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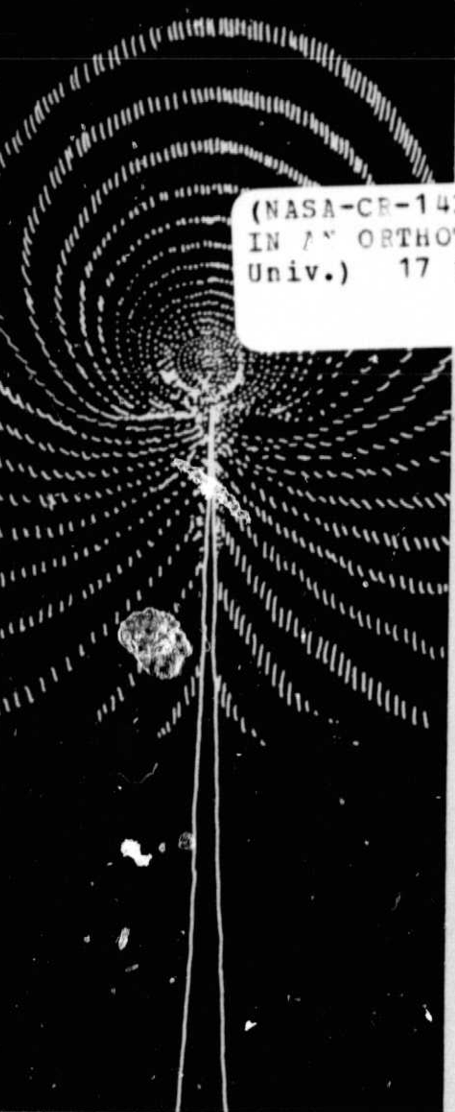

A NOTE ON A BROKEN LAYER IN AN ORTHOTROPIC LAMINATE COMPOSITE

BY

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A NOTE ON A BROKEN LAYER IN AN ORTHOTROPIC LAMINATE COMPOSITE

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ABSTRACT

An orthotropic laminate composite containing a completely broken layer is considered. The problem is formulated in terms of integral transforms and then reduced to a singular integral equation which is solved numerically. The strength of stress singularity at the crack tip is determined from a characteristic equation which is obtained by studying the dominant part of the singular integral equation near the end points. The stress intensity factors are given for various material properties.

1. INTRODUCTION

In a previous work by Arin [1], the problem of an orthotropic laminate composite containing a layer with a crack and bonded to two half-planes of dissimilar materials has been solved and the effect of the material properties as well as various crack sizes on the stress intensity factor at the tip of the crack illustrated. It has been shown that there are basically two types of orthotropic materials. The results were given for two different cases (Materials Type I and II). It was concluded that the decrease in the stress intensity factors was apparent in the case of a matrix of stiffer material. This is meaningful from the point of view of crack arrest.

In this paper, the limiting case for which the crack reaches the interface will be investigated. It can be shown that the Fredholm kernels in [1] become unbounded as the crack touches the interface, hence requiring a separate treatment of the problem. Extracting the additional singularities, the singular integral equation of [1] can be solved in a similar manner. Here, only the material Type I will be considered. The solution for material Type II can be obtained similarly. The notation and certain results of [1] will be used wherever necessary. Both the plane strain and the generalized plane stress conditions are studied simultaneously.

The isotropic counterpart of this problem has been solved by Gupta [2] and Ashbaugh [3].

