ERRORS IN HADAMARD SPECTROSCOPY OR IMAGING CAUSED BY IMPERFECT MASKS

Ming Hing Tai and Martin Harwit
Center for Radiophysics and Space Research
Cornell University
Ithaca, New York

and

Neil J. A. Sloane
Bell Laboratories
Murray Hill, New Jersey

ABSTRACT

An analysis is given of the errors in Hadamard spectroscopy that are caused by transparent slits in the mask being systematically wider or else narrower than they should be. It is shown that if the input spectrum consists of a single line, the distorted spectrum that is actually calculated consists of this line, plus four small blips. When the transparent slits are too wide, these blips are of equal height and the same sign, one pair surrounding the line, and another pair displaced a certain distance from it. When the slits are too narrow, the displaced blips have the same amplitude but are negative.

The response to an arbitrary input spectrum is then determined from this. The same method of analysis may also be used to handle other types of errors.

(NASA-CR-142932) ERRORS IN HADAMARD SPECTROSCOPY OR IMAGING CAUSED BY IMPERFECT MASKS (Cornell Univ.) 37 p HC $3.75 CSCL 14B
N75-25128
Unclas
G3/35 24161

ERRORS IN HADAMARD SPECTROSCOPY OR IMAGING
CAUSED BY IMPERFECT MASKS

Introduction

In the course of running a series of laboratory calibration spectra with a mercury vapor source at 1.7 microns, we observed a consistently occurring pair of negative dips, displaced a fixed number of spectral element positions to one side of the mercury vapor doublet Fig. 1(a). These looked like absorption features except that the intensity at these positions of our derived spectrum was actually negative rather than just zero.

On further analysis, we found that moving the grating shifted both the spectral lines and these negative dips by equal amounts. However, when the spectral lines were moved to the extreme left, the two dips reappeared at the extreme right of the spectrum. This immediately suggested that we were dealing with a coding rather than with an optical error. The encoding mask has a cyclic code and a coding error could therefore be simply cyclic in this fashion, whereas stray light effects, for example, would not be as likely to produce a cycling error.

A further piece of evidence, showing that light intensity did play an important role, was obtained when the mercury vapor doublet was moved so far to the left edge of the spectrum that one of the lines no longer was transmitted through the spectral mask. Not only did that component of the doublet then disappear from the spectrum, but the corresponding negative dip also vanished. From this we concluded that the error was dependent both on the light intensity and on our encoding scheme.

This still left a number of possibilities open. After a number of attempts to find the correct explanation, we finally thought of the following possibility. During the manufacture of the masks, whether by deposition of metal, or by removal of metal through an etching process, it is possible to obtain a systematic error that leaves each of the opaque portions of the mask either too wide or too narrow by a fixed amount. Essentially there is too much metal deposited, or too much metal etched away at each edge separating an open or a closed slit, and each of these edges is therefore displaced by a fixed amount, independent of whether the open slit is wide or narrow. For example, if the open slits are too narrow, then the light passing through an open slit position may have intensity values I_o, $I_o(1-\epsilon)$ or $I_o(1-2\epsilon)$ depending on whether this particular open position is bounded by two other open slits, has a closed slit bounding it on one side only, or bounds on closed slits on both sides. A computer simulation of this type of defect showed, in fact, that we had hit on the correct solution. Fig. 1(b) shows the computer response when provided with synthetic data expected for a single spectral line and a perfectly constructed 255-element S-matrix encoded mask[1]. Fig. 1(c) shows the same output when the computer is provided with synthetic data corresponding to a mask in which each transmitting slot is too small by 0.1 slit widths at each edge. Fig. 1(d) shows the output for a similarly defective mask when the open slits are too wide by the same amount.

While the source of the spectral errors had thus been located, we felt the need for a more general treatment of this class of defect, since the displacement of the spectral dips presumably could be quite different for differing encoding masks. We also wished to know what would happen if the transmitting slits were systematically too wide rather than too narrow. Finally, since image encoding masks can make use of the same codes used in spectroscopy, we felt that a general mathematical analysis would also be germane to a wider range of optical problems outside of spectroscopy.

The remaining sections of this paper therefore deal with this analysis. The results are remarkably simple. Independent of the particular S-matrix mask to be used there are always precisely four false blips in the response to any single element impulse. The amplitude of these blips is always the same for a fixed narrowing or widening of the transmitting slits. Two of the blips always surround the main impulse and a pair of adjacent blips always is some distance removed from the impulse. We show how to compute the distance of this displacement. The amplitude of the displaced blips is positive when the transparent slits are too wide and is negative when the slits are too narrow. In contrast, the two blips surrounding the main impulse always are positive.

(I) Mathematical analysis if no errors occur in the mask

We shall assume that there are $n = 2^m - 1$ unknowns. Let

$x_0, x_1, \ldots, x_{n-1}$ denote the unknowns,

$y_0, y_1, \ldots, y_{n-1}$ denote the measurements,

$z_0, z_1, \ldots, z_{n-1}$ denote the estimates of the unknowns.

Also we write $\underline{x} = (x_0, \ldots, x_{n-1})^T$, $\underline{y} = (y_0, \ldots, y_{n-1})^T$, $\underline{z} = (z_0, \ldots, z_{n-1})^T$, where the T denotes transpose. These quantities are related as follows. First, $\underline{y}$ is a Hadamard transform of $\underline{x}$, given by

$$\underline{y} = S\underline{x} \qquad (1)$$

where S is a circulant matrix whose general definition is given below in (III). As an example, if $n = 15$, a typical matrix S is shown in Fig. 2. The blank entries are zero.

Fig. 2

An Example of the Matrix S When n = 15

S =

			1			1	1		1		1	1	1	1	v_1
		1			1	1		1		1	1	1	1		v_2
	1			1	1		1		1	1	1	1			v_3
1			1	1		1		1	1	1	1				$v_1 \oplus v_4$
		1	1		1		1	1	1	1				1	$v_1 \oplus v_2$
	1	1		1		1	1	1	1				1		$v_2 \oplus v_3$
1	1		1		1	1	1	1				1			$v_1 \oplus v_3 \oplus v_4$
1		1		1	1	1	1				1			1	$v_2 \oplus v_4$
	1		1	1	1	1				1			1	1	$v_1 \oplus v_3$
1		1	1	1	1				1			1	1		$v_1 \oplus v_2 \oplus v_4$
	1	1	1	1				1			1	1		1	$v_1 \oplus v_2 \oplus v_3$
1	1	1	1				1			1	1		1		$v_1 \oplus v_2 \oplus v_3 \oplus v_4$
1	1	1				1			1	1		1		1	$v_2 \oplus v_3 \oplus v_4$
1	1				1			1	1		1		1	1	$v_3 \oplus v_4$
1				1			1	1		1		1	1	1	v_4

Equation (1) assumes that there is no detector error, and that the mask is made correctly. Second, $\underline{z}$ is the inverse Hadamard transform of $\underline{y}$, given by

$$\underline{z} = S^{-1}\underline{y}.$$

Of course in this case

$$\underline{z} = S^{-1}S\underline{x} = \underline{x},$$

and the unknowns have been determined exactly.

