shape of an equipotential surface for the spheroidal body is symmetric about the axis of rotation and is defined by

$$r = r_o\,(1 + \alpha_2 P_{20} + \alpha_4 P_{40} + \ldots)$$

where

r is the radial distance from the body's center to a point on the equipotential surface with latitude ϕ

r_o is the equatorial radius of that surface

α_2 and α_4 are coefficients to be determined

By rewriting r in terms of ϕ, and using the binomial expansion for r^n, one can write U in terms of powers of $\sin^2\phi$. The coefficients α_2 and α_4 may be expressed in terms of f and ω, whence after combining coefficients of powers of $\sin^2\phi$ in the expression for U, the three parameters μ, J_2, and J_4 may be obtained to order f^2:

$$\mu = a_e^2 g_e \left[1 - f + \left(\frac{3}{2} - \frac{15}{4}\right) m\right]$$

$$J_2 = \frac{2}{3} f \left(1 - \frac{1}{2} f\right) - \frac{1}{3} m \left(1 - \frac{3}{2} m - \frac{2}{7} f\right)$$

$$J_4 = -\frac{4}{35} f(7f - 5m)$$

where to first order in f, according to Heiskanen and Moritz (p. 74, ref. 69),

$$m = \frac{\omega^2 a_e}{g_e}$$

and where g_e is the gravitational acceleration at the equator.

C.2. Legendre Polynomials

Legendre polynomials arise as particular solutions to Legendre's equation:

$$\sin\theta\, g''(\theta) + \cos\theta\, g'(\theta) + \left[n(n+1)\sin\theta - \frac{m^2}{\sin\theta}\right] g(\theta) = 0$$

where $g(\theta)$ is some function of the independent variable θ, n and m are dimensionless constants, and $'$ and $''$ denote first and second derivatives with respect to θ. Making the substitutions, $t = \cos\theta$ and $G(t) = g(\theta)$, a solution for G(t) is found to be

$$P_{nm}(t) = \frac{1}{2^n n!} (1 - t^2)^{m/2} \frac{d^{n+m}}{dt^{n+m}} (t^2 - 1)^n, \quad (m \neq 0)$$

$$P_n(t) = P_{no}(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n, \quad (m = 0)$$

where t is cos θ and θ is the colatitude.

The functions $P_n(t)$ are polynomials in t. These polynomials may also be obtained from the recursion formula

$$P_n(t) = \frac{1 - n}{n} P_{n-2}(t) + \frac{2n - 1}{n} tP_{n-1}(t)$$

$$P_o(t) = 1$$

by making use of the trigonometric identities:

$$\cos^2 \theta = \frac{1}{2} \cos 2\theta + \frac{1}{2}$$

$$\cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta, \text{ etc.}$$

The Legendre polynomials with m = 0 are summarized in table C-1.

TABLE C-1.
LEGENDRE POLYNOMIALS OF ZERO ORDER

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
4	$(35\cos^4\theta - 30\cos^2\theta + 3)/8$
5	$(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)/8$
6	$(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)/16$
7	$(429\cos^7\theta - 693\cos^5\theta + 315\cos^3\theta - 35\cos\theta)/16$
8	$(6435\cos^8\theta - 12012\cos^6\theta + 6930\cos^4\theta - 1260\cos^2\theta + 35)/128$
9	$(12155\cos^9\theta - 25740\cos^7\theta + 18018\cos^5\theta - 4620\cos^3\theta + 315\cos\theta)/128$
10	$(46189\cos^{10}\theta - 109395\cos^8\theta + 90090\cos^6\theta - 30030\cos^4\theta + 3465\cos^2\theta - 63)/256$

Associated Legendre functions can be derived from Legendre polynomials by means of the equation

$$P_{nm}(t) = \left(1 - t^2\right)^{m/2} \frac{d^m P_n(t)}{dt^m}$$

Associated Legendre functions through n = 10 are given in Table C-2.

TABLE C-2.
ASSOCIATED LEGENDRE FUNCTIONS (1 of 5)

n	m	$P_{nm}(\cos\theta)$
1	1	$\sin\theta$
2	1	$3\sin\theta\cos\theta$
3	1	$\sin\theta\left(\frac{15}{2}\cos^2\theta - \frac{3}{2}\right)$
2	2	$3\sin^2\theta$
3	2	$15\sin^2\theta\cos\theta$
3	3	$15\sin^3\theta$
4	1	$\sin\theta\left(\frac{35}{2}\cos^3\theta - \frac{15}{2}\cos\theta\right)$
4	2	$\sin^2\theta\left(\frac{105}{2}\cos^2\theta - \frac{15}{2}\right)$
4	3	$105\sin^3\theta\cos\theta$
4	4	$105\sin^4\theta$
5	1	$\sin\theta\left(\frac{315}{8}\cos^4\theta - \frac{105}{4}\cos^2\theta + \frac{15}{8}\right)$
5	2	$\sin^2\theta\left(\frac{315}{2}\cos^3\theta - \frac{105}{2}\cos\theta\right)$
5	3	$\sin^3\theta\left(\frac{945}{2}\cos^2\theta - \frac{105}{2}\right)$
5	4	$945\sin^4\theta\cos\theta$
5	5	$945\sin^5\theta$

TABLE C-2.
ASSOCIATED LEGENDRE FUNCTIONS (2 of 5)

n	m	$P_{nm}(\cos\theta)$
6	1	$\sin\theta\left(\frac{693}{8}\cos^5\theta - \frac{315}{4}\cos^3\theta + \frac{105}{8}\cos\theta\right)$
6	2	$\sin^2\theta\left(\frac{3465}{8}\cos^4\theta - \frac{945}{4}\cos^2\theta + \frac{105}{8}\right)$
6	3	$\sin^3\theta\left(\frac{3465}{2}\cos^3\theta - \frac{945}{2}\cos\theta\right)$
6	4	$\sin^4\theta\left(\frac{10395}{2}\cos^2\theta - \frac{945}{2}\right)$
6	5	$10395\sin^5\theta\cos\theta$
6	6	$10395\sin^6\theta$
7	1	$\sin\theta\left(\frac{3003}{16}\cos^6\theta - \frac{3465}{16}\cos^4\theta + \frac{945}{16}\cos^2\theta - \frac{35}{16}\right)$
7	2	$\sin^2\theta\left(\frac{9009}{8}\cos^5\theta - \frac{3465}{4}\cos^3\theta + \frac{945}{8}\cos\theta\right)$
7	3	$\sin^3\theta\left(\frac{45045}{8}\cos^4\theta - \frac{10395}{4}\cos^2\theta + \frac{945}{8}\right)$
7	4	$\sin^4\theta\left(\frac{45045}{2}\cos^3\theta - \frac{10395}{2}\cos\theta\right)$
7	5	$\sin^5\theta\left(\frac{135135}{2}\cos^2\theta - \frac{10395}{2}\right)$
7	6	$135135\sin^6\theta\cos\theta$
7	7	$135135\sin^7\theta$

TABLE C-2.
ASSOCIATED LEGENDRE FUNCTIONS (3 of 5)

n	m	$P_{nm}(\cos\theta)$
8	1	$\sin\theta\left(\frac{6435}{16}\cos^7\theta - \frac{9009}{16}\cos^5\theta + \frac{3465}{16}\cos^3\theta - \frac{315}{16}\cos\theta\right)$
8	2	$\sin^2\theta\left(\frac{45045}{16}\cos^6\theta - \frac{45045}{16}\cos^4\theta + \frac{10395}{16}\cos^2\theta - \frac{315}{16}\right)$
8	3	$\sin^3\theta\left(\frac{135135}{8}\cos^5\theta - \frac{45045}{4}\cos^3\theta + \frac{10395}{8}\cos\theta\right)$
8	4	$\sin^4\theta\left(\frac{675675}{8}\cos^4\theta - \frac{135135}{4}\cos^2\theta + \frac{10395}{8}\right)$
8	5	$\sin^5\theta\left(\frac{675675}{2}\cos^3\theta - \frac{135135}{2}\cos\theta\right)$
8	6	$\sin^6\theta\left(\frac{2027025}{2}\cos^2\theta - \frac{135135}{2}\right)$
8	7	$2027025\sin^7\theta\cos\theta$
8	8	$2027025\sin^8\theta$
9	1	$\sin\theta\left(\frac{109395}{128}\cos^8\theta - \frac{45045}{32}\cos^6\theta + \frac{45045}{64}\cos^4\theta\right.$ $\left. - \frac{3465}{32}\cos^2\theta + \frac{315}{128}\right)$
9	2	$\sin^2\theta\left(\frac{109395}{16}\cos^7\theta - \frac{135135}{16}\cos^5\theta + \frac{45045}{16}\cos^3\theta - \frac{3465}{16}\cos\theta\right)$
9	3	$\sin^3\theta\left(\frac{765765}{16}\cos^6\theta - \frac{675675}{16}\cos^4\theta + \frac{135135}{16}\cos^2\theta - \frac{3465}{16}\right)$
9	4	$\sin^4\theta\left(\frac{2297295}{8}\cos^5\theta - \frac{675675}{4}\cos^3\theta + \frac{135135}{8}\cos\theta\right)$