2. FORMULATION OF THE PROBLEM AND THE SOLUTION

The singular integral equation of the problem of an orthotropic laminate composite containing a layer with a crack of length 2a is given as [1]:

$$\frac{1}{\pi}\int_{-a}^{a}\frac{\phi(t)dt}{t-x}+\int_{-a}^{a}k(x,t)\phi(t)dt=-\frac{(1-\nu_{xy}\nu_{yx})}{2\gamma_{20}E_y}p(x),\quad |x|<a$$

subject to

$$\int_{-a}^{a}\phi(t)dt=0 \tag{2.1}$$

where

$$\phi(t) = \frac{\partial v_1(t,0)}{\partial t} \quad , \quad |t|<a \quad , \quad y = 0 \qquad (2.2)$$

and $v_1(x,y)$ is the displacement component in y direction corresponding to the layer. Here, Ex, Ey are the Young's Moduli; ν_{xy}, ν_{yx} are Poisson's ratios and G_{xy} is the shear modulus. The bielastic constants γ_i are given in Appendix A and [1] and the kernel k(x,t) is given in Appendix B. The input function p(x) is the crack surface pressure and corresponds to the perturbation problem. (2.1) is given for the generalized plane stress. The plane strain case is obtained by simply interchanging certain constants (see [1]).

However, the Fredholm kernel k(x,t) becomes unbounded for a = h which represents a broken laminate. It can easily be shown that in this case, the part of k(x,t) which contributes to further singularities can be expressed as (see Appendix B)

$$k_s(x,t) = \frac{1}{\pi\gamma_{20}} \int_0^\infty [k_1^*(x,\eta) e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_1|}$$

$$+ k_2^*(x,\eta) e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_3|}] d\eta \qquad (2.3)$$

where $k_1^*(x,\eta)$ and $k_2^*(x,\eta)$ are the asymptotic expressions of $k_1(x,\eta)$ and $k_2(x,\eta)$ as $\eta \to \infty$.

Hence, equations (2.1) can now be written as

$$\frac{1}{\pi} \int_{-h}^{h} \frac{\phi(t)dt}{t-x} + \int_{-h}^{h} k_s(x,t)\phi(t)dt + \int_{-h}^{h} k_f(x,t)\phi(t)dt$$

$$= -\frac{(1-\nu_{xy}\nu_{yx})}{2\gamma_{20}Ey} p(x) \quad , \quad |x|<h$$

$$\int_{-h}^{h} \phi(t)dt = 0 \qquad (2.4)$$

where

$$k_f(x,t) = k(x,t) - k_s(x,t) = \frac{1}{\pi\gamma_{20}} \int_0^\infty \{[k_1(x,\eta) - k_1^*(x,\eta)]$$

$$e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_1|} + [k_2(x,\eta) - k_2^*(x,\eta)]e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_3|}$$

$$\} d\eta \qquad (2.5)$$

It is quite easy to show that $k_f(x,t)$ is a bounded kernel and thus can be evaluated numerically from (2.5).

On the other hand, $k_s(x,t)$ can be obtained in closed form. From the Appendix B, the asymptotic expressions of $k_1(x,\eta)$ and $k_2(x,\eta)$ can be given as follows:

$$k_1^*(x,\eta) = \gamma_{51}e^{-|\omega_1|\eta h}\cosh(\omega_1\eta x) + \gamma_{52}e^{-|\omega_3|\eta h}\cosh(\omega_3\eta x)$$

$$k_2^*(x,\eta) = \gamma_{53}e^{-|\omega_1|\eta h}\cosh(\omega_1\eta x) + \gamma_{54}e^{-|\omega_3|\eta h}\cosh(\omega_3\eta x) \qquad (2.6)$$

Hence, after intermediate manipulations we arrive at

$$k_s(x,t) = \frac{1}{2\pi\gamma_{20}} \sum_{j=1}^{3} c_j \left[\frac{1}{t-(a_jh+b_jx)} + \frac{1}{t-(a_jh-b_jx)}\right]$$

$$j = 1,..4 \qquad (2.7)$$

where bielastic constants a_j, b_j and c_j are given in Appendix A.