(II) Errors caused when the transmitting slits are too wide

Now suppose that the transparent slits in the mask (which correspond to the 1's in S) are slightly larger than they should be. Thus some radiation manages to get through slits which should be opaque, if they are adjacent to transparent slits. More precisely, consider an arbitrary row of S, say

$$\ldots\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ \ldots\ .$$

We assume that

every 0 which is adjacent to two 0's is unchanged,

every 0 which is adjacent to a single 1 is replaced by ϵ, &

every 0 which is adjacent to two 1's is replaced by 2ϵ.

Here ϵ is a small positive number which measures the fraction of radiation which manages to get through a slit which should be opaque and which is adjacent to exactly one transparent slit. Thus the preceding row would be changed to

$$\ldots\ 1\ 1\ \epsilon\ \epsilon\ 1\ 2\epsilon\ 1\ 1\ \epsilon\ 0\ 0\ \epsilon\ 1\ 1\ \ldots\ .$$

Let S* denote the distorted matrix obtained from S in this way. Figure 3 shows the distorted matrix corresponding to Fig. 2.

Fig. 3

The Distorted Matrix S^* When $n = 15$

$$
S^* = \left[\begin{array}{ccccccccccccccc}
\epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 \\
0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon \\
\epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 \\
1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon \\
\epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 \\
\epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon \\
1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon \\
1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 \\
2\epsilon & 1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 \\
1 & 2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon \\
2\epsilon & 1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 \\
1 & 1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon \\
1 & 1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 \\
1 & 1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 \\
1 & \epsilon & 0 & \epsilon & 1 & \epsilon & \epsilon & 1 & 1 & 2\epsilon & 1 & 2\epsilon & 1 & 1 & 1
\end{array}\right]
$$

Now we actually measure (instead of Eq. (1))

$$\underline{y}^* = S^*\underline{x} \qquad (2)$$

and the estimate of the unknowns is

$$\underline{z}^* = S^{-1}\underline{y}^* = S^{-1}S^*\underline{x}.$$

We wish to discover how $\underline{z}^*$ differs from $\underline{x}$.

In fact it is enough to determine $\underline{z}^*$ when $\underline{x}$ has a single nonzero component. For suppose the input

$$\underline{x} = \underline{x}^{(1)} = (1,0,0,\ldots,0)^T$$

produces the response

$$\underline{z}^* = S^{-1}S^*\underline{x}^{(1)} = \underline{z}^{(1)} \quad \text{(say)}. \tag{3}$$

We call $\underline{z}^{(1)}$ the <u>impulse</u> <u>response</u>. Then since every row of S (and S*) is a cyclic shift to the left of the previous row, it follows that

$$\underline{x} = (0,1,0,\ldots,0)^T$$

produces the response

$$\underline{z}^* = \mathcal{S}\underline{z}^{(1)},$$

and

$$\underline{x} = (0,0,1,\ldots,0)^T$$

produces

$$\underline{z}^* = \mathcal{S}^2\underline{z}^{(1)},$$

and so on. Here $\mathcal{S}$ denotes a cyclic shift of one place to the left. Then an arbitrary input

$$\underline{x} = (x_0, x_1, \ldots, x_{n-1})^T \tag{4}$$

produces the response

$$\underline{z}^* = x_0\underline{z}^{(1)} + x_1\mathcal{S}\underline{z}^{(1)} + \ldots + x_{n-1}\mathcal{S}^{n-1}\underline{z}^{(1)}. \tag{5}$$

Therefore, once the impulse response $\underline{z}^{(1)}$ is known, the response to an arbitrary input (4) is determined (by (5)).

Before giving the analysis of the general case, let us calculate $\underline{z}^{(1)}$ for the case n = 15. Figure 4 shows S^{-1}, where - stands for - 1.

Fig. 4

The Matrix S^{-1} When n = 15

$S^{-1} = \frac{1}{8}$

-	-	-	1	-	-	1	1	-	1	-	1	1	1	1
-	-	1	-	-	1	1	-	1	-	1	1	1	1	-
-	1	-	-	1	1	-	1	-	1	1	1	1	-	-
1	-	-	1	1	-	1	-	1	1	1	1	-	-	-
-	-	1	1	-	1	-	1	1	1	1	-	-	-	1
-	1	1	-	1	-	1	1	1	1	-	-	-	1	-
1	1	-	1	-	1	1	1	1	-	-	-	1	-	-
1	-	1	-	1	1	1	1	-	-	-	1	-	-	1
-	1	-	1	1	1	1	-	-	-	1	-	-	1	1
1	-	1	1	1	1	-	-	-	1	-	-	1	1	-
-	1	1	1	1	-	-	-	1	-	-	1	1	-	1
1	1	1	1	-	-	-	1	-	-	1	1	-	1	-
1	1	1	-	-	-	1	-	-	1	1	-	1	-	1
1	1	-	-	-	1	-	-	1	1	-	1	-	1	1
1	-	-	-	1	-	-	1	1	-	1	-	1	1	1

Then $\underline{z}^{(1)}$ is found from Eq. (3) to be

$$\underline{z}^{(1)} = (1-\epsilon, \tfrac{1}{2}\epsilon, 0, \tfrac{1}{2}\epsilon, \tfrac{1}{2}\epsilon, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tfrac{1}{2}\epsilon). \qquad (6)$$

(III) The matrix S

The theory of the matrix S is intimately connected with that of finite fields. However, we shall give a simple, self-contained description in several steps which assumes no previous knowledge of finite fields. The definition of S given here is in fact the same as that given in references 1, 2 and 3.

(i) Choose a primitive irreducible polynomial. To begin with, we need to choose a primitive irreducible polynomial

$$p(x) = 1 + a_1 x + a_2 x^2 + \ldots + a_{m-1} x^{m-1} + x^m, \; a_i = 0 \text{ or } 1,$$

of degree m, where $n = 2^m - 1$. The definition of such a polynomial need not be stated here. For convenience Fig. 5 gives a short table of such polynomials

Fig. 5

Primitive Irreducible Polynomials

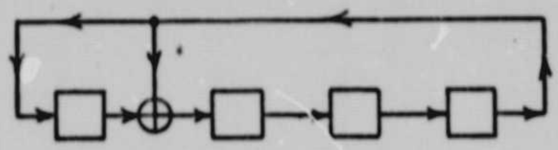
m	p(x)	m	p(x)
1	$1+x$	7	$1+x+x^7$
2	$1+x+x^2$	8	$1+x^4+x^5+x^6+x^8$
3	$1+x+x^3$	9	$1+x^4+x^9$
4	$1+x+x^4$	10	$1+x^3+x^{10}$
5	$1+x^2+x^5$	11	$1+x^2+x^{11}$
6	$1+x+x^6$	12	$1+x^3+x^4+x^7+x^{12}$

Extensive tables of such polynomials are available in the literature [4,5,6,7,8]. For our example with n = 15, we take $p(x) = 1+x+x^4$.