TABLE C-2.
ASSOCIATED LEGENDRE FUNCTIONS (4 of 5)

n	m	$P_{nm}(\cos\theta)$
9	5	$\sin^5\theta\left(\frac{11486475}{8}\cos^4\theta - \frac{2027025}{4}\cos^2\theta + \frac{135135}{8}\right)$
9	6	$\sin^6\theta\left(\frac{11486475}{2}\cos^3\theta - \frac{2027025}{2}\cos\theta\right)$
9	7	$\sin^7\theta\left(\frac{34459425}{2}\cos^2\theta - \frac{2027025}{2}\right)$
9	8	$34459425\sin^8\theta\cos\theta$
9	9	$34459425\sin^9\theta$
10	1	$\sin\theta\left(\frac{230945}{128}\cos^9\theta - \frac{109395}{32}\cos^7\theta + \frac{135135}{64}\cos^5\theta - \frac{15015}{32}\cos^3\theta + \frac{3465}{128}\cos\theta\right)$
10	2	$\sin^2\theta\left(\frac{2078505}{128}\cos^8\theta - \frac{765765}{32}\cos^6\theta + \frac{675675}{64}\cos^4\theta - \frac{45045}{32}\cos^2\theta + \frac{3465}{128}\right)$
10	3	$\sin^3\theta\left(\frac{2078505}{16}\cos^7\theta - \frac{2297295}{16}\cos^5\theta + \frac{675675}{16}\cos^3\theta - \frac{45045}{16}\cos\theta\right)$
10	4	$\sin^4\theta\left(\frac{14549535}{16}\cos^6\theta - \frac{11486475}{16}\cos^4\theta + \frac{2027025}{16}\cos^2\theta - \frac{45045}{16}\right)$

TABLE C-2.
ASSOCIATED LEGENDRE FUNCTIONS (5 of 5)

n	m	$P_{nm}(\cos\theta)$
10	5	$\sin^5\theta\left(\frac{43648605}{8}\cos^5\theta - \frac{11486475}{4}\cos^3\theta + \frac{2027025}{8}\cos\theta\right)$
10	6	$\sin^6\theta\left(\frac{218243025}{8}\cos^4\theta - \frac{34459425}{4}\cos^2\theta + \frac{2027025}{8}\right)$
10	7	$\sin^7\theta\left(\frac{218243025}{2}\cos^3\theta - \frac{34459425}{2}\cos\theta\right)$
10	8	$\sin^8\theta\left(\frac{654729075}{2}\cos^2\theta - \frac{34459425}{2}\right)$
10	9	$654729075\sin^9\theta\cos\theta$
10	10	$654729075\sin^{10}\theta$

APPENDIX D
SPHERICAL HARMONICS AND ORTHOGONALITY

The general solution for the gravitational potential exterior to a massive body satisfies Laplace's equation (ref. 67).

The expression for V in terms of spherical harmonics may be derived from the expression for Laplace's equation in spherical coordinates:

$$r^2 \nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\cos\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{\cos^2\theta}\frac{\partial^2 V}{\partial\lambda^2} = 0 \qquad \text{(D-1)}$$

where r = radial distance from center of mass

θ = colatitude ($\pi/2$ - latitude)

λ = longitude (eastward)

To solve Laplace's equation, the variables r, θ, λ are separated by the form:

$$V(r, \theta, \lambda) = \rho(r)\, Y(\theta, \lambda) \qquad \text{(D-2)}$$

whence, applying the boundary condition $\lim_{r\to\infty} (V) = 0$, the solutions

$$\rho_n(r) = A_n\, r^{-(n+1)}$$

and (D-3)

$$Y_n(\theta, \lambda) = \sum_{m=0}^{n}\left(C'_{nm} \cos m\lambda + S'_{nm} \sin m\lambda\right) P_{nm}(\cos\theta)$$

are found where A_n, C'_{nm}, and S'_{nm} are coefficients whose values are to be determined; P_{nm} ($\cos\theta$) are the Legendre functions (appendix C); and the functions Y_n (θ, λ) are known as surface harmonics. The general solution for V is given as a summation of the above solutions for P_n and Y_n:

$$V(r, \theta, \lambda) = \sum_{n=0}^{\infty} \frac{A_n}{r^{n+1}} \sum_{m=0}^{n}\left(C'_{nm} \cos m\lambda + S'_{nm} \sin m\lambda\right) P_{nm}(\cos\theta) \qquad \text{(D-4)}$$

where $V(r, \theta, \lambda)$ is the gravitational potential in spherical coordinates (sec. 2.3.2 and appendix F). Note that in this formulation the product $A_o\, C_{oo}$ is equivalent to μ as defined in section 2.1.

The individual components of the surface spherical harmonics are represented by

$$\Phi^c_{nm} = C_{nm} \cos m\lambda \; P_{nm} (\cos \theta)$$

and

$$\Phi^s_{nm} = S_{nm} \sin m\lambda \; P_{nm} (\cos \theta)$$

An important property of the functions Φ^c_{nm}, Φ^s_{nm} is that they are orthogonal, that is, the integrated product of two different functions, Φ^c_{nm} and Φ^s_{nm}, over the surface of the sphere, σ, is zero.

$$\left.\begin{aligned} &\iint_\sigma \Phi^c_{nm} (\theta, \lambda)\, \Phi^c_{sr} (\theta, \lambda)\, d\sigma = 0 \\ &\iint_\sigma \Phi^s_{nm} (\theta, \lambda)\, \Phi^s_{sr} (\theta, \lambda)\, d\sigma = 0 \\ &\iint_\sigma \Phi^c_{nm} (\theta, \lambda)\, \Phi^s_{sr} (\theta, \lambda)\, d\sigma = 0 \end{aligned}\right\} \text{if } s \neq n \text{ or } r \neq m \qquad \text{(D-5)}$$

and the integral of the product of two identical functions ($s = n$ and $r = m$) is

$$\begin{aligned} &\iint_\sigma \left[\Phi^c_{no} (\theta, \lambda)\right]^2 d\sigma = \iint_\sigma \left[\Phi^s_{no} (\theta, \lambda)\right]^2 d\sigma = \frac{4\pi}{n + 1} \\ &\iint_\sigma \left[\Phi^c_{nm} (\theta, \lambda)\right]^2 d\sigma = \iint_\sigma \left[\Phi^s_{nm} (\theta, \lambda)\right]^2 d\sigma = \frac{2\pi}{2n + 1} \frac{(n + m)!}{(n - m)!} \end{aligned} \qquad \text{(D-6)}$$

Orthogonality makes the spherical harmonics the natural means of representing a function over a spherical surface (analogous to the use of Fourier series for functions in a rectilinear space). The orthogonality property means that the effect of each term of the harmonic series is unique. Thus terms or groups can be studied as independent subsets of the complete series. Further, the spherical harmonic expansion model is not limited to the Earth; it has been applied to the Moon and some of the planets.

In the case of the Earth, the spherical harmonic model is discussed in section 3.3.

APPENDIX E
SPHERICAL HARMONIC REPRESENTATION OF THE GRAVITATIONAL POTENTIAL

E.1. Zonal Harmonics

The expression below:

$$(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm} (\cos \theta)$$

is a surface spherical harmonic of degree n and order m. If m = 0, the harmonic is a constant multiple of the Legendre function P_{no} $(\cos \theta)$. The function, P_{no} $(\cos \theta)$, has n distinct zeros between θ =0 and $\theta = \pi$ (between $-\pi/2$ and $\pi/2$ latitude) arranged symmetrically about $\theta = \frac{1}{2}\pi$ (fig. E-1).