In dimensionless variables (2.4) becomes

$$\frac{1}{\pi}\int_{-1}^{1} \left\{\frac{1}{\tau-\chi} + \frac{1}{2\gamma_{20}} \sum_{j=1}^{3} c_j\left[\frac{1}{\tau-(a_j+b_j\chi)} + \frac{1}{\tau-(a_j-b_j\chi)}\right]\right\}$$

$$\phi_o(\tau)d\tau + h\int_{-1}^{1} k_f(h\chi,h\tau)\phi_o(\tau)d\tau = g(\chi) \quad , \quad |\chi|<1$$

$$\int_{-1}^{1} \phi_o(\tau)d\tau = 0 \qquad (2.8)$$

where

$$\tau = t/h \quad , \quad \chi = x/h$$

$$\phi_o(\tau) = \phi(t) \quad , \quad g(\chi) = -\frac{(1-\nu_{xy}\nu_{yx})}{2\gamma_{20}E_y} p(h\chi) \quad , \tag{2.9}$$

The solution of (2.8) can be expressed as follows [5]:

$$\phi_o(\tau) = \frac{F(\tau)}{(1-\tau^2)^{\gamma}} = \frac{h^{2\gamma}F(t/h)e^{\pi i\gamma}}{(t-h)^{\gamma}(t+h)^{\gamma}} = \phi(t) \quad , \quad |\tau|<1 \tag{2.10}$$

where the unknown function $F(\tau)$ is Hölder continuous in the interval $-1\leq\tau\leq1$ and $o<Re(\gamma)<1$.

The strength of singularity γ can be determined by investigating the dominant part of (2.8) or (2.4) near the end points (see [5]). Defining the sectionally holomorphic function

$$\Phi(z) = \frac{1}{\pi}\int_{-h}^{h}\frac{\phi(t)dt}{t-z} \tag{2.11}$$

following results can be obtained:

$$\phi(z_1) = (\frac{h}{2})^{\gamma}\cot\pi\gamma\left[\frac{F(-1)}{(h+x)^{\gamma}} - \frac{F(1)}{(h-x)^{\gamma}}\right] + \Phi_1(z_1) \quad , \quad -h < z_1=x < h$$

$$\Phi(z_{2j}) = -(\frac{h}{2b_j})^{\gamma}\frac{F(1)}{\sin\pi\gamma(h+x)^{\gamma}} + \Phi_2(z_{2j}) \quad , \quad h < z_{2j}=a_jh+b_jx < (1+2b_j)h$$

$$\Phi(z_{3j}) = -(\frac{h}{2b_j})^{\gamma}\frac{F(1)}{\sin\pi\gamma(h-x)^{\gamma}} + \Phi_3(z_{3j}) \quad , \quad h < z_{3j}=a_jh-b_jx < (1+2b_j)h \tag{2.12}$$

$$j = 1,2,3$$

where Φ_i, $i = 1,2,3$ are bounded functions everywhere except the end points where they behave as follows:

$$\Phi_1(z_1) < \frac{\Phi_1^*(z_1)}{|z_1\pm h|^{\alpha_1}} \, , \quad \Phi_2(z_{2j}) < \frac{D_1}{|z_{2j}-h|^{\alpha_2}} \quad ,$$

$$\phi_3(z_{3j}) < \frac{D_2}{|z_{3j}-h|^{\alpha_3}} \qquad (2.13)$$

where α_j and D_j are real constants and $\alpha_j < \mathrm{Re}(\gamma)$. Also, $\Phi_1^*(z_1)$ satisfies the Hölder condition near and at the end points.

Substitution from (2.12) into the dominant part of either (2.4) or (2.8) and considering that $F(\tau)$ is an odd function, the following characteristic equation is obtained to determine γ:

$$\cos\pi\gamma + \frac{1}{2\gamma_{20}} \sum_{j=1}^{3} \frac{c_j}{b_j^{\gamma}} = 0 \qquad (2.14)$$

which can be shown to have at least one real root.