(ii) Form a shift register. Next, we form a linear feedback shift register with m stages, whose feedback connections are described by p(x). For example, the shift register corresponding to $p(x) = 1+x+x^4$ is shown in Fig. 6. Here -□- is a delay element which holds a 0 or a 1, and ⊕ is a mod 2 adder.[9]

Fig. 6

Shift Register Corresponding to $1+x+x^4$



(iii) Form the matrix M. We load this shift register with the initial state (1,0,0,...,0) and let it cycle. Because p(x) was chosen to be a primitive irreducible polynomial, the shift register will go through 2^m-1 states before repeating. It has period 2^m-1. During one period, the successive states of the register give all 2^m-1 nonzero m-tuples.

For example, the first few states of the shift register of Fig. 6 are shown in Fig. 7.

Fig. 7

Successive States of the Shift Register of Fig. 6

State				Number
1	0	0	0	0
0	1	0	0	1
0	0	1	0	2
0	0	0	1	3
1	1	0	0	4
0	1	1	0	5
0	0	1	1	6
1	1	0	1	7
	...			...

We form an m x (2^m-1) matrix M whose columns are the successive states of this shift register. For n = 15, for example, we find from Fig. 6 & 7 that

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array} \\ \left[\begin{array}{ccccccccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array} . \quad (7)$$

In general we have

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ \cdots \\ v_{m-2} \\ v_{m-1} \\ v_m \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 \ldots m-1 \ldots & \ldots \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 \ldots & 1 & \ldots & \ldots \\ 0 & 0 & 0 \ldots & 0 & \cdots & \cdots \\ \cdot & \cdot & \cdot \ldots & \cdot & \cdots & \cdots \\ 0 & 0 & 1 \ldots & 0 \ldots & & \ldots \\ 0 & 1 & 0 \ldots & 0 \ldots & & \ldots \\ 1 & 0 & 0 \ldots & 0 \ldots & & \ldots \end{array}\right] \end{array} . \quad (8)$$

The columns of M consist of all 2^m-1 distinct nonzero binary m-tuples. The rows are labeled $v_1, v_2, \ldots, v_m$ as shown.

Note that in Eq. (7), $v_2 = \mathcal{L}v_1$, $v_3 = \mathcal{L}^2 v_1$, and

$$v_4 = \mathcal{L}^3 v_1 \oplus v_1$$

where $\oplus$ denotes componentwise addition modulo 2, without carries. Also $v_4 = \mathcal{R}v_1 = \mathcal{L}^{-1}v_1$, where $\mathcal{R}$ denotes a cyclic shift of one place to the right. Therefore

$$\mathcal{S}^3 v_1 \oplus v_1 \oplus \mathcal{S}^{-1} v_1 = 0,$$

or

$$(\mathcal{S}^4 \oplus \mathcal{S} \oplus 1) v_1 = 0. \tag{9}$$

In the general case, the same argument applied to Eq. (8) shows that

$$(\mathcal{S}^m \oplus a_{m-1}\mathcal{S}^{m-1} \oplus \ldots \oplus a_1\mathcal{S} \oplus a_0 1) v_1 = 0, \tag{10}$$

i.e.,

$$p(\mathcal{S}) v_1 = 0 \tag{11}$$

(iv) The connection with Galois fields. (This section can be omitted by readers only interested in the statement of the result and not in its derivation.) The successive contents of the shift register in fact give all nonzero elements of the Galois field $GF(2^m)$. These elements can be written in two ways, either as successive powers

$$1, \alpha, \alpha^2, \ldots, \alpha^{2^m-2}$$

of α, where α is a zero of the polynomial $p(x)$, and $\alpha^{2^m-1} = 1$; or equivalently as a polynomial in α of degree less than m, by reading off the contents of the shift register. In the example with $m = 4$ we have

$$
\begin{aligned}
1 &= 1 \\
\alpha &= \alpha \\
\alpha^2 &= \alpha^2 \\
\alpha^3 &= \alpha^3 \\
\alpha^4 &= 1+\alpha \\
\alpha^5 &= \alpha+\alpha^2 \\
\alpha^6 &= \alpha^2+\alpha^3 \\
\alpha^7 &= 1+\alpha+\alpha^3 \\
\cdots & \quad \cdots
\end{aligned}
$$

Thus M is actually a logarithm table for the field $GF(2^m)$! E.g. column 7 of Eq. (7) says that

$$1 + \alpha + \alpha^3 = \alpha^7,$$

or

$$\log_\alpha(1 + \alpha + \alpha^3) = 7.$$

(v) Definition of the matrix S. We can now define S, which is the $(2^m-1) \times (2^m-1)$ matrix whose first row is v_1 (the first row of M), and each successive row is a cyclic left shift of the previous row by one place. If ρ_i denotes the i^{th} row of S (calling the top row the 0^{th} row), we have

$$\rho_i = \mathcal{L}^i v_1, \text{ for } i = 0,\ldots,2^m-2.$$

Indeed Fig. 2 is obtained in this way from Eq. (7).

It can be shown[1] that

$$S^{-1} = \frac{2}{n+1} (2S-J), \qquad (12)$$

where J is the all-ones matrix. In fact by adding a top row and a left column of 0's to S and then changing 1's to -1's and 0's to +1's, we obtain a Hadamard matrix H satisfying $H^2 = (n+1)I$. Figure 4 shows S^{-1} when n = 15.

(vi) Labeling the rows of S. It turns out[10], and this is a key point, that the rows of S consist of all linear combinations of the rows $v_1, \ldots, v_m$ of M, with coefficients 0 or 1 but not all 0. I.e., the rows of S are equal to the vectors

$$v_1, \ldots, v_m, v_1 \oplus v_2, v_1 \oplus v_3, \ldots, v_1 \oplus v_2 \oplus \cdots \oplus v_m$$

arranged in some order. The rows of S can therefore be labeled with the corresponding sum of the v_i's. For example, the rows of S are labeled in this way in Fig. 2. We also use the same labeling for the components of $\underline{z}$. Thus if the i^{th} row of S is labeled $v_r \oplus v_s \oplus \cdots \oplus v_t$, the i^{th} component of $\underline{z}$ is labeled in the same way.