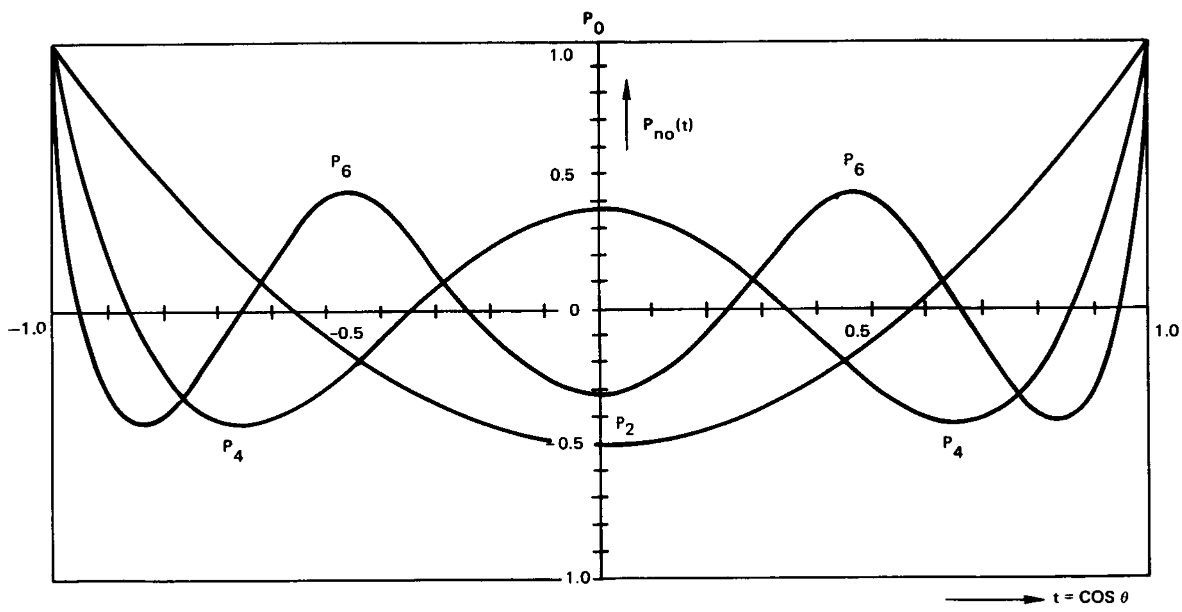

Figure E-1a.– P_{no} $(\cos \theta)$ for n Even

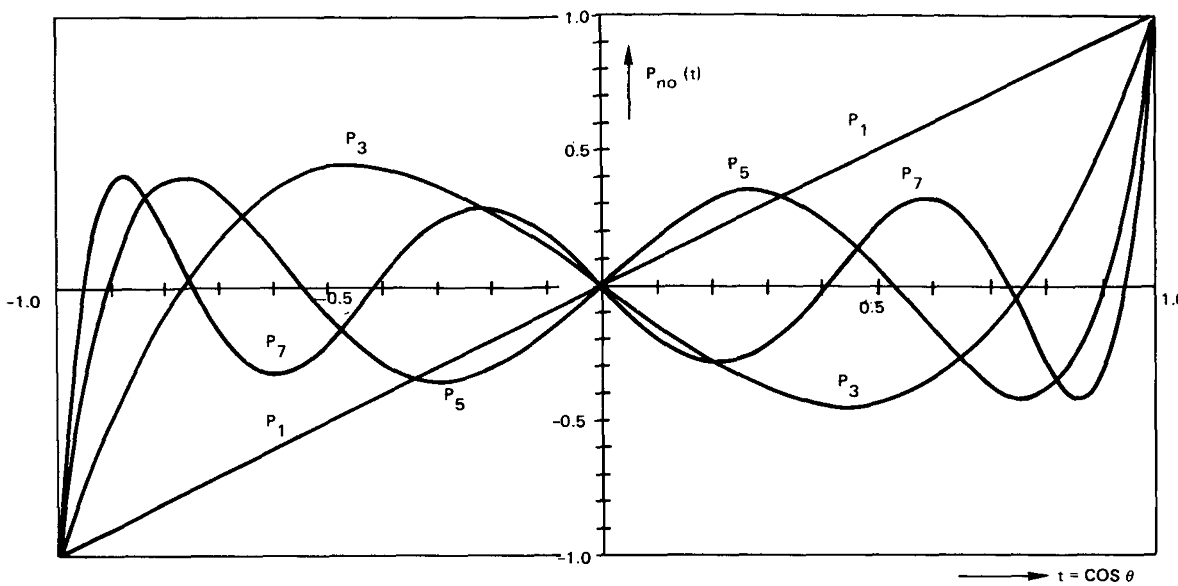

Figure E-1b.- P_{no} $(\cos \theta)$ for n Odd

In a coordinate system having its origin at the center of a sphere, the function P_{no} $(\cos \theta)$ vanishes on n circles of constant latitude (fig. E-2). Similarly, the locus of points on the sphere at which the function has a constant value consists of a number of parallel circles. Because of this division of the sphere into latitude zones in which P_{no} $(\cos \theta)$ alternates in sign, the functions, P_{no} $(\cos \theta)$, are called zonal harmonics.

Many authors have analyzed the zonal harmonics of the Earth's gravitational potential, e.g., Kozai (ref. 70) and King-Hele (ref. 71) have represented the potential at a point, (r, θ), by

$$U = \frac{\mu}{r}\left[1 - \sum_{n=0}^{\infty} J_n \left(\frac{a_e}{r}\right)^n P_n (\cos \theta)\right] \tag{E-1}$$

In what follows, the standard convention of omitting the o subscript from P_{no} will be adopted, so that, in general $J_n \equiv -C_{no}$.

In pre-1958 literature (refs. 24 and 25), only even order zonal harmonics were believed to be significant in the shape and potential of the Earth. This assumption would be true if the Earth were an equilibrium figure of rotation. Even though it was admitted that this assummption was not perfect, the Earth was believed to be sufficiently near equilibrium to make all odd J_n negligible. Henriksen (ref. 72) and Cook (ref. 36) were among the early researches in the interpretation of satellite results to accept the possibility that the

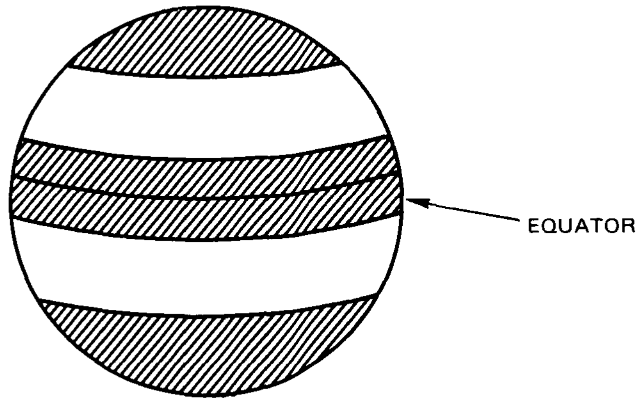

Figure E-2. – Alternate Positive and Negative Regions of P_{40} $(\cos\theta) = 0$
(Shaded Areas are Positive)

flattening of the Earth is not necessarily that of a body in perfect pressure equilibrium; this led to the adjustment of the value of the second harmonic. O'Keefe, Eckels, and Squires (ref. 73) were the first to report the existence of odd zonal harmonics. They attributed the 80-day periodic variation of the eccentricity in the orbit of Vanguard 1 to the presence of the third zonal harmonic in the Earth's field. The announcement of this result gave rise to the term "pear-shaped" in describing the Earth's shape.

Zonal harmonics of even degree give rise to secular perturbations of the orbital elements Ω and ω, and both even and odd zonals give rise to long period perturbations of e, i, Ω and ω. Therefore, their influence can be detected in changes of orbital parameters that are integrated over many revolutions of satellites. The discussion in section 2.2 of the relationship between the zonals J_2 and J_4 and the secular changes in Ω and ω may be generalized to include numbered zonals of higher degree (ref. 27).

E.2 Tesseral Harmonics

If $0 < m < n$, then the associated Legendre functions, P_{nm} $(\cos\theta)$, change their sign n-m times in the colatitude interval $0 \leqslant \theta \leqslant \pi$ and the surface harmonic representation takes the form

$$\left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right) P_{nm} (\cos \theta)$$

The functions $\cos m\lambda$ and $\sin m\lambda$ have 2m zeros in the longitude interval $0 \leqslant \lambda \leqslant 2\pi$. The geometrical representation of such a harmonic is shown in figure E-3 where the sphere is divided into compartments (tesserae) which are alternately positive and negative; the harmonics are called tesseral harmonics.

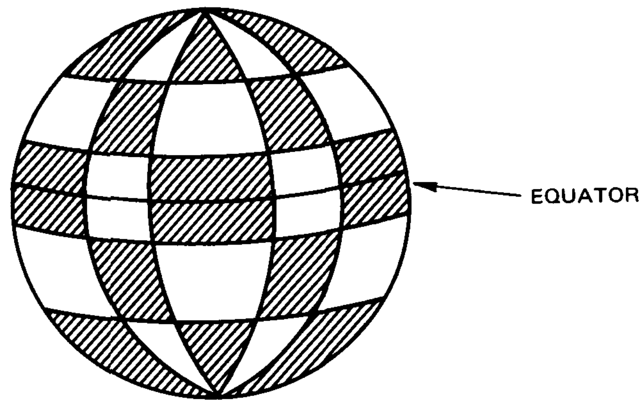

Figure E-3. – Alternate Positive and Negative Regions of
$P_{10,5}(\cos\theta)\{C_{10,5}\cos 5\lambda + S_{10,5}\sin 5\lambda\}$
(Shaded Areas are Positive)

The perturbations caused by tesseral harmonics usually have short periods (sub-multiples of the planet's period of rotation).

E.3 Sectorial Harmonics

If m = n, then the surface harmonic takes the form

$$\left(C_{nn}\cos n\lambda + S_{nn}\sin n\lambda\right)\sin^n\theta$$

which is represented on a sphere in figure E-4. In this form the harmonic oscillates in sign within longitude bands separated by n great circles passing through $\theta = 0$ and $\theta = \pi$, i.e., the poles. Because the sphere is thus divided into 2n sectors, the designation, sectorial harmonics, is used for this type of harmonics. Except for resonance effects, perturbations on satellite motion that result from the sectorial harmonics have short periods (submultiples of a day). The determination of the sectorial harmonics is similar to that of tesseral harmonics.