After determining γ, (2.8) can be solved using the numerical method given in [4], i.e.

$$\frac{1}{\pi} \sum_{j=1}^{N} F(\tau_j) W_j \left[\frac{1}{\tau_j - \chi_i} + \pi h k_s(h\chi_i, h\tau_j) + \pi h k_f(h\chi_i, h\tau_j)\right] = g(\chi_i) \quad i = 1,..,N-1$$

$$\sum_{j=1}^{N} W_j F(\tau_j) = 0 \qquad (2.15)$$

where

$$P_N^{(-\gamma,-\gamma)}(\tau_j) = 0 \qquad j = 1,..,N$$

$$P_{N-1}^{(1-\gamma,1-\gamma)}(\chi_i) = 0 \qquad i = 1,..,N-1 \qquad (2.16)$$

and the W_j are the corresponding weights. From (2.15) N unknowns $F(\tau_j)$, $j = 1,..,N$ can be determined.

3. THE STRESS INTENSITY FACTOR

The stress intensity factor at the crack tip will be defined as follows [$\tau_{xy2}(x,0) = 0$]:

$$K = \lim_{\substack{x\to h \\ x>h}} [2(x-h)]^{\gamma} \sigma_{y2}(x,0) \tag{3.1}$$

where $\sigma_{y2}(x,y)$ corresponds to the matrix. $\sigma_{y2}(x,0)$ can be obtained in terms of $\phi(t)$ starting from the expressions given in [1]. Hence, after intermediate steps we obtain,

$$\frac{\pi\gamma_{19}(1-\nu^*_{xy}\nu^*_{yx})}{E^*_y} \sigma_{y2}(x,0) = \int_{-h}^{h} \phi(t)dt \int_0^\infty \frac{d\eta}{f(\eta)}$$

$$\{e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_1|}[\gamma^*_{16}f_{13}(\eta)e^{-|\omega^*_3|(x-h)\eta} + \frac{\gamma^*_{15}}{\gamma_{21}} f_{17}(\eta)e^{-|\omega^*_1|(x-h)\eta}]$$

$$+ e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_3|}[\gamma^*_{16}f_{14}(\eta)e^{-|\omega^*_3|(x-h)\eta} + \frac{\gamma^*_{15}}{\gamma_{21}} f_{18}(\eta)$$

$$e^{-|\omega^*_1|(x-h)\eta}]\} \tag{3.2}$$

where

$$f_{11}(\eta) = \frac{\beta_7}{\gamma_{21}} + \gamma_{22} \tanh(\omega_1\eta h)$$

$$f_{12}(\eta) = \frac{\beta_8}{\gamma_{21}} + \gamma_{22} \tanh(\omega_3\eta h)$$

$$f_{13}(\eta) = \gamma_{53}f(\eta) + f_7(\eta)f_{11}(\eta) + f_8(\eta)f_{12}(\eta)$$

$$f_{14}(\eta) = \gamma_{54}f(\eta) + f_9(\eta)f_{11}(\eta) + f_{10}(\eta)f_{12}(\eta) \tag{3.3}$$

$$f_{15}(\eta) = \beta_7 + \beta^*_8 \operatorname{sign}(\omega^*_3)\cdot\tanh(\omega_1\eta h)$$

$$f_{16}(\eta) = \beta_8 + \beta^*_8 \operatorname{sign}(\omega^*_3) \tanh(\omega_3\eta h)$$

$$f_{17}(\eta) = \gamma_{51}f(\eta) - f_7(\eta)f_{15}(\eta) - f_8(\eta)f_{16}(\eta)$$

$$f_{18}(\eta) = \gamma_{52}f(\eta) - f_9(\eta)f_{15}(\eta) - f_{10}(\eta)f_{16}(\eta) \tag{3.4}$$

However, the right hand side of (3.2) becomes unbounded near the tip of the crack. Hence, the dominant part which contributes to the singular behavior can be extracted in a manner similar to the procedure used to obtain (2.4). Thus, $\sigma_{y2}(x,0)$ can be expressed as:

$$\frac{(1-\nu^*_{xy}\nu^*_{yx})}{E^*_y}\,\sigma_{y2}(x,0) = \sum_{j=1}^{4} \frac{c^*_j}{\pi} \int_{-h}^{h} \frac{\phi(t)dt}{t-(a^*_j h+b^*_j x)} + \sigma^o_{y2}(x,0) \tag{3.5}$$

where $\sigma^o_{y2}(x,0)$ is a bounded function.