(IV) The impulse response $\underline{z}^{(1)}$

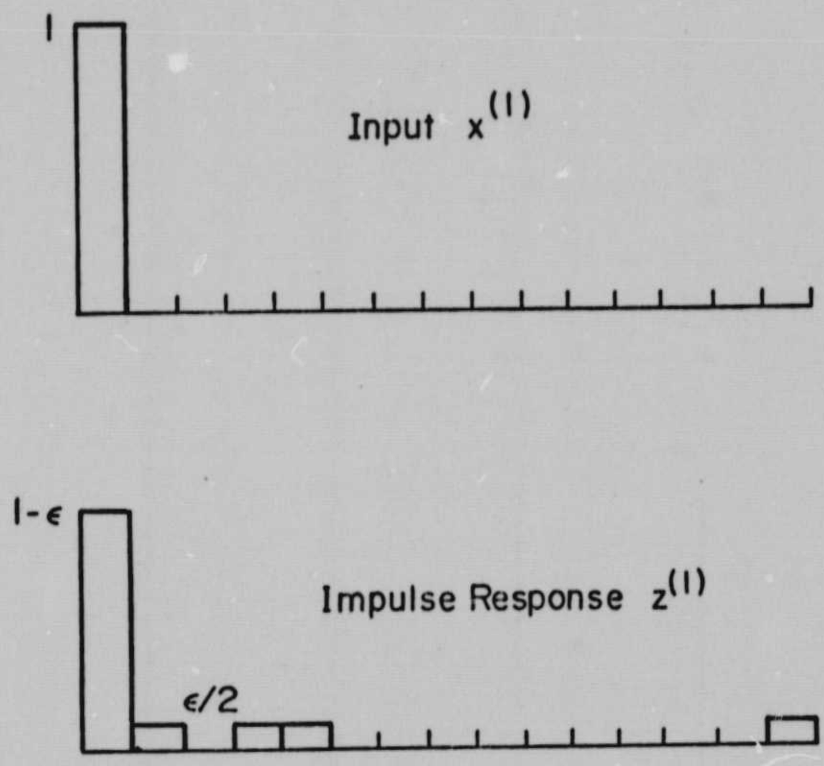
We can now specify the impulse response. If $\underline{x}^{(1)} = (1,0,\ldots,0)^T$ then the impulse response $\underline{z}^{(1)} = S^{-1}S{*}\underline{x}^{(1)}$ is given by

$$\underline{z}^{(1)} = (1-\epsilon,\tfrac{\epsilon}{2},0,\ldots,0,\tfrac{\epsilon}{2},\tfrac{\epsilon}{2},0,\ldots,0,\tfrac{\epsilon}{2})^T, \qquad (13)$$

where the second and third components $\frac{\epsilon}{2}$ occur at the coordinates which are labeled $v_1 + v_m$ and $v_1 + v_2$. In words, $\underline{z}^{(1)}$ has the dominant component $1-\epsilon$ (where $\underline{x}^{(1)}$ had the component 1), surrounded by two blips of height $\frac{\epsilon}{2}$, plus a second pair of blips of the same height which are displaced a certain distance. The proof of this will be given in section V. See Fig. 8, which shows $\underline{z}$ when $n = 15$ (in agreement with Eq. (6)).

Fig. 8

Impulse response $\underline{z}^{(1)}$ when $n = 15$



To discover exactly how far the second pair of blips is displaced, we may proceed as follows. Because of Eq. (10) & (11), there is an isomorphism[11] between the rows of S and the nonzero elements of the field $GF(2^m)$, as follows:

row	Corresponding element of $GF(2^m)$
$\rho_0 = v_1$	$\leftrightarrow 1$
$\rho_1 = v_2 = \mathcal{S}v_1$	$\leftrightarrow \alpha$
$\rho_i = \mathcal{S}^i v_1$	$\leftrightarrow \alpha^i$ for $0 \le i \le 2^m - 2$
$\rho_{2^m-2} = v_m = \mathcal{S}^{-1} v_1$	$\leftrightarrow \alpha^{-1}$.

Therefore the second and third blips in the impulse response occur at the i^{th} and $(i+1)^{st}$ coordinates of $\underline{z}^{(1)}$ (starting the count at 0), where i is the solution of

$$1 \oplus \alpha^{-1} = \alpha^i \qquad (14)$$

in the field $GF(2^m)$. Indeed, for this value of i, we have

$$v_1 \oplus v_m = v_1 \oplus \mathcal{S}^{-1} v_1 = \mathcal{S}^i v_1 = \rho_i .$$

Fortunately the columns of M provide the information needed to solve (14). For example, when n = 15, we have

$$1 \oplus \alpha^{-1} = 1 \oplus \alpha^{14} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \oplus \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \alpha^3,$$

so the solution is $i = 3$. Therefore the blips occur in the 3^{rd} and 4^{th} coordinates of $\underline{z}^{(1)}$ (calling the left-most coordinate the 0^{th}), in agreement with Eq. (6) and Fig. 8.

The case n = 255

S matrices of size 255×255 have been used in Ref. 2 and 3, corresponding to the polynomial

$$p(x) = 1+x^4+x^5+x^6+x^8.$$

The matrix M obtained from this is an 8×255 matrix, and is shown in the appendix (turned on its side and cut into six pieces). Using this appendix Eq. (14) becomes

$$1 \oplus \alpha^{-1} = 1 \oplus \alpha^{254} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \alpha^{230}.$$

Therefore the impulse response is

$$\underline{z}^{(1)} = (1-\epsilon, \tfrac{\epsilon}{2}, 228\ 0\text{'s}, \tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}, 22\ 0\text{'s}, \tfrac{\epsilon}{2})$$

(V) Derivation of the impulse response

If $\underline{x}^{(1)} = (1,0,\ldots,0)^T$ then from Eq. (2), since S^* is symmetric, the transpose of $\underline{y}^*$ is

$$\begin{aligned}(\underline{y}^*)^T &= 1^{st} \text{ row of } S^* \\ &= v_1 + \epsilon w, \text{ (say)},\end{aligned}$$

where w is a vector of 0's, 1's and 2's, and + denotes real vector addition. Observe that w is given by

$$w = \bar{v}_1 * \mathcal{L}v_1 + \bar{v}_1 * \mathcal{R}v_1,$$

where the bar denotes the complement of a vector and * stands for componentwise multiplication,

$$= \bar{v}_1 * v_2 + \bar{v}_1 * v_m$$

$$= w_1 + w_2 \quad \text{(say)},$$

where $w_1 = \bar{v}_1 * v_2$, $w_2 = \bar{v}_1 * v_m$. Then

$$\underline{z}^{(1)} = S^{-1}\underline{y}^* = S^{-1}(v_1^T + \epsilon w_1^T + \epsilon w_2^T)$$