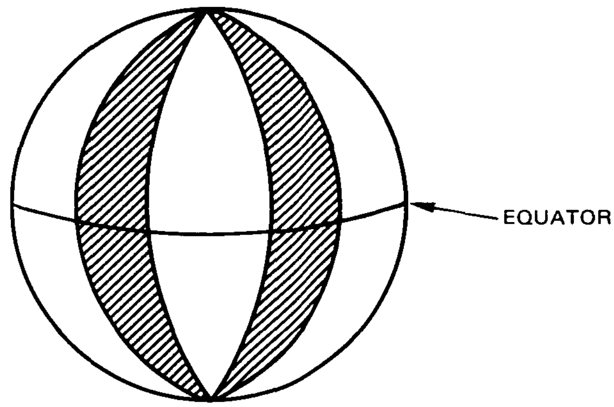

Figure E-4. – Alternate Positive and Negative Sectors for
$(C_{55} \cos 5\lambda + S_{55} \sin 5\lambda) \sin 5\theta$
(Shaded Areas are Positive)

E.4 Resonant Harmonics

The orbits most nearly commensurate with the Earth's rotation have 12- and 24-hour periods. The constraints placed on the low-order harmonics of the geopotential by analysis of these orbits were presented by Wagner in reference 74. The high altitude resonant data provide verification of the low-order gravity terms which reflect average dynamic effects over large areas. In the same way, surface gravity data over limited regions give strength to the terms of high order and degree (ref. 26).

The actual number of orbits significantly perturbed by resonance phenomena is much greater than formerly supposed; this is an important consideration for mission planners concerned with tracking of Earth satellites. The altitudes and periods of Earth resonant orbits are listed in table E-1. Likewise, resonance may be important for planning orbits of other planets.

The discovery of the resonance of satellite orbits with the gravitational potentials has provided a means of obtaining values for harmonic coefficients whose contributions, otherwise, might have been too small to detect. The main investigators in the field have been Wagner, working mainly with low-order resonant harmonics; Gaposchkin (ref. 43), working with 9th, 12th, 13th, and 14th order; Anderle (ref. 40), with 13th order; Yionoulis (ref. 75), 13th order; Douglas and Marsh (ref. 76), with 13th order; and King-Hele, et al. (ref. 77) with 15th order resonant coefficients.

E.5 The Gravitational Potential Using Spherical Harmonics

The general expression for the gravitational potential at a point (r, θ, λ), external to a planet, including zonal, tesseral, and sectorial harmonics is

$$U = \frac{\mu}{r}\left[1 + \sum_{n=2}^{\infty} \left(\frac{a_e}{r}\right)^n \sum_{m=o}^{n} \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right) P_{nm} (\cos \theta)\right] \qquad \text{(E-2)}$$

where a_e is the equatorial radius of the planet.

Equations D-4 and E-2 are different because in equation D-4 the integration constants have been written in terms of a_e and μ (the central force coefficient) and in equation E-2 the harmonic coefficients C_{nm} and S_{nm} are dimensionless. In equation E-2, terms of degree 1 are omitted because they represent terms that result in an offset of the center of mass of the body relative to the center of the coordinate system to which the expression for U is referenced. In practice, such offsets are usually negligible or zero. In addition, the terms C_{21} and S_{21} are set to zero because of the coincidence of the Earth's rotation axis and its axis of maximum moment of inertia.

TABLE E-1
ALTITUDES AND PERIODS OF RESONANT SATELLITE ORBITS

Resonant Order m	Satellite Altitude		
	Period (min)	Naut. Miles	km
1	1436.0	19,320	35,781
2	718.0	10,895	20,178
5	287.2	3,340	6,186
8	179.5	2,248	4,163
9	159.56	1,816	3,363
10	143.6	1,461	2,706
11	130.05	1,146	2,122
12	119.67	899	1,665
13	110.5	671	1,243
14	102.57	475	880

APPENDIX F
NORMALIZATION AND CONVERSION

F.1 Normalization

The conventional harmonics C_{nm} and S_{nm} are sometimes replaced by normalized harmonics, C_{nm} and S_{nm}. Normalization has the effect of giving the coefficients C_{nm} and S_{nm} a clear physical interpretation. For any two harmonic perturbations that are observed to have the same magnitude, the following expression has the same value

$$\sqrt{\bar{C}_{nm}^{2} + \bar{S}_{nm}^{2}}$$

This relationship is seen more clearly in the formulation of the potential in terms of the coefficients J_n^m and λ_n^m where the magnitude of the perturbation is completely contained in J_n^m and the phase angle is λ_n^m. Thus,

$$\bar{J}_n^m = \sqrt{\bar{C}_{nm}^{2} + \bar{S}_{nm}^{2}}$$

and perturbations of equal magnitude would be represented by equal values of J_n^m.

The relation between normalized and unnormalized coefficients in the C_{nm} and S_{nm} formulation is given in reference 27 by

$$\bar{C}_{nm} = \frac{C_{nm}}{K_{nm}}$$

$$\bar{S}_{nm} = \frac{S_{nm}}{K_{nm}}$$

where

$$K_{nm} = \left[K(2n + 1) \frac{(n - m)!}{(n + m)!} \right]^{1/2}$$

and where

$$K = 1 \text{ when } m = 0$$

$$K = 2 \text{ when } m \neq 0$$

The selection of this particular expression for K_{nm} may be seen to arise from the results of the integrals in equation D-6. A table coefficients for normalization of harmonic coefficients is given in table F-1.

F.2 Conversion

Committee 7 of the International Astronomical Union (ref. 78) recommended the following form and notation for the general expression for the Earth's gravitational potential:

$$V = \frac{\mu}{r} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r} \right)^n P_n^m (\sin \phi) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right] \right\} \qquad \text{(F-1)}$$

where μ = GM = product of universal gravitational constant times the mass of attracting body

a_e = mean equatorial radius of Earth

r = distance from geocenter to point of observation

ϕ = geocentric latitude

λ = east longitude

n,m = indices indicating degree and order

Table F-1
NORMALIZATIONS FACTORS (1 OF 2)