To determine the stress intensity factor K, the behavior of the Cauchy integral in (3.5) near the end points has to be investigated. Following the method given by Muskhelishvili [5], the sectionally holomorphic function defined in (2.11) can be expressed near x = h as follows ($h<x<\infty$):

$$\Phi(z_{4j}) = -\left(\frac{h}{2b^*_j}\right)^{\gamma} \frac{F(1)}{\sin\pi\gamma\,(x-h)^{\gamma}} + \Phi_o(z_{4j}) \quad ,$$

$$h < z_{4j}=a^*_j h+b^*_j x < \infty \tag{3.6}$$

Here, $\Phi_o(z_{4j})$ is bounded everywhere except the end points, and near x = h

$$|\Phi_o(z_{4j})| < \frac{C}{|z_{4j}-h|^{\alpha_o}} \quad , \quad \alpha_o<\mathrm{Re}(\gamma)$$

where C and α_o are real constants. Hence from (3.1), (3.5) and (3.6) we obtain

$$K = -\frac{h^{\gamma}F(1)E_y^*}{(1-\nu_{xy}^*\nu_{yx}^*)\sin\pi\gamma} \sum_{j=1}^{4} \frac{c_j^*}{(b_j^*)^{\gamma}} \quad \text{, generalized plane stress} \tag{3.7}$$

For plane strain $\frac{1-\nu_{xy}^*\nu_{yx}^*}{E_y^*} = E_x^*\,\Delta^*$ should be replaced by $1/A_{22}^*$ (see [1]).

4. NUMERICAL RESULTS AND CONCLUSION

The numerical results will be given for $p(x) = p_o$ = constant which corresponds to uniform crack surface pressure. $K/p_o h^{\gamma}$ values as well as γ will be given for different material combinations.

The following materials will be selected:

(A) Boron-Epoxy:

$E_x = 3.5 \times 10^6$psi , $E_y = 3.24 \times 10^7$psi
$\nu_{yx} = 0.23$, $G_{xy} = 1.23 \times 10^6$psi
and for the plane strain
$E_z = 3.5 \times 10^6$psi , $\nu_{zx} = \nu_{zy} = 0.23$

(B) Boron-Epoxy:

$E_x = 2.72 \times 10^7$psi, $E_y = 5.5 \times 10^6$psi
$\nu_{yx} = 0.22$, $G_{xy} = 7.0 \times 10^5$psi
and for the plane strain
$E_z = 2.72 \times 10^7$psi $\nu_{zx} = \nu_{zy} = 0.22$

(C) Boron-Epoxy:

$E_x = 5.5 \times 10^6$psi , $E_y = 2.72 \times 10^7$psi
$\nu_{yx} = 0.1084$, $G_{xy} = 7.0 \times 10^5$psi
and for the plane strain
$E_z = 5.5 \times 10^6$psi , $\nu_{zx} = \nu_{zy} = 0.1084$

(D) Glass-Fiber (20% volume fraction):

$E_x = 6.6 \times 10^5$psi , $E_y = 2.52 \times 10^6$psi
$\nu_{yx} = 0.32$, $G_{xy} = 2.9 \times 10^5$psi
and for the plane strain
$E_z = 6.6 \times 10^5$psi , $\nu_{zx} = \nu_{zy} = 0.32$

TABLE 1

$K/p_o h^\gamma$ and γ for various material combinations.