$$= S^{-1}v_1^T + \epsilon 2^{1-m}(2S-J)(w_1^T + w_2^T) \quad \text{from (12)}$$

$$= \underline{x}^{(1)} + \epsilon 2^{1-m}(Gw_1^T + Gw_2^T) \qquad (15)$$

where $G = 2S-J$ is a matrix of +1's and -1's. If ρ is a typical row of S, let

$$\lambda_1 = \langle \rho, w_1 \rangle = \rho w_1^T, \quad \lambda_2 = \langle \rho, w_2 \rangle = \rho w_2^T$$

denote the usual real inner products. Then[12] the component of $Gw_1^T + Gw_2^T$ corresponding to ρ is

$$2(\lambda_1 + \lambda_2) - 2^{m-1}$$

To show this, let $\underline{a}$ be an arbitrary row of G, and let $\underline{\rho} = \frac{1}{2}(\underline{a} + \underline{1})$ be the corresponding row of S, where $\underline{1}$ denotes a vector of all 1's. Also let $\underline{w}$ be an arbitrary vector of 0's and 1's, containing say h 1's. Suppose there are i coordinates where $\underline{\rho}$ and $\underline{w}$ are both 0, j coordinates where $\underline{\rho}$ is 1 and $\underline{w}$ is 0, and k coordinates where $\underline{\rho}$ is 0 and $\underline{w}$ is 1, as shown in the following picture.

$$\begin{array}{ccccc} 1\ldots1 & 1\ldots1 & -\ldots- & -\ldots- & = \underline{a} \\ 1\ldots1 & 1\ldots1 & 0\ldots0 & 0\ldots0 & = \underline{\rho} \\ \underbrace{1\ldots1}_{i} & \underbrace{0\ldots0}_{j} & \underbrace{1\ldots1}_{k} & 0\ldots0 & = \underline{w} \end{array}$$

Then $\underline{aw}^T = i-k$. But $i = \langle\underline{\rho},\underline{w}\rangle$ and $i+k = h$, so

$$\underline{aw}^T = 2\langle\underline{\rho},\underline{w}\rangle - h \quad . \tag{16}$$

This implies that the component of $Gw_1{}^T + Gw_2{}^T$ corresponding to $\underline{\rho}$ is

$$2(\lambda_1 + \lambda_2) - 2^{m-1} \quad . \tag{17}$$

The values of the inner products λ, λ_2 and, from Eq. (15), of the components of $\underline{z}^{(1)}$ are easily calculated, and are shown in Fig. 9.

Figure 9

The components of $Gw_1^T + Gw_2^T$, and of $\underline{z}^{(1)}$

row $\underline{\rho}$ of s	$\lambda_1 = \langle \underline{\rho}, w_1 \rangle$	$\lambda_2 = \langle \underline{\rho}, w_2 \rangle$	component of $Gw_1^T + Gw_2^T$, from (17)	component of $\underline{z}^{(1)}$, from (15)
v_1	0	0	-2^{m-1}	$1-\epsilon$
v_2	2^{m-2}	2^{m-3}	2^{m-2}	$\frac{1}{2}\epsilon$
v_m	2^{m-3}	2^{m-2}	2^{m-2}	$\frac{1}{2}\epsilon$
v_1+v_2	2^{m-2}	2^{m-3}	2^{m-2}	$\frac{1}{2}\epsilon$
v_1+v_m	2^{m-3}	2^{m-2}	2^{m-2}	$\frac{1}{2}\epsilon$
all other rows	2^{m-3}	2^{m-3}	0	0

This completes the proof that the impulse response $\underline{z}^{(1)}$ is given by Eq. (13).

VI. Errors caused by the transparent slits being too narrow

Now let us consider the case where the transparent slits are too narrow. More precisely, we assume that in each row of S,

> every 1 which is adjacent to two 1's is unchanged,
>
> every 1 which is adjacent to a single 1 is replaced by $1 - \epsilon$, and
>
> every 1 which is adjacent to two 0's is replaced by $1 - 2\epsilon$,

while the 0's are unchanged. Let S^{**} be the distorted matrix obtained from S in this way. Fig. 10 shows the distorted matrix corresponding to Fig. 2, where a denotes $1 - \epsilon$ and b denotes $1 - 2\epsilon$.

Figure 10

The Distorted Matrix S^{**} When n = 15

S^{**} =

			b			a	a		b		a	1	1	a
		b			a	a		b		a	1	1	a	
	b			a	a		b		a	1	1	a		
b			a	a		b		a	1	1	a			
		a	a		b		a	1	1	a				b
	a	a		b		a	1	1	a				b	
a	a		b		a	1	1	a				b		
a		b		a	1	1	a				b			a
	b		a	1	1	a				b			a	a
b		a	1	1	a				b			a	a	
	a	1	1	a				b			a	a		b
a	1	1	1				b			a	a		b	
1	1	a				b			a	a		b		a
1	a				b			a	a		b		a	1
a				b			a	a		b		a	1	1

Now the input

$$\underline{x} = \underline{x}^{(1)} = (1,0,0,\ldots,0)^T$$

produces as output the impulse response

$$\underline{z}^2 = S^{-1} S^{**} \underline{x}^{(1)} \quad .$$

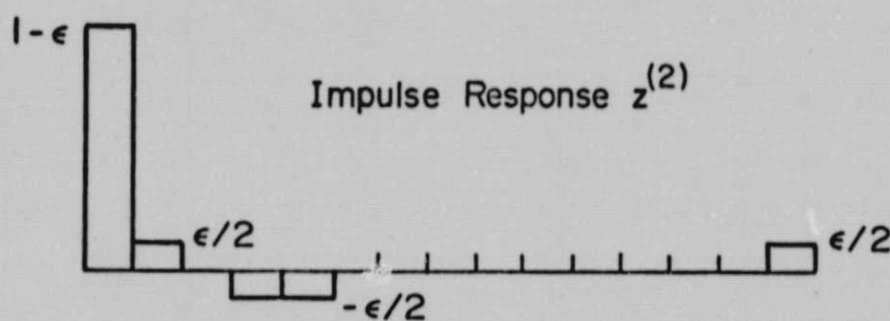
This is given by

$$\underline{z}^{(2)} = (1-\epsilon, \tfrac{\epsilon}{2}, 0, \ldots, 0, -\tfrac{\epsilon}{2}, -\tfrac{\epsilon}{2}, 0, \ldots, 0, \tfrac{\epsilon}{2})^T, \quad (18)$$

where the components $\frac{\epsilon}{2}$ occur at the coordinates which are labeled $v_1 + v_m$ and $v_1 + v_2$. Fig. 11 shows $\underline{z}^{(2)}$ when $n = 15$. Note that $\underline{z}^{(2)}$ is the same as $\underline{z}^{(1)}$ except for the sign of the second pair of blips.