n	m	K_{nm}
0	0	1.000000×10^{0}
1	0	1.732049×10^{0}
2	0	2.236065×10^{0}
3	0	2.645748×10^{0}
4	0	2.999997×10^{0}
5	0	3.316620×10^{0}
6	0	3.605545×10^{0}
7	0	3.872976×10^{0}
8	0	4.123096×10^{0}
9	0	4.358893×10^{0}
10	0	4.582563×10^{0}
11	0	4.795825×10^{0}
12	0	4.999991×10^{0}
13	0	5.196136×10^{0}
14	0	5.385150×10^{0}
15	0	5.567744×10^{0}
16	0	5.744547×10^{0}
17	0	5.916055×10^{0}
18	0	6.082737×10^{0}
19	0	6.244961×10^{0}
20	0	6.403098×10^{0}
21	0	6.557402×10^{0}
22	0	6.708173×10^{0}
23	0	6.855634×10^{0}
24	0	6.999969×10^{0}
25	0	7.141410×10^{0}
26	0	7.280082×10^{0}
27	0	7.416167×10^{0}
28	0	7.549786×10^{0}
29	0	7.681094×10^{0}
30	0	7.810185×10^{0}
1	1	1.732049×10^{0}
2	1	1.290996×10^{0}
3	1	1.080125×10^{0}
4	1	0.948683×10^{0}
5	1	0.856350×10^{0}
6	1	0.786795×10^{0}
7	1	0.731923×10^{0}
8	1	0.687186×10^{0}
9	1	0.649788×10^{0}
10	1	0.617909×10^{0}
11	1	0.590321×10^{0}
12	1	0.566140×10^{0}
13	1	0.544706×10^{0}
14	1	0.525537×10^{0}
15	1	0.508265×10^{0}
16	1	0.492590×10^{0}
17	1	0.478287×10^{0}
18	1	0.465165×10^{0}
19	1	0.453061×10^{0}
20	1	0.441854×10^{0}
21	1	0.431444×10^{0}
22	1	0.421741×10^{0}
23	1	0.412661×10^{0}
24	1	0.404140×10^{0}
25	1	0.396137×10^{0}
26	1	0.388584×10^{0}
27	1	0.381447×10^{0}
28	1	0.374691×10^{0}
29	1	0.368285×10^{0}
30	1	0.362191×10^{0}
2	2	0.645498×10^{0}
3	2	0.341566×10^{0}
4	2	0.223608×10^{0}
5	2	0.161835×10^{0}
6	2	0.124403×10^{0}
7	2	0.996025×10^{-1}
8	2	0.821343×10^{-1}
9	2	0.692671×10^{-1}
10	2	0.594585×10^{-1}
11	2	0.517748×10^{-1}
12	2	0.456205×10^{-1}
13	2	0.406000×10^{-1}
14	2	0.364396×10^{-1}
15	2	0.329458×10^{-1}
16	2	0.299782×10^{-1}
17	2	0.274318×10^{-1}
18	2	0.252270×10^{-1}
19	2	0.233030×10^{-1}
20	2	0.216118×10^{-1}
21	2	0.201164×10^{-1}
22	2	0.187856×10^{-1}
23	2	0.175958×10^{-1}
24	2	0.165268×10^{-1}
25	2	0.155616×10^{-1}
26	2	0.146871×10^{-1}
27	2	0.138915×10^{-1}
28	2	0.131654×10^{-1}
29	2	0.125004×10^{-1}
30	2	0.118696×10^{-1}
3	3	0.139444×10^{0}
4	3	0.597615×10^{-1}
5	3	0.330344×10^{-1}
6	3	0.207340×10^{-1}
7	3	0.140859×10^{-1}
8	3	0.101100×10^{-1}
9	3	0.755765×10^{-2}
10	3	0.583042×10^{-2}
11	3	0.461248×10^{-2}
12	3	0.372490×10^{-2}
13	3	0.306032×10^{-2}
14	3	0.255128×10^{-2}
15	3	0.215375×10^{-2}
16	3	0.183809×10^{-2}
17	3	0.158377×10^{-2}
18	3	0.137624×10^{-2}
19	3	0.120496×10^{-2}
20	3	0.106218×10^{-2}
21	3	0.942028×10^{-3}
22	3	0.840116×10^{-3}
23	3	0.753033×10^{-3}
24	3	0.678096×10^{-3}
25	3	0.613218×10^{-3}
26	3	0.556707×10^{-3}
27	3	0.507252×10^{-3}
28	3	0.463733×10^{-3}
29	3	0.425271×10^{-3}
30	3	0.391136×10^{-3}
4	4	0.211289×10^{-1}
5	4	0.776630×10^{-2}
6	4	0.378549×10^{-2}
7	4	0.212352×10^{-2}
8	4	0.130519×10^{-2}
9	4	0.855740×10^{-3}
10	4	0.588959×10^{-3}
11	4	0.421060×10^{-3}
12	4	0.310409×10^{-3}
13	4	0.234716×10^{-3}
14	4	0.181311×10^{-3}
15	4	0.142637×10^{-3}
16	4	0.113994×10^{-3}
17	4	0.923673×10^{-4}
18	4	0.757591×10^{-4}
19	4	0.628135×10^{-4}
20	4	0.525850×10^{-4}
21	4	0.444078×10^{-4}
22	4	0.377990×10^{-4}
23	4	0.324053×10^{-4}
24	4	0.279642×10^{-4}
25	4	0.242773×10^{-4}
26	4	0.211939×10^{-4}
27	4	0.185965×10^{-4}
28	4	0.163955×10^{-4}
29	4	0.145185×10^{-4}
30	4	0.129096×10^{-4}
5	5	0.246225×10^{-2}
6	5	0.807062×10^{-3}
7	5	0.353921×10^{-3}
8	5	0.180998×10^{-3}
9	5	0.102280×10^{-3}
10	5	0.620818×10^{-4}
11	5	0.397863×10^{-4}
12	5	0.266173×10^{-4}
13	5	0.184411×10^{-4}
14	5	0.131538×10^{-4}
15	5	0.961646×10^{-5}
16	5	0.718094×10^{-5}
17	5	0.546177×10^{-5}
18	5	0.422193×10^{-5}
19	5	0.331052×10^{-5}
20	5	0.262926×10^{-5}
21	5	0.211228×10^{-5}
22	5	0.171461×10^{-5}
23	5	0.140496×10^{-5}
24	5	0.116115×10^{-5}
25	5	0.967237×10^{-6}
26	5	0.811546×10^{-6}
27	5	0.685463×10^{-6}
28	5	0.582578×10^{-6}
29	5	0.497987×10^{-6}
30	5	0.427945×10^{-6}
6	6	0.232979×10^{-3}
7	6	0.694097×10^{-4}
8	6	0.279287×10^{-4}
9	6	0.132044×10^{-4}
10	6	0.694095×10^{-5}
11	6	0.393943×10^{-5}
12	6	0.237127×10^{-5}
13	6	0.149578×10^{-5}
14	6	0.980422×10^{-6}
15	6	0.663598×10^{-6}
16	6	0.461600×10^{-6}
17	6	0.328765×10^{-6}
18	6	0.239018×10^{-6}
19	6	0.176955×10^{-6}
20	6	0.133138×10^{-6}
21	6	0.101627×10^{-6}
22	6	0.785889×10^{-7}
23	6	0.614936×10^{-7}
24	6	0.486356×10^{-7}
25	6	0.388450×10^{-7}
26	6	0.313061×10^{-7}
27	6	0.254404×10^{-7}
28	6	0.208334×10^{-7}
29	6	0.171818×10^{-7}
30	6	0.142648×10^{-7}
7	7	0.185505×10^{-4}
8	7	0.509905×10^{-5}
9	7	0.190586×10^{-5}
10	7	0.841714×10^{-6}
11	7	0.415254×10^{-6}
12	7	0.222091×10^{-6}
13	7	0.126416×10^{-6}
14	7	0.756412×10^{-7}
15	7	0.471596×10^{-7}
16	7	0.304374×10^{-7}
17	7	0.202336×10^{-7}
18	7	0.137996×10^{-7}
19	7	0.962516×10^{-8}
20	7	0.684786×10^{-8}
21	7	0.495888×10^{-8}
22	7	0.364839×10^{-8}
23	7	0.272300×10^{-8}
24	7	0.205892×10^{-8}
25	7	0.157538×10^{-8}
26	7	0.121858×10^{-8}
27	7	0.952089×10^{-9}
28	7	0.750771×10^{-9}
29	7	0.597116×10^{-9}
30	7	0.478692×10^{-9}
8	8	0.127476×10^{-5}
9	8	0.326856×10^{-6}
10	8	0.114543×10^{-6}
11	8	0.476332×10^{-7}
12	8	0.222090×10^{-7}
13	8	0.112620×10^{-7}
14	8	0.609526×10^{-8}
15	8	0.347669×10^{-8}
16	8	0.207098×10^{-8}
17	8	0.127969×10^{-8}
18	8	0.815992×10^{-9}
19	8	0.534732×10^{-9}
20	8	0.358927×10^{-9}
21	8	0.246104×10^{-9}
22	8	0.171988×10^{-9}
23	8	0.122266×10^{-9}
24	8	0.862753×10^{-10}
25	8	0.646386×10^{-10}
26	8	0.479451×10^{-10}
27	8	0.359852×10^{-10}
28	8	0.273252×10^{-10}
29	8	0.209268×10^{-10}
30	8	0.161923×10^{-10}
9	9	0.771410×10^{-7}
10	9	0.185815×10^{-7}
11	9	0.614941×10^{-8}
12	9	0.242321×10^{-8}
13	9	0.107378×10^{-8}
14	9	0.518871×10^{-9}

Table F-2
NORMALIZATIONS FACTORS (2 OF 2)