Material		Generalized Plane Stress				Plane Strain			
Matrix	Layer	γ	$K/p_o h^\gamma$			γ	$K/p_o h^\gamma$		
			1	2	3		1	2	3
A	D	0.2798	4.694	4.740	5.110	0.3384	2.946	2.970	3.198
D	A	0.7500	0.350	0.347	0.285	0.7626	0.344	0.342	0.280
C	B	0.3761	2.755	2.764	2.848	0.3704	2.910	2.920	3.009
B	C	0.623[illegible]	0.473	0.472	0.451	0.6290	0.460	0.459	0.437
C	D	0.2972	4.159	4.196	4.496	0.3785	1.985	1.998	2.120
D	C	0.7208	0.362	0.360	0.316	0.6875	0.448	0.446	0.400

Both γ and $K/p_o h^\gamma$ are given in Table 1 for various material combinations. As seen from the equation (3.7), the stress intensity factor is determined in terms of F(1). Since the unknowns obtained from (2.15) are $F(\tau_j)$ (j = 1,..,N), F(1) is found by an extrapolation. Hence, three different values for $K/p_o h^\gamma$ are given in Table 1 to illustrate the effect of extrapolation. Columns 1, 2 and 3 are obtained by extrapolating $F(\tau_j)$ (j = 1,..,N), (j = 1,2,3) and (j = 2,3,4) respectively. From these results, it appears that $F(\tau_1)$ (as well as $F(\tau_N)$) has a significant effect on the stress intensity factor. It is also observed that the γ and the $K/p_o h^\gamma$ values are quite sensitive to the material properties.

ACKNOWLEDGEMENT

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For Material Type I:

Dimensionless bielastic constants γ_i (i = 46,70):
(see reference [1] for the constants $\gamma_1 - \gamma_{45}$).

$$\gamma_{46} = [\gamma_{27} + \gamma_{26}\ \mathrm{sign}(\omega_1)][\gamma_{25} + \gamma_{24}\ \mathrm{sign}(\omega_3)]$$

$$- [\gamma_{29} + \gamma_{28}\ \mathrm{sign}(\omega_3)][\gamma_{23} + \gamma_{24}\ \mathrm{sign}(\omega_1)]$$

$$\gamma_{47} = \gamma_{38} + \gamma_{39}\ \mathrm{sign}(\omega_3)$$

$$\gamma_{48} = \gamma_{40} + \gamma_{41}\ \mathrm{sign}(\omega_1)$$

$$\gamma_{49} = \gamma_{42} + \gamma_{43}\ \mathrm{sign}(\omega_3)$$

$$\gamma_{50} = \gamma_{44} + \gamma_{45}\ \mathrm{sign}(\omega_1)$$

$$\gamma_{51} = \gamma_7\ \gamma_{47}/(\gamma_{19}\cdot\gamma_{46})$$

$$\gamma_{52} = \gamma_8\ \gamma_{48}/(\gamma_{19}\gamma_{46})$$

$$\gamma_{53} = \gamma_7\ \gamma_{49}/(\gamma_{19}\gamma_{46})$$

$$\gamma_{54} = \gamma_8\ \gamma_{50}/(\gamma_{19}\gamma_{46})$$

$$\gamma_{55} = \beta_9\ \mathrm{sign}(\omega_1) - \beta_8^*\ \mathrm{sign}(\omega_3^*)$$

$$\gamma_{56} = \frac{\gamma_{11}}{\gamma_{12}}\ [\beta_8^*\ \mathrm{sign}(\omega_3^*) - \beta_{10}\ \mathrm{sign}(\omega_3)]$$

$$\gamma_{57} = \gamma_{22} - \frac{1}{\gamma_{21}}\ \beta_9\ \mathrm{sign}(\omega_1)$$

$$\gamma_{58} = \frac{\gamma_{11}}{\gamma_{12}}\ [-\ \gamma_{22} + \frac{1}{\gamma_{21}}\ \beta_{10}\ \mathrm{sign}(\omega_3)]$$

$$\gamma_{59} = \frac{\beta_7}{\gamma_{21}} + \gamma_{22}\ \mathrm{sign}(\omega_1)$$

$$\gamma_{60} = \frac{\beta_8}{\gamma_{21}} + \gamma_{22}\ \mathrm{sign}(\omega_3)$$