Figure 11

Impulse response $\underline{z}^{(2)}$ when $n = 15$



The proof of Eq. (18) is similar to the proof of Eq. (13), except that instead of Eq. (15) one has

$$\underline{z}^{(2)} = \underline{x}^{(1)} - \epsilon 2^{1-m} (2Gv_1^T - Gw_3^T - Gw_4^T) ,$$

where $w_3 = v_1 * v_2$ and $w_4 = v_1 * v_m$. Also instead of Eq. (17), one has that the component of $2Gv_1^T - Gw_3^T - Gw_4^T$ corresponding to $\underline{\rho}$ is

$$4\langle \underline{\rho}, v_1 \rangle - 2\langle \underline{\rho}, w_3 \rangle - 2\langle \underline{\rho}, w_4 \rangle - 2^{m-1} .$$

The details of the proof are left to the reader.

The part of this study that was done at Cornell University was supported by NASA contract NGR 33-010-210.

REFERENCES

1. N. J. A. Sloane, T. Fine, P. G. Phillips, and M. Harwit, Appl. Optics, 8, 2103 (1969).

2. J. A. Decker, Jr., Appl. Optics, 10, 510 (1971).

3. M. H. Tai, D. A. Briotta, Jr., N. Kamath, and M. Harwit, A practical multi-spectrum Hadamard transform spectrometer,

4. W. W. Peterson and E. J. Weldon, Jr., Error-Correcting Codes, 2nd Edition, (M.I.T. Press, Cambridge, Mass. 1972).

5. J. D. Alanen and D. E. Knuth, Sankyā, Series A, 26, 305 (1964).

6. R. W. Marsh, Tables of Irreducible Polynomials over GF(2) Through Degree 19 (Office of Technical Services, Dept. of Commerce, Wash., D.C., October 24, 1957).

7. S. Mossige, Math. Comp., 26, 1007 (1972).

8. W. Stahnke, Math. Comp., 27, 977 (1973).

9. Strictly speaking this is a half-adder, since there are no carries. For more about shift registers, see for example Ch. 7 of Ref. 4.

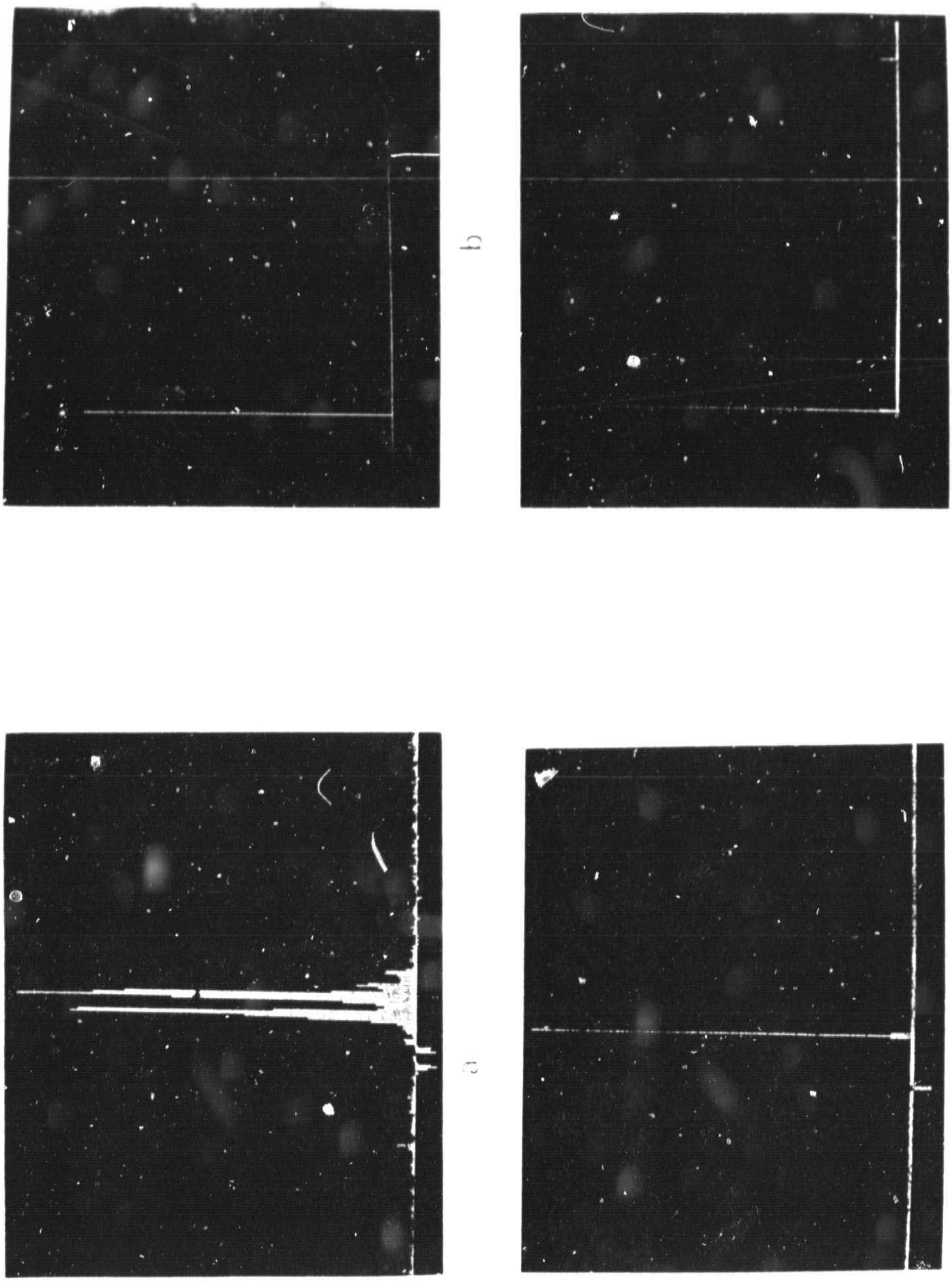
10. This is a standard result in coding theory, and expresses the fact that the rows $v_1, \ldots, v_m$ of M generate a cyclic code $\mathcal{C}$, of length $n = 2^m - 1$, dimension m, and Hamming distance 2^{m-1} between codewords. The rows of S are exactly the nonzero codewords of $\mathcal{C}$. See for example Ref. 4, or the forthcoming book F. J. MacWilliams and N. J. A. Sloane, Combinatorial Coding Theory (North-Holland, Amsterdam, in preparation).

11. In other words the code $\mathcal{C}$ is isomorphic to $GF(2^m)$.

12. In fact we are here finding the weight distribution of the cosets of $\mathcal{C}$ which contain w_1 and w_2. For more about the weight distribution of these cosets see N. J. A. Sloane and R. J. Dick, IEEE Internat. Conf. on Commun., Montreal 1971, Vol. 7, pp. 36-2 to 36-6 (IEEE Press, N.Y., 1972); and E. R. Berlekamp and L. R. Welch, IEEE Trans. Info. Theory, IT-18, 203 (1972). Another application of these cosets in studying Hadamard transforms is given by E. R. Berlekamp, Bell Syst. Tech. J., 49, 969 (1970).