n	m	K_{nm}	n	m	K_{nm}	n	m	K_{nm}
15	9	0.268229 X 10 -9	20	13	0.218162 X 10 -15	28	18	0.274189 X 10 -24
16	9	0.146441 X 10 -9	21	13	0.108376 X 10 -15	29	18	0.134954 X 10 -24
17	9	0.636563 X 10 -10	22	13	0.562193 X 10 -16	30	18	0.686113 X 10 -25
18	9	0.496598 X 10 -10	23	13	0.302813 X 10 -16	19	19	0.386179 X 10 -21
19	9	0.304689 X 10 -10	24	13	0.168586 X 10 -16	20	19	0.634034 X 10 -22
20	9	0.192405 X 10 -10	25	13	0.966519 X 10 -17	21	19	0.145192 X 10 -22
21	9	0.124622 X 10 -10	26	13	0.568850 X 10 -17	22	19	0.401774 X 10 -23
22	9	0.825566 X 10 -11	27	13	0.342828 X 10 -17	23	19	0.126716 X 10 -23
23	9	0.558066 X 10 -11	28	13	0.211099 X 10 -17	24	19	0.441195 X 10 -24
24	9	0.384166 X 10 -11	29	13	0.132559 X 10 -17	25	19	0.166213 X 10 -24
25	9	0.268863 X 10 -11	30	13	0.847506 X 10 -18	26	19	0.668287 X 10 -25
26	9	0.191017 X 10 -11	14	14	0.137925 X 10 -13	27	19	0.263906 X 10 -25
27	9	0.137594 X 10 -11	15	14	0.264805 X 10 -14	28	19	0.126473 X 10 -25
28	9	0.100376 X 10 -11	16	14	0.705439 X 10 -15	29	19	0.587310 X 10 -26
29	9	0.740872 X 10 -12	17	14	0.226003 X 10 -15	30	19	0.282950 X 10 -26
30	9	0.552784 X 10 -12	18	14	0.821551 X 10 -16	20	20	0.100250 X 10 -22
10	10	0.415492 X 10 -8	19	14	0.328317 X 10 -16	21	20	0.160337 X 10 -23
11	10	0.946871 X 10 -9	20	14	0.141415 X 10 -16	22	20	0.357931 X 10 -24
12	10	0.298273 X 10 -9	21	14	0.647661 X 10 -17	23	20	0.966197 X 10 -25
13	10	0.111951 X 10 -9	22	14	0.312330 X 10 -17	24	20	0.297451 X 10 -25
14	10	0.473656 X 10 -10	23	14	0.157426 X 10 -17	25	20	0.101155 X 10 -25
15	10	0.219009 X 10 -10	24	14	0.824581 X 10 -18	26	20	0.372421 X 10 -26
16	10	0.108550 X 10 -10	25	14	0.446772 X 10 -18	27	20	0.146412 X 10 -26
17	10	0.569212 X 10 -11	26	14	0.249459 X 10 -18	28	20	0.608499 X 10 -27
18	10	0.312829 X 10 -11	27	14	0.143094 X 10 -18	29	20	0.265321 X 10 -27
19	10	0.178919 X 10 -11	28	14	0.841039 X 10 -19	30	20	0.120650 X 10 -27
20	10	0.105916 X 10 -11	29	14	0.505379 X 10 -19	21	21	0.247406 X 10 -24
21	10	0.646130 X 10 -12	30	14	0.309876 X 10 -19	22	21	0.385965 X 10 -25
22	10	0.404768 X 10 -12	15	15	0.483469 X 10 -15	23	21	0.840965 X 10 -26
23	10	0.259634 X 10 -12	16	15	0.895907 X 10 -16	24	21	0.221709 X 10 -26
24	10	0.170114 X 10 -12	17	15	0.230664 X 10 -16	25	21	0.666996 X 10 -27
25	10	0.113615 X 10 -12	18	15	0.715068 X 10 -17	26	21	0.221773 X 10 -27
26	10	0.772139 X 10 -13	19	15	0.251811 X 10 -17	27	21	0.798746 X 10 -28
27	10	0.533168 X 10 -13	20	15	0.975843 X 10 -18	28	21	0.307339 X 10 -28
28	10	0.373564 X 10 -13	21	15	0.407966 X 10 -18	29	21	0.125074 X 10 -28
29	10	0.265272 X 10 -13	22	15	0.181539 X 10 -18	30	21	0.534251 X 10 -29
30	10	0.190729 X 10 -13	23	15	0.851266 X 10 -19	22	22	0.581862 X 10 -26
11	11	0.202298 X 10 -9	24	15	0.417539 X 10 -19	23	22	0.866459 X 10 -27
12	11	0.439785 X 10 -10	25	15	0.212988 X 10 -19	24	22	0.168732 X 10 -27
13	11	0.131934 X 10 -10	26	15	0.112464 X 10 -19	25	22	0.486450 X 10 -28
14	11	0.473656 X 10 -11	27	15	0.612386 X 10 -20	26	22	0.143154 X 10 -28
15	11	0.192084 X 10 -11	28	15	0.342783 X 10 -20	27	22	0.465839 X 10 -29
16	11	0.852845 X 10 -12	29	15	0.196717 X 10 -20	28	22	0.164280 X 10 -29
17	11	0.406579 X 10 -12	30	15	0.115485 X 10 -20	29	22	0.619213 X 10 -30
18	11	0.205382 X 10 -12	16	16	0.158376 X 10 -16	30	22	0.246952 X 10 -30
19	11	0.108888 X 10 -12	17	16	0.283926 X 10 -17	23	23	0.130701 X 10 -27
20	11	0.601559 X 10 -13	18	16	0.708031 X 10 -18	24	23	0.194660 X 10 -28
21	11	0.344386 X 10 -13	19	16	0.212817 X 10 -18	25	23	0.405379 X 10 -29
22	11	0.203404 X 10 -13	20	16	0.727351 X 10 -19	26	23	0.102254 X 10 -29
23	11	0.123497 X 10 -13	21	16	0.273823 X 10 -19	27	23	0.294625 X 10 -30
24	11	0.768489 X 10 -14	22	16	0.111309 X 10 -19	28	23	0.939128 X 10 -31
25	11	0.488918 X 10 -14	23	16	0.481930 X 10 -20	29	23	0.324549 X 10 -31
26	11	0.317348 X 10 -14	24	16	0.220064 X 10 -20	30	23	0.119934 X 10 -31
27	11	0.209773 X 10 -14	25	16	0.105187 X 10 -20	24	24	0.260969 X 10 -29
28	11	0.140992 X 10 -14	26	16	0.523230 X 10 -21	25	24	0.409498 X 10 -30
29	11	0.962250 X 10 -15	27	16	0.269589 X 10 -21	26	24	0.834901 X 10 -31
30	11	0.666051 X 10 -15	28	16	0.143325 X 10 -21	27	24	0.206260 X 10 -31
12	12	0.897695 X 10 -11	29	16	0.783748 X 10 -22	28	24	0.582410 X 10 -32
13	12	0.186583 X 10 -11	30	16	0.439644 X 10 -22	29	24	0.182000 X 10 -32
14	12	0.536315 X 10 -12	17	17	0.486934 X 10 -18	30	24	0.616856 X 10 -33
15	12	0.184833 X 10 -12	18	17	0.846249 X 10 -19	25	25	0.579119 X 10 -31
16	12	0.720768 X 10 -13	19	17	0.204783 X 10 -19	26	25	0.826676 X 10 -32
17	12	0.308226 X 10 -13	20	17	0.597882 X 10 -20	27	25	0.165151 X 10 -32
18	12	0.141727 X 10 -13	21	17	0.198654 X 10 -20	28	25	0.400013 X 10 -33
19	12	0.691437 X 10 -14	22	17	0.727639 X 10 -21	29	25	0.110761 X 10 -33
20	12	0.354472 X 10 -14	23	17	0.288008 X 10 -21	30	25	0.339571 X 10 -34
21	12	0.189577 X 10 -14	24	17	0.121510 X 10 -21	26	26	0.114638 X 10 -32
22	12	0.105178 X 10 -14	25	17	0.541028 X 10 -22	27	26	0.160413 X 10 -33
23	12	0.602598 X 10 -15	26	17	0.252324 X 10 -22	28	26	0.314273 X 10 -34
24	12	0.355234 X 10 -15	27	17	0.122540 X 10 -22	29	26	0.746752 X 10 -35
25	12	0.214819 X 10 -15	28	17	0.616775 X 10 -23	30	26	0.202932 X 10 -35
26	12	0.132923 X 10 -15	29	17	0.320501 X 10 -23	27	27	0.218291 X 10 -34
27	12	0.839755 X 10 -16	30	17	0.171390 X 10 -23	28	27	0.299651 X 10 -35
28	12	0.540676 X 10 -16	18	18	0.141041 X 10 -19	29	27	0.576131 X 10 -36
29	12	0.354209 X 10 -16	19	18	0.238056 X 10 -20	30	27	0.134397 X 10 -36
30	12	0.235761 X 10 -16	20	18	0.559966 X 10 -21	28	28	0.400421 X 10 -36
13	13	0.365921 X 10 -12	21	18	0.159049 X 10 -21	29	28	0.539599 X 10 -37
14	13	0.729831 X 10 -13	22	18	0.514521 X 10 -22	30	28	0.101885 X 10 -37
15	13	0.201670 X 10 -13	23	18	0.183628 X 10 -22	29	29	0.708528 X 10 -38
16	13	0.669232 X 10 -14	24	18	0.708659 X 10 -23	30	29	0.937925 X 10 -39
17	13	0.251668 X 10 -14	25	18	0.291705 X 10 -23	30	30	0.121086 X 10 -39
18	13	0.103920 X 10 -14	26	18	0.126798 X 10 -23			
19	13	0.461986 X 10 -15	27	18	0.577661 X 10 -24			

C_{nm}, S_{nm} = dimensionless numerical coefficients

P_n^n $(\sin\phi)$ = associated Legendre function

Variations in form normally occur in the dimensionless numerical coefficients C_{nm} and S_{nm}; occasionally they are found in the form of V. For example, P_n^m $(\cos\theta)$ is often used in place of P_n^m $(\sin\phi)$ where θ represents colatitude. It may also be noted that the notations P_n^m, P_{nm}, and $P_{n,m}$ are all equivalent.

The following list gives the potential forms used by various investigators and the required scaling and variation for reducing them to the international form (foregoing equation F-1).

1. Form Used by Moritz, Cambridge Research Laboratory, and Others

$$U = \frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{a_e}{r}\right)^n \left[J_n P_n (\sin\phi) + \sum_{m=1}^{n} (J_{nm} \cos m\lambda \right.\right.$$
$$\left.\left. + K_{nm} \sin m\lambda) P_n^m (\sin\phi)\right]\right\} \tag{F-2}$$

The important difference between this equation and equation F-1 is the sign ahead of the summation:

$$C_{n,m} = - J_{n,m} \qquad S_{n,m} = - K_{n,m}$$

$$C_{n,o} = - J_n \qquad S_{n,o} = 0$$

2. Form Used by Jeffreys, O'Keefe, and Others

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{1}{r}\right)^{n+1} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, P_n^m (\sin\phi)$$

$$C_{n,m} = A_{n,m} / \left(\mu a_e^n \right)$$

$$S_{n,m} = B_{n,m} / \left(\mu a_e^n \right)$$

3. Form Used by Kozai and Newton and Others

$$U = \frac{\mu}{r} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r} \right)^n J_n^m P_n^m (\sin \phi) \cos m \left(\lambda - \lambda_n^m \right) \right]$$

The conversion equations are

$$C_{n,m} = J_n^m \cos m\lambda_n^m$$

$$S_{n,m} = J_n^m \sin m\lambda_n^m$$

4. Form Used by Mueller and Others

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r} \right)^{n+1} \left(a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right) P_n^m (\sin \phi)$$