$$\gamma_{61} = \beta_7 + \beta_8^*\ \mathrm{sign}(\omega_3^*)\ \mathrm{sign}(\omega_1)$$

$$\gamma_{62} = \beta_8 + \beta_8^*\ \mathrm{sign}(\omega_3^*)\ \mathrm{sign}(\omega_3)$$

$$\gamma_{63} = \gamma_{57}\gamma_{46} + \gamma_{47}\gamma_{59} + \gamma_{48}\gamma_{60}$$

$$\gamma_{64} = \gamma_{58}\gamma_{46} + \gamma_{49}\gamma_{59} + \gamma_{50}\gamma_{60}$$

$$\gamma_{65} = \gamma_{55}\gamma_{46} - \gamma_{61}\gamma_{47} - \gamma_{62}\gamma_{48}$$

$$\gamma_{66} = \gamma_{56}\gamma_{46} - \gamma_{61}\gamma_{49} - \gamma_{62}\gamma_{50}$$

$$\gamma_{67} = \gamma_{16}^*\gamma_{63}/(\gamma_{46}\gamma_{19})$$

$$\gamma_{68} = \gamma_{65}\cdot\gamma_{15}^*/(\gamma_{46}\gamma_{19}\gamma_{21})$$

$$\gamma_{69} = \gamma_{16}^*\gamma_{64}/(\gamma_{46}\gamma_{19})$$

$$\gamma_{70} = \gamma_{66}\gamma_{15}^*/(\gamma_{46}\gamma_{19}\gamma_{21})$$

Bielastic constants c_i, b_i, and a_i $(i = 1,3)$:

$$c_1 = -\frac{|\omega_1|}{\sqrt{\beta_5}}\gamma_{51} \quad , \quad b_1 = \frac{\omega_1{}^2}{\sqrt{\beta_5}}$$

$$c_2 = -\frac{|\omega_1|\gamma_{52}+|\omega_3|\gamma_{53}}{\sqrt{\beta_5}} \quad , \quad b_2 = \frac{|\omega_1||\omega_3|}{\sqrt{\beta_5}}$$

$$c_3 = -\frac{|\omega_3|}{\sqrt{\beta_5}}\gamma_{54} \quad , \quad b_3 = \frac{\omega_3{}^2}{\sqrt{\beta_5}}$$

$$a_i = 1 + \rho_i$$

Bielastic constants c_i^*, b_i^*, a_i^* $(i = 1,4)$:

$$c_1^* = -\frac{|\omega_1|}{\sqrt{\beta_5}}\gamma_{67} \quad , \quad b_1^* = \frac{|\omega_1||\omega_3^*|}{\sqrt{\beta_5}}$$

$$c_2^* = -\frac{|\omega_1|}{\sqrt{\beta_5}}\gamma_{68} \quad , \quad b_2^* = \frac{|\omega_1||\omega_1^*|}{\sqrt{\beta_5}}$$

$$c_3^* = -\frac{|\omega_3|}{\sqrt{\beta_5}}\gamma_{69} \quad , \quad b_3^* = \frac{|\omega_3||\omega_3^*|}{\sqrt{\beta_5}}$$

$$c_4^* = -\frac{|\omega_3|}{\sqrt{\beta_5}}\gamma_{70} \quad , \quad b_4^* = \frac{|\omega_3||\omega_1^*|}{\sqrt{\beta_5}}$$

$$a_i^* = 1 - b_i^*$$

APPENDIX B

Fredholm kernel (Bielastic constants γ_i are defined in [1]):

$$k(x,t) = \frac{1}{\pi\gamma_{20}} \int_0^\infty [k_1(x,\eta) e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_1|}$$

$$+ k_2(x,\eta) e^{-(h-t)\eta\sqrt{\beta_5}/|\omega_3|}] \, d\eta$$

$$k_1(x,\eta) = \frac{1}{2\gamma_{19} f(\eta)} [\gamma_7 f_5(x,\eta) f_7(\eta) + \gamma_8 f_6(x,\eta) f_8(\eta)]$$