Figure 1 Caption

Spectrum of the 1.7 micron mercury vapor doublet (a). Peaks extending downward from the top of the picture are negative values which ordinarily would fall the same amount below the abscissa, but, in our form of display, appear at the top of the graph. (b) Shows the response we would obtain to a single spectral line with a perfect mask. (c) Shows the response for a single line with the radiation simulated as passing through a mask with slits too narrow because each opaque mask element protrudes into the adjacent transparent slot by a tenth of a slot width. (d) Shows the effect of simulating slits that are systematically too wide. Note that the main spectral line has been placed in different positions for the synthetic runs (b), (c) and (d).

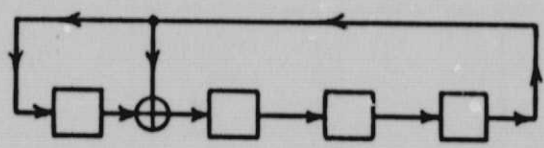

b
d
a
c

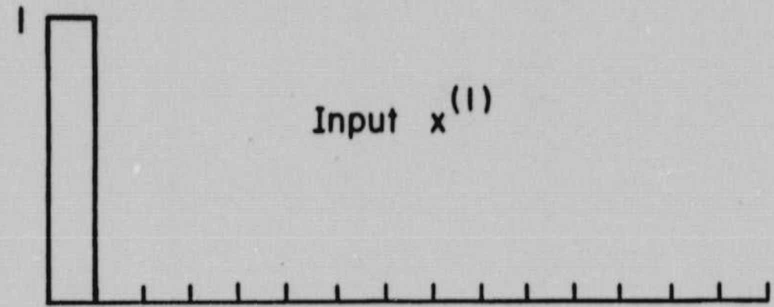
S =

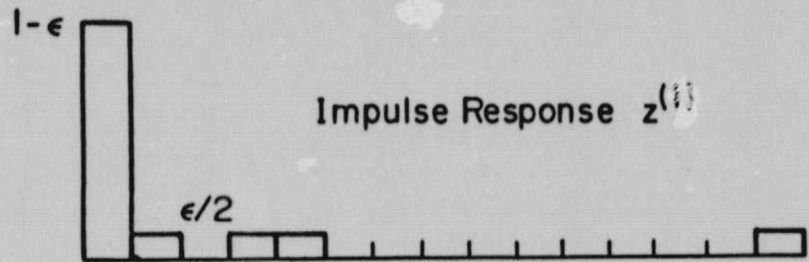
			1			1	1		1		1	1	1	1	V_1
		1			1	1		1		1	1	1	1		V_2
	1			1	1		1		1	1	1	1			V_3
1			1	1		1		1	1	1	1				$V_1 \oplus V_4$
		1	1		1		1	1	1	1				1	$V_1 \oplus V_2$
	1	1		1		1	1	1	1				1		$V_2 \oplus V_3$
1	1		1		1	1	1	1				1			$V_1 \oplus V_3 \oplus V_4$
1		1		1	1	1	1				1			1	$V_2 \oplus V_4$
	1		1	1	1	1				1			1	1	$V_1 \oplus V_3$
1		1	1	1	1				1			1	1		$V_1 \oplus V_2 \oplus V_4$
	1	1	1	1				1			1	1		1	$V_1 \oplus V_2 \oplus V_3$
1	1	1	1				1			1	1		1		$V_1 \oplus V_2 \oplus V_3 \oplus V_4$
1	1	1				1			1	1		1		1	$V_2 \oplus V_3 \oplus V_4$
1	1				1			1	1		1		1	1	$V_3 \oplus V_4$
1				1			1	1		1		1	1	1	V_4

$$
S^{*} = \begin{bmatrix}
\varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 \\
0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon \\
\varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 \\
1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon \\
\varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 \\
\varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon \\
1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon \\
1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 \\
2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 \\
1 & 2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon \\
2\varepsilon & 1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 \\
1 & 1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon \\
1 & 1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 \\
1 & 1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 \\
1 & \varepsilon & 0 & \varepsilon & 1 & \varepsilon & \varepsilon & 1 & 1 & 2\varepsilon & 1 & 2\varepsilon & 1 & 1 & 1
\end{bmatrix}
$$

$$S^{-1} = \frac{1}{8}\left[\begin{array}{ccccccccccccccc}
- & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 \\
- & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - \\
- & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & + & - \\
1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - \\
- & - & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 \\
- & 1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - \\
1 & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - \\
1 & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 \\
1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - \\
- & 1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - \\
1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 \\
1 & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 \\
1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & 1
\end{array}\right]$$

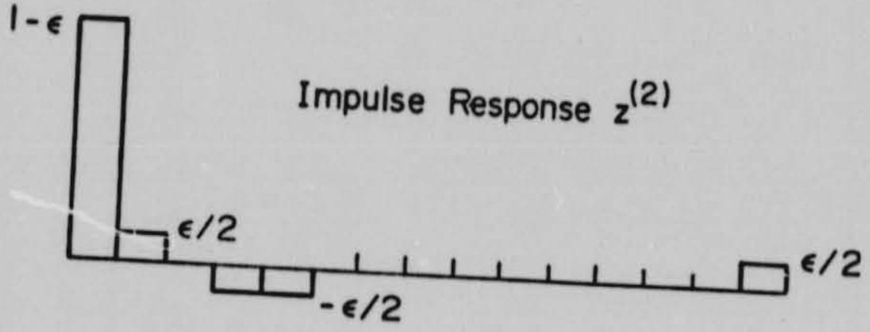



1
Input $x^{(1)}$


1-ϵ
Impulse Response $z^{(1)}$
ϵ/2

$S^{**} =$

			b			a	a		b		a	l	l	a
		b			a	a		b		a	l	l	a	
	b			a	a		b		a	l	l	a		
b			a	a		b		a	l	l	a			
		a	a		b		a	l	l	a				b
	a	a		b		a	l	l	a				b	
a	a		b		a	l	l	a				b		
a		b		a	l	l	a				b			a
	b		a	l	l	a				b			a	a
b		a	l	l	a				b			a	a	
	a	l	l	a				b			a	a		b
a	l	l	l				b			a	a		b	
l	l	a				b			a	a		b		a
l	a				b			a	a		b		a	l
a				b			a	a		b		a	l	l