The conversion is

$$C_{n,m} = \left(\frac{a_e}{\mu} \right) a_{n,m}$$

$$S_{n,m} = \left(\frac{a_e}{\mu} \right) b_{n,m}$$

5. Special Form for Zonals Only

This form is used only when zonal terms are considered adequate to describe the potential. The form recommended for this case by committee 7 of the International Astronomical Union (Hagihara, ref. 78) is

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{a_e}{r}\right)^n P_n (\sin \phi)\right]$$

This is the same as the equation F-1 without the tesseral and sectorial terms.

$$C_{n,o} = - J_n \qquad \text{and} \qquad S_{n,o} = 0$$

6. Early Form Used by JPL and Others

$$U = \frac{\mu}{r}\left[1 - \frac{2}{3} J \left(\frac{a_e}{r}\right)^2 P_2 (\sin \phi) - \frac{2}{5}\left(\frac{a_e}{r}\right)^3 P_3 (\sin \phi)\right.$$

$$\left. + \frac{8}{35} D \left(\frac{a_e}{r}\right)^4 P_4 (\sin \phi)\right]$$

The conversion relations are

$$C_{2,0} = -\frac{2}{3} J \qquad S_{2,0} = 0$$

$$C_{3,0} = -\frac{2}{5} H \qquad S_{3,0} = 0$$

$$C_{4,0} = \frac{35}{8} D \qquad S_{4,0} = 0$$

The coefficients in front of J, H, and D are actually combined with the Legendre polynomial, which is not explicitly given in the expression for U.

7. Form Used by Sterne, Baker, Herrick, and Others

$$U = \frac{\mu}{r}\left[1 - \frac{2}{3} J\left(\frac{a_e}{r}\right)^2 P_2 (\sin \phi) - \frac{2}{3} H\left(\frac{a_e}{r}\right)^3 P_3 (\sin \phi) + \frac{4}{15} K\left(\frac{a_e}{r}\right)^4 P_4 (\sin \phi)\right]$$

The conversion is

$$C_{2,0} = -\frac{2}{3} J \qquad S_{2,0} = 0$$

$$C_{3,0} = -\frac{2}{5} H \qquad S_{3,0} = 0$$

$$C_{4,0} = \frac{4}{15} K \qquad S_{4,0} = 0$$

Here again the coefficients in front of J, H and K are combined with the Legendre polynomial.

8. Form Used by RCA at Air Force Eastern Test Range

$$U = \frac{\mu}{r}\left[1 - \frac{\alpha}{3r^2} P_2 (\sin \phi) + \frac{\beta}{5r^4} P_4 (\sin \phi)\right]$$

The conversion relations are

$$C_{2,0} = -\alpha/\left(3a_e^2\right) \qquad S_{2,0} = 0$$

$$C_{4,0} = \beta/\left(5a_e^4\right) \qquad S_{4,0} = 0$$

9. Form Used by JHU Applied Physics Laboratory

$$U = \frac{\mu}{r}\left\{1 + \sum_{\ell=2}^{\infty} \sum_{m=0}^{\ell} P_\ell^m (\sin\phi)\left[C_{\ell m} \cos m\lambda + S_{\ell m} \sin m\lambda\right]\right\}$$

APL's unnormalized coefficients are identical to those used in equation F-1 where the degree index n has been replaced by ℓ. The normalization factors used by APL, however, differ from those presented in this appendix.

$$(\bar{C}_{\ell m}, \bar{S}_{\ell m}) = \sqrt{\frac{(\ell + m)!}{(\ell - m)!}}\,(C_{\ell m}, S_{\ell m})$$

APPENDIX G
PROCEDURE FOR ESTIMATING PERTURBATION MAGNITUDES AND TRUNCATING

The selection of the size of an Earth gravity model is influenced by the magnitudes of the perturbations caused by each term. For gravity harmonics of degree n, these will vary with $r^{-(n+1)}$ where r is the geocentric radius. The relative magnitudes of the perturbations caused by terms of degree n are shown graphically in figure G-1, which is an expansion of a similar figure in reference 79. For purposes of comparison, the gravitational effects of the Sun and Moon and the effects of atmospheric drag also are shown. In using the figure, it should be assumed that all geocentric radius vectors point in the direction of the Sun and that the Moon lies between the Earth and the Sun. The curves of drag perturbations assume circular orbital velocity at each altitude.

As long as the perturbations caused by the individual terms of the spherical harmonic series remain small, i.e., conditions of perfect or "deep" resonance are avoided, it is possible to obtain expressions for the perturbations of the orbital elements caused by each V_{nmpq}. Assuming that $\dot{\omega}$, $\dot{M}$ and $\dot{\Omega}$ are all independent of time, Kaula (ref. 26) obtained:

$$\Delta a_{nmpq} = 2F_{nmp} G_{npq} (n - 2p + q) \bar{D}_{nmpq}$$

$$\Delta e_{nmpw} = F_{nmp} G_{npq} \sqrt{1 - e^2} \left[\sqrt{1 - e^2} (n - 2p + q) - (n - 2p) \right] \frac{\bar{D}_{nmpq}}{ae} \qquad \text{(G-1)}$$

$$\Delta i_{nmpq} = F_{nmp} G_{npq} \left[(n - 2p) \cos (i) - m \right] \frac{\bar{D}_{nmpq}}{a\sqrt{1 - e^2} \sin i}$$

where p is the inclination function subscript ($p = 0, 1, 2 \ldots\ldots n$).
q is the eccentricity function subscript ($q = n - 2p$).

Equation G-1 can be used to evaluate the along track, cross track, and normal orbit element perturbations caused by each term of the harmonic gravity potential. A computer program called HAP is available from the Geodynamics Branch, NASA Goddard Space Flight Center. On the basis of equation G-1, HAP provides rapid perturbation estimates for any desired orbit of low or moderate eccentricity.

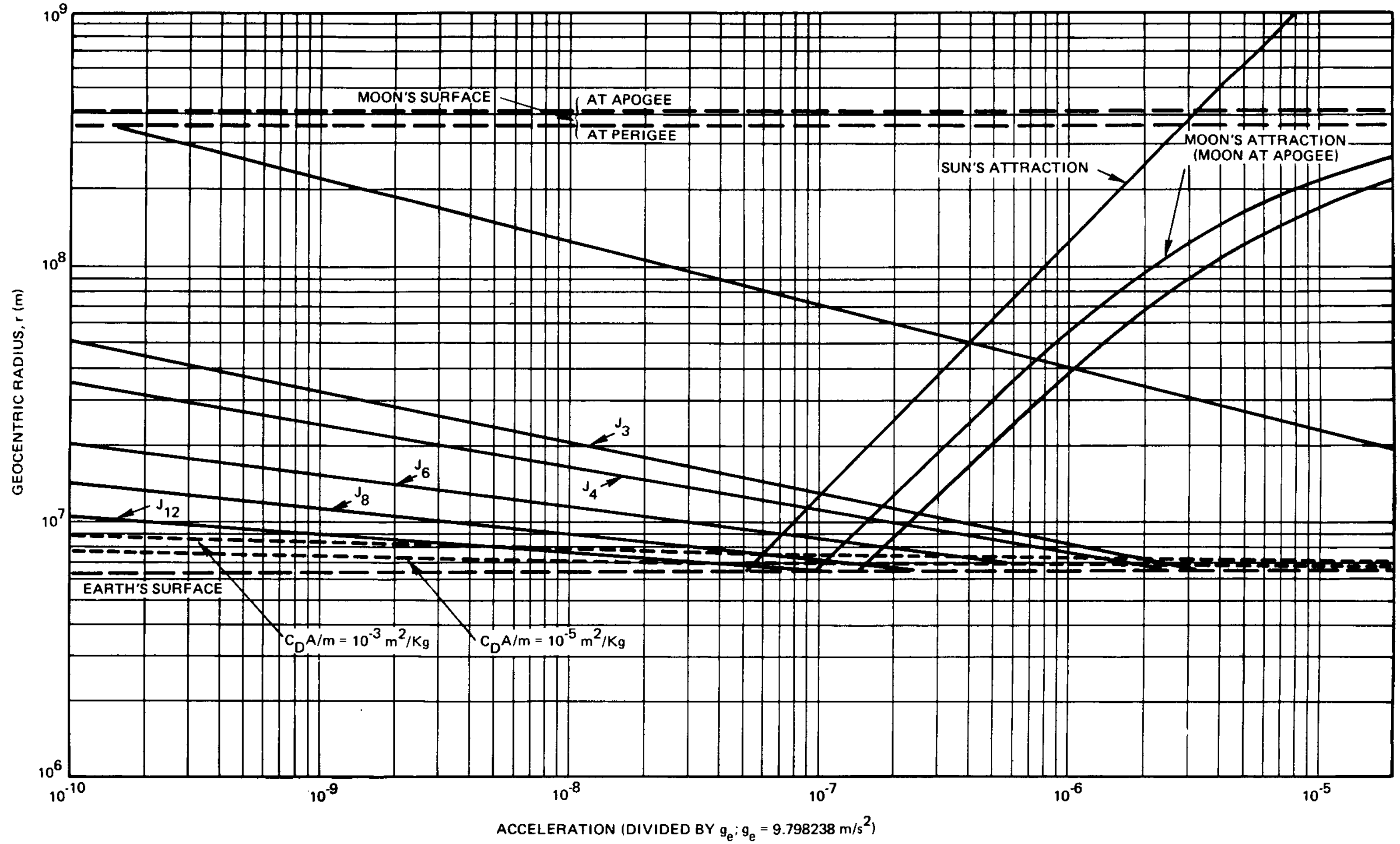

Figure G-1. Magnitudes of Noncentral Force Perturbations in Near-Earth Environment With Earth, Moon, and Sun Assumed Collinear (1 of 2)