$$k_2(x,\eta) = \frac{1}{2\gamma_{19} f(\eta)} [\gamma_7 f_5(x,\eta) f_9(\eta) + \gamma_8 f_6(x,\eta) f_{10}(\eta)]$$

$$f(\eta) = f_3(\eta) f_1(\eta) - f_2(\eta) f_4(\eta)$$

$$f_1(\eta) = \gamma_{25} + \gamma_{24} \tanh(\omega_3 \eta h)$$

$$f_2(\eta) = \gamma_{29} + \gamma_{28} \tanh(\omega_3 \eta h)$$

$$f_3(\eta) = \gamma_{27} + \gamma_{26} \tanh(\omega_1 \eta h)$$

$$f_4(\eta) = \gamma_{23} + \gamma_{24} \tanh(\omega_1 \eta h)$$

$$f_5(\eta, x) = \cosh(\omega_1 \eta x)/\cosh(\omega_1 \eta h)$$

$$f_6(\eta, x) = \cosh(\omega_3 \eta x)/\cosh(\omega_3 \eta h)$$

$$f_7(\eta) = \gamma_{38} + \gamma_{39} \tanh(\omega_3 \eta h)$$

$$f_8(\eta) = \gamma_{40} + \gamma_{41} \tanh(\omega_1 \eta h)$$

$$f_9(\eta) = \gamma_{42} + \gamma_{43} \tanh(\omega_3 \eta h)$$

$$f_{10}(\eta) = \gamma_{44} + \gamma_{45} \tanh(\omega_1 \eta h)$$

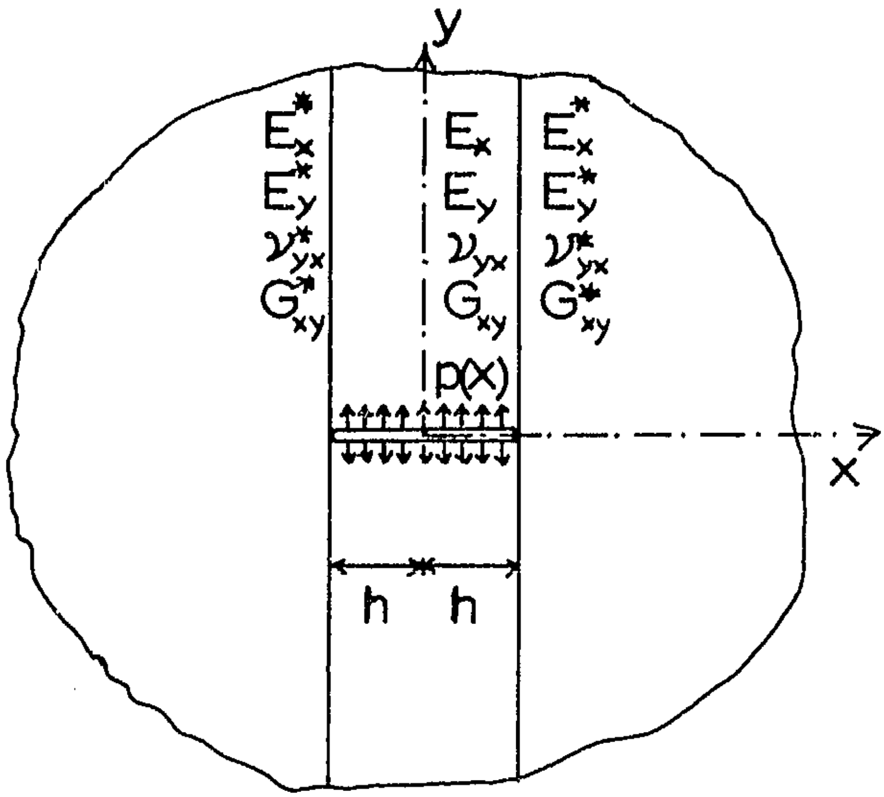

Figure 1. Geometry of the problem.