1-ε
Impulse Response z(2)
ε/2
-ε/2
ε/2

APPENDIX: The Galois Field GF(2^8), defined by $x^8+x^6+x^5+x^4+1 = 0$

0	1	2	3	4	5	6	7	log	0	1	2	3	4	5	6	7	log	0	1	2	3	4	5	6	7	log
								-∞	1		1		1	1			42	1	1		1	1		1	1	85
1								0		1		1		1	1		43	1	1	1				1	1	86
	1							1			1		1		1	1	44	1	1	1	1	1	1	1	1	87
		1						2	1			1	1		1	1	45	1	1	1	1				1	88
			1					3	1	1					1	1	46	1	1	1	1		1	1		89
				1				4	1	1	1		1	1	1	1	47		1	1	1	1		1	1	90
					1			5	1	1	1	1	1			1	48	1		1	1			1	1	91
						1		6	1	1	1	1			1		49	1	1		1		1	1	1	92
							1	7		1	1	1	1			1	50	1	1	1			1		1	93
1				1	1	1		8	1		1	1			1		51	1	1	1	1	1	1			94
	1				1	1	1	9		1		1	1			1	52		1	1	1	1	1	1		95
1		1		1	1		1	10	1		1				1		53			1	1	1	1	1	1	96
1	1		1	1				11		1		1				1	54	1			1				1	97
	1	1		1	1			12	1		1			1	1		55	1	1				1	1		98
		1	1		1	1		13		1		1			1	1	56		1	1				1	1	99
			1	1		1	1	14	1		1			1	1	1	57	1		1	1	1	1	1	1	100
1						1	1	15	1	1		1	1	1		1	58	1	1		1				1	101
1	1			1	1	1	1	16	1	1	1						59	1	1	1			1	1		102
1	1	1		1			1	17		1	1	1					60		1	1	1			1	1	103
1	1	1	1	1		1		18			1	1	1				61	1		1	1		1	1	1	104
	1	1	1	1	1		1	19				1	1	1			62	1	1		1		1		1	105
1		1	1					20					1	1	1		63	1	1	1			1			106
	1		1	1				21						1	1	1	64		1	1	1			1		107
		1		1	1			22	1				1	1		1	65			1	1	1			1	108
			1		1	1		23	1	1			1				66	1			1			1		109
				1		1	1	24		1	1			1			67		1			1			1	110
1				1		1	1	25			1	1			1		68	1		1		1		1		111
1	1			1		1	1	26				1	1			1	69		1		1		1		1	112
1	1	1		1		1	1	27	1						1		70	1		1			1			113
1	1	1	1	1		1	1	28		1						1	71		1		1			1		114
1	1	1	1			1	1	29	1		1		1	1	1		72			1		1			1	115
1	1	1	1		1	1	1	30		1		1		1	1	1	73	1			1	1		1		116
1	1	1	1		1		1	31	1		1			1		1	74		1			1	1		1	117
1	1	1	1		1			32	1	1		1	1	1			75	1		1		1				118
	1	1	1	1		1		33		1	1		1	1	1		76		1		1		1			119
		1	1	1	1		1	34			1	1		1	1	1	77			1		1		1		120
1			1					35	1			1		1		1	78				1		1		1	121
	1			1				36	1	1				1			79	1					1			122
		1			1			37		1	1				1		80		1					1		123
			1			1		38			1	1				1	81			1					1	124
				1			1	39	1			1		1	1		82	1			1	1	1	1		125
1				1		1		40		1			1		1	1	83		1			1	1	1	1	126
	1				1		1	41	1		1		1		1	1	84	1		1		1			1	127

The Galois Field GF(2^8), continued

0	1	2	3	4	5	6	7	log
1	1		1	1		1		128
	1	1		1	1		1	129
1		1	1	1				130
	1		1	1	1			131
		1		1	1	1		132
			1		1	1	1	133
1					1		1	134
1	1			1	1			135
	1	1			1	1		136
		1	1			1	1	137
1			1		1	1	1	138
1	1				1		1	139
1	1	1		1	1			140
	1	1	1		1	1		141
		1	1	1		1	1	142
1			1			1	1	143
1	1				1	1	1	144
1	1	1		1	1		1	145
1	1	1	1	1				146
	1	1	1	1	1			147
		1	1	1	1	1		148
			1	1	1	1	1	149
1							1	150
1	1			1	1	1		151
	1	1			1	1	1	152
1		1	1	1	1		1	153
1	1		1					154
	1	1		1				155
		1	1		1			156
			1	1		1		157
				1	1		1	158
1				1				159
	1				1			160
		1				1		161
			1				1	162
1					1	1		163
	1					1	1	164
1		1		1	1	1	1	165
1	1		1	1			1	166
1	1	1				1		167
	1	1	1				1	168
1		1	1		1	1		169
	1		1	1		1	1	170

0	1	2	3	4	5	6	7	log
1		1				1	1	171
1	1		1	1	1	1	1	172
1	1	1					1	173
1	1	1	1	1	1	1		174
	1	1	1	1	1	1	1	175
1		1	1				1	176
1	1		1		1	1		177
	1	1		1		1	1	178
1		1	1	1		1	1	179
1	1		1			1	1	180
1	1	1			1	1	1	181
1	1	1	1	1	1		1	182
1	1	1	1					183
	1	1	1	1				184
		1	1	1	1			185
			1	1	1	1		186
				1	1	1	1	187
1				1			1	188
1	1			1		1		189
	1	1			1		1	190
1		1	1	1	1			191
	1		1	1	1	1		192
		1		1	1	1	1	193
1			1	1			1	194
1	1					1		195
	1	1					1	196
1		1	1	1	1	1		197
	1		1	1	1	1	1	198
1		1					1	199
1	1		1	1	1	1		200
	1	1		1	1	1	1	201
1		1	1	1			1	202
1	1		1			1		203
	1	1		1			1	204
1		1	1	1		1		205
	1		1	1	1		1	206
1		1						207
	1		1					208
		1		1				209
			1		1			210
				1		1		211
					1		1	212
1				1	1			213

0	1	2	3	4	5	6	7	log
	1				1	1		214
		1				1	1	215
1			1	1	1	1	1	216
1	1						1	217
1	1	1		1	1	1		218
	1	1	1		1	1	1	219
1		1	1		1		1	220
1	1		1		1			221
	1	1		1		1		222
		1	1		1		1	223
1			1		1			224
	1			1		1		225
		1			1		1	226
1			1	1	1			227
	1			1	1	1		228
		1			1	1	1	229
1			1	1	1		1	230
1	1							231
	1	1						232
		1	1					233
			1	1				234
				1	1			235
					1	1		236
						1	1	237
1				1	1	1	1	238
1	1			1			1	241
1	1	1		1		1		240
	1	1	1		1		1	241
1		1	1		1			242
	1		1	1		1		243
		1		1	1		1	244
1			1	1				245
	1			1	1			246
		1			1	1		247
			1			1	1	248
1					1	1	1	249
1	1			1	1		1	250
1	1	1		1				251
	1	1	1		1			252
		1	1	1		1		253
			1	1	1		1	254
1								255