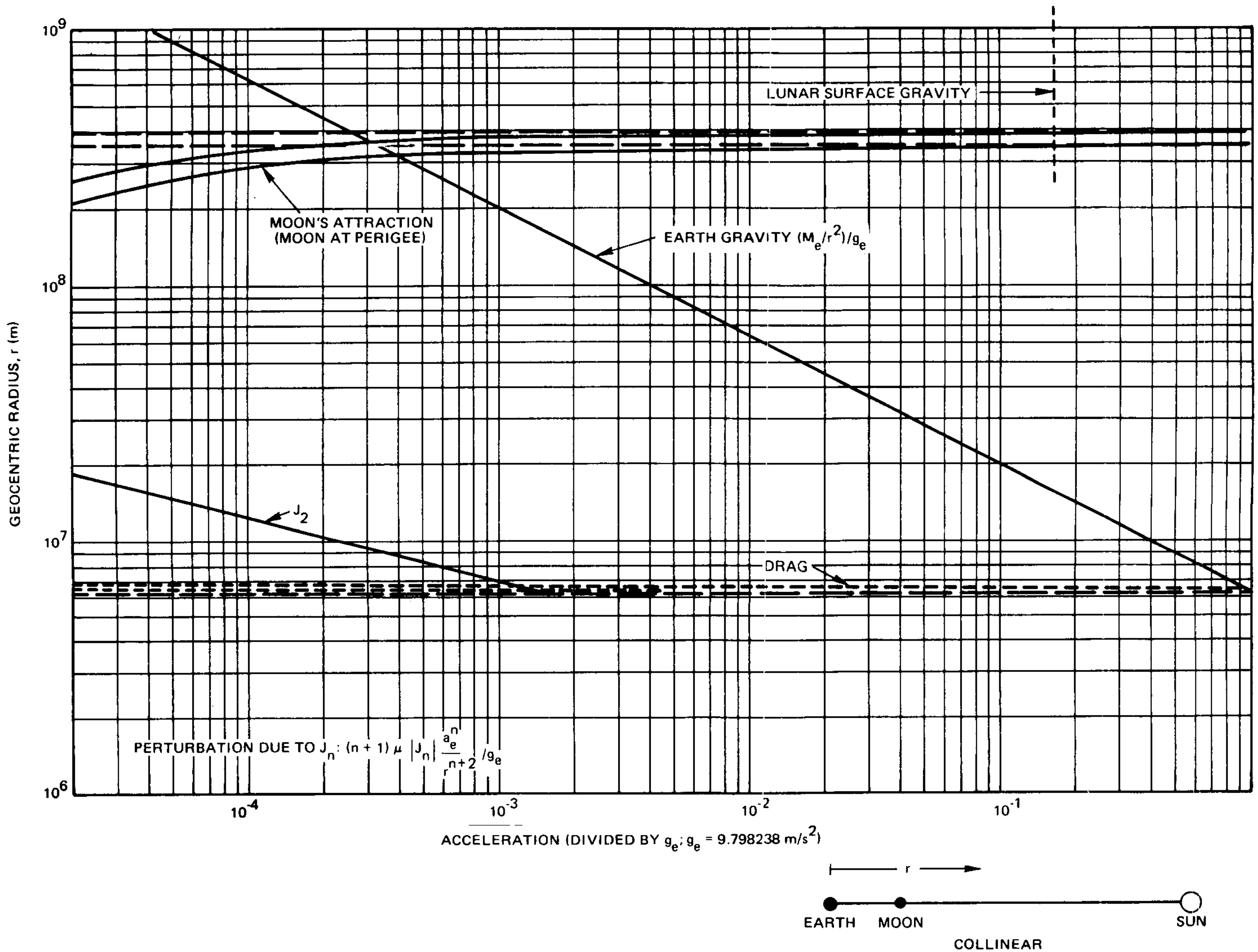

Figure G-1. Magnitudes of Noncentral Force Perturbations in Near-Earth Environment With Earth, Moon, and Sun Assumed Collinear (2 of 2)

NASA SPACE VEHICLE DESIGN CRITERIA MONOGRAPHS

ENVIRONMENT

SP-8005	Solar Electromagnetic Radiation, revised May 1971
SP-8010	Models of Mars' Atmosphere (1974) revised December 1974
SP-8011	Models of Venus Atmosphere (1972), revised September 1972
SP-8013	Meteoroid Environment Model–1969 (Near Earth to Lunar Surface), March 1969
SP-8017	Magnetic Fields–Earth and Extraterrestrial, March 1969
SP-8020	Mars Surface Models (1968), May 1969
SP-8021	Models of Earth's Atmosphere (90 to 2500 km), revised March 1973
SP-8023	Lunar Surface Models, May 1969
SP-8037	Assessment and Control of Spacecraft Magnetic Fields, September 1970
SP-8038	Meteoroid Environment Model–1970 (Interplanetary and Planetary), October 1970
SP-8049	The Earth's Ionosphere, March 1971
SP-8067	Earth Albedo and Emitted Radiation, July 1971
SP-8069	The Planet Jupiter (1970), December 1971
SP-8084	Surface Atmospheric Extremes (Launch and Transportation Areas), revised June 1974
SP-8085	The Planet Mercury (1971), March 1972
SP-8091	The Planet Saturn (1970), June 1972
SP-8092	Assessment and Control of Spacecraft Electromagnetic Interference, June 1972
SP-8103	The Planets Uranus, Neptune, and Pluto (1971), November 1972

SP-8105	Spacecraft Thermal Control, May 1973
SP-8111	Assessment and Control of Electrostatic Charges, May 1974
SP-8116	The Earth's Trapped Radiation Belts, March 1975
SP-8117	Gravity Fields of the Solar System, April 1975
SP-8118	Interplanetary Charged Particle Models (1974), March 1975

STRUCTURES

SP-9011	Buffeting During Atmospheric Ascent, revised November 1970
SP-8002	Flight-Loads Measurements During Launch and Exit, revised June 1972
SP-8003	Flutter, Buzz, and Divergence, July 1964
SP-8004	Panel Flutter, revised June 1972
SP-8006	Local Steady Aerodynamic Loads During Launch and Exit, May 1965
SP-8007	Buckling of Thin-Walled Circular Cylinders, revised August 1968
SP-8008	Prelaunch Ground Wind Loads, November 1965
SP-8009	Propellant Slosh Loads, August 1968
SP-8012	Natural Vibration Modal Analysis, September 1968
SP-8014	Entry Thermal Protection, August 1968
SP-8019	Buckling of Thin-Walled Truncated Cones, September 1968
SP-8022	Staging Loads, February 1969
SP-8029	Aerodynamic and Rocket-Exhaust Heating During Launch and Ascent, May 1969
SP-8031	Slosh Suppression, May 1969
SP-8032	Buckling of Thin-Walled Doubly Curved Shells, August 1969
SP-8035	Wind Loads During Ascent, June 1970
SP-8040	Fracture Control of Metallic Pressure Vessels, May 1970
SP-8042	Meteoroid Damage Assessment, May 1970
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SP-8044	Qualification testing, May 1970
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SP-8055	Prevention of Coupled Structure-Propulsion Instability (Pogo), October 1970
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SP-8057	Structural Design Criteria Applicable to a Space Shuttle, revised March 1972
SP-8060	Compartment Venting, November 1970
SP-8061	Interaction with Umblicals and Launch Stand, August 1970
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GUIDANCE AND CONTROL

SP-8015	Guidance and Navigation for Entry Vehicles, November 1968
SP-8016	Effects of Structural Flexibility on Spacecraft Control Systems, April 1969
SP-8018	Spacecraft Magnetic Torques, March 1969
SP-8024	Spacecraft Gravitational Torques, May 1969
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SP-8096	Space Vehicle Gyroscope Sensor Applications, October 1972
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SP-8102	Space Vehicle Accelerometer Applications, December 1972

CHEMICAL PROPULSION

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SP-8041	Captive-Fired Testing of Solid Rocket Motors, March 1971
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SP-8075	Solid Propellant Processing Factors in Rocket Motor Design, October 1971
SP-8076	Solid Propellant Grain Design and Internal Ballistics, March 1972
SP-8080	Liquid Rocket Pressure Regulators, Relief Valves, Check Valves, Burst Disks, and Explosive Valves, March 1973
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SP-8100	Liquid Rocket Engine Turbopump Gears, March 1974
SP-8101	Liquid Rocket Engine Turbopump Shafts and Couplings, September 1972
SP-8110	Liquid Rocket Engine Turbines, January 1974

NASA-Langley, 